Contents

Isomorphisms $A \cong TA \oplus (A/TA)$ are not Natural in $A \in \mathbf{Ab}$

Epimorphism $f : A \twoheadrightarrow B$ means $hf = kf \implies h = k$ which corresponds to the surjective property. We can see this by setting $h : B \to B/\operatorname{im} f$ and $k : B \to \operatorname{im} f$. Then since f is epic, $h = k$. Let $b \in B$, then $h(b) = k(b) \implies b \in \operatorname{im} f$ and so f is surjective.

Monomorphism $f : A \rightarrowtail B$ means $fh = fk \implies h = k$ which corresponds to the injective property.

We see the canonical quotient map $A \twoheadrightarrow A/TA$ is clearly surjective and so epic.

The inclusion map $A/TA \rightarrowtail TA \oplus (A/TA)$ is injective/monic by $a + TA \to (0, a + TA)$.

Lastly we have $A \cong TA \oplus (A/TA)$, giving us the composite

$$A \twoheadrightarrow A/TA \rightarrowtail TA \oplus (A/TA) \cong A$$

Our aim is to show this endomorphism is not natural.

Every Natural Endomorphism of $1_{\mathbf{Ab}}$ is Multiplication by some $n \in \mathbb{Z}$

Clearly for $\alpha : 1_{\mathbf{Ab}} \implies 1_{\mathbf{Ab}}$, this $n = 1$.

Observe that homomorphisms $\mathbb{Z} \xrightarrow{a} A$ correspond bijectively to elements $a \in A$ by $1 \to a$.

We have the following diagram, noting that since α is supposed to be natural, the diagram should commute:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}} = [n]"] \arrow[d, "a"'] & \mathbb{Z} \arrow[d, "a"] \\
A \arrow[r, "\alpha_A"] & A
\end{tikzcd}$$

which means we define $\alpha_A = n \cdot a$.

The Endofunctor $A \to TA \oplus (A/TA)$

Let us define $F : \mathbf{Ab} \to \mathbf{Ab}$ for a morphism $f : A \to B$ giving us a morphism

$$Ff : TA \oplus (A/TA) \to TB \oplus (B/TB)$$

- $f(TA) \subseteq TB$, so $f|_{TA} : TA \to TB$ is a valid restriction.
- We can define $g : A/TA \to B/TB$ by $g(a + TA) = f(a) + TB$.
  - This is well defined since $a + TA = a' + TA \implies a - a' \in TA$, but by above that $f(TA) \subseteq TB$, so $f(a - a') \in TB \implies g(a + TA)\ g(a' + TA)$.
- Now put $Ff = f|_{TA} \oplus g$ which is: $Ff(a, a' + TA) = (f(a), f(a') + TB)$.

All 3 operations: taking torsion, modding out by torsion, and taking direct sums are valid functors.

Saying there's natural isomorphisms $A \cong TA \oplus (A/TA)$ means there's a natural isomorphism $\alpha : F \implies 1_{\mathbf{Ab}}$.

$\beta : 1_{\mathbf{Ab}} \Longrightarrow G$ is a Natural Transformation

$$\beta : 1_{\mathbf{Ab}} \Longrightarrow G$$
$$\beta_A : A \to GA = A/TA$$

Let $f : A \to B$

$$\begin{tikzcd}
a \arrow[r, "\beta_A"] \arrow[d, "f"'] & Ga \arrow[d, "Gf=g"] \\
b \arrow[r, "\beta_B"] & Gb
\end{tikzcd}$$

so we expect that

$$\beta_B f a = g \beta_A a$$

where $g(a + TA) = f(a) + TB$ so

$$\begin{aligned}
\beta_B f a &= f(a) + TB \\
g \beta_A a &= g(a + TA) \\
&= f(a) + TB
\end{aligned}$$

with the other transformation being the inclusion $\iota_A(a + TA) = (0, a + TA)$.

"Natural Endomorphism of the Identity Functor"

> "the hypothesized natural isomorphism would define a natural endomorphism of the identity functor on **Ab**."

We showed there's a natural transformation $\beta : 1_{\mathbf{Ab}} \Longrightarrow G$ which is the composition $A \to A/TA \to TA \oplus (A/TA)$.

We seek to prove there's a natural transformation $\alpha : G \Longrightarrow 1_{\mathbf{Ab}}$.

The composition $\alpha \circ \beta : 1_{\mathbf{Ab}} \Longrightarrow G \Longrightarrow 1_{\mathbf{Ab}}$ must also be natural.

We now see this is impossible.

