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Preface

Ich sehe jetzt alles mit anderen Augen!
Die Tiefen des Daseins sind unermeßlich!
Mein lieber Freund!
Es gibt manches auf der Welt,
Das läßt sich nicht sagen.
Jedoch, jedoch, jedoch, jedoch, jedoch!
Mut ist in mir, Mut Freund!
Die Welt ist lieblich
Und nicht fürchterlich dem Mutigen.

"Ariadne auf Naxos", by Richard Strauss,
libretto by Hugo von Hofmannsthal

In 1968, massive political and social upheaval shook the world, most noticeably in Paris. These events influenced a generation to which one of the authors of this book belongs. A lesser-known event in France that year is the publication of "Dix exposés sur la cohomologie des schémas" by J. Giraud, A. Grothendieck, S.L. Kleiman, M. Raynaud and J. Tate. Included there, with the kind permission of N. Bourbaki, were two talks by Grothendieck in the Bourbaki seminar, entitled "Le groupe de Brauer I" and "Le groupe de Brauer II", followed by the 100-page "Le groupe de Brauer III". More than fifty years later, it remains the principal source on Grothendieck's generalisation of the Brauer group of fields to the Brauer group of schemes, in the language of étale cohomology. While masterfully written, with the fresh appeal of a newly designed theory, Grothendieck's two seminar talks and a long paper are hardly a textbook.

Our first motivation for writing this book was to complement Grothendieck's foundational text with a more accessible modern exposition, and to give proofs of some results not easily found in the literature. Our second motivation was to describe recent developments in the theory of the Brauer–Manin obstruction and local-to-global principles, as well as new geometric applications of the Brauer group.

What we now call the Brauer group of a field was defined by Richard Brauer in [Bra32, p. 243]. He showed that this is a torsion abelian group [Bra32, Sätze 1, 2]. Under the name of "R. Brauersche Algebrenklassengruppe", the Brauer group of a number field features in the paper by Hasse [Has33] dedicated to Emmy Noether's fiftieth birthday. (See [Roq05] for the early history of the Brauer group.) Precursors of Grothendieck's work on the Brauer group of a scheme were the works of Azumaya [Az51] and Auslander and Goldman [AG60] on the Brauer group of a commutative ring.

Soon after the publication of "Le groupe de Brauer I, II, III" it became clear that the Brauer group of an algebraic variety is a very useful tool. In his 1970 ICM address [Man71], Manin defined a natural pairing between the Brauer group of a variety X over a number field k and the space of its adelic points $X(\mathbf{A}_k)$. He pointed out that this pairing generalises pairings in the theory of abelian varieties (Cassels–Tate pairing on the Tate–Shafarevich group, maps in the Cassels–Tate dual sequence) and in the theory of algebraic tori (Voskresenskiĭ). He also showed how several known counter-examples to the Hasse principle building on reciprocity laws could be interpreted in terms of this pairing. The Brauer–Manin obstruction revolutionised the theory of Diophantine equations by enabling one to study local-to-global principles for rational points beyond the narrow confines of varieties satisfying the Hasse principle and weak approximation.

In a separate development, Artin and Mumford [AM72] used the birational invariance of the Brauer group to construct examples of unirational but not rational varieties over the field of complex numbers. This gave a negative answer to the Lüroth problem in dimension at least 3, by a method different from those of Clemens–Griffiths and Iskovskikh–Manin, found about the same time. In 1984 the unramified Brauer group was used by Saltman who found examples of finite subgroups $G \subset \mathrm{GL}(n,\mathbb{C})$ such that the quotient $\mathrm{GL}(n,\mathbb{C})/G$ is not rational. This gives a negative answer to a problem of Emmy Noether motivated by the inverse Galois problem.

In the 1970s and 1980s, Colliot-Thélène and Sansuc developed the theory of descent and universal torsors, and linked it to the Brauer–Manin obstruction. For geometrically rational surfaces X over a number field k, in 1979 they asked whether the Brauer–Manin obstruction correctly describes the closure of the set of rational points $X(k)$ in $X(\mathbf{A}_k)$. This was supported by a conjectural argument involving Schinzel's hypothesis, and also by a theorem about certain intersections of two quadrics proved jointly with D. Coray. In 1987, that theorem was vastly generalised in a joint work with Swinnerton-Dyer.

This was the origin of the conjecture that $X(k)$ should be dense in the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ for arbitrary smooth, projective, rationally connected varieties. In contrast, in 1997 Skorobogatov constructed a bielliptic surface X over $\mathbb{Q}$ which is a counter-example to the Hasse principle that cannot be explained by the Brauer–Manin obstruction. Stronger versions of the Brauer–Manin obstruction were soon proposed by Harari and Skorobogatov, but a more radical counter-example found by Poonen in 2010 shows that for general varieties these obstructions are insufficient too.

Also in the 1980s a related question was raised: does a natural analogue of the Brauer–Manin obstruction control zero-cycles on arbitrary smooth projective varieties? This question has connections with algebraic K-theory. Important results were obtained by Salberger for conic bundles.

More recently, the birational invariance of the Brauer group has become one of the ingredients of the specialisation method discovered by Voisin and developed by Colliot-Thélène and Pirutka, and later by Schreieder and others. This method was used by Hassett, Pirutka and Tschinkel to give examples of algebraic families of smooth projective varieties over the field of complex numbers, some of which are rational, whereas some others are not even stably rational.

Contents

Let us give a brief outline of the contents of this book. We refer to the introductions to individual chapters for more details.

Chapters 1 and 2 contain preliminary material on Galois and étale cohomology. For obvious reasons many results here are stated without proofs, though we give a proof of compatibility of two definitions of the residue map for the Brauer group of a discretely valued field.

Chapter 3 starts with definitions of the two Brauer groups of a scheme: the Brauer group defined in terms of Azumaya algebras, which we call the Brauer–Azumaya group, and the cohomological Brauer group, which we call the Brauer–Grothendieck group. Grothendieck denoted the Brauer–Azumaya group of a scheme X by $\mathrm{Br}(X)$ and the cohomological Brauer group by $\mathrm{Br}'(X)$. This book is mainly concerned with the latter group, which is why we reserve the notation $\mathrm{Br}(X)$ for the cohomological Brauer group (and usually refer to it as 'the Brauer group') and denote the Brauer–Azumaya group by $\mathrm{Br}_{\mathrm{Az}}(X)$. We make initial observations towards comparing the two groups. Other fundamental subjects discussed in this chapter are localisation and the purity theorem for the Brauer group.

In Chapter 4 we reproduce de Jong's proof of a theorem of Gabber which says that a natural homomorphism from $\mathrm{Br}_{\mathrm{Az}}(X)$ to the torsion subgroup of $\mathrm{Br}(X)$ is an isomorphism for a quasi-projective scheme over an affine scheme. This requires the use of stacks, to which we give a short introduction.

In Chapters 5, 6, 7 we focus on the Brauer group of a smooth variety over a field and its behaviour under extension of the ground field. In Chapter 5, we describe the structure of this group and methods to compute it, both in the general case and for classes of varieties satisfying additional geometric conditions. In Chapter 6 we define the unramified Brauer group and prove that the Brauer group of a smooth and proper variety is a birational invariant.

Chapter 7 deals with Severi–Brauer varieties and projective quadrics, and ends with computations of Brauer groups of some affine hypersurfaces.

Chapter 8 contains various results on the Brauer group of singular varieties, which show that properties familiar in the smooth case do not extend to arbitrary varieties.

In Chapter 9 we collect results on the Brauer group and on the unramified Brauer group of a variety equipped with an action of a linear algebraic group, such as a torsor or a homogeneous space. We discuss theorems of Saltman and of Bogomolov that can be used to give negative answers to Noether's problem.

Chapters 10, 11 and 12 are devoted to the Brauer group of a family of varieties. The subject of Chapter 10 is schemes over a local ring and varieties over a local field. Here we also discuss split fibres and explore their properties. In Chapter 11, after defining the vertical Brauer group of a morphism, we explain how to compute the Brauer group of a conic bundle over a 1- or 2-dimensional base. We present the Artin–Mumford examples from this birational point of view. Chapter 12 contains an exposition of the specialisation method with applications to the behaviour of stable rationality in a smooth family.

The next group of chapters concerns arithmetic applications. The Brauer–Manin obstruction is introduced and studied in Chapter 13. Chapter 14 contains an exposition of several results stating that for some classes of varieties the Brauer–Manin obstruction precisely describes the closure of the set of rational points inside the topological space of adelic points. We discuss Schinzel's Hypothesis (H), applications of results in additive combinatorics due to Green, Tao and Ziegler to rational points and sketch a proof of a theorem of Harpaz and Wittenberg about families of rationally connected varieties. In the last part of this chapter we also give an overview of the theory of obstructions to the local-to-global principles for rational points. Chapter 15 deals with zero-cycle analogues of these themes: we explain Salberger's method and sketch the general result of Harpaz and Wittenberg.

The last chapter, Chapter 16, concerns finiteness properties of the Brauer group of abelian varieties, K3 surfaces, and varieties dominated by products of curves when the ground field is finitely generated over its prime subfield. We recall Tate's conjecture and its Brauer group variant over a field finitely generated over the prime field, in particular over a finite field. The treatment of K3 surfaces necessitates a detour via an interpretation of their moduli spaces as Shimura varieties and the Kuga–Satake construction. We give complete proofs of the Tate conjecture and the finiteness of the Brauer group for K3 surfaces modulo the Brauer group of the ground field in the case of characteristic zero.

The reader will not fail to notice that the style of this book varies from chapter to chapter, from a more in-depth treatment to a survey. The authors are aware of these and other imperfections, as well as omissions of a number of important subjects. In this book we only fleetingly discuss descent and torsors, for which we refer to [Sko01]. Other subjects which could have been but are not included:

- unramified cohomology in higher degrees;
- the Brauer group and differentials in characteristic p;
- Swinnerton-Dyer's method for rational points on a pencil of genus 1 curves;
- the integral Brauer–Manin obstruction.

We recommend Poonen's recent book [Po18] as an extremely helpful and comprehensive introduction to rational points. Another book on the Brauer group of varieties was recently published by Gorchinskiy and Shramov [GSh18].

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Contents

Notation

The symbol $\simeq$ stands for a (not necessarily canonical) isomorphism. The symbol $\cong$ denotes a canonical isomorphism.

For an abelian group A we denote by $A[n]$ the n-torsion subgroup of A, that is, $A[n] = \{x \in A | nx = 0\}$. If ℓ is a prime number, we denote by $A\{\ell\}$ the ℓ-primary torsion subgroup of A, i.e. the union of $A[\ell^i]$ for all $i \geq 1$. We write A_{tors} for the torsion subgroup of A, i.e. the union of $A[n]$ for all $n \geq 1$. If p is a prime or $p = 1$, we write $A(p')$ for the union of $A\{\ell\}$ for all primes $\ell \neq p$.

We denote by R^* the group of units of a ring R. For a commutative ring R, an R-module M and an element $r \in R$ we often write M/r for M/rM. In particular, for an integer n we write $\mathbb{Z}/n$ instead of $\mathbb{Z}/n\mathbb{Z}$.

For a field k we write $\bar{k}$ for a fixed algebraic closure of k, and $k_{\mathrm{s}} \subset \bar{k}$ for the separable closure of k in $\bar{k}$. Let $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$ be the absolute Galois group of k. The characteristic exponent of k is 1 if $\mathrm{char}(k) = 0$ and p if $\mathrm{char}(k)$ is a prime number p.

The p-cohomological dimension $\mathrm{cd}_p(G)$ of a profinite group G, where p is a prime, is the smallest integer n such that $\mathrm{H}^i(G, M)\{p\} = 0$ for all G-modules M such that $M = M_{\mathrm{tors}}$ and all $i > n$. The cohomological dimension $\mathrm{cd}(G)$ of a profinite group G is the supremum of its p-cohomological dimensions over all primes p.

The cohomological dimension $\mathrm{cd}(k)$ of a perfect field k is the cohomological dimension of its absolute Galois group Γ.

For a scheme X over a field k, we write $\overline{X} = X \times_k \bar{k}$ and $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. A variety over k is defined as a separated scheme of finite type over k. In particular, a variety X over k is quasi-compact (i.e., it is a finite union of affine open subsets), and the intersection of two affine open subsets of X is affine.

A scheme X is called quasi-separated if the diagonal morphism $X \to X \times_{\mathbb{Z}} X$ is quasi-compact; equivalently the intersection of two affine open subsets of X is a finite union of affine open subsets of X.

We adopt the convention that an integral scheme is by definition non-empty, see [Stacks, Def. 01OK].

Given a smooth integral variety U over a field k, by a smooth compactification of U we understand a smooth, proper, integral variety X over k, which contains U as a Zariski open set. If $\mathrm{char}(k) = 0$, then such a variety X exists by Hironaka's theorem.

Let X be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called finite locally free if every point $x \in X$ has a Zariski open neighbourhood $U \subset X$ such that $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$ for some n.

Chapter 1
Galois cohomology

This chapter begins with a brief introduction to quaternion algebras over a field. After recalling basic facts about central simple algebras, we discuss the classical definition of the Brauer group of a field as the group of equivalence classes of such algebras. We state several standard results about Galois cohomology and descent, and then give the cohomological definition of the Brauer group of a field and construct a natural isomorphism between the resulting groups. For a thorough treatment of central simple algebras and the Brauer groups of fields we refer to the book by P. Gille and T. Szamuely [GS17] from which we borrowed some of the material for this chapter. Various aspects of the theory of simple algebras and the Brauer group can be found in Bourbaki's *Algèbre*, Ch. VIII [BouVIII], and in the books by J-P. Serre [SerCL, SerCG], A.A. Albert [Alb31], I. Reiner [Rei03] and I.N. Herstein [Her68].

In this chapter we also state several results about cyclic algebras and the vanishing of the Brauer group for specific fields, such as finite fields, function fields in one variable over an algebraically closed field, and C_1-fields.

In Section 1.4 we discuss the Brauer group of discretely valued fields and the associated crucial notion of residue. There are several approaches to the definition of the residue; we explain how two of them are related to each other. We finish by proving a theorem of D.K. Faddeev which describes the Brauer group of the field of rational functions $k(t)$, where k is a perfect field.

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1.1 Quaternion algebras and conics

In this section, unless mentioned otherwise, k is a field of characteristic not equal to 2. Note, however, that many of the results stated here have analogues over a field of characteristic 2.

1.1.1 Quaternions

To $a, b \in k^*$ one can attach a non-commutative associative k-algebra in the following way.

Definition 1.1.1 *A* **quaternion algebra** *over k is a k-algebra isomorphic to the 4-dimensional associative algebra $Q_k(a,b)$ with basis 1, i, j, ij and the multiplication table*

$$i^2 = a, \; j^2 = b, \; ij = -ji,$$

where $a, b \in k^*$.

The algebra $M_2(k)$ of (2×2)-matrices with entries in k is spanned by

$$1 = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ij = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and so is isomorphic to $Q_k(1,1)$.

Switching i and j shows that $Q_k(a,b) \simeq Q_k(b,a)$.

For a field extension $k \subset K$ there is a natural isomorphism

$$Q_k(a,b) \otimes_k K \xrightarrow{\sim} Q_K(a,b).$$

Exercise 1.1.2 The map $k \to Q_k(a,b)$ sending x to $x \cdot 1$ identifies k with the centre of $Q_k(a,b)$. The two-sided ideals of $Q_k(a,b)$ are 0 and $Q_k(a,b)$.

The algebra $Q_{\mathbb{R}}(-1,-1)$ is the algebra of Hamilton's quaternions $\mathbb{H}$. This is a division algebra: every non-zero element of $\mathbb{H}$ is invertible.

A natural question is: for which $a, b \in k^*$ is $Q_k(a,b)$ a division algebra?

Definition 1.1.3 *Let Q be a quaternion algebra over the field k. A* **pure quaternion** *in Q is 0 or an element $q \in Q$ such that $q \notin k$ but $q^2 \in k$.*

If $Q \cong Q_k(a,b)$, then the pure quaternions are precisely the elements of the form $yi + zj + wij$. To see this, square $x + yi + zj + wij$, then there are some cancellations, and if $2x \neq 0$, then $y = z = w = 0$. Thus each quaternion $q \in Q$ is uniquely written as $q = q_s + q_p$, where $q_s \in k$ is a scalar and q_p is a pure quaternion.

It is an easy exercise to show that the pure quaternions in $M_2(k)$ are precisely the traceless matrices.

Definition 1.1.4 *The* **conjugate** *of* $q = q_s + q_p \in Q$ *is* $\bar{q} = q_s - q_p$. *The* **norm** *of* q *is* $\mathrm{N}(q) = q\bar{q} = \bar{q}q \in k$. *The* **trace** *of* q *is* $\mathrm{Tr}(q) = q + \bar{q} \in k$.

For any $q_1, q_2 \in Q$ we have

$$\overline{q_1 q_2} = \bar{q}_2\,\bar{q}_1, \quad \mathrm{N}(q_1 q_2) = \mathrm{N}(q_1)\mathrm{N}(q_2), \quad \mathrm{Tr}(q_1 + q_2) = \mathrm{Tr}(q_1) + \mathrm{Tr}(q_2).$$

Exercise 1.1.5 If $q \in Q$ is a pure quaternion such that q^2 is not a square in k, then 1 and q span a quadratic field extension of k which is a maximal subfield of Q.

The quaternion k-algebras $Q_k(a,b)$ and $Q_k(c,d)$ are isomorphic if and only if there exist anti-commuting pure quaternions $I, J \in Q_k(a,b)$ such that $I^2 = c$, $J^2 = d$. Then $1, I, J, IJ$ is a basis of the k-vector space $Q_k(a,b)$. In particular, for any $u, v \in k^*$ we have $Q_k(au^2, bv^2) \simeq Q_k(a,b)$.

Lemma 1.1.6 *If* $c \in k^*$ *is a norm from* $k(\sqrt{a})^*$, *then* $Q_k(a,b) \simeq Q_k(a,bc)$.

Proof. Write $c = x^2 - ay^2$ with $x, y \in k$. Set $J = xj + yij \in Q_k(a,b)$. One checks that $Ji = -iJ$ and $J^2 = -\mathrm{N}(J) = bc$. □

Lemma 1.1.7 *The invertible elements in* $Q_k(a,b)$ *are exactly the elements with non-zero norm.*

Proof. For $q \in Q_k(a,b)$, we have $q\bar{q} = \mathrm{N}(q) \in k$. If $\mathrm{N}(q) \neq 0$, then $q\bar{q}/\mathrm{N}(q) = 1$ hence q is invertible. If $\mathrm{N}(q) = 0$ and $q \neq 0$, then $\bar{q} \neq 0$ and $q\bar{q} = 0$ hence q is a zero divisor. □

The norm on $Q_k(a,b)$ is the diagonal quadratic form $\langle 1, -a, -b, ab\rangle$. This leads us to the following criterion.

Proposition 1.1.8 *Let* $Q = Q_k(a,b)$, *where* $a, b \in k^*$. *The following statements are equivalent:*

(i) *Q is not a division algebra;*
(ii) *Q is isomorphic to the matrix algebra $M_2(k)$;*
(iii) *the diagonal quadratic form $\langle 1, -a, -b\rangle$ represents zero in k;*
(iv) *the norm form $\mathrm{N} = \langle 1, -a, -b, ab\rangle$ represents zero in k;*
(v) *b is in the image of the norm homomorphism $k(\sqrt{a})^* \to k^*$.*

Proof. First assume that $a \in k^{*2}$. The equivalence of all statements but (ii) is clear, and these statements hold in this case. To prove the equivalence with (ii), by Lemma 1.1.6 we can assume that $a = 1$. The matrix algebra $M_2(k)$ is spanned by

$$1 = \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \quad ij = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix},$$

and so is isomorphic to $Q_k(1,b)$.

Now assume that $a \in k^*$ is not a square. Then (i) is equivalent to (iv) since $\mathrm{N}(q) = q\bar{q}$ (see Lemma 1.1.7). Next, (iv) implies (v) because the ratio of two non-zero norms is a norm. It is clear that (v) implies (iii), which implies (iv) since N is the diagonal quadratic form $\langle 1, -a, -b, ab \rangle$. So (iii), (iv) and (v) are all equivalent to (i). Lemma 1.1.6 shows that under the assumption of (v) the algebra $Q_k(a, b)$ is isomorphic to $Q_k(a, b^2) \cong Q_k(a, 1)$, so we use the result of the first part of the proof to prove the equivalence with (ii). □

If the conditions of this proposition are satisfied, then one says that $Q_k(a, b)$ is *split*. If K is a field extension of k such that the K-algebra $Q_K(a, b) = Q_k(a, b) \otimes_k K$ is split, then one says that K *splits* $Q_k(a, b)$.

Since any quaternion algebra $Q_k(a, b)$ is split by a separable closure k_{s} of k, by Proposition 1.1.8 we see that $Q_k(a, b)$ is a (k_{s}/k)-*form* of the 2×2-matrix algebra, which means that

$$Q_k(a, b) \otimes_k k_{\mathrm{s}} \simeq M_2(k_{\mathrm{s}}).$$

For example, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$.

Proposition 1.1.9 *Any quaternion algebra Q split by $k(\sqrt{a})$ contains this field and can be written as $Q = Q_k(a, c)$ for some $c \in k^*$. Conversely, if Q contains $k(\sqrt{a})$, then Q is split by $k(\sqrt{a})$.*

Proof. If the algebra Q is split, take $c = 1$. Assume Q is not split, hence is a division algebra. In particular, a is not a square in k. There exist $q_0, q_1 \in Q$, not both equal to 0, such that $\mathrm{N}(q_0 + q_1\sqrt{a}) = 0$. Since Q is a division algebra, we have $q_0 \neq 0$ and $q_1 \neq 0$. We have

$$\mathrm{N}(q_0 + q_1\sqrt{a}) = \mathrm{N}(q_0) + a\mathrm{N}(q_1) + \sqrt{a}(q_0\bar{q}_1 + q_1\bar{q}_0) = 0,$$

hence $\mathrm{N}(q_0) + a\mathrm{N}(q_1) = 0$ and $q_0\bar{q}_1 + q_1\bar{q}_0 = 0$. Set $q_2 = q_0\bar{q}_1$. We have

$$q_2^2 = q_0\bar{q}_1 q_0\bar{q}_1 = -q_0\bar{q}_1 q_1\bar{q}_0 = -\mathrm{N}(q_0)\mathrm{N}(q_1) = a\mathrm{N}(q_1)^2.$$

Let $I = q_2/\mathrm{N}(q_1)$. Then $I^2 = a$. Since a is not a square in k, we see that $I \notin k$. The conjugation by I is a k-linear transformation of Q. It preserves the subspace of pure quaternions, since it preserves the condition $z^2 \in k$. The order of this linear transformation is 2 because $I \notin k$, hence I is not in the centre of Q. Thus the -1-eigenspace is non-zero, so we can find a non-zero pure quaternion $J \in Q$ such that $IJ + JI = 0$. We have $J^2 = c \in k$, since J is pure. This is enough to conclude that $Q \simeq Q_k(a, c)$.

The converse follows from the fact that $k(\sqrt{a}) \otimes k(\sqrt{a})$ contains zero-divisors (the norm form $x^2 - ay^2$ represents zero in $k(\sqrt{a})$). Hence the same is true for $Q \otimes_k k(\sqrt{a})$. □

Corollary 1.1.10 *The quadratic field extensions K of k that split a quaternion division algebra over k are exactly the quadratic subfields of this algebra.*

1.1.2 Conics

Definition 1.1.11 *Let Q be a quaternion algebra over k. Let $Q_{\mathrm{pure}} \subset Q$ be the 3-dimensional subspace of pure quaternions. The norm on Q induces a non-degenerate quadratic form on the vector space Q_{pure}. The conic attached to Q is defined as the smooth conic given by this quadratic form in the projective plane $\mathbb{P}(Q_{\mathrm{pure}}) \simeq \mathbb{P}^2_k$. It is denoted by $C(Q)$.*

Thus the conic attached to the quaternion algebra $Q_k(a,b)$ is the plane algebraic curve $C(a,b) \subset \mathbb{P}^2_k$ given by the equation

$$-ax^2 - by^2 + abz^2 = 0.$$

Up to a change of variables, this conic is also given by the equation

$$ax^2 + by^2 = z^2.$$

By Proposition 1.1.8 the conic $C(Q)$ has a k-point if and only if the quaternion algebra Q is split.

Remark 1.1.12 (1) Since the characteristic of k is not 2, every smooth conic can be given by a diagonal quadratic form, and so is attached to some quaternion algebra.

(2) The projective line is isomorphic to the conic $xz - y^2 = 0$ via the map $(X : Y) \mapsto (X^2 : XY : Y^2)$.

(3) If a smooth conic C has a k-point, then $C \cong \mathbb{P}^1_k$. (The projection from a k-point gives rise to a rational parameterisation of C, which is an isomorphism.)

(4) Thus the function field $k(C)$ of a smooth conic C is a purely transcendental extension of k if and only if C has a k-point.

Exercise 1.1.13 (1) Check that $Q_k(a, 1-a)$ and $Q_k(a, -a)$ are split.

(2) Check that if $k = \mathbb{F}_q$ is a finite field, then all quaternion k-algebras are split. (By assumption q is not a power of 2. Write $ax^2 = 1 - by^2$ and use a counting argument for x and y to prove the existence of a solution in $\mathbb{F}_q$.)

(3) A quaternion k-algebra Q is split if and only if the quaternion $k(t)$-algebra $Q_{k(t)}$ is split. (Take a $k(t)$-point on the associated conic $C(Q) \subset \mathbb{P}^2_k$ represented by three polynomials not all divisible by t and reduce modulo t.)

(4) Q is split over the function field $k(C(Q))$ of the associated conic $C(Q)$. (Consider the generic point of $C(Q)$.)

The following theorem of Max Noether [Noe70] is a special case of Tsen's theorem (Theorem 1.2.14 below). It plays an important rôle in the classification of complex algebraic surfaces. The proof given here is due to Tsen.

Theorem 1.1.14 (Max Noether) *Assume that k is an algebraically closed field. Then all quaternion $k(t)$-algebras are split.*

Proof. By Proposition 1.1.8, it is enough to show that any conic over $k(t)$ has a point (this is Max Noether's statement). We can assume that the coefficients of the corresponding quadratic form are polynomials in t of degree at most m. We look for a solution (X, Y, Z), where X, Y and Z are polynomials in t (not all of them zero) of degree at most n for some large integer n. The coefficients of these polynomials can be thought of as points of the projective space $\mathbb{P}^{3n+2}$. The solutions bijectively correspond to the points of a closed subset of $\mathbb{P}^{3n+2}$ given by $2n+m+1$ homogeneous quadratic equations. Since k is algebraically closed this set is non-empty when $3n+2 \geq 2n+m+1$, by a standard result from algebraic geometry. (If an irreducible variety X is not contained in a hypersurface H, then $\dim(X \cap H) = \dim(X) - 1$. This implies that on intersecting X with r hypersurfaces the dimension drops at most by r, see [Sha74, Ch. 1]). □

The following theorem is due to Witt [Wit35, §2].

Theorem 1.1.15 (Witt) *Let k be field of characteristic not equal to 2. Two quaternion algebras over k are isomorphic if and only if the conics attached to them are isomorphic.*

Proof. We follow the proof of [GS17, Thm. 1.4.2]. Recall that $C(Q)$ denotes the smooth conic attached to the quaternion algebra Q. An isomorphism of quaternion algebras $Q \cong Q'$ induces an isomorphism of their vector spaces of pure quaternions respecting the norm form. Hence it induces an isomorphism $C(Q) \cong C(Q')$.

Let us prove that if $C(Q) \cong C(Q')$, then $Q \cong Q'$. If Q is split, then $C(Q)$ has a k-point. Thus $C(Q')$ also has a k-point. But then the norm form of Q' represents zero, and this implies that Q' is split.

Assume from now on that neither algebra is split. Write $Q = Q_k(a, b)$ and write C for the conic $C(Q') \cong C(Q) = C(a, b)$ given by the equation

$$ax^2 + by^2 = z^2. \tag{1.1}$$

Let $K = k(\sqrt{a})$ and let $K(C)$ be the function field of the conic $C_K = C \times_k K$. The conic C has a K-point, hence Q' is split by K. By Proposition 1.1.9 we can write $Q' = Q_k(a, c)$ for some $c \in k^*$. By Exercise 1.1.13 (4), Q' is split by the function field $k(C)$. By Proposition 1.1.8 this implies that $c \in k^* \subset k(C)^*$ is contained in the image of the norm map

$$c \in \mathrm{Im}[K(C)^* \to k(C)^*].$$

Let $\sigma \in \mathrm{Gal}(K/k) \cong \mathbb{Z}/2$ be the generator. Then we can write $c = f\sigma(f)$, where f is a rational function on the conic C_K. One can replace f with $f\sigma(g)g^{-1}$ for any $g \in K(C)^*$ without changing c.

The group $\mathrm{Div}(C_K)$ of divisors on $C_K \cong \mathbb{P}^1_K$ is freely generated by the closed points of C_K. This is a module over $\mathbb{Z}/2 = \langle \sigma \rangle$ with a σ-stable basis. The divisors of functions are exactly the divisors of degree 0. The divisor

$D = \text{div}(f)$ is an element of $\text{Div}(C_K)$ satisfying $(1+\sigma)D = 0$. By comparing the multiplicities of points in the support of D we deduce that there is a divisor $G \in \text{Div}(C_K)$ such that $D = (1-\sigma)G$. Let $P = (1:0:\sqrt{a}) \in C(K)$ (see equation (1.1)). If $n = \deg(G)$ the divisor $G - nP \in \text{Div}(C_K)$ has degree 0. Since $C_K \cong \mathbb{P}^1_K$, this implies $G - nP = \text{div}(g)$ for some $g \in K(C)^*$. We have

$$\text{div}(f\sigma(g)g^{-1}) = D+\sigma G-G+n(P-\sigma P) = n(P-\sigma P) = n\ \text{div}\left(\frac{z-\sqrt{a}x}{y}\right).$$

The last equality is readily checked using equation (1.1). It follows that

$$f\sigma(g)g^{-1} = e\left(\frac{z-\sqrt{a}x}{y}\right)^n \in K(C)^*$$

for some $e \in K^*$. Taking norms, we obtain

$$c = f\sigma(f) = \text{N}_{K/k}(e)\left(\frac{z^2-ax^2}{y^2}\right)^n = \text{N}_{K/k}(e)b^n \in k(C)^*,$$

hence $c = \text{N}_{K/k}(e)b^n \in k^*$. Thus $Q' = Q_k(a,c) = Q_k(a, \text{N}_{k(\sqrt{a})/k}(e)b^n)$ for some integer n. By Lemma 1.1.6, Q' is isomorphic to $Q_k(a,b)$ or to $Q_k(a,1)$. Since Q' is not split, we must have $Q' \cong Q_k(a,b)$. □

1.2 The language of central simple algebras

In this section k denotes an arbitrary field (possibly of characteristic 2).

Recall that if V and W are vector spaces over k, then $V \otimes_k W$ is the linear span of vectors $v \otimes w$, $v \in V$, $w \in W$, subject to the axioms

$$(v_1+v_2)\otimes w = v_1\otimes w + v_2 \otimes w, \quad v\otimes(w_1+w_2) = v\otimes w_1 + v\otimes w_2,$$

and

$$c(v\otimes w) = (cv)\otimes w = v\otimes(cw) \quad \text{for any} \quad c \in k.$$

This turns $V \otimes_k W$ into a k-vector space. If (e_i) is a basis of V, and (f_j) is a basis of W, then $(e_i \otimes f_j)$ is a basis of $V \otimes_k W$. The vector spaces $(V \otimes_k U) \otimes_k W$ and $V \otimes_k (U \otimes_k W)$ are canonically isomorphic.

Given two k-algebras A and B, one defines the structure of a k-algebra on $A \otimes_k B$ by the rule $(x \otimes y)\cdot(x' \otimes y') = (xx') \otimes (yy')$.

1.2.1 Central simple algebras

Quaternion algebras and matrix algebras are particular cases of central simple algebras.

Definition 1.2.1 *An associative k-algebra A is called* **simple** *if $A \neq 0$ and the only two-sided ideals of A are 0 and A. An associative k-algebra A is called* **central** *if its centre is k. A* **central simple algebra** *is a finite-dimensional k-algebra that is both central and simple.*

Remark 1.2.2 (1) Any finite-dimensional central division algebra is a central simple algebra.

(2) For any integer $n \geq 1$ the algebra of $n \times n$ matrices $M_n(k)$ is a central simple k-algebra. More generally, if D is a finite-dimensional central division k-algebra, then $M_n(D)$ is a central simple k-algebra [GS17, Example 2.1.2].

(3) We have $M_m(k) \otimes_k M_n(k) \simeq M_{mn}(k)$.

Later we shall use the following important property of matrix algebras.

Proposition 1.2.3 *Any automorphism of the k-algebra $M_n(k)$ is induced by conjugation by an invertible matrix. This invertible matrix is well defined up to multiplication by a scalar matrix.*

Proof. [GS17, Lemma 2.4.1, Cor. 2.4.2]. □

The structure of central simple algebras is described by a theorem of Wedderburn.

Theorem 1.2.4 (Wedderburn) *Let A be a central simple algebra over k. There exists a finite-dimensional central division algebra D over k such that $A \simeq D \otimes_k M_n(k) \cong M_n(D)$.*

The integer n is well defined, and the algebra D is well defined up to a non-unique isomorphism. Proofs of this fundamental theorem can be found in [BouVIII, §5, no. 4, Cor. 2], [Her68, Thm. 2.1.6], [GS17, Thm. 2.1.3].

Corollary 1.2.5 *Any central simple algebra over an algebraically closed field k is isomorphic to a matrix algebra $M_n(k)$.*

Proof. We need to prove that a finite-dimensional central division k-algebra D coincides with its centre k. Choose any $x \in D$. Let $I \subset k[t]$ be the ideal consisting of polynomials vanishing on x. This is a non-zero ideal, generated by some $f(t) \in k[t]$. Since D is a division algebra, $f(t)$ is irreducible. As k is algebraically closed, $f(t)$ has degree 1, hence $x \in k$. □

Lemma 1.2.6 *Let k be a field and let A be a finite-dimensional k-algebra. Let K/k be a finite field extension. Then A is a central simple k-algebra if and only if $A \otimes_k K$ is a central simple K-algebra.*

Proof. This is [GS17, Lemma 2.2.2]. □

Theorem 1.2.7 *Let k be a field and let A be a finite-dimensional k-algebra. Then A is a central simple k-algebra if and only if there exist a positive integer n and a finite field extension K/k such that $A\otimes_k K$ is isomorphic to $M_n(K)$. Moreover, if this is so, then one can choose K separable over k.*

Proof. See [GS17, Thm. 2.2.1, Thm. 2.2.7]. See also [Alb31, Ch. IV, §7, Thm. 18] and [BouVIII, §10, no. 3, Prop. 4]. □

For a central simple algebra A over a field k, a field extension K/k such that $A\otimes_k K$ is isomorphic to $M_n(K)$ is called *a splitting field* of A. Then one says that A *splits over* K.

Theorem 1.2.7 and Remarks 1.2.2 (2) and (3) immediately imply that the tensor product $A\otimes_k B$ of two central simple algebras is again a central simple algebra. It also immediately implies that the dimension of a central simple algebra over its centre is d^2, where d is a positive integer. This integer d is called the *degree* of the algebra.

Two central simple algebras A and B are called *equivalent* if there are positive integers n and m such that $A\otimes_k M_n(k)\cong B\otimes_k M_m(k)$. The relation is transitive by Remark 1.2.2 (3). The equivalence class of k consists of the matrix algebras of all sizes (see the comment after Theorem 1.2.4). The following theorem is [Bra32, Satz 1].

Theorem 1.2.8 (Brauer) *The tensor product equips the set of equivalence classes of central simple algebras over k with the structure of an abelian group. It is called the* **Brauer group** *of k and is denoted by* $\mathrm{Br}(k)$.

Proof. That the equivalence classes form a set follows from the fact that a central simple algebra A of degree d over k can be given by a basis of A over k and the multiplication table. The neutral element of $\mathrm{Br}(k)$ is the class of k. Associativity follows from the associativity of the tensor product. Commutativity follows from the isomorphisms $A\otimes_k B\xrightarrow{\sim} B\otimes_k A$ given by $x\otimes y\mapsto y\otimes x$. The inverse element of the class of A is the equivalence class of the *opposite* algebra A^{op}. Indeed, $A\otimes_k A^{\mathrm{op}}$ is a central simple algebra, and there is a non-zero homomorphism $A\otimes_k A^{\mathrm{op}}\to\mathrm{End}_k(A)$ that sends $a\otimes b$ to $x\mapsto axb$. It is injective since a central simple algebra has no proper two-sided ideals, and hence is an isomorphism by dimension count. □

We write the group operation in $\mathrm{Br}(k)$ additively.

Theorem 1.2.4 implies that any class $\alpha\in\mathrm{Br}(k)$ is represented by a central division k-algebra D which is well defined up to a non-unique isomorphism. In particular, the degree of D is well defined. It is called the *index* of (any algebra in) the class α.

The following theorem goes back to the early days and was proved without Galois cohomology. See [Alb31, Thm. IV.17] and [KMRT, Ch. II, Cor. 10.5]. See also [GS17, Cor. 2.8.5].

Theorem 1.2.9 *For any field k the Brauer group* $\mathrm{Br}(k)$ *is a torsion group. More precisely, for any central simple k-algebra A of index n, the class of $A^{\otimes n}$ in* $\mathrm{Br}(k)$ *is zero.*

The order of the class of A in $\mathrm{Br}(k)$ is called the *exponent* of A.

From Theorem 1.2.4 it follows that two central simple algebras of the same dimension and the same class in $\mathrm{Br}(k)$ are isomorphic. We deduce that cancellation holds: $A\otimes_k B\cong A\otimes_k C$ implies $B\cong C$.

By Corollary 1.2.5, the Brauer group of an algebraically closed field is zero. By Theorem 1.2.7 this also holds for a separably closed field. Since $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite-dimensional division $\mathbb{R}$-algebras (and $\mathbb{C}$ is not central), we see from Theorem 1.2.4 that $\mathrm{Br}(\mathbb{R})=\mathbb{Z}/2$. This calculation also immediately follows from Theorem 1.3.5 below, using that the algebraic closure of $\mathbb{R}$ is $\mathbb{C}$.
Given a field extension K/k there is a natural *restriction* map

$$\mathrm{res}_{K/k}\colon \mathrm{Br}(k)\longrightarrow \mathrm{Br}(K)$$

which sends the class of a central simple k-algebra A to the class of $A\otimes_k K$. The kernel of $\mathrm{res}_{K/k}$ is denoted by $\mathrm{Br}(K/k)$ and is called the *relative Brauer group*.

Lemma 1.2.10 *Assume* $\mathrm{char}(k)\neq 2$. *For any $a,b,b'\in k^*$ we have the following properties:*

(i) $Q_k(a,b)\otimes_k Q_k(a,b')\cong Q_k(a,bb')\otimes_k M_2(k)$.
(ii) $Q_k(a,b)\otimes_k Q_k(a,b)\cong M_4(k)$.

Proof (See [GS17, Lemma 1.5.2]) The vector subspace of $Q_k(a,b)\otimes_k Q_k(a,b')$ spanned by $1\otimes 1$, $i\otimes 1$, $j\otimes j'$, $ij\otimes j'$ is $A_1=Q_k(a,bb')$. Similarly, the span of $1\otimes 1$, $1\otimes j'$, $i\otimes i'j'$, $-b(i\otimes i')$ is $A_2=Q_k(b',-a^2b')$. The conic associated to $Q_k(b',-a^2b')$ clearly has a k-point, so $A_2\cong M_2(k)$. The canonical homomorphism

$$A_1\otimes_k A_2\longrightarrow Q_k(a,b)\otimes_k Q_k(a,b')$$

defined by the product in $Q_k(a,b)\otimes_k Q_k(a,b')$ is surjective. By a dimension count, it is an isomorphism. This proves (i), and (ii) follows. □

We continue to assume that $\mathrm{char}(k)\neq 2$. Given $a,b\in k^*$ we write (a,b) for the class of $Q_k(a,b)$ in $\mathrm{Br}(k)$. By Lemma 1.2.10 (ii) we have $(a,b)\in\mathrm{Br}(k)[2]$. We have already seen that $(au^2,bv^2)=(a,b)$ for any $u,v\in k^*$. Lemma 1.2.10 (i) shows that associating to $a,b\in k^*$ the class $(a,b)\in\mathrm{Br}(k)[2]$ induces a bilinear map

$$k^*/k^{*2}\times k^*/k^{*2}\longrightarrow \mathrm{Br}(k)[2].$$

By Proposition 1.1.8 we have $(a,b)=0$ if and only if the conic $ax^2+by^2=z^2$ has a rational point. In particular, we have $(a,-a)=0$, and $(a,b)=0$ if $a+b=1$.

A deep theorem of Merkurjev says that the 2-torsion subgroup $\mathrm{Br}(k)[2]$ is generated by the classes (a,b); moreover, the kernel of the surjective map $k^*/k^{*2} \otimes k^*/k^{*2} \to \mathrm{Br}(k)[2]$ is generated by the elements $(a, 1-a)$, where $a \in k \setminus \{0,1\}$. In other words, $\mathrm{Br}(k)[2] \simeq K_2^M(k)/2$, where $K_2^M(k)$ is the second Milnor K-group of the field k, see [GS17, Ch. 8].

1.2.2 Cyclic algebras

Quaternion algebras are a special case of the following construction. Let K/k be a Galois extension of fields such that the Galois group $G = \mathrm{Gal}(K/k)$ is cyclic of order n. Let σ be a generator of G and let $\chi\colon G \xrightarrow{\sim} \mathbb{Z}/n$ be the character sending σ to $1 \in \mathbb{Z}/n$. Let $b \in k^*$.

The *cyclic algebra* $D_k(\chi, b)$ is the k-algebra generated by the field K and a symbol y subject to the relations $y^n = b$ and $\lambda y = y\sigma(\lambda)$ for any $\lambda \in K$, cf. [GS17, Prop. 2.5.2]. This is a central simple k-algebra of degree n, which contains K as a maximal subfield. Conversely, any central simple k-algebra of degree n with a maximal subfield K which is cyclic of degree n over k is isomorphic to $D_k(\chi, b)$ for some $b \in k^*$, see [GS17, Prop. 2.5.3]. Note that this construction also works if n is a prime p and $\mathrm{char}(k) = p > 0$, see [GS17, Cor. 2.5.5]. We write (χ, b) for the class of $D_k(\chi, b)$ in $\mathrm{Br}(k)$.

When $\mathrm{char}(k)$ does not divide n and k contains all n-th roots of 1, one can describe the cyclic algebra $D_k(\chi, b)$ without mentioning the Galois action. Let $\omega \in \mu_n$ be a primitive root. For $a, b \in k^*$ let $(a,b)_\omega$ be the k-algebra with generators x, y and relations $x^n = a$, $y^n = b$, $xy = \omega yx$. One checks that this is a central simple k-algebra. Permuting x and y we obtain an isomorphism

$$(a,b)_\omega \cong (b,a)_{\omega^{-1}}.$$

Assume that $K = k[t]/(t^n - a)$ is a field. Then K is a cyclic extension of k of degree n. Let $\sqrt[n]{a} \in K$ be the image of t in K. There is a unique element $\sigma \in G = \mathrm{Gal}(K/k)$ such that $\sigma(\sqrt[n]{a}) = \omega \sqrt[n]{a}$. If $\chi\colon G \to \mathbb{Z}/n$ is the character that sends σ to $1 \in \mathbb{Z}/n$, then the k-algebras $(a,b)_\omega$ and $D_k(\chi, b)$ are isomorphic [GS17, Cor. 2.5.5].

1.2.3 C_1-fields

The point of view of central simple algebras allows one to prove the triviality of the Brauer group of several types of fields which are fundamental for arithmetic and geometry.

Definition 1.2.11 (Lang) *A field k is called a C_1-field if any homogeneous form of degree d in $n > d$ variables with coefficients in k has a non-trivial zero in k.*

One easily checks that any finite field extension of a C_1-field is a C_1-field [GS17, Lemma 6.2.4].

Theorem 1.2.12 *If k is a C_1-field, then* $\mathrm{Br}(k) = 0$.

Proof. A central simple k-algebra A is equipped with a *reduced norm*, which is a multiplicative function $\mathrm{Nrd}_A\colon A \to k$. Let d be the degree of A. Choosing a basis of the vector space A over k one can write Nrd_A as a homogeneous form of degree d in d^2 variables with coefficients in k. (By Theorem 1.2.7, after extending the ground field from k to k_{s} the algebra $A \otimes_k k_{\mathrm{s}}$ can be identified with the matrix algebra $M_d(k_{\mathrm{s}})$. Under this identification, the reduced norm becomes the determinant.) Let $A = D$ be a division algebra. If $D \neq k$, then Nrd is a homogeneous polynomial of degree d in $d^2 > d$ variables. (For all this, see [GS17, §2.6, §6.2].) Thus if k is a C_1-field, then $D = k$, so that $\mathrm{Br}(k) = 0$. □

Theorem 1.2.13 *If k is a finite field, then k is a C_1-field and* $\mathrm{Br}(k) = 0$.

Proof. By Wedderburn's Little Theorem every finite ring with no zero-divisors is a field. In particular, the only finite-dimensional central division k-algebra is k itself. This gives $\mathrm{Br}(k) = 0$. The stronger statement that a finite field is a C_1-field is the Chevalley–Warning theorem [GS17, Thm. 6.2.6]. □

Theorem 1.2.14 (Tsen) *Let k be a field of transcendence degree* 1 *over an algebraically closed field. Then k is a C_1-field and* $\mathrm{Br}(k) = 0$.

Proof. This is proved in [GS17, Thm. 6.2.8]. The proof is an extension of the proof of Theorem 1.1.14. □

For fields of transcendence degree 1 over a separably closed field, see Proposition 3.8.2.

A local ring R with maximal ideal $\mathfrak{m}$ and residue field k is *henselian* if it satisfies the following property: for any monic polynomial $P(t) \in R[t]$ whose reduction modulo $\mathfrak{m}$ is a product $\overline{P}(t) = q(t)s(t)$ of coprime monic polynomials $q(t), s(t) \in k[t]$, there exist monic polynomials $Q(t), S(t) \in R[t]$ such that $\overline{Q}(t) = q(t)$, $\overline{S}(t) = s(t)$ and $P(t) = Q(t)S(t)$.

There are several equivalent definitions of a henselian local ring, see [Ray70b, Ch. I, §1], [Ray70b, Ch. VII, Prop. 3], [BLR90, §2.3] and [Stacks, Section 09XI]. In particular, a local ring R is henselian if for every monic polynomial $P(t) \in R[t]$ every simple root in k of the reduction of $P(t)$ modulo $\mathfrak{m}$ lifts to a root of $P(t)$ in R. Using Newton's approximation one proves that any complete local ring is henselian. (See [Stacks, Section 04GE].)

Define the *completion* of R at $\mathfrak{m}$ as $\widehat{R} = \varprojlim(R/\mathfrak{m}^n)$. A local ring R is $\mathfrak{m}$-*adically complete* if the canonical map $R \to \widehat{R}$ is an isomorphism. If $\mathfrak{m}$ is finitely generated, then the completion $\widehat{R}$ of R at $\mathfrak{m}$ is a complete local ring with maximal ideal $\mathfrak{m}\widehat{R}$ and residue field k, see [Stacks, Section 00M9].

See Section 2.6 for the definition and basic properties of excellent rings.

Theorem 1.2.15 *Let R be a henselian discrete valuation ring with algebraically closed residue field k. Let K be the fraction field of R.*

(i) *If R is excellent, for example if* $\mathrm{char}(K) = 0$ *or if R is complete, then K is a C_1-field, and hence* $\mathrm{Br}(K) = 0$.
(ii) *In general, we have* $\mathrm{Br}(K) = 0$.

Proof. (i) See Lang's thesis [Lang52], see also [Shatz, Thm. 27, p. 116]. The excellence property is needed to ensure that the field of fractions $\widehat{K}$ of the completion $\widehat{R}$ is a separable extension of K.

(ii) There are several other ways to establish $\mathrm{Br}(K) = 0$ under the assumption that R is complete [SerCL, Ch. XII, §1, §2]. As pointed out in [Mil80, Ch. III, Example 2.22 (a)], these proofs also give $\mathrm{Br}(K) = 0$ for R henselian with algebraically closed residue field. □

See Proposition 1.4.5 for the case when the residue field is separably closed but not algebraically closed.

Corollary 1.2.16 *Let R be a henselian discrete valuation ring with* **perfect** *residue field k and field of fractions K of characteristic zero. Let K_{nr} be the maximal unramified extension of K. Then K_{nr} is a C_1-field.*

Proof. The field K_{nr} is the field of fractions of the strict henselisation of R, which is a henselian discrete valuation ring with algebraically closed residue field. Since $\mathrm{char}(K) = 0$, the result is a special case of Theorem 1.2.15. □

Remark 1.2.17 Let K be a henselian discretely valued field and let $\widehat{K}$ be the completion of K. Then the natural map $\mathrm{Br}(K) \to \mathrm{Br}(\widehat{K})$ is an isomorphism. For a proof see Proposition 7.1.8.

1.3 The language of Galois cohomology

1.3.1 Group cohomology and Galois cohomology

We now assume that the reader is familiar with the cohomology theory of abstract groups, which can be found in many places in the literature, for example in [AW65], [SerCG], [SerCL], [GS17], [Wei94] and [Har17].

Let G be a group and let M be a G-module. The group $\mathrm{H}^0(G, M) := M^G$ is the subgroup of G-invariant elements of M. Higher cohomology groups $\mathrm{H}^n(G, M)$, $n \geq 1$, are the right derived functors of the functor from the category of G-modules to the category of abelian groups that sends M to M^G.

They can be computed using the standard projective resolution $P_\bullet \to \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, as the cohomology groups of the complex $\mathrm{Hom}_G(P_\bullet, M)$. This leads to the definition in terms of homogeneous cocycles, which can be restated as a definition in terms of inhomogeneous cocycles.

We refer to the books mentioned above for the following aspects of the cohomology of groups:

- its relation with the cohomology of subgroups: restriction, inflation, and corestriction in the case of a subgroup $H \subset G$ of finite index, Shapiro's lemma;
- long exact sequences coming from the Hochschild–Serre spectral sequence;
- cup-products and their properties with respect to boundary maps in exact cohomology sequences;
- cohomology of cyclic groups, Herbrand's quotient theorem.

Let G be a group that acts on a (not necessarily commutative) group A preserving the group structure. In this case we call A a G-group. We denote the result of applying $g \in G$ to $a \in A$ by ${}^g a$. A 1-*cocycle* is a function $a = \{a_g\} \colon G \to A$ which satisfies the condition

$$a_{gh} = a_g \cdot {}^g a_h$$

for all $g, h \in G$. Two cocycles $\{a_g\}$ and $\{b_g\}$ are called *equivalent* if there exists an element $c \in A$ such that for any $g \in G$ one has

$$a_g = c^{-1} \cdot b_g \cdot {}^g c.$$

The 1-*cohomology set* $\mathrm{H}^1(G, A)$ is defined as the set of equivalence classes of 1-cocycles $G \to A$. A cocycle $c^{-1} \cdot {}^g c$, where $c \in A$, is called trivial. The class of trivial cocycles is the *distinguished* point of $\mathrm{H}^1(G, A)$, so $\mathrm{H}^1(G, A)$ is naturally a pointed set.

Now suppose that G is a profinite group and the action of G on A is *continuous* when A is given the discrete topology. Equivalently, the stabilisers of elements of A are open subgroups of G, i.e. closed subgroups of G of finite index. One defines the continuous cohomology pointed set $\mathrm{H}^1(G, A)$ as the direct limit of the pointed sets $\mathrm{H}^1(G/U, A^U)$, where $U \subset G$ ranges over all open normal subgroups – any such subgroup being of finite index in G. Alternatively, one defines $\mathrm{H}^1(G, A)$ as the set of equivalence classes of *continuous* cocycles $G \to A$. Note that for a profinite group G the continuous cohomology set does not necessarily coincide with the abstract cohomology set. Unless otherwise mentioned, we shall only use continuous cohomology sets in this book.

If M is a continuous discrete G-module, then the continuous cohomology group $\mathrm{H}^i(G, M)$ is defined for all $i \geq 0$ as the direct limit of $\mathrm{H}^i(G/U, M^U)$ over the set of open normal subgroups $U \subset G$.

A short exact sequence of continuous discrete G-groups

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1,$$

where A is normal in B, gives rise to an exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \to \mathrm{H}^1(G,A) \to \mathrm{H}^1(G,B) \to \mathrm{H}^1(G,C).$$

If A is central in B, it extends to an exact sequence of pointed sets

$$1 \to A^G \to B^G \to C^G \to \mathrm{H}^1(G,A) \to \mathrm{H}^1(G,B) \to \mathrm{H}^1(G,C) \to \mathrm{H}^2(G,A).$$

If B is abelian, it can be continued to the usual long exact sequence.

An important particular case is when k is a field with separable closure k_s and the absolute Galois group $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$ acts on the group of k_s-points of an algebraic group A over k. The pointed set $\mathrm{H}^1(\Gamma, A(k_\mathrm{s}))$ does not depend on the choice of k_s; it is well defined up to a canonical isomorphism [SerCG, Ch. II, §1, 1.1] and is denoted by $\mathrm{H}^1(k,A)$. The map $K \mapsto \mathrm{H}^1(K, A \times_k K)$ defines a functor from the category of field extensions of k to the category of pointed sets.

If A is a commutative algebraic group over k and $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$, the abelian group $\mathrm{H}^i(\Gamma, A(k_\mathrm{s}))$ is well defined for any integer $i \geq 0$, up to canonical isomorphism [SerCG, Ch. II, §1, 1.1]; it is denoted by $\mathrm{H}^i(k,A)$. The map $K \mapsto \mathrm{H}^i(K, A \times_k K)$ is a functor from the category of field extensions K of k to the category of abelian groups.

We shall mostly deal with the case of the projective linear group, so let us recall its definition. The group $\mathrm{PGL}_n(k)$ is defined by the exact sequence of groups

$$1 \longrightarrow k^* \longrightarrow \mathrm{GL}_n(k) \longrightarrow \mathrm{PGL}_n(k) \longrightarrow 1,$$

where the second map is the embedding of the central subgroup of scalar matrices. The multiplicative group $\mathbb{G}_{m,k}$ represents the functor associating to a commutative k-algebra R the group of invertible elements R^*. The algebraic group $\mathrm{GL}_{n,k}$ represents the functor associating to a commutative k-algebra R the group $\mathrm{GL}_n(R)$. (In particular, $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k}$.) Finally, the algebraic group $\mathrm{PGL}_{n,k}$ is defined by the exact sequence of algebraic k-groups

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow \mathrm{GL}_{n,k} \longrightarrow \mathrm{PGL}_{n,k} \longrightarrow 1. \tag{1.2}$$

1.3.2 Galois descent

A general reference for Galois descent is [BLR90, §6.2, Example B], see also [SerCL, Ch. X], [PR94, Section 2.2], [Sko01, Ch. 2], [GS17, Ch. 2.3], [Ols16, Ch. 4] and [Po18, Ch. 4].

Let K/k be a finite Galois extension of fields with Galois group $\mathrm{Gal}(K/k)$. The descent problem deals with the following question: when can a scheme X' over K be descended to k, that is, does there exist a scheme X over k such that $X' \cong X \times_k K$? Grothendieck explored the analogy with the classical case, where a topological space or a differentiable manifold can be constructed by glueing together open subsets via transition functions which satisfy a compatibility condition on triple intersections. A 'descent datum' is an analogue of this for schemes. (Descent data can be defined more generally for any category fibred over a category with finite fibred products, see [Ols16, §4.2] or Section 4.1.2 below.) In [BLR90, pp. 140–141] it is shown that giving a 'descent datum' on a K-scheme X' with respect to K/k is equivalent to giving an action of $\mathrm{Gal}(K/k)$ on X' that is compatible with the action of $\mathrm{Gal}(K/k)$ on K by automorphisms. This descent problem is 'effective' (that is, there is a scheme X over k such that $X' \cong X \times_k K$) when X' is quasi-separated and the $\mathrm{Gal}(K/k)$-orbit of every point of X' is contained in a quasi-affine open subscheme of X'. In particular, Galois descent is effective for quasi-projective varieties over a field.

Let X be a variety over k. Let K/k be a Galois extension (not necessarily finite) with Galois group $\mathrm{Gal}(K/k)$. A k-variety Y is called a *(K/k)-form of X* if there is an isomorphism $Y \times_k K \cong X \times_k K$ of K-varieties. Using effectivity of Galois descent one shows that if X is a quasi-projective variety over k, then the (K/k)-forms of X are classified, up to isomorphism, by the elements of the Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{Aut}(X \times_k K))$ in such a way that the isomorphism class of X corresponds to the distinguished point. See [Po18, §4.4, §4.5] for a detailed proof of this classical result.

For example, (k_{s}/k)-forms of a projective space are called *Severi–Brauer varieties*. It is not hard to see that the Severi–Brauer varieties of dimension 1 are precisely the smooth projective conics. Indeed, the linear system attached to the anticanonical sheaf on such a curve embeds it as a conic in $\mathbb{P}^2_k$.

By a theorem of Châtelet, a Severi–Brauer variety is isomorphic to $\mathbb{P}^{n-1}_k$ if and only if it has a k-point, see Section 7.1 for this and other results on Severi–Brauer varieties. Note that the automorphism functor of $\mathbb{P}^{n-1}_k$ is represented by the group k-scheme $\mathrm{PGL}_{n,k}$.

