Contents

Categorification of the Orbit-Stabilizer Theorem

The left G-set for a set S is $X = \{g \cdot s | g \in S, s \in S\}$. Let $X : \mathbf{B}G \to \mathbf{Set}$ be such a left G-set. The translation groupoid (action groupoid in other sources) $\mathbf{T}_G X$ has elements of the set X as objects.

For example let $R \times K^n \to Ax \in K^n$ where R is the group of $\pi/2$ rotation matrices be such a left G-set. Then $\mathbf{B}G$ has a single object s and morphisms I, G, G^2, G^3 where $R = \langle G \rangle$.

Then $X : \mathbf{B}G \to \mathbf{Set}$ is a functor taking s to the left G-set and the same morphisms as before.

$$G = \left\{ I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, G^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, G^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Applying this group to the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get the left G-set

$$X_{\mathbf{u}} : s \to \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

And applying it to $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get

$$X_{\mathbf{v}} : s \to \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The **translation groupoid** for these sets consists of the following diagrams

$$\begin{tikzcd}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \arrow[r, "G"] \arrow[d, "G"'] \arrow[dr, "G^2"] & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \arrow[d, "G"] \arrow[dl, "G^2"'] \\
\begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \end{pmatrix} \arrow[l, "G"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \arrow[r, "G"] \arrow[d, "G"'] \arrow[dr, "G^2"] & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \arrow[d, "G"] \arrow[dl, "G^2"'] \\
\begin{pmatrix} -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} \arrow[l, "G"]
\end{tikzcd}$$

(including the inverses $G^{-1} = G^3$ and $(G^2)^{-1} = G^2$)

Morphisms $g : x \to y$ are elements of $g \in G$ such that $g \cdot x = y$.

The orbit $O_x = \{g \cdot x | g \in G\}$. There are two orbits which we can observe from the diagrams. They are

$$O_{\mathbf{u}} = \{\mathbf{u}, G\mathbf{u}, G^2\mathbf{u}, G^3\mathbf{u}\}$$

$$O_{\mathbf{v}} = \{\mathbf{v}, G\mathbf{v}, G^2\mathbf{v}, G^3\mathbf{v}\}$$

We can pick one of these diagrams to form the skeleton $\mathbf{skT}_G X$ which is equivalent to any of these orbit partitions.

Take an $x \in X$ as a representative and the orbit O_x. Then by the previous section's mentioned equivalency of categories $\mathbf{skC} \hookrightarrow \mathbf{C}$ we must have

$$\mathrm{Hom}_{\mathbf{skT}_G X}(O_x, O_y) \cong \mathrm{Hom}_{\mathbf{T}_G X}(x, x) = G_x$$

These are the automorphisms of x, the $g \in G$ such that $g \cdot x = x$.

The **skeleton** of the translation groupoid $\mathbf{skT}_G X$ is simply the two objects $O_{\mathbf{u}}, O_{\mathbf{v}}$ with no arrows between them.