Impossibility of Hypothesis

Let $A = \mathbb{Z}, B = \mathbb{Z}/2n\mathbb{Z}$ and $\phi : A \to A/TA \to TA \oplus (A/TA)$.

$$\begin{tikzcd}
B \arrow[r, "\alpha_B"] & B \\
\mathbb{Z} \arrow[u, "b"] \arrow[r, "\alpha_{\mathbb{Z}}={[n]}"] \arrow[d, "a"'] & \mathbb{Z} \arrow[u, "b"'] \arrow[d, "a"] \\
A \arrow[r, "\alpha_A"] & A
\end{tikzcd}$$

We see that $\phi(A) = (0, \mathbb{Z}) \Longrightarrow n = \pm 1$. Likewise $\phi(B) = (0, 0) \Longrightarrow n = 0_{\mathbb{Z}/2n\mathbb{Z}}$ but $n \neq 0 \in \mathbb{Z}/2n\mathbb{Z}$ which is a contradiction.

Ex

1.4.ii: Natural Transform between Parallel Functors in BA

We view the group A as a category with one object s_A with the Hom-set $\mathrm{End}(s_A) = A$ and composition by the group law. Likewise for B.

If $F, G : A \to B$ are functors, then they both consist of the following data: $F(s_A) = s_B$ and for all $g_1, g_2 \in \mathrm{End}(s_A), F(g_1)F(g_2) = F(g_1 g_2) \in \mathrm{End}(s_B)$ (likewise for G).

A natural transformation $\alpha : F \Longrightarrow G$ consists of the following data:

1. $\alpha_{s_A} \in \mathrm{End}(s_B) = B$
2. For all $h \in \mathrm{End}(s_A) = A$, we have $G(h)\alpha_{s_A} = \alpha_{s_A} F(h)$.

Noting that since B is a group there exists $\alpha_{s_A}^{-1}$, we can rewrite (2) as

$$G(h) = \alpha_{s_A} F(h) \alpha_{s_A}^{-1}$$

That is that the group elements $G(h), F(h) \in B$ are conjugates of each other.

1.4.v: Comma Category Natural Transform $\alpha : F\,\mathrm{dom} \Longrightarrow G\,\mathrm{cod}$

Comma Category

$$\begin{tikzcd}
d \arrow[r] \arrow[d, "h"] & Fd \arrow[r, "f"] \arrow[d, "Fh"] & Ge \arrow[d, "Gk"] & e \arrow[l] \arrow[d, "k"] \\
d' \arrow[r] & Fd' \arrow[r, "f"] & Ge' & e' \arrow[l] \\
D & C & C & E
\end{tikzcd}$$

$$f : Fd \to Ge \in C$$

Comma category $F \downarrow G$:

- objects (d, e, f)
- morphisms $(d, e, f) \to (d', e', f')$

$$\mathrm{dom} : F \downarrow G \to D$$

$$\mathrm{cod} : F \downarrow G \to E$$

Globular Diagram

$$\begin{tikzcd}
F \downarrow G \arrow[r, "\mathrm{cod}"] \arrow[d, "\mathrm{dom}"] & E \arrow[d, "G"] \\
D \arrow[r, "F"] \arrow[ur, Rightarrow, "\alpha"] & C
\end{tikzcd}$$

$$\alpha : F\,\mathrm{dom} \Longrightarrow G\,\mathrm{cod}$$

$$F\,\mathrm{dom} : F \downarrow G \xrightarrow{\mathrm{dom}} D \xrightarrow{F} C$$

$$G\,\mathrm{cod} : F \downarrow G \xrightarrow{\mathrm{cod}} E \xrightarrow{G} C$$

$$\begin{tikzcd}
F \downarrow G \arrow[r, bend left, "F\,\mathrm{dom}"] \arrow[r, bend right, "G\,\mathrm{cod}"'] \arrow[r, phantom, "\Downarrow\alpha"] & C
\end{tikzcd}$$

Natural Transformation $\alpha : F\,\mathrm{dom} \Longrightarrow G\,\mathrm{cod}$

$$F\,\mathrm{dom}, G\,\mathrm{cod} : F \downarrow G \rightrightarrows C$$

$$\alpha : F\,\mathrm{dom} \Longrightarrow G\,\mathrm{cod}$$

Data of natural transform is:

For each $(d, e, f) \in F \downarrow G$,

$$\alpha_{(d,e,f)} : F\,\mathrm{dom}(d, e, f) \to G\,\mathrm{cod}(d, e, f) \in C$$

Let

$$z : (d, e, f) \to (d', e', f') \in F \downarrow G$$

$$\begin{tikzcd}
F\operatorname{dom}(d,e,f) \arrow[r, "\alpha_{(d,e,f)}"] \arrow[d, "F\operatorname{dom} z"'] & G\operatorname{cod}(d,e,f) \arrow[d, "G\operatorname{cod} z"] \\
F\operatorname{dom}(d',e',f') \arrow[r, "\alpha_{(d',e',f')}"] & G\operatorname{cod}(d',e',f')
\end{tikzcd}$$

which is equivalent to

$$\begin{tikzcd}
Fd \arrow[r, "\alpha_{(d,e,f)}"] \arrow[d, "Fh"'] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "\alpha_{(d',e',f')}"] & Ge'
\end{tikzcd}$$

so we can take $\alpha_{(d,e,f)} : F\operatorname{dom}(d,e,f) = Fd \to G\operatorname{cod}(d,e,f) = Ge \in C$ to be $\alpha_{(d,e,f)} = f : Fd \to Ge \in G$ which commutes as expected.