More generally, suppose that we have a quasi-projective variety X over k endowed with an action of a group k-scheme A. By definition, each cohomology class in $\mathrm{H}^1(k, A)$ contains a 1-cocycle $c\colon \Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k) \to A(k_{\mathrm{s}})$; it comes from a 1-cocycle $c\colon \mathrm{Gal}(K/k) \to A(K)$ for some finite Galois extension $k \subset K$. The cocycle c defines a twisted action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ which is the composition of the action on $X \times_k K$ via the second factor with the action of $c(g) \in A(K)$. The cocycle condition is equivalent to this being an action of $\mathrm{Gal}(K/k)$ on $X \times_k K$ compatible with the action of $\mathrm{Gal}(K/k)$ on K by automorphisms. By effectivity of Galois descent, there exists a quasi-projective variety X^c over k such that the K-varieties $X \times_k K$ and $X^c \times_k K$ are isomorphic; this isomorphism identifies the action of $\mathrm{Gal}(K/k)$ on $X^c \times_k K$ via the second factor with the twisted by c action of $\mathrm{Gal}(K/k)$ on $X \times_k K$. The variety X^c is called the *twist* of X by c. By construction, it is a (k_s/k)-form of X. Replacing c by an equivalent cocycle gives rise to a variety isomorphic to X^c. Particular cases of this situation include the following (see [Sko01, pp. 12–13], [Po18, §4.5]).

(a) Twists of the vector space k^n by a 1-cocycle with coefficients in $A = \mathrm{GL}_{n,k}$ are isomorphic to k^n, cf. [Po18, §1.3].
(b) Twists of the matrix algebra $M_n(k)$ by a 1-cocycle with coefficients in $A = \mathrm{PGL}_{n,k}$ are central simple algebras of degree n. Moreover, by [SerCL, Ch. X, §5, Prop. 8], this gives a bijection between the isomorphism classes of central simple algebras of degree n and the pointed set $\mathrm{H}^1(k, \mathrm{PGL}_{n,k})$.
(c) Torsors of an algebraic k-group A are obtained by twisting A by a 1-cocycle with coefficients in A acting on itself by left translations. In this case A represents the automorphism functor of A considered together with its right action on itself, i.e., of A as a right A-torsor. Using effectivity of Galois descent one shows that the isomorphism classes of right A-torsors over k bijectively correspond to the elements of $\mathrm{H}^1(k, A)$. (This is the easy case of [BLR90, §6.5, Thm. 1], see also [Sko01, p. 13].) For example, the affine conic $x^2 - ay^2 = c$ is a torsor for the norm 1 torus given by $x^2 - ay^2 = 1$. Also, a smooth projective curve of genus 1 is a torsor for its Jacobian.
(d) Suppose that an algebraic k-group A acts on an algebraic k-group G by automorphisms. Twisting G by a 1-cocycle $\Gamma \to A$ one obtains a (k_s/k)-form of G. For example, the group of invertible elements of a central simple k-algebra of degree n is the group of k-points of a twist of $\mathrm{GL}_{n,k}$ by a 1-cocycle with values in $A = \mathrm{PGL}_{n,k}$. For any commutative algebraic group one defines quadratic twists by taking $A = \{\pm 1\}$, where -1 sends x to x^{-1}. For example, the quadratic twists of $\mathbb{G}_{m,k}$ are the norm tori $x^2 - ay^2 = 1$, where $a \in k^*$. The quadratic twists of an elliptic curve $y^2 = x^3 + ax + b$ are the elliptic curves $cy^2 = x^3 + ax + b$, where $c \in k^*$.

Looking closer at the case of vector spaces one deduces the triviality of 1-cocycles with coefficients in $\mathrm{GL}_{n,k}$.

Theorem 1.3.1 (Speiser) *For any Galois extension of fields K/k with Galois group G we have* $\mathrm{H}^1(G, \mathrm{GL}_{n,k}(K)) = \{1\}$.

Proof. Let us show that every 1-cocycle $c\colon G \to \mathrm{GL}_{n,k}(K)$ is trivial.

Let V be a k-vector space of dimension n. The twist of V by c is a vector space V^c over k of dimension n such that there is an isomorphism of K-vector spaces $\psi\colon V \otimes_k K \xrightarrow{\sim} V^c \otimes_k K$ which identifies the action of an element $g \in G$ on $V \otimes_k K$ which sends $v \otimes x$ to $c_g(v \otimes {}^g x)$ with the action on $V^c \otimes_k K$ via the second factor. All k-vector spaces of dimension n are isomorphic, so we can choose an isomorphism of k-vector spaces $\varphi\colon V^c \to V$.

We obtain the following commutative diagram where the horizontal arrows are isomorphisms of K-vector spaces and the vertical arrows describe the action of $g \in G$:

$$\begin{array}{ccccc} V \otimes_k K & \xrightarrow[\psi]{\sim} & V^c \otimes_k K & \xrightarrow[\varphi\otimes\mathrm{id}]{\sim} & V \otimes_k K \\ \downarrow{\scriptstyle c_g(\mathrm{id}\otimes g)} & & \downarrow{\scriptstyle \mathrm{id}\otimes g} & & \downarrow{\scriptstyle \mathrm{id}\otimes g} \\ V \otimes_k K & \xrightarrow[\psi]{\sim} & V^c \otimes_k K & \xrightarrow[\varphi\otimes\mathrm{id}]{\sim} & V \otimes_k K \end{array}$$

Let σ be the composition of horizontal arrows. The commutativity of the diagram gives $c_g = \sigma^{-1}(g\sigma g^{-1}) = \sigma^{-1} \cdot {}^g\sigma$ for any $g \in G$, so c is trivial. □

This theorem is often proved by a direct cocycle computation, see [SerCL, Ch. X, Prop. 3]. See also [GS17, Example 2.3.4] and [Po18, Prop. 1.3.15].

Theorem 1.3.2 (Hilbert's theorem 90) *For any Galois extension of fields K/k with Galois group G we have* $\mathrm{H}^1(G, K^*) = 0$.

This is a particular case of Speiser's theorem for $n = 1$. For later use let us record a corollary of this theorem: given field extensions $k \subset K \subset L$ with L/k and K/k Galois, there is a short exact sequence

$$0 \to \mathrm{H}^2(\mathrm{Gal}(K/k), K^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(L/k), L^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(L/K), L^*) \quad (1.3)$$

where the first arrow is inflation and the second arrow is restriction.

Applying Hilbert's theorem 90 to (1.2) we see that for any field extension K/k the group of K-points of $\mathrm{PGL}_{n,k}$ is precisely $\mathrm{PGL}_n(K)$. Proposition 1.2.3 then shows that the natural map

$$\mathrm{PGL}_n(K) \longrightarrow \mathrm{Aut}_{K\text{-alg}}(M_n(K))$$

is an isomorphism of groups, where K-alg is the category of K-algebras. When K is a Galois extension of k, this isomorphism respects the Galois action on both sides. This shows that the automorphism functor of the matrix algebra $M_n(k)$ (which is a functor from the category of field extensions of k to the category of groups) is represented by the algebraic group $\mathrm{PGL}_{n,k}$.

Theorem 1.3.3 (Skolem–Noether) *All automorphisms of a central simple algebra over a field are inner automorphisms.*

Proof. Let A be a central simple algebra over a field k. Choose a finite Galois extension K/k that splits A (see Theorem 1.2.7). The homomorphism $A^* \to \mathrm{Aut}_{k\text{-alg}}(A)$ sending an element to the conjugation by this element extends to a similar map over K. Let $G = \mathrm{Gal}(K/k)$. We then have the exact sequence of G-groups

$$1 \longrightarrow K^* \longrightarrow (A \otimes_k K)^* \longrightarrow \mathrm{Aut}_{K\text{-alg}}(A \otimes_k K) \longrightarrow 1,$$

where surjectivity of the third map follows from Proposition 1.2.3. The long exact cohomology sequence gives an exact sequence of pointed sets

$$1 \longrightarrow k^* \longrightarrow A^* \longrightarrow \mathrm{Aut}_{k\text{-alg}}(A) \longrightarrow \mathrm{H}^1(G, K^*).$$

Since $\mathrm{H}^1(G, K^*) = 0$ by Hilbert's theorem 90, the homomorphism

$$A^* \longrightarrow \mathrm{Aut}_{k\text{-alg}}(A)$$

is surjective. □

There is actually a more general result.

Theorem 1.3.4 (Skolem–Noether) *Let k be a field, let B be a simple k-algebra and let A be a central simple algebra over k. Then all non-zero k-homomorphisms $B \to A$ are injective and can be obtained from one another by conjugations in A.*

Proof. See [Rei03, Thm. 7.21]. □

1.3.3 Cohomological description of the Brauer group

Let K/k be a finite Galois extension of fields with Galois group G. Recall that a central simple algebra of degree n over k is split by K, i.e., is a (K/k)-form of $M_n(k)$, if and only if there exists an isomorphism of K-algebras $A \otimes_k K \cong M_n(K)$. Let us denote by $\mathrm{Az}_{n,K}$ the set of isomorphism classes of central simple algebras of degree n over k which are split by K. As discussed in the previous section, we have a bijection of pointed sets

$$\mathrm{Az}_{n,K} \xrightarrow{\sim} \mathrm{H}^1(G, \mathrm{PGL}_n(K)).$$

Since $\mathrm{H}^1(G, \mathrm{GL}_n(K)) = \{1\}$ by Theorem 1.3.1, the exact sequence of pointed cohomology sets attached to (1.2),

$$\mathrm{H}^1(G, \mathrm{GL}_n(K)) \longrightarrow \mathrm{H}^1(G, \mathrm{PGL}_n(K)) \longrightarrow \mathrm{H}^2(G, K^*),$$

gives rise to maps

$$\mathrm{Az}_{n,K} \longrightarrow \mathrm{H}^2(G, K^*)$$

with trivial kernel. One easily checks that for given n and m there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & k^* & \to & \mathrm{GL}_n(k) & \to & \mathrm{PGL}_n(k) & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & k^* & \to & \mathrm{GL}_{nm}(k) & \to & \mathrm{PGL}_{nm}(k) & \to & 1 \end{array}$$

where the middle vertical map sends a matrix M to the matrix with m diagonal blocks equal to M and zero elsewhere. Replacing k by K and taking Galois cohomology we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{H}^1(G,\mathrm{PGL}_n(K)) & \to & \mathrm{H}^2(G,K^*) \\ \downarrow & & \| \\ \mathrm{H}^1(G,\mathrm{PGL}_{nm}(K)) & \to & \mathrm{H}^2(G,K^*) \end{array}$$

The left vertical map can be identified with the map $\mathrm{Az}_{n,K}{\to}\mathrm{Az}_{nm,K}$ sending A to $A\otimes_k M_m(k)$. Passing to the limit over n we obtain a map of pointed sets

$$\mathrm{Br}(K/k)\longrightarrow \mathrm{H}^2(G,K^*)$$

with trivial kernel. Using Theorem 1.2.7 and passing to the limit over finite Galois extensions K/k, we get a map of pointed sets

$$\mathrm{Br}(k)\longrightarrow \mathrm{H}^2(k,k_{\mathrm{s}}^*)$$

with trivial kernel. One then establishes the following properties.

- These maps are homomorphisms of groups, hence they are injective. See [GS17, Prop. 2.7.9].
- These maps are surjective. This is proved by a cocycle computation using the classical construction of crossed products, see [SerCL, Ch. X, §5, Prop. 9]. An elegant cocycle-free proof is given in [GS17, Thm. 4.4.1].

We summarise this as the following theorem.

Theorem 1.3.5 *For a field k and a Galois extension of fields K/k there are natural isomorphisms of abelian groups*

$$\mathrm{Br}(K/k)\xrightarrow{\sim}\mathrm{H}^2(\mathrm{Gal}(K/k),K^*)$$

and

$$\mathrm{Br}(k)\xrightarrow{\sim}\mathrm{H}^2(k,k_{\mathrm{s}}^*).$$

The second isomorphism is functorial with respect to arbitrary field extensions of k, see [SerCL, Ch. 10, §4].

The cohomological description of the Brauer group is very useful. For example, it gives a quick proof of Theorem 1.2.9.

Corollary 1.3.6 *For any field k the Brauer group* $\mathrm{Br}(k)$ *is a torsion group.*

Proof. The group $\mathrm{Br}(k)$ is the direct limit of $\mathrm{Br}(K/k)=\mathrm{H}^2(\mathrm{Gal}(K/k),K^*)$, where K/k is a finite Galois extension. But if G is finite, then $\mathrm{H}^i(G,M)$, where M is any G-module and $i\geq 1$, is annihilated by the order of G. (Indeed, the composition of the restriction to a subgroup $H\subset G$ with the corestriction is the multiplication by the index $[G:H]$. One applies this to the case when H consists of the identity element of G.) □

Theorem 1.3.7 *If k is a field of characteristic $p>0$, then the group* $\mathrm{Br}(k)$ *is p-divisible. Moreover, if k is perfect, then* $\mathrm{Br}(k)\{p\}=0$.

Proof. Let k_s be a separable closure of k. The map $x\mapsto x^p$ gives rise to the exact sequence of Γ-modules

$$0\longrightarrow k_\mathrm{s}^*\longrightarrow k_\mathrm{s}^*\longrightarrow M\longrightarrow 1,$$

where $pM=0$. The p-cohomological dimension of a field of characteristic $p>0$ is at most 1 [SerCG, Ch. II, §2.2, Prop. 3], hence $\mathrm{H}^2(k,M)=0$. This implies that $p\mathrm{Br}(k)=\mathrm{Br}(k)$. If k is perfect, then $M=0$ and the map $x\mapsto x^p$ is an automorphism of the Galois module k_s^*. Thus multiplication by p is an automorphism of the group $\mathrm{Br}(k)$, hence $\mathrm{Br}(k)\{p\}=0$. □

Let $k\subset K$ be an arbitrary field extension. The map

$$\mathrm{res}_{K/k}\colon \mathrm{Br}(k)\longrightarrow \mathrm{Br}(K)$$

defined by associating to a central simple k-algebra A the central simple K-algebra $A\otimes_k K$ coincides with the cohomological restriction map

$$\mathrm{H}^2(k,k_\mathrm{s}^*)\longrightarrow \mathrm{H}^2(K,K_\mathrm{s}^*).$$

Let us spell out the formalism of corestriction in the special case of the Brauer group and finite separable extensions of fields. For a more general context, which includes not necessarily separable field extensions, see Section 3.8. Let $K\subset k_\mathrm{s}$ be a separable finite field extension of k. We have an isomorphism

$$\mathrm{Br}(K)\cong \mathrm{H}^2(K,k_\mathrm{s}^*)\cong \mathrm{H}^2(k,(k_\mathrm{s}\otimes_k K)^*)$$

obtained using Shapiro's lemma and the fact that $(k_\mathrm{s}\otimes_k K)^*$ is the direct product of finitely many copies of k_s^* indexed by the embeddings of $K\hookrightarrow k_\mathrm{s}$, so the $\mathrm{Gal}(k_\mathrm{s}/k)$-module $(k_\mathrm{s}\otimes_k K)^*$ is induced from the $\mathrm{Gal}(k_\mathrm{s}/K)$-module k_s^*. The norm $\mathrm{N}_{K/k}\colon K\to k$ gives rise to a map of Galois modules $(k_\mathrm{s}\otimes_k K)^*\to k_\mathrm{s}^*$, hence to a homomorphism $\mathrm{H}^2(k,(k_\mathrm{s}\otimes_k K)^*)\to \mathrm{H}^2(k,k_\mathrm{s}^*)$. This defines a *corestriction* (or *transfer*) map

$$\mathrm{cores}_{K/k}\colon \mathrm{Br}(K)\cong \mathrm{H}^2(K,k_\mathrm{s}^*)\cong \mathrm{H}^2(k,(k_\mathrm{s}\otimes_k K)^*)\longrightarrow \mathrm{H}^2(k,k_\mathrm{s}^*)=\mathrm{Br}(k).$$

Since $\mathrm{N}_{K/k}(x)=x^n$ for $x\in k$, where $n=[K:k]$, the composition

$$\mathrm{cores}_{K/k}\circ\mathrm{res}_{K/k}\colon \mathrm{Br}(k)\longrightarrow \mathrm{Br}(K)\longrightarrow \mathrm{Br}(k)$$

is the multiplication by the degree $[K:k]$.

1.3.4 Cyclic algebras, cup-products and the Kummer sequence

Let G be a cyclic group of order n. Fix a generator σ of G. Let

$$\chi \in \mathrm{Hom}(G, \mathbb{Z}/n) \cong \mathrm{H}^1(G, \mathbb{Z}/n)$$

be the homomorphism sending σ to $1 \in \mathbb{Z}/n$. The exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{[n]} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

induced by multiplication by n on $\mathbb{Z}$ gives rise to an isomorphism

$$d\colon \mathrm{H}^1(G, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^2(G, \mathbb{Z})[n],$$

and so defines the class $d(\chi) \in \mathrm{H}^2(G, \mathbb{Z})[n]$. For any G-module A the cup-product with $d(\chi) \in \mathrm{H}^2(G, \mathbb{Z})$ induces an isomorphism

$$\mathrm{H}^i(G, A) \xrightarrow{\sim} \mathrm{H}^{i+2}(G, A)$$

for $i \geq 1$. For $i = 0$ it induces an isomorphism

$$A^G/N_G A = \widehat{\mathrm{H}}^0(G, A) \xrightarrow{\sim} \mathrm{H}^2(G, A),$$

where $N_G = \sum_{g \in G} g \in \mathbb{Z}[G]$. The first identification here is the definition of the Tate cohomology group $\widehat{\mathrm{H}}^0(G, A)$. For all this, see [GS17, Prop. 3.4.11].

For a Galois field extension K/k with cyclic Galois group $\mathrm{Gal}(K/k) \simeq \mathbb{Z}/n$ with generator σ, the previous considerations give an isomorphism

$$k^*/\mathrm{N}_{K/k}(K^*) \xrightarrow{\sim} \mathrm{H}^2(G, K^*) = \mathrm{Ker}[\mathrm{Br}(k) \to \mathrm{Br}(K)]. \tag{1.4}$$

The map here is the cup-product with $d(\chi) \in \mathrm{H}^2(G, \mathbb{Z})$, so it depends on the choice of a generator $\sigma \in G$.

Recall that for an element $a \in k^*$ we denote by $(\chi, a) \in \mathrm{Br}(k)$ the class of the cyclic algebra $D_k(\chi, a)$, see Section 1.2.2. By [GS17, Prop. 4.7.3, Cor. 4.7.4] we have

$$(\chi, a) = a \cup d(\chi) = d(\chi) \cup a \in \mathrm{Br}(k). \tag{1.5}$$

The second equality is due to the fact that a and $d(\chi)$ are elements of cohomology groups of even degree.

Proposition 1.3.8 *Let k be a field, let $a \in k^*$ and let $\chi\colon \Gamma \to \mathbb{Z}/n$ be a homomorphism. Let $K \subset k_{\mathrm{s}}$ be the invariant subfield of the kernel of χ. The class $(\chi, a) \in \mathrm{Br}(k)$ of the cyclic algebra $D_k(\chi, a)$ is zero if and only if $a \in k^*$ is a norm for the extension K/k.*

Proof. This follows from (1.4) and (1.5). □

Let n be a positive integer invertible in k. Then the map $x \mapsto x^n$ on k_s^* is surjective and hence gives rise to an exact sequence of Galois modules

$$1 \longrightarrow \mu_n \longrightarrow k_s^* \longrightarrow k_s^* \longrightarrow 1, \tag{1.6}$$

called the *Kummer exact sequence*. Taking Galois cohomology, and using Hilbert's theorem 90, we obtain isomorphisms

$$k^*/k^{*n} \xrightarrow{\sim} \mathrm{H}^1(k,\mu_n) \quad \text{and} \quad \mathrm{H}^2(k,\mu_n) \xrightarrow{\sim} \mathrm{Br}(k)[n].$$

The first of these isomorphisms (the connecting map in the long exact sequence of Galois cohomology) associates to an element $a \in k^*$ the class of the 1-cocycle which sends $g \in \Gamma$ to $g(b)b^{-1} \in \mu_n(k_s)$, where $b \in k_s^*$ is such that $b^n = a$. We shall denote this class by $(a)_n$.

The cup-product gives rise to the pairing

$$\mathrm{H}^1(k,\mathbb{Z}/n) \times \mathrm{H}^1(k,\mu_n) \longrightarrow \mathrm{H}^2(k,\mu_n) \cong \mathrm{Br}(k)[n].$$

In Section 1.4.4 we shall prove an important formula

$$(\chi, a) = \chi \cup (a)_n \in \mathrm{Br}(k). \tag{1.7}$$

Continue to assume that n is invertible in k and also assume that $\mu_n(k_s) \subset k$, so that μ_n is isomorphic to $\mathbb{Z}/n$ as a Γ-module. Since $\mathrm{H}^1(k,\mu_n) \cong k^*/k^{*n}$ we see that every cyclic field extension of k of degree n is of the form $k(\sqrt[n]{a})$ for some $a \in k^*$. The cup-product pairing

$$\cup\colon k^*/k^{*n} \times k^*/k^{*n} = \mathrm{H}^1(k,\mu_n) \times \mathrm{H}^1(k,\mu_n) \longrightarrow \mathrm{H}^2(k,\mu_n^{\otimes 2})$$

is anticommutative, that is, we have $a \cup b = -b \cup a$. Choose an isomorphism $\mu_n \xrightarrow{\sim} \mathbb{Z}/n$, which is equivalent to choosing a primitive root of unity $\omega \in k$ (sent to $1 \in \mathbb{Z}/n$). This induces an isomorphism $\mu_n^{\otimes 2} \xrightarrow{\sim} \mu_n$, hence an isomorphism

$$\mathrm{H}^2(k,\mu_n^{\otimes 2}) = \mathrm{H}^2(k,\mu_n) \otimes \mu_n \xrightarrow{\sim} \mathrm{H}^2(k,\mu_n) = \mathrm{Br}(k)[n].$$

The inverse map sends a class $\alpha \in \mathrm{H}^2(k,\mu_n)$ to $\alpha \otimes \omega$. For $a, b \in k^*$ we denote the image of (a,b) under the composite map

$$k^* \times k^* \longrightarrow k^*/k^{*n} \times k^*/k^{*n} \longrightarrow \mathrm{H}^2(k,\mu_n^{\otimes 2}) \longrightarrow \mathrm{H}^2(k,\mu_n) = \mathrm{Br}(k)[n]$$

by $(a,b)_\omega$. Under the isomorphism $\mathrm{H}^2(k,\mu_n^{\otimes 2}) = \mathrm{H}^2(k,\mu_n) \otimes \mu_n$ the class $a \cup b$ corresponds to $(a,b)_\omega \otimes \omega$.

The class of $(a,b)_\omega$ is the class of the algebra defined in Section 1.2.2, see [GS17, Prop. 4.7.1]. The equality $(a,b)_\omega = -(b,a)_\omega$ follows from the equality $a \cup b = -b \cup a$.

For any integer $n > 1$, by treating odd and even integers separately one checks that both $-a$ and $1-a$ are norms for the extension $k[t]/(t^n - a)$ of k. Thus $a \cup (-a) = 0$ and $a \cup (1-a) = 0$.

When $n = 2$ is invertible in k we recover the case of quaternion algebras. The bilinearity of the cup-product then gives various properties that we proved in a more explicit way in Section 1.1.

1.4 Galois cohomology of discretely valued fields

Let R be a discrete valuation ring with field of fractions K and residue field k. Let ℓ be a prime number invertible in R. The literature contains various constructions of residue maps

$$\mathrm{Br}(K)\{\ell\} \longrightarrow \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

When k is *perfect* of characteristic $p > 0$, there are constructions of a residue map

$$\mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

which also take care of the p-primary subgroup of $\mathrm{Br}(K)$.

One approach that we do not pursue here is via the Merkurjev–Suslin theorem, which gives an isomorphism $\mathrm{K}_2(F)/n \cong \mathrm{H}^2(F, \mu_n^{\otimes 2})$ valid for any field F and any integer n invertible in F (see, e.g., [GS17, Ch. 8]). When, moreover, $\mu_n \subset F$, we obtain an isomorphism $\mathrm{K}_2(F)/n \xrightarrow{\sim} \mathrm{Br}(F)[n]$, which depends on the choice of a primitive n-th root of unity in F. Thus if $\mu_n \subset K$ and $(\mathrm{char}(K), n) = 1$ we can combine the Merkurjev–Suslin isomorphism with the tame symbol $\mathrm{K}_2(K)/n \to k^*/k^{*n}$ to obtain a composite map

$$\mathrm{Br}(K)[n] \cong \mathrm{K}_2(K)/n \xrightarrow{\text{tame}} k^*/k^{*n}$$

without assuming that k is perfect or has characteristic coprime to n.

The classical case is that of local fields, i.e. *complete* discretely valued fields K with *finite* (hence perfect) residue field k. Then either K is a finite extension of the p-adic field $\mathbb{Q}_p$, or K is the field of formal power series in one variable over a finite field. In these cases local class field theory gives an isomorphism often called the *local invariant*. Its construction goes back to the 1930s and is due to Hasse and Witt [Wit37], and so predates Galois cohomology. This approach uses Brauer classes of central simple algebras over local fields and maximal orders in such algebras; the key fact is that a central division ring over K contains a maximal subfield which is unramified over K, see [SerCL, Ch. XII, §2] and [Rei03, Ch. 8]. We do not go in this direction here but concentrate instead on the cohomological constructions with finite and infinite coefficients. We discuss the existence and give precise definitions of residue maps $\mathrm{Br}(K) \to \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ for complete, or at least *henselian* discrete valuation rings R with field of fractions K and residue field k. Once a residue map of a given kind is defined for complete discretely valued fields, we can define it for any field K with a discrete valuation $v\colon K^*\to\mathbb{Z}$ and the associated residue field k. Namely, if $\widehat{K}_v$ is the completion of K in the v-adic topology, we define the residue for K at v as the composition

$$\mathrm{Br}(K) \longrightarrow \mathrm{Br}(\widehat{K}_v) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}),$$

where the first arrow is the restriction map. The resulting residue map has the same functoriality properties as the residue map for $\widehat{K}_v$. Here, completion can be replaced by henselisation.

Our exposition in this section is based on Chapters 6 and 7 of [GS17] and Chapters II and III of Serre's lectures [Ser03]. As mentioned in [Ser03, §7.13], the properties discussed there for complete local fields also hold for henselian discretely valued fields.

1.4.1 Serre residue

Let K be a henselian discretely valued field with residue field k. Let n be a positive integer not divisible by $\mathrm{char}(k)$. We shall define a residue map $r\colon \mathrm{Br}(K)[n]\to\mathrm{H}^1(k,\mathbb{Z}/n)$ as the composition

$$\mathrm{Br}(K)[n] \xrightarrow{\sim} \mathrm{H}^2(K, \mu_n) \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n),$$

where the first map is the inverse of the isomorphism $\mathrm{H}^2(K,\mu_n) \xrightarrow{\sim} \mathrm{Br}(K)[n]$ provided by the Kummer sequence (1.6). Thus our task is to define a residue map $r\colon \mathrm{H}^2(K,\mu_n) \longrightarrow \mathrm{H}^1(k,\mathbb{Z}/n)$. We shall actually define this map in a more general situation.

Theorem 1.4.1 *Let G be a profinite group and let N be a closed normal subgroup of G. Let C be a discrete G-module. Define $\Gamma = G/N$.*

(i) *Suppose that $\mathrm{H}^i(N,C) = 0$ for $i > 1$. Then there is a long exact sequence*

$$\ldots\to\mathrm{H}^i(\Gamma, C^N)\to\mathrm{H}^i(G,C)\to\mathrm{H}^{i-1}(\Gamma,\mathrm{H}^1(N,C))\to\mathrm{H}^{i+1}(\Gamma,C^N)\to\ldots \qquad (1.8)$$

(ii) *In addition to the assumptions of* (i) *assume that N acts trivially on C, so that C can be considered as a Γ-module. If, moreover, the exact sequence*

$$1 \longrightarrow N \longrightarrow G \longrightarrow \Gamma \longrightarrow 1 \qquad (1.9)$$

is split, then for each $i \geq 1$ there is a split exact sequence

$$0 \longrightarrow \mathrm{H}^i(\Gamma, C) \longrightarrow \mathrm{H}^i(G,C) \longrightarrow \mathrm{H}^{i-1}(\Gamma, \mathrm{Hom}(N,C)) \longrightarrow 0. \qquad (1.10)$$

Here $\mathrm{Hom}(N,C)$ denotes the group of continuous homomorphisms $N \to C$, i.e. homomorphisms with finite image and open kernel.

Proof. (i) We have the Hochschild–Serre spectral sequence

$$E_2^{pq} = \mathrm{H}^p(\Gamma, \mathrm{H}^q(N,C)) \Rightarrow \mathrm{H}^{p+q}(G,C).$$

The assumption of (i) implies that this spectral sequence gives rises to the exact sequence (1.8).

(ii) We have $C^N = C$. Let $\sigma\colon \Gamma \to G$ be a homomorphism such that the composition $\Gamma \to G \to \Gamma$ is the identity map. The composition of the inflation $\mathrm{H}^i(\Gamma,C) \to \mathrm{H}^i(G,C)$ with restriction $\sigma^*\colon \mathrm{H}^i(G,C) \to \mathrm{H}^i(\Gamma,C)$ is the identity. This implies the injectivity of $\mathrm{H}^i(\Gamma,C) \to \mathrm{H}^i(G,C)$ for $i \geq 0$. Thus we obtain the exact sequences (1.10). The same argument gives that these sequences are split. □

Let R be a *henselian* discrete valuation ring with field of fractions K and residue field k. We denote the characteristic exponent of k by p (so $p=1$ when $\mathrm{char}(k)=0$). We have a chain of field extensions

$$K \subset K_{\mathrm{nr}} \subset K_{\mathrm{t}} \subset K_{\mathrm{s}},$$

where K_{s} is a separable closure of K, K_{nr} is the maximal unramified extension of K, and K_{t} is the maximal tamely ramified extension of K. Let $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$ and $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) \cong \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let $I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}})$ be the inertia group and let $N = \mathrm{Gal}(K_{\mathrm{t}}/K_{\mathrm{nr}})$ be the tame inertia group. By Hensel's lemma, the field K_{nr} contains all n-th roots of 1, for n coprime to p.

The field K_{t} is obtained from K_{nr} by adjoining the n-th roots of a fixed uniformiser $\pi \in K$ for all n coprime to p. Indeed, let L be a finite tame extension of K_{nr} and let $e = [L : K_{\mathrm{nr}}]$ be its degree, which is prime to p. Let $\pi_1 \in L$ be a uniformiser. We have $\pi = u\pi_1^e$, where u is a unit in L. By Hensel's lemma, any unit in L is an e-th power. Thus we can find an element π_1 such that $\pi = \pi_1^e$. By Eisenstein's criterion, $L = K_{\mathrm{nr}}(\pi^{1/e})$.

Hence the profinite group N is isomorphic to $\widehat{\mathbb{Z}}$ if $p=1$, and to the quotient of $\widehat{\mathbb{Z}}$ by its maximal pro-p-subgroup if $p>1$. It follows that $\mathrm{cd}(N) \leq 1$, that is, for any discrete torsion Galois module C we have $\mathrm{H}^i(N,C)=0$ for any $i \geq 2$. The wild inertia subgroup $\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}})$ is trivial if $p=1$, otherwise it is a pro-p-group. Thus for any continuous discrete torsion G-module C with trivial p-torsion, one has $\mathrm{H}^i(\mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{t}}),C)=0$ for $i>0$. The Hochschild–Serre spectral sequence then gives that $\mathrm{H}^i(I,C)=0$ for all $i \geq 2$.

For each $n>1$ coprime to p choose an n-th root π_n of π in K_{s} in such a way that $(\pi_{mn})^m = \pi_n$ for all m and n. Let K' be the extension of K generated by all the roots π_n. It is clear that K_{nr} and K' are linearly disjoint over K, and $K_{\mathrm{t}} = K_{\mathrm{nr}}K'$. This implies that the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{Gal}(K_{\mathrm{t}}/K) \longrightarrow \Gamma \longrightarrow 1$$

is split, where we have $\mathrm{Gal}(K_\mathrm{t}/K) = G/\mathrm{Gal}(K_\mathrm{s}/K_\mathrm{t})$. The action of $\mathrm{Gal}(K_\mathrm{t}/K)$ on the abelian group N by conjugations gives rise to an action of $\Gamma \cong \mathrm{Gal}(K_\mathrm{t}/K')$, hence to a G-module structure on N. We have an isomorphism of Γ-modules $N \cong \varprojlim \mu_n$, where the limit is over the positive integers n coprime to p. Here the map $N \to \mu_n$ sends $h \in N$ to $h(\pi_n)/\pi_n$.

For $p > 1$, the p-cohomological dimension of Γ is at most 1 [SerCG, Ch. 2, §2, Prop. 3]. Thus every homomorphism $\Gamma \to G/\mathrm{Gal}(K_\mathrm{s}/K_\mathrm{t})$ lifts to a homomorphism $\Gamma \to G$, see [SerCG, Ch. 1, §3, Prop. 16]. Hence the following exact sequence is also split:

$$1 \longrightarrow I \longrightarrow G \longrightarrow \Gamma \longrightarrow 1.$$

We conclude that for every discrete, torsion Γ-module C with trivial p-torsion, Theorem 1.4.1(ii) gives rise to split exact sequences for all $i \geq 1$

$$0 \longrightarrow \mathrm{H}^i(k, C) \longrightarrow \mathrm{H}^i(K, C) \xrightarrow{r} \mathrm{H}^{i-1}(k, C(-1)) \longrightarrow 0. \tag{1.11}$$

Here $C(-1) := \mathrm{Hom}(N, C) = \mathrm{H}^1(N, C)$ with its natural Γ-action.

Remark 1.4.2 By construction, for $i = 1$ the map r is the restriction map

$$\mathrm{res}_{K_\mathrm{nr}/K} \colon \mathrm{H}^1(K, C) \longrightarrow \mathrm{H}^1(K_\mathrm{nr}, C)^\Gamma.$$

Definition 1.4.3 (i) *Let K be a henselian discretely valued field with residue field k. Let p be the characteristic exponent of k. Let $\Gamma = \mathrm{Gal}(K_\mathrm{nr}/K) = \mathrm{Gal}(k_\mathrm{s}/k)$. Let C be a continuous, discrete, torsion Γ-module with trivial p-torsion. For $i \geq 1$ the map*

$$r \colon \mathrm{H}^i(K, C) \longrightarrow \mathrm{H}^{i-1}(k, C(-1))$$

in the exact sequence (1.11) is called the **Serre residue**. *An element x of $\mathrm{H}^i(K, C)$ is called* **unramified** *if $r(x) = 0$.*

(ii) *Let F be a field with a discrete valuation $v \colon F^* \to \mathbb{Z}$ and the associated residue field k. Let K be the completion of F in the v-adic topology. Let ℓ be a prime invertible in k. The* **Serre residue** *with coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ attached to the valuation v is the composition*

$$\mathrm{Br}(F)\{\ell\} \longrightarrow \mathrm{Br}(K)\{\ell\} \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell),$$

where the first arrow is the restriction map.

Similarly, one defines the Serre residue with coefficients $\mathbb{Z}/\ell^n$ for any positive integer n. These residue maps satisfy obvious compatibility properties.

Suppose we have $nC = 0$, where n is a positive integer not divisible by $\mathrm{char}(k)$. We have a cup-product pairing of Galois cohomology groups of K

$$\cup \colon \mathrm{H}^1(K, \mu_n) \times \mathrm{H}^{i-1}(K, C(-1)) \longrightarrow \mathrm{H}^i(K, C). \tag{1.12}$$

The exact sequence (1.11) allows one to identify $\mathrm{H}^{i-1}(k, C(-1))$ with a subgroup of $\mathrm{H}^{i-1}(K, C(-1))$. This gives rise to the pairing

$$\cup\colon \mathrm{H}^1(K, \mu_n) \times \mathrm{H}^{i-1}(k, C(-1)) \longrightarrow \mathrm{H}^i(K, C). \tag{1.13}$$

The pairing (1.12) is functorial in K, so (1.13) is too (see [Ser03, Prop. 8.2]).

Let us discuss the case $C = \mu_n$, where $(n, p) = 1$.

For $i = 1$ one obtains a split exact sequence

$$0 \longrightarrow k^*/k^{*n} \longrightarrow K^*/K^{*n} \xrightarrow{r} \mathbb{Z}/n \longrightarrow 0. \tag{1.14}$$

Proposition 1.4.4 *The Serre residue r in this sequence is induced by the valuation $v\colon K^* \to \mathbb{Z}$.*

Proof. By Remark 1.4.2, this map is the natural map $K^*/K^{*n} \to K^*_{\mathrm{nr}}/K^{*n}_{\mathrm{nr}}$. The identification of $K^*_{\mathrm{nr}}/K^{*n}_{\mathrm{nr}}$ with

$$\mathbb{Z}/n \cong \mathrm{Hom}(\mu_n, \mu_n) \cong \mathrm{Hom}(N, \mu_n) \cong \mathrm{Hom}(I, \mu_n)$$

is such that the image of π corresponds to the homomorphism $N \to \mu_n$ by which N acts on the n-th roots of π. This is exactly the isomorphism used in the identification of N with the inverse limit of μ_n, for n coprime to char(k), so the image of π in $K^*_{\mathrm{nr}}/K^{*n}_{\mathrm{nr}}$ corresponds to $1 \in \mathbb{Z}/n$. □

For $i = 2$ one obtains a split exact sequence

$$0 \longrightarrow \mathrm{H}^2(k, \mu_n) \longrightarrow \mathrm{H}^2(K, \mu_n) \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow 0, \tag{1.15}$$

which, in view of the Kummer exact sequence (1.6), can be rewritten as follows:

$$0 \longrightarrow \mathrm{Br}(k)[n] \longrightarrow \mathrm{Br}(K)[n] \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow 0. \tag{1.16}$$

Proposition 1.4.5 *Let R be a strictly henselian discrete valuation ring with fraction field K and (separably closed) residue field k. Let $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$. Let p be the characteristic exponent of k. Then we have the following statements.*

(i) *For any prime $\ell \neq p$, any ℓ-primary torsion G-module C and any integer $i \geq 2$, we have $\mathrm{H}^i(K, C) = 0$. In other words, $\mathrm{cd}_\ell(K) \leq 1$.*
(ii) *For any $i \geq 1$ the group $\mathrm{H}^i(K, \mathbb{G}_m)$ is a p-primary torsion group (so the group is trivial when $p = 1$).*
(iii) *The Brauer group $\mathrm{Br}(K)$ is a p-primary torsion group.*
(iv) *If k is algebraically closed, then $\mathrm{cd}(K) \leq 1$ and $\mathrm{H}^i(K, K_{\mathrm{s}}^*) = 0$ for $i \geq 1$.*

Proof. Part (i) is an immediate consequence of the exact sequence (1.11). Statement (ii) then follows from the Kummer sequence (1.6) and statement (iii) is just the special case $i = 2$.

We owe the following proof of (iv) to L. Moret-Bailly. Quite generally, let R be a discrete valuation ring with field of fractions K. Let L/K be an arbitrary finite field extension, and let S be the integral closure of R in L. (Note that if R is not excellent and L/K is not separable, then S is not necessarily a finitely generated R-module.) The ring S is a semilocal Dedekind domain, and for each maximal ideal $q \subset S$, the quotient S/q is finite over $R/(q \cap R)$. This is a special case of the Krull–Akizuki theorem [BouAC, Ch. 7, §2, no. 5]. If, moreover, R is henselian, then since S is integral over R and has no zero-divisors, a limit argument shows that it is a henselian local ring [Ray70b, Ch. I, §2, Prop. 2, p. 7]. In the case considered in (iv), the residue fields of R and hence of S are algebraically closed. By Theorem 1.2.15 we thus have $\mathrm{Br}(L) = 0$ for any finite field extension L/K. By [SerCG, Ch. II, §3.1, Prop. 5], this implies $\mathrm{cd}(K) \leq 1$, which in turn implies $\mathrm{H}^i(K, K_s^*) = 0$ for all $i \geq 1$. □

Proposition 1.4.6 *Let R be a henselian discrete valuation ring with field of fractions K and residue field k. Let $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K)$ and let C be a Γ-module of exponent n invertible in R. Let π be a uniformiser of R and let $(\pi)_n$ be the image of π under the map $K^* \to \mathrm{H}^1(K, \mu_n)$ given by the Kummer sequence (1.6). Any $\alpha \in \mathrm{H}^i(K, C)$ is uniquely written as*

$$\alpha = \alpha_0 + (\pi)_n \cup \alpha_1,$$

where $\alpha_0 \in \mathrm{H}^i(k, C)$ and $\alpha_1 \in \mathrm{H}^{i-1}(k, C(-1))$. Moreover, $\alpha_1 = r(\alpha)$.

Proof. See [Ser03, Ch. II, Prop. 7.11, p. 18]. □

Using this, one proves the following general formula [Ser03, II.6.5, Exercise 7.12]. Let A, B, C be n-torsion Γ-modules such that there is a Γ-equivariant pairing $A \times B \to C$. It induces the pairing

$$\cup \colon \mathrm{H}^p(K, A) \times \mathrm{H}^q(K, B) \longrightarrow \mathrm{H}^{p+q}(K, C).$$

For $\alpha \in \mathrm{H}^p(K, A)$ and $\beta \in \mathrm{H}^q(K, B)$, one has

$$r(\alpha \cup \beta) = r(\alpha) \cup \beta + (-1)^p \alpha \cup r(\beta) + r(\alpha) \cup r(\beta) \cup (-1)_n \in \mathrm{H}^{p+q-1}(K, C(-1)),$$

where $(-1)_n \in \mathrm{H}^1(K, \mu_n)$ denotes the class of -1 in $K^*/K^{*n} \cong \mathrm{H}^1(K, \mu_n)$. All terms of this formula are elements of $\mathrm{H}^{p+q-1}(K, C(-1))$ with the convention that the Serre residue r is composed with the injective map in (1.11), and similarly for A and B in place of C.

Here are some applications of this formula to residues for an arbitrary (not necessarily henselian) discrete valuation ring R with field of fractions K and residue field k, where $n > 1$ is an integer invertible in R.

- The cup-product followed by the Serre residue

$$\mathrm{H}^1(K, \mu_n) \times \mathrm{H}^1(K, \mu_n) \xrightarrow{\cup} \mathrm{H}^2(K, \mu_n^{\otimes 2}) \xrightarrow{r} \mathrm{H}^1(k, \mu_n)$$

gives rise to the skew-symmetric pairing

$$K^*/K^{*n} \times K^*/K^{*n} \longrightarrow k^*/k^{*n}. \quad (1.17)$$

The above formula for the Serre residue of the cup-product shows that the value of this pairing on the classes of $a, b \in K^*$ is the image in k^*/k^{*n} of the following element of R^*:

$$(-1)^{v(a)v(b)} b^{v(a)}/a^{v(b)} \in R^*. \quad (1.18)$$

- If we consider

$$\mathrm{H}^1(K, \mathbb{Z}/n) \times \mathrm{H}^1(K, \mu_n) \xrightarrow{\cup} \mathrm{H}^2(K, \mu_n) \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n),$$

then for any $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n) \subset \mathrm{H}^1(K, \mathbb{Z}/n)$ and any $a \in K^*$ we obtain

$$r(\chi \cup (a)_n) = -v(a)\chi \in \mathrm{H}^1(k, \mathbb{Z}/n), \quad (1.19)$$

where $(a)_n$ is the image of a in K^*/K^{*n}.

- However, if we consider

$$\mathrm{H}^1(K, \mu_n) \times \mathrm{H}^1(K, \mathbb{Z}/n) \xrightarrow{\cup} \mathrm{H}^2(K, \mu_n) \xrightarrow{r} \mathrm{H}^1(k, \mathbb{Z}/n),$$

then for any $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n) \subset \mathrm{H}^1(K, \mathbb{Z}/n)$ and any $a \in K^*$ we obtain

$$r((a)_n \cup \chi) = v(a)\chi \in \mathrm{H}^1(k, \mathbb{Z}/n). \quad (1.20)$$

This implies that the map $s\colon \mathrm{H}^1(k, \mathbb{Z}/n) \to \mathrm{H}^2(K, \mu_n)$ given by

$$s(\chi) = (\pi)_n \cup \chi, \quad (1.21)$$

where $(\pi)_n$ is the image of π in K^*/K^{*n}, is a section of r.

1.4.2 Extensions of rings

Let R be a discrete valuation ring with field of fractions K and residue field k. Let L be a finite separable extension of K. Then the integral closure B of R in L is a semilocal Dedekind domain which is a finitely generated R-module [SerCL, Ch. I, §4, Prop. 8]. Let $\mathfrak{m}_i$, for $i = 1, \ldots, n$, be the maximal ideals of B. Let $k_i = B/\mathfrak{m}_i$ be the residue field at $\mathfrak{m}_i$. Let e_i be the ramification index of $\mathfrak{m}_i$ over K.

Proposition 1.4.7 *Let ℓ be a prime invertible in R. Then one has commutative diagrams*

$$\begin{array}{ccc} \mathrm{Br}(L)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k_i,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \uparrow{\scriptstyle \mathrm{res}_{K/L}} & & \uparrow{\scriptstyle e_i\mathrm{res}_{k/k_i}} \\ \mathrm{Br}(K)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array} \qquad \begin{array}{ccc} \mathrm{Br}(L)\{\ell\} & \xrightarrow{r} & \bigoplus_{i=1}^n \mathrm{H}^1(k_i,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \downarrow{\scriptstyle \mathrm{cores}_{K/L}} & & \downarrow{\scriptstyle \sum \mathrm{cores}_{k/k_i}} \\ \mathrm{Br}(K)\{\ell\} & \xrightarrow{r} & \mathrm{H}^1(k,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

These diagrams are compatible with similar commutative diagrams for the Serre residue with finite coefficients $r\colon \mathrm{Br}(K)[\ell^n]\to\mathrm{H}^1(k,\mathbb{Z}/\ell^n)$, for $n\geq 1$.

The corestriction map $\mathrm{cores}_{k'/k}\colon \mathrm{H}^i(k',C)\to\mathrm{H}^i(k,C)$ is defined for any finite extension of fields k'/k and any torsion Galois module C with trivial p-torsion, where p is the characteristic exponent of k, as the composition of Galois corestriction and multiplication by the degree of inseparability of k'/k. See [Ser03, Ch. II, 8.5] or [GS17, Remark 6.9.2].

Proof of Proposition 1.4.7. We give a sketch and refer to [Ser03, §8] for details.

Let $\widehat{R}$ be the completion of R and let $\widehat{K}$ be the completion of K. Let $\widehat{B}_i$ be the completion of B with respect to the discrete valuation defined by $\mathfrak{m}_i$. Similarly, let $\widehat{L}_i$ be the completion of L at $\mathfrak{m}_i$. Clearly, $\widehat{K}$ is the field of fractions of $\widehat{R}$ and $\widehat{L}_i$ is the field of fractions of $\widehat{B}_i$. By [SerCL, Ch. II, §3] we have

$$L\otimes_K\widehat{K}\xrightarrow{\sim}\prod_{i=1}^n\widehat{L}_i,\qquad B\otimes_R\widehat{R}\xrightarrow{\sim}\prod_{i=1}^n\widehat{B}_i.$$

It is enough to prove the proposition in the case when R is complete. For the first diagram, using Proposition 1.4.6, it suffices to check commutativity for $(\pi_R)_{\ell^n}\cup\chi\in\mathrm{Br}(K)[\ell^n]$, where $\chi\in\mathrm{H}^1(k,\mathbb{Z}/\ell^n)$ and π_R is a uniformiser of R. This follows from the functoriality of the pairing (1.13) with respect to extensions of the field K.

Checking the commutativity of the second diagram reduces to the following two cases: unramified extensions L/K (that is, $e(L/K)=1$ and the residue field extension k_L/k is separable), and extensions L/K with k_L/k purely inseparable. In the first case, one considers $(\pi_R)_{\ell^n}\cup\chi$, where $\chi\in\mathrm{H}^1(k_L,\mathbb{Z}/\ell^n)$. In the second case it is enough to consider the elements of $\mathrm{Br}(L)[\ell^n]$ of the form $(\pi_B)_{\ell^n}\cup\chi$, where $\chi\in\mathrm{H}^1(k,\mathbb{Z}/\ell^n)$. The result then follows from the standard "projection formulae". □

Proposition 1.4.8 *Let $K\subset L$ be an unramified extension of henselian discretely valued fields with residue fields $k\subset k_L$. Let $\alpha\in\mathrm{Br}(K)\{\ell\}$, where ℓ is invertible in k. Suppose that $\mathrm{res}_{L/K}(\alpha)\in\mathrm{Br}(L)$ is unramified, so that $\mathrm{res}_{L/K}(\alpha)$ is the image of an element $\beta\in\mathrm{Br}(k_L)$ under the injective map $\mathrm{Br}(k_L)\{\ell\}\to\mathrm{Br}(L)\{\ell\}$ from the exact sequence (1.16). Then β is contained in the image of the restriction map $\mathrm{res}_{k_L/k}\colon\mathrm{Br}(k)\to\mathrm{Br}(k_L)$.*

Proof. Take any n such that $\ell^n\alpha = 0$. By Proposition 1.4.6, α is uniquely written as

$$\alpha = \alpha_0 + (\pi)_{\ell^n} \cup \alpha_1,$$

where $\pi \in K$ is a uniformiser, $(\pi)_{\ell^n} \in \mathrm{H}^1(K, \mu_{\ell^n})$ is the image of $\pi \in K^*$ under the connecting map in the Kummer exact sequence, $\alpha_0 \in \mathrm{Br}(k)[\ell^n]$ and $\alpha_1 \in \mathrm{H}^1(k, \mathbb{Z}/\ell^n)$. Moreover, $\alpha_1 = r_K(\alpha)$ is the residue of α. By the compatibility of pairings for K and L (see (1.13)) the image of $(\pi)_{\ell^n} \cup \alpha_1$ in $\mathrm{Br}(L)$ is $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$, where π is understood as an element of L.

Since $\mathrm{res}_{L/K}(\alpha_0)$ and $\mathrm{res}_{L/K}(\alpha)$ are unramified, $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$ is also unramified. As L is unramified over K, the uniformiser $\pi \in K$ is also a uniformiser of L. Therefore, the residue map $r_L\colon \mathrm{Br}(L)[\ell^n] \to \mathrm{H}^1(k_L, \mathbb{Z}/\ell^n)$ sends $(\pi)_{\ell^n} \cup \mathrm{res}_{k_L/k}(\alpha_1)$ to $\mathrm{res}_{k_L/k}(\alpha_1) \in \mathrm{H}^1(k_L, \mathbb{Z}/\ell^n)$, so this last element is zero. Hence $(\pi)_{\ell^n} \cup \alpha_1$ goes to zero in $\mathrm{Br}(L)$, so that $\mathrm{res}_{L/K}(\alpha)$ is the image of $\mathrm{res}_{k_L/k}(\alpha_0)$. □

Corollary 1.4.9 *Let $R \subset B$ be an unramified extension of (not necessarily henselian) discrete valuation rings with fraction fields $K \subset L$ and residue fields $\kappa \subset \lambda$. Let $\alpha \in \mathrm{Br}(K)\{\ell\}$, where ℓ is a prime invertible in R. Suppose that the image of α in $\mathrm{Br}(L)$ is unramified, so it is the image of a (well defined) element $\beta \in \mathrm{Br}(B)$. Then the image of β under the natural map $\mathrm{Br}(B) \to \mathrm{Br}(\lambda)$ is contained in the image of the restriction map $\mathrm{Br}(\kappa) \to \mathrm{Br}(\lambda)$.*

Proof. The statement only concerns the value of β at the closed point $\mathrm{Spec}(\lambda)$ of $\mathrm{Spec}(B)$, so we can assume without loss of generality that R and B are henselian. In this case the statement follows from Proposition 1.4.8. □

1.4.3 Witt residue

Let R be a henselian discrete valuation ring with fraction field K and residue field k. Let p be the characteristic exponent of k. As in Section 1.4.1 we have inclusions of discretely valued fields

$$K \subset K_{\mathrm{nr}} \subset K_{\mathrm{t}} \subset K_{\mathrm{s}}.$$

The residue field of both K_{nr} and K_{t} is the separable closure k_{s} of k. We have $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) \cong \mathrm{Gal}(k_{\mathrm{s}}/k)$.

By Proposition 1.4.5, both $\mathrm{Br}(K_{\mathrm{nr}})$ and $\mathrm{Br}(K_{\mathrm{t}})$ are p-primary torsion groups. (Note that $\mathrm{Br}(K_{\mathrm{nr}}) = 0$ if k is perfect by Theorem 1.2.15.) Since $N = \mathrm{Gal}(K_{\mathrm{t}}/K_{\mathrm{nr}})$ is the inverse limit of $\mathbb{Z}/n$, where $(n,p) = 1$, we have $\mathrm{H}^2(N, K_{\mathrm{t}}^*)\{p\} = 0$. Thus by Hilbert's theorem 90 the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(N, \mathrm{H}^q(K_{\mathrm{t}}, K_{\mathrm{s}}^*)) \Rightarrow \mathrm{H}^{p+q}(K_{\mathrm{nr}}, K_{\mathrm{s}}^*)$$

shows that the restriction map $\mathrm{Br}(K_{\mathrm{nr}}) \to \mathrm{Br}(K_{\mathrm{t}})$ is injective.

Definition 1.4.10 *Define the* **tame** *(or* **tamely ramified***) subgroup of the Brauer group as*

$$\mathrm{Br}_{\mathrm{t}}(K) := \mathrm{Ker}[\mathrm{Br}(K){\to}\mathrm{Br}(K_{\mathrm{nr}})] = \mathrm{Ker}[\mathrm{Br}(K){\to}\mathrm{Br}(K_{\mathrm{t}})].$$

We have $\mathrm{Br}_{\mathrm{t}}(K)\{\ell\} = \mathrm{Br}(K)\{\ell\}$ for any prime $\ell \neq p$, and $\mathrm{Br}_{\mathrm{t}}(K) = \mathrm{Br}(K)$ if k is perfect. By Hilbert's theorem 90, the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma, \mathrm{H}^q(K_{\mathrm{nr}}, K_{\mathrm{s}}^*)) \Rightarrow \mathrm{H}^{p+q}(K, K_{\mathrm{s}}^*) \tag{1.22}$$

gives an isomorphism $\mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{t}}(K)$. Composing it with the Galois equivariant map $v\colon K_{\mathrm{nr}}^*{\to}\mathbb{Z}$ given by the valuation we obtain

$$\mathrm{Br}_{\mathrm{t}}(K) \xleftarrow{\sim} \mathrm{H}^2(\Gamma, K_{\mathrm{nr}}^*) \xrightarrow{v_*} \mathrm{H}^2(\Gamma, \mathbb{Z}) \xleftarrow{\sim} \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}), \tag{1.23}$$

where the isomorphism $\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(\Gamma, \mathbb{Z})$ comes from Galois cohomology of the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0. \tag{1.24}$$

We have a canonical isomorphism $\mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z})$.

Definition 1.4.11 (i) *Let K be a henselian discretely valued field with residue field k. The composed map in* (1.23)

$$r_W\colon \mathrm{Br}_{\mathrm{t}}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

is called the **Witt residue**. *For any prime ℓ invertible in k it defines a map*

$$r_W\colon \mathrm{Br}(K)\{\ell\} \longrightarrow \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell). \tag{1.25}$$

When k is perfect, it defines a map

$$r_W\colon \mathrm{Br}(K) \longrightarrow \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}). \tag{1.26}$$

(ii) *Let F be a field with a discrete valuation $v\colon F^*{\to}\mathbb{Z}$ and the associated residue field k. Let K be the completion of F in the v-adic topology. If ℓ is a prime invertible in k, we define the* **Witt residue**

$$r_W\colon \mathrm{Br}(F)\{\ell\} \longrightarrow \mathrm{H}^1(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

as the composition of the restriction map $\mathrm{Br}(F)\{\ell\}{\to}\mathrm{Br}(K)\{\ell\}$ *with* (1.25). *If k is perfect, we define the* **Witt residue** $r_W\colon \mathrm{Br}(K){\to}\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ *as the composition of the restriction map* $\mathrm{Br}(F){\to}\mathrm{Br}(K)$ *with* (1.26).

We note that the choice of a uniformiser gives a section of the homomorphism $v\colon K_{\mathrm{nr}}^*{\to}\mathbb{Z}$, and hence of r_W. In particular, the Witt residue map r_W is surjective (for the kernel of the Witt residue, see Theorem 3.6.2). This section can be described in terms of the cup-product. Since Γ is a quotient of

$\mathrm{Gal}(K_s/K)$, we can view a continuous character $\chi\colon \Gamma\to\mathbb{Q}/\mathbb{Z}$ as a character of $\mathrm{Gal}(K_s/K)$. Applying the connecting homomorphism in the long exact sequence attached to the exact sequence of $\mathrm{Gal}(K_s/K)$-modules (1.24) we obtain $d(\chi)\in \mathrm{H}^2(K,\mathbb{Z})$. For any $a\in K^*$ the cup-product $d(\chi)\cup a$ under the pairing

$$\mathrm{H}^2(K,\mathbb{Z})\times \mathrm{H}^0(K,K_s^*)\longrightarrow \mathrm{Br}(K) \tag{1.27}$$

is an element of $\mathrm{Br}(K)$, see also [SerCL, Ch. XIV, §1]. In Section 1.3.4 this element was denoted by (χ,a), see (1.5). By (1.7), if $n\chi=0$ for a positive integer n, we have $d(\chi)\cup a=\chi\cup(a)_n$. Using (1.19) we obtain

$$r_W\big(d(\chi)\cup a\big)=v(a)\chi. \tag{1.28}$$

Thus if $\pi\in R$ is a uniformiser, then the map

$$s_W(\chi)=d(\chi)\cup\pi \tag{1.29}$$

is a section of r_W.

We refer to [SerCL, Ch. XII, Exercise 3] for the following result.

Theorem 1.4.12 *Let R be a complete discrete valuation ring with fraction field K and residue field k. There is a split exact sequence*

$$0\longrightarrow \mathrm{Br}(k)\longrightarrow \mathrm{Br}_{\mathrm{t}}(K)\longrightarrow \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\longrightarrow 0,$$

where the third arrow is the Witt residue.

Remark 1.4.13 Let R be a complete discrete valuation ring with fraction field K of characteristic zero and residue field k of characteristic $p>0$. Assuming that K contains the p-th roots of 1, Kato [Kat82] constructed a filtration on $\mathrm{Br}(K)[p]$ whose smallest term is identified with the direct sum $\mathrm{Br}(k)[p]\oplus \mathrm{H}^1(k,\mathbb{Z}/p)$. The higher quotients of the filtration involve the group of absolute differentials $\Omega^1_{k/\mathbb{Z}}$ of k, i.e. the group of differentials Ω^1_{k/k^p}, which vanishes if k is perfect. See also [CT99a, Thm. 4.3.1]. The group $\mathrm{Br}(k)[p]$ also features as the Galois cohomology group $\mathrm{H}^1(k,\nu(1))$, cf. [CT99a, §1].

Similarly, for a complete discrete valuation ring R with fraction field K and residue field k both of characteristic $p>0$, Izhboldin [Izh96, Thm. 2.5, Cor. 2.6] constructed a filtration on $\mathrm{Br}(K)[p]$ whose lowest term is identified with $\mathrm{Br}(k)[p]\oplus \mathrm{H}^1(k,\mathbb{Z}/p)$. The higher quotients of the filtration involve the group $\Omega^1_{k/Z}=\Omega^1_{k/k^p}$, which vanishes if k is perfect.

In both cases, the lowest term of these filtrations is the p-torsion subgroup of $\mathrm{Br}_{\mathrm{t}}(K)$.

1.4.4 Compatibility of residues

In this section we compare the Serre residue with the Witt residue.

Theorem 1.4.14 *Let R be a henselian discrete valuation ring with fraction field K and residue field k. Let n be an integer invertible in R. The composition*

$$\mathrm{H}^2(K,\mu_n) \xrightarrow{\sim} \mathrm{Br}(K)[n] \xrightarrow{r_W} \mathrm{H}^1(k,\mathbb{Z}/n),$$

where the first arrow is given by the Kummer sequence (1.6), is equal to the negative of the Serre residue $r\colon \mathrm{H}^2(K,\mu_n)\to\mathrm{H}^1(k,\mathbb{Z}/n)$.

Proof. This was proved by Serre in his 1991–1992 course at Collège de France, cf. the appendix to the thesis of E. Frossard [Fro95, Lemme A.3.2]. See also [GS17, Prop. 6.8.9].

The idea is to use explicit splittings of the residue maps r and r_W given by their respective sections s and s_W, see (1.21) and (1.29). Let $\chi\in\mathrm{H}^1(k,\mathbb{Z}/n)=\mathrm{Hom}(\Gamma,\mathbb{Z}/n)$. We need to show that the Brauer class given by $s(\chi)=(\pi)_n\cup\chi$ is the negative of $s_W(\chi)=d(\chi)\cup\pi$. The proof of this property works more generally for any field K of characteristic coprime to n. Let $G=\mathrm{Gal}(K_\mathrm{s}/K)$. We shall show that for any character $\chi\in\mathrm{Hom}(G,\mathbb{Z}/n)$ and any $a\in K^*$ the image of $(a)_n\cup\chi\in\mathrm{H}^2(G,\mu_n)$ in $\mathrm{H}^2(G,K_\mathrm{s}^*)$ equals $-d(\chi)\cup a$. Recall that we denote by $(a)_n$ the image of $a\in K^*$ in $\mathrm{H}^1(K,\mu_n)=K^*/K^{*n}$ under the connecting map of the Kummer sequence. In particular, this will prove formula (1.7), which is the equality $\chi\cup(a)_n=d(\chi)\cup a$.

We refer the reader to [BouX, Ch. 7] for a careful exposition of Ext-groups of modules over a ring and their relations to exact sequences of modules. By [BouX, §7.4, Prop. 3] the canonical pairing between Ext-groups can be computed by splicing extensions, which is also known as Yoneda pairing. We shall use the following well-known properties, see [HS70, Ch. IV, §9] or [BouX, §7.6, Prop. 5]. Suppose we are given a G-module M, an integer $m\geq 0$, and an exact sequence of G-modules

$$\theta\colon\qquad 0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0.$$

Let $[\theta]$ be the class of this extension in $\mathrm{Ext}^1_G(C,A)$.

(a) The connecting homomorphism $\mathrm{Ext}^m_G(M,C)\to\mathrm{Ext}^{m+1}_G(M,A)$ in the *second* argument is the product with $[\theta]\in\mathrm{Ext}^1_G(C,A)$. For $m\geq 1$ this homomorphism sends the class of an m-fold extension of M by C to the class of the splicing of this extension with θ. For $m=0$ it sends a homomorphism of G-modules $f\colon M\to C$ to the class of the pullback of θ by f.

(b) The connecting homomorphism $\mathrm{Ext}^m_G(A,M)\to\mathrm{Ext}^{m+1}_G(C,M)$ in the *first* argument is the product with $(-1)^{m+1}[\theta]$. For $m\geq 1$ it sends the class of an m-fold extension of A by M to the class of its splicing with θ, multiplied by $(-1)^{m+1}$. For $m=0$ it sends a homomorphism of G-modules $g\colon A\to M$ to the *negative* of the class of the push-forward of θ by g.

We have a canonical isomorphism of functors $\mathrm{Ext}^n_G(\mathbb{Z},\cdot) = \mathrm{H}^n(K,\cdot)$. By [BouX, §7.5, Thm. 1 (a)] any class in $\mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n)$ is the class of an extension of G-modules $\mathbb{Z}$ by $\mathbb{Z}/n$. Let the first short exact sequence in

$$0\to\mathbb{Z}/n\to E_\chi\to\mathbb{Z}\to 0, \qquad 0\to\mathbb{Z}\to\mathbb{Z}\to\mathbb{Z}/n\to 0 \tag{1.30}$$

be an extension with class

$$\chi \in \mathrm{Hom}_{\mathrm{cont}}(G,\mathbb{Z}/n) = \mathrm{H}^1(K,\mathbb{Z}/n) = \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n).$$

The second short exact sequence is obtained from the multiplication by n map $[n]\colon \mathbb{Z}\to\mathbb{Z}$. We denote it by M_n, and write $[M_n]$ for the class of M_n in $\mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z})$. Given $a \in K^*$, we let $f_a\colon \mathbb{Z}\to K_{\mathrm{s}}^*$ be the map of G-modules sending 1 to a.

We write $M_n \cup E_\chi$ for the 2-fold extension of $\mathbb{Z}$ by $\mathbb{Z}$ obtained by splicing the short exact sequences in (1.30). We write $f_{a*}M_n$ for the extension of $\mathbb{Z}/n$ by K_{s}^*, which is the push-forward of M_n via f_a. Similarly, $f_{a*}(M_n \cup E_\chi) = f_{a*}M_n \cup E_\chi$ is the push-forward of $M_n \cup E_\chi$ by f_a. We use square brackets to denote the classes of these extensions in the relevant Ext-groups.

The first two rows in the following diagram of pairings are Yoneda pairings:

$$\begin{array}{ccccccc}
\mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z}) & \times & \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^2_G(\mathbb{Z},\mathbb{Z}) & \cong & \mathrm{H}^2(G,\mathbb{Z}) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_G(\mathbb{Z}/n,K_{\mathrm{s}}^*) & \times & \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n) & \longrightarrow & \mathrm{Ext}^2_G(\mathbb{Z},K_{\mathrm{s}}^*) & \cong & \mathrm{H}^2(G,K_{\mathrm{s}}^*) \\
\epsilon\uparrow\simeq & & \| & & & & \uparrow \\
\mathrm{H}^1(G,\mu_n) & \times & \mathrm{H}^1(G,\mathbb{Z}/n) & & \xrightarrow{\cup} & & \mathrm{H}^2(G,\mu_n)
\end{array}$$

The upper vertical maps denoted by arrows are given by the push-forward via $f_a\colon \mathbb{Z}\to K_{\mathrm{s}}^*$. It is thus clear that the upper part of this diagram commutes. The map ϵ is the edge map $\mathrm{H}^1(G,\mathrm{Hom}(\mathbb{Z}/n,K_{\mathrm{s}}^*))\to\mathrm{Ext}^1_G(\mathbb{Z}/n,K_{\mathrm{s}}^*)$ from the spectral sequence

$$\mathrm{H}^p(G,\mathrm{Ext}^q(\mathbb{Z}/n,K_{\mathrm{s}}^*)) \Rightarrow \mathrm{Ext}^{p+q}_G(\mathbb{Z}/n,K_{\mathrm{s}}^*).$$

In the category of abelian groups we have $\mathrm{Ext}^q(\mathbb{Z}/n,K_{\mathrm{s}}^*) = 0$ for $q \geq 1$ since K_{s}^* is divisible by n, hence ϵ is an isomorphism. The pairing in the bottom row is the cup-product pairing, which is defined via the tensor product $\mu_n \otimes_{\mathbb{Z}} \mathbb{Z}/n \xrightarrow{\sim} \mu_n$. The commutativity of the lower part of the diagram, i.e., the equality of the 'internal product' to the 'Yoneda-edge-product', is proved in [GH70, Prop. 4.5] and in [Mil80, Prop. V.1.20].

The upper pairing of the diagram applied to $[M_n] \in \mathrm{Ext}^1_G(\mathbb{Z}/n,\mathbb{Z})$ and $[E_\chi] \in \mathrm{Ext}^1_G(\mathbb{Z},\mathbb{Z}/n)$ gives $[M_n \cup E_\chi] \in \mathrm{Ext}^2_G(\mathbb{Z},\mathbb{Z})$, by [BouX, §7.4, Prop. 3]. By property (a) above applied in the case $m = 1$, this equals the image of $[E_\chi]$ under the connecting homomorphism in the long exact sequence of $\mathrm{Ext}^n_G(\mathbb{Z},\cdot)$'s in the second argument. This sequence is the same as the long exact sequence of $\mathrm{H}^n(G,\cdot)$, so we conclude that $[M_n \cup E_\chi] = d(\chi)$. It follows that the middle pairing of the diagram sends $[f_{a*}M_n]$ and $[E_\chi]$ to

$$[f_{a*}M_n \cup E_\chi] = f_{a*}(d(\chi)) = d(\chi) \cup a = a \cup d(\chi).$$

The bottom pairing sends $(a)_n$ and χ to $(a)_n \cup \chi$. By definition χ goes to $[E_\chi]$, so to prove that $d(\chi) \cup a$ is the image of $-(a)_n \cup \chi$ it remains to show that the edge map ϵ sends $(a)_n$ to $-[f_{a*}M_n]$.

To check this consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathbb{Z}, K_{\mathrm{s}}^*) & \longrightarrow & \mathrm{Ext}^1_G(\mathbb{Z}/n, K_{\mathrm{s}}^*) \\ || & & \epsilon \uparrow\simeq \\ \mathrm{H}^0(G, \mathrm{Hom}(\mathbb{Z}, K_{\mathrm{s}}^*)) & \longrightarrow & \mathrm{H}^1(G, \mathrm{Hom}(\mathbb{Z}/n, K_{\mathrm{s}}^*)) \end{array}$$

Here the upper horizontal arrow is the connecting homomorphism in the long exact sequence of Ext's in the first argument associated to the exact sequence M_n. The lower horizontal arrow is the connecting homomorphism in the long exact sequence of cohomology attached to the Kummer exact sequence (1.6). The commutativity of the last diagram is proved by a standard derived category argument based on the representation of the left derived functor $\mathbf{R}\mathrm{Hom}_G(\cdot, K_{\mathrm{s}}^*)$ from the bounded derived category of continuous discrete G-modules to abelian groups as the composition of the derived functors of $\mathrm{Hom}(\cdot, K_{\mathrm{s}}^*)$ and $M \mapsto M^G$. By property (b) above applied in the case $m = 0$, the upper arrow sends a to $-[f_{a*}M_n]$.

We conclude that $\epsilon((a)_n) = -[f_{a*}M_n]$. □

1.5 The Faddeev exact sequences

Let k be a *perfect* field with algebraic closure $k_{\mathrm{s}} = \bar{k}$ and Galois group $\Gamma = \Gamma_k = \mathrm{Gal}(\bar{k}/k)$. To a monic irreducible polynomial $P(t) \in k[t]$ we attach a free $\mathbb{Z}$-module $\mathbb{Z}_P$ generated by the roots of $P(t)$ in $\bar{k}$ with a natural action of Γ permuting these generators. It is clear that the Γ-module $\mathbb{Z}_P$ is induced from the trivial $\mathrm{Gal}(\bar{k}/k(P))$-module $\mathbb{Z}$, where $k(P) = k[t]/(P(t))$. In particular, by Shapiro's lemma, we have $\mathrm{H}^n(\Gamma_k, \mathbb{Z}_P) \cong \mathrm{H}^n(\Gamma_{k(P)}, \mathbb{Z})$ for all $n \geq 0$. For $n = 2$ we have a canonical isomorphism

$$\mathrm{H}^2(\Gamma_k, \mathbb{Z}_P) \cong \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z}).$$

The valuations attached to the roots of $P(t)$ give rise to a map of Γ-modules $\bar{k}(t)^* \to \mathbb{Z}_P$, which has a section sending the generator of $\mathbb{Z}_P$ corresponding to a root $\varepsilon \in \bar{k}$ to $t - \varepsilon$. Using this notation we rewrite the natural exact sequence of Γ-modules

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(t)^* \longrightarrow \mathrm{Div}(\mathbb{A}^1_{\bar{k}}) \longrightarrow 0 \qquad (1.31)$$

as a split exact sequence of Γ-modules

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(t)^* \longrightarrow \bigoplus_{P(t)} \mathbb{Z}_P \longrightarrow 0, \tag{1.32}$$

where the sum is over all monic irreducible polynomials $P(t) \in k[t]$.

Proposition 1.5.1 (D.K. Faddeev) *Let k be a perfect field. There is a split exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(k(t)) \longrightarrow \bigoplus_{P(t)} \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0, \tag{1.33}$$

where the direct sum is over all monic irreducible polynomials $P(t) \in k[t]$. The second arrow is given by the inclusion of fields $k \subset k(t)$. For each $P(t)$, the map $\mathrm{Br}(k(t)) \to \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z})$ is the Witt residue attached to the valuation of $k(t)$ for which $P(t)$ is a uniformiser.

Proof. Applying $\mathrm{H}^2(\Gamma_k, \cdot)$ to (1.32) we obtain (1.33), once we identify the middle term with $\mathrm{Br}(k(t))$. The natural isomorphism $\Gamma_k \cong \mathrm{Gal}(\bar{k}(t)/k(t))$ gives rise to the inflation map

$$\mathrm{inf}\colon \mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\mathrm{Gal}(\overline{k(t)}/k(t)), \overline{k(t)}^*) \cong \mathrm{Br}(k(t)).$$

It is enough to prove that this map is an isomorphism. Indeed, inf fits into the Hochschild–Serre spectral sequence

$$\mathrm{H}^p(\Gamma_k, \mathrm{H}^q(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t), \overline{k(t)}^*)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Gal}(\overline{k(t)}/k(t)), \overline{k(t)}^*).$$

We have $\mathrm{H}^1(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t)), \overline{k(t)}^*) = 0$ (Theorem 1.3.2, Hilbert's theorem 90) and $\mathrm{H}^2(\mathrm{Gal}(\overline{k(t)}/\bar{k}(t)), \overline{k(t)}^*) \cong \mathrm{Br}(\bar{k}(t)) = 0$ (Theorem 1.2.14, Tsen's theorem). The spectral sequence now implies that inf is an isomorphism.

It remains to check the compatibility with the Witt residue. Let $k[t]_P$ be the localisation of $k[t]$ at the principal prime ideal $(P(t))$, let $k[t]_P^{\mathrm{h}}$ be the henselisation of $k[t]_P$. It is a henselian discrete valuation ring with residue field $k(P)$. The integral closure of k in $k[t]_P^{\mathrm{h}}$ is a field of representatives for $k(P)$ inside $k[t]_P^{\mathrm{h}}$, that is, the reduction map induces an isomorphism between this field and the residue field $k(P)$. Henceforth we denote this field by $k(P)$.

Let $K \subset \overline{k(t)}$ be the fraction field of $k[t]_P^{\mathrm{h}}$. Let K_{nr} be the maximal unramified extension of K. We note that $K_{\mathrm{nr}} = K \otimes_{k(P)} \bar{k}$ and $\mathrm{Gal}(K_{\mathrm{nr}}/K) = \Gamma_{k(P)}$. The map

$$\mathrm{Br}(k(t)) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z})$$

comes from $\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \to \mathrm{H}^2(\Gamma_k, \mathbb{Z}_P)$ which factors as

$$\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\Gamma_k, (K \otimes_k \bar{k})^*) \longrightarrow \mathrm{H}^2(\Gamma_k, \mathbb{Z}_P).$$

Since $\mathrm{H}^2(\Gamma_k, (K \otimes_k \bar{k})^*) \cong \mathrm{H}^2(\Gamma_{k(P)}, (K \otimes_{k(P)} \bar{k})^*) \cong \mathrm{H}^2(\Gamma_{k(P)}, K_{\mathrm{nr}}^*)$ by Shapiro's lemma, our map can also be written as

$$\mathrm{H}^2(\Gamma_k, \bar{k}(t)^*) \longrightarrow \mathrm{H}^2(\Gamma_{k(P)}, K_{\mathrm{nr}}^*) \longrightarrow \mathrm{H}^2(\Gamma_{k(P)}, \mathbb{Z}).$$

Here the second map is induced by the valuation, so, by definition, it is the Witt residue. □

Theorem 1.5.2 *Let k be a perfect field. There is an exact sequence*

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}(k(t)) \to \bigoplus_{x \in (\mathbb{P}^1_k)^{(1)}} \mathrm{H}^1(k(x), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to 0, \qquad (1.34)$$

where the direct sum is over all closed points of $\mathbb{P}^1_k$. The third map is the direct sum of Witt residues. The fourth map is the sum of corestrictions $\mathrm{cores}_{k(x)/k}$ *over all closed points of $\mathbb{P}^1_k$.*

Proof. Instead of (1.31) we now consider the exact sequence of Γ-modules

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(t)^* \longrightarrow \mathrm{Div}(\mathbb{P}^1_{\bar{k}}) \longrightarrow \mathbb{Z} \longrightarrow 0, \qquad (1.35)$$

where the fourth arrow is given by the degree. This sequence can be obtained by splicing two exact sequences of Γ-modules, both of which are split:

$$0 \longrightarrow \bar{k}^* \longrightarrow \bar{k}(t)^* \longrightarrow \mathrm{Div}_0(\mathbb{P}^1_{\bar{k}}) \longrightarrow 0,$$

where $\mathrm{Div}_0(\mathbb{P}^1_{\bar{k}})$ is the degree 0 subgroup of $\mathrm{Div}(\mathbb{P}^1_{\bar{k}})$, and

$$0 \longrightarrow \mathrm{Div}_0(\mathbb{P}^1_{\bar{k}}) \longrightarrow \mathrm{Div}(\mathbb{P}^1_{\bar{k}}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Applying $\mathrm{H}^2(\Gamma_k, -)$ to (1.35) we obtain (1.34). The identification of the third arrow of (1.34) with the Witt residues follows from the last sentence of Theorem 1.5.1. The fourth map in (1.34) is the sum of maps

$$\mathrm{H}^2(\Gamma_k, \mathbb{Z}_{k(x)}) \longrightarrow \mathrm{H}^2(\Gamma_k, \mathbb{Z}),$$

each of which is induced by the summation map $\mathbb{Z}_{k(x)} \to \mathbb{Z}$. This implies the last statement of the theorem. □

The exact sequence (1.33) is split and it is instructive to write down an element of $\mathrm{Br}(k(t))$ that lifts a character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{k(P)}, \mathbb{Q}/\mathbb{Z})$ for a given monic irreducible polynomial $P(t)$.

Let τ_P be the image of t in $k(P) = k[t]/(P(t))$. Then $t - \tau_P \in k(P)(t)$. The character χ gives rise to a character of $\mathrm{Gal}\big(\overline{k(P)(t)}/k(P)(t)\big)$. Let A_χ in $\mathrm{Br}(k(P)(t))$ be the class of the cyclic algebra associated to this character and $t - \tau_P \in k(P)(t)$. By (1.5) this class can also be described as follows. Let $d(\chi) \in \mathrm{H}^2(k(P), \mathbb{Z})$ be the image of χ under the connecting homomorphism d in the long exact sequence of cohomology groups attached to the exact sequence of $\Gamma_{k(P)}$-modules (1.24). We also denote by $d(\chi)$ the image of this element in $\mathrm{H}^2(k(P)(t),\mathbb{Z})$ under the restriction from $k(P)$ to $k(P)(t)$. Then $A_\chi \in \mathrm{Br}(k(P)(t))$ is the cup-product $d(\chi)\cup(t-\tau_P)$ with respect to the pairing (1.27):

$$\mathrm{H}^2(k(P)(t),\mathbb{Z})\times \mathrm{H}^0(k(P)(t),\overline{k(P)(t)}^*)\longrightarrow \mathrm{Br}(k(P)(t)).$$

It is clear that A_χ is unramified on $\mathbb{P}^1_{k(P)}$ away from the $k(P)$-point $t=\tau_P$ and the point at infinity, i.e., the residues of A_χ at all other closed points of $\mathbb{P}^1_{k(P)}$ are trivial. By (1.28), the Witt residue of A_χ at $t=\tau_P$ is

$$r_W(A_\chi)=v(t-\tau_P)\chi=\chi\in \mathrm{H}^1(k(P),\mathbb{Z}/n).$$

A similar formula shows that the Witt residue of $A_\chi\in \mathrm{Br}(k(P)(t))$ at the point at infinity of $\mathbb{P}^1_{k(P)}$ is $-\chi$.

Let us abbreviate the notation for the corestriction map from $k(P)(t)$ to $k(t)$ as $\mathrm{cores}_{k_P/k}$. Using Proposition 1.4.7 we see that $\mathrm{cores}_{k_P/k}(A_\chi)$ is an element of $\mathrm{Br}(k(t))$ unramified away from the closed point P, which is the zero set of $P(t)$, and possibly the point at infinity. More precisely, the Witt residue of $\mathrm{cores}_{k(P)/k}(A_\chi)$ at P is χ and the Witt residue at ∞ is $-\mathrm{cores}_{k(P)/k}(\chi)$.

Let $A\in \mathrm{Br}(k(t))$ be an arbitrary element. Let χ_P be the Witt residue of A at the closed point P of $\mathbb{P}^1_k$. Let S be the set of closed points $P\in\mathbb{A}^1_k$ for which $\chi_P\neq 0$. Then $A-\sum_{P\in S}\mathrm{cores}_{k(P)/k}(A_{\chi_P})$ is unramified over $\mathbb{A}^1_k$. Faddeev's exact sequence (1.33) now shows that

$$A=\sum_{P\in S}\mathrm{cores}_{k(P)/k}(A_{\chi_P})+A_0,$$

for some $A_0\in\mathrm{Br}(k)$.

Remark 1.5.3 Izhboldin computed the group $\mathrm{Br}(k(t))[p]$ for any field k of characteristic $p>0$, not necessarily perfect, see [Izh96, Thm. 4.5].

Chapter 2
Étale cohomology

The étale topology and étale cohomology were invented by A. Grothendieck in the beginning of the 1960s, after Serre's discussion of local triviality for principal homogeneous spaces in algebraic geometry [Ser58]. More general "topologies" and associated cohomology theories were developed by A. Grothendieck, M. Artin and J.-L. Verdier.

Original references for this material are Artin's Harvard seminar [Art62] and the Artin–Grothendieck–Verdier seminar [SGA4]. There is a gentle introduction by Deligne in [SGA4$\frac{1}{2}$, Arcata]. See also [Tam94]. A basic book on the subject, written by J.S. Milne, is [Mil80]. For a thorough, modern treatment we refer to the Stacks project [Stacks]; it follows the same line of exposition as [Art62], which remains an excellent reference. A concise and very readable introduction can be found in [Po18, Ch. 6].

In the first two sections of this chapter we introduce notation and terminology, and state basic results about étale sheaves and étale cohomology. It would not be realistic to give proofs; instead, we try to help the reader navigate through some of the above references.

The third section reports on purity results for étale cohomology with torsion coefficients and on residues in the étale cohomological context.

In the next two sections we discuss the first cohomology group of the multiplicative group, which is the Picard group, and then the relative Picard group and the Picard scheme. The Brauer group of a field k naturally appears when one considers the obstruction for a Galois invariant element of the geometric Picard group of a smooth projective variety X over k to come from an element of the Picard group of X.

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2.1 Topologies, sites, sheaves

2.1.1 Grothendieck topologies

Let $\mathcal{C}$ be a small category. Consider the set of all families of morphisms in $\mathcal{C}$ with a common target $\{U_i \to U\}_{i\in I}$. A *Grothendieck topology* on $\mathcal{C}$ is a subset $\mathcal{T}$ of this set satisfying the following axioms. The elements of $\mathcal{T}$ are referred to as "coverings".

(i) If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering.
(ii) If $\{U_i \to U\}_{i\in I}$ is a covering and for each $i \in I$ we have a covering $\{V_{ij} \to U_i\}$ for j in a set I_i, then $\{V_{ij} \to U\}$ is a covering, where $V_{ij} \to U$ is the composition $V_{ij} \to U_i \to U$.
(iii) For an *arbitrary* morphism $V \to U$ in $\mathcal{C}$ and a covering $\{U_i \to U\}_{i\in I}$, the fibred products $U_i \times_U V$ exist in $\mathcal{C}$ and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

The pair consisting of a small category $\mathcal{C}$ together with a set $\mathcal{T}$ of coverings satisfying the above axioms is called a *site*. The category $\mathcal{C}$ is referred to as the underlying category of the site. This is the general presentation of [Art62, SGA4, Stacks, Po18].

Milne [Mil80] discusses the following construction of a site, which is also enough for our purposes. Consider a scheme X and a full subcategory C_X of the category Sch/X of schemes over X. In particular, the morphisms in C_X are morphisms of X-schemes. Consider a subclass E of morphisms in C_X satisfying the following properties. The elements of E are referred to as "open sets".

(i) Every isomorphism is in E.
(ii) A composition of morphisms in E is in E.
(iii) If $V \to U$ is in E and $W \to U$ is an *arbitrary* morphism in C_X, then $V \times_U W \to W$ is in E.

By definition, a family of morphisms $\{f_i \colon U_i \to U\}_{i\in I}$ in C_X is a covering if each f_i is in E and $\cup_{i\in I} f_i(U_i) = U$. Such families define a Grothendieck topology on C_X. The pair consisting of C_X and the family of all such coverings for all U in C_X is a site denoted by X_E. The category C_X is the underlying category of the site X_E.

The property of a morphism $f \colon V \to U$ to be étale is local on V. In particular, although every étale morphism is locally of finite presentation, hence locally of finite type, one does not assume it to be quasi-compact, so it is not necessarily of finite type. See [Stacks, Section 02GH] for properties of étale morphisms.

In this book the following sites X_E will be used.

X_{zar} is the small Zariski site of X, that is, C_X is the category of open subschemes of X, and E is the class of open embeddings.

X_{Zar} is the big Zariski site, that is, C_X is the category Sch/X of all schemes over X, and E is the class of open embeddings.

$X_{\text{ét}}$ is the small étale site, that is, C_X is the category of schemes that are étale over X, and E is the class of étale maps. Note that in the case of $X_{\text{ét}}$ all maps in C_X are automatically étale (unlike for $X_{\text{Ét}}$ or X_{fppf}).

$X_{\text{Ét}}$ is the big étale site, that is, C_X is the category Sch/X of all schemes over X, and E is the class of étale maps.

X_{fppf} is the big flat site, that is, C_X is the category Sch/X of all schemes over X, and E is the class of flat morphisms that are locally of finite presentation.

We shall only consider sites X_E of one the above types (although the fpqc site will be fleetingly mentioned in Chapter 4). We write C_X for the underlying category of X-schemes of the site X_E.

Let $\pi\colon X'\to X$ be a morphism of schemes. Let X_E and $X'_{E'}$ be sites. One says that π induces a *continuous map of sites* $X'_{E'}\to X_E$, sometimes also denoted by π, if the following properties are satisfied:

(i) For any $Y\in C_X$, $Y\times_X X'$ is in $C_{X'}$.
(ii) For any covering $\{U_i\to Y\}_{i\in I}$ in X_E, the family $\{U_i\times_X X'\to Y\times_X X'\}_{i\in I}$ is a covering in $X'_{E'}$.

Such a definition makes sense for more general sites $(\mathcal{C},\mathcal{T})$ and $(\mathcal{C}',\mathcal{T}')$, when one is given a functor $\mathcal{C}\to\mathcal{C}'$, see [Po18, Def. 6.2.7].

For the sites listed above (whose underlying categories are categories of schemes over a base scheme X) the second condition reduces to the condition that for any "open set" $V\to U$ in X_E, the fibred product $V\times_X X'\to U\times_X X'$ is an "open set" in $X'_{E'}$.

For example, the identity map on a scheme X defines continuous maps of sites

$$X_{\mathrm{fppf}}\longrightarrow X_{\text{Ét}}\longrightarrow X_{\text{ét}}\longrightarrow X_{\mathrm{zar}}.$$

Here is another basic example. Any morphism of schemes $X'\to X$ induces a continuous map of sites $X'_E\to X_E$ where E is any one of zar, Zar, ét, Ét, fppf.

2.1.2 Presheaves and sheaves

Let X be a scheme. A *presheaf* of sets on a site X_E with underlying category C_X is by definition a presheaf of sets on C_X, i.e. a contravariant functor $\mathcal{P}$ from C_X to the category of sets. For Y in C_X, we refer to $\mathcal{P}(Y)$ as the set of sections of $\mathcal{P}$ over Y.

Presheaves of abelian groups on X_E form an abelian category, where a sequence of presheaves is exact if and only if the corresponding sequence of sections over Y is an exact sequence of abelian groups, for any Y in C_X. We denote this abelian category by $\mathrm{P}(X_E)$. For example,

$\mathbb{G}_{a,X}$ is the presheaf such that $\mathbb{G}_{a,X}(Y) = \mathrm{H}^0(Y, \mathcal{O}_Y)$,
$\mathbb{G}_{m,X}$ is the presheaf such that $\mathbb{G}_{m,X}(Y) = \mathrm{H}^0(Y, \mathcal{O}_Y^*)$.
$\mu_{n,X}$, for $n \geq 1$, is the presheaf such that $\mu_n(Y) = \{x \in \mathrm{H}^0(Y, \mathcal{O}_Y^*) | x^n = 1\}$.

A presheaf $\mathcal{P}$ of sets on X_E is called a *sheaf* on X_E if for any scheme Y in C_X and any covering $\{U_i \to Y\}_{i \in I}$, any section over Y is uniquely determined by its restrictions to the U_i, and any family of sections over the U_i which agree on $U_i \times_Y U_j$ comes from a section over Y.

Given a presheaf of sets $\mathcal{P}$ on X_E, there is a sheaf of sets $a\mathcal{P}$ on X_E associated to the presheaf $\mathcal{P}$. The construction of this sheaf is non-trivial [Art62, Ch. II], [Mil80, Thm. II.2.11], [Stacks, Section 00W1]. One can give an explicit construction of $a\mathcal{P}$ in terms of the 0-th Čech cohomology presheaf $\check{\mathcal{H}}^0(\mathcal{P})$, namely, $a\mathcal{P} = \check{\mathcal{H}}^0(\check{\mathcal{H}}^0(\mathcal{P}))$. (Applying $\check{\mathcal{H}}^0$ to a presheaf produces a separated presheaf, that is, a presheaf whose sections are uniquely determined by their restrictions to the U_i. Applying $\check{\mathcal{H}}^0$ to a separated presheaf produces a sheaf, see [Mil80, Remark III.2.2 (c)].)

One defines the category $\mathrm{S}(X_E)$ of sheaves of abelian groups on X_E as the full subcategory of $\mathrm{P}(X_E)$: a homomorphism between two sheaves on X_E is just a homomorphism of the underlying presheaves on X_E.

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of abelian groups on X and $\varphi \colon \mathcal{F} \to \mathcal{G}$ be a morphism in $\mathrm{S}(X_E)$. The kernel $\mathrm{Ker}(\varphi)$ in $\mathrm{S}(X_E)$ is the same as the kernel of the morphism of presheaves $\varphi \colon \mathcal{F} \to \mathcal{G}$ (which is a sheaf). However, the cokernel presheaf $\mathrm{Coker}(\varphi)$ is not always a sheaf (for example, the cokernel of the differentiation on the sheaf of holomorphic functions on $\mathbb{C} \smallsetminus \{0\}$ is not a sheaf). The cokernel in $\mathrm{S}(X_E)$ is the sheaf associated to the presheaf $\mathrm{Coker}(\varphi)$. This makes $\mathrm{S}(X_E)$ into an abelian category.[1] The inclusion functor $i \colon \mathrm{S}(X_E) \to \mathrm{P}(X_E)$ is left exact, and $a \colon \mathrm{P}(X) \to \mathrm{S}(X_E)$ is the left adjoint of i. We have an isomorphism of (bi-)functors

$$\mathrm{Hom}_{\mathrm{S}(X_E)}(a\mathcal{P}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{P}(X_E)}(\mathcal{P}, i\mathcal{F}),$$

see [Mil80], Remark II.2.14 (a) and Thm. II.2.15. The functor

$$a \colon \mathrm{P}(X_E) \to \mathrm{S}(X_E)$$

is exact [Mil80, Thm. II.2.15 (a)].

If G is a commutative group scheme over X, then the functor represented by G, that is, the functor associating to a scheme Y in C_X the abelian group $\mathrm{Hom}_X(Y, G)$, is not only a presheaf but is actually a sheaf for each of the sites X_E mentioned above, by [Mil80, Cor. II.1.7]. In particular, the presheaves $\mathbb{G}_{a,X}$ and $\mathbb{G}_{m,X}$ are sheaves because they are represented by the pullbacks to X of the additive group $\mathbb{Z}$-scheme $\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[x])$ and the multiplicative group $\mathbb{Z}$-scheme $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$, respectively. This also holds for $\mu_{n,X}$.

[1] Thus $\mathrm{S}(X_E)$ is an abelian category and is also a full subcategory of $\mathrm{P}(X_E)$, but $\mathrm{S}(X_E)$ is *not* an abelian subcategory of $\mathrm{P}(X_E)$, because the notion of cokernel is not the same.

When this causes no confusion, for a commutative group $\mathbb{Z}$-scheme G we sometimes write G for the sheaf represented by the pullback of G to X.

A sheaf of abelian groups $\mathcal{F}$ on $X_{\text{ét}}$ is called a *torsion sheaf* if for any étale morphism $U\to X$, where U is quasi-compact, the group $\mathcal{F}(U)$ is a torsion group [Mil80, Ch. VI, §1].

2.1.3 Direct and inverse images

A continuous map of sites $\pi\colon X'_{E'}\to X_E$ defines a functor $\pi_p\colon \mathrm{P}(X'_{E'})\to\mathrm{P}(X_E)$ which associates to a presheaf $\mathcal{P}$ on $X'_{E'}$ the presheaf on X_E which sends $Y\in C_X$ to $\mathcal{P}(Y\times_X X')\in C_{X'}$. It is obvious that π_p is an exact functor.

For a presheaf $\mathcal{P}$ on X_E and an object $Y'\in C_{X'}$, define

$$\pi^p(\mathcal{P})(Y') = \varinjlim \mathcal{P}(Y),$$

where the limit is over all commutative diagrams

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

where $Y\in C_X$. One then proves [Art62, Ch. I, Thm. (2.1)], [Mil80, Ch. II, §2] that π^p is a functor $\mathrm{P}(X_E)\to\mathrm{P}(X'_{E'})$ which is left adjoint to π_p:

$$\mathrm{Hom}_{\mathrm{P}(X_E)}(\mathcal{P}_1,\pi_p\mathcal{P}_2) = \mathrm{Hom}_{\mathrm{P}(X'_{E'})}(\pi^p\mathcal{P}_1,\mathcal{P}_2).$$

The functor π_p is exact. The functor π^p is right exact. If $\pi\colon X'\to X$ belongs to the category C_X, then $\pi^p(\mathcal{P})$ is the restriction of $\mathcal{P}$ to $C_{X'}$.

For each of the five sites in Section 2.1.1, the functor π^p is left exact, hence π^p is exact. See [Mil80, Prop. II.2.6]. For $X_{\text{ét}}$, see [Art62, Thm. II.4.14] and [Art62, Cor. III.1.6]. The key point is that finite inverse limits exist in C_X. In particular, if $U\to W$ and $V\to W$ are in C_X, then $U\times_W V$ is in C_X. This would not hold for the small flat site (which is why this site is not used so we did not define it). Indeed, if $U\to X$, $V\to X$ and $W\to X$ are flat, and we have morphisms $U\to W$ and $V\to W$ in C_X (which need not be "open sets"), then the fibred product $U\times_W V$ may not be flat over X.

It is easy to see that π_p sends sheaves to sheaves [Mil80, Prop. II.2.7]. In general, this does not hold for π^p.

For $\pi\colon X'_{E'}\to X_E$ a continuous map of sites, and $\mathcal{F}$ a sheaf on $X'_{E'}$, the direct image sheaf $\pi_*\mathcal{F}$ is defined as $\pi_p\mathcal{F}$, that is, $\pi_*\mathcal{F}(Y) = \mathcal{F}(Y\times_X X')$, where Y is in C_X. Note that since the cokernels in $\mathrm{P}(X_E)$ and $\mathrm{S}(X_E)$ are not the same, π_* is not in general an exact functor.

The inverse image $\pi^*\mathcal{G}$ of a sheaf $\mathcal{G}$ on X_E is defined as $a\pi^p\mathcal{G}$, so one can write $\pi^* = a\pi^p i$. For each of the five sites X_E listed in Section 2.1.1, the functor π^* on $\mathrm{S}(X_E)$ is exact, since π^p is exact on $\mathrm{P}(X_E)$.

If $E' = E$ and the morphism $\pi\colon X' \to X$ is in C_X, then π^* is the restriction of $\mathcal{F}$ to X'_E.

In particular, if G_X is the sheaf on X_E represented by a commutative group scheme G over X, then $\pi^* G_X = G_{X'}$ when $\pi\colon X' \to X$ is in C_X. For example, this holds for the big étale site. (But $\pi^* G_X$ in general differs from $G_{X'}$ for the small étale site, unless π is étale or G is a commutative étale group X-scheme. For example, take $X = \mathrm{Spec}(k)$, where k is an algebraically closed field. Let $X' = \mathbb{A}^1_k$ and let $\pi\colon X' \to X$ be the structure morphism. The sheaf $\mathbb{G}_{m,k}$ on $X_{\text{ét}}$ is the constant sheaf associated to k^*, so $\pi^*\mathbb{G}_{m,k}$ is also a constant sheaf, hence $\pi^*\mathbb{G}_{m,k}$ and $\mathbb{G}_{m,X'}$ are not isomorphic.)

The functors π_* and π^* are adjoint:

$$\mathrm{Hom}_{\mathrm{S}(X_E)}(\mathcal{F}, \pi_*\mathcal{F}') = \mathrm{Hom}_{\mathrm{S}(X'_{E'})}(\pi^*\mathcal{F}, \mathcal{F}').$$

Torsors

Let $\mathcal{G}$ be a sheaf of groups on X_E. A sheaf of sets $\mathcal{F}$ on X_E with a right action $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ is called a right $\mathcal{G}$-*torsor* if every Y in C_X has a covering $\{U_i \to Y\}_{i \in I}$ such that $\mathcal{F}(U_i) \neq \emptyset$ for all $i \in I$, and the right action of $\mathcal{G}(Y)$ on $\mathcal{F}(Y)$ is simply transitive whenever $\mathcal{F}(Y) \neq \emptyset$. A torsor $\mathcal{F}$ is trivial if $\mathcal{F}(X) \neq \emptyset$.

Let G be a group X-scheme. An X-scheme $\mathcal{T}$ with a right action of G

$$m\colon \mathcal{T} \times_X G \longrightarrow \mathcal{T}$$

is called a right G-*torsor* over X_E, or an X_E-*torsor* for G, or a *principal homogeneous set* of G, if the morphism

$$\mathcal{T} \times_X G \longrightarrow \mathcal{T} \times_X \mathcal{T}$$

given by $(t, g) \mapsto (t, m(t, g))$ is an isomorphism of X-schemes and there is a covering $\{U_i \to X\}_{i \in I}$ with a G-equivariant isomorphism

$$G \times_X U_i \xrightarrow{\sim} \mathcal{T} \times_X U_i$$

for each $i \in I$. The last condition is equivalent to the existence of a section of $\mathcal{T} \times_X U_i \to U_i$. A torsor with a section is called trivial. For more on torsors see [Gir71, Ch. III, §1], [SGA3, IV], [Mil80, Ch. III, §4], [Stacks, Section 0497], and [Sko01, Ch. 2].

Assume that the group X-scheme G is flat and locally of finite presentation over X. Let $\mathcal{G}$ be the sheaf of groups on X_E defined by G. If G is an *affine* group X-scheme, then, by flat descent, any $\mathcal{G}$-torsor over X_{fppf} is representable by an X-scheme which is a G-torsor over X_{fppf}, see [Mil80, Thm. III.4.3 (a)].

If G is *smooth* over X, then any X_{fppf}-torsor is an $X_{\text{ét}}$-torsor, that is, it is locally trivial for the étale topology on X, see [Mil80, Ch. III, §4].

2.1.4 Sheaves on the small étale site

Let $x = \mathrm{Spec}(k(x))$ be a point of a scheme X. The local ring of X at x is denoted by $\mathcal{O}_{X,x}$. We have

$$\mathcal{O}_{X,x} = \varinjlim \mathcal{O}(U),$$

where the injective limit is taken over all open subsets $U \subset X$ containing x.

We let

$$\mathcal{O}^{\mathrm{h}}_{X,x} = \varinjlim \mathcal{O}(U),$$

where U is an étale X-scheme equipped with a lifting $x \hookrightarrow U$ of $x \hookrightarrow X$. The superscript h says that $\mathcal{O}^{\mathrm{h}}_{X,x}$ is the *henselisation* of the local ring $\mathcal{O}_{X,x}$. The residue field of the local ring $\mathcal{O}^{\mathrm{h}}_{X,x}$ is $k(x)$.

Now let $\bar{x}\to X$ be a geometric point, i.e. a morphism $\mathrm{Spec}(k(\bar{x}))\to X$, where $k(\bar{x})$ is algebraically closed. One says that $\bar{x}$ lies over the point x and one writes $\bar{x} \mapsto x \in X$ if x is the image of $\bar{x}$ in X; then $k(x) \subset k(\bar{x})$. Define

$$\mathcal{O}^{\mathrm{sh}}_{X,x} = \varinjlim \mathcal{O}(U),$$

where U is an étale X-scheme equipped with a lifting $\bar{x}\to U$ of $\bar{x}\to X$. This is the analogue for the étale topology of the local ring for the Zariski topology. The superscript sh says that $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is strictly henselian; it is the *strict henselisation* of the local ring $\mathcal{O}_{X,x}$. The residue field of the local ring $\mathcal{O}^{\mathrm{sh}}_{X,x}$ is the separable closure of $k(x)$ in $k(\bar{x})$. Speaking of "the" strict henselisation is a common abuse of language, which we shall keep in this book. If $k(x)$ is not separably closed, the ring extension $\mathcal{O}^{\mathrm{sh}}_{X,x}$ of $\mathcal{O}_{X,x}$ is defined up to a non-unique isomorphism. Replacing $\bar{x}\to X$ by a different geometric point over x produces a local ring isomorphic to $\mathcal{O}^{\mathrm{sh}}_{X,x}$; this isomorphism is determined by the induced isomorphism of residue fields, which are two separable closures of $k(x)$, hence they are isomorphic but in a non-unique way.

For more on henselisation and strict henselisation see [Ray70b, Ch. VIII], [BLR90, §2.3] and [Stacks, Section 0BSK].

The *stalk* of a presheaf $\mathcal{P}$ on $X_{\text{ét}}$ at a geometric point $u\colon \bar{x} \mapsto x \in X$ is defined as

$$\mathcal{P}_{\bar{x}} = \varinjlim \mathcal{P}(U),$$

where U is étale over X such that u factors through $U \to X$, see [Mil80, Ch. II, §2, pp. 59–60]. The stalk of $\mathcal{P}$ is canonically isomorphic to the stalk of the sheafification $\mathcal{P}_{\bar{x}} = (a\mathcal{P})_{\bar{x}}$.

For example, it is clear from the definition that we have

$$(\mathbb{G}_{a,X})_{\bar{x}} = \mathcal{O}^{\mathrm{sh}}_{X,x}, \qquad (\mathbb{G}_{m,X})_{\bar{x}} = (\mathcal{O}^{\mathrm{sh}}_{X,x})^*.$$

Let $\mathcal{F}$ be sheaf on $X_{\text{ét}}$. A section $s \in \mathcal{F}(X)$ is non-zero if and only if there is a geometric point $\bar{x} \mapsto x \in X$ such that the image of s in $\mathcal{F}_{\bar{x}}$ is non-zero [Art62, Prop. III.1.6], [Mil80, Prop. II.2.10]. Similarly, a sequence of sheaves on $X_{\text{ét}}$ is exact if and only if the corresponding sequence of stalks is exact for each geometric point $\bar{x} \mapsto x \in X$ [Mil80, Thm. II.2.15].

Proposition 2.1.1 *Let $\pi\colon X' \to X$ be a finite morphism of schemes. The direct image functor $\pi_*\colon \mathrm{S}(X'_{\text{ét}}) \to \mathrm{S}(X_{\text{ét}})$ is exact.*

Proof. See [Mil80, Cor. II.3.6] and [Art62, Cor. III.4.11]. □

Let $\pi\colon X' \to X$ be a morphism, and let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$. If x' is a point of X' that maps to $x \in X$, then we can choose a geometric point over x' to be also a geometric point over x, that is, $\bar{x} = \bar{x}'$. This formally implies that we have $(\pi^*\mathcal{F})_{\bar{x}'} = \mathcal{F}_{\bar{x}}$ (see also [Mil80, Thm. II.3.2 (a)]).

Now let $\pi\colon X' \to X$ be quasi-compact and quasi-separated. Let $\mathcal{F}$ be a sheaf on $X'_{\text{ét}}$. Then the stalk of $\pi_*\mathcal{F}$ at a geometric point $\bar{x} \mapsto x \in X$ can be computed at the strict henselisation of X at x:

$$(\pi_*\mathcal{F})_{\bar{x}} = \widetilde{\mathcal{F}}(X' \times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x})), \tag{2.1}$$

where $\widetilde{\mathcal{F}}$ is the inverse image of $\mathcal{F}$ with respect to the first projection

$$X' \times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x}) \longrightarrow X',$$

see [Mil80, Thm. II.3.2 (b)], [SGA4, Thm. VIII.5.2].

2.2 Cohomology

2.2.1 Definition and basic properties

Let X be a scheme and let X_E be one of the sites mentioned in Section 2.1.1. The abelian category $\mathrm{S}(X_E)$ of sheaves of abelian groups on X_E has enough injectives [Mil80, Prop. III.1.1], [Stacks, Thm. 01DP], which makes it possible to define the cohomology groups $\mathrm{H}^n_E(X, \mathcal{F})$ as the right derived functors of the left exact sections functor $\mathcal{F} \mapsto \mathcal{F}(X)$, see [Mil80, Def. III.1.5]. If $\pi\colon X'_{E'} \to X_E$ is a continuous map of sites, then the higher derived image sheaves $(R^n\pi_*)(\mathcal{F})$ are the right derived functors of the left exact functor $\pi_*\colon \mathrm{S}(X'_{E'}) \to \mathrm{S}(X_E)$.

One proves [Mil80, Prop. III.1.13] that $(R^n\pi_*)(\mathcal{F})$ is the sheaf on X_E associated to the presheaf that sends $U \in C_X$ to the group $\mathrm{H}^n_E(U \times_X X', \mathcal{F})$.

If $\mathcal{G}$ is a sheaf on X_E, then the functor $\mathrm{Hom}_X(\mathcal{G}, \cdot)$ on $\mathrm{S}(X_E)$ is left exact; so one defines $\mathrm{Ext}^n_X(\mathcal{G}, \cdot)$ as its right derived functors. Since $\mathcal{F}(X) = \mathrm{Hom}_X(\mathbb{Z}_X, \mathcal{F})$, where $\mathbb{Z}_X$ is the constant sheaf associated to the constant presheaf $\mathbb{Z}$, one has $\mathrm{Ext}^n_X(\mathbb{Z}_X, \mathcal{F}) = \mathrm{H}^n_E(X, \mathcal{F})$.

Let us consider the small étale site $X_{\text{ét}}$. For a commutative ring R we shall sometimes write $\mathrm{H}^i(R, -)$ for $\mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(R), -)$.

If $\pi\colon X' \to X$ is a quasi-compact and quasi-separated morphism of schemes with associated continuous map of sites $X'_{\text{ét}} \to X_{\text{ét}}$, then the stalk of $(R^n\pi_*)(\mathcal{F})$ at a geometric point $\bar{x} \mapsto x \in X$ can be described in the same way as we described $(\pi_*\mathcal{F})_{\bar{x}}$ in (2.1):

$$(R^n\pi_*)(\mathcal{F})_{\bar{x}} = \mathrm{H}^n_{\text{ét}}(X' \times_X \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,x}), \widetilde{\mathcal{F}}),$$

where $\widetilde{\mathcal{F}}$ is as in the end of the previous section. See [Stacks, Thm. 03Q9].

If $\pi\colon X' \to X$ is a proper morphism, then a corollary of the *proper base change theorem* says that for a torsion sheaf $\mathcal{F}$ on $X'_{\text{ét}}$, the stalk of $(R^n\pi_*)(\mathcal{F})$ at a geometric point $\bar{x} \mapsto x \in X$ is $\mathrm{H}^n_{\text{ét}}(X'_{\bar{x}}, \mathcal{F})$, where

$$X'_{\bar{x}} = \pi^{-1}(\bar{x}) = X' \times_X \mathrm{Spec}(k(\bar{x}))$$

is the geometric fibre of π at $\bar{x}$. See [Stacks, Lemma 0DDF] and [Mil80, Cor. VI.2.5].

By a corollary of the *smooth base change theorem* [Mil80, Cor. VI.4.2], if $\pi\colon X' \to X$ is a smooth and proper morphism, and m is prime to the residual characteristics of X, then the sheaf $(R^n\pi_*)(\mathbb{Z}/m)$ is locally constant for every $n \geq 0$. Since the stalk of this sheaf at a geometric point $\bar{x}$ is naturally identified with $\mathrm{H}^n_{\text{ét}}(X'_{\bar{x}}, \mathbb{Z}/m)$, we see that if X is connected, the groups $\mathrm{H}^n_{\text{ét}}(X'_{\bar{x}}, \mathbb{Z}/m)$ are isomorphic for all geometric points $\bar{x}$. These results have many applications. For example, if $\pi\colon X \to \mathrm{Spec}(R)$ is a smooth and proper morphism, where R is a discrete valuation ring with fraction field K and residue field k, and m is prime to $\mathrm{char}(k)$ and $\mathrm{char}(K)$, then the restriction of the representation of $\mathrm{Gal}(K_{\mathrm{s}}/K)$ in $\mathrm{H}^n_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/m)$ to the inertia group is trivial.

Non-abelian cohomology

Let X be a scheme. Let G be a flat group X-scheme which is locally of finite type. We do not assume that G is abelian. Let $\mathcal{G}$ be the sheaf of groups on X_{fppf} defined by G. By [Mil80, Prop. III.4.6] the isomorphism classes of $\mathcal{G}$-torsors over X_{fppf} are in a natural bijection with the pointed Čech cohomology set $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X,G)$. If, moreover, G/X is affine and smooth, then G-torsors over $X_{\text{ét}}$, up to isomorphism, are classified by the elements of $\check{\mathrm{H}}^1_{\text{ét}}(X,G)$. If G is abelian, then $\mathrm{H}^1_{\text{ét}}(X,G) \cong \check{\mathrm{H}}^1_{\text{ét}}(X,G)$ by [Mil80, Cor. III.2.10]. We shall sometimes use the notation $\mathrm{H}^1_{\text{ét}}(X,G)$ for the pointed set of isomorphism classes of étale G-torsors over X, whether G is abelian or not. (This agrees with the use of this notation in Giraud's book, see [Gir71, Déf. 2.4.2].)

2.2.2 Passing to the limit

Let I be an increasing filtering ordered set. Choose an element $0 \in I$. Suppose that we have a projective system $\{X_i\}_{i\in I}$ of quasi-compact and quasi-separated schemes X_i with affine transition morphisms $X_i \to X_j$ for all $i,j \in I$ such that $i \geq j$. Then there is a projective limit scheme $X = \varprojlim X_i$ [SGA4, VII.5.1], [Stacks, Lemma 01YX].

The following statements are proved in [EGA, IV$_3$, §8], [SGA4, VII, Cor. 5.9, Cor. 5.10], see also [Mil80, Lemma III.1.16, Remark III.1.17 (a)].

(i) Let $\mathcal{F}_0$ be a sheaf on $X_{0,\text{ét}}$. For each $i \in I$ such that $i \geq 0$ let $\mathcal{F}_i$ be the inverse image of $\mathcal{F}_0$ on $X_{i,\text{ét}}$. Let $\mathcal{F}$ be the inverse image of $\mathcal{F}_0$ on $X_{\text{ét}}$. Then for any $n \geq 0$ the natural homomorphism

$$\varinjlim \mathrm{H}^n_{\text{ét}}(X_i,\mathcal{F}_i) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(X,\mathcal{F})$$

is an isomorphism.

(ii) Let G_0 be a commutative group scheme locally of finite presentation over X_0. For each $i \in I$ such that $i \geq 0$ define $G_i = G_0 \times_{X_0} X_i$. Let $G = G_0 \times_{X_0} X$. Then for any $n \geq 0$ the natural homomorphism

$$\varinjlim \mathrm{H}^n_{\text{ét}}(X_i,G_i) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(X,G)$$

is an isomorphism. In particular, we have natural isomorphisms

$$\varinjlim \mathrm{H}^n_{\text{ét}}(X_i,\mathbb{G}_{m,X_i}) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(X,\mathbb{G}_{m,X}).$$

Note that (ii) is not an instance of (i), since the pullback of the étale sheaf represented by a non-étale group scheme is not computed by base change. (See the example at the end of Section 2.1.3.)

2.2.3 Étale and Galois cohomology

Let k be a field. Consider the small étale site $\mathrm{Spec}(k)_{\text{ét}}$. The underlying category of $\mathrm{Spec}(k)_{\text{ét}}$ consists of étale schemes over k, that is, disjoint unions of $\mathrm{Spec}(K)$, where K is a finite separable field extension of k.

There is an equivalence of categories between the category of sheaves of abelian groups on $\mathrm{Spec}(k)_{\text{ét}}$ and the category of continuous discrete Γ-modules. See [Stacks, Theorem 03QT]. Let us describe the relevant functors.

To a sheaf $\mathcal{F}$ on $\mathrm{Spec}(k)_{\text{ét}}$ one associates its stalk. It is the continuous discrete Γ-module

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}(k'),$$

where k' runs through the finite separable field extensions of k and the limit is taken over compatible k-embeddings into k_{s}. Indeed, we can assume that k' is Galois over k, so that Γ acts on each $\mathcal{F}(k')$, and thus on $M_{\mathcal{F}}$. This module is discrete because $M_{\mathcal{F}}$ is the union of the invariants with respect to all open subgroups of Γ.

In the opposite direction, a continuous discrete Γ-module M defines a presheaf $\mathcal{F}_M$ on $\mathrm{Spec}(k)_{\text{ét}}$ by the formula

$$\mathcal{F}_M\Big(\coprod_{i\in I}\mathrm{Spec}(k_i)\Big) = \prod_{i\in I} M^{\mathrm{Gal}(k_{\mathrm{s}}/k_i)},$$

where the fields $k_i \subset k_{\mathrm{s}}$ are finite extensions of k. One checks that $\mathcal{F}_M$ is a sheaf [Mil80, Lemma II.1.8].

For a discrete Γ-module M the cohomology group $\mathrm{H}^n(\Gamma, M)$ for $n \geq 0$ is the inductive limit of $\mathrm{H}^n(\Gamma/U, M^U)$, where U ranges over all open normal subgroups of Γ, see [SerCG, Ch. I, §2]. By [SerCG, II, §1, 1.1], the group $\mathrm{H}^n(\Gamma, M_{\mathcal{F}})$ does not depend on the choice of k_{s} up to a unique isomorphism; it is called the *Galois cohomology group* and is denoted by $\mathrm{H}^n(k, M_{\mathcal{F}})$.

There is a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(k), \mathcal{F}) \cong \mathrm{H}^n(k, M_{\mathcal{F}}),$$

since these groups are the right derived functors of $M \mapsto M^{\Gamma}$. Similarly, the Ext group $\mathrm{Ext}^n_{\mathrm{Spec}(k)}(\mathcal{F}, \mathcal{F}')$ in the category of étale sheaves on $\mathrm{Spec}(k)$ is canonically identified with the Ext group $\mathrm{Ext}^n_k(M_{\mathcal{F}}, M_{\mathcal{F}'})$ in the category of discrete Γ-modules.