Categorification of the Orbit-Stabilizer Theorem

> The orbit of x is "everything that can be reached from x by an action of something in G".
>
> The stabilizer of x is "the set of all elements of G which don't move x when they act on x".

The left G-set for a set S is $X = \{g \cdot s | g \in S, s \in S\}$. Let $X : \mathbf{B}G \to \mathbf{Set}$ be such a left G-set. The translation groupoid (action groupoid in other sources) $\mathbf{T}_G X$ has elements of the set X as objects.

For example let $R \times K^n \to Ax \in K^n$ where R is the group of $\pi/2$ rotation matrices be such a left G-set. Then $\mathbf{B}G$ has a single object s and morphisms I, G, G^2, G^3 where $R = \langle G \rangle$.

Then $X : \mathbf{B}G \to \mathbf{Set}$ is a functor taking s to the left G-set and the same morphisms as before.

$$G = \left\{ I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, G^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, G^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Applying this group to the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we get the left G-set

$$X_{\mathbf{u}} : s \to \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

And applying it to $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get

$$X_{\mathbf{v}} : s \to \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

The **translation groupoid** for these sets consists of the following diagrams

$$\begin{tikzcd}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} \arrow[r, "G"] \arrow[d, "G"'] \arrow[dr, "G^2"] & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \arrow[d, "G"] \arrow[dl, "G^2"'] \\
\begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} -1 \\ 0 \end{pmatrix} \arrow[l, "G"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} 1 \\ 1 \end{pmatrix} \arrow[r, "G"] \arrow[d, "G"'] \arrow[dr, "G^2"] & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \arrow[d, "G"] \arrow[dl, "G^2"'] \\
\begin{pmatrix} -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} \arrow[l, "G"]
\end{tikzcd}$$

(including the inverses $G^{-1} = G^3$ and $(G^2)^{-1} = G^2$)

Morphisms $g : x \to y$ are elements of $g \in G$ such that $g \cdot x = y$.

The orbit $O_x = \{g \cdot x | g \in G\}$. There are two orbits which we can observe from the diagrams. They are

$$O_{\mathbf{u}} = \{\mathbf{u}, G\mathbf{u}, G^2\mathbf{u}, G^3\mathbf{u}\}$$

$$O_{\mathbf{v}} = \{\mathbf{v}, G\mathbf{v}, G^2\mathbf{v}, G^3\mathbf{v}\}$$

We can pick one of these diagrams to form the skeleton $\mathbf{skT}_G X$ which is equivalent to any of these orbit partitions.

Take an $x \in X$ as a representative of its orbit O_x. Then by the previous section's mentioned equivalency of categories $\mathbf{skC} \hookrightarrow \mathbf{C}$ we must have

$$\mathrm{Hom}_{\mathbf{skT}_G X}(O_x, O_y) \cong \mathrm{Hom}_{\mathbf{T}_G X}(x, x) = G_x$$

These are the automorphisms of x, the $g \in G$ such that $g \cdot x = x$. G_x is the stabilizer of x with respect to G.

The **skeleton** of the translation groupoid $\mathbf{skT}_G X$ is simply the two objects $O_{\mathbf{u}}, O_{\mathbf{v}}$ with no arrows between them. They are discrete, and there's no morphisms between the objects.

This argument implies any pair of elements in the same orbit, must have isomorphic stabilizers.

The skeleton of the translation groupoid, as a category, is the disjoint union of the stabilizer groups, indexed by the orbits O_x. In our example, the stabilizer groups are trivial.

The set of morphisms for the translation groupoid with domain x is isomorphic to G. That is $\mathrm{Hom}_{\mathbf{T}_G X}(x, y)$ for all $y \in O_x$. Each of these hom-sets is isomorphic to $\mathrm{Hom}_{\mathbf{T}_G X}(x, x) = G_x$.

$$|G| = |O_x||G_x|$$

D_4

$$\begin{aligned} G &= \{a, b : a^4 = e, b^2 = e, bab^{-1} = a^{-1}\} \\ &= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\} \end{aligned}$$