Now assume that $\pi\colon X \to \mathrm{Spec}(k)$ is a quasi-compact and quasi-separated scheme over a field k. Define $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. Let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$, and let $\widetilde{\mathcal{F}}$ be the inverse image of $\mathcal{F}$ with respect to the morphism $X^{\mathrm{s}}_{\text{ét}} \to X_{\text{ét}}$. The sheaf $\pi_*\mathcal{F}$ on $\mathrm{Spec}(k)_{\text{ét}}$ corresponds to the Γ-module

$$(\pi_*\mathcal{F})_{k_{\mathrm{s}}} = \varinjlim \mathcal{F}(X \times_k k') \cong \widetilde{\mathcal{F}}(X^{\mathrm{s}}),$$

where k'/k ranges over finite subextensions of k_{s}/k. In the same way, the sheaf $(R^n\pi_*)(\mathcal{F})$ corresponds to the Γ-module

$$(R^n\pi_*)(\mathcal{F})_{k_{\mathrm{s}}} = \varinjlim \mathrm{H}^n_{\text{ét}}(X \times_k k', \mathcal{F}) \cong \mathrm{H}^n_{\text{ét}}(X^{\mathrm{s}}, \widetilde{\mathcal{F}}).$$

For the two isomorphisms, see [Mil80, Thm. III.1.15]. If $\mathcal{F}$ is defined by a commutative group scheme G which is locally of finite presentation over X, then $\widetilde{\mathcal{F}}$ is defined by the group scheme $G \times_X X^{\mathrm{s}}$ over X^{s} [Mil80, III.1.17].

2.2.4 Standard spectral sequences

Recall that when we have three abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and left exact functors $F\colon \mathcal{A}\to\mathcal{B}$ and $G\colon \mathcal{B}\to\mathcal{C}$ such that F sends injective objects in $\mathcal{A}$ to G-acyclic objects in $\mathcal{B}$, then there is a convergent first quadrant *Grothendieck spectral sequence* of composed functors

$$E_2^{p,q} = (R^pG)(R^qF)A \Rightarrow R^{p+q}(GF)A, \tag{2.2}$$

where $A \in \mathrm{Ob}(\mathcal{A})$, see [Wei94, Thm. 5.8.3], [Mil80, Appendix B]. Let $\mathcal{D}^+(\mathcal{A})$ denote the derived category of bounded below complexes in the abelian category $\mathcal{A}$ (for which we refer to [Wei94, Ch. X]). The above spectral sequence can be viewed as the spectral sequence of composed functors between derived categories $\mathbf{R}F\colon \mathcal{D}^+(\mathcal{A})\to\mathcal{D}^+(\mathcal{B})$ and $\mathbf{R}G\colon \mathcal{D}^+(\mathcal{B})\to\mathcal{D}^+(\mathcal{C})$. In this interpretation (2.2) comes from the fact that $\mathbf{R}(GF)$ is the composition $\mathbf{R}G \circ \mathbf{R}F$, see [Wei94, Thm. 10.8.3].

Suppose that we have continuous maps of sites

$$X''_{E''} \xrightarrow{\pi'} X'_{E'} \xrightarrow{\pi} X_E,$$

and $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the categories of sheaves of abelian groups on $X''_{E''}$, $X'_{E'}$, X_E, respectively. Then one has the Leray spectral sequence [Mil80, Thm. III.1.18 (b)]

$$E_2^{p,q} = (R^p\pi_*)(R^q\pi'_*)\mathcal{F} \Rightarrow R^{p+q}(\pi\pi')_*\mathcal{F}. \tag{2.3}$$

Similarly, for a continuous map of sites $\pi\colon X'_{E'}\to X_E$ we obtain the spectral sequence [Mil80, Thm. III.1.18 (a)]

$$E_2^{p,q} = \mathrm{H}^p_E(X, (R^q\pi_*)(\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}_E(X', \mathcal{F}), \tag{2.4}$$

where $\mathcal{F}$ is a sheaf on $X'_{E'}$.

This is proved by showing that for any continuous map of sites π, the direct image map π_* sends flabby sheaves to flabby sheaves [Mil80, Cor. III.2.13 (b)]. As also mentioned in [Mil80, Remark III.1.20], when π^* is exact on sheaves, which is the case for the five sites X_E under consideration here (Section 2.1.1), then π_* has an exact left adjoint functor hence π_* sends injectives to injectives, and this gives the spectral sequences.

Applications of these spectral sequences are many.

(1) Assume that X is a quasi-compact and quasi-separated scheme over a field k. Let $\Gamma = \mathrm{Gal}(k_s/k)$. Let us apply (2.4) to $\pi\colon X_{\text{ét}} \to \mathrm{Spec}(k)_{\text{ét}}$ and a sheaf $\mathcal{F}$ on $X_{\text{ét}}$. Let $\widetilde{\mathcal{F}}$ be the inverse image of $\mathcal{F}$ with respect to the morphism $X^{\mathrm{s}}_{\text{ét}} \to X_{\text{ét}}$. As we have seen in Section 2.2.3, the sheaf $(R^n\pi_*)(\mathcal{F})$ on $\mathrm{Spec}(k)_{\text{ét}}$ corresponds to the Γ-module $\mathrm{H}^n_{\text{ét}}(X^{\mathrm{s}}, \widetilde{\mathcal{F}})$. Therefore, we obtain the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \widetilde{\mathcal{F}})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathcal{F}).$$

If the sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is defined by a commutative group X-scheme G which is locally of finite presentation, then we obtain the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, G \times_X X^{\mathrm{s}})) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, G). \tag{2.5}$$

(2) Let $\pi\colon X_{\text{Ét}} \to X_{\text{ét}}$ be the continuous map of sites induced by the identity on X. For a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ there is a canonical isomorphism $\mathrm{H}^n_{\text{ét}}(X, \mathcal{F}) = \mathrm{H}^n_{\text{Ét}}(X, \pi^*\mathcal{F})$, see [Tam94, Thm. II. 3.3.1] or [Mil80, Prop. III.3.1]. Since π is induced by the identity on X, the functor π_* is clearly exact. Thus for any sheaf $\mathcal{G}$ on $X_{\text{Ét}}$ the spectral sequence (2.4) gives a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(X, \pi_*\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^n_{\text{Ét}}(X, \mathcal{G}).$$

In particular, if $\mathcal{G}$ is a sheaf on $X_{\text{Ét}}$ represented by a commutative group scheme G over X, then $\pi_*\mathcal{G}$ is the sheaf on $X_{\text{ét}}$ obtained by restricting $\mathcal{G}$ from the category of all X-schemes to the category of étale X-schemes, so $\pi_*\mathcal{G}$ is the sheaf on $X_{\text{ét}}$ represented by G. Thus we obtain a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(X, G_X) \xrightarrow{\sim} \mathrm{H}^n_{\text{Ét}}(X, G_X). \tag{2.6}$$

For for any morphism $f\colon X \to Y$ and any commutative group scheme G over Y this allows one to define a natural map

$$f^*\colon \mathrm{H}^i_{\text{ét}}(Y, G_Y) \longrightarrow \mathrm{H}^i_{\text{ét}}(X, G_X), \tag{2.7}$$

where G_X is the sheaf defined by the group scheme $G \times_Y X$ over X. Indeed, in view of the canonical isomorphism (2.6) we can replace the small étale site by the big étale site. Then $f\colon X \to Y$ is in the underlying category of Y, so $f^*G_Y = G_X$, see Section 2.1.3. The adjunction morphism $G_Y \to f_*f^*G_Y = f_*G_X$ gives rise to the first arrow in

$$\mathrm{H}^i_{\text{Ét}}(Y, G_Y) \longrightarrow \mathrm{H}^i_{\text{Ét}}(Y, f_*G_X) \longrightarrow \mathrm{H}^i_{\text{Ét}}(X, G_X),$$

where the second arrow comes from the spectral sequence (2.4) attached to $f\colon X\to Y$. The map in (2.7) is defined as the composition of these two maps.

(3) Now let $\pi\colon X_{\mathrm{fppf}}\to X_{\text{ét}}$ be the continuous map of sites induced by the identity on X. We refer to [Gro68, Thm. 11.7] (see also [Mil80, Thm. III.3.9] and [Mil80, Rem. 3.11 (b)]) for the following fact. If G is a smooth commutative group scheme over X, then $(R^i\pi_*)(G)=0$ for $i>0$. The Leray spectral sequence then gives isomorphisms

$$\mathrm{H}^n_{\text{ét}}(X,G_X)\xrightarrow{\sim}\mathrm{H}^n_{\mathrm{fppf}}(X,G_X). \tag{2.8}$$

2.3 Cohomological purity

In this section we are concerned with the small étale site on a scheme and Galois cohomology over a field. We often drop the subscript ét.

2.3.1 Absolute purity with torsion coefficients

Let X be a scheme. We write $\mathcal{D}^+(X_{\text{ét}})$ for the derived category of bounded below complexes of étale sheaves of abelian groups on X. Similarly, for a positive integer n we write $\mathcal{D}^+(X_{\text{ét}},\mathbb{Z}/n)$ for the derived category of bounded below complexes of étale $\mathbb{Z}/n$-sheaves on X (for the isomorphism between the corresponding derived functors see [Mil80, Ex. III.2.25]). A standard reference for derived categories is the book [Wei94].

Let $\mathcal{F}$ be a sheaf of abelian groups on X. Suppose that we have a closed immersion $i\colon Z\to X$. Let $U\subset X$ be the complement to Z. To an étale morphism $V\to X$ one associates the abelian group $\mathrm{Ker}[\mathcal{F}(V)\to\mathcal{F}(V_U)]$ of sections supported in Z. The associated sheaf vanishes on U. It is the image under i_* of a sheaf on Z which is denoted by $i^!\mathcal{F}$. The functor from X-sheaves to abelian groups that sends $\mathcal{F}$ to

$$(i^!\mathcal{F})(Z)=\mathrm{Ker}[\mathcal{F}(X)\to\mathcal{F}(U)]$$

is left exact. Its derived functors are denoted by $\mathrm{H}^n_Z(X,\mathcal{F})$ and called the *cohomology groups with support* in Z. This gives rise to a long exact sequence

$$\ldots\to\mathrm{H}^i_Z(X,\mathcal{F})\longrightarrow\mathrm{H}^i_{\text{ét}}(X,\mathcal{F})\longrightarrow\mathrm{H}^i_{\text{ét}}(U,\mathcal{F})\longrightarrow\mathrm{H}^{i+1}_Z(X,\mathcal{F})\to\ldots \tag{2.9}$$

At the level of sheaves we get the functor $\mathbf{R}i^!\colon\mathcal{D}^+(X_{\text{ét}})\to\mathcal{D}^+(Z_{\text{ét}})$. The cohomology sheaves $(R^ni^!)(\mathcal{F})$ of $(\mathbf{R}i^!)\mathcal{F}$ are denoted by $\mathcal{H}^n_Z(X,\mathcal{F})$. Note that $i_*\mathcal{H}^n_Z(X,\mathcal{F})$ is the sheaf associated to the presheaf sending V to $\mathrm{H}^n_{Z\times_X V}(V,\mathcal{F}|_V)$. By definition, the sheaves $\mathcal{H}^n_Z(X,\mathcal{F})$ are the derived functors of the functor from X-sheaves to Z-sheaves sending $\mathcal{F}$ to $i^!\mathcal{F}$. There is a

Grothendieck spectral sequence of composed functors involving $\mathbf{R}i^!$ and the derived functor of the sections functor $\Gamma(Z,\cdot)$:

$$E_2^{pq} = \mathrm{H}^p_{\text{ét}}(Z, \mathcal{H}^q_Z(X,\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}_Z(X,\mathcal{F}), \tag{2.10}$$

see [SGA4, V, Prop. 4.4], [Art62, Ch. III.2], [Mil80, pp. 73, 91, 241].

Assume that n is invertible on X. Write $(\mathbb{Z}/n)(0)_X = (\mathbb{Z}/n)_X$.

For $c > 0$ one defines the sheaf $(\mathbb{Z}/n)(c)_X :\ = \mu_{n,X}^{\otimes c}$.

For $c < 0$ one defines $(\mathbb{Z}/n)(c)_X$ as the sheaf which associates to an étale $Y \to X$ the group $\mathrm{Hom}_Y((\mathbb{Z}/n)(-c)_Y, (\mathbb{Z}/n)_Y)$.

For a sheaf $\mathcal{F}$ of $\mathbb{Z}/n$-modules, one defines $\mathcal{F}(c) :\ = \mathcal{F} \otimes_{\mathbb{Z}/n} (\mathbb{Z}/n)(c)$.

See [Mil80, Ch. II, §3, pp. 78–79] for a general definition of Hom sheaves and tensor product sheaves.

Theorem 2.3.1 (Gabber) *Let X be a regular scheme, let $i\colon Z \hookrightarrow X$ be a closed regular subscheme of codimension c everywhere, let ℓ be a prime different from the residual characteristics of X and let m be a positive integer. In $\mathcal{D}^+(Z_{\text{ét}}, \mathbb{Z}/\ell^m)$ we have an isomorphism $\mathbb{Z}/\ell^m \xrightarrow{\sim} (\mathbf{R}i^!)(\mathbb{Z}/\ell^m)(c)[2c]$. In particular, we have*

$$\mathcal{H}^n_Z(X, \mathbb{Z}/\ell^m) = 0 \text{ for } n \neq 2c, \quad (\mathbb{Z}/\ell^m)(-c)_Z \xrightarrow{\sim} \mathcal{H}^{2c}_Z(X, \mathbb{Z}/\ell^m).$$

Proof. See [Rio14, Thm. 3.1.1, p. 323]. For schemes locally of finite type over a perfect field, the theorem was proved in [SGA4, XVI, Cor. 3.9]. In [Mil80, Thm. VI.5.1] it is proved for schemes smooth over a base scheme. □

Remark 2.3.2 In the case $c = 1$ the isomorphism in this theorem can be described as follows [Mil80, Ch. VI, §6]. Assume that $Z \subset X$ is an integral regular divisor. Let $U = X \setminus Z$. Then (2.9) for $\mathcal{F} = \mathbb{G}_{m,X}$ gives an exact sequence

$$\mathrm{H}^0_{\text{ét}}(U, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_Z(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^1_{\text{ét}}(U, \mathbb{G}_m),$$

where $\mathrm{H}^1_Z(X, \mathbb{G}_m)$ is identified with $\mathbb{Z}$ so that the first arrow is the valuation defined by the local ring $\mathcal{O}_{X,Z}$, the second arrow is the map that 'forgets the support', and the third arrow is the restriction to U. The image of $1 \in \mathbb{Z}$ under the map induced by the Kummer sequence

$$\mathrm{H}^1_Z(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_Z(X, \mu_{\ell^m})$$

is called the *fundamental class* $s_{Z/X}$. The order of $s_{Z/X}$ is ℓ^m, hence $s_{Z/X}$ generates $\mathcal{H}^2_Z(X, \mu_{\ell^m})$. The isomorphism in Theorem 2.3.1 is obtained as the Tate twist of the map $(\mathbb{Z}/\ell^m)_Z \to \mathcal{H}^2_Z(X, \mu_{\ell^m})$ that sends 1 to the image of $s_{Z/X}$.

The following diagram is *anticommutative*:

$$\begin{array}{ccc} \mathrm{H}^0_{\text{ét}}(U,\mathbb{G}_m) & \longrightarrow & \mathrm{H}^1_Z(X,\mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathrm{H}^1_{\text{ét}}(U,\mu_{\ell^m}) & \longrightarrow & \mathrm{H}^2_Z(X,\mu_{\ell^m}) \end{array} \tag{2.11}$$

see [SGA4$\frac{1}{2}$, Cycle, §2.1]. Here the vertical arrows come from the Kummer sequence, and the horizontal arrows come from (2.9). Assume that $Z\subset X$ is given by the equation $f=0$, where $f\in \mathrm{H}^0_{\text{ét}}(U,\mathbb{G}_m)$. The image of f in $\mathrm{H}^1_{\text{ét}}(U,\mu_{\ell^m})$ is the class of the μ_{ℓ^m}-torsor $t^{\ell^m}=f$. The anticommutativity of (2.11) implies that $s_{Z/X}$ is the *negative* of the image of this torsor in $\mathrm{H}^2_Z(X,\mu_{\ell^m})$.

We record a useful corollary of Theorem 2.3.1. By a *strict normal crossing divisor* we understand an effective divisor $D=D_1+\ldots+D_r$ in a regular scheme X such that each divisor D_i is irreducible and regular and all multiple intersections are transversal. Transversality means that at each point $x\in D$ the local equations of the components D_i containing x form a part of a regular system of parameters for the local ring $\mathcal{O}_{X,x}$. The following corollary of Gabber's absolute purity theorem is proved in [Rio14, Cor. 3.1.4, p. 324].

Corollary 2.3.3 (Gabber) *Let X be a regular scheme and $j\colon U\to X$ be an open immersion such that $X\smallsetminus U$ is a strict normal crossing divisor with the irreducible components $D_1,\ldots,D_r$. Let ℓ be a prime different from the residual characteristics of X. For $n\geq 1$ we have canonical isomorphisms of X-sheaves*

$$(R^n j_*)(\mathbb{Z}/\ell^m)=\bigwedge^n (R^1 j_*)(\mathbb{Z}/\ell^m)=\bigwedge^n\Big(\bigoplus_{i=1}^r(\mathbb{Z}/\ell^m)(-1)_{D_i}\Big).$$

2.3.2 The Gysin exact sequence

Let ℓ be a prime invertible on X. Let $i\colon Z\hookrightarrow X$ be a closed subscheme, let $U=X\smallsetminus Z$ and let $j\colon U\to X$ be the natural open immersion. The functor j^* has a left adjoint functor $j_!$ which is exact [Mil80, p. 78]. This implies that we have an exact sequence of étale sheaves on X:

$$0\longrightarrow \mathbb{Z}_U\longrightarrow \mathbb{Z}_X\longrightarrow \mathbb{Z}_Z\longrightarrow 0, \tag{2.12}$$

where $\mathbb{Z}_U=j_!j^*\mathbb{Z}$ and $\mathbb{Z}_Z=i_*i^*\mathbb{Z}$, see [Mil80, p. 92].

Applying the functor $\mathcal{E}xt(\cdot, \mathcal{F})$, defined as the derived functor of the internal $\mathcal{H}om$, to (2.12), gives rise to a long exact sequence which breaks down into isomorphisms

$$(R^{n-1}j_*)(j^*\mathcal{F}) \xrightarrow{\sim} \mathcal{H}^n_Z(X, \mathcal{F}), \quad n \geq 2, \tag{2.13}$$

see [Mil80, p. 242].

Assume that X is a regular scheme and $i\colon Z \hookrightarrow X$ is a closed regular subscheme of codimension $c \geq 1$ everywhere. We have

$$j_*(\mathbb{Z}/\ell^m) \cong \mathbb{Z}/\ell^m.$$

From Theorem 2.3.1 and from (2.13) we obtain canonical isomorphisms

$$(R^{2c-1}j_*)(\mathbb{Z}/\ell^m) \cong (\mathbb{Z}/\ell^m)(-c)_Z \tag{2.14}$$

and

$$(R^n j_*)(\mathbb{Z}/\ell^m) = 0 \text{ for } n \neq 0,\ 2c-1.$$

In view of these isomorphisms the spectral sequence

$$E_2^{pq} = \mathrm{H}^p_{\text{ét}}(X, (R^q j_*)(\mathbb{Z}/\ell^m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(U, \mathbb{Z}/\ell^m) \tag{2.15}$$

gives rise to the *Gysin exact sequence*

$$\begin{aligned} \ldots \longrightarrow \mathrm{H}^{n-2c}_{\text{ét}}(Z, (\mathbb{Z}/\ell^m)(-c)) \longrightarrow \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/\ell^m) \longrightarrow \mathrm{H}^n_{\text{ét}}(U, \mathbb{Z}/\ell^m) \\ \longrightarrow \mathrm{H}^{n-2c+1}_{\text{ét}}(Z, (\mathbb{Z}/\ell^m)(-c)) \longrightarrow \ldots \end{aligned} \tag{2.16}$$

Here we used the canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(X, (\mathbb{Z}/\ell^m)_Z) \cong \mathrm{H}^n_{\text{ét}}(X, i_*(\mathbb{Z}/\ell^m)) \cong \mathrm{H}^n_{\text{ét}}(Z, \mathbb{Z}/\ell^m)$$

coming from the spectral sequence $\mathrm{H}^p_{\text{ét}}(X, R^q i_*(\mathbb{Z}/\ell^m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(Z, \mathbb{Z}/\ell^m)$. Indeed, i_* is an exact functor, because the closed immersion $i\colon Z \to X$ is a finite morphism.

Alternatively, the Gysin sequence can be obtained as follows. Consider the long exact sequence (2.9)

$$\ldots \to \mathrm{H}^n_Z(X, \mathbb{Z}/\ell^m) \to \mathrm{H}^n_{\text{ét}}(X, \mathbb{Z}/\ell^m) \to \mathrm{H}^n_{\text{ét}}(U, \mathbb{Z}/\ell^m) \to \mathrm{H}^{n+1}_Z(X, \mathbb{Z}/\ell^m) \to \ldots.$$

Then the spectral sequence (2.10) in view of Theorem 2.3.1 gives a canonical isomorphism

$$\mathrm{H}^{n-2c}_{\text{ét}}(Z, \mathbb{Z}/\ell^m(-c)) \xrightarrow{\sim} \mathrm{H}^n_Z(X, \mathbb{Z}/\ell^m). \tag{2.17}$$

2.3.3 Cohomology of henselian discrete valuation rings

Let A be a henselian discrete valuation ring with fraction field K and residue field k. If we set

$$X = \mathrm{Spec}(A), \quad Z = \mathrm{Spec}(k), \quad U = \mathrm{Spec}(K),$$

then $i\colon \mathrm{Spec}(k)\to\mathrm{Spec}(A)$ is a closed immersion of regular schemes of codimension $c = 1$, so this is a particular case of the situation considered in the previous section. By Section 2.2.3 the étale cohomology groups of $\mathrm{Spec}(k)$ and $\mathrm{Spec}(K)$ coincide with Galois cohomology groups of k and K, respectively. We now describe how to interpret the étale cohomology of $\mathrm{Spec}(A)$ in terms of Galois cohomology.

As before, let $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$, $I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}})$, $\Gamma = \mathrm{Gal}(K_{\mathrm{nr}}/K) \cong G/I$, where $K_{\mathrm{nr}} \subset K_{\mathrm{s}}$ is the maximal unramified extension of K, so K_{nr} is the field of fractions of the strict henselisation A^{sh}. The category of étale sheaves on $\mathrm{Spec}(A)$ is equivalent to the category of triples (M, N, φ), where M is a Γ-module, N is a G-module, and $\varphi\colon M\to N^I$ is a homomorphism of Γ-modules [Mil80, Example II.3.12]. A morphism of triples

$$(M, N, \varphi) \longrightarrow (M', N', \varphi')$$

is a pair consisting of a map of Γ-modules $M\to M'$ and a map of G-modules $N\to N'$ such that the obvious resulting diagram is commutative. To a sheaf $\mathcal{F}$ on $\mathrm{Spec}(A)$ one associates the triple $(i^*\mathcal{F}, j^*\mathcal{F}, \varphi)$, where φ is the natural morphism $i^*\mathcal{F}\to i^*j_*j^*\mathcal{F}$. This agrees with the definition of triples, because the stalk of the $\mathrm{Spec}(A)$-sheaf j_*N at $\mathrm{Spec}(k_{\mathrm{s}})$ is computed at the strict henselisation, see (2.1), thus the $\mathrm{Spec}(k)$-sheaf i^*j_*N corresponds to the Γ-module N^I. In particular, the $\mathrm{Spec}(A)$-sheaf j_*M, where M is a G-module, corresponds to the triple (M^I, M, id).

Let $\mathcal{F}(M, N, \varphi)$ be the sheaf on $\mathrm{Spec}(A)$ corresponding to the triple (M, N, φ). It can be constructed as the fibred product of i_*M and j_*N over $i_*i^*j_*N$, see [Mil80, Thm. II.3.10]. The constant $\mathrm{Spec}(A)$-sheaf $\mathbb{Z}$ corresponds to the triple $(\mathbb{Z}, \mathbb{Z}, \mathrm{id})$, thus the group of sections of $\mathcal{F}(M, N, \varphi)$ is M^Γ. It follows that

$$\mathrm{H}^i(\mathrm{Spec}(A), \mathcal{F}(M, N, \varphi)) \cong \mathrm{H}^i(k, M). \tag{2.18}$$

2.3.4 Gysin residue and functoriality

We continue the discussion of the previous section and keep the same notation.

Let ℓ be a prime not equal to $\mathrm{char}(k)$. Then μ_{ℓ^m}, where m is a positive integer, is an étale sheaf on $\mathrm{Spec}(A)$. By (2.18), for any $n \geq 1$ we have an isomorphism

$$\mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(A), \mu_{\ell^m}) \cong \mathrm{H}^n(k, \mu_{\ell^m}).$$

Thus, after twisting, the Gysin sequence (2.16) becomes the exact sequence

$$\ldots \to \mathrm{H}^n(k, \mu_{\ell^m}) \to \mathrm{H}^n(K, \mu_{\ell^m}) \xrightarrow{\partial_A} \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m) \to \mathrm{H}^{n+1}(k, \mu_{\ell^m}) \to \ldots \tag{2.19}$$

Definition 2.3.4 (i) *The maps $\partial_A\colon \mathrm{H}^n(K, \mu_{\ell^m}) \to \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m)$ in the exact sequence* (2.19)*, for positive integers m and n, are called the* **Gysin residue** *maps. We write $\partial = \partial_A$ when the context is clear.*

(ii) *Let F be a field with a discrete valuation $v\colon F^* \to \mathbb{Z}$ and associated residue field k. For any prime ℓ invertible in k we define the* **Gysin residue** *$\partial_v\colon \mathrm{H}^n(F, \mu_{\ell^m}) \to \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m)$ by precomposing the Gysin residue for the completion K of F at v with the restriction map $\mathrm{H}^n(F, \mu_{\ell^m}) \to \mathrm{H}^n(K, \mu_{\ell^m})$.*

Theorem 2.3.5 *Let A be a henselian discrete valuation ring with fraction field K and residue field k, and let ℓ be a prime invertible in A. For each n and m, the Gysin residue $\partial\colon \mathrm{H}^n(K, \mu_{\ell^m}) \to \mathrm{H}^{n-1}(k, \mathbb{Z}/\ell^m)$ is the negative of the Serre residue r.*

Proof. We check that the sequences (2.19) and (1.11) come from identical spectral sequences. In our case the spectral sequence (2.15) has the form

$$\mathrm{H}^p_{\text{ét}}(\mathrm{Spec}(A), (R^q j_*)(\mu_{\ell^m})) \Rightarrow \mathrm{H}^{p+q}(K, \mu_{\ell^m}) \tag{2.20}$$

whereas the Hochschild–Serre spectral sequence is

$$\mathrm{H}^p(\mathit{\Gamma}, \mathrm{H}^q(I, \mu_{\ell^m})) \Rightarrow \mathrm{H}^{p+q}(G, \mu_{\ell^m}).$$

On the one hand, the Hochschild–Serre spectral sequence is the spectral sequence of composed functors: the functor $M \mapsto M^I$ from continuous G-modules to continuous $\mathit{\Gamma}$-modules, followed by the functor of $\mathit{\Gamma}$-invariants. On the other hand, the spectral sequence (2.20) is the spectral sequence of composed functors j_* from $\mathrm{Spec}(K)$-sheaves to $\mathrm{Spec}(A)$-sheaves, followed by the functor of sections from $\mathrm{Spec}(A)$-sheaves to abelian groups. As we have seen in the previous section, the dictionary between étale $\mathrm{Spec}(A)$-sheaves and triples interprets the first of these as the functor sending a G-module M to the triple (M^I, M, id). The functor of sections sends this to M^G, which shows that the spectral sequences are indeed identical. We obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^i(\mathit{\Gamma}, \mu_{\ell^m}) & \to & \mathrm{H}^i(G, \mu_{\ell^m}) & \xrightarrow{r} & \mathrm{H}^{i-1}(\mathit{\Gamma}, \mathrm{Hom}(I, \mu_{\ell^m})) & \to & 0 \\
 & & \| & & \| & & \| & & \\
0 & \to & \mathrm{H}^i(k, \mu_{\ell^m}) & \to & \mathrm{H}^i(K, \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^{i-1}(k, (R^1 j_*)(\mu_{\ell^m})) & \to & 0
\end{array}$$

Thus, to compare the Serre residue r with the Gysin residue ∂ it remains to compare the isomorphism $\mathrm{Hom}(I, \mu_{\ell^m}) \cong \mathbb{Z}/\ell^m$ from Section 1.4.1 with the isomorphism $(R^1j_*)(\mu_{\ell^m}) \cong (\mathbb{Z}/\ell^m)_{\mathrm{Spec}(k)}$ in (2.14). This can be done over the strict henselisation A_s of A, which is a henselian discrete valuation ring with residue field k_s and fraction field K_{nr}. We have a diagram of isomorphisms

$$\begin{array}{ccccc} \mathrm{H}^1(K_{\mathrm{nr}}, \mu_{\ell^m}) & \xrightarrow{r} & \mathrm{Hom}(I, \mu_{\ell^m}) & \xrightarrow{\sim} & \mathbb{Z}/\ell^n \\ \| & & \| & & \| \\ \mathrm{H}^1(K_{\mathrm{nr}}, \mu_{\ell^m}) & \longrightarrow & (R^1j_*)(\mu_{\ell^m}) & \xrightarrow{\sim} & \mathbb{Z}/\ell^n \end{array}$$

whose left-hand square commutes. We need to show that the right-hand square anticommutes.

Recall that $(\ell, \mathrm{char}(k)) = 1$, so K_{nr} contains the roots of unity of degree ℓ^m. The isomorphism $\mathrm{Hom}(I, \mu_{\ell^m}) \cong \mathbb{Z}/\ell^m$ from Section 1.4.1 is such that $1 \in \mathbb{Z}/\ell^m$ corresponds to the homomorphism

$$I = \mathrm{Gal}(K_s/K_{\mathrm{nr}}) \longrightarrow \mathrm{Gal}(K_{\mathrm{nr}}(\pi^{1/\ell^m})/K_{\mathrm{nr}}) \cong \mu_{\ell^m}$$

given by the action of I on the K_s-points of the μ_{ℓ^m}-torsor $t^{\ell^m} = \pi$, where π is a uniformiser. As explained in Remark 2.3.2, the Gysin map

$$\mathrm{H}^1(K_{\mathrm{nr}}, \mu_{\ell^m}) \longrightarrow \mathrm{H}^2_{\mathrm{Spec}(k_s)}(\mathrm{Spec}(A_s), \mu_{\ell^m}) \cong \mathrm{H}^0(k_s, \mathbb{Z}/\ell^m) \cong \mathbb{Z}/\ell^m$$

sends the *negative* of the class of the torsor $t^{\ell^m} = \pi$ to the fundamental class of $\mathrm{Spec}(k_s) \subset \mathrm{Spec}(A_s)$, which gives $1 \in \mathbb{Z}/\ell^m$. This proves our claim. See also [Rio14, p. 324]. □

We now make some observations regarding the functoriality of the Gysin sequence.

Let $f\colon X' \to X$ be a morphism of integral regular schemes. Let $Z \subset X$ and $Z' \subset X'$ be regular, integral, closed subschemes of codimension 1 such that $f(Z') \subset Z$. Let $U = X \setminus Z$ and $U' = X' \setminus Z'$. Assume that $f(U') \subset U$, so that there is a commutative diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow f & & \downarrow f & & \downarrow f \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z \end{array}$$

Since $f(X')$ is not contained in Z, we have a well-defined divisor $f^{-1}(Z) \subset X'$ supported on Z'. Thus we can write $f^{-1}(Z) = d\,Z'$, where d is a positive integer. Explicitly, since X is regular, any point of Z has an affine open neighbourhood $V \subset X$ such that $Z \cap V$ is the zero set of a regular function on V. If π is a local equation of $Z \subset X$ in such an open set V, where $V \cap f(X') \neq \emptyset$, then π gives rise to a non-zero rational function on X'; moreover, $v_{Z'}(\pi) = d$, where $v_{Z'}$ is the valuation of the discrete valuation ring $\mathcal{O}_{X',Z'}$.

Lemma 2.3.6 *Let ℓ be a prime invertible on X. There is a commutative diagram*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}^n_{\text{ét}}(X', \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^n_{\text{ét}}(U', \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^{n-1}_{\text{ét}}(Z', \mathbb{Z}/\ell^m) & \longrightarrow & \dots \\ & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle [d]f^*} & & \\ \dots & \longrightarrow & \mathrm{H}^n_{\text{ét}}(X, \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^n_{\text{ét}}(U, \mu_{\ell^m}) & \longrightarrow & \mathrm{H}^{n-1}_{\text{ét}}(Z, \mathbb{Z}/\ell^m) & \longrightarrow & \dots \end{array}$$

Proof. By the construction of the Gysin sequence, the bottom row comes from the spectral sequence of composed functors $\mathbf{R}j_*\colon \mathcal{D}^+(U_{\text{ét}}) \to \mathcal{D}^+(X_{\text{ét}})$ and the sections functor $\mathbf{R}\Gamma\colon \mathcal{D}^+(X_{\text{ét}}) \to \mathcal{D}^+(\mathrm{Ab})$, where Ab is the category of abelian groups (and similarly for the top row). From the functoriality of the spectral sequence and the purity theorem we obtain the commutative diagram as above, where we only need to identify the map from $\mathrm{H}^{n-1}_{\text{ét}}(Z, \mathbb{Z}/\ell^m)$ to $\mathrm{H}^{n-1}_{\text{ét}}(Z', \mathbb{Z}/\ell^m)$.

The canonical isomorphism

$$\mathrm{H}^{n-1}_{\text{ét}}(Z, \mathbb{Z}/\ell^m) \xrightarrow{\sim} \mathrm{H}^{n+1}_Z(X, \mu_{\ell^m}),$$

see (2.17), is obtained by applying $\mathrm{H}^{n-1}_{\text{ét}}(Z, -)$ to the isomorphism

$$(\mathbb{Z}/\ell^m)_Z \xrightarrow{\sim} \mathcal{H}^2_Z(X, \mu_{\ell^m})$$

that sends 1 to the image of the fundamental class $s_{Z/X} \in \mathrm{H}^2_Z(X, \mu_{\ell^m})$. By Remark 2.3.2 we have a commutative diagram, where the arrows pointing right come from the Kummer sequence:

$$\begin{array}{ccccc} \mathrm{H}^1_{\text{ét}}(X', \mathbb{G}_m) & \longleftarrow & \mathrm{H}^1_{Z'}(X', \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_{Z'}(X', \mu_{\ell^m}) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) & \longleftarrow & \mathrm{H}^1_Z(X, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^2_Z(X, \mu_{\ell^m}) \end{array}$$

We have an isomorphism $\mathrm{H}^1_Z(X, \mathbb{G}_m) \cong \mathbb{Z}$ identifying $1 \in \mathbb{Z}$ with the element $\mathrm{cl}_Z \in \mathrm{H}^1_Z(X, \mathbb{G}_m)$ which is locally given by the image of a local equation of Z in X under the natural map $\mathrm{H}^0_{\text{ét}}(U, \mathbb{G}_m) \to \mathrm{H}^1_Z(X, \mathbb{G}_m)$, see Remark 2.3.2. By definition, $s_{Z/X}$ is the image of cl_Z. Since $f^*(\mathrm{cl}_Z) = d\,\mathrm{cl}_{Z'}$, our claim follows from the commutativity of the last diagram. □

2.4 H^1 with coefficients $\mathbb{Z}$ and $\mathbb{G}_m$

Let X be a scheme. Recall that $\mathbb{Z}_X$ is the étale sheaf on X associated to the constant presheaf $\mathbb{Z}$.

Lemma 2.4.1 *Let X be a scheme. Let L be a field and let $f\colon \mathrm{Spec}(L)\to X$ be a morphism. We have the following properties:*

(i) $\mathrm{H}^1_{\text{ét}}(X, f_*\mathbb{Z}_L) = 0$*;*
(ii) $\mathrm{H}^1_{\text{ét}}(X, f_*\mathbb{G}_{m,L}) = 0$*;*
(iii) $R^1 f_*\mathbb{Z}_L = 0$*;*
(iv) $R^1 f_*\mathbb{G}_{m,L} = 0$.

If $\mathcal{F}$ is a sheaf on $\mathrm{Spec}(L)_{\text{ét}}$, then, for any $i \geq 1$,

(v) *the sheaf $R^i f_*\mathcal{F}$ is a torsion sheaf;*
(vi) *if, in addition, X is quasi-compact and quasi-separated, then the group $\mathrm{H}^i_{\text{ét}}(X, f_*\mathcal{F})$ is a torsion group.*

Proof. The spectral sequence (2.4) gives an injective map

$$\mathrm{H}^1_{\text{ét}}(X, f_*(\mathcal{F})) \hookrightarrow \mathrm{H}^1_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F}).$$

Statements (i) and (ii) then follow since $\mathrm{H}^1(L, \mathbb{Z}_L) = 0$ and $\mathrm{H}^1(L, \mathbb{G}_{m,L}) = 0$ (Hilbert's theorem 90).

The sheaf $R^1 f_*\mathbb{Z}_L$ is associated to the presheaf sending an étale $U\to X$ to $\mathrm{H}^1_{\text{ét}}(U \times_X \mathrm{Spec}(L), \mathbb{Z}_L)$. But this group is zero, because $U \times_X \mathrm{Spec}(L)$ is either empty or the disjoint union of spectra of fields, and $\mathrm{H}^1_{\text{ét}}(E, \mathbb{Z}_E) = 0$ when E is a field. This proves (iii). A similar argument, which uses Hilbert's theorem 90, proves (iv).

Let us prove that the stalk of $R^i f_*\mathcal{F}$ at each geometric point $s\colon \bar{x} \mapsto x \in X$ is a torsion abelian group. By passing to an affine open neighbourhood of x we can assume that X is affine. The sheaf $R^i f_*\mathcal{F}$ is associated to the presheaf $\mathcal{P}$ which sends an étale morphism $U\to X$ to $\mathrm{H}^i_{\text{ét}}(U \times_X \mathrm{Spec}(L), \mathcal{F})$. Recall that the sheafification $a\mathcal{P} = R^i f_*\mathcal{F}$ has the same stalks as $\mathcal{P}$, so it is enough to prove that $\mathcal{P}_{\bar{x}}$ is a torsion abelian group. By definition, the stalk $\mathcal{P}_{\bar{x}}$ is the direct limit of $\mathrm{H}^i_{\text{ét}}(U \times_X \mathrm{Spec}(L), \mathcal{F})$, where $\pi\colon U\to X$ is étale and $s = \pi s_U$ for some morphism $s_U\colon \bar{x}\to U$. We can replace U by an affine open $U' \subset U$ such that s_U factors through U'. The morphism $U'\to X$ is quasi-compact [EGA, I, Ch. 1, 6.6.1, p. 152], hence so is $U' \times_X \mathrm{Spec}(L)\to\mathrm{Spec}(L)$, thus $U' \times_X \mathrm{Spec}(L)$ is a finite disjoint union of spectra of fields. Thus $\mathrm{H}^i_{\text{ét}}(U' \times_X \mathrm{Spec}(L), \mathcal{F})$ is a *finite* direct product of Galois cohomology groups. For $i \geq 1$ each of these groups is a torsion group, hence so is their product. This implies our claim. To show that $R^i f_*\mathcal{F}$ is a torsion sheaf we need to show that $(R^i f_*\mathcal{F})(V)$ is a torsion abelian group for any $V\to X$, where V is quasi-compact. There is a natural injective map $(R^i f_*\mathcal{F})(V)\to \prod (R^i f_*\mathcal{F})_{\bar{x}}$, where the product is taken over the geometric points of V. Let $\sigma \in (R^i f_*\mathcal{F})(V)$. The image of σ

in $(R^i f_* \mathcal{F})_{\bar{x}}$ extends to some étale open set, so we obtain an étale covering $\{V_i \to V\}$, $i \in I$, such that the restriction of σ to each V_i has finite order. Since V is quasi-compact, we can assume that I is finite. It follows that σ has finite order. This proves (v).

In our case the spectral sequence (2.4) takes the form

$$E_2^{pq} = \mathrm{H}^p_{\text{ét}}(X, R^q f_* \mathcal{F}) \Rightarrow \mathrm{H}^n_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F}).$$

The combination of (v) and a standard limit argument [Stacks, Lemma 0DDC] for quasi-compact and quasi-separated schemes shows that the terms E_2^{pq} are torsion groups for $p \geq 0$ and $q \geq 1$. It follows that the kernel of the natural map

$$\mathrm{H}^i_{\text{ét}}(X, f_* \mathcal{F}) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(L), \mathcal{F})$$

is a torsion group. But $\mathrm{H}^i(\mathrm{Spec}(L), \mathcal{F})$ is also a torsion group for $i \geq 1$, so statement (vi) follows. □

Proposition 2.4.2 *Let X be a noetherian normal scheme. Then we have $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_X) = 0$.*

Proof. A locally noetherian, normal, connected scheme is integral [Stacks, Lemma 033N]. We can thus assume that X is integral. Let $i \colon \eta \to X$ be the generic point of X. We claim that the natural map $\mathbb{Z}_X \to i_* \mathbb{Z}_\eta$ is an isomorphism. Indeed, let $U \to X$ be an étale morphism such that U is connected. Then U is normal [EGA, IV$_4$, Prop. 18.10.7] and connected, hence U is integral and dominates X. Thus the generic fibre of $U \to X$ is connected. This shows that the map $\mathbb{Z}_X \to i_* \mathbb{Z}_\eta$ is an isomorphism. Then Lemma 2.4.1 (i) gives $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}_X) = 0$. □

2.5 The Picard group and the Picard scheme

Definition 2.5.1 *The* **Picard group** $\mathrm{Pic}(X)$ *of a scheme X is the group of invertible coherent sheaves of $\mathcal{O}_X$-modules, considered up to isomorphism.*

By this definition we have

$$\mathrm{Pic}(X) = \mathrm{H}^1_{\text{zar}}(X, \mathcal{O}_X^*) = \mathrm{H}^1_{\text{zar}}(X, \mathbb{G}_{m,X}).$$

Let $\pi \colon X_{\text{ét}} \to X_{\text{zar}}$ be the continuous morphism induced by the identity on X. We have $(R^1 \pi_*)(\mathbb{G}_m) = 0$; this is Grothendieck's version of Hilbert's theorem 90, see [Mil80, Prop. III.4.9]. The Leray spectral sequence then entails a canonical isomorphism

$$\mathrm{Pic}(X) = \mathrm{H}^1_{\text{zar}}(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_{m,X}). \tag{2.21}$$

The same is true for $\mathrm{H}^1_{\mathrm{fppf}}(X, \mathbb{G}_{m,X})$. Alternatively, to an invertible sheaf $\mathcal{L}$ one directly associates a torsor T for $\mathbb{G}_{m,X}$ defined by

$$T(U) = \mathrm{Isom}_U(\mathcal{O}_U, f^*\mathcal{L}),$$

where $f\colon U \to X$ is étale. This gives an equivalence of the category of invertible sheaves of $\mathcal{O}_X$-modules and the category of étale X-torsors for $\mathbb{G}_{m,X}$, see [SGA4$\frac{1}{2}$, Arcata, Prop. II.2.3].

The rest of this section is based on Kleiman's excellent survey [Kle05], see also [BLR90, Ch. 8]. Fix a noetherian base scheme S and let $f\colon X \to S$ be a separated morphism of finite type. For an S-scheme T we write $X_T = X \times_S T$ and write $f_T\colon X_T \to T$ for the projection to T.

The *relative Picard functor* $\mathrm{Pic}_{X/S}$ from the category Sch/S to abelian groups is defined as follows:

$$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X_T)/f_T^*\mathrm{Pic}(T).$$

Let $\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$, $\mathrm{Pic}_{(X/S)\,\text{Ét}}$, $\mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$ be the associated sheaves on the big sites S_{Zar}, $S_{\text{Ét}}$, S_{fppf}.

Proposition 2.5.2 *Assume that for any S-scheme T the canonical adjunction morphism $\mathcal{O}_T \to f_{T*}f_T^*\mathcal{O}_S \cong f_{T*}\mathcal{O}_{X_T}$ is an isomorphism. Then the following natural maps of presheaves on the category of schemes locally of finite type over S are injective:*

$$\mathrm{Pic}_{X/S} \hookrightarrow \mathrm{Pic}_{(X/S)\,\mathrm{Zar}} \hookrightarrow \mathrm{Pic}_{(X/S)\,\text{Ét}} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}, \tag{2.22}$$

and the last map is an isomorphism. The first two maps in (2.22) are isomorphisms if f has a section. The second map is an isomorphism if f has a section locally in the Zariski topology.

Proof. This is [Kle05, Thm. 2.5]. We sketch the proof given in [Kle05, Remark 2.11] because it is a good illustration of the use of the spectral sequence (2.4).

Take an S-scheme T. The Zariski sheaf on T, which is associated to the presheaf sending Z/T to $\mathrm{H}^1_{\mathrm{Zar}}(X_Z, \mathbb{G}_{m,X_Z})$, is $R^1f_{T*}\mathbb{G}_{m,X_T}$. (Here f_{T*} is the direct image with respect to the Zariski topology.) Hence

$$\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) = \mathrm{H}^0_{\mathrm{Zar}}(T, R^1f_{T*}\mathbb{G}_{m,X_T}).$$

The morphism $f_T\colon X_T \to T$ gives rise to the spectral sequence (2.4):

$$\mathrm{H}^p_{\mathrm{Zar}}(T, R^qf_{T*}\mathbb{G}_{m,X_T}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}).$$

The assumption $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$ implies that $\mathbb{G}_{m,T} \xrightarrow{\sim} f_*\mathbb{G}_{m,X_T}$. Hence the exact sequence of low degree terms of the spectral sequence is

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X_T) \to \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) \to \mathrm{H}^2_{\mathrm{Zar}}(T, \mathbb{G}_{m,T}) \to \mathrm{H}^2_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}),$$

proving the injectivity of $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$. A section of f induces a retraction of each canonical map

$$\mathrm{H}^n_{\mathrm{Zar}}(T, \mathbb{G}_{m,T}) \longrightarrow \mathrm{H}^n_{\mathrm{Zar}}(X_T, \mathbb{G}_{m,X_T}),$$

which is therefore injective. This implies that the first map in (2.22) is an isomorphism.

Using (2.21), the same arguments apply to the étale and fppf topologies. Hence we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\mathrm{Zar}}(T) & \to & \mathrm{H}^2_{\mathrm{Zar}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\mathrm{Zar}}(X_T, \mathbb{G}_m) \\
\| & & \| & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\acute{\mathrm{E}}\mathrm{t}}(T) & \to & \mathrm{H}^2_{\acute{\mathrm{E}}\mathrm{t}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\acute{\mathrm{E}}\mathrm{t}}(X_T, \mathbb{G}_m) \\
\| & & \| & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}(T) & \to & \mathrm{Pic}(X_T) & \to & \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}(T) & \to & \mathrm{H}^2_{\mathrm{fppf}}(T, \mathbb{G}_m) & \to & \mathrm{H}^2_{\mathrm{fppf}}(X_T, \mathbb{G}_m)
\end{array}$$

The injectivity of $\mathrm{Pic}_{X/S} \to \mathrm{Pic}_{(X/S)\,\acute{\mathrm{E}}\mathrm{t}}$ formally implies the injectivity of

$$\mathrm{Pic}_{(X/S)\,\mathrm{Zar}} \longrightarrow \mathrm{Pic}_{(X/S)\,\acute{\mathrm{E}}\mathrm{t}},$$

since the latter map is obtained from the former by passing from presheaves to associated Zariski sheaves, and this operation preserves injectivity by the exactness of the functor a from presheaves to sheaves [Mil80, Thm. II.2.15 (a)].

In view of (2.8), the Five Lemma applied to the two lower rows of the diagram gives an isomorphism $\mathrm{Pic}_{(X/S)\,\acute{\mathrm{E}}\mathrm{t}} \xrightarrow{\sim} \mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$. □

Remark 2.5.3 If $f \colon X \to S$ is flat and proper, and the geometric fibres of f are reduced and connected, then for any morphism $T \to S$ the map $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism. (See [Kle05, Exercise 9.3.11].) This applies for instance when $S = \mathrm{Spec}(k)$ is the spectrum of a field and X is a proper, geometrically integral variety over k.

The following proposition shows that the condition that $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism holds for any *flat* S-scheme T if it holds for $T = S$.

Proposition 2.5.4 *Let $f \colon X \to S$ be a separated morphism of noetherian schemes such that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism. Then for any flat scheme $T \to S$ the map $\mathcal{O}_T \to f_{T*}\mathcal{O}_{X_T}$ is an isomorphism.*

Proof. The statement is local on S and T. We may thus assume $S = \mathrm{Spec}(A)$ and $T = \mathrm{Spec}(B)$ with B flat over A. Since X is separated, we can write X as a finite union $X = \cup_i X_i$ of affine open sets $X_i = \mathrm{Spec}(A_i)$ with affine intersections $X_{ij} = \mathrm{Spec}(A_{ij})$. We have the obvious exact sequence of A-modules

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X) \longrightarrow \prod_i A_i \longrightarrow \prod_{ij} A_{ij}.$$

The hypothesis that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an isomorphism then gives the exactness of the sequence of A-modules

$$0 \longrightarrow A \longrightarrow \prod_i A_i \longrightarrow \prod_{ij} A_{ij}.$$

Since B is flat over A, we have an exact sequence of B-modules

$$0 \longrightarrow B \longrightarrow \prod_i A_i \otimes_A B \longrightarrow \prod_{ij} A_{ij} \otimes_A B.$$

The scheme $X_T = X \times_S T$ is covered by open subsets $X_i \times_S T = \mathrm{Spec}(A_i \otimes_A B)$ with intersections $X_{ij} \times_S T = \mathrm{Spec}(A_{ij} \otimes_A B)$, hence we have an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X_T, \mathcal{O}_{X_T}) \longrightarrow \prod_i A_i \otimes_A B \longrightarrow \prod_{ij} A_{ij} \otimes_A B.$$

Comparing the last two exact sequences, we find that

$$\mathrm{H}^0(T, \mathcal{O}_T) \cong B \cong \mathrm{H}^0(X_T, \mathcal{O}_{X_T}).$$

Thus $\mathcal{O}_T(T) \to f_{T*}\mathcal{O}_{X_T}(T)$ is an isomorphism. The same argument works for any Zariski open subset of T. This gives an isomorphism $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$. $\Box$

Remark 2.5.5 This result is a particular case of the following general statement. Let $f\colon X \to S$ be a quasi-compact and quasi-separated morphism and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then the formation of the direct image sheaves $R^i f_*\mathcal{F}$, where $i \geq 0$, commutes with flat base change over S. See [EGA, III, Prop. 1.4.15] and [Stacks, Lemma 02KH].

If any of the functors $\mathrm{Pic}_{X/S}$, $\mathrm{Pic}_{(X/S)\,\mathrm{Zar}}$, $\mathrm{Pic}_{(X/S)\,\text{Ét}}$, $\mathrm{Pic}_{(X/S)\,\mathrm{fppf}}$ is representable, then the representing scheme (which is uniquely determined) is called the *Picard scheme* and is denoted by $\mathbf{Pic}_{X/S}$.

The main existence theorem for $\mathbf{Pic}_{X/S}$ is the following result of Grothendieck, see [Kle05, Thm. 4.8] for a slightly stronger statement.

Theorem 2.5.6 *Assume that $f\colon X \to S$ is projective and flat with integral geometric fibres. Then the scheme $\mathbf{Pic}_{X/S}$ representing $\mathrm{Pic}_{(X/S)\,\text{Ét}}$ exists, is separated and is locally of finite type over S.*

Another important result of Grothendieck is the following theorem [Kle05, Thm. 4.18.2, Cor. 4.18.3].

Theorem 2.5.7 *Assume that S is integral and $X \to S$ is proper. Then there is a non-empty open subset $V \subset S$ such that $\mathbf{Pic}_{X_V/V}$ exists, represents $\mathrm{Pic}_{(X_V/V)\,\mathrm{fppf}}$, and is a disjoint union of quasi-projective schemes. In particular, this holds for $S = \mathrm{Spec}(k)$, where k is a field.*

Corollary 2.5.8 *Let X be a proper variety over a field k. Assume that it is geometrically reduced and geometrically connected. Then for any k-scheme T there is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Pic}_{X/k}(T) \longrightarrow \mathbf{Pic}_{X/k}(T) \longrightarrow \mathrm{H}^2_{\text{ét}}(T, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(X_T, \mathbb{G}_m). \quad (2.23)$$

If $X(k) \neq \emptyset$, then $\mathbf{Pic}_{X/k}(T) = \mathrm{Pic}_{X/k}(T)$ for any k-scheme T, so that the Picard group scheme $\mathbf{Pic}_{X/k}$ represents the relative Picard functor $\mathrm{Pic}_{X/k}$.

Proof. By the representability of $\mathrm{Pic}_{(X/k)\,\text{Ét}}$ we obtain (2.23) from the middle row of the commutative diagram in the proof of Proposition 2.5.2, using canonical isomorphisms

$$\mathrm{H}^2_{\text{ét}}(T, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^2_{\text{Ét}}(T, \mathbb{G}_m), \qquad \mathrm{H}^2_{\text{ét}}(X_T, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^2_{\text{Ét}}(X_T, \mathbb{G}_m),$$

see (2.6). If $X(k) \neq \emptyset$, then the morphism $X_T \to T$ has a section, so that the map $\mathrm{H}^2_{\text{ét}}(T, \mathbb{G}_m) \longrightarrow \mathrm{H}^2_{\text{ét}}(X_T, \mathbb{G}_m)$ is injective. □

Corollary 2.5.9 *Let X be a proper variety over a field k. Assume that it is geometrically reduced and geometrically connected. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbf{Pic}_{X/k}(k) \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m). \quad (2.24)$$

If K is a finite Galois field extension of k with Galois group $G = \mathrm{Gal}(K/k)$ such that $X(K) \neq \emptyset$, then we have a canonical isomorphism

$$\mathbf{Pic}_{X/k}(k) \cong \mathrm{Pic}(X_K)^G.$$

Proof. The exact sequence (2.24) is obtained from (2.23) by taking $T = \mathrm{Spec}(k)$. Taking $T = \mathrm{Spec}(K)$ in (2.23), we obtain a compatible exact sequence

$$0 \longrightarrow \mathrm{Pic}(X_K) \longrightarrow \mathbf{Pic}_{X/k}(K) \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{H}^2_{\text{ét}}(X_K, \mathbb{G}_m).$$

This is also a sequence of G-modules. Since $X(K) \neq \emptyset$, Corollary 2.5.8 gives an isomorphism $\mathrm{Pic}(X_K) \xrightarrow{\sim} \mathbf{Pic}_{X/k}(K)$. For the group k-scheme $\mathbf{Pic}_{X/k}$, we have $\mathbf{Pic}_{X/k}(k) = \mathbf{Pic}_{X/k}(K)^G$. □

2.6 Excellent rings

The aim of this section is to give the definitions of excellent rings and excellent schemes which feature mainly in Chapters 3 and 10. We refer the reader to [EGA, IV_2, §7.8], [Mat86, §32] and [RL14] for a detailed treatment.

Following Bourbaki [BouV, §15, no. 2, Déf. 1], a commutative k-algebra A is called *separable* if the ring $A \otimes_k L$ is reduced (i.e., has no non-zero nilpotents) for any field extension L/k. By [BouV, §15, no. 2, Prop. 3] A is a separable k-algebra if and only if $A \otimes_k \bar{k}$ is a separable $\bar{k}$-algebra, which is equivalent to $A \otimes_k \bar{k}$ being reduced [BouV, §15, no. 5, Thm. 3 (c)].

Let $k \subset K$ be fields. The k-algebra K is separable if and only if for any subextension $k \subset K' \subset K$ with K' finitely generated over k, the field K' is a finite separable extension of a purely transcendental extension of k, see [Stacks, Def. 030O], [Stacks, Lemma 030W].

A noetherian ring A is *regular* if the localisation A_P at every prime ideal $P \subset A$ is a regular local ring. A noetherian k-algebra A is *geometrically regular* if $A \otimes_k k'$ is regular for every finite field extension $k \subset k'$.

A commutative ring R is called *catenary* if for any prime ideals $P \subset P'$ of R there exists a finite chain of prime ideals

$$P = I_0 \subsetneq I_1 \subsetneq I_2 \subset \ldots \subsetneq I_n = P'$$

with no prime ideal of R strictly contained between I_m and I_{m+1} for $m = 0, \ldots, n-1$, and all such chains have the same length. A noetherian local domain R is catenary if and only if for every prime ideal $P \subset R$ we have $\dim(R/I) + \dim(R_P) = \dim(R)$, see [Mat86, Thm. 31.4].

A commutative ring R is called *universally catenary* if R is noetherian and every finitely generated R-algebra is catenary. (See [Mat86, §15]. For a noetherian R the last condition is equivalent to the condition that $R[x_1, \ldots, x_n]$ is catenary for any $n \geq 1$.)

A noetherian ring R is called a G-*ring* (G for Grothendieck) if all *formal fibres* of R are regular. This means that for every prime ideal $P \subset R$ the fibres of the morphism $\mathrm{Spec}(\widehat{R_P}) \to \mathrm{Spec}(R_P)$ are geometrically regular. Here $\widehat{R_P}$ is the completion of the localisation R_P of R at P.

Definition 2.6.1 *A noetherian ring A is* **excellent** *if A is a universally catenary G-ring such that for every finitely generated A-algebra B the set of regular points of* $\mathrm{Spec}(B)$ *is an open subset of* $\mathrm{Spec}(B)$.

The class of excellent rings is closed under localisations, finitely generated extensions and taking quotients. Any field is excellent. Any complete noetherian local ring is excellent. The excellence property of a noetherian local ring is preserved by henselisation [EGA, IV_4, Cor. 18.7.6].

Any Dedekind ring with fraction field of characteristic zero is excellent.

If R is an excellent integral domain with field of fractions K, then the integral closure of R in any finite field extension of K is a finitely generated R-module [EGA, IV_2, (7.8.3) (vi)].