Let our set X be the verices of a square

$$X = \{1, 2, 3, 4\}$$

The orbit of each vertex x is the entire group since vertices can be sent to any other vertex through rotations,

$$O_x = \{1, 2, 3, 4\}$$

The stabilizer for 3 is

$$G_3 = \{e, b\}$$

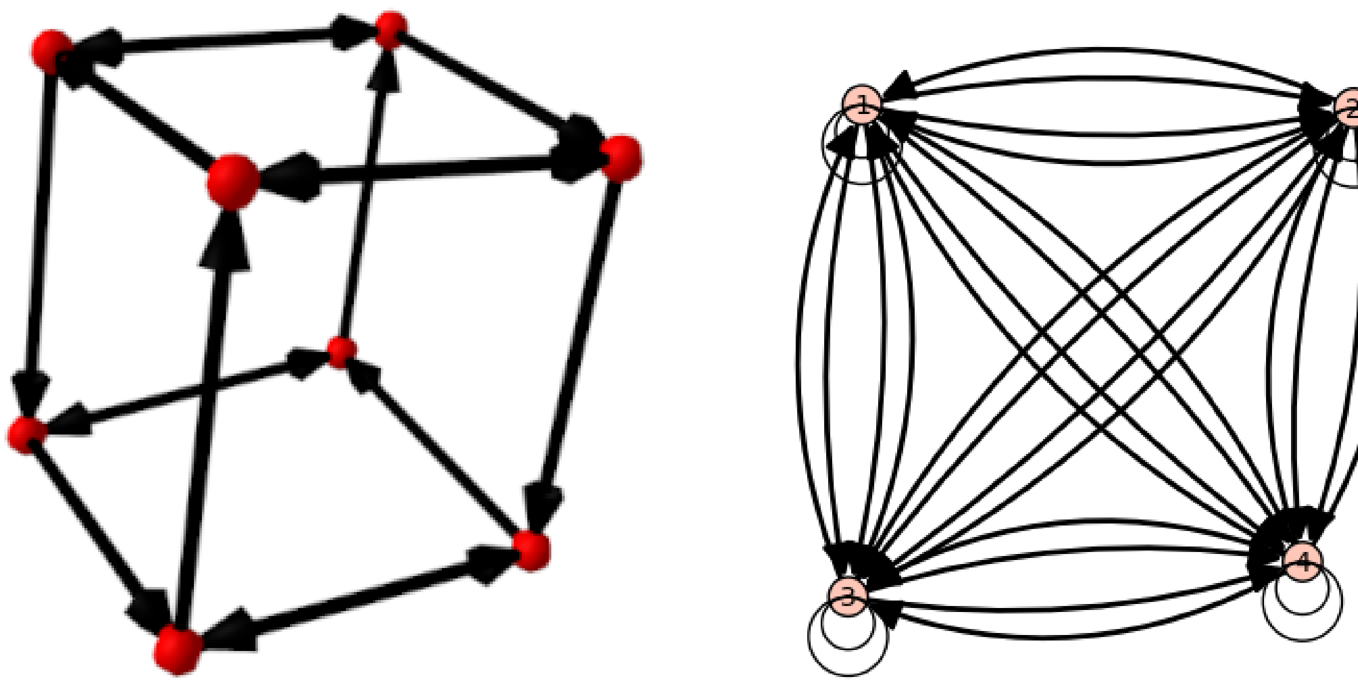
We now formalize this as per above.

$$X : \mathbf{B}G \to \mathbf{Set}$$

$$X(s) = \{1, 2, 3, 4\}$$

with X mapping morphisms identically. We now prove X is functorial. Let $g : s \to s$ define an arrow in $\mathbf{B}G$. Define the morphism $F(g) : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ by $F(g)(s) = m \cdot s$. Then

$$\begin{aligned} X(g_1 \circ g_2)(s) &= (m_1 m_2) \cdot s \\ &= m_1 \cdot (m_2 \cdot s) \\ &= X(m_1)(X(m_2)(s)) \\ &= (X(m_1) \circ X(m_2))(s) \end{aligned}$$



The entire group, and its skeletal translation groupoid.

The translation groupoid is the dihedral group applied to all sets S where $|S| = 4$. Which is an infinite collection of discrete identical graphs.

Each node has exactly 2 automorphisms:

```
1 ()
2 ()
3 ()
4 ()
1 (2,4)
3 (2,4)
2 (1,3)
4 (1,3)
```

Which are the stabilizers. All elements in the same orbit thus have isomorphic stabilizers.

The disjoint union of the stabilizer groups is:

$$\{(O_1, \{e, b\}), (O_3, \{e, b\}), (O_2, \{e, a^2b\}), (O_4, \{e, a^2b\})\}$$

The set of morphisms with domain x is isomorphic to G.

Another way of looking at it is the disjoint union of hom-sets $\mathrm{Hom}_{\mathbf{T}_G X}(x, y)$ for $y \in O_x$ where each set is isomorphic to $\mathrm{Hom}_{\mathbf{T}_G X}(x, x) = G_x$.

$$|G| = |O_x||G_x|$$