Definition 2.6.2 *A locally noetherian scheme is* **excellent** *if it has an open covering by affine open sets* $\mathrm{Spec}(A_i)$*, where each A_i is an excellent ring.*

If X is an excellent scheme, then the set of regular points of X is an open subset of X. If $X' \to X$ is a morphism locally of finite type and X is excellent, then X' is also excellent. See [EGA, IV_2, Prop. 7.8.6].

Chapter 3
Brauer groups of schemes

There are two ways to generalise the Brauer group of fields to schemes. The definition of the Brauer group of a field k in terms of central simple algebras over k readily extends to schemes as the group of equivalence classes of Azumaya algebras. We call it the Brauer–Azumaya group, see Section 3.1. The Brauer–Azumaya group $\mathrm{Br}_{\mathrm{Az}}(X)$ is a torsion group if X is quasi-compact (for example, noetherian) or if X has finitely many connected components. The cohomological description $\mathrm{Br}(k) \cong \mathrm{H}^2(k, k_{\mathrm{s}}^*)$ also extends and gives rise to the Brauer–Grothendieck group $\mathrm{Br}(X) := \mathrm{H}^2_{\mathrm{\acute{e}t}}(X, \mathbb{G}_{m,X})$, see Section 3.2. In Section 3.3 we discuss a natural injective map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$. There exist integral, normal, noetherian schemes such that $\mathrm{Br}(X)$ is not a torsion group; in fact, this is already the case for some normal complex surfaces, see Chapter 8. The focus of Section 3.4 is the Brauer group of henselian local rings. In Section 3.5 we prove a theorem of Grothendieck that the Brauer–Grothendieck group $\mathrm{Br}(X)$ of a regular, integral, noetherian scheme X is naturally a subgroup of the Brauer group of its field of functions. In particular, $\mathrm{Br}(X)$ is then a torsion group.

The purity theorem for the Brauer group of a regular, integral, noetherian scheme X is discussed in Section 3.6 in the special case of schemes of dimension 1, and in Section 3.7 in the general case. For torsion of order invertible on X the purity theorem can be stated and proved in terms of residues at the generic points of the irreducible divisors on X. We state the absolute purity theorem for the Brauer group of a regular scheme, whose proof was recently completed by Česnavičius after work of Gabber. This leads to a description of the Brauer group of a regular integral scheme in terms of discrete valuations of its function field.

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3.1 The Brauer–Azumaya group

The following theorem is due to Azumaya [Az51] (over a local ring), Auslander and Goldman [AG60] (over an arbitrary commutative ring), and Grothendieck (over a scheme), see [Gro68, I, Thm. 5.1] and [Mil80, Ch. IV, §2].

Theorem 3.1.1 *Let X be a scheme and let A be an $\mathcal{O}_X$-algebra which is a finite locally free $\mathcal{O}_X$-module. The following conditions are equivalent:*

(i) *For each $x \in X$ the fibre $A \otimes k(x)$ is a central simple algebra over the residue field $k(x)$.*
(ii) *The natural map $A \otimes_{\mathcal{O}_X} A^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(A)$ is an isomorphism.*
(iii) *For each $x \in X$ there exist an integer $n \geq 1$, a Zariski open set $U \subset X$ with $x \in U$ and a surjective étale morphism $U' \to U$ such that we have $A_{U'} \cong M_n(\mathcal{O}_{U'})$.*
(iv) *For each $x \in X$ there exist an integer $n \geq 1$, a Zariski open set $U \subset X$ with $x \in U$ and a surjective finite étale morphism $U' \to U$ such that we have $A_{U'} \cong M_n(\mathcal{O}_{U'})$.*

An $\mathcal{O}_X$-algebra A satisfying these equivalent conditions is called an *Azumaya algebra*. The integer n is called the *degree* of A at the point x.

Proposition 3.1.2 *Let A be an Azumaya algebra of degree n over a scheme X. Then $A^{\otimes n} \simeq \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(F)$ for some locally free $\mathcal{O}_X$-module F of rank n^n.*

Proof. This is sketched by Grothendieck in [Gro68, I, end of proof of Prop. 1.4]. In the affine case, a more explicit proof was given by D. Saltman [Sal81]. That proof is extended to arbitrary schemes in [Stacks, Lemma 0A2L]. □

Azumaya algebras A and B over X are called *equivalent* if there exist finite locally free $\mathcal{O}_X$-modules P and Q and an isomorphism of $\mathcal{O}_X$-algebras

$$A \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(P) \simeq B \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(Q).$$

Tensor product makes the set of equivalence classes of Azumaya $\mathcal{O}_X$-algebras a commutative monoid with the class of $\mathcal{O}_X$ as the identity element. By Theorem 3.1.1 (ii), it is an abelian group.

Definition 3.1.3 *The set of equivalence classes of Azumaya $\mathcal{O}_X$-algebras is called the* **Brauer–Azumaya group** $\mathrm{Br}_{\mathrm{Az}}(X)$.

Under very mild conditions, the Brauer–Azumaya group is a torsion group.

Corollary 3.1.4 *Let X be a scheme. The order of the class in* $\mathrm{Br}_{\mathrm{Az}}(X)$ *represented by an Azumaya algebra of degree n divides n. If X is quasi-compact or has only finitely many connected components, then* $\mathrm{Br}_{\mathrm{Az}}(X)$ *is a torsion group.*

Proof. Let A be an Azumaya algebra over X. Write $[A]$ for the class of A in $\mathrm{Br}_{\mathrm{Az}}(X)$. If the degree of A is n, then by Proposition 3.1.2 we have $[A^{\otimes n}] = n[A] = 0$.

The scheme X is the union of the Zariski open sets $X_n \subset X$ such that A has degree n on X_n, for $n \geq 1$. If X is quasi-compact or has only finitely many connected components, then only finitely many of the open sets X_n are non-empty. Thus $[A] \in \mathrm{Br}_{\mathrm{Az}}(X)$ has finite order for any Azumaya algebra A over X. See also [Gro68, I.1.4, I.2], [Mil80, Prop. IV.2.7], [KO74b, IV.6.1]. □

A generalisation of the Skolem–Noether theorem leads to a bijection of pointed sets between the set of isomorphism classes of Azumaya algebras of degree n over X and the étale Čech cohomology set $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathrm{PGL}_{n,X})$, see [Mil80, Proof of Thm. IV.2.5, Step 1, p. 122]. This pointed set classifies PGL_n-torsors over $X_{\text{ét}}$, see Section 2.2.1.

The equivalence of (iii) and (iv) in Theorem 3.1.1 is due to the following remarkable fact: if R is a local ring, then any $\mathrm{PGL}_{n,R}$-torsor is split by a finite étale extension of R. More generally, we have the following theorem.

Theorem 3.1.5 *Let R be a semilocal ring and let G be a semisimple group scheme over R. Then any G-torsor over R is split by a finite étale extension of R. The same holds if G is a reductive group scheme over a semilocal ring R which is also assumed to be noetherian and normal.*

Proof. See [SGA3, XXIV, Thm. 4.1.5, Cor. 4.1.6]. □

3.2 The Brauer–Grothendieck group

Grothendieck's definition of the (cohomological) Brauer group formally resembles his formula for the Picard group (2.21).

Definition 3.2.1 *The* **Brauer–Grothendieck group** *of a scheme X is*

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}).$$

By (2.6) and (2.8) there are canonical isomorphisms

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{H}^2_{\text{Ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{H}^2_{\text{fppf}}(X, \mathbb{G}_{m,X}).$$

Remark 3.2.2 Our notation may be different from the notation elsewhere in the literature. Sometimes $\mathrm{Br}(X)$ is used to denote the Brauer–Azumaya group (which we denote by $\mathrm{Br}_{\mathrm{Az}}(X)$) or the torsion subgroup of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_{m,X})$.

For an affine scheme $X = \mathrm{Spec}(R)$, where R is a commutative ring, one often writes $\mathrm{Br}(R) := \mathrm{Br}(X)$. In the particular case $X = \mathrm{Spec}(k)$, where k is a field, we obtain the classical description of the Brauer group of a field in terms of equivalence classes of continuous 2-cocycles of its absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$, where k_{s} is a separable closure of k:

$$\mathrm{Br}(k) = \mathrm{H}^2(k, k_{\mathrm{s}}^*) = \mathrm{H}^2(\Gamma, k_{\mathrm{s}}^*).$$

One may also consider the Zariski cohomological Brauer group of a scheme X. Let us denote it by $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m)$. Write $\pi\colon X_{\text{ét}} \to X_{\mathrm{zar}}$ for the morphism of sites. Then we have $\mathbb{G}_{m,\mathrm{zar}} = \pi_*\mathbb{G}_m$ and $R^1\pi_*(\mathbb{G}_m) = 0$. From the spectral sequence (2.4) we get an injective map

$$\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m).$$

Note, however, that it need not be an isomorphism. Indeed, if X is noetherian, integral and locally factorial, then $\mathrm{H}^2_{\mathrm{zar}}(X, \mathbb{G}_m) = 0$, see Remark 3.5.2.

A morphism of schemes $f\colon X \to Y$ gives rise to a morphism (2.7). In the case of $G = \mathbb{G}_m$ we obtain

$$f^*\colon \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y}) \longrightarrow \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}). \tag{3.1}$$

For $n = 2$ this gives a natural map of Brauer groups $f^*\colon \mathrm{Br}(Y) \to \mathrm{Br}(X)$, which is sometimes referred to as the *restriction* map. If K is a field and $M\colon \mathrm{Spec}(K) \to X$ is a K-point of X, then one writes $A(M) = M^*(A) \in \mathrm{Br}(K)$ and refers to $A(M)$ as the *value*, or specialisation, of A at M.

3.2.1 The Kummer exact sequence

The Brauer group is linked to étale cohomology with finite coefficients by the Kummer exact sequence

$$1 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^{\ell^n}} \mathbb{G}_{m,X} \longrightarrow 1.$$

Here ℓ is a prime invertible on X and n is a positive integer. The associated long exact sequence of cohomology gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow 0. \tag{3.2}$$

In degree 1 the Kummer sequence gives an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathbb{G}_m)/\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n} \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{Pic}(X)[\ell^n] \longrightarrow 0,$$

where $\mathrm{H}^0(X, \mathbb{G}_m)^{\ell^n}$ stands for the group of ℓ^n-powers of invertible regular functions on X. In degree 3 we have another useful exact sequence

$$0 \longrightarrow \mathrm{Br}(X)/\ell^n \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)[\ell^n] \longrightarrow 0. \tag{3.3}$$

One can drop the restriction that ℓ is invertible on X by using the site X_{fppf} instead of $X_{\text{ét}}$. For any integer $n \geq 1$, the sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_{m,X} \xrightarrow{x \mapsto x^n} \mathbb{G}_{m,X} \longrightarrow 1$$

is an exact sequence on X_{fppf}. In view of the isomorphisms (2.8)

$$\mathrm{H}^i_{\text{ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{fppf}}(X, \mathbb{G}_{m,X}),$$

it gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X)/n \longrightarrow \mathrm{H}^2_{\mathrm{fppf}}(X, \mu_n) \longrightarrow \mathrm{Br}(X)[n] \longrightarrow 0. \tag{3.4}$$

There are also fppf analogues of the other two exact sequences.

Theorem 1.3.7, which concerns fields of characteristic $p > 0$, admits a generalisation to arbitrary commutative rings.

Theorem 3.2.3 *Let A be a commutative ring of characteristic $p > 0$. Let $X = \mathrm{Spec}(A)$. Then $\mathrm{Br}(X) = p\,\mathrm{Br}(X)$.*

Proof. Let X be a scheme of characteristic $p > 0$. (This means that we have $p = 0$ in the structure sheaf $\mathcal{O}_X$.) The absolute Frobenius map $\mathbb{G}_{m,X} \to \mathbb{G}_{m,X}$ given by $x \mapsto x^p$ gives rise to an exact sequence of sheaves on $X_{\text{ét}}$:

$$1 \longrightarrow \mu_{p,X} \longrightarrow \mathbb{G}_{m,X} \longrightarrow \mathbb{G}_{m,X} \longrightarrow M \longrightarrow 1,$$

where M is a p-torsion sheaf. By [SGA4, X, Thm. 5.1], for a noetherian scheme X of characteristic p we have $\mathrm{cd}_p(X) \leq \mathrm{cdqc}(X) + 1$, where $\mathrm{cd}_p(X)$ is the étale cohomological dimension of p-torsion sheaves on X and $\mathrm{cdqc}(X)$ is the quasi-coherent cohomological dimension of X. In particular, for $X = \mathrm{Spec}(R)$, where R is a noetherian ring of characteristic p, we have $\mathrm{cd}_p(X) \leq 1$. Applying this to the above exact sequence of sheaves we deduce $\mathrm{Br}(X) = p\,\mathrm{Br}(X)$ when R is noetherian.

An arbitrary commutative ring A of characteristic p is a filtering direct limit of finitely generated $\mathbb{F}_p$-algebras. The limit statement for the Brauer group in Section 2.2.2 allows us to deduce the result for A. □

3.2.2 The Mayer–Vietoris exact sequence

We now state a particular case of the Mayer–Vietoris sequence for the étale sheaf $\mathbb{G}_{m,X}$ on X, see [Stacks, Lemma 0A50].

Theorem 3.2.4 *Let X be a scheme and let $X = U \cup V$ be a Zariski open covering. Write $W = U \cap V$. Then there is an infinite exact sequence*

$$\begin{aligned} 0 &\longrightarrow \Gamma(X,\mathcal{O}_X^*) \longrightarrow \Gamma(U,\mathcal{O}_U^*) \oplus \Gamma(V,\mathcal{O}_V^*) \longrightarrow \Gamma(W,\mathcal{O}_W^*) \\ &\longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(W) \\ &\longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(W) \longrightarrow \mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m) \longrightarrow \cdots \end{aligned}$$

Here the arrows like $\mathrm{Pic}(X) \to \mathrm{Pic}(U) \oplus \mathrm{Pic}(V)$ are the restriction maps, and the arrows like $\mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \to \mathrm{Pic}(W)$ are differences of the restriction maps.

If the open set U is locally factorial, for instance if U is regular, then the restriction map $\mathrm{Pic}(U) \to \mathrm{Pic}(W)$ is surjective. In this case Theorem 3.2.4 gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(W). \tag{3.5}$$

This can be compared with Theorem 3.5.7 below.

3.2.3 Passing to the reduced subscheme

Proposition 3.2.5 *Let X be a noetherian scheme. Let $X_{\mathrm{red}} \subset X$ be the reduced subscheme.*

(i) *If X is affine, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is an isomorphism.*
(ii) *If $\dim(X) \leq 1$, then $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is an isomorphism.*
(iii) *If $\dim(X) \leq 2$, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is surjective.*

Proof. Cf. [De75], [CTOP02, Lemma 1.6]. There are closed immersions

$$X_{\mathrm{red}} = X_0 \subset X_1 \subset \ldots \subset X_n = X$$

and ideals $\mathcal{I}_j \subset \mathcal{O}_{X_j}$, for $j = 1, \ldots, n$, such that $\mathcal{O}_{X_{j-1}} = \mathcal{O}_{X_j}/\mathcal{I}_j$ and $\mathcal{I}_j^2 = 0$. On each X_j we have an exact sequence of sheaves for the étale topology

$$0 \longrightarrow \mathcal{I}_j \longrightarrow \mathbb{G}_{m,X_j} \longrightarrow r_*\mathbb{G}_{m,X_{j-1}} \longrightarrow 1,$$

where $r\colon X_{j-1} \to X_j$ is the given closed immersion, the coherent ideal $\mathcal{I}_j$ is viewed as a sheaf for the étale topology, and the map $\mathcal{I}_j \to \mathbb{G}_{m,X_j}$ is given by $x \mapsto 1 + x$. For any i we have $\mathrm{H}^i_{\text{ét}}(X_j, \mathcal{I}_j) = \mathrm{H}^i_{\text{zar}}(X_j, \mathcal{I}_j)$. If X is affine, then all these groups vanish for $i \geq 1$. If $\dim(X) \leq 1$, then these groups vanish for $i \geq 2$. If $\dim(X) \leq 2$, these groups vanish for $i \geq 3$. Thus

$$\mathrm{H}^2_{\text{ét}}(X_j, \mathbb{G}_{m,X_j}) \longrightarrow \mathrm{H}^2_{\text{ét}}(X_j, r_*\mathbb{G}_{m,X_{j-1}})$$

is an isomorphism if X is affine or if $\dim(X) \leq 1$. If $\dim(X) \leq 2$, then this map is surjective. Since r is a closed immersion, we have $R^i r_*(\mathcal{F}) = 0$ for $i \geq 1$ and any sheaf $\mathcal{F}$. Hence the Leray spectral sequence for the morphism $r\colon X_{j-1} \to X_j$ and the sheaf $\mathbb{G}_m$ gives an isomorphism

$$\mathrm{H}^2_{\text{ét}}(X_j, r_*\mathbb{G}_{m,X_{j-1}}) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{j-1}, \mathbb{G}_{m,X_j}).$$

Thus the natural map $\mathrm{Br}(X_j) \to \mathrm{Br}(X_{j-1})$ is an isomorphism if X is affine or $\dim(X) \leq 1$; it is surjective if $\dim(X) \leq 2$. □

As we shall see in Section 8.1, the map $\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{red}})$ is not necessarily injective for $\dim(X) \geq 2$.

Proposition 3.2.6 *Let X be a noetherian scheme. Let n be a positive integer invertible on X. Then we have the following statements.*

(i) *The natural map* $\mathrm{Br}(X)/n \to \mathrm{Br}(X_{\mathrm{red}})/n$ *is injective.*
(ii) *The natural map* $\mathrm{Br}(X)[n] \to \mathrm{Br}(X_{\mathrm{red}})[n]$ *is surjective.*
(iii) *If X is a scheme over $\mathbb{Q}$, then the natural map* $\mathrm{Br}(X)_{\mathrm{tors}} \to \mathrm{Br}(X_{\mathrm{red}})_{\mathrm{tors}}$ *is surjective.*

Proof. If $\mathcal{F}$ is a coherent sheaf on X, then multiplication by n on $\mathrm{H}^i_{\mathrm{zar}}(X, \mathcal{F}) \cong \mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ is an isomorphism for any $i \geq 0$. The arguments from the proof of Proposition 3.2.5 then give an exact sequence

$$A \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_{\mathrm{red}}) \longrightarrow B,$$

where A and B are uniquely n-divisible. The three statements then follow by diagram chase. Statements (i) and (ii) can also be established using the Kummer sequence and the fact that the canonical morphism $X_{\mathrm{red}} \to X$ induces the identity map on étale cohomology groups with coefficients μ_n when n is invertible on X, cf. [SGA4, VIII, §1]. □

3.3 Comparing the two Brauer groups, I

For a field k we gave in Section 1.3.3 a classical construction of a natural injective map $\mathrm{Br}(k) \to \mathrm{H}^2(k, k_{\mathrm{s}}^*)$, which is in fact an isomorphism. It can be generalised to schemes and gives a natural injective map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$. For a general scheme X this map is not surjective; we shall discuss its image in Chapter 4.

After some indications of Grothendieck in [Gro68, I, §2], Milne sketches two constructions of this map in [Mil80, Thm. IV.2.5]. The first of them, which we explain in this section, requires the additional assumption that X is quasi-compact and any finite set of points of X is contained in an affine open set. (This holds, for example, if X is quasi-projective over an affine scheme.) This hypothesis allows one to use a theorem of Artin on the joins of

Hensel rings. The second construction uses gerbes and Giraud's work [Gir71], and works for any scheme X; we shall discuss it in Section 4.2 below.

Theorem 3.3.1 *Let X be a quasi-compact scheme such that any finite set of points of X is contained in an affine open set. Then there is an injective homomorphism* $\mathrm{Br}_{\mathrm{Az}}(X)\to\mathrm{Br}(X)_{\mathrm{tors}}$*, which is functorial in X.*

Proof. Let us first consider Azumaya algebras over X that have the same degree at every point of X. As recalled in Section 3.1, there is a natural bijection of pointed sets between the set of isomorphism classes of Azumaya algebras of degree n over X and the Čech cohomology set $\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X})$.

By [Mil80, Cor. IV.2.4], there is a natural exact sequence of group schemes over $X_{\text{ét}}$

$$1 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \mathrm{GL}_{n,X} \longrightarrow \mathrm{PGL}_{n,X} \longrightarrow 1, \tag{3.6}$$

where $\mathbb{G}_{m,X} \hookrightarrow \mathrm{GL}_{n,X}$ is the central subgroup of scalar matrices. Under the assumption on X, this sequence gives rise to an exact sequence of pointed Čech cohomology sets [Mil80, p. 143]

$$\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{GL}_{n,X}) \longrightarrow \check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X}) \longrightarrow \check{\mathrm{H}}^2_{\text{ét}}(X,\mathbb{G}_{m,X}).$$

By [Mil80, Cor. III.2.10] there is a canonical embedding

$$\check{\mathrm{H}}^2_{\text{ét}}(X,\mathbb{G}_{m,X}) \hookrightarrow \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_{m,X}) = \mathrm{Br}(X).$$

(Note that under our assumption on X, this map is actually an isomorphism [Mil80, Thm. III.2.17].) Hence the boundary map attached to (3.6) gives rise to a map of pointed sets

$$\delta_n\colon \check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X}) \longrightarrow \mathrm{Br}(X),$$

whose kernel is the image of $\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{GL}_{n,X})\to\check{\mathrm{H}}^1_{\text{ét}}(X,\mathrm{PGL}_{n,X})$, i.e. the classes of Azumaya algebras of the form $\mathcal{E}nd_{\mathcal{O}_X\text{-mod}}(P)$, where P is a locally free $\mathcal{O}_X$-module of rank n.

For Azumaya algebras A and B of degrees n and m, respectively, one shows that

$$\delta_{mn}(A\otimes_{\mathcal{O}_X} B) = \delta_n(A) + \delta_m(B) \in \mathrm{Br}(X).$$

In particular, this implies that $\delta_n(A)$ depends only on the class of A in $\mathrm{Br}_{\mathrm{Az}}(X)$. Thus for a class $\alpha\in\mathrm{Br}_{\mathrm{Az}}(X)$ represented by an Azumaya algebra A of constant degree n we can define $\delta(\alpha)=\delta_n(A)\in\mathrm{Br}(X)$. If $\beta\in\mathrm{Br}_{\mathrm{Az}}(X)$ is also represented by an Azumaya algebra of constant degree, then we have $\delta(\alpha+\beta)=\delta(\alpha)+\delta(\beta)$.

An arbitrary element $\alpha\in\mathrm{Br}_{\mathrm{Az}}(X)$ is represented by an Azumaya algebra A over X which does not necessarily have the same degree everywhere. The scheme X is the disjoint union of the open subsets $X_n\subset X$ such that A has degree n on X_n, for $n\geq 1$. Since X is quasi-compact, we have $X_n=\emptyset$ for almost all n. Thus $X = \coprod_{n=1}^{r} X_n$. Let $\alpha_n \in \mathrm{Br}_{\mathrm{Az}}(X_n)$ be the restriction of $\alpha \in \mathrm{Br}_{\mathrm{Az}}(X)$. We define $\delta(\alpha) \in \mathrm{Br}(X) = \prod_{n=1}^{r} \mathrm{Br}(X_n)$ as $(\delta(\alpha_n))$.

For any $\alpha, \beta \in \mathrm{Br}_{\mathrm{Az}}(X)$ the quasi-compact scheme X is a disjoint union of finitely many open subsets X_n such that the restrictions $\alpha_n, \beta_n \in \mathrm{Br}_{\mathrm{Az}}(X_n)$ are represented by Azumaya algebras of constant degrees. This implies that in $\mathrm{Br}(X)$ we have $\delta(\alpha+\beta) = \delta(\alpha) + \delta(\beta)$, hence $\delta \colon \mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)$ is a homomorphism with trivial kernel, and so is injective.

Since $\mathrm{Br}_{\mathrm{Az}}(X)$ is a torsion group by Corollary 3.1.4, the image of δ belongs to the torsion subgroup $\mathrm{Br}(X)_{\mathrm{tors}}$. □

The following theorem was proved by Gabber in his thesis [Gab81, Ch. II, Thm. 1].

Theorem 3.3.2 (Gabber) *If X is an affine scheme, then*

$$\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)_{\mathrm{tors}}$$

is an isomorphism.

We do not prove this result here. Gabber's proof uses gerbes (see Section 4.1.4 below); it proceeds by reduction to the case of local rings. A more elementary proof was later given by Knus and Ojanguren [KO81]. A proof in the language of twisted sheaves (see Section 4.1.5 below) can be found in [Lie08, Cor. 3.1.4.2]. As we shall see in Chapter 4, the theorem holds more generally for quasi-projective schemes over an affine scheme.

The following result was first proved directly in terms of Azumaya algebras by Knus, Ojanguren and Saltman [KOS76, Cor. 4.4].

Proposition 3.3.3 *Let A be a commutative ring of characteristic $p > 0$. Then $\mathrm{Br}_{\mathrm{Az}}(A) = p\,\mathrm{Br}_{\mathrm{Az}}(A)$.*

Proof. This follows from Theorems 3.3.2 and 3.2.3. □

3.4 Localising elements of the Brauer group

Lemma 3.4.1 *Let X be a scheme. For any element $\alpha \in \mathrm{Br}(X)$ there exists an étale cover $f \colon U \to X$ such that $f^*\alpha = 0 \in \mathrm{Br}(U)$.*

Proof. This is a special case of a general statement: for any scheme X, any site X_E with underlying category C_X, any sheaf F on X_E, any $U \in C_X$, any $i > 0$ and any cohomology class $\alpha \in \mathrm{H}^i_E(U, F)$ there exists a covering $\{U_j \to U\}_{j \in J}$ such that the restriction of α to each $\mathrm{H}^i_E(U_j, F)$ is zero [Mil80, Prop. III.2.9, Remark III.2.11 (a)]. Take $U = \coprod_{j \in J} U_j$. □

Theorem 3.4.2 (Azumaya) *Let R be a henselian local ring with residue field k.*

(i) *The embedding of the closed point* $\mathrm{Spec}(k)\to\mathrm{Spec}(R)$ *induces an isomorphism* $\mathrm{Br}(R)\xrightarrow{\sim}\mathrm{Br}(k)$.
(ii) *If R is a strictly henselian local ring, that is, if k is separably closed, then* $\mathrm{Br}(R)=0$.

Proof. For any smooth commutative group R-scheme G we have an isomorphism $\mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(R),G)\xrightarrow{\sim}\mathrm{H}^i(k,G\times_R k)$ when $i\geq 1$, see [Mil80, Remark III.3.11 (a)] and [Gro68, III, Thm. 11.7]. For $G=\mathbb{G}_m$ and $i=2$ we get the desired statement $\mathrm{Br}(R)\xrightarrow{\sim}\mathrm{Br}(k)$.

(ii) follows from (i). Alternatively, by [Mil80, Thm. I.4.2 (d)] an étale morphism $U\to\mathrm{Spec}(R)$ has a section provided U contains a k-point which goes to the closed point of $\mathrm{Spec}(R)$. Thus (ii) is a consequence of Lemma 3.4.1. □

The original theorem proved by Azumaya says that the evaluation map $\mathrm{Br}_{\mathrm{Az}}(R)\to\mathrm{Br}(k)$ is an isomorphism. We briefly outline the proof of this result given in [Mil80, Cor. IV.2.13]. Let $\alpha\in\mathrm{Br}(R)$. Since R is local henselian, Lemma 3.4.1 implies that there exists a *finite* étale extension $R\subset R'$ of henselian local rings such that α goes to 0 under the natural map $\mathrm{Br}(R)\to\mathrm{Br}(R')$. This implies that $\alpha\in\mathrm{Br}_{\mathrm{Az}}(R)$, so the embedding of $\mathrm{Br}_{\mathrm{Az}}(R)$ into $\mathrm{Br}(R)$ is an isomorphism. Since the map $\mathrm{Br}_{\mathrm{Az}}(X)\to\mathrm{Br}(X)$ is functorial, Theorem 3.4.2 and the result for fields $\mathrm{Br}_{\mathrm{Az}}(k)\xrightarrow{\sim}\mathrm{Br}(k)$ (Theorem 1.3.5) then imply that the evaluation map $\mathrm{Br}_{\mathrm{Az}}(R)\to\mathrm{Br}(k)$ is an isomorphism.

Corollary 3.4.3 *Let R be a henselian noetherian local ring with maximal ideal $\mathfrak{m}$. Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then the natural map* $\mathrm{Br}(R)\to\mathrm{Br}(\widehat{R})$ *is an isomorphism.*

Proof. Since R is noetherian, by [Mat86, Thm. 8.13] or [Stacks, Lemma 05GG], the ring $\widehat{R}$ is a complete local ring with maximal ideal $\mathfrak{m}\widehat{R}$, and $\widehat{R}/\mathfrak{m}\widehat{R}\cong R/\mathfrak{m}$. Thus $\widehat{R}$ is henselian, see [Mil80, Prop. I.4.5] or [Stacks, Lemma 04GM]. By Theorem 3.4.2, the natural map $\mathrm{Br}(R)\to\mathrm{Br}(\widehat{R})$ is identified with $\mathrm{Br}(R/\mathfrak{m})\xrightarrow{\sim}\mathrm{Br}(\widehat{R}/\mathfrak{m}\widehat{R})$. □

Corollary 3.4.4 *Let k be a field, let X be a k-scheme and let $P\in X(k)$ be a k-point. For any $\alpha\in\mathrm{Br}(X)$ with $\alpha(P)=0\in\mathrm{Br}(k)$ there exist an étale morphism $f\colon U\to X$ and a k-point $M\in U(k)$ such that $f(M)=P$ and $f^*(\alpha)=0\in\mathrm{Br}(U)$.*

Proof. Let R be the henselisation of the local ring of X at P. By Theorem 3.4.2 (i) the image of α under the natural map $\mathrm{Br}(X)\to\mathrm{Br}(R)$ is zero. The ring R is a filtering direct limit of rings R_i, each of them equipped with an étale map $f_i\colon\mathrm{Spec}(R_i)\to X$ and a k-point M_i such that $f_i(M_i)=P$. The group $\mathrm{Br}(R)$ is the direct limit of the groups $\mathrm{Br}(R_i)$, see Section 2.2.2. Thus α goes to zero in $\mathrm{Br}(R_i)$ for some i, so we can take $U=\mathrm{Spec}(R_i)$. □.

Lemma 3.4.5 *Let k be a field and let X be a variety over k. Let $A \in \mathrm{Br}(X)$. There exists an integer $n \geq 1$ such that nA vanishes in each residue field of X.*

Proof. Suppose this has been proved for all varieties of dimension at most d. Let X be a variety of dimension $d+1$. To prove the result for X we may assume that X is reduced and irreducible. Let $k(X)$ be the function field of X. By Section 2.2.2, the torsion group $\mathrm{Br}(k(X))$ is the direct limit of the groups $\mathrm{Br}(U)$, where U is non-empty and open in X. Thus there exists a non-empty open set $U \subset X$ such that the restriction of A to U is an element of $\mathrm{Br}(U)$ annihilated by some positive integer n. Let $Z = X \setminus U$. By the induction hypothesis there exists an integer $m > 0$ such that the restriction of mA to the residue fields of Z is zero. Thus the restriction of nmA to the residue fields of X is zero. □

3.5 Going over to the generic point

The following definition was suggested in [SGA2, Exposé XIII, §6] and adopted in [CTS78].

Definition 3.5.1 *A locally noetherian scheme X is* **geometrically locally factorial** *if for any étale morphism $U \to X$ each local ring of U is factorial, that is, a unique factorisation domain.*

Equivalently, strict henselisations of the local rings of X are factorial.

It is clear that any geometrically locally factorial scheme is normal. By [BouAC, Ch. VII, §2, Thm. 1] any prime ideal of height 1 in a factorial ring is principal, hence any Weil divisor of a geometrically locally factorial scheme is a Cartier divisor. A theorem of Auslander–Buchsbaum implies that a regular scheme is geometrically locally factorial, see [SGA2, Thm. XI.3.13]. Grothendieck proved that a noetherian local ring which is a complete intersection and whose localisations $R_{\mathfrak{p}}$ at primes $\mathfrak{p}$ of height up to 3 are regular is geometrically locally factorial, see [SGA2, Cor. XI.3.14].

Let X be a normal, integral, noetherian scheme. Let $j\colon \mathrm{Spec}(F) \hookrightarrow X$ be the generic point of X. There is a natural exact sequence of sheaves on the small étale site $X_{\text{ét}}$, which describes the embedding of the group of invertible regular functions into the group of non-zero rational functions as the kernel of the divisor map:

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D \in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)}, \tag{3.7}$$

see [Mil80, Example II.3.9]. Here $i_D\colon \mathrm{Spec}(k(D)) \hookrightarrow X$ is the embedding of the generic point of an irreducible divisor $D \subset X$; the direct sum ranges over all such divisors. On a connected étale open set $U \to X$ the map $j_*\mathbb{G}_{m,F}\to i_{D*}\mathbb{Z}_{k(D)}$ can be described as follows. Let D' be an irreducible divisor on U contained in $D\times_X U$. Since the scheme X is normal, the scheme U is also normal, hence the local ring $\mathcal{O}_{U,D'}$ is a discrete valuation ring with valuation $v_{D'}\colon \mathcal{O}_{U,D'}\smallsetminus\{0\}\to\mathbb{N}$. The group of sections $\mathrm{H}^0_{\text{ét}}(U,j_*\mathbb{G}_{m,F})$ is the multiplicative group of the function field of U. The map $\mathrm{H}^0_{\text{ét}}(U,j_*\mathbb{G}_{m,F})\to\mathrm{H}^0_{\text{ét}}(U,i_{D*}\mathbb{Z}_{k(D)})$ sends a function f to the integer $v_{D'}(f)$.

Now assume, in addition, that the noetherian scheme X is geometrically locally factorial. Then Weil divisors are the same as Cartier divisors, i.e., any divisor locally at each point is given by one equation. Thus (3.7) extends to an exact sequence of sheaves on $X_{\text{ét}}$

$$0\longrightarrow\mathbb{G}_{m,X}\longrightarrow j_*\mathbb{G}_{m,F}\longrightarrow\bigoplus_{D\in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)}\longrightarrow 0. \tag{3.8}$$

Remark 3.5.2 The exact sequence (3.8) restricted to the small Zariski site X_{zar} is a flasque resolution of the Zariski sheaf $\mathbb{G}_{m,X}$. Recall that a Zariski sheaf $\mathcal{F}$ on X is *flasque* if for any Zariski open set $U\subset X$ the restriction map $\mathrm{H}^0(X,\mathcal{F})\to\mathrm{H}^0(U,\mathcal{F})$ is surjective. As remarked by Grothendieck in [Gro57], this implies $\mathrm{H}^i_{\text{zar}}(X,\mathbb{G}_{m,X})=0$ for $i\geq 2$. This argument can be applied to any scheme X which is locally factorial (in the usual sense, i.e., for the Zariski topology) and not necessarily regular.

Lemma 3.5.3 *Let X be a geometrically locally factorial, integral, noetherian scheme, for example, a regular, integral, noetherian scheme. Then the groups $\mathrm{H}^n_{\text{ét}}(X,\mathbb{G}_{m,X})$ are torsion for $n\geq 2$. In particular, the Brauer group* $\mathrm{Br}(X)$ *is a torsion group.*

Proof. This follows from Lemma 2.4.1 and the long exact sequence of cohomology attached to (3.8). □

Lemma 3.5.4 *Let X be a geometrically locally factorial (for example, regular), integral, noetherian scheme with generic point $j\colon\mathrm{Spec}(F)\hookrightarrow X$. If $D\subset X$ is an irreducible divisor, we denote its generic point by* $\mathrm{Spec}(k(D))$. *There is an exact sequence*

$$0\longrightarrow\mathrm{Br}(X)\longrightarrow\mathrm{H}^2_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\longrightarrow\bigoplus_{D\in X^{(1)}}\mathrm{H}^1(k(D),\mathbb{Q}/\mathbb{Z}). \tag{3.9}$$

Proof. By Lemma 2.4.1 the long exact sequence of cohomology groups attached to (3.8) gives

$$0\longrightarrow\mathrm{Br}(X)\longrightarrow\mathrm{H}^2_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\longrightarrow\bigoplus_{D\in X^{(1)}}\mathrm{H}^2_{\text{ét}}(X,i_{D*}(\mathbb{Z}_{k(D)})).$$

By the same Lemma 2.4.1 the spectral sequence

$$\mathrm{H}^p_{\text{ét}}(X,(R^q i_{D*})(\mathbb{Z}_{k(D)}))\Rightarrow\mathrm{H}^{p+q}(k(D),\mathbb{Z})$$

gives an injective map $\mathrm{H}^2_{\text{ét}}(X, i_{D*}(\mathbb{Z}_{k(D)}))\to\mathrm{H}^2(k(D),\mathbb{Z})$. Multiplication by any non-zero integer is an automorphism of the abelian group $\mathbb{Q}$; however, any Galois cohomology group of positive degree is a torsion group [SerCG, Cor. 2.2.3], so $\mathrm{H}^n(k(D),\mathbb{Q}) = 0$ for $n > 0$. Thus the long exact sequence associated to the exact sequence of trivial Galois modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0 \tag{3.10}$$

gives an isomorphism $\mathrm{H}^1(k(D),\mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2(k(D),\mathbb{Z})$. This gives (3.9). □

For the Brauer–Azumaya group of a regular affine scheme, the following theorem was proved by Auslander and Goldman [AG60, Thm. 7.2]. The general case was proved by Grothendieck [Gro68, II, Cor. 1.8].

Theorem 3.5.5 *Let X be a geometrically locally factorial, integral, noetherian scheme with generic point* $\mathrm{Spec}(F)$*, for example a regular, integral, noetherian scheme. The natural map* $\mathrm{Br}(X)\to\mathrm{Br}(F)$ *is injective. For any non-empty open subset $U \subset X$ this map factors through the natural map* $\mathrm{Br}(X)\to\mathrm{Br}(U)$*, which is therefore also injective.*

Proof. By Lemma 2.4.1 the spectral sequence

$$\mathrm{H}^p_{\text{ét}}(X, (R^q j_*)(\mathbb{G}_{m,F})) \Rightarrow \mathrm{H}^{p+q}(F, \mathbb{G}_{m,F}) \tag{3.11}$$

implies that $\mathrm{H}^2_{\text{ét}}(X, j_*\mathbb{G}_{m,F})$ is a subgroup of $\mathrm{H}^2(F,\mathbb{G}_{m,F}) = \mathrm{Br}(F)$. Now (3.9) shows that $\mathrm{Br}(X)$ is naturally a subgroup of $\mathrm{Br}(F)$. □

Corollary 3.5.6 *Let X be a geometrically locally factorial, integral, noetherian scheme with generic point* $\mathrm{Spec}(F)$*. Let $\alpha \in \mathrm{Br}(F)$. If $\alpha \in \mathrm{Br}(\mathcal{O}_{X,x})$ for all points $x \in X$ or, equivalently, if there exists a Zariski open covering $X = \cup_i U_i$ such that $\alpha \in \mathrm{Br}(U_i) \subset \mathrm{Br}(F)$, then $\alpha \in \mathrm{Br}(X)$.*

Proof. This follows from the exact sequence (3.5) and Theorem 3.5.5. □

The definition of an excellent scheme was recalled in Section 2.6.

Theorem 3.5.7 [Ber05] *Let X be an excellent, noetherian, integral scheme. Let $U \subset X$ be a non-empty open subscheme. Assume that U contains every singular point of X. Then the restriction homomorphism* $\mathrm{Br}(X)\to\mathrm{Br}(U)$ *is injective.*

Proof. Let V be the set of regular points. Since X is excellent, this is an open set. Then $X = U \cup V$. Let $W = U \cap V$. Since V is regular, the restriction map $\mathrm{Pic}(V)\to\mathrm{Pic}(W)$ is surjective. By the Mayer–Vietoris sequence (Theorem 3.2.4) the diagonal restriction map $\mathrm{Br}(X)\to\mathrm{Br}(U) \oplus \mathrm{Br}(V)$ is injective. If $\alpha \in \mathrm{Br}(X)$ has a trivial image in $\mathrm{Br}(U)$, then it has a trivial image at each generic point of U, hence it has a trivial image in $\mathrm{Br}(V)$. Indeed, as V is regular, the restriction map to the generic point is injective (Theorem 3.5.5). Thus $\alpha = 0 \in \mathrm{Br}(X)$. □

Remark 3.5.8 In Section 8.6 we give counter-examples to the injectivity of the restriction map $\mathrm{Br}(R)\to\mathrm{Br}(K)$, where R is an integral local ring which is a local complete intersection and K is the field of fractions of R. In the second counter-example R is normal of dimension 2, in the third counter-example R is regular in codimension 2, but not in codimension 3. The ring R is not geometrically locally factorial.

3.6 Schemes of dimension 1

3.6.1 Regular schemes of dimension 1

This section follows [Gro68, III, §2] and [Mil80, III, Example 2.22]. Proposition 1.4.5, whose proof uses the Krull–Akizuki Theorem, enables one to recover all results stated in [Gro68, III, §2] without the excellence assumption added in [Mil80, III, Example 2.22].

Theorem 3.6.1 *Let X be an integral, regular, noetherian scheme of dimension* 1 *with generic point* $\mathrm{Spec}(F)$*.*

(i) *For any prime ℓ invertible on X there is an exact sequence*

$$0\to\mathrm{Br}(X)\{\ell\}\to\mathrm{Br}(F)\{\ell\}\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^1(k(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m)\{\ell\}\to\ldots$$
$$\ldots\to\mathrm{H}^i_{\text{ét}}(X,\mathbb{G}_m)\{\ell\}\to\mathrm{H}^i(F,\mathbb{G}_m)\{\ell\}\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^{i-1}(k(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\ldots$$

where $k(x)$ is the residue field of the point $x\in X$. For each $x\in X^{(1)}$ the map $\mathrm{Br}(F)\{\ell\}\to\mathrm{H}^1(k(x),\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ *is the Witt residue.*

(ii) *If for each closed point $x\in X$ the residue field $k(x)$ is perfect, then there is an exact sequence*

$$0\to\mathrm{Br}(X)\to\mathrm{Br}(F)\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m)\to\mathrm{H}^3(F,\mathbb{G}_m)\to\ldots$$
$$\ldots\to\mathrm{H}^i_{\text{ét}}(X,\mathbb{G}_m)\to\mathrm{H}^i(F,\mathbb{G}_m)\to\bigoplus_{x\in X^{(1)}}\mathrm{H}^{i-1}(k(x),\mathbb{Q}/\mathbb{Z})\to\ldots$$

For each $x\in X^{(1)}$ the map $\mathrm{Br}(F)\to\mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})$ *is the Witt residue.*

The Witt residue r_W was introduced in Definition 1.4.11. We have

$$r_W=-r=\partial,$$

where r is the Serre residue (Definition 1.4.3) and ∂ is the Gysin residue (Definition 2.3.4), see Theorems 1.4.14 and 2.3.5.

Proof of Theorem 3.6.1. The exact sequence of sheaves (3.8)

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,F} \longrightarrow \bigoplus_{D\in X^{(1)}} i_{D*}\mathbb{Z}_{k(D)} \longrightarrow 0$$

gives rise to the long exact sequence of étale cohomology groups

$$\ldots \to \mathrm{H}^i_{\text{ét}}(X,\mathbb{G}_m)\to \mathrm{H}^i_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\to \mathrm{H}^i_{\text{ét}}(X,\textstyle\bigoplus_{x\in X^{(1)}} i_{x*}\mathbb{Z})\to \mathrm{H}^{i+1}_{\text{ét}}(X,\mathbb{G}_m)\to \ldots$$

Since $\dim(X) = 1$, each inclusion $i_x\colon x\to X$ is a closed immersion, hence a finite morphism. Thus for any sheaf $\mathcal{F}$ on x we have $R^q i_{x*}(\mathcal{F}) = 0$ for $i \geq 1$. Therefore, we can rewrite the above sequence as follows:

$$\ldots \to \mathrm{H}^i_{\text{ét}}(X,\mathbb{G}_m)\to \mathrm{H}^i_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^i(k(x),\mathbb{Z})\to \mathrm{H}^{i+1}_{\text{ét}}(X,\mathbb{G}_m)\to \ldots$$

In particular, we have a long exact sequence

$$0\to \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_m)\to \mathrm{H}^2_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\to \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})\to \mathrm{H}^3_{\text{ét}}(X,\mathbb{G}_m)\to \ldots$$

By Lemma 2.4.1 we have $R^1j_*\mathbb{G}_{m,F} = 0$. For $q \geq 2$ the stalk of $R^qj_*\mathbb{G}_{m,F}$ at the generic point of X is the Galois cohomology group $\mathrm{H}^q(F_{\mathrm{s}},\mathbb{G}_m)$, where F_{s} is a separable closure of F, hence this stalk is zero. The stalk at a geometric point $\bar{x}$ above a closed point $x \in X$ is $\mathrm{H}^q(F_x^{\mathrm{sh}},\mathbb{G}_m)$, where F_x^{sh} is the field of fractions of $\mathcal{O}^{\mathrm{sh}}_{X,x}$. By Proposition 1.4.5 (ii) this group is p_x-primary, where p_x is the characteristic exponent of the residue field $k(x)$. If $k(x)$ is perfect, then $\mathrm{H}^q(F_x^{\mathrm{sh}},\mathbb{G}_m) = 0$ for all $q \geq 1$, by Proposition 1.4.5 (iv). If this holds for all x, then $R^qj_*\mathbb{G}_{m,F} = 0$ all $q \geq 1$.

From the spectral sequence $\mathrm{H}^p_{\text{ét}}(X, R^qj_*\mathbb{G}_{m,F}) \Rightarrow \mathrm{H}^{p+q}(F,\mathbb{G}_{m,F})$, see (3.11), we then deduce the following statements.

- For $q \geq 2$ the natural map $\mathrm{H}^q_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\to \mathrm{H}^q(F,\mathbb{G}_m)$ induces an isomorphism of the ℓ-primary subgroups, for each prime ℓ invertible on X.
- The natural map $\mathrm{H}^q_{\text{ét}}(X,j_*\mathbb{G}_{m,F})\to \mathrm{H}^q(F,\mathbb{G}_m)$ is an isomorphism for all $q \geq 2$ if for each closed point $x \in X$ the residue field $k(x)$ is perfect.

This gives the exact sequences in the proposition.

To identify the map $\mathrm{Br}(F)\to \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z})$ with the Witt residue in (ii) and with its variant on the ℓ-primary part in case (i) (where the residue fields need not be perfect), we can assume that $X = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$. Let $K = F_x^{\mathrm{h}}$ be the field of fractions of $\mathcal{O}^{\mathrm{h}}_{X,x}$. We follow the arguments from the proof of Theorem 2.3.5 using similar notation. Let K_{s} be a separable closure of K. Then F_x^{sh} coincides with the maximal unramified extension K_{nr} of K in K_{s}. Define $G = \mathrm{Gal}(K_{\mathrm{s}}/K)$, $I = \mathrm{Gal}(K_{\mathrm{s}}/K_{\mathrm{nr}})$, and

$$\Gamma = \mathrm{Gal}(k(x)_{\mathrm{s}}/k(x)) \cong \mathrm{Gal}(K_{\mathrm{nr}}/K) \cong G/I.$$

As discussed in Section 2.3.3, the category of étale sheaves on $\mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$ is equivalent to the category of triples (M, N, φ), where M is a Γ-module, N is a G-module, and $\varphi\colon M \to N^I$ is a homomorphism of Γ-modules. Under this equivalence, the sheaf $j_*\mathbb{G}_{m,K}$ corresponds to the triple $(K^*_{\mathrm{nr}}, K^*_{\mathrm{s}}, \mathrm{id})$, the sheaf $i_*\mathbb{Z}_{k(x)}$ corresponds to $(\mathbb{Z}, 0, 0)$, and the map $j_*\mathbb{G}_{m,K} \to i_*\mathbb{Z}_{k(x)}$ is given by the valuation $K^*_{\mathrm{nr}} \to \mathbb{Z}$, see [Mil80, Example II.3.15]. According to (2.18) there is a canonical isomorphism

$$\mathrm{H}^2(\mathcal{O}^{\mathrm{h}}_{X,x}, j_*\mathbb{G}_{m,K}) \cong \mathrm{H}^2(\Gamma, K^*_{\mathrm{nr}}).$$

Under this isomorphism, the map

$$\mathrm{H}^2(\mathcal{O}^{\mathrm{h}}_{X,x}, j_*\mathbb{G}_{m,K}) \longrightarrow \mathrm{H}^2(k(x), \mathbb{Z}) \cong \mathrm{H}^1(k(x), \mathbb{Q}/\mathbb{Z})$$

becomes the Witt residue $\mathrm{H}^2(\Gamma, K^*_{\mathrm{nr}}) \to \mathrm{H}^2(\Gamma, \mathbb{Z}) \cong \mathrm{H}^1(\Gamma, \mathbb{Q}/\mathbb{Z})$. □

The following theorem gives a description of the Brauer group of a henselian discrete valuation field K in the case when the residue field k is perfect. It can be compared to a similar description (1.16), where n is coprime to the characteristic of k but k is not necessarily perfect.

Theorem 3.6.2 (Witt) [Wit37] *Let R be a henselian discrete valuation ring with fraction field K and perfect residue field k. Then there is a split exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(K) \xrightarrow{r_W} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0. \tag{3.12}$$

Proof. By the functoriality of étale cohomology the embedding of the closed point $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ gives rise to the specialisation map $\mathrm{Br}(R) \to \mathrm{Br}(k)$. This map is an isomorphism by Theorem 3.4.2. Now (3.12) follows from Theorem 3.6.1 in view of the surjectivity of the Witt residue, see Section 1.4.3. That the sequence is split follows from the comments after Definition 1.4.11. □

Corollary 3.6.3 *Let R be a henselian discrete valuation ring with fraction field K and* finite *residue field k. Then $\mathrm{Br}(K) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$.*

Proof. By Theorem 1.2.13 (Wedderburn) we have $\mathrm{Br}(k) = 0$. In this case the Galois group Γ is the profinite completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$ generated by the Frobenius automorphism. Hence $\mathrm{Hom}_{\mathrm{cont}}(\Gamma, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. □

In particular, when $K = F_v$ is the completion of a global field F at a non-archimedean place v we obtain an isomorphism

$$\mathrm{inv}_v\colon \mathrm{Br}(F_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z},$$

called the *local invariant*. For example, if F_v is the field of p-adic numbers $\mathbb{Q}_p$, $p \neq 2$, and $a \in \mathbb{Z}_p^*$, by formula (1.18), $\mathrm{inv}_p(a, p) = 0$ if and only if the Legendre symbol $\left(\frac{a}{p}\right) = 1$.

There are other cases when Theorem 3.6.1 gives rise to a short exact sequence.

Theorem 3.6.4 *Let A be a semilocal Dedekind domain of dimension* 1 *with field of fractions K. Let ℓ be a prime invertible in A. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(A)\{\ell\} \longrightarrow \mathrm{Br}(K)\{\ell\} \xrightarrow{\{\partial_{\mathfrak{p}}\}} \bigoplus_{\mathfrak{p}} \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0,$$

where $\mathfrak{p}$ ranges over the maximal ideals of A.

Proof. By Theorem 3.6.1 it remains to prove the surjectivity of the third map in the sequence. Choose a maximal ideal $\mathfrak{p} \subset A$ and let $x \in \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. The group $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is the union of subgroups $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/\ell^m)$, so x is in $\mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/n)$ for some $n = \ell^m$. It is enough to find an element $\alpha \in \mathrm{Br}(K)[n]$ such that $\partial_{\mathfrak{p}}(\alpha) = x$ and $\partial_{\mathfrak{p}'}(\alpha) = 0$ for all maximal ideals $\mathfrak{p}' \neq \mathfrak{p}$ of A.

Let $A_{\mathfrak{p}}$ be the localisation of A at $\mathfrak{p}$ and let $A_{\mathfrak{p}}^{\mathrm{h}}$ be the henselisation of the local ring $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}^{\mathrm{h}}$ is a henselian local ring, the specialisation map

$$\mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1(A/\mathfrak{p}, \mathbb{Z}/n)$$

is an isomorphism. Let $\widetilde{x} \in \mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ be the inverse image of x under this isomorphism.

Consider a finite separable field extension $K \subset L$ with the following two properties: if B is the integral closure of $A_{\mathfrak{p}}$ in L, then the embedding of the closed point $\mathrm{Spec}(A/\mathfrak{p}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ factors as

$$\mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A_{\mathfrak{p}})$$

and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A_{\mathfrak{p}})$ is étale at the image of $\mathrm{Spec}(A/\mathfrak{p})$ in $\mathrm{Spec}(B)$. Let $\mathfrak{q} \subset B$ be the prime ideal such that $\mathrm{Spec}(B/\mathfrak{q})$ is this image of $\mathrm{Spec}(A/\mathfrak{p})$, and let $B_{\mathfrak{q}}$ be the localisation of B at $\mathfrak{q}$. Then, as was recalled in Section 2.1.4, we have

$$A_{\mathfrak{p}}^{\mathrm{h}} = \varinjlim B_{\mathfrak{q}}.$$

There is an isomorphism of residue fields $A/\mathfrak{p} = A_{\mathfrak{p}}/\mathfrak{p} \cong B_{\mathfrak{q}}/\mathfrak{q} = B/\mathfrak{q}$. Since L is separable over K, the A-algebra B is a finitely generated A-module. By the Krull–Akizuki theorem [BouAC, Ch. 7, §2, no. 5], B is a semilocal Dedekind domain, so has finitely many maximal ideals [SerCL, Ch. I, §4].

Since $\mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ is the inductive limit of $\mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n)$ (see Section 2.2.2), our element $\widetilde{x} \in \mathrm{H}^1(A_{\mathfrak{p}}^{\mathrm{h}}, \mathbb{Z}/n)$ comes from an element $\rho \in \mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n)$ for some ring B as above. The injective map $\mathrm{H}^1(B_{\mathfrak{q}}, \mathbb{Z}/n) \to \mathrm{H}^1(L, \mathbb{Z}/n)$ allows us to consider ρ as an element of $\mathrm{H}^1(L, \mathbb{Z}/n)$.

By the independence of valuations we can choose an element $t \in B$ such that the valuation of t at $\mathfrak{q}$ is 1 and $t \equiv 1 \bmod \mathfrak{q}'$ for each maximal ideal $\mathfrak{q}' \subset B$, $\mathfrak{q}' \neq \mathfrak{q}$. Let $\beta \in \mathrm{H}^2(L, \mu_n) = \mathrm{Br}(L)[n]$ be the cup-product of the class of t in $L^*/L^{*n} \cong \mathrm{H}^1(L, \mu_n)$ and the class $\rho \in \mathrm{H}^1(L, \mathbb{Z}/n)$. By Proposition 1.4.7, corestriction gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(L)[n] & \xrightarrow{r} & \bigoplus_{J\subset B} \mathrm{H}^1(B/J, \mathbb{Z}/n) \\ \downarrow{\scriptstyle \mathrm{cores}_{L/K}} & & \downarrow{\scriptstyle \mathrm{cores}_{(B/J)/(A/I)}} \\ \mathrm{Br}(K)[n] & \xrightarrow{r} & \bigoplus_{I\subset A} \mathrm{H}^1(A/I, \mathbb{Z}/n) \end{array}$$

where the horizontal maps are Serre residues, I ranges over the maximal ideals of A, and J ranges over the maximal ideals of B. We have $\partial_{\mathfrak{q}}(\beta) = x$ and $\partial_{\mathfrak{q}'}(\beta) = 0$ when $\mathfrak{q}' \subset B$ is a maximal ideal $\mathfrak{q}' \neq \mathfrak{q}$. Now let $\alpha = \mathrm{cores}_{L/K}(\beta)$. From the diagram we obtain $\partial_{\mathfrak{p}}(\alpha) = x$ and $\partial_{\mathfrak{p}'}(\alpha) = 0$ when $\mathfrak{p}' \subset A$ is a maximal ideal $\mathfrak{p}' \neq \mathfrak{p}$. □

Remark 3.6.5 A similar proof gives surjectivity of the maps

$$\{\partial_{\mathfrak{p}}\}\colon \mathrm{H}^i(K, \mu_n^{\otimes j}) \longrightarrow \bigoplus_{\mathfrak{p}\subset A} \mathrm{H}^{i-1}(A/\mathfrak{p}, \mu_n^{\otimes j-1})$$

in Theorem 3.6.1 for all $i \geq 2$. The long exact localisation sequence then breaks up into short exact sequences

$$0 \longrightarrow \mathrm{H}^i(A, \mu_n^{\otimes j}) \longrightarrow \mathrm{H}^i(K, \mu_n^{\otimes j}) \longrightarrow \bigoplus_{\mathfrak{p}\subset A} \mathrm{H}^{i-1}(A/\mathfrak{p}, \mu_n^{\otimes j-1}) \longrightarrow 0,$$

where n is invertible in A, for any $i, j \in \mathbb{Z}$, $i \geq 1$, see [CTKH97, Cor. B.3.3]. It also works for various other theories such as Milnor K-theory with torsion coefficients (H. Gillet).

3.6.2 Singular schemes of dimension 1

The following proposition clarifies some points in [Gro68, II, §1].

Proposition 3.6.6 *Let X be a noetherian 1-dimensional scheme. The Brauer group $\mathrm{Br}(X)$ is a torsion group. If $\alpha \in \mathrm{Br}(X)$ vanishes when evaluated at each generic point of X and also at each singular point of X, then $\alpha = 0$.*

Proof. By Proposition 3.2.5 (ii), the map $\mathrm{Br}(X)\to\mathrm{Br}(X_{\mathrm{red}})$ is an isomorphism. Thus we can assume that X is reduced. Let us write $x = \mathrm{Spec}(k(x))$ for a closed point of X, and $y = \mathrm{Spec}(k(y))$ for any of the finitely many points of X of dimension 1. Let $i_x\colon x\to X$ and $i_y\colon y\to X$ be the natural morphisms.

Then we have an exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \prod_y i_{y*}\mathbb{G}_{m,k(y)} \longrightarrow \bigoplus_x i_{x*}F_x \longrightarrow 0, \tag{3.13}$$

where F_x is an étale sheaf on x which is the constant sheaf $\mathbb{Z}$, except possibly when x is a singular point of X (if X is excellent, then there are only finitely many of these).

Using Lemma 2.4.1 (ii) for the fields $k(y)$, we deduce from (3.13) an exact sequence

$$0 \longrightarrow \bigoplus_x \mathrm{H}^1(k(x), F_x) \longrightarrow \mathrm{Br}(X) \longrightarrow \prod_y \mathrm{Br}(k(y)). \tag{3.14}$$

Note that $\mathrm{H}^1(x, F_x) = 0$ if x is a regular point, since $\mathrm{H}^1(k(x), \mathbb{Z}) = 0$. From this exact sequence we conclude that $\mathrm{Br}(X)$ is a torsion group.

Let $X_x = \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,x})$ be the henselisation of X at a singular point x. Then we have a similar exact sequence

$$0 \longrightarrow \mathrm{H}^1(k(x), F_x) \longrightarrow \mathrm{Br}(X_x) \longrightarrow \prod_{y_x} \mathrm{Br}(k(y_x)),$$

where the product is over the generic points y_x of X_x. The two sequences are compatible via the maps induced by the natural morphism $X_x \to X$.

If $\alpha \in \mathrm{Br}(X)$ vanishes at each generic point of X, then α is the image of a well-defined element $\{\zeta_x\} \in \oplus_x \mathrm{H}^1(k(x), F_x)$, where the sum is over the singular points of X. By Theorem 3.4.2 the evaluation map $\mathrm{Br}(X_x) \to \mathrm{Br}(k(x))$ is an isomorphism. Thus if α also vanishes when evaluated at the closed point x, then the image of α in $\mathrm{Br}(X_x)$ is zero, hence $\zeta_x = 0$. This proves the proposition. □.

Remark 3.6.7 If an excellent 1-dimensional scheme X is affine one may give a proof of Proposition 3.6.6 in terms of Azumaya algebras, using conductors and patching diagrams [CTOP02, Prop. 1.12]. See also [Chi74] and [KO74a]. For X not necessarily affine, the result then follows from Theorem 3.5.7 and the fact that the set of singular points of X is contained in an affine open subset.

Lemma 3.6.8 *Let X be a noetherian separated scheme of dimension 1. Then X has an ample invertible sheaf.*

Proof. See [Stacks, Prop. 09NZ]. □

Proposition 3.6.9 *Let X be a noetherian separated scheme of dimension ≤ 1. The natural inclusion $\mathrm{Br}_{\mathrm{Az}}(X) \hookrightarrow \mathrm{Br}(X)$ is an equality.*

Proof. By Proposition 3.6.6, $\mathrm{Br}(X)$ is a torsion group. Now Lemma 3.6.8 and Gabber's theorem 4.2.1 give the result. □

Remark 3.6.10 As we shall see in Section 8.1, there exist 2-dimensional reduced varieties X such that $\mathrm{Br}(X)$ is not a torsion group.

Let X be a reduced, excellent, noetherian scheme of dimension 1, and let $\widetilde{X}\to X$ be the normalisation of X. If one lets $\widetilde{x}$ run through the closed points of $\widetilde{X}$ above the singular points $x\in X$, one obtains an obvious complex

$$\mathrm{Br}(X)\longrightarrow \mathrm{Br}(\widetilde{X})\oplus\bigoplus_x \mathrm{Br}(k(x))\longrightarrow\bigoplus_{\widetilde{x}}\mathrm{Br}(k(\widetilde{x})),$$

where x runs through the closed points of X. The proposition implies that the first map here is injective. One may wonder whether the complex is exact. This has been studied from the Azumaya point of view in [Chi74] and [KO74a]. In the case of a curve over a field k of characteristic zero, this will be established in Section 8.5. The proof there relies on a closer inspection of the sheaves F_x.

3.7 Purity for the Brauer group

The results of this section were obtained by Grothendieck in the case of smooth varieties over a field for the torsion prime to the characteristic of the field. Thanks to Gabber's absolute purity (Theorem 2.3.1) we can state Grothendieck's purity theorem for the Brauer group in a more general form.

Theorem 3.7.1 *Let X be a regular, integral, noetherian scheme, let $Z\subset X$ be a regular*[1] *closed subset of pure codimension c. Let $U\subset X$ be the open set $X\smallsetminus Z$. Let ℓ be a prime invertible on X.*

(i) *If $c\geq 2$, then the restriction map $\mathrm{Br}(X)\{\ell\}\to\mathrm{Br}(U)\{\ell\}$ is an isomorphism.*

(ii) *If $c=1$ and $D_1,\ldots,D_m$ are the connected components of Z, then the Gysin exact sequences*

$$\mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n})\longrightarrow\mathrm{H}^2_{\text{ét}}(U,\mu_{\ell^n})\longrightarrow\bigoplus_{i=1}^m\mathrm{H}^1_{\text{ét}}(D_i,\mathbb{Z}/\ell^n)$$

for all $n\geq 1$ give rise to exact sequences

$$0\longrightarrow\mathrm{Br}(X)\{\ell\}\longrightarrow\mathrm{Br}(U)\{\ell\}\longrightarrow\bigoplus_{i=1}^m\ \mathrm{H}^1_{\text{ét}}(D_i,\mathbb{Q}_\ell/\mathbb{Z}_\ell),\qquad(3.15)$$

$$0\longrightarrow\mathrm{Br}(X)\{\ell\}\longrightarrow\mathrm{Br}(U)\{\ell\}\longrightarrow\bigoplus_{i=1}^m\ \mathrm{H}^1(k(D_i),\mathbb{Q}_\ell/\mathbb{Z}_\ell).\qquad(3.16)$$

[1] This regularity condition is missing in [Gro68, Ch. III, §6, formula (6.4) and Thm. 6.1].

Proof. The exact sequence (3.2) based on the Kummer sequence gives rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Pic}(X)/\ell^n & \to & \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n}) & \to & \mathrm{Br}(X)[\ell^n] & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Pic}(U)/\ell^n & \to & \mathrm{H}^2_{\text{ét}}(U,\mu_{\ell^n}) & \to & \mathrm{Br}(U)[\ell^n] & \to 0 \end{array}$$

Since X is regular, the left-hand vertical map is surjective, and the right-hand vertical map $\mathrm{Br}(X)[\ell^n]\to\mathrm{Br}(U)[\ell^n]$ is injective by Theorem 3.5.5.

The snake lemma applied to the above commutative diagram combined with the Gysin exact sequence (2.16) gives the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Br}(X)[\ell^n] \longrightarrow \mathrm{Br}(U)[\ell^n] \longrightarrow \mathrm{H}^{3-2c}_{\text{ét}}(Z,\mu_{\ell^n}) \\ \longrightarrow \mathrm{H}^3_{\text{ét}}(X,\mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(U,\mu_{\ell^n}). \end{aligned} \tag{3.17}$$

We now apply Gabber's purity theorem. If $c \geq 2$, then $\mathrm{H}^{3-2c}_{\text{ét}}(Z,\mu_{\ell^n}) = 0$. In this case the restriction map $\mathrm{Br}(X)\{\ell\}\to\mathrm{Br}(U)\{\ell\}$ is an isomorphism. If $c = 1$, then the regular divisor Z is the disjoint union of its irreducible components $D_1,\ldots,D_m$, and $\mathrm{H}^{3-2c}_{\text{ét}}(D,\mu_{\ell^n}) = \bigoplus_{i=1}^m \mathrm{H}^1_{\text{ét}}(D_i,\mathbb{Z}/\ell)$.

Taking the limit as $n\to\infty$ we obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(U)\{\ell\} \longrightarrow \textstyle\bigoplus_{i=1}^m \mathrm{H}^1_{\text{ét}}(D_i,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ \longrightarrow \mathrm{H}^3_{\text{ét}}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^3_{\text{ét}}(U,\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)). \end{aligned} \tag{3.18}$$

Exact sequence (3.16) follows, since for D_i regular (or even normal), the restriction map $\mathrm{H}^1_{\text{ét}}(D_i,\mathbb{Q}_\ell/\mathbb{Z}_\ell)\to\mathrm{H}^1(k(D_i),\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is injective. □

Theorem 3.7.2 *Let X be an excellent, regular, integral, noetherian scheme, let $U \subset X$ be a dense open set and let $Z := X \setminus U$. Let ℓ be a prime invertible on X. Let c be the codimension of Z in X.*

(i) *If $c \geq 2$, then the restriction map $\mathrm{Br}(X)\{\ell\}\to\mathrm{Br}(U)\{\ell\}$ is an isomorphism.*

(ii) *If $c = 1$ and $D_1,\ldots,D_m$ are the irreducible components of Z of codimension 1 in X, then we have an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(U)\{\ell\} \longrightarrow \bigoplus_{i=1}^m \mathrm{H}^1(k(D_i),\mathbb{Q}_\ell/\mathbb{Z}_\ell). \tag{3.19}$$

Proof. The hypothesis that X is excellent ensures that for any closed $Z \subset X$ the singular locus of Z is closed. For an arbitrary proper closed reduced subscheme $Z \subset X$ we define a descending chain of reduced closed subschemes

$$Z = Z_0 \supset Z_1 \supset Z_2 \supset \ldots$$

as follows. For $n \geq 1$ define Z_n as the union of the singular locus of Z_{n-1} and the union of the irreducible components of Z_{n-1} which have codimension at least $n+1$ in X. Then Z is the disjoint union of locally closed regular subschemes $Z_{n-1} \smallsetminus Z_n$ for $n \geq 1$. We note that $Z_{n-1} \smallsetminus Z_n$ is either empty or of pure codimension n in $X \smallsetminus Z_n$.

Unless Z_0 is regular and of pure codimension 1 in X, the last non-empty complement $Z_{n-1} \smallsetminus Z_n$, where $n \geq 2$, is a closed regular subscheme of X of pure codimension n, thus removing it from X does not affect the ℓ-primary torsion of the Brauer group, as we have seen in the proof of Theorem 3.7.1. Repeating the operation we end up with an isomorphism $\mathrm{Br}(X)\{\ell\} \cong \mathrm{Br}(X \smallsetminus Z_1)\{\ell\}$. If $Z = Z_1$, we are done. Otherwise, we can apply the exact sequence (3.16) to the regular subscheme $Z \smallsetminus Z_1$ of $X \smallsetminus Z_1$ to obtain (3.19). □

Theorem 3.7.3 *Let X be an excellent, regular, integral, noetherian scheme with generic point* $\mathrm{Spec}(F)$ *and let ℓ be a prime invertible on X. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Br}(X)\{\ell\} \longrightarrow \mathrm{Br}(F)\{\ell\} \xrightarrow{\{\partial_D\}} \bigoplus_{D \in X^{(1)}} \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell), \qquad (3.20)$$

where $k(D)$ denotes the residue field at the generic point of D. For each D the Gysin residue $\partial_D \colon \mathrm{Br}(F)\{\ell\} \to \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ *coincides with the Witt residue.*

Proof. The sequence is obtained by passing to the direct limit over all open sets $U \subset X$ in (3.19). Let us justify the assertion regarding the maps

$$\partial_D \colon \mathrm{Br}(F)\{\ell\} \longrightarrow \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

We have been using the Gysin sequence in étale cohomology with finite coefficients. The embedding $i_D \colon \mathrm{Spec}(k(D)) \to X$ of the generic point of an integral divisor $D \subset X$ factors as

$$\mathrm{Spec}(k(D)) \to \mathrm{Spec}(\widehat{\mathcal{O}_{X,D}}) \to \mathrm{Spec}(\mathcal{O}^{\mathrm{h}}_{X,D}) \to \mathrm{Spec}(\mathcal{O}_{X,D}) \to X,$$

where $\widehat{\mathcal{O}_{X,D}}$ is the completion and $\mathcal{O}^{\mathrm{h}}_{X,D}$ is the henselisation of the local ring $\mathcal{O}_{X,D}$ (the henselisation and the completion of a noetherian local ring do not affect the residue field). The Gysin residue $\partial_D \colon \mathrm{Br}(F)\{\ell\} \to \mathrm{H}^1(k(D), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ can be computed with respect to any of these three discrete valuation rings with residue field $k(D)$. By Theorems 2.3.5 and 1.4.14 it equals the Witt residue. □

Remark 3.7.4 Theorem 3.6.1 deals with regular schemes of dimension 1 using étale cohomology with coefficients $\mathbb{G}_m$. The residue there is also identified with the Witt residue.

It is important to understand the functorial behaviour of residues.

Theorem 3.7.5 *Let X be a regular, integral, noetherian scheme. Let Y be a regular irreducible divisor in X. Let X' be a regular, integral, noetherian scheme and let $f\colon X' \to X$ be a morphism such that $f(X')$ is not contained in Y. The divisor $f^{-1}(Y) \subset X'$ can be written as a finite sum $\sum_{t\in T} r_t Z_t$, where $Z_t \subset X'$ is an irreducible divisor and r_t is a positive integer, for $t \in T$.*

Let ℓ be a prime invertible on X. For any $\alpha \in \mathrm{Br}(X \smallsetminus Y)\{\ell\}$ and any $t \in T$ the residue $\partial_{Z_t}(f^*(\alpha))$ is the image of $r_t\partial_Y(\alpha)$ under the composite map

$$\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1_{\text{ét}}(Z_t, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^1(k(Z_t), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Proof. Let $U = X \smallsetminus Y$ and let $U' = f^{-1}(U) = X' \smallsetminus f^{-1}(Y)$. Let Z' be a regular dense open subset of Z_t. By removing a closed subset from X' we can assume that $X' \smallsetminus U' = Z'$.

Let $m \geq 1$ be such that $\ell^m\alpha = 0$. Then α comes from some element $\widetilde{\alpha} \in \mathrm{H}^2_{\text{ét}}(U, \mu_{\ell^m})$. We have $f^*\widetilde{\alpha} \in \mathrm{H}^2_{\text{ét}}(U', \mu_{\ell^m})$. As (X, Y) and (X', Z') are regular pairs of codimension 1, we have the associated Gysin sequences. The commutative diagram from Lemma 2.3.6 implies that $\partial_{Z'}(f^*\widetilde{\alpha}) = r(f^*\partial_Y(\widetilde{\alpha}))$. The proof is finished by taking the restriction to the generic point $\mathrm{Spec}(k(Z_t)) = \mathrm{Spec}(k(Z'))$. □

The following general result, many special cases of which had been established earlier, was recently proved by Česnavičius [Čes19].

Theorem 3.7.6 *Let X be a regular, integral, noetherian scheme. Let $U \subset X$ be an open subset whose complement is of codimension at least 2. Then the restriction map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(U)$$

is an isomorphism.

For the ℓ-primary subgroup of the Brauer group, where ℓ is a prime invertible on X, this is a special case of Theorem 3.7.2, itself a consequence of Gabber's purity theorem. Česnavičius' proof uses the result in dimension ≤ 2 (Auslander–Goldman, Grothendieck [Gro68, II, Thm. 2.1]), the result in dimension 3 (Gabber [Gab81, Thm. 2', p. 131]), Theorem 3.7.2 and other results by Gabber, as well as Scholze's recent theory of perfectoid spaces and tilting equivalence to handle p-torsion in the local unequal characteristic case. For a further extension of this purity result to singular schemes, see [ČS19, Thm. 7.2.8].

As an easy consequence of Theorem 3.7.6, we obtain

Theorem 3.7.7 *Let X be a regular, integral, noetherian scheme with function field F. Then $\mathrm{Br}(X) \subset \mathrm{Br}(F)$ is the subgroup*

$$\bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x}).$$

Proof. The inclusion $\mathrm{Br}(X) \subset \bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x}) \subset \mathrm{Br}(F)$ is clear. Suppose that $\alpha \in \mathrm{Br}(F)$ is contained in $\bigcap_{x\in X^{(1)}} \mathrm{Br}(\mathcal{O}_{X,x})$. Using the fact that the

Brauer group commutes with limits (Section 2.2.2), one finds a non-empty open set $U \subset X$ and an element $\beta \in \mathrm{Br}(U)$ such that β maps to $\alpha \in \mathrm{Br}(F)$. Let U be a maximal open subset of X with this property. Suppose that there exists a codimension 1 point $x \in X$ which is not in U. Since α is in the image of $\mathrm{Br}(\mathcal{O}_{X,x})$, there exists an open set $V \subset X$ containing x and an element $\gamma \in \mathrm{Br}(V)$ that maps to $\alpha \in \mathrm{Br}(F)$. Consider the Mayer–Vietoris exact sequence (Theorem 3.2.4)

$$\mathrm{Br}(U \cup V) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(U \cap V).$$

Since X is regular, by Theorem 3.5.5 the map $\mathrm{Br}(U\cap V)\to\mathrm{Br}(F)$ is injective. Thus there exists an element $\delta \in \mathrm{Br}(U \cup V)$ that restricts to α. Since $x \notin U$, we have a contradiction. Thus the complement to U in X has codimension at least 2. By the purity theorem (Theorem 3.7.6) the inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(U)$ is an equality. □

This immediately implies

Proposition 3.7.8 *Let X be a regular, integral, noetherian scheme with function field F. Let $A_i \subset F$, for $i \in I$, be the discrete valuation rings with fraction field F which lie over X, that is, such that the map* $\mathrm{Spec}(F)\to X$ *factors through* $\mathrm{Spec}(F)\to\mathrm{Spec}(A)$. *Then*

$$\mathrm{Br}(X) = \bigcap_{i\in I} \mathrm{Br}(A_i) \subset \mathrm{Br}(F).$$

Proposition 3.7.9 *Let S be a scheme, let X be a regular, integral, noetherian scheme with function field F, and let $X\to S$ be a proper morphism. Let $A_i \subset F$, $i \in I$, be the discrete valuation rings with fraction field F which lie over S, that is, such that the composition* $\mathrm{Spec}(F)\to X\to S$ *factors through* $\mathrm{Spec}(F)\to\mathrm{Spec}(A)$. *Then*

$$\mathrm{Br}(X) = \bigcap_{i\in I} \mathrm{Br}(A_i) \subset \mathrm{Br}(F).$$

Proof. The morphism $X\to S$ is proper, in particular, it is separated and of finite type. By the valuative criterion of properness [EGA, II, Thm. 7.3.8], [Stacks, Lemma 0BX5] there exists a unique morphism $\mathrm{Spec}(A)\to X$ such that the composition $\mathrm{Spec}(F)\to X\to S$ factors as

$$\mathrm{Spec}(F) \longrightarrow \mathrm{Spec}(A) \longrightarrow X \longrightarrow S.$$

It remains to apply Proposition 3.7.8. □

This proposition can be applied to a smooth, proper, integral variety X over a field k to deduce the birational invariance of $\mathrm{Br}(X)$, see Proposition 6.2.7.

Proposition 3.7.10 *Let S be a scheme. Let X and Y be regular, integral, proper, noetherian S-schemes, with function fields F_X and F_Y, respectively. Suppose that there exists an S-isomorphism $g\colon F_X \xrightarrow{\sim} F_Y$. Then the induced isomorphism $\mathrm{Br}(F_X) \xrightarrow{\sim} \mathrm{Br}(F_Y)$ restricted to the subgroup $\mathrm{Br}(X)$ is an isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Br}(Y)$ compatible with natural maps $\mathrm{Br}(S)\to\mathrm{Br}(X)$ and $\mathrm{Br}(S)\to\mathrm{Br}(Y)$.*

Proof. Note that in Proposition 3.7.9 the collection of A_i, $i \in I$, is defined solely in terms of the morphism $\mathrm{Spec}(F)\to S$. Therefore, an isomorphism of S-schemes $\mathrm{Spec}(F_Y) \cong \mathrm{Spec}(F_X)$ gives rise to the desired isomorphisms. □

3.8 The Brauer group and finite morphisms

Let X be a connected scheme. Let $f\colon Y\to X$ be a finite locally free morphism of schemes. This means that locally for the Zariski topology on X the morphism is of the form $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$, where B a free A-module of finite rank. Since X is connected, this rank is constant; let us denote it by d. If X is locally noetherian, the hypothesis on f is equivalent to f being flat and finite.

The norm of an element $b \in B$ is the determinant of the matrix that gives the multiplication by b on B with respect to some A-basis of B. It does not depend on the basis. The norm is multiplicative; the norm of $a \in A$ is a^d. We obtain a map of quasi-coherent sheaves $f_*\mathcal{O}_Y\to\mathcal{O}_X$. The composition of the canonical map $\mathcal{O}_X\to f_*\mathcal{O}_Y$ with $f_*\mathcal{O}_Y\to\mathcal{O}_X$ sends u to u^d, cf. [Mum66, Lecture 10]. Recall that the étale sheaf $\mathbb{G}_{m,X}$ is defined by setting $\mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U)^*$ for any étale morphism $U\to X$, and similarly for $\mathbb{G}_{m,Y}$. We thus obtain natural morphisms of étale sheaves

$$\mathbb{G}_{m,X} \longrightarrow f_*\mathbb{G}_{m,Y} \longrightarrow \mathbb{G}_{m,X},$$

whose composition sends u to u^d. By the finiteness of f, the functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact [Mil80, Cor. II.3.6], [SGA4, Prop. VIII.5.5]. Thus the Leray spectral sequence (2.4) gives an isomorphism $\mathrm{H}^n_{\text{ét}}(X, f_*\mathbb{G}_{m,Y}) \xrightarrow{\sim} \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y})$ which identifies the canonical map (3.1) with $\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X})\to\mathrm{H}^n_{\text{ét}}(X, f_*\mathbb{G}_{m,Y})$. We thus obtain the *restriction* and *corestriction* maps

$$\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X}) \xrightarrow{\mathrm{res}_{Y/X}} \mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y}) \xrightarrow{\mathrm{cores}_{Y/X}} \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X})$$

whose composition is multiplication by d. Here the restriction $\mathrm{res}_{Y/X}$ is the canonical map $f^*\colon \mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_{m,X})\to\mathrm{H}^n_{\text{ét}}(Y, \mathbb{G}_{m,Y})$. For $n = 2$ we obtain the restriction and corestriction maps of Brauer groups

$$\mathrm{res}_{Y/X}\colon \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y), \quad \mathrm{cores}_{Y/X}\colon \mathrm{Br}(Y)\to\mathrm{Br}(X).$$

The following proposition, which will be used in Section 6.4, is a standard formalism that applies to various functors.

Proposition 3.8.1 *Let Y and X be schemes and let $f\colon Y\to X$ be a finite locally free morphism of constant rank. Let $i\colon V\to X$ be a morphism and let $W = V\times_X Y$. Let $j\colon W\to Y$ and $g\colon W\to V$ be the natural projections; here g is a finite locally free morphism of constant rank. The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Br}(Y) & \xrightarrow{j^*} & \mathrm{Br}(W) \\ \downarrow{\scriptstyle \mathrm{cores}_{Y/X}} & & \downarrow{\scriptstyle \mathrm{cores}_{W/V}} \\ \mathrm{Br}(X) & \xrightarrow{i^*} & \mathrm{Br}(V) \end{array}$$

Proof. We have $fj = ig$, hence $f_*j_*\mathbb{G}_{m,W} = i_*g_*\mathbb{G}_{m,W}$. There is a commutative diagram of étale sheaves on X

$$\begin{array}{ccc} f_*\mathbb{G}_{m,Y} & \longrightarrow & i_*g_*\mathbb{G}_{m,W} \\ \downarrow & & \downarrow \\ \mathbb{G}_{m,X} & \longrightarrow & i_*\mathbb{G}_{m,V} \end{array} \tag{3.21}$$

where the left vertical arrow is the norm map associated to f and the right vertical arrow is induced by the norm map $g_*\mathbb{G}_{m,W}\to\mathbb{G}_{m,V}$. Applying cohomology to (3.21), we see that the bottom left square of the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^2_{\text{ét}}(Y,\mathbb{G}_{m,Y}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(Y,j_*\mathbb{G}_{m,W}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(W,\mathbb{G}_{m,W}) \\ \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} & & \uparrow{\scriptstyle\cong} \\ \mathrm{H}^2_{\text{ét}}(X,f_*\mathbb{G}_{m,Y}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X,i_*g_*\mathbb{G}_{m,W}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(V,g_*\mathbb{G}_{m,W}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_{m,X}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X,i_*\mathbb{G}_{m,V}) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(V,\mathbb{G}_{m,V}) \end{array}$$

The right-hand horizontal and the top vertical arrows are natural maps $E_2^{2,0}\to E^2$ in the spectral sequence attached to a morphism. In the case of top vertical maps these are finite morphisms f and g, hence the functor f_* from the category of étale sheaves on Y to the category of étale sheaves on X is exact [Mil80, Cor. II.3.6], and the same applies to g_*. Thus the top vertical maps are isomorphisms. The bottom vertical maps are induced by the norm maps $f_*\mathbb{G}_{m,Y}\to\mathbb{G}_{m,X}$ and $g_*\mathbb{G}_{m,W}\to\mathbb{G}_{m,V}$. All this ensures that the whole diagram is commutative.

Retaining the four corners of the last diagram we obtain the commutative diagram of the proposition. □

The above definitions of restriction and corestriction can be applied to the case when X is a scheme over a field k. A finite (not necessarily separable) extension $k \subset L$ gives rise to a finite locally free morphism $X \times_k L \to X$ of rank $[L : k]$, so we obtain the restriction and corestriction maps

$$\mathrm{Br}(X) \xrightarrow{\mathrm{res}_{L/k}} \mathrm{Br}(X \times_k L) \xrightarrow{\mathrm{cores}_{L/k}} \mathrm{Br}(X)$$

whose composition is multiplication by $[L : k]$.

In the particular case $X = \mathrm{Spec}(k)$ we obtain the corestriction map

$$\mathrm{cores}_{L/k} \colon \mathrm{Br}(L) \longrightarrow \mathrm{Br}(k).$$

The composition $\mathrm{cores}_{L/k} \circ \mathrm{res}_{L/k}$ is multiplication by $[L : k]$ on $\mathrm{Br}(k)$.

One application is the following proposition.

Proposition 3.8.2 *Let K be a field of transcendence degree 1 over a separably closed field k of characteristic $p > 0$. Then $\mathrm{Br}(K)$ is a p-primary divisible torsion group.*

Proof. Note that $\mathrm{Br}(K) = p\,\mathrm{Br}(K)$ by Theorem 1.3.7. Take any $\alpha \in \mathrm{Br}(K)$. There is an integral curve C over k such that $K = k(C)$. Let $\bar{k}$ be an algebraic closure of k and let $\overline{C} = C \times_k \bar{k}$. Since k is separably closed, the curve $\overline{C}$ is irreducible [EGA, IV_2, Prop. 4.5.9 (c)], but $\overline{C}$ is not reduced if K is not a separable k-algebra, i.e., if $K \otimes_k \bar{k}$ is not reduced. The function field of the reduced $\bar{k}$-curve $(\overline{C})_{\mathrm{red}}$ is $(K \otimes_k \bar{k})_{\mathrm{red}}$, which is the quotient of $K \otimes_k \bar{k}$ by its nilradical. By Tsen's Theorem 1.2.14, the image of α in $\mathrm{Br}((K \otimes_k \bar{k})_{\mathrm{red}})$ is zero. By Proposition 3.2.5 (i), the image of α in $\mathrm{Br}(K \otimes_k \bar{k})$ is zero. Since $\mathrm{Br}(K \otimes_k \bar{k})$ is the inductive limit of the groups $\mathrm{Br}(K \otimes_k L)$, where $L \subset \bar{k}$ is a finite field extension of k, there is a finite extension L/k such that α goes to zero in $\mathrm{Br}(K \otimes_k L)$. Since k is separably closed, L is a purely inseparable extension of k, hence $[L : k] = p^n$ for some $n \geq 0$. By the corestriction-restriction formula, the composition

$$\mathrm{Br}(K) \longrightarrow \mathrm{Br}(K \otimes_k L) \longrightarrow \mathrm{Br}(K)$$

is multiplication by $[L : k] = p^n$, so $p^n \alpha = 0$. □

Remark 3.8.3 If the field k is separably closed but not algebraically closed, then $\mathrm{Br}(\mathbb{A}^1_k) \neq 0$ by Theorem 5.6.1 (vi) below. From Theorem 3.5.5 we deduce that $\mathrm{Br}(k(\mathbb{A}^1_k)) \neq 0$.

Proposition 3.8.4 *Let X and Y be regular, integral, noetherian schemes and let $f \colon Y \to X$ be a dominant, generically finite morphism of degree d. Then the kernel of the natural map $f^* \colon \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is killed by d. In particular, for any integer $n > 1$ coprime to d the map $f^* \colon \mathrm{Br}(X)[n] \to \mathrm{Br}(Y)[n]$ is injective.*

Proof. By Theorem 3.5.5 the embedding of the generic point $\mathrm{Spec}(k(X))$ in X induces an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$, and similarly for Y. Since the composition of restriction and corestriction

$$\mathrm{cores}_{k(Y)/k(X)} \circ \mathrm{res}_{k(Y)/k(X)} \colon \mathrm{Br}(k(X)) \longrightarrow \mathrm{Br}(k(Y)) \longrightarrow \mathrm{Br}(k(X))$$

is the multiplication by d, the kernel of the natural map $f^* \colon \mathrm{Br}(X) \to \mathrm{Br}(Y)$ is killed by d, so our statement follows. □

Theorem 3.8.5 *Let X and Y be regular, integral, noetherian schemes and let $f\colon Y \to X$ be a finite flat morphism of degree d such that $k(Y)$ is a Galois extension of $k(X)$ with Galois group G. Then we have*

$$d\,\mathrm{Br}(Y)^G \subset f^*\mathrm{Br}(X) \subset \mathrm{Br}(Y)^G.$$

In particular, for any integer $n > 1$ coprime to $d = |G|$ the natural map $f^*\colon \mathrm{Br}(X)[n] \to \mathrm{Br}(Y)[n]^G$ is an isomorphism.

Proof. For a non-empty affine open set $\mathrm{Spec}(A) \subset X$, the inverse image in Y is an affine scheme $\mathrm{Spec}(B)$. The ring B is regular hence normal, is finite over A, and its fraction field is $k(Y)$. Hence B is the integral closure of A in $k(Y)$. Thus the action of G on $k(Y)$ induces an action of G on B. Covering X by affine open sets, we see that the action of G on $k(Y)$ induces an action of G on Y. This induces an action of G on $\mathrm{Br}(Y)$.

We claim that the composition

$$\mathrm{res}_{Y/X} \circ \mathrm{cores}_{Y/X} \colon \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y)$$

is given by the formula

$$\alpha \mapsto \sum_{\sigma \in G} \sigma^*(\alpha).$$

Since X and Y are regular and noetherian, the embedding of the generic point into X induces an injective map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$, and there is a similar injective map for Y. The claim is thus reduced to a similar claim for a finite Galois extension of fields, which is well known, see [GS17, Ch. 3, Exercise 3].

Thus for $\alpha \in \mathrm{Br}(Y)^G$ we obtain

$$\mathrm{res}_{Y/X} \circ \mathrm{cores}_{Y/X}(\alpha) = \sum_{\sigma \in G} \sigma^*(\alpha) = d\alpha \in \mathrm{Br}(Y).$$

Thus $d\alpha = f^*(\mathrm{cores}_{Y/X}(\alpha))$ belongs to $f^*(\mathrm{Br}(X)) \subset \mathrm{Br}(Y)$.

In the last statement of the theorem, the surjectivity is clear since $\mathrm{Br}(Y)^G[n] \subset d\mathrm{Br}(Y)^G$. The injectivity follows from Proposition 3.8.4. □

The following lemma will be used in Section 6.4.

Lemma 3.8.6 *Let k be a field and let A be a finite-dimensional commutative k-algebra. Write $A \simeq \prod_{i=1}^m A_i$, where each A_i is a local k-algebra. For $i = 1, \ldots, m$, let k_i be the residue field of A_i, and let $n_i = \dim_k(A_i)/[k_i : k]$. For $\alpha \in \mathrm{Br}(A)$ write $\alpha_i \in \mathrm{Br}(k_i)$ for the image of α under the evaluation map $\mathrm{Br}(A) \to \mathrm{Br}(k_i)$. Then we have*

$$\mathrm{cores}_{A/k}(\alpha) = \sum_{i=1}^m n_i(\mathrm{cores}_{k_i/k}(\alpha_i)) \ \in \ \mathrm{Br}(k).$$

Proof. It is clearly enough to consider the case when A is a local k-algebra. Let $\mathfrak{m} \subset A$ be the maximal ideal and let $\kappa = A/\mathfrak{m}$ be the residue field. We need to calculate the map $\det\colon A^* \to k^*$ and the induced map

$$\det_*\colon \mathrm{H}^2(k, (A \otimes_k k_\mathrm{s})^*) \longrightarrow \mathrm{H}^2(k, k_\mathrm{s}^*). \tag{3.22}$$

The k-vector space A is a finite direct sum of $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ for $i \geq 0$. Each summand $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is naturally a κ-vector space. Let $r_i = \dim_\kappa(\mathfrak{m}^i/\mathfrak{m}^{i+1})$. There is a k-vector space V_i, with $\dim_k(V_i) = r_i$, such that there is an isomorphism of κ-vector spaces $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq V_i \otimes_k \kappa$. Let $r = \sum_{i \geq 0} r_i$.

For a k-vector space V write $\mathrm{GL}(V, k)$ for the group of k-linear transformations of V. The subgroup $A^* \subset \mathrm{GL}(A, k)$ preserves the flag $\mathfrak{m} \supset \mathfrak{m}^2 \supset \ldots$, hence the determinant of an element $x \in A^*$ is the product of $\det(x_i)$, where x_i is the image of x in $\mathrm{GL}(\mathfrak{m}^i/\mathfrak{m}^{i+1}, k)$, for all $i \geq 0$. But x acts on $\mathfrak{m}^i/\mathfrak{m}^{i+1} \simeq V_i \otimes_k \kappa$ via the factor κ, on which it acts by multiplication by x mod $\mathfrak{m}$. Hence the map $\det\colon A^* \to k^*$ is the composition of the following three maps: reduction modulo $\mathfrak{m}$, the norm $\mathrm{N}_{\kappa/k}$, and raising to the power r.

The induced map (3.22) is therefore the composition of the corresponding induced maps. The first of these is an isomorphism $\mathrm{Br}(A) \xrightarrow{\sim} \mathrm{Br}(\kappa)$, by Azumaya's theorem (Theorem 3.4.2 (i), which is applicable since an Artinian local ring A is complete). The norm map $\mathrm{N}_{\kappa/k}\colon (\kappa \otimes_k k_\mathrm{s})^* \to k_\mathrm{s}^*$ induces the corestriction map $\mathrm{cores}_{\kappa/k}\colon \mathrm{Br}(\kappa) \to \mathrm{Br}(k)$. This gives the desired formula in the local case, thus finishing the proof. □

Chapter 4
Comparing the two Brauer groups, II

This section is devoted to the proof of Gabber's Theorem 4.2.1, which describes the image of the natural map $\mathrm{Br}_{\mathrm{Az}}(X)\to\mathrm{Br}(X)$. In Section 4.2 we reproduce a proof of this theorem found by de Jong. It uses the language of stacks and gerbes, which we briefly introduce in Section 4.1.

4.1 The language of stacks

Our goal here is to give a very short list of key concepts with some examples. This is not a replacement for a detailed introduction to stacks, algebraic spaces and gerbes, for which we refer the reader to a very helpful book by Olsson [Ols16], see also [SGA1, Ch. VI], [Gir71], [LMB00], [Vis05] and [Stacks].

4.1.1 Fibred categories

We start with the definition of a fibred category, see [Ols16, §3.1] and [Stacks, Section 02XJ].

Let C be a category. (We shall mostly be interested in the case when C is the category Sch/S of schemes over a base scheme S.)

Definition 4.1.1 *A* **category over** C *is a pair* (F, p) *where* F *is a category and* $p\colon F\to C$ *is a functor. For an object* U *of* C *the* **fibre** $F(U)$ *over* U *is the category whose objects are the objects* u *of* F *over* U*, i.e. such that* $p(u)=U$*, and whose morphisms are morphisms in* F *that lift* $\mathrm{id}\colon U\to U$*.*

Definition 4.1.2 *A morphism* $\phi\colon u\to v$ *in* F *is* **cartesian** *if for any object* w *in* F*, a morphism* $\psi\colon w\to v$ *and a factorisation* $p(w)\xrightarrow{h}p(u)\xrightarrow{p(\phi)}p(v)$ *of* $p(\psi)$*, there exists a unique morphism* $\lambda\colon w\to u$ *in* F *such that* $p(\lambda)=h$ *and* $\phi\circ\lambda=\psi$*.*

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In this case u is called a *pullback of v along $f = p(\phi)$* and is denoted by $u = f^*v$; it is unique up to a unique isomorphism.

Definition 4.1.3 *A* **fibred category over** *C is a category $p\colon F \to C$ over C such that for every morphism $f\colon U \to V$ in C and for every $v \in F(V)$ there exist an object $u \in F(U)$ and a cartesian morphism $\phi\colon u \to v$ which lifts f, i.e., $p(\phi) = f$.*

A morphism of fibred categories $p\colon F \to C$ to $q\colon G \to C$ is a functor $g\colon F \to G$ sending cartesian morphisms to cartesian morphisms such that there is an equality of functors $p = q \circ g$.

Example 4.1.4 Let X be an object of a category C. Write C/X for the localisation of C at the object X. This is the category whose objects are the pairs (Y, f) where Y is an object of C and f is a morphism $Y \to X$, and the morphisms are the morphisms in C that make the obvious triangles commutative. The forgetful functor $C/X \to C$ is a fibred category.

Example 4.1.5 Let $F\colon C^{\rm op} \to (Sets)$ be a contravariant functor from a category C to the category of sets. Let $\mathcal{F}$ be the category of pairs (U, x), where U is an object of C and $x \in F(U)$. A morphism $(U', x') \to (U, x)$ is a morphism $g\colon U' \to U$ such that $F(g)x = x'$. It is easy to check that the functor $\mathcal{F} \to C$ sending (U, x) to U is a fibred category. This allows one to view presheaves of sets as categories fibred in sets, see [Ols16, Prop. 3.2.8]. We shall return to this example in the particular case when C is the category of schemes over a base scheme S.

Categories fibred in groupoids

The references are [Ols16, §3.4] and [Stacks, Section 003S].

Definition 4.1.6 *A fibred category $p\colon F \to C$ is a* **category fibred in groupoids** *if the fibre $F(U)$ is a groupoid for every U in C, i.e., every morphism in $F(U)$ is an isomorphism.*

Equivalently, $p\colon F \to C$ is a category fibred in groupoids if and only if every morphism in F is cartesian [Ols16, Exercise 3.D, p. 85]. For a given object x of F, the functor p gives rises to an equivalence of the localised categories F/x and $C/p(x)$.

Example 4.1.7 To a group G one canonically associates a groupoid $\mathcal{G}$ with one object and morphisms given by the elements of the group. A homomorphism of groups $p\colon G_1 \to G_2$ gives rise to a functor $p\colon \mathcal{G}_1 \to \mathcal{G}_2$. It makes $\mathcal{G}_1$ a category fibred over $\mathcal{G}_2$ if and only if p is surjective. Then $\mathcal{G}_1$ is a category fibred in groupoids. In this case, the fibre of $p\colon \mathcal{G}_1 \to \mathcal{G}_2$ is the groupoid associated to the kernel of p.

Let $p\colon F\to C$ be a category fibred in groupoids. For X in C and for objects x_1 and x_2 in $F(X)$ define the functor

$$\underline{\mathrm{Isom}}(x_1, x_2)\colon (C/X)^{\mathrm{op}} \longrightarrow (Sets)$$

that associates to $f\colon Y\to X$ the set $\mathrm{Isom}_{F(Y)}(f^*x_1, f^*x_2)$, for some chosen pullbacks f^*x_1 and f^*x_2 along f. The definition of a category fibred in groupoids then implies that a morphism $g\colon Z\to Y$ gives rise to a canonical map

$$\underline{\mathrm{Isom}}(x_1, x_2)(f\colon Y\to X) \longrightarrow \underline{\mathrm{Isom}}(x_1, x_2)(fg\colon Z\to X),$$

so this is indeed a functor. Up to a canonical isomorphism it does not depend on the choice of pullbacks.

As a particular case, for an object x of $F(X)$ we get a functor

$$\underline{\mathrm{Aut}}_x := \underline{\mathrm{Isom}}(x, x)\colon (C/X)^{\mathrm{op}} \longrightarrow (Groups).$$

The construction of Example 4.1.5 associated to $\underline{\mathrm{Isom}}(x_1, x_2)$ a category fibred in sets. Defining morphisms on a set to be the identity maps on its elements, we obtain a category fibred in groupoids associated to $\underline{\mathrm{Isom}}(x_1, x_2)$.

Yoneda's lemma

For an S-scheme X we have the functor of points $h_X\colon (Sch/S)^{\mathrm{op}}\to(Sets)$ defined by $h_X(Y) = \mathrm{Hom}_S(Y, X)$. Yoneda's lemma says that the functor of points is a fully faithful functor $Sch/S\to\mathrm{Hom}((Sch/S)^{\mathrm{op}}, (Sets))$, hence it gives an embedding of Sch/S into the category of contravariant functors from Sch/S to $(Sets)$. Moreover, for any functor $F\colon (Sch/S)^{\mathrm{op}}\to(Sets)$ we have a bijection

$$\mathrm{Hom}(h_X, F) \xrightarrow{\sim} F(X)$$

given by evaluating on the object $\mathrm{id}\colon X\to X$ of $h_X(X)$. This allows one to replace an S-scheme X by its functor of points h_X, which is an object of a larger category. In what follows we often say a 'scheme' instead of a 'functor $(Sch/S)^{\mathrm{op}}\to(Sets)$ representable by a scheme', i.e., a functor isomorphic to h_X where X is an S-scheme.

This operation can be refined as follows. As we have seen, for an S-scheme X the category Sch/X of X-schemes is a category fibred over Sch/S, via the functor that forgets X. This is a replacement for h_X. The 2-Yoneda lemma [Ols16, §3.2], [Stacks, Lemma 004B] says that if $p\colon F\to Sch/S$ is another fibred category, then the functor

$$\xi\colon \mathrm{HOM}_{Sch/S}(Sch/X, F) \longrightarrow F(X)$$

that sends a morphism of fibred categories to the value of this morphism on the object $\mathrm{id}\colon X\to X$ of Sch/X, is an equivalence of categories.

Sheaves on a category fibred in groupoids over a site

Let $p\colon \mathcal{X}\to S_{\text{Ét}}$ be a category fibred in groupoids over the category of schemes over a base scheme S equipped with the étale topology.

Firstly, the site $S_{\text{Ét}}$ induces a site $\mathcal{X}_{\text{Ét}}$ where the coverings are families of morphisms $\{x_i\to x\}_{i\in I}$ in $\mathcal{X}$ such that $\{p(x_i)\to p(x)\}_{i\in I}$ is a covering in $S_{\text{Ét}}$, see [Stacks, Lemma 06NU]. Thus we can talk about sheaves on $\mathcal{X}_{\text{Ét}}$. The functor p induces an equivalence of the localised categories $\mathcal{X}/x$ and $Sch/p(x)$; moreover, $\mathcal{X}_{\text{Ét}}/x$ and $p(x)_{\text{Ét}}$ are equivalent sites [Stacks, Lemma 06W0].

Secondly, we define the structure sheaf $\mathcal{O}_{\mathcal{X}}$ as follows. The structure sheaf $\mathcal{O}$ on $S_{\text{Ét}}$ associates to an S-scheme T the ring $\Gamma(T,\mathcal{O}_T)$. Define $\mathcal{O}_{\mathcal{X}}$ as the sheaf of rings on $\mathcal{X}_{\text{Ét}}$ such that $\mathcal{O}_{\mathcal{X}}(x) = \mathcal{O}(p(x))$, for any object x of $\mathcal{X}$. Thus we can talk about sheaves of $\mathcal{O}_{\mathcal{X}}$-modules. The sites $\mathcal{X}_{\text{Ét}}/x$ and $p(x)_{\text{Ét}}$ are also equivalent as ringed sites, i.e., if one also takes into account their structure sheaves. Now the definitions of various classes of $\mathcal{O}_{\mathcal{X}}$-modules are the standard definitions on a ringed site, see [Stacks, Def. 03DL].

Definition 4.1.8 *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules.*

(i) *$\mathcal{F}$ is* **locally free** *if for every object x in $\mathcal{X}$ there is an étale covering $\{x_i\to x\}_{i\in I}$ such that the restriction of $\mathcal{F}$ to each x_i is a free $\mathcal{O}_{x_i}$-module.*
(ii) *$\mathcal{F}$ is* **finite locally free** *if for every object x in $\mathcal{X}$ there is an étale covering $\{x_i\to x\}_{i\in I}$ such that the restriction of $\mathcal{F}$ to each x_i is isomorphic to $\mathcal{O}_{x_i}^{\oplus n}$ for some n.*
(iii) *$\mathcal{F}$ is of* **finite type** *if for every object x in $\mathcal{X}$ there is an étale covering $\{x_i\to x\}_{i\in I}$ such that the restriction of $\mathcal{F}$ to each x_i is isomorphic to a quotient of $\mathcal{O}_{x_i}^{\oplus n}$ for some n.*
(iv) *$\mathcal{F}$ is* **quasi-coherent** *if for every object x in $\mathcal{X}$ there is an étale covering $\{x_i\to x\}_{i\in I}$ such that the restriction of $\mathcal{F}$ to each x_i is isomorphic to the cokernel of a map of free $\mathcal{O}_{x_i}$-modules.*
(v) *$\mathcal{F}$ is* **coherent** *if $\mathcal{F}$ is of finite type, and for any object x in $\mathcal{X}$ and any n the kernel of any map $\mathcal{O}_x^{\oplus n}\to\mathcal{F}$ is of finite type.*

4.1.2 Stacks

The references for this section are [Ols16, §4.2, §4.6].

Let $p\colon F\to C$ be a category fibred in groupoids, where C has finite fibred products. For a set of morphisms $\{X_i\to X\}_{i\in I}$ in C one defines $F(\{X_i\to X\}_{i\in I})$ to be the category of *descent data*, consisting of objects E_i of $F(X_i)$, for $i\in I$, and isomorphisms $\sigma_{ij}\colon \mathrm{pr}_1^*(E_i)\to\mathrm{pr}_2^*(E_j)$ in $F(X_i\times_X X_j)$, for each $i,j\in I$, satisfying the standard compatibility condition on triple intersections. If the natural functor $F(X)\to F(\{X_i\to X\}_{i\in I})$ is an equivalence of categories, then one says that the set of morphisms $\{X_i \to X\}_{i\in I}$ is of *effective descent* for F.

Now let C be a site, i.e. a category with a Grothendieck topology on it. For example, we can consider $S_{\text{Ét}}$, which is the category Sch/S of schemes with the big étale topology over a base scheme S.

Definition 4.1.9 *A category fibred in groupoids* $p\colon F \to C$ *is a* **stack** *if for any object* X *of* C *every covering family* $\{X_i \to X\}_{i\in I}$ *is of effective descent for* F.

Equivalently [Ols16, Prop. 4.6.2], for any covering of any X in C any descent datum with respect to this covering is effective, and $\underline{\mathrm{Isom}}(x_1, x_2)$ is a sheaf on C/X, for any x_1 and x_2 in $F(X)$. In particular, $\underline{\mathrm{Aut}}_x$ is a sheaf on C/X, for any x in $F(X)$.

Example 4.1.10 *The stack associated to a sheaf on a site.* A set is canonically turned into a groupoid by defining morphisms to be the identity maps on the elements of this set. In Example 4.1.5 we have seen that a functor $f\colon C^{\mathrm{op}} \to (Sets)$ naturally gives rise to a category fibred in sets over C, whose fibre over X is the set $f(X)$. Hence it can be seen as a category fibred in groupoids. If C is a site, then this fibred category over C is a stack if and only if f is a sheaf [Vis05, Prop. 4.9].

Example 4.1.11 *The stack associated to an S-scheme.* The 2-Yoneda lemma allows one to replace an S-scheme X by the fibred category $Sch/X \to Sch/S$. One immediately checks that this is a category fibred in groupoids, more precisely, in sets with the identity maps. Moreover, it is a stack for the various topologies on Sch/S since, by a theorem of Grothendieck, h_X is a sheaf in the fpqc topology, hence also in fppf and big étale topologies [Vis05, Thm. 2.55]. This is also trivially true for the big Zariski topology, since morphisms of schemes can be obtained by glueing morphisms on Zariski open coverings.

4.1.3 Algebraic spaces and algebraic stacks

The definition of algebraic stacks [Ols16, §8.1] uses algebraic spaces, so we need to recall their definition too, see [Ols16, Ch. 5]. See [Ols16, §3.4] for the definition of the (2-categorical) fibred product of categories fibred in groupoids.

Definition 4.1.12 *Let S be a scheme. A morphism of sheaves of sets* $F \to G$ *on* $S_{\text{Ét}}$ *is* **representable by schemes** *if for any S-scheme T and a morphism* $T \to G$ *the fibred product* $F \times_G T$ *is a scheme.*

(Recall that we write T for the sheaf h_T.) If F and G are representable sheaves, say $F = h_X$ and $G = h_Y$, then, by Yoneda's lemma, any morphism

$F \to G$ is induced by a morphism of schemes $X \to Y$, which implies that $F \to G$ is representable by schemes.

Let F be a sheaf of sets on $S_{\text{Ét}}$. An important observation is that if the diagonal morphism $F \to F \times_S F$ is representable by schemes, then any S-morphism $T \to F$, where T is an S-scheme, is representable too. This follows from the isomorphism $T \times_F Z \cong (T \times_S Z) \times_{F \times F} F$, for any S-scheme Z and any S-morphism $Z \to F$.

Let (P) be a stable property of morphisms of schemes. This means that for every covering $\{U_i \to U\}$ of S-schemes, $U \to S$ has property (P) if and only if $U_i \to S$ has property (P) for each i. If F and G are functors $(Sch/S)^{\text{op}} \to (Sets)$, then a morphism of functors $F \to G$ has property (P) if it is representable by schemes, i.e., for every $T \in Sch/S$ and any morphism $T \to G$ the fibred product functor $F \times_G T$ is isomorphic to h_Y for some $Y \in Sch/S$, and the resulting morphism of schemes $Y \to T$ has property (P).

If in the above characterisation of S-schemes as big Zariski sheaves with certain additional properties (Example 4.1.11) we replace the Zariski topology with the big étale topology, we obtain the definition of an algebraic space [Ols16, Def. 5.1.10].

Definition 4.1.13 *A sheaf of sets X on $S_{\text{Ét}}$ is an* **algebraic space** *over S if*

(1) *the diagonal $\Delta\colon X \to X \times_S X$ is representable by schemes, and*
(2) *there is a surjective étale S-morphism $U \to X$, where U is an S-scheme.*

Condition (1) implies that the morphism $U \to X$ in condition (2) is representable by schemes, so the property 'surjective étale' makes sense.

Alternatively, one can define algebraic spaces as quotients of schemes by étale equivalence relations [Ols16, §5.2]. (In particular, this leads to examples of algebraic spaces which are quotients of schemes by free group actions, which may not be schemes.)

Like schemes, algebraic spaces are sheaves for the fpqc and hence for fppf topology on Sch/S. The fpqc property is a recent result of Gabber, see [Stacks, Section 03W8]. The fppf property is an earlier result of M. Artin.

Consider stacks over $S_{\text{Ét}}$. Since an algebraic space is a big étale sheaf, it gives rise to a stack (see Example 4.1.10).

Definition 4.1.14 *A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is* **representable by algebraic spaces** *if for every algebraic space V and every morphism $V \to \mathcal{Y}$ the fibred product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space.*

Definition 4.1.15 *A stack $\mathcal{X}$ over $S_{\text{Ét}}$ is called* **algebraic** *(or an Artin stack) if*

(1) *the diagonal $\Delta\colon \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces, and*
(2) *there exists a smooth surjective S-morphism from an S-scheme to $\mathcal{X}$.*

An algebraic stack $\mathcal{X}$ is a Deligne–Mumford stack if there is a surjective étale S-morphism from an S-scheme to $\mathcal{X}$.

Property (1) is equivalent to the following property: for every S-scheme U and any two objects u_1 and u_2 in $\mathcal{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space [Ols16, Lemma 8.1.8].

Example 4.1.16 *(Quotient stacks)* Important examples of algebraic stacks over $S_{\text{Ét}}$ are quotient stacks [Ols16, Example 8.1.12]. If G is a smooth group S-scheme that acts on an algebraic space X over S, then $[X/G]$ is defined as the stack whose objects are triples $(T, \mathcal{P}, \pi)$, where T is an S-scheme, $\mathcal{P}$ is a sheaf of torsors for $G \times_S T$ on the big étale site of T, and $\pi\colon \mathcal{P} \to X \times_S T$ is a $G \times_S T$-equivariant morphism of sheaves. Using the criterion in the previous paragraph, one shows that $[X/G]$ is an algebraic stack. A smooth covering is obtained from the map $X \to [X/G]$ defined by the trivial G_X-torsor over X.

In the particular case when $X = S$ and G acts trivially on S, the quotient stack $[S/G]$ is called the *classifying stack* of G and is denoted by BG (or, more precisely, by $B_S G$).

We summarise the logical links between the concepts we discussed above in the following diagram:

$$\begin{array}{ccc} (S\text{-schemes}) & \hookrightarrow & (S\text{-algebraic spaces}) \\ & & \downarrow \\ (\text{algebraic stacks over } S_{\text{Ét}}) & \hookleftarrow & (\text{Deligne–Mumford stacks over } S_{\text{Ét}}) \\ \downarrow & & \\ (\text{stacks over } S_{\text{Ét}}) & \hookrightarrow & (\text{categories fibred in groupoids over } Sch/S) \end{array}$$

Let us finally mention that the category of quasi-coherent sheaves on an algebraic stack is an abelian category. Moreover, it is a Grothendieck category, in particular, it has direct sums, tensor products, direct and inverse limits. The dual of a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules locally of finite presentation (for example, finite locally free) is quasi-coherent. See [Stacks, Section 06WU].

4.1.4 Gerbes

The references for this section are [Ols16, §12.2], [deJ], [Lie08], see also [Gir71, Ch. III, §2, Ch. IV, §2].

Let G be a sheaf of *abelian* groups on $S_{\text{Ét}}$. For an S-scheme X, by an abuse of notation, we write G for the sheaf of abelian groups on $X_{\text{Ét}}$ induced by G.

Definition 4.1.17 *A G-***gerbe** *over $S_{\text{Ét}}$ is a stack $p\colon F\to S_{\text{Ét}}$ together with an isomorphism of sheaves of groups on $S_{\text{Ét}}$*

$$\iota_x\colon G \xrightarrow{\sim} \underline{\mathrm{Aut}}_x$$

for every object x in F such that the following conditions hold.

(G1) *Objects exist locally: every S-scheme Y has a covering $\{f_i\colon Y_i\to Y\}$ such that all the $F(Y_i)$ are non-empty.*

(G2) *Any two objects are locally isomorphic: for any objects y and y' in $F(Y)$ there exists a covering $\{f_i\colon Y_i\to Y\}$ such that f_i^*y and f_i^*y' are isomorphic in $F(Y_i)$ for all i.*

(G3) *For every S-scheme Y if $\sigma\colon y\to y'$ is an isomorphism in $F(Y)$, then the induced isomorphism $\sigma\colon \underline{\mathrm{Aut}}_y\to\underline{\mathrm{Aut}}_{y'}$ is compatible with the isomorphisms ι_x, that is, $\iota_{y'}=\sigma\iota_y$.*

By (G1) and (G2) the sheaf $\underline{\mathrm{Isom}}(x_1,x_2)$ is a G-torsor on Sch/X, for every S-scheme X and every x_1 and x_2 in $F(X)$, see [Ols16, Remark 12.2.3].

A *morphism of G-gerbes* is defined as a morphism of stacks $f\colon F'\to F$ such that for every object x of F' the composition $G\xrightarrow{\iota_x}\underline{\mathrm{Aut}}_x\xrightarrow{f_*}\underline{\mathrm{Aut}}_{f(x)}$ is equal to $G\xrightarrow{\iota_{f(x)}}\underline{\mathrm{Aut}}_{f(x)}$. Any morphism of G-gerbes is in fact an isomorphism [Ols16, Lemma 12.2.4], so to prove that two G-gerbes are isomorphic it is enough to construct a G-morphism between them.

A G-gerbe $p\colon F\to S_{\text{Ét}}$ is called *trivial* if it has a global object, i.e., $F(S)$ is non-empty. In this case it is isomorphic to the classifying stack B_SG.

Remark 4.1.18 If G is a smooth group S-scheme, for example $G=\mathbb{G}_m$, then any G-gerbe on $S_{\text{Ét}}$ is an algebraic stack [Ols16, Exercise 12.E].

The gerbe of liftings of a torsor

Let us give an example of a gerbe.

Consider an exact sequence of sheaves of groups on $S_{\text{Ét}}$ (where G is abelian but not necessarily H and K)

$$1\longrightarrow G\xrightarrow{a}H\xrightarrow{b}K\longrightarrow 1. \tag{4.1}$$

A K-torsor P over S gives rise to the G-gerbe over $S_{\text{Ét}}$ whose objects are the liftings of P to an H-torsor.

More precisely, consider the fibred category $\mathcal{S}_P$ over $S_{\text{Ét}}$ whose objects are triples (X, R, ϵ), where X is an S-scheme, R is an H-torsor over X, and ϵ is an isomorphism of the push-forward of R along b (the quotient of R by G) with $P \times_S X$. A morphism $(X', R', \epsilon') \to (X, R, \epsilon)$ is a pair consisting of a morphism of S-schemes $f\colon X' \to X$ and an isomorphism of H-torsors $\widetilde{f}\colon f^*R \to R'$ over X' such that the following diagram commutes:

$$\begin{array}{ccccc} b_* f^* R & \xrightarrow{b_* \widetilde{f}} & b_* R' & \xrightarrow{\epsilon'} & P \times_S X' \\ \cong \downarrow & & & & \downarrow \text{id} \\ f^* b_* R & \xrightarrow{f^* \epsilon} & f^*(P \times_S X) & \xrightarrow{\cong} & P \times_S X' \end{array}$$

It is clear that $\mathcal{S}_P$ is a category fibred in groupoids over $S_{\text{Ét}}$, via the forgetful functor sending (X, R, ϵ) to X, and for any object x of $\mathcal{S}_P$ the sheaf $\underline{\text{Aut}}_x$ is canonically isomorphic to G over $X_{\text{Ét}}$. Using the effectivity of descent for sheaves and for morphisms of sheaves one shows that $\mathcal{S}_P$ is a G-gerbe [Ols16, Prop. 12.2.6].

The gerbe associated to a cohomology class

Using the previous construction, one associates a G-gerbe to any cohomology class $\alpha \in \mathrm{H}^2_{\text{Ét}}(S, G) \cong \mathrm{H}^2_{\text{ét}}(S, G)$. Namely, consider an exact sequence (4.1), where H and K are sheaves of *abelian* groups and H is *injective*. The boundary map induces an isomorphism

$$\mathrm{H}^1_{\text{Ét}}(S, K) \xrightarrow{\sim} \mathrm{H}^2_{\text{Ét}}(S, G).$$

Thus α gives rise to a K-torsor P over S (understood as a sheaf over S); it is unique up to isomorphism. To P we associate the gerbe $\mathcal{S}_P$ of liftings of P to an H-torsor as above. Embedding G into another injective sheaf H' gives rise to an isomorphic gerbe. Indeed, write $K' = H'/G$. Let P' be a K'-torsor that goes to α. Since H' is injective, we get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & K & \longrightarrow & 1 \\ & & \text{id} \downarrow & & r \downarrow & & s \downarrow & & \\ 1 & \longrightarrow & G & \longrightarrow & H' & \longrightarrow & K' & \longrightarrow & 1 \end{array}$$

Then P' is isomorphic to the K'-torsor $s_* P$, since both torsors give rise to the same $\alpha \in \mathrm{H}^2_{\text{Ét}}(S, G)$. So we can assume that $P' = s_* P$. By the commutativity of the diagram, the push-forward s_* of K-torsors to K'-torsors is compatible with the push-forward r_* of H-torsors to H'-torsors; this gives a morphism of G-gerbes $\mathcal{S}_P \to \mathcal{S}_{P'}$, which is necessarily an isomorphism. We denote by $\mathcal{S}_\alpha$ a G-gerbe isomorphic to $\mathcal{S}_P$.

A theorem from Giraud's book [Gir71, Thm. IV.3.4.2 (i)] says that this gives an isomorphism between $\mathrm{H}^2_{\text{Ét}}(S,G)$ and the group of isomorphism classes of G-gerbes over $S_{\text{Ét}}$. See also [Ols16, Thm. 12.2.8].

Suppose that

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 1 \tag{4.2}$$

is an exact sequence of sheaves of abelian groups. Consider the map that sends a section of D to its inverse image under $C\to D$. This inverse image is a B/A-torsor; so we obtain a map $\mathrm{H}^0_{\text{Ét}}(S,D)\to\mathrm{H}^1_{\text{Ét}}(S,B/A)$. (According to [Gir71, III.3.5.5.1], this map is the negative of the map defined using injective resolutions when torsors are assumed to be right torsors; in the case of left torsors the two maps coincide.) Next, associating to a B/A-torsor the gerbe of its liftings to a B-torsor defines a map $\mathrm{H}^1_{\text{Ét}}(S,B/A)\to\mathrm{H}^2_{\text{Ét}}(S,A)$, which is in fact a homomorphism, see [Gir71, IV.3.4.1.1]. By [Gir71, Thm. IV.3.4.2 (ii)], the above identification of $\mathrm{H}^2_{\text{Ét}}(S,A)$ with the isomorphism classes of A-gerbes over $S_{\text{Ét}}$ is such that the composition

$$\mathrm{H}^0_{\text{Ét}}(S,D) \longrightarrow \mathrm{H}^1_{\text{Ét}}(S,B/A) \longrightarrow \mathrm{H}^2_{\text{Ét}}(S,A) \tag{4.3}$$

is the negative of the map defined using injective resolutions (with the same convention that torsors are right torsors).

Let us now assume that B and C in (4.2) are *injective*. Then the first map in (4.3) is surjective with kernel the image of $\mathrm{H}^0_{\text{Ét}}(S,C)$, and the second map is an isomorphism. Thus we can lift $\alpha\in\mathrm{H}^2_{\text{Ét}}(S,A)$ to a section $\tau\in\mathrm{H}^0_{\text{Ét}}(S,D)$. Define the category $\mathcal{S}_\tau$ whose objects are pairs (X,σ), where X is an S-scheme and $\sigma\in\Gamma(X,C)$ lifts the restriction of τ to X. A morphism $(X,\sigma)\to(X',\sigma')$ is a pair consisting of a morphism of S-schemes $f\colon X\to X'$ and a section $\rho\in\Gamma(X,B)$ that maps to $\sigma-f^*\sigma'$. This implies that the composition of morphisms

$$(X,\sigma)\xrightarrow{(f,\rho)}(X',\sigma')\xrightarrow{(f',\rho')}(X'',\sigma'')$$

is the morphism $(f'\circ f,\rho+f^*\rho')$. The forgetful functor $\mathcal{S}_\tau\to Sch/S$ makes $\mathcal{S}_\tau$ a category fibred in groupoids (the restriction of sheaves with respect to $X'\to X$ defines a natural pullback functor), which is in fact a stack. It is also possible to prove directly that this stack is algebraic. Indeed, there is an étale morphism $U\to S$ that trivialises α. Then there is a lifting $\sigma\in\Gamma(U,C)$ of τ; it gives rise to a smooth surjective morphism $U\to\mathcal{S}_\tau$. Finally, an automorphism of (X,σ) is a pair (id,ρ), where $\rho\in\Gamma(X,A)$. Thus $\underline{\mathrm{Aut}}_{(X,\sigma)}\cong A_X$, and so $\mathcal{S}_\tau$ is an A-gerbe.

Let P be a B/A-torsor over S representing the class in $\mathrm{H}^1_{\text{Ét}}(S,B/A)$ corresponding to $\alpha\in\mathrm{H}^2_{\text{Ét}}(S,A)$ under the second map in (4.3) (which is an isomorphism). The gerbes $\mathcal{S}_P$ and $\mathcal{S}_\tau$ are isomorphic. Indeed, since B is injective, every B-torsor is trivial, so given a triple (X,R,ϵ) as above, a section of R over X gives rise to a section of P over X, which is a lifting of τ (since P

is the inverse image of τ in C). In this way we obtain a morphism of A-gerbes, which must be an isomorphism.

4.1.5 Twisted sheaves

For each $\alpha \in \mathrm{H}^2_{\text{Ét}}(S, \mathbb{G}_m)$ we denote by $\pi\colon \mathcal{S}_\alpha \to S$ a $\mathbb{G}_m$-gerbe $\mathcal{S}_P$ over $S_{\text{Ét}}$ as above; it is well defined up to isomorphism. By Giraud's theorem, this gives a bijection between the group $\mathrm{H}^2_{\text{Ét}}(S, \mathbb{G}_m)$ and the set of isomorphism classes of $\mathbb{G}_m$-gerbes over $S_{\text{Ét}}$. By Remark 4.1.18, $\mathcal{S}_\alpha$ is an algebraic stack over $S_{\text{Ét}}$.

Twisted sheaves on a gerbe were studied by Giraud [Gir71] and more recently by Lieblich [Lie08].

Let n be an integer. A quasi-coherent sheaf of $\mathcal{O}_{\mathcal{S}_\alpha}$-modules $\mathcal{F}$ on $\mathcal{S}_\alpha$ is called an *n-twisted sheaf* if for any object x of $\mathcal{S}_\alpha$, the natural action of the group of sections of $\underline{\mathrm{Aut}}_x \cong \mathbb{G}_{m,x}$ on the group of sections of $\mathcal{F}$ over x is via the character $t \mapsto t^n$.

We refer to [Ols16, Lemma 12.3.3] (which only treats finite locally free twisted sheaves) and [Lie08, Lemma 3.1.1.7] for the following facts.

- The tensor product of an n-twisted sheaf and an m-twisted sheaf is an $(n+m)$-twisted sheaf.
- The dual of an n-twisted finite locally free sheaf is a $(-n)$-twisted finite locally free sheaf.
- The functor π^* sends finite locally free $\mathcal{O}_S$-modules to 0-twisted finite locally free sheaves on $\mathcal{S}_\alpha$, and induces an equivalence of these categories. In particular, if $\mathcal{E}$ is an n-twisted finite locally free sheaf on the gerbe $\mathcal{S}_\alpha$, then the sheaf $\mathcal{E}nd(\mathcal{E})$ is a 0-twisted finite locally free sheaf and hence isomorphic to $\pi^*\mathcal{A}$ for a unique finite locally free sheaf of $\mathcal{O}_S$-algebras.

For a given $\alpha \in \mathrm{H}^2_{\text{Ét}}(S, \mathbb{G}_m)$, there is a closely related notion of an α-twisted sheaf whose definition does not use gerbes. This approach has been developed by A. Căldăraru in his thesis. Assume that there is an étale covering $\{U_i \to S\}_{i\in I}$ that trivialises α such that α is represented by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$, where we use the standard notation

$$U_{ij}\colon = U_i \times_S U_j, \qquad U_{ijk}\colon = U_i \times_S U_j \times_S U_k.$$

By a theorem of Artin, this holds for any α when S is noetherian and every finite subset of S is contained in an affine open set, for example, when S is quasi-projective over the spectrum of a noetherian ring, see [Art71, Cor. 4.2]. Fix such an étale covering $\{U_i \to S\}_{i\in I}$. For $\alpha \in \mathrm{H}^2_{\text{Ét}}(S, \mathbb{G}_m)$, an *$\alpha$-twisted sheaf* with respect to this covering is given by quasi-coherent sheaves of $\mathcal{O}_{U_i}$-modules $\mathcal{M}_i$ together with isomorphisms $\varphi_{ij}\colon \mathcal{M}_j|_{U_{ij}} \xrightarrow{\sim} \mathcal{M}_i|_{U_{ij}}$ such that restricting to U_{ijk} we have

$$\varphi_{ij} \circ \varphi_{jk} = \alpha_{ijk}\varphi_{ik}. \tag{4.4}$$

Note that in general an α-twisted sheaf is not a sheaf on a scheme in the usual sense. If $\beta_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$ is another Čech cocycle, defining a class $\beta \in \mathrm{H}^2_{\acute{\mathrm{E}}\mathrm{t}}(S, \mathbb{G}_m)$, then the naturally defined tensor product of an α-twisted sheaf and a β-twisted sheaf (with respect to the same covering $\{U_i \to S\}_{i\in I}$) is an $(\alpha + \beta)$-twisted sheaf. The dual of a finite locally free α-twisted sheaf is a $(-\alpha)$-twisted sheaf.

The following lemma is an expanded version of [deJ, Lemma 2.10].

Lemma 4.1.19 *Let S be a scheme, let $\alpha \in \mathrm{H}^2_{\acute{\mathrm{E}}\mathrm{t}}(S, \mathbb{G}_m)$ and let $\mathcal{S}_\alpha$ be a $\mathbb{G}_m$-gerbe over S defined by α. Suppose that there is an étale covering $\{U_i \to S\}_{i\in I}$ such that α is represented by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$. Then the category of α-twisted sheaves on S with respect to the covering $\{U_i \to S\}_{i\in I}$ is equivalent to the category of 1-twisted sheaves on $\mathcal{S}_\alpha$.*

Proof. Choose an exact sequence (4.2) with $A = \mathbb{G}_m$ and B, C injective. Choose a section $\tau \in \mathrm{H}^0_{\acute{\mathrm{E}}\mathrm{t}}(S, D)$ that lifts $\alpha \in \mathrm{H}^2_{\acute{\mathrm{E}}\mathrm{t}}(S, \mathbb{G}_m)$, and then take $\mathcal{S}_\alpha$ to be the $\mathbb{G}_m$-gerbe $\mathcal{S}_\tau$. Since α restricts to 0 on each U_i, our section τ lifts to a section $\sigma_i \in \Gamma(U_i, C)$. By the definition of the gerbe $\mathcal{S}_\tau$, the pairs (U_i, σ_i) are objects of $\mathcal{S}_\tau$, as well as the pairs obtained by replacing U_i by any U_{ij} or U_{ijk}.

Since α is represented by the Čech cocycle $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$, the difference $\sigma_i - \sigma_j \in \Gamma(U_{ij}, C)$ is the image of some $\rho_{ij} \in \Gamma(U_{ij}, B)$ such that

$$\alpha_{ijk} = \rho_{ij} + \rho_{jk} - \rho_{ik} \in \Gamma(U_{ijk}, \mathbb{G}_m). \tag{4.5}$$

By the definition of $\mathcal{S}_\tau$, the pair (id, ρ_{ij}) is a morphism $(U_{ij}, \sigma_i) \to (U_{ij}, \sigma_j)$ in $\mathcal{S}_\tau$.

Let $\mathcal{F}$ be a quasi-coherent 1-twisted sheaf on $\mathcal{S}_\tau$. By Definition 4.1.8 (iv) the restriction of $\mathcal{F}$ to each (U_i, σ_i) is a quasi-coherent sheaf $\mathcal{M}_i$ of $\mathcal{O}_{U_i}$-modules. For each i and j let

$$\varphi_{ij} \colon \mathcal{M}_j \otimes \mathcal{O}_{U_{ij}} \xrightarrow{\sim} \mathcal{M}_i \otimes \mathcal{O}_{U_{ij}} \tag{4.6}$$

be an isomorphism on U_{ij} induced by (id, ρ_{ij}). The composition

$$\varphi_{ij} \circ \varphi_{jk} \colon \mathcal{M}_k \otimes \mathcal{O}_{U_{ijk}} \xrightarrow{\sim} \mathcal{M}_j \otimes \mathcal{O}_{U_{ijk}} \xrightarrow{\sim} \mathcal{M}_i \otimes \mathcal{O}_{U_{ijk}}$$

is induced by the composition of morphisms $(U_{ijk}, \sigma_i) \to (U_{ijk}, \sigma_j) \to (U_{ijk}, \sigma_k)$ given by $(\mathrm{id}, \rho_{jk}) \circ (\mathrm{id}, \rho_{ij}) = (\mathrm{id}, \rho_{ik} + \alpha_{ijk})$, where the equality follows from (4.5). Hence $\varphi_{ij} \circ \varphi_{jk}$ differs from φ_{ik} by the action of $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$ on $\mathcal{M}_i \otimes \mathcal{O}_{U_{ijk}}$. But $\mathcal{F}$ is a 1-twisted sheaf on $\mathcal{S}_\tau$, so $\mathbb{G}_m$ acts on the sections of $\mathcal{F}$ by the tautological character. This gives the desired formula (4.4).

Conversely, suppose that we are given an α-twisted sheaf with respect to the covering $\{U_i \to S\}_{i\in I}$. This is a collection of quasi-coherent $\mathcal{O}_{U_i}$-modules $\mathcal{M}_i$ together with isomorphisms (4.6) satisfying (4.4). The localisation of the gerbe $\mathcal{S}_\alpha$ with respect to the object (U_i, σ_i) is trivial, hence isomorphic to the classifying stack $B_{U_i}\mathbb{G}_m$, see Example 4.1.16. An explicit isomorphism associates to the pair (U_i, σ_i) the $\mathbb{G}_m$-torsor $\mathcal{P}_i$ over U_i which is the inverse image of σ_i under the map $B \to C$. The translation by ρ_{ij} defines an isomorphism between the restrictions of $\mathcal{P}_i$ and $\mathcal{P}_j$ to U_{ij}. The category of quasi-coherent sheaves on $B_{U_i}\mathbb{G}_m$ is equivalent to the category of $\mathbb{G}_m$-equivariant quasi-coherent sheaves on U_i, which are just the quasi-coherent sheaves on U_i equipped with a fibre-wise action of $\mathbb{G}_m$ (cf. [Ols16, Exercise 9.H]). Let $\mathcal{F}_i$ be the 1-twisted quasi-coherent sheaf of $\mathcal{O}_{(U_i,\sigma_i)}$-modules obtained from $\mathcal{M}_i$ with the tautological action of $\mathbb{G}_m$. Recall that we have a morphism $(\mathrm{id}, \rho_{ij})\colon (U_{ij}, \sigma_i) \to (U_{ij}, \sigma_j)$. Now φ_{ij} gives rise to an isomorphism of 1-twisted sheaves of $\mathcal{O}_{(U_{ij},\sigma_i)}$-modules:

$$\widetilde{\varphi}_{ij}\colon (\mathrm{id}, \rho_{ij})^* \mathcal{F}_j|_{(U_{ij},\sigma_j)} \xrightarrow{\sim} \mathcal{F}_i|_{(U_{ij},\sigma_i)}.$$

Using (4.4) and (4.5) we verify that this gives a glueing datum of quasi-coherent 1-twisted sheaves $\mathcal{F}_i$ of $\mathcal{O}_{(U_i,\sigma_i)}$-modules with respect to the covering of $\mathcal{S}_\tau$ by the objects $(U_i, \sigma_i)_{i\in I}$. Hence these sheaves glue together and give rise to a quasi-coherent 1-twisted sheaf on $\mathcal{S}_\tau$. □

4.2 de Jong's proof of Gabber's theorem

The fundamental result linking the Brauer–Azumaya group to the Brauer–Grothendieck group is the following Theorem 4.2.1, which is due to Gabber.

See `https://mathoverflow.net/questions/158614` for some information about Gabber's original proof.

As mentioned in Theorem 3.3.2, the affine case was already established by Gabber in his thesis [Gab81, Ch. II, Thm. 1]. Further partial results were obtained by Gabber and by Hoobler. A proof which reduces the general case to the affine case was given by de Jong [deJ]; it is this proof that we present in this section.

Theorem 4.2.1 (Gabber) *Let X be a scheme. There is a natural injective map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)_{\mathrm{tors}}$. If X has an ample invertible sheaf, for example, if X is a quasi-projective scheme over an affine scheme, then this map is an isomorphism.*

By definition [Stacks, Def. 01PS], to say that an invertible sheaf L is ample on X means that X is *quasi-compact* and for any $x \in X$ there is a section $s \in \mathrm{H}^0(X, L^{\otimes n})$ for some $n \geq 1$ such that $s(x) \neq 0$ and the open subset $s \neq 0$ is affine. By [Stacks, Lemma 09MP] if X has an ample invertible sheaf then X is *separated*. The separatedness assumption is necessary. Indeed, there exists a non-separated, normal scheme X over $\mathbb{C}$ with torsion elements in $\mathrm{Br}(X)$ that are not in the image of $\mathrm{Br}_{\mathrm{Az}}(X)$, see [EHKV01] and [Ber05].

Let us construct a map $\mathrm{Br}_{\mathrm{Az}}(X)\to\mathrm{Br}(X)$ that associates to an Azumaya algebra over X a certain $\mathbb{G}_m$-gerbe over X. This construction is used in the second proof of [Mil80, Thm. IV.2.5].

To an Azumaya algebra A on X one attaches the category $\mathcal{X}(A)$ whose objects are triples $(T,\mathcal{M},j)$, where T is an X-scheme, $\mathcal{M}$ is a finite locally free $\mathcal{O}_T$-module, and j is an isomorphism $j\colon \mathcal{E}nd(\mathcal{M}) \xrightarrow{\sim} A_T$. A morphism of triples $(T,\mathcal{M},j)\to(T',\mathcal{M}',j')$ is a pair (f,i) consisting of a morphism of X-schemes $f\colon T\to T'$ and an isomorphism $i\colon f^*\mathcal{M}' \xrightarrow{\sim} \mathcal{M}$ compatible with j and j'. Note that there is a natural map $\mathbb{G}_m(T)\to\mathrm{Aut}(T,\mathcal{M},j)$ sending u to (id_T,u).

Proposition 4.2.2 *The forgetful functor $\pi\colon \mathcal{X}(A)\to Sch/X$ is a $\mathbb{G}_m$-gerbe for the étale topology.*

Proof. [Ols16, Prop. 12.3.6] One checks that $\mathcal{X}(A)$ is a stack. The verification that $\mathcal{X}(A)$ is a $\mathbb{G}_m$-gerbe can be done locally, so one can assume that $A=\mathcal{E}nd(\mathcal{O}_X^n)$. Furthermore, we can assume that $\mathcal{M}=\mathcal{O}_T^n$. After localising again, we can assume that j comes from the conjugation by an element of $\mathrm{Aut}(\mathcal{O}_T^n)$. Thus any object in $\mathcal{X}(A)$ is locally isomorphic to $(T,\mathcal{O}_T^n,\mathrm{id})$, so any two objects are locally isomorphic. Now the automorphism sheaf of the object $(T,\mathcal{O}_T^n,\mathrm{id})$ is $\mathbb{G}_m$ acting by scalar multiplication on $\mathcal{O}_T^n$. □

Since the isomorphism classes of $\mathbb{G}_m$-gerbes over X are classified by the elements of $\mathrm{H}^2_{\text{Ét}}(X,\mathbb{G}_m)$, this gives a map $\mathrm{Br}_{\mathrm{Az}}(X)\to\mathrm{Br}(X)$. The class in $\mathrm{Br}(X)$ associated to A can be described as follows. Assume that A is an Azumaya algebra over X of degree n. Consider (3.6) as an exact sequence of sheaves of groups for the étale topology. Let P be the functor on Sch/X sending $Y\to X$ to $\mathrm{Isom}_{\mathcal{O}_Y}(M_n(\mathcal{O}_Y),A_Y)$. Using essentially the Noether–Skolem theorem one shows that this functor is a PGL_n-torsor on Sch/X. Then the class associated to A is the image of the class of this torsor under the map $\mathrm{H}^1_{\text{Ét}}(X,\mathrm{PGL}_n)\to\mathrm{H}^2_{\text{Ét}}(X,\mathbb{G}_m)$ which sends a PGL_n-torsor to the gerbe of its liftings to a GL_n-torsor, as defined in Section 4.1.4, see [Ols16, Lemma 12.3.9].

To any cohomology class $\alpha\in\mathrm{Br}(X)$ one associates a $\mathbb{G}_m$-gerbe $\mathcal{X}_\alpha$ over X (well defined up to isomorphism) using the construction of the gerbe associated to a cohomology class in Section 4.1.4. Namely, one takes (4.1) to be the extension (3.6). One wants to show that $\mathcal{X}_\alpha$ is isomorphic to $\mathcal{X}(A)$ for some A.

We refer to [Ols16, Ch. 9], [LMB00, Ch. 13] and [Stacks, Ch. 06TF], [Stacks, Ch. 073P] for the theory of quasi-coherent sheaves on algebraic stacks. In particular, the definition of the push-forward of quasi-coherent sheaves with respect to a quasi-compact and quasi-separated morphism of algebraic stacks can be found in [Stacks, Section 070A].

The gerbe $\mathcal{X}(A)$ has a tautological finite locally free 1-twisted sheaf $\mathcal{M}$ together with an isomorphism $\mathcal{E}nd(\mathcal{M})\cong\pi^*A$ of algebras over $\mathcal{X}(A)$. Then $A\cong\pi_*\mathcal{E}nd(\mathcal{M})$.

Proposition 4.2.3 *A $\mathbb{G}_m$-gerbe $\mathcal{X}$ over $X_{\text{Ét}}$ is isomorphic to the gerbe $\mathcal{X}(A)$ for some Azumaya algebra A on X if and only if $\mathcal{X}$ has a finite locally free 1-twisted $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{M}$ of positive rank. In this case $A = \pi_*\mathcal{E}nd(\mathcal{M})$ is an Azumaya algebra on X, the adjunction map $\pi^*A \to \mathcal{E}nd(\mathcal{M})$ is an isomorphism, and $\mathcal{X} \cong \mathcal{X}(A)$.*

Proof. See [Ols16, Prop. 12.3.11] or [Lie08, Prop. 3.1.2.1 (i)]. □

The goal is thus to show that if X has an ample invertible sheaf, then for any $\alpha \in \mathrm{Br}(X)_{\mathrm{tors}}$ the $\mathbb{G}_m$-gerbe $\mathcal{X} = \mathcal{X}_\alpha$ has a finite locally free 1-twisted $\mathcal{O}_{\mathcal{X}}$-module of positive rank. Recall that by Lemma 4.1.19 the category of 1-twisted sheaves on $\mathcal{X}$ is equivalent to the category of α-twisted sheaves on X with respect to a given étale covering of S which trivialises α. This induces an equivalence of the full subcategory of 1-twisted finite locally free sheaves on $\mathcal{X}$ with the full subcategory of α-twisted finite locally free sheaves on X. So our task is to construct a finite locally free α-twisted sheaf on X.

To avoid confusion, we shall denote the α-twisted sheaves on X by F, G, H, and the corresponding 1-twisted sheaves on $\mathcal{X} = \mathcal{X}_\alpha$ by $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$.

Let L be an ample invertible sheaf on X.

By 'absolute noetherian approximation' [TT90, Thm. C.9] one can represent (X, L) as a filtering inverse limit of pairs (X_i, L_i), where X_i is separated and of finite type over $\mathbb{Z}$ and L_i is an ample invertible sheaf on X_i, with affine transition morphisms $X_i \to X_j$. By Section 2.2.2 the group $\mathrm{H}^n_{\text{ét}}(X, \mathbb{G}_m)$ is naturally isomorphic to the direct limit of the groups $\mathrm{H}^n_{\text{ét}}(X_i, \mathbb{G}_m)$. Hence $\mathrm{Br}(X)_{\mathrm{tors}}$ is the direct limit of the groups $\mathrm{Br}(X_i)_{\mathrm{tors}}$. Thus without loss of generality we can assume that X is a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$ with an ample invertible sheaf. In particular, X is noetherian. Since X is of finite type, [Stacks, Lemma 01Q1] implies that X is a quasi-projective scheme over an affine scheme. In particular, a theorem of Artin [Mil80, Thm. III.2.17] applies, hence there is an étale covering $\{U_i \to S\}$ that trivialises α, so that α is represented by a Čech cocycle $\alpha_{ijk} \in \Gamma(U_{ijk}, \mathbb{G}_m)$.

In the course of the proof X will be repeatedly replaced by X_R for some ring R which is finite and flat over $\mathbb{Z}$. This is justified by the following lemma due to Gabber [Gab81, Ch. II, Lemma 4] and Hoobler [Hoo82, Prop. 3].

Lemma 4.2.4 *Let $\varphi\colon Y \to X$ be a finite surjective locally free morphism of schemes. Then $\alpha \in \mathrm{H}^2_{\text{Ét}}(X, \mathbb{G}_m)$ comes from an Azumaya algebra on X if and only if $\alpha_Y \in \mathrm{H}^2_{\text{Ét}}(Y, \mathbb{G}_m)$ comes from an Azumaya algebra on Y.*

Proof. If α_Y comes from an Azumaya algebra on Y, then there is a finite locally free 1-twisted sheaf $\mathcal{F}$ on the $\mathbb{G}_m$-gerbe $\mathcal{Y}$. Then the direct image $\varphi_*\mathcal{F}$ is a finite locally free 1-twisted sheaf on the $\mathbb{G}_m$-gerbe $\mathcal{X}$. □

If X is locally noetherian, then $\varphi\colon Y \to X$ is a finite locally free morphism if and only if φ is finite and flat, see [Stacks, Lemma 02KB].

The proof of Theorem 4.2.1

There is a section $s \in \mathrm{H}^0(X, L^{\otimes m})$ for some $m \geq 1$ such that the open set X_s is affine. By Theorem 3.3.2 the restriction of α to X_s is represented by an Azumaya algebra A. Hence there is a finite locally free α-twisted sheaf F_s on X_s. By taking direct sums on the connected components we can assume that F_s has constant rank.

Write $j\colon \mathcal{X}_s \to \mathcal{X}$ for the open immersion of $\mathbb{G}_m$-gerbes defined by the inclusion $X_s \hookrightarrow X$. By Lemma 4.1.19, F_s corresponds to a finite locally free 1-twisted sheaf $\mathcal{F}_s$ on the $\mathbb{G}_m$-gerbe $\mathcal{X}_s$. The push-forward sheaf $j_*\mathcal{F}_s$ is a quasi-coherent 1-twisted sheaf on the $\mathbb{G}_m$-gerbe $\mathcal{X}$. By [LMB00, Prop. 15.4] every quasi-coherent sheaf on a noetherian algebraic stack is a filtering direct limit of *coherent* subsheaves. A coherent subsheaf of a 1-twisted sheaf is clearly a 1-twisted sheaf. This allows one to find a coherent 1-twisted subsheaf $\mathcal{F} \subset j_*\mathcal{F}_s$ such that $j^*\mathcal{F} = \mathcal{F}_s$.

We can ensure that X_s contains any given finite set of closed points (see [EGA, II, Cor. 4.5.4]), so the coherent α-twisted sheaf F corresponding to $\mathcal{F}$ is finite locally free at each of these points.

A quasi-coherent sheaf of $\mathcal{O}_X$-modules is finite locally free if and only if it is flat and of finite type [Stacks, Lemma 05P2]. Thus the task is to ensure that our coherent α-twisted sheaf F is flat. Let $\mathrm{Sing}(F)$ be the closed set of points of X at which F is not flat. What we have obtained now is the case $c = 1$ of the following statement.

(H_c) *For any finite set T of closed points of X, after a finite flat ring extension of the base ring R, there exists a coherent α-twisted sheaf F which is finite locally free at T, of constant positive rank outside of* $\mathrm{Sing}(F)$*, and such that*

$$\mathrm{codim}_X(\mathrm{Sing}(F)) \geq c.$$

The strategy of the proof is to use finite flat ring extensions R' of R to increase c. In doing so we replace T by its inverse image in $X \times_R R'$. In view of Lemma 4.2.4, the theorem will be proved if one can make $c = \dim(X) + 1$.

Recall that $\alpha \in \mathrm{Br}(X)_{\mathrm{tors}}$. Let n be a positive integer such that $n\alpha = 0$.

Step 1

Assume that (H_c) holds for a finite set of closed points $T \subset X$. The claim of this step is that, after replacing R by a finite flat ring extension, there exist $n+1$ coherent α-twisted sheaves $F_0, \ldots, F_n$ (where $n\alpha = 0$) and finite sets of closed points $S_0, \ldots, S_n$ in X with the following properties:

(1) each F_i is finite locally free at each of the points of T;
(2) each F_i has constant positive rank on $X \smallsetminus \mathrm{Sing}(F_i)$;

(3) for each $i = 0, \ldots, n$ we have $\mathrm{codim}_X(\mathrm{Sing}(F_i)) \geq c$, and each irreducible component of $\mathrm{Sing}(F_i)$ of codimension c contains a point of S_i;
(4) for any $i \neq j$ the sheaf F_j is finite locally free at all the points of S_i.

Indeed, (H_c) ensures the existence of an F_0 which is locally free at T. Choose a closed point in each irreducible component of $\mathrm{Sing}(F_0)$ of codimension c; let $S_0 \subset X$ be the set of these points. Define $T_0 := T \cup S_0$. Now (H_c) ensures the existence of an F_1 which is locally free at T_0. If a codimension c irreducible component of $\mathrm{Sing}(F_1)$ is contained in $\mathrm{Sing}(F_0)$, then it is a codimension c irreducible component of $\mathrm{Sing}(F_0)$, but this is not possible because F_1 is locally free at some closed point of this component. Thus we can choose a closed point in each codimension c irreducible component of $\mathrm{Sing}(F_1)$ such that this point is not in $\mathrm{Sing}(F_0)$. Let $S_1 \subset X$ be the set of these points, and let $T_1 := T_0 \cup S_1$. The pairs (F_0, S_0) and (F_1, S_1) satisfy properties (1) to (4) with $n = 1$. Next, one constructs F_2 and so on. If $F_0, \ldots, F_{j-1}$ are already constructed so that properties (1) to (4) are satisfied, one constructs F_j which is locally free at all the points of $T \cup S_0 \cup \ldots \cup S_{j-1}$ and chooses S_j in $\mathrm{Sing}(F_j)$ outside of the union of the $\mathrm{Sing}(F_i)$ for $i = 0, \ldots, j-1$.

Step 2

Replacing each F_i by $F_i^{\oplus m_i}$ for appropriate positive integers m_i we ensure that there is a positive integer r such that the rank of F_i on $X \smallsetminus \mathrm{Sing}(F_i)$ is r for each $i = 0, \ldots, n$. Later on we shall assume that r is large. Define

$$G_1 = (F_0 \oplus \ldots \oplus F_n)^{\oplus r^n}, \qquad G_2 = F_0 \otimes \ldots \otimes F_n.$$

It is clear that G_1 is a coherent α-twisted sheaf; in fact, G_2 is also a coherent α-twisted sheaf since $n\alpha = 0$. It follows that

$$H = \mathcal{H}om(G_1, G_2)$$

is a coherent 0-twisted sheaf on X, so is just a coherent $\mathcal{O}_X$-module. Recall that L is an ample invertible sheaf on X. Replacing X by X_R preserves the ampleness of L.

Let ψ be a section of $H \otimes L^{\otimes N}$ over X for some positive integer N, and let F be the kernel of the map $\psi\colon G_1 \to G_2 \otimes L^{\otimes N}$.

Let U be the complement to $\bigcup_{i=0}^n \mathrm{Sing}(F_i)$ in X. The aim of Step 2 is to give conditions for F to be finite locally free of positive rank on a larger open set than U. More precisely, one gives conditions ensuring that $\mathrm{codim}_X(\mathrm{Sing}(F)) \geq c+1$, in terms of pullbacks at closed points of X.

The fibre of a coherent sheaf on the $\mathbb{G}_m$-gerbe $\mathcal{X}$ at a geometric point $\bar{x} = \mathrm{Spec}(\kappa(\bar{x})) \in X$ is defined by choosing a lifting of the morphism $\bar{x} \to X$ to a morphism $\bar{x} \to \mathcal{X}$, which is possible as $\alpha \in \mathrm{Br}(X)$ is annihilated by the restriction to the algebraically closed residue field $\kappa(\bar{x})$. This fibre is a finite-dimensional vector space over $\kappa(\bar{x})$. (Different liftings give rise to non-canonically isomorphic fibres.) This is used in (a) below, via the equivalence of 1-twisted sheaves on $\mathcal{X}$ and α-twisted sheaves on X as in Lemma 4.1.19. The same works for a closed point with a finite residue field, by the triviality of the Brauer group of a finite field. This is used in (b) below.

Claim. *Let*

$$F = \mathrm{Ker}[\psi\colon G_1 \longrightarrow G_2 \otimes (L^{\otimes N})],$$

where $\psi \in \Gamma(X, H \otimes (L^{\otimes N}))$ *for some positive integer* N*. Assume that the following conditions are satisfied.*

(a) *For every geometric point* $\bar{x} = \mathrm{Spec}(\kappa(\bar{x})) \in U$ *the pullback to* $\bar{x}$ *gives a surjective map of* $\kappa(\bar{x})$*-vector spaces*

$$\psi_{\bar{x}}\colon G_1 \otimes \kappa(\bar{x}) \longrightarrow G_2 \otimes (L^{\otimes N}) \otimes \kappa(\bar{x}).$$

(b) *For any* $i = 0, \ldots, n$ *and any* $s = \mathrm{Spec}(\kappa(s)) \in S_i$ *the composition*

$$F_i^{\oplus r^n} \otimes \kappa(s) \hookrightarrow G_1 \otimes \kappa(s) \longrightarrow G_2 \otimes (L^{\otimes N}) \otimes \kappa(s)$$

is an isomorphism of $\kappa(s)$*-vector spaces, but the following composition is zero:*

$$(\oplus_{j \neq i} F_j)^{\oplus r^n} \otimes \kappa(s) \hookrightarrow G_1 \otimes \kappa(s) \longrightarrow G_2 \otimes (L^{\otimes N}) \otimes \kappa(s).$$

Then F *is an* α*-twisted sheaf on* X *such that* $\mathrm{Sing}(F) \subset \bigcup_{i=0}^n \mathrm{Sing}(F_i)$ *and* $\mathrm{Sing}(F)$ *is disjoint from* $S = \cup_{i=0}^n S_i$*. In particular,* $\mathrm{codim}_X(\mathrm{Sing}(F)) \geq c+1$*.*

This shows that if ψ satisfying (a) and (b) exists, then (H_c) implies (H_{c+1}).

Proof of Claim. It is clear that $F = \mathrm{Ker}(\psi)$ is a coherent α-twisted sheaf on X. The last sentence of the statement is a consequence of the fact that each codimension c irreducible component of $\mathrm{Sing}(F_i)$ contains a point of S_i.

Condition (a) implies that the restriction of F to the open subscheme $U \subset X$ is the kernel of a surjective map of finite locally free sheaves. Locally such a map has a section, so its kernel is finite locally free.

Let us prove that condition (b) implies that F is finite locally free at each $x \in S$. Let $\mathcal{O}_{X,x}$ be the local ring at x and let $\mathcal{O}_{X,x}^{\mathrm{h}}$ be the henselisation of $\mathcal{O}_{X,x}$. The Brauer group $\mathrm{Br}(\mathcal{O}_{X,x}^{\mathrm{h}})$ is canonically isomorphic to the Brauer group of the residue field $\mathrm{Br}(\kappa(x))$, see Theorem 3.4.2 (i). Since $\kappa(x)$ is finite, we have $\mathrm{Br}(\kappa(x)) = 0$. It follows that there is a finite étale extension of local rings $\mathcal{O}_{X,x} \subset B$ with trivial residue field extension such that the image of α in $\mathrm{Br}(B)$ is zero. Thus there is a lifting $\mathrm{Spec}(B) \to \mathcal{X}$ of $\mathrm{Spec}(B) \to X$ so that each $\mathcal{F}_i$ pulls back to a coherent sheaf on $\mathrm{Spec}(B)$. This sheaf is associated to a finitely generated B-module M_i.

We have $x \in S_i$ for some i. Then for $j \neq i$ we can arrange that the B-module M_j is free of rank r. Let us write $M = M_i$. Then

$$H \otimes B = \mathrm{Hom}_B(M^{\oplus r^n}, M^{\oplus r^n}) \oplus \mathrm{Hom}_B(B^{\oplus nr^{n+1}}, M^{\oplus r^n}). \tag{4.7}$$

Write $\psi \otimes B = \psi_1 \oplus \psi_2$. The residue field of the local ring B is $\kappa(x)$, so by Nakayama's lemma condition (b) gives that ψ_1 is an isomorphism and $\psi_2 = 0$. Hence $F \otimes B = \mathrm{Ker}(\psi \otimes B)$ is the direct summand $B^{\oplus nr^{n+1}} \subset G_1 \otimes B$. Thus F is finite locally free at each point of S. This proves the claim. □

For each $s \in S$ we can tensor (4.7) for $x = s$ with $\kappa(s)$ and obtain the following decomposition for the fibre $H \otimes \kappa(s)$ of the sheaf H at s:

$$H \otimes \kappa(s) = \mathrm{End}_{\kappa(s)}((M \otimes \kappa(s))^{\oplus r^n}) \oplus \mathrm{Hom}_{\kappa(s)}(\kappa(s)^{\oplus nr^{n+1}}, (M \otimes \kappa(s))^{\oplus r^n}).$$

With respect to this decomposition define $\psi_s = \mathrm{id} \oplus 0$.

For a geometric point $\bar{x} \in U$, we have an analogue of the previous formula with $M \otimes \kappa(\bar{x}) \simeq \kappa(\bar{x})^r$. Hence the fibre of H at $\bar{x}$ is the $\kappa(\bar{x})$-vector space of matrices of size $(n+1)r^{n+1} \times r^{n+1}$. Condition (a) at $\bar{x}$ is satisfied if $\psi_{\bar{x}}$ avoids the subset of matrices of rank less than r^{n+1}. By linear algebra, this is a closed homogeneous subset of codimension

$$(n+1)r^{n+1} - r^{n+1} + 1 > nr^{n+1}.$$

Here homogeneous means stable under multiplication of matrix entries by any common multiple in $\kappa(\bar{x})^*$. We can make r arbitrarily large and thus ensure that this codimension is greater than $\dim(X) + 1$.

Step 3

It remains to show that if N is sufficiently large, then there exists a section ψ satisfying conditions (a) and (b) above. This is a purely algebraic-geometric statement, so this part of the proof has nothing to do with either Brauer elements or gerbes.

Let R be a ring which is finite and flat over $\mathbb{Z}$, and let X be a quasi-projective scheme over R with an invertible sheaf L. Let H be a coherent $\mathcal{O}_X$-module whose restriction to an open subscheme $U \subset X$ is finite locally free. For any $x \in X$ choose an isomorphism between $L_x = L \otimes \kappa(x)$ and $\kappa(x)$.

Suppose that for every $u \in U$ we are given a closed homogeneous subset C_u of the fibre $H_u = H \otimes \kappa(u)$ of codimension greater than $\dim(X) + 1$. *Suppose also that for a finite set of closed points $S \subset X \setminus U$ we are given $\psi_s \in H_s$ for each $s \in S$. Then there exist a positive integer N, a finite flat extension of rings $R \subset R'$ and a section $\psi \in \Gamma(X_{R'}, H \otimes L^{\otimes N})$ such that $\psi_u \notin C_u \otimes L_u^{\otimes N}$ for $u \in U_{R'}$, and for each closed point s' of $X_{R'}$ over a point $s \in S$ the value of ψ at s' is a non-zero multiple of ψ_s.*

Using the isomorphism $L_u \simeq \kappa(u)$ we identify C_u in H_u with $C_u \otimes L_u^{\otimes N}$ in $H_u \otimes L_u^{\otimes N}$.

For the proof we may assume that $R = \mathbb{Z}$.

Let $\mathcal{I}_S \subset \mathcal{O}_X$ be the sheaf of ideals defined by S. For all N sufficiently large one can find sections $\Psi_i \in \Gamma(X, \mathcal{I}_S \otimes H \otimes L^{\otimes N})$, for i in a finite set I, such that the map of sheaves $\mathcal{O}_X^I \to \mathcal{I}_S \otimes H \otimes L^{\otimes N}$ sending 1_i to Ψ_i is surjective. In particular, the sections Ψ_i generate the sheaf $H \otimes L^{\otimes N}$ over U. See [EGA, II, Prop. 4.5.5]. By increasing N further, for each $s \in S$ one finds a section

$$\Psi_s \in \Gamma(X, \mathcal{I}_{S \smallsetminus \{s\}} \otimes H \otimes L^{\otimes N})$$

whose value at s is ψ_s.

Let $\mathbb{A} = \mathrm{Spec}(\mathbb{Z}[x_i, y_s; i \in I, s \in S])$ be the affine space over $\mathbb{Z}$ of relative dimension $|I| + |S|$. Write $\mathbb{A} \times X$ for $\mathbb{A} \times_{\mathbb{Z}} X$ and consider the universal section

$$\Psi = \sum_{i \in I} x_i \Psi_i + \sum_{s \in S} y_s \Psi_s$$

of the pullback of $H \otimes L^{\otimes N}$ to $\mathbb{A} \times X$. The value $\Psi_{a,u}$ of Ψ at $(a, u) \in \mathbb{A} \times U$ is an element of $H_u \otimes L^{\otimes N} \simeq H_u$. Let $Z \subset \mathbb{A} \times U$ be the closed subset defined by the condition $\Psi_{a,u} \in C_u$. The values of the sections Ψ_i, for $i \in I$, generate the $\kappa(u)$-vector space H_u, hence the dimension of each fibre of the natural projection $Z \to U$ is at most $|I| + |S| - \mathrm{codim}_{H_u}(C_u)$. By assumption we have $\mathrm{codim}_{H_u}(C_u) > \dim(X) + 1$, hence $\dim(Z) < |I| + |S| - 1 = \dim(\mathbb{A}) - 2$. Thus the Zariski closure Z' of the projection of Z to $\mathbb{A}$ has codimension at least 2.

Let $\pi\colon \mathbb{A} \to \mathrm{Spec}(\mathbb{Z})$ and $p\colon X \to \mathrm{Spec}(\mathbb{Z})$ be the structure morphisms. For each $s \in S$ define $Z_s \subset \mathbb{A}$ to be the closed subscheme defined by the ideal $(p(s), y_s)$. To finish the proof we need to find a point in $\mathbb{A}(R)$ outside of the codimension 2 closed subset $Z' \cup \bigcup_{s \in S} Z_s$, for some finite flat extension $\mathbb{Z} \subset R$. Note that π induces a surjective morphism $\mathbb{A} \smallsetminus (Z' \cup \bigcup_{s \in S} Z_s) \to \mathrm{Spec}(\mathbb{Z})$. The result then follows from Rumely's local-to-global principle [Rum86] in the form of [Mor89, Thm. 1.7]: an irreducible scheme V which is separated and of finite type over a ring of integers $\mathcal{O}_K$ of a number field K has a point in the ring of *all* algebraic integers if the structure morphism $V \to \mathrm{Spec}(\mathcal{O}_K)$ is surjective with geometrically irreducible generic fibre V_K. It is clear that such a point is defined over a finite extension of $\mathbb{Z}$. □

Chapter 5
Varieties over a field

In this chapter we describe a general technique for computing the Brauer group $\mathrm{Br}(X)$ of a smooth projective variety X over a field k. Let k_{s} be a separable closure of k and let $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$. The Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$ acts on the *geometric Picard group* $\mathrm{Pic}(X^{\mathrm{s}})$ and on the *geometric Brauer group* $\mathrm{Br}(X^{\mathrm{s}})$. One would like to understand the kernel and the cokernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$. This can be done (with some success) using a Leray spectral sequence which involves Galois cohomology groups with coefficients in $\mathrm{Pic}(X^{\mathrm{s}})$ and $\mathrm{Br}(X^{\mathrm{s}})$. The structure of $\mathrm{Pic}(X^{\mathrm{s}})$ is discussed in the first section, and the structure of $\mathrm{Br}(X^{\mathrm{s}})$ is the subject of the second section. In the third section, we consider the action of the absolute Galois group on the Tate module of $\mathrm{Br}(X^{\mathrm{s}})$. This will be used in Chapter 16. The spectral sequence and its differentials, with applications to the computation of $\mathrm{Br}(X)$, are discussed in the fourth section. In Section 5.5 we consider general geometric hypotheses on X that allow one to obtain more precise results about $\mathrm{Br}(X)$. In Section 5.6 we discuss the Brauer groups of curves. The last section of this chapter concerns the computation of the Picard and Brauer groups of a product of two varieties.

5.1 The Picard group of a variety

Basic references on the Picard group are the book by Bosch, Lütkebohmert and Raynaud [BLR90], and Kleiman's contribution [Kle05] to [FGI+05].

Let k be a field with algebraic closure $\bar{k}$. Let k_{s} be the separable closure of k in $\bar{k}$. In this section we assume that X is a *proper*, *geometrically reduced* and *geometrically connected* variety over k. Recall that we write $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$ and $\overline{X} = X \times_k \bar{k}$.

We have $k = \mathrm{H}^0(X, \mathcal{O}_X)$, hence, by Proposition 2.5.4, $\mathcal{O}_T = f_{T*}(\mathcal{O}_{X\times_k T})$ for any k-scheme T, where $f_T\colon X \times_k T \to T$ is the base change of the structure morphism $f\colon X \to \mathrm{Spec}(k)$. Proposition 2.5.2 then gives that the natural map

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between the functors

$$\mathrm{Pic}_{(X/k)\,\text{ét}} \xrightarrow{\sim} \mathrm{Pic}_{(X/k)\,\text{fppf}}$$

is an isomorphism. By a fundamental result of Grothendieck (Theorem 2.5.7), this functor is representable by a commutative group scheme $\mathbf{Pic}_{X/k}$ which is a disjoint union of quasi-projective k-schemes. There is a functorial injective map $\mathrm{Pic}(X)\to\mathbf{Pic}_{X/k}(k)$, which is an isomorphism if $X(k)\neq\emptyset$, see Corollary 2.5.9.

Let $\mathbf{Pic}^0_{X/k}\subset\mathbf{Pic}_{X/k}$ be the connected component of 0, see [SGA3, VI_A, §2]. This is the smallest connected open subgroup of $\mathbf{Pic}_{X/k}$. It is a group k-scheme of finite type. For any field extension K/k there is an isomorphism $\mathbf{Pic}^0_{X/k}\times_k K\cong\mathbf{Pic}^0_{X_K/K}$. We define

$$\mathrm{Pic}^0(X):=\mathrm{Pic}(X)\cap\mathbf{Pic}^0_{X/k}(k)\subset\mathbf{Pic}_{X/k}(k).$$

The following property will not be used in this book, but is worth mentioning. An invertible sheaf on X is algebraically equivalent to 0 [Kle05, Def. 9.5.9] if and only if the corresponding point in $\mathrm{Pic}(X)$ belongs to $\mathrm{Pic}^0(X)$ [Kle05, Prop. 9.5.10].

The *Néron–Severi group k-scheme* is defined as the quotient

$$\mathbf{NS}_{X/k}:=\mathbf{Pic}_{X/k}/\mathbf{Pic}^0_{X/k}. \tag{5.1}$$

It is étale over k, see [SGA3, VI_A, 5.5]. In particular, we have $\mathbf{NS}_{X/k}(k_{\mathrm{s}})=\mathbf{NS}_{X/k}(\bar{k})$. By a theorem of Néron and Severi it is a finitely generated abelian group, see [SGA6, XIII, 5.1]. For any field K containing k_{s} the natural map $\mathbf{NS}_{X/k}(k_{\mathrm{s}})\to\mathbf{NS}_{X/k}(K)$ is an isomorphism.

Let us define the *Néron–Severi group* of X as

$$\mathrm{NS}(X):=\mathrm{Pic}(X)/\mathrm{Pic}^0(X).$$

Thus $\mathrm{NS}(\overline{X})=\mathbf{NS}_{X/k}(\bar{k})$. The positive integer $\rho=\dim_{\mathbb{Q}}(\mathrm{NS}(\overline{X})\otimes\mathbb{Q})$ is called the *Picard number* of X.

The tangent space to $\mathbf{Pic}_{X/k}$ at 0 is the coherent cohomology group $\mathrm{H}^1(X,\mathcal{O}_X)$ [Kle05, Thm. 9.5.11]. It follows that

$$\dim(\mathbf{Pic}_{X/k})\leq\dim(\mathrm{H}^1(X,\mathcal{O}_X)),$$

and equality holds if and only if $\mathbf{Pic}_{X/k}$ is smooth. If the characteristic of k is zero, then $\mathbf{Pic}_{X/k}$ is smooth by Cartier's theorem, so $\mathbf{Pic}_{X/k}$ has the same dimension $\dim(\mathrm{H}^1(X,\mathcal{O}_X))$ at every point. As recalled in [Kle05, Rem. 9.5.15, Prop. 9.5.19], Mumford proved in [Mum66, Ch. 27] that for an algebraically closed field $\bar{k}$ the tangent space to $\mathbf{Pic}_{X/\bar{k},\mathrm{red}}$ at 0 is the intersection of the kernels of Bockstein homomorphisms

$$\mathrm{H}^1(X,\mathcal{O}_X)\longrightarrow\mathrm{H}^2(X,\mathcal{O}_X)$$

defined by Serre using cohomology with values in Witt sheaves. Thus, over any field k, the k-scheme $\mathbf{Pic}_{X/k}$ is smooth whenever X satisfies either $\mathrm{H}^1(X,\mathcal{O}_X)=0$ or $\mathrm{H}^2(X,\mathcal{O}_X)=0$, e.g., if X is a curve. See [BLR90, Ch. 8, §4, Prop. 2] for a quick proof that $\mathrm{H}^2(X,\mathcal{O}_X)=0$ implies the smoothness of $\mathbf{Pic}_{X/k}$. Another important case is that $\mathbf{Pic}_{X/k}$ is smooth when X is an abelian variety over k, see [EGM, Thm. 6.18].

5.1.1 Picard variety

If the k-variety X is *projective, geometrically integral* and *geometrically normal*, then $\mathbf{Pic}^0_{X/k}$ is projective [Kle05, Thm. 9.5.4]. Following [FGA6, Prop. 3.1, Cor. 3.2, p. 236-16], we associate to such a variety X an abelian variety A called the *Picard variety* of X.

If $\mathbf{Pic}_{X/k}$ is smooth, for instance if $\mathrm{char}(k)=0$, we define $A=\mathbf{Pic}^0_{X/k}$. Now let us consider the general case. (The reader should be aware of a delicate point in the proof. Namely, for an algebraic group G over an *imperfect* field k, the reduced scheme G_{red} need not be an algebraic group over k. For examples due to Raynaud, see [SGA3, VI_A, Examples 1.3.2].) Let $G=\mathbf{Pic}^0_{X/k}$. The reduced subgroup scheme $(G\times_k\bar{k})_{\mathrm{red}}$ of $G\times_k\bar{k}$ is smooth and projective over $\bar{k}$, so it is an abelian variety over $\bar{k}$. The group $G\times_k\bar{k}$ fits into an exact sequence of commutative group $\bar{k}$-schemes

$$0\longrightarrow (G\times_k\bar{k})_{\mathrm{red}}\longrightarrow G\times_k\bar{k}\longrightarrow L\longrightarrow 0,$$

where L a finite group $\bar{k}$-scheme satisfying $L(\bar{k})=1$. Let $n>0$ be the degree of $L/\bar{k}$. Let $A\subset G$ be the scheme theoretic image of the homomorphism $[n]\colon G\to G$. (By definition, this is the smallest closed subscheme of G through which $[n]$ factors.) Then A is a proper, commutative group k-scheme which satisfies

$$A\times_k\bar{k}\cong(G\times_k\bar{k})_{\mathrm{red}}.$$

Thus A is geometrically reduced, hence is an abelian variety over k [FGA6, Prop. 3.1, Cor. 3.2, p. 236-16]. The underlying topological spaces of A and G are the same, so the k-variety A is equal to $\mathbf{Pic}^0_{X/k,\mathrm{red}}$, which is therefore a subgroup of $\mathbf{Pic}^0_{X/k}$ and an abelian variety, and in particular is geometrically reduced. The abelian variety $A=\mathbf{Pic}^0_{X/k,\mathrm{red}}$ over k is called the *Picard variety* of X. It satisfies $A(F)=\mathbf{Pic}^0_{X/k}(F)$ for any field F containing k.

We summarise the basic properties of the Picard scheme of a projective variety over a field in the following theorem.

Theorem 5.1.1 *Let X be a projective, geometrically integral and geometrically normal variety over a field k.*

(i) *The group k-scheme $\mathbf{Pic}^0_{X/k}$ is projective and geometrically connected. The tangent space to $\mathbf{Pic}^0_{X/k}$ at 0 is the coherent cohomology group $\mathrm{H}^1(X,\mathcal{O}_X)$. There is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathbf{Pic}^0_{X/k}(k_\mathrm{s}) \longrightarrow \mathrm{Pic}(X^\mathrm{s}) \longrightarrow \mathrm{NS}(X^\mathrm{s}) \longrightarrow 0.$$

(ii) *The reduced subscheme $\mathbf{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety over k, and is a subgroup of $\mathbf{Pic}^0_{X/k}$.*

(iii) *If $\mathrm{H}^1(X,\mathcal{O}_X)=0$ or $\mathrm{H}^2(X,\mathcal{O}_X)=0$, or if $\mathrm{char}(k)=0$, then $\mathbf{Pic}^0_{X/k}$ is smooth, so $\mathbf{Pic}^0_{X/k}=\mathbf{Pic}^0_{X/k,\mathrm{red}}$ is an abelian variety of dimension $\dim(\mathrm{H}^1(X,\mathcal{O}_X))$.*

(iv) *The abelian group $\mathrm{NS}(\overline{X})$ is finitely generated. If $\mathrm{char}(k)=p>0$, then the cokernel of the natural map $\mathrm{NS}(X^\mathrm{s})\to\mathrm{NS}(\overline{X})$ is a finite p-group.*

(v) *For $\ell\neq\mathrm{char}(k)$, we have $\mathrm{NS}(X^\mathrm{s})\{\ell\}\cong\mathrm{H}^2_{\text{ét}}(X^\mathrm{s},\mathbb{Z}_\ell(1))\{\ell\}$.*

Proof. Let us prove the second claim of (iv). Each element $\alpha\in\mathrm{NS}(\overline{X})$ lifts to a Cartier divisor D on X_K, where K is a finite, purely inseparable extension of k_s. Let $q=p^n=[K:k_\mathrm{s}]$. Since the morphism $X_K\to X^\mathrm{s}$ is finite and flat, hence locally free, there exists an open covering $X^\mathrm{s}=\cup_{i\in I}U_i$ and a rational function $f_i\in K(U_i)^*$ for $i\in I$ such that $D\cap(U_i\times_{k_\mathrm{s}}K)$ is the divisor of f_i. Then $f_i/f_j\in K[U_i\cap U_j]^*$. We have $f_i^q\in k(U_i)^*$ with $f_i^q/f_j^q\in k[U_i\cap U_j]^*$ for each i and j, hence the family (U_i,f_i^q) defines a Cartier divisor on X^s. It equals p^nD, hence $p^n\alpha\in\mathrm{NS}(X^\mathrm{s})$. Since $\mathrm{NS}(\overline{X})$ is generated by finitely many elements, our claim is proved. For property (v), see (5.13) below. □

Example 5.1.2 There are smooth, projective, geometrically integral surfaces X over an algebraically closed field k such that the group k-scheme $\mathbf{Pic}^0_{X/k}$ is not reduced, hence not smooth. Such are the so-called *non-classical Enriques surfaces* that exist when $\mathrm{char}(k)=2$. These are minimal surfaces of Kodaira dimension zero such that $\mathrm{H}^1_{\text{ét}}(X,\mathbb{Q}_\ell)=0$ and $\dim(\mathrm{H}^2_{\text{ét}}(X,\mathbb{Q}_\ell))=10$ (where $\ell\neq2$) and $\dim(\mathrm{H}^1(X,\mathcal{O}_X))=1$. For these surfaces $\mathbf{Pic}^0_{X/k}$ is $\alpha_2=\mathrm{Spec}(k[t]/(t^2))$ or $\mu_2=\mathrm{Spec}(k[t]/(t^2-1))$, depending on whether the action of Frobenius on $\mathrm{H}^1(X,\mathcal{O}_X)$ is trivial or not. (The classical Enriques surfaces have $\dim(\mathrm{H}^1(X,\mathcal{O}_X))=0$, and hence their Picard scheme is smooth.) See [Dol16] for a detailed treatment and explicit examples.

Corollary 5.1.3 *Let X be a projective, geometrically integral and geometrically normal variety over a field k.*

(i) *If $\mathrm{H}^1(X,\mathcal{O}_X)=0$, then the groups $\mathrm{Pic}(X^\mathrm{s})$, $\mathrm{Pic}(\overline{X})$, $\mathrm{NS}(X^\mathrm{s})$ and $\mathrm{NS}(\overline{X})$ are all equal. In this case this is a finitely generated abelian group.*

(ii) *Assume that $\mathrm{char}(k)=0$. Then $\overline{X}$ has no non-trivial finite, connected, abelian étale cover if and only if $\mathrm{H}^1(X,\mathcal{O}_X)=0$ and $\mathrm{NS}(\overline{X})$ is torsion-free.*

Proof. We only need to prove (ii). By the Kummer sequence, the variety $\overline{X}$ has a non-trivial finite, connected, abelian étale cover if and only if $\mathrm{Pic}(\overline{X})$ has non-trivial torsion, cf. [Mil80, Cor. III.4.19]. □

5.1.2 Albanese variety and Albanese torsor

We continue to assume that X is a projective, geometrically integral and geometrically normal variety over a field k, so that the Picard scheme $\mathbf{Pic}_{X/k}$ exists (see Section 2.5). If, *in addition*, $\mathbf{Pic}_{X/k}$ represents the relative Picard functor $\mathrm{Pic}_{X/k}$, then it is a formal consequence of Yoneda's lemma that the product $X\times_k \mathbf{Pic}_{X/k}$ carries a *universal* invertible sheaf $\mathcal{P}$. This is a sheaf with the following property: for any k-scheme T and any invertible sheaf $\mathcal{L}$ on $X\times_k T$ there exists a unique morphism of k-schemes $h\colon T\to\mathbf{Pic}_{X/k}$ such that $\mathcal{L}\simeq(\mathrm{id},h)^*\mathcal{P}\otimes p_2^*\mathcal{N}$, where $\mathcal{N}$ is an invertible sheaf on T and $p_2\colon X\times_k T\to T$ is the natural projection. (See [Kle05, Ex. 9.4.3].) The sheaf $\mathcal{P}$ is unique up to tensoring with the pullback of an invertible sheaf on $\mathbf{Pic}_{X/k}$. By Corollary 2.5.8, the condition that $\mathbf{Pic}_{X/k}$ represents $\mathrm{Pic}_{X/k}$ is satisfied when X has a k-point. In this case the universal sheaf can be made unique by normalising at such a k-point x_0, i.e., by imposing the condition that the restriction of $\mathcal{P}$ to $x_0\times\mathbf{Pic}_{X/k}$ is trivial. If X is an abelian variety, then $\mathcal{P}$ normalised at $0\in X(k)$ is the usual Poincaré sheaf.

Let $A=\mathbf{Pic}^0_{X/k,\mathrm{red}}$ be the Picard variety of X/k; as recalled above, it is an abelian variety over k. The dual abelian variety $A^\vee=\mathbf{Pic}^0_{A/k}$ is called the *Albanese variety* of X and is denoted by $\mathrm{Alb}_{X/k}$. If X has a k-point x_0, then the sheaf $\mathcal{P}$ on $X\times_k\mathbf{Pic}^0_{X/k}$ normalised at x_0 induces a sheaf on $X\times_k A$ normalised at x_0, hence gives rise to a morphism $X\to\mathrm{Alb}_{X/k}$ which sends x_0 to $0\in A^\vee(k)$.

In general, if X does not necessarily have a k-point, we can find a K-point on X, where K is a finite separable extension of k. By Galois descent, the K-morphism $X\times_k K\to\mathrm{Alb}_{X/k}\times_k K$, normalised so that the chosen K-point goes to 0, descends to a k-morphism $u\colon X\to\mathrm{Alb}^1_{X/k}$, where $\mathrm{Alb}^1_{X/k}$ is a k-torsor for $\mathrm{Alb}_{X/k}$, called the *Albanese torsor*. This morphism $u\colon X\to\mathrm{Alb}^1_{X/k}$ has the following universal property: if T is a k-torsor for an abelian variety B over k and $f\colon X\to T$ is a morphism of k-varieties, then there exists a unique morphism of k-varieties $g\colon\mathrm{Alb}^1_{X/k}\to T$ and a unique morphism of abelian varieties $\alpha\colon\mathrm{Alb}_{X/k}\to B$ such that $f=g\circ u$, where g is compatible with α. In particular, the Albanese torsor is well defined up to translation by a k-point of $\mathrm{Alb}_{X/k}$, whereas the triple $(\mathrm{Alb}_{X/k},\mathrm{Alb}^1_{X/k},u)$ is unique up to a unique isomorphism. See [FGA6] (the statement of Thm. 3.3 (iii), p. 236–17) and [Lang83a]; for a more recent reference see [Witt08]. For a helpful discussion of the Albanese torsor over a not necessarily perfect ground field see `https://mathoverflow.net/questions/260982`. As explained there, the universal property of the Albanese torsor holds for any proper, geometrically reduced and geometrically connected scheme.

5.2 The geometric Brauer group

Let X be a variety over a field k.

Definition 5.2.1 *The group* $\mathrm{Br}(X^{\mathrm{s}})$ *is called the* **geometric** *Brauer group of* X*.*

In this section we investigate the structure of $\mathrm{Br}(X^{\mathrm{s}})$, keeping track of the action of the Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$ on it.

Proposition 5.2.2 *Let* X *be a variety over a separably closed field* k *of characteristic exponent* p*. Let* S *be a non-empty* k*-scheme. The kernel of the map*

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X \times_k S)$$

is a p*-primary torsion group. If* k *is algebraically closed or if* S *is smooth over* k*, then this map is injective.*

Proof. Suppose that $\alpha \in \mathrm{Br}(X)$ is in the kernel of the map. We may replace S by a non-empty affine open set, say $S = \mathrm{Spec}(R)$. The non-zero k-algebra R is a direct filtering limit of non-zero k-algebras A_i of finite type. By Section 2.2.2, $\mathrm{Br}(X_R)$ is the direct limit of the Brauer groups $\mathrm{Br}(X_{A_i})$. Thus there exists a k-algebra of finite type A such that α goes to zero in $\mathrm{Br}(X_A)$. Let m be a maximal ideal of A. The quotient field $K = A/m$, which is a finitely generated k-algebra, is a finite extension of k. Since k is separably closed, the degree $[K : k]$ is a power of p. The homomorphism $A{\to}A/m = K$ induces a homomorphism $\mathrm{Br}(X_A){\to}\mathrm{Br}(X_K)$. Thus α is in the kernel of the map $\mathrm{Br}(X){\to}\mathrm{Br}(X_K)$. A corestriction argument (Section 3.8) gives that α is annihilated by $[K : k]$, which is a power of p and is 1 if k is algebraically closed.

If S is smooth over a separably closed field k, then S has a k-point, so $\mathrm{Br}(X){\to}\mathrm{Br}(X \times_k S)$ is injective in this case. □

We shall see (Theorem 5.6.1) that if k is a field of characteristic p which is separably closed, but not algebraically closed, then the kernel of the map $\mathrm{Br}(\mathbb{A}^1_k){\to}\mathrm{Br}(\mathbb{A}^1_{\bar{k}})$ contains a non-trivial p-torsion subgroup.

Recall that if p is a prime or $p = 1$, then for an abelian group A we write $A(p')$ for the union of ℓ-primary torsion subgroups $A\{\ell\}$ for all primes $\ell \neq p$.

Proposition 5.2.3 *Let* X *be a variety over a separably closed field* k *of characteristic exponent* p*. Then for any separably closed field* K *containing* k *(for example, an algebraic closure of* k*) the map* $\mathrm{Br}(X)(p'){\to}\mathrm{Br}(X_K)(p')$ *is an isomorphism.*

Proof. It is enough to prove that for any prime $\ell \neq p$ and for any $n \geq 1$ the restriction map $\mathrm{Br}(X)[\ell^n] \to \mathrm{Br}(X_K)[\ell^n]$ is an isomorphism. The smooth base change theorem in étale cohomology [Mil80, Cor. VI.4.3] gives isomorphisms

$$\mathrm{H}^i_{\text{ét}}(X, \mu_{\ell^n}) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X_K, \mu_{\ell^n}), \quad i \geq 0.$$

Comparing the Kummer sequences (3.2) for X and X_K, we deduce the surjectivity of $\mathrm{Br}(X)[\ell^n] \to \mathrm{Br}(X_K)[\ell^n]$. The injectivity of this map follows from Proposition 5.2.2. □

Corollary 5.2.4 *Let X and Y be geometrically integral varieties over a field k of characteristic exponent p. Let k_{s} be a separable closure of k and let $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Then the natural map of Γ-modules*

$$\mathrm{Br}(X^{\mathrm{s}})(p') \oplus \mathrm{Br}(Y^{\mathrm{s}})(p') \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})(p')$$

is split injective.

Proof. Let $k(Y)$ be the function field of Y and let $k_{\mathrm{s}}(Y)$ be the function field of Y^{s}. Let L be a separable closure of $k_{\mathrm{s}}(Y)$. The group $\Gamma \cong \mathrm{Gal}(k_{\mathrm{s}}(Y)/k(Y))$ is a quotient of $\Gamma_1 = \mathrm{Gal}(L/k(Y))$. The composition of natural maps

$$\mathrm{Br}(X^{\mathrm{s}})(p') \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})(p') \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times_{k_{\mathrm{s}}} L)(p')$$

is an isomorphism by Proposition 5.2.3. It respects the action of Γ_1, so is an isomorphism of Γ_1-modules, hence of Γ-modules. We note that

$$\mathrm{Br}(Y^{\mathrm{s}})(p') \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times Y^{\mathrm{s}})(p') \longrightarrow \mathrm{Br}(X^{\mathrm{s}} \times_{k_{\mathrm{s}}} L)(p')$$

is the zero map, since it factors through $\mathrm{Br}(L) = 0$. Reversing the roles of X and Y we prove the statement. □

Theorem 5.2.5 *Let X be a proper, geometrically reduced and geometrically connected variety over a separably closed field k.*

(i) *There is an embedding of p-primary torsion groups*

$$\mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})] \hookrightarrow \mathrm{H}^1_{\mathrm{fppf}}(k, \mathbf{Pic}_{X/k}).$$

(ii) *If either $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$ is injective.*

Here we write $\bar{k}$ for an algebraic closure of the separably closed field k and $\overline{X} = X \times_k \bar{k}$.

Proof. Let $p\colon X \to \mathrm{Spec}(k)$ be the structure map. The hypothesis on X implies that for any k-scheme T the map $\mathcal{O}_T \to p_*\mathcal{O}_{X_T}$ is an isomorphism, see Remark 2.5.3. It follows that the natural map $\mathbb{G}_{m,k} \xrightarrow{\sim} p_*\mathbb{G}_{m,X}$ is an isomorphism of sheaves for the fppf topology on $\mathrm{Spec}(k)$.

Since the group scheme $\mathbb{G}_{m,k}$ is smooth and k is separably closed, we have $\mathrm{H}^i_{\mathrm{fppf}}(k,\mathbb{G}_m) \simeq \mathrm{H}^i_{\text{ét}}(k,\mathbb{G}_m) = 0$ for any $i > 0$, see (2.8). By the same result we also have an isomorphism

$$\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X,\mathbb{G}_{m,X}) \cong \mathrm{H}^2_{\mathrm{fppf}}(X,\mathbb{G}_{m,X}). \tag{5.2}$$

Since $\mathrm{H}^i_{\mathrm{fppf}}(k,p_*\mathbb{G}_{m,X}) = \mathrm{H}^i_{\mathrm{fppf}}(k,\mathbb{G}_m) = 0$ for $i > 0$, the Leray spectral sequence

$$\mathrm{H}^p_{\mathrm{fppf}}(k,R^q p_*\mathbb{G}_{m,X}) \Rightarrow \mathrm{H}^{p+q}_{\mathrm{fppf}}(X,\mathbb{G}_{m,X})$$

gives rise to the exact sequence

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(k,R^1 p_*\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^2_{\mathrm{fppf}}(X,\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^0(k,R^2 p_*\mathbb{G}_{m,X}).$$

Since X is proper over a field k, the fppf sheaf $R^1 p_*\mathbb{G}_{m,X}$ is representable by a k-group scheme $\mathbf{Pic}_{X/k}$, see Theorem 2.5.7. Thus, using (5.2), we can rewrite the above exact sequence as follows:

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{fppf}}(k,\mathbf{Pic}_{X/k}) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^0(k,R^2 p_*\mathbb{G}_{m,X}).$$

Since $R^2 p_*\mathbb{G}_{m,X}$ is a sheaf for the fppf topology, the last group is a subgroup of $\mathrm{H}^0(\bar{k},R^2 p_*\mathbb{G}_{m,X})$, so we get a natural map $\mathrm{Br}(X)\to\mathrm{H}^0(\bar{k},R^2 p_*\mathbb{G}_{m,X})$, which coincides with the composition

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \mathrm{H}^0(\bar{k},R^2 p_*\mathbb{G}_{m,X}).$$

This formally implies statement (i).

The definition of the Néron–Severi group k-scheme (5.1) and the property $\mathbf{NS}_{X/k}(k_{\mathrm{s}}) = \mathbf{NS}_{X/k}(\bar{k})$ imply that for $k = k_{\mathrm{s}}$ the k-group scheme $\mathbf{Pic}_{X/k}$ is an extension of the constant group of finite type $\mathbf{NS}_{X/k}(k)$ by the connected group k-scheme $\mathbf{Pic}^0_{X/k}$. If either $\mathrm{H}^1(X,\mathcal{O}_X) = 0$ or $\mathrm{H}^2(X,\mathcal{O}_X) = 0$, then $\mathbf{Pic}^0_{X/k}$ is smooth by Theorem 5.1.1 (iii). Using (2.8) again, we obtain $\mathrm{H}^i_{\mathrm{fppf}}(k,\mathbf{Pic}_{X/k}) \cong \mathrm{H}^i_{\text{ét}}(k,\mathbf{Pic}_{X/k}) = 0$ for all $i \geq 1$. □

Theorem 5.2.6 *Let X be a variety over a separably closed field k. Let $n \geq 1$ be an integer invertible in k. Then the group $\mathrm{H}^i_{\text{ét}}(X,\mathbb{Z}/n)$ is finite for all $i \geq 0$.*

Proof. See [SGA4$\frac{1}{2}$, Finitude, Thm. 1.1]. □

Let A be an abelian group and let ℓ be a prime number. The ℓ-adic *Tate module* $T_\ell(A)$ is defined by

$$T_\ell(A) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A) = \varprojlim_n A[\ell^n],$$

where the transition maps $A[\ell^{n+1}]\to A[\ell^n]$ are multiplications by ℓ. It is easy to check that $T_\ell(A)$ is a torsion-free $\mathbb{Z}_\ell$-module. There are natural injective maps $T_\ell(A)/\ell^n \hookrightarrow A[\ell^n]$. If the group $A[\ell]$ is finite, then the $\mathbb{Z}_\ell$-module $T_\ell(A)$

is finitely generated. By Nakayama's lemma [AM69, Ch. 2, Prop. 2.8], we have $T_\ell(A) \simeq \mathbb{Z}_\ell^r$ where $r \leq \dim_{\mathbb{F}_\ell}(A[\ell])$. If, moreover, A is an ℓ-primary torsion abelian group, then $T_\ell(A) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is the divisible subgroup $A_{\rm div}$ of A.

Let X be a variety over a field k and let ℓ be a prime invertible in k. For any integer j the sheaves $\mu_{\ell^n}^{\otimes j}$ on $X_{\text{ét}}$, $n \geq 1$, form an inverse system with respect to the natural surjective maps $\mu_{\ell^{n+1}}^{\otimes j} \to \mu_{\ell^n}^{\otimes j}$. Then one shows that

$$\mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Z}_\ell(j)) := \varprojlim_n \mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mu_{\ell^n}^{\otimes j})$$

is isomorphic to $\varprojlim_n \mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Z}_\ell(j))/\ell^n$, from which one deduces that it is a finitely generated $\mathbb{Z}_\ell$-module [Mil80, Lemma V.1.11]. The group $\mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Z}_\ell(j))$ carries a continuous action of the Galois group $\Gamma = \mathrm{Gal}(k_{\rm s}/k)$. One defines

$$\mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Q}_\ell(j)) := \mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Z}_\ell(j)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Remark 5.2.7 For ℓ different from $\mathrm{char}(k)$, let $b_{i,\ell} = \dim_{\mathbb{Q}_\ell} \mathrm{H}^i_{\text{ét}}(X^{\rm s}, \mathbb{Q}_\ell)$ be the i-th ℓ-adic *Betti number* of $X^{\rm s}$. If $\mathrm{char}(k) > 0$ and X is proper and smooth over k, then for a given i the Betti number $b_{i,\ell}$ does not depend on ℓ when k is a finite field, as follows from Deligne's results on the Weil conjectures [Del80, Cor. 3.3.9]. The case of an arbitrary field of positive characteristic follows from this by a spreading out argument, invariance of étale cohomology under extensions of separably closed ground fields, and the proper base change theorem. If $\mathrm{char}(k) = 0$, the Betti numbers do not depend on ℓ for any k-variety. Indeed, this follows from the comparison theorem between ℓ-adic étale and Betti cohomology [SGA4, XVI, Thm. 4.1]

$$\mathrm{H}^i(X, \mathbb{Q}) \otimes \mathbb{Q}_\ell \simeq \mathrm{H}^i_{\text{ét}}(X, \mathbb{Q}_\ell),$$

where X is a variety over $\mathbb{C}$, and from invariance of étale cohomology under extensions of separably closed ground fields.

For each $n \geq 1$ the Kummer sequence gives rise to the exact sequence (3.4) of Γ-modules:

$$0 \longrightarrow \mathrm{Pic}(X^{\rm s})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\rm s}, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\rm s})[\ell^n] \longrightarrow 0. \qquad (5.3)$$

These sequences are compatible when n varies. By Theorem 5.2.6 these are sequences of finite groups. In particular we have:

Corollary 5.2.8 *Let X be a variety over a field k and let n be a positive integer not divisible by* $\mathrm{char}(k)$. *Then the group* $\mathrm{Br}(X^{\rm s})[n]$ *is finite.*

Let ℓ be a prime number. An ℓ-primary torsion abelian group A is of *cofinite type* if it is isomorphic to a direct sum of a finite number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ and a finite abelian ℓ-primary group. It is the Pontryagin dual of a finitely generated $\mathbb{Z}_\ell$-module.

Recall that $\mathrm{Br}(X^{\mathrm{s}})(p')$ is the prime-to-p torsion subgroup of $\mathrm{Br}(X^{\mathrm{s}})$. By Proposition 5.2.3 we have $\mathrm{Br}(X^{\mathrm{s}})(p') \cong \mathrm{Br}(\overline{X})(p')$.

By definition, the Picard number ρ is the rank of $\mathrm{NS}(X^{\mathrm{s}})$.

Proposition 5.2.9 *Let X be a projective, geometrically integral and geometrically normal variety over a field k of characteristic exponent p. Let $\ell \neq p$ be a prime. Let $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}}$ be the maximal divisible subgroup of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$. Then the following statements hold.*

(i) *There is an exact sequence of Γ-modules, which are ℓ-primary torsion groups of cofinite type*

$$0 \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\} \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{5.4}$$

and $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$ is finite. There is an isomorphism

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}} \cong \left(\frac{\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))}{\mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell}\right) \otimes_{\mathbb{Z}_\ell} (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_{2,\ell}-\rho}. \tag{5.5}$$

(ii) *Suppose that X is a smooth, projective and geometrically integral variety over k. Then $\mathrm{Br}(X^{\mathrm{s}})$ is a torsion group and the Betti number $b_{2,\ell}$ does not depend on ℓ. There is an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Br}(X^{\mathrm{s}})(p')_{\mathrm{div}} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})(p') \longrightarrow \bigoplus_{\ell \neq p} \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow 0, \tag{5.6}$$

where the direct sum is a finite group. If, moreover, $k \subset \mathbb{C}$, then the finite abelian group $\bigoplus_\ell \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$ is isomorphic to the torsion subgroup of $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z})$, and $b_{2,\ell} = \dim_{\mathbb{C}} \mathrm{H}^2(X(\mathbb{C}), \mathbb{C})$.

(iii) *Suppose that $\mathrm{char}(k) = 0$ and X is a smooth, projective and geometrically integral variety over k. Then we have a natural exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors} \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q} \\ \longrightarrow \textstyle\bigoplus_\ell \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}} \longrightarrow 0. \end{aligned} \tag{5.7}$$

Proof. (i) Since the terms of (5.3) are finite groups, taking the inverse limit in (5.3) preserves exactness, so we obtain exact sequences of Γ-modules which are finitely generated $\mathbb{Z}_\ell$-modules

$$0 \longrightarrow \varprojlim_n \mathrm{Pic}(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \tag{5.8}$$

Passing to the direct limit in (5.3) we obtain exact sequences of Γ-modules which are ℓ-primary torsion abelian groups:

$$0 \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\} \longrightarrow 0. \tag{5.9}$$

The groups are of cofinite type, see the proof of (5.4) below.

By Theorem 5.1.1 (i) we have an exact sequence

$$0 \longrightarrow \mathbf{Pic}^0_{X/k}(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \longrightarrow 0.$$

Multiplication by ℓ^n on the Picard variety A of X is a finite étale morphism, because A is an abelian variety and $\ell \neq \mathrm{char}(k)$, hence it is surjective on k_{s}-points. Thus $\mathbf{Pic}^0_{X/k}(k_{\mathrm{s}}) = A(k_{\mathrm{s}})$ is divisible by ℓ^n. Hence we obtain compatible isomorphisms of Γ-modules

$$\mathrm{Pic}(X^{\mathrm{s}})/\ell^n \cong \mathrm{NS}(X^{\mathrm{s}})/\ell^n \tag{5.10}$$

for all integers n. The abelian group $\mathrm{NS}(X^{\mathrm{s}})$ is finitely generated by Theorem 5.1.1 (iv). Thus from (5.9) we get the exact sequence of ℓ-primary torsion groups of cofinite type

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\} \longrightarrow 0. \tag{5.11}$$

From (5.8) we get the exact sequence of finitely generated $\mathbb{Z}_\ell$-modules

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \tag{5.12}$$

Since $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ is torsion-free, this gives an isomorphism of torsion subgroups

$$\mathrm{NS}(X^{\mathrm{s}})\{\ell\} \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}, \tag{5.13}$$

and the isomorphism (5.5). The maps in (5.11), (5.12), (5.13) respect the action of the Galois group Γ.

Let us explain how to obtain (5.4). Consider a commutative diagram of sheaves on $X_{\text{ét}}$ with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_{\ell^m} & \longrightarrow & \mu_{\ell^{m+n}} & \longrightarrow & \mu_{\ell^n} & \longrightarrow & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_{\ell^m} & \longrightarrow & \mu_{\ell^{m+n+1}} & \longrightarrow & \mu_{\ell^{n+1}} & \longrightarrow & 1
\end{array}$$

Taking the inverse limit over m in the associated long exact sequences of étale cohomology groups (using their finiteness) we obtain a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) & \xrightarrow{\ell^n} & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \\
\downarrow{=} & & \downarrow{\ell} & & \downarrow & & \downarrow{=} \\
\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) & \xrightarrow{\ell^{n+1}} & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^{n+1}}) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))
\end{array}$$

Now taking the direct limit over n one obtains an exact sequence

$$\ldots \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$$
$$\longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell(1)) \longrightarrow \ldots,$$

hence an exact sequence

$$0\rightarrow\frac{\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell(1))}{\mathrm{Im}(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1)))} \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))_{\mathrm{tors}}\rightarrow 0.$$

Passing to the quotients by $\mathrm{NS}(X^{\mathrm{s}})\otimes\mathbb{Q}_\ell/\mathbb{Z}_\ell$ in the first and second terms, we obtain (5.4).

(ii) If X is smooth over k, then for any field extension $k\subset K$ the group $\mathrm{Br}(X\times_k K)$ is a torsion group (Lemma 3.5.3). If X is projective and smooth, then for a given i the Betti number $b_{i,\ell}$ does not depend on ℓ (Remark 5.2.7). This proves the first claim. The étale cohomology groups of a variety over k_{s} with coefficients in a torsion sheaf of order not divisible by $\mathrm{char}(k)$ do not change under extension of k_{s} to a bigger separably closed field, for example $\bar{k}$, see [Mil80, Cor. VI.4.3]. This implies the second claim. If X is projective, smooth and geometrically integral over k, then by a special case of a theorem of Gabber [Gab83], for almost all ℓ the group $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))$ is torsion-free. If k has characteristic zero, this is also a consequence of the comparison theorem between étale cohomology and classical Betti cohomology, see [Mil80, Thm. III.3.12]. Now the third claim follows from (i). Finally, in the case $k_{\mathrm{s}}\subset\mathbb{C}$ we have $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))=\mathrm{H}^3_{\text{ét}}(X\times_k\mathbb{C},\mathbb{Z}_\ell(1))$. The comparison theorem [Mil80, Thm. III.3.12] says that the latter group is isomorphic to the Betti cohomology group $\mathrm{H}^3(X\times_k\mathbb{C},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}_\ell(1)$.

These arguments also prove (iii). □

We need to recall Poincaré duality for a smooth, proper, connected variety X^{s} of dimension d over k_{s}. For any $m\geq 1$ we have compatible pairings of abelian groups

$$\begin{array}{ccccc}
\mathbb{Z}/\ell^m & \times & \mathrm{Hom}_{\mathbb{Z}/\ell^m}(\mathbb{Z}/\ell^m,\mathbb{Z}/\ell^m) & \longrightarrow & \mathbb{Z}/\ell^m \\
\downarrow & & \uparrow & & \downarrow \\
\mathbb{Z}/\ell^{m+1} & \times & \mathrm{Hom}_{\mathbb{Z}/\ell^{m+1}}(\mathbb{Z}/\ell^{m+1},\mathbb{Z}/\ell^{m+1}) & \longrightarrow & \mathbb{Z}/\ell^{m+1}
\end{array}$$

For any $i=0,\ldots,2d$ they give rise to compatible bilinear pairings

$$\begin{array}{ccccc}
\mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}/\ell^m) & \times & \mathrm{Ext}^{2d-i}_{X^{\mathrm{s}},\mathbb{Z}/\ell^m}(\mathbb{Z}/\ell^m,\mathbb{Z}/\ell^m(d)) & \rightarrow & \mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}/\ell^m(d)) \\
\downarrow & & \uparrow & & \downarrow \\
\mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}/\ell^{m+1}) & \times & \mathrm{Ext}^{2d-i}_{X^{\mathrm{s}},\mathbb{Z}/\ell^{m+1}}(\mathbb{Z}/\ell^{m+1},\mathbb{Z}/\ell^{m+1}(d)) & \rightarrow & \mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}/\ell^{m+1}(d))
\end{array}$$

Here $\mathrm{Ext}^n_{X^{\mathrm{s}},\mathbb{Z}/\ell^m}(-,-)$ is taken in the category of étale sheaves of $\mathbb{Z}/\ell^m$-modules on X^{s}.

The spectral sequence

$$\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}}, \mathcal{E}xt^q_{X^{\mathrm{s}},\,\mathbb{Z}/\ell^m}(\mathbb{Z}/\ell^m, \mathbb{Z}/\ell^m(d))) \Rightarrow \mathrm{Ext}^{p+q}_{X^{\mathrm{s}},\,\mathbb{Z}/\ell^m}(\mathbb{Z}/\ell^m, \mathbb{Z}/\ell^m(d))$$

degenerates since $\mathcal{E}xt^q_{X^{\mathrm{s}},\,\mathbb{Z}/\ell^m}(\mathbb{Z}/\ell^m, \mathbb{Z}/\ell^m) = 0$ for $q \geq 1$, see [Mil80, Exercise III.1.31 (c)]. This allows us to rewrite the previous diagram as

$$\begin{array}{ccccc} \mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^m) & \times & \mathrm{H}^{2d-i}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^m(d)) & \longrightarrow & \mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^m(d)) \\ \downarrow & & \uparrow & & \downarrow \\ \mathrm{H}^i_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^{m+1}) & \times & \mathrm{H}^{2d-i}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^{m+1}(d)) & \longrightarrow & \mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^{m+1}(d)) \end{array}$$

The Poincaré duality theorem [Mil80, Thm. VI.11.1] gives an isomorphism

$$\mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}/\ell^m(d)) \xrightarrow{\sim} \mathbb{Z}/\ell^m,$$

sending the image of the fundamental class of any k_{s}-point $s_{P/X}$ to 1, and making the above pairings perfect dualities [Mil80, Cor. VI.11.2]. These pairings coincide with the pairings given by the cup-product [Mil80, Prop. V.1.20].

Proposition 5.2.10 *Let X be a smooth, proper, geometrically integral surface over a field k. Then for every prime $\ell \neq \mathrm{char}(k)$ there is a natural isomorphism of finite Γ-modules*

$$\mathrm{Br}(X^{\mathrm{s}})\{\ell\}/\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}} \cong \mathrm{Hom}(\mathrm{NS}(X^{\mathrm{s}})\{\ell\}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Proof. As recalled above, the Poincaré duality theorem for the surface X^{s} gives a perfect duality pairing

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^m}) \times \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^m}) \longrightarrow \mathrm{H}^4_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^m}^{\otimes 2}) \cong \mathbb{Z}/\ell^m \subset \mathbb{Q}_\ell/\mathbb{Z}_\ell, \quad (5.14)$$

hence a canonical isomorphism

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^m}) \cong \mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^m}), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

The pairings (5.14) for m and $m+1$ are compatible with respect to the natural inclusion $\mu_{\ell^m} \hookrightarrow \mu_{\ell^{m+1}}$ in the first argument, the natural surjective map $\mu_{\ell^{m+1}} \to \mu_{\ell^m}$ given by raising to the power ℓ in the second argument, and the natural injective map $\mathbb{Z}/\ell^m \hookrightarrow \mathbb{Z}/\ell^{m+1}$. Thus going over to the limit one obtains an isomorphism of abelian groups of cofinite type

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \cong \mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)), \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

Taking the quotients of these groups by their maximal divisible subgroups (as given by sequence (5.2)), we obtain an isomorphism of finite groups

$$\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \cong \mathrm{Hom}(\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

This gives a perfect duality pairing of finite abelian groups

$$\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \times \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

All these constructions are Galois equivariant. The result now follows from the isomorphism (5.13) and the exact sequence (5.4). □

After classical work of Godeaux and of Campedelli, surfaces X over $\mathbb{C}$ with $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ and $\mathrm{NS}(X)_{\mathrm{tors}} \neq 0$ have been much discussed in the literature, see [BPV84, Ch. VII, §11] and [BCGP12].

Remark 5.2.11 Proposition 5.2.9 gives a precise formula for the size of the Brauer group of a smooth projective variety X over $\mathbb{C}$. In practice, it is very hard to explicitly represent the elements of $\mathrm{Br}(X)$ by Azumaya algebras over X. It is also hard to represent their images in $\mathrm{Br}(\mathbb{C}(X))$ by central simple algebras or sums of symbols – which they are according to the Merkurjev–Suslin theorem [GS17, Thm. 8.6.5].

5.3 The Tate module of the Brauer group as a Galois representation

Let k be a field with separable closure k_{s} and absolute Galois group $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Let X be a smooth, proper, geometrically integral variety over k. Let ℓ be a prime such that $\ell \neq \mathrm{char}(k)$. From Theorem 5.2.9 we know that $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ is an abelian group of cofinite type. More precisely, it is an extension of a finite abelian group by the divisible subgroup

$$\mathrm{Br}^0(X^{\mathrm{s}})\{\ell\} = T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}.$$

Let $V_\ell(\mathrm{Br}(X^{\mathrm{s}})) = T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. This is a vector space over $\mathbb{Q}_\ell$ of dimension $b_2 - \rho$. Write cl_ℓ for the ℓ-adic cycle class map $\mathrm{NS}(X^{\mathrm{s}}) \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$, defined as the second map in the exact sequence (5.12). Tensoring the terms of (5.12) with $\mathbb{Q}_\ell$ we obtain an exact sequence of Γ-modules

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \xrightarrow{\mathrm{cl}_\ell} \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \quad (5.15)$$

In view of the isomorphism (5.10) the exact sequence (5.3) is an exact sequence of Γ-modules

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}})/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n}) \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n] \longrightarrow 0. \quad (5.16)$$

We set $\nu := \mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}$ and write ν_ℓ for the ℓ-primary subgroup of ν. Recall that we have canonical isomorphisms (5.13):

$$\nu_\ell \cong \mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}} \otimes \mathbb{Z}_\ell \cong \mathrm{NS}(X^{\mathrm{s}})\{\ell\} \cong \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}.$$

Theorem 5.3.1 *Let X be a smooth, projective, geometrically integral variety over a field k.*

(i) *There exist a positive integer N and a Γ-submodule $M_\ell \subset T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ for every prime $\ell \neq \mathrm{char}(k)$, such that $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))/M_\ell$ is annihilated by N and the pullback of (5.12) to M_ℓ is a split exact sequence of Γ-modules. In particular, the ℓ-adic cycle class map $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ is split injective for almost all primes ℓ, making the Γ-module $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ a direct summand of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ for these primes.*

(ii) *For all primes $\ell \neq \mathrm{char}(k)$, the exact sequence of Γ-modules (5.15) splits.*

(iii) *For almost all primes ℓ the exact sequence (5.16) is a split exact sequence of Γ-modules, for all $n \geq 1$. In particular, for almost all primes ℓ the Γ-module $\mathrm{NS}(X^{\mathrm{s}})/\ell^n$ is a direct summand of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n})$, for all $n \geq 1$.*

(iv) *For almost all primes ℓ and all positive integers n there is a split exact sequence of abelian groups*

$$0 \longrightarrow (\mathrm{NS}(X^{\mathrm{s}})/\ell^n)^\Gamma \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_{\ell^n})^\Gamma \longrightarrow \mathrm{Br}(X^{\mathrm{s}})[\ell^n]^\Gamma \longrightarrow 0. \tag{5.17}$$

Recall that the cup-product gives rise to a bilinear pairing

$$\cup: \quad \mathrm{H}^{2i}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(i)) \times \mathrm{H}^{2j}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(j)) \longrightarrow \mathrm{H}^{2(i+j)}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(i+j)).$$

The intersection of cycles is a symmetric bilinear pairing between Chow groups

$$\mathrm{CH}^i(X^{\mathrm{s}}) \times \mathrm{CH}^j(X^{\mathrm{s}}) \longrightarrow \mathrm{CH}^{i+j}(X^{\mathrm{s}}).$$

The two pairings are compatible via the cycle class map

$$\mathrm{cl}_\ell \colon \mathrm{CH}^i(X^{\mathrm{s}}) \longrightarrow \mathrm{H}^{2i}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(i)),$$

namely, for any $a \in \mathrm{CH}^i(X^{\mathrm{s}})$ and $b \in \mathrm{CH}^j(X^{\mathrm{s}})$ we have $\mathrm{cl}_\ell(a) \cup \mathrm{cl}_\ell(b) = \mathrm{cl}_\ell\big((a \cdot b)_{X^{\mathrm{s}}}\big)$, see [Lau76, Cor. 7.2.1]. Thus, for a fixed $L \in \mathrm{NS}(X^{\mathrm{s}})$, the $\mathbb{Z}_\ell$-valued symmetric bilinear form on $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ given by

$$x \cup y \cup \mathrm{cl}_\ell(L)^{d-2} \in \mathrm{H}^{2d}_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(d)) \xrightarrow{\sim} \mathbb{Z}_\ell$$

restricts to an integral symmetric bilinear form $(a \cdot b \cdot L^{d-2})_{X^{\mathrm{s}}}$ on $\mathrm{NS}(X^{\mathrm{s}})$. The last form clearly descends to a form on $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors} = \mathrm{NS}(X^{\mathrm{s}})/\nu$.

The proof of Theorem 5.3.1 is based on the following general fact.

Lemma 5.3.2 *Let X be a smooth, projective, geometrically integral variety over a field k such that $d = \dim(X) \geq 2$. Let $\ell \neq \mathrm{char}(k)$ be a prime. Let $L \in \mathrm{NS}(X^{\mathrm{s}})$ be the class of an ample line bundle on X. Then the integral symmetric bilinear form $\mathrm{cl}_\ell(x) \cup \mathrm{cl}_\ell(y) \cup L^{d-2} = (x \cdot y \cdot L^{d-2})_{X^{\mathrm{s}}}$ on $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$ is non-degenerate, i.e., its kernel is trivial.*

Proof. Let $\mathrm{NS}(X^{\mathrm{s}})^L \subset \mathrm{NS}(X^{\mathrm{s}})$ be the kernel of the map $\mathrm{NS}(X^{\mathrm{s}}) \to \mathbb{Z}$ given by $x \mapsto (x \cdot L^{d-1})_{X^{\mathrm{s}}}$. It is clear that $\nu \subset \mathrm{NS}(X^{\mathrm{s}})^L$. Since $L^d > 0$ and $\mathrm{NS}(X^{\mathrm{s}})^L$ is the orthogonal complement to L, it is enough to show that the restriction of our form to $\mathrm{NS}(X^{\mathrm{s}})^L/\mathrm{tors}$ is negative definite. If $d = 2$ this statement is a consequence of the Hodge index theorem when $\mathrm{char}(k) = 0$, but it is actually true in any characteristic [Gro58].

The case $d \geq 3$ is reduced to the case $d = 2$ as follows.

The field k_{s} is infinite, so by the Bertini theorem there is a smooth hyperplane section $Y \subset X^{\mathrm{s}}$ defined over k_{s}. By the hyperplane (weak) Lefschetz theorem, the restriction map

$$r\colon \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}_\ell(1))$$

is an isomorphism for $d \geq 4$ and is injective for $d = 3$, see [Kat04, Thm. B.4]. For any $x, y \in \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ we have

$$x \cup y \cup \mathrm{cl}_\ell(L)^{d-2} = r(x) \cup r(y) \cup \mathrm{cl}_\ell(L|_Y)^{d-3} \ \in \ \mathbb{Z}_\ell.$$

Similarly, for $x, y \in \mathrm{NS}(X^{\mathrm{s}})$ we have

$$(x \cdot y \cdot L^{d-2})_{X^{\mathrm{s}}} = (r(x) \cdot r(y) \cdot (L|_Y)^{d-3})_Y \ \in \ \mathbb{Z}.$$

The natural restriction map $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \to \mathrm{NS}(Y) \otimes \mathbb{Z}_\ell$ is identified with the map $r\colon \mathrm{cl}_\ell(\mathrm{NS}(X^{\mathrm{s}})) \otimes \mathbb{Z}_\ell \to \mathrm{cl}_\ell(\mathrm{NS}(Y)) \otimes \mathbb{Z}_\ell$, which is injective since $d \geq 3$. Applying this argument $d-2$ times we obtain a smooth k_{s}-surface $S \subset X^{\mathrm{s}}$ such that the natural map $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell \to \mathrm{NS}(S) \otimes \mathbb{Z}_\ell$ is injective. This gives rise to an injective map $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors} \subset \mathrm{NS}(S)/\mathrm{tors}$. Moreover, the restriction of the intersection form $(x \cdot y)_S$ on $\mathrm{NS}(S)/\mathrm{tors}$ to $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$ is our original form $(x \cdot y \cdot L^{d-2})_{X^{\mathrm{s}}}$.

Define $\mathrm{NS}(S)^L \subset \mathrm{NS}(S)$ as the orthogonal complement to $L|_S$ with respect to the intersection pairing $(x \cdot y)_S$. Then $\mathrm{NS}(X^{\mathrm{s}})^L/\mathrm{tors} \subset \mathrm{NS}(S)^L/\mathrm{tors}$. Since the form $(x \cdot y)_S$ on $\mathrm{NS}(S)^L/\mathrm{tors}$ is negative definite, the form $(x \cdot y \cdot L^{d-2})_{X^{\mathrm{s}}}$ on $\mathrm{NS}(X^{\mathrm{s}})^L/\mathrm{tors}$ is negative definite, hence non-degenerate. □

Proof of Theorem 5.3.1. Let $d = \dim(X)$. For $d = 1$ all statements are trivial, so we can assume $d \geq 2$. Let $L \in \mathrm{NS}(X^{\mathrm{s}})$ be the class of an ample line bundle defined over k. Thus $L \in \mathrm{NS}(X^{\mathrm{s}})^\Gamma$, hence the symmetric bilinear form $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$ on $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ is Γ-invariant.

Let us prove (i). Let M'_ℓ be the orthogonal complement to $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$ in $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))$ with respect to $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$. Since this form is Γ-invariant, M'_ℓ is a Γ-submodule. Note that M'_ℓ contains ν_ℓ, the torsion subgroup of $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell$. The discriminant $\delta \in \mathbb{Z}$ of the integral symmetric bilinear form $(x \cdot y \cdot L^{d-2})_{X^{\mathrm{s}}}$ on $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$, which is the restriction of $x \cup y \cup \mathrm{cl}_\ell(L)^{d-2}$, is non-zero by Lemma 5.3.2. Let $N = \delta \cdot |\nu|$ and let $M_\ell = |\nu_\ell| M'_\ell$. Then $M_\ell \cap (\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell) = 0$ and we have

$$N\, \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \subset M_\ell \oplus (\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell) \subset \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)).$$

The restriction of the surjective map $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))\to T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ in (5.12) to M_ℓ is an isomorphism, hence the pullback of (5.12) to M_ℓ is a split short exact sequence of Γ-modules. This proves (i), from which (ii) follows immediately.

Let us prove (iii). From (i) we deduce that the Γ-module $\mathrm{NS}(X^{\mathrm{s}})/\ell^n$ is a direct summand of the Γ-module $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))/\ell^n$ for almost all ℓ. We have an exact sequence

$$0 \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))/\ell^n \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu_{\ell^n}) \longrightarrow \mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))[\ell^n] \longrightarrow 0.$$

By a theorem of Gabber [Gab83], the $\mathbb{Z}_\ell$-module $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell)$ has no torsion for almost all ℓ. Since $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell)$ and $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))$ are isomorphic as abelian groups, for almost all ℓ we have $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))[\ell] = 0$, hence $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mu_{\ell^n}) = \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))/\ell^n$. This proves (iii).

Finally, (iv) follows directly from (iii) and (5.16). □

Remark 5.3.3 (1) A slightly different argument allows one to prove Theorem 5.3.1 without using Lemma 5.3.2, so there is no need to choose an ample divisor L. Let $\overline{X} = X \times \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Consider the intersection pairing

$$\mathrm{CH}^1(\overline{X}) \times \mathrm{CH}^{d-1}(\overline{X}) \longrightarrow \mathbb{Z}. \tag{5.18}$$

Let $\mathrm{Num}^1(\overline{X})$ be the quotient of $\mathrm{CH}^1(\overline{X}) = \mathrm{Pic}(\overline{X})$ by the left kernel of this paring. By Matsusaka's theorem the kernel of the natural map $\mathrm{NS}(\overline{X})\to\mathrm{Num}^1(\overline{X})$ is the finite group $\nu = \mathrm{NS}(\overline{X})_{\mathrm{tors}}$. This implies that one can find a Γ-invariant free abelian group $W \subset \mathrm{CH}^{d-1}(\overline{X})$ such that the intersection of cycles defines a non-degenerate pairing $\mathrm{NS}(\overline{X})/\mathrm{tors}\times W\to\mathbb{Z}$ (in the sense that the right and left kernels of this pairing are trivial). This gives a natural embedding $\mathrm{NS}(\overline{X})/\mathrm{tors} \subset \mathrm{Hom}(W,\mathbb{Z})$ with finite cokernel, say of order δ. Let $W_\ell^\perp \subset \mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ be the orthogonal complement to the image of W in $\mathrm{H}^{2d-2}_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ with respect to the cup-product pairing. Define $M_\ell = |\nu_\ell| W_\ell^\perp$. Then $M_\ell \cap (\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell) = 0$ and

$$N\,\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1)) \subset M_\ell \oplus (\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell) \subset \mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1)),$$

where $N = \delta\cdot|\nu|$, and we conclude as before. (Since $\ell \neq \mathrm{char}(k)$, the difference between k_{s} and $\bar{k}$ is not important here, see Proposition 5.2.3.)

(2) If $\mathrm{char}(k) = 0$, there is a canonical splitting of (5.15). Let $\mathrm{Num}_1(\overline{X})$ be the quotient of $\mathrm{CH}^{d-1}(\overline{X})$ by the right kernel of (5.18). By a theorem of Lieberman, in this case homological and numerical equivalences coincide in dimension 1, so $\mathrm{Num}_1(\overline{X})$ is the image of $\mathrm{CH}^{d-1}(\overline{X})$ in the quotient of $\mathrm{H}^{2d-2}_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$ by its torsion subgroup. We have a non-degenerate, Γ-invariant pairing

$$\mathrm{NS}(\overline{X})/\mathrm{tors} \;\times\; \mathrm{Num}_1(\overline{X}) \longrightarrow \mathbb{Z}.$$

The above arguments then apply. See [CTS13b, §1.1] and [Qin, Prop. 2.1] for details and references to the literature.

5.4 Algebraic and transcendental Brauer groups

5.4.1 The Picard group and the algebraic Brauer group

For a variety X over a field k there is a natural filtration on the Brauer group

$$\mathrm{Br}_0(X) \subset \mathrm{Br}_1(X) \subset \mathrm{Br}(X),$$

which is defined as follows.

Definition 5.4.1 *Let*

$$\mathrm{Br}_0(X) = \mathrm{Im}[\mathrm{Br}(k) \to \mathrm{Br}(X)], \quad \mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})].$$

The **algebraic** *Brauer group of X is the subgroup $\mathrm{Br}_1(X) \subset \mathrm{Br}(X)$. The* **transcendental** *Brauer group of X is the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$.*

A particular case of the Leray spectral sequence (2.5) for the structure morphism $X \to \mathrm{Spec}(k)$ is the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m), \tag{5.19}$$

which is contravariant functorial in the k-variety X. It gives rise to the functorial exact sequence of terms of low degree

$$0 \longrightarrow \mathrm{H}^1(k, k_{\mathrm{s}}[X]^*) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{H}^2(k, k_{\mathrm{s}}[X]^*) \longrightarrow \mathrm{Br}_1(X)$$
$$\longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}[X]^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \tag{5.20}$$

Proposition 5.4.2 *Let X be a variety over a field k such that $k_{\mathrm{s}}[X]^* = k_{\mathrm{s}}^*$. Then there is an exact sequence*

$$\begin{aligned} &0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X) \\ &\longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{Ker}[\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)]. \end{aligned} \tag{5.21}$$

This sequence is contravariantly functorial in X.

Proof. This follows from (5.20), as $\mathrm{H}^1(k, k_{\mathrm{s}}^*) = 0$ by Hilbert's theorem 90. □

The assumption of Proposition 5.4.2 holds for any proper, geometrically connected and geometrically reduced k-variety X. It also holds for $X = \mathbb{A}^n_k$.

Remark 5.4.3 Let X be a variety over a field k such that $k_{\mathrm{s}}[X]^* = k_{\mathrm{s}}^*$.

(1) If X has a k-point or, more generally, if X has a zero-cycle of degree 1, then each of the maps $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ and $\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3_{\text{ét}}(X, \mathbb{G}_m)$ in (5.21) has a retraction, hence is injective. (Then $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathrm{s}})^{\Gamma}$ is an isomorphism.) Indeed, a k-point on X defines a section of the structure morphism

$X \to \mathrm{Spec}(k)$. A standard restriction-corestriction argument (see Section 3.8) reduces the case when X has a zero-cycle of degree 1 to the case when X has a k-point.

(2) The map $\mathrm{Br}_1(X) \to \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ is surjective when there exists a variety Y over k such that $k_{\mathrm{s}}[Y]^* = k_{\mathrm{s}}^*$ and $\mathrm{H}^1(k, \mathrm{Pic}(Y^{\mathrm{s}})) = 0$, and there is a morphism $Y \to X$. This follows by comparing (5.21) for X and Y. These conditions on Y are satisfied for proper, geometrically connected and geometrically reduced k-varieties Y such that $\mathrm{Pic}(Y^{\mathrm{s}})$ is a permutation Γ-module. This holds, for example, when Y is a smooth projective quadric of dimension at least 1 or a Brauer–Severi variety.

(3) If k is a number field, then $\mathrm{H}^3(k, k_{\mathrm{s}}^*) = 0$, see [CF67, Ch. VII, §11.4, p. 199]. Thus for a variety X over a number field k such that $k_{\mathrm{s}}[X]^* = k_{\mathrm{s}}^*$ we have an isomorphism $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$.

Proposition 5.4.4 *Let k be a field finitely generated over $\mathbb{Q}$. For any geometrically integral variety X over k, the kernel of the restriction map* $\mathrm{Br}(k) \to \mathrm{Br}(X)$ *is a finite group.*

Proof. Let $U \subset X$ be the smooth locus of X. By Hironaka's theorem there exists a smooth, projective, geometrically integral variety Y over k which contains U as an open set. The kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is contained in the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(Y))$, where $k(Y)$ is the function field of Y. By Theorem 3.5.5, the latter kernel is contained in the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(Y)$, which is a quotient of $\mathrm{Pic}(Y^{\mathrm{s}})^{\Gamma}$, by (5.21).

Let A be the Picard variety of Y. By Theorem 5.1.1 (i) the Γ-module $\mathrm{Pic}(Y^{\mathrm{s}})$ is an extension of the Néron–Severi group $\mathrm{NS}(Y^{\mathrm{s}})$, which is finitely generated, by $A(k_{\mathrm{s}})$. The group of k-points $A(k) = A(k_{\mathrm{s}})^{\Gamma}$ is finitely generated, by the theorem of Mordell–Weil–Néron, see [Con06, Cor. 7.2]. Thus $\mathrm{Pic}(Y^{\mathrm{s}})^{\Gamma}$ is a finitely generated abelian group, and hence so is the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(Y)$. By a restriction–corestriction argument, this kernel is annihilated by the degree of any closed point on Y, and thus is finite. □

5.4.2 Geometric interpretation of differentials

Proposition 5.4.5 *Let X be a smooth and geometrically integral variety over a field k. For each $n \geq 0$ the differential*

$$\mathrm{H}^n(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^{n+2}(k, k_{\mathrm{s}}[X]^*) \tag{5.22}$$

from the spectral sequence (5.19) coincides, up to sign, with the connecting map defined by the 2-extension of Γ-modules

$$0 \longrightarrow k_{\mathrm{s}}[X]^* \longrightarrow k_{\mathrm{s}}(X)^* \longrightarrow \mathrm{Div}(X^{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow 0. \tag{5.23}$$

Proof. This follows from the general description of connecting maps given in [Sko07, Prop. 1.1], combined with [Sko01, Thm. 2.3.4 (a)]. □

Remark 5.4.6 The differential (5.22) can be seen as the map attached to the exact triangle

$$p_*(\mathbb{G}_{m,X}) \longrightarrow \tau_{[0,1]}(\mathbf{R}p_*(\mathbb{G}_{m,X})) \longrightarrow (R^1p_*)(\mathbb{G}_{m,X})[-1]$$

in the bounded below derived category $\mathcal{D}^+(k)$ of Γ-modules. Here we write $p\colon X\to\mathrm{Spec}(k)$ for the structure morphism, $\mathbf{R}p_*: \mathcal{D}^+(X)\to\mathcal{D}^+(k)$ is the derived functor from the bounded below derived category $\mathcal{D}^+(X)$ of étale sheaves on X to $\mathcal{D}^+(k)$, and $\tau_{[0,1]}$ is the truncation functor. Proposition 5.4.5 then follows from the fact that $\tau_{[0,1]}(\mathbf{R}p_*(\mathbb{G}_{m,X}))$ is represented by the 2-term complex $k_{\mathrm{s}}(X)^*\to\mathrm{Div}(X^{\mathrm{s}})$, as proved in [BvH09, Lemma 2.3].

Example 5.4.7 Let k be a field of characteristic 0 which contains a primitive cube root of 1. Let a, b, c be independent variables and let $K = k(a, b, c)$. Let $X \subset \mathbb{P}^3_K$ be the diagonal cubic surface given by the homogeneous equation

$$x^3 + ay^3 + bz^3 + ct^3 = 0.$$

By [CTKS87, Prop. 1], one has $\mathrm{H}^1(K, \mathrm{Pic}(X^{\mathrm{s}})) \simeq \mathbb{Z}/3$. By rather involved cocycle calculations, Uematsu [Uem14] shows that the map

$$\mathrm{H}^1(K, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^3(K, K_{\mathrm{s}}^*)$$

is injective. Thus in Proposition 5.4.5, for $n = 1$, the differential can be non-zero.

For any k-variety X, the spectral sequence (5.19) gives rise to a *complex*

$$\mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})).$$

Assume that $k_{\mathrm{s}}^* = k_{\mathrm{s}}[X]^*$. From the general structure of spectral sequences we see that if $\mathrm{H}^3(k, k_{\mathrm{s}}^*) = 0$ (e.g., k is a number field) or if X has a k-point (or a zero-cycle of degree 1), then, in view of Remark 5.4.3 (1), the above complex fits into an exact sequence

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})). \tag{5.24}$$

Thus $\mathrm{Br}(X)/\mathrm{Br}_1(X) = \mathrm{Ker}(\beta)$. For concrete calculations of the Brauer group one would like to be able to compute the map β.

Proposition 5.4.8 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic zero. Let $N(X^{\mathrm{s}}) = \mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$. The composition*

$$(\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}})^{\Gamma} \hookrightarrow \mathrm{Br}(X^{\mathrm{s}})^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}})) \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}})) \tag{5.25}$$

coincides with the connecting map $(\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}})^{\Gamma} \to \mathrm{H}^2(k, N(X^{\mathrm{s}}))$ *defined by the exact sequence* (5.7).

Proof. This is [CTS13b, Cor. 3.5]. □

Remark 5.4.9 By Theorem 5.1.1, if $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then $\mathbf{Pic}^0_{X/k} = 0$ and $\mathrm{Pic}(X^{\mathrm{s}}) \cong \mathrm{NS}(X^{\mathrm{s}})$, and if all the groups $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ are torsion-free, then $\mathrm{NS}(X^{\mathrm{s}}) \cong N(X^{\mathrm{s}})$. By Proposition 5.2.9 (i), if $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ is torsion-free for all ℓ, then $\mathrm{Br}(X^{\mathrm{s}})$ is divisible. Hence when all these hypotheses are satisfied, the composite map in the above proposition computes β. Thus the above description covers many important cases.

There is a variant of this description of (5.25) which involves comparison theorems between ℓ-adic and classical cohomology as well as between the corresponding cycle class maps. Here we discuss the important case of surfaces and refer to [CTS13b, Prop. 4.1] for a somewhat more technical description in the case of higher-dimensional varieties.

Let X be a smooth, projective, geometrically integral surface over a subfield k of $\mathbb{C}$. Let k_{s} be the algebraic closure of k in $\mathbb{C}$. Since $\mathrm{NS}(X^{\mathrm{s}}) \cong \mathrm{NS}(X_{\mathbb{C}})$, we have $N(X^{\mathrm{s}}) \cong N(X_{\mathbb{C}})$. Let us write $\mathrm{H}^2 = \mathrm{H}^2(X_{\mathbb{C}}, \mathbb{Z})/\mathrm{tors}$. For a surface X, Poincaré duality gives rise to a perfect (unimodular) pairing

$$\mathrm{H}^2 \times \mathrm{H}^2 \longrightarrow \mathbb{Z}$$

given by the cup-product. By the Hodge index theorem, the restriction of this pairing to $N(X_{\mathbb{C}})$ has a non-zero discriminant. A classical argument based on the exponential exact sequence shows that $N(X_{\mathbb{C}})$ is a saturated subgroup of H^2, in the sense that the quotient is torsion-free.

Let $T(X_{\mathbb{C}})$ be the *transcendental lattice* of $X_{\mathbb{C}}$ defined as the orthogonal complement to $N(X_{\mathbb{C}})$ in H^2 with respect to the cup-product pairing. Thus $T(X_{\mathbb{C}})$ is a saturated subgroup of H^2, and $N(X_{\mathbb{C}}) \cap T(X_{\mathbb{C}}) = 0$. Write

$$N(X_{\mathbb{C}})^* = \mathrm{Hom}(N(X_{\mathbb{C}}), \mathbb{Z}), \qquad T(X_{\mathbb{C}})^* = \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Z}).$$

The cup-product gives rise to the injective maps

$$N(X_{\mathbb{C}}) \hookrightarrow N(X_{\mathbb{C}})^*, \qquad T(X_{\mathbb{C}}) \hookrightarrow T(X_{\mathbb{C}})^*.$$

By the unimodularity of the pairing on H^2 we have canonical isomorphisms of finite abelian groups

$$N(X_{\mathbb{C}})^*/N(X_{\mathbb{C}}) \cong \mathrm{H}^2/(N(X_{\mathbb{C}}) \oplus T(X_{\mathbb{C}})) \cong T(X_{\mathbb{C}})^*/T(X_{\mathbb{C}}).$$

We deduce an exact sequence

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

By the comparison theorem between classical and étale cohomology, for $i \geq 0$ we have isomorphisms $\mathrm{H}^i(X_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell(1) \cong \mathrm{H}^i(\overline{X}, \mathbb{Z}_\ell(1))$ compatible with the cycle class map and the cup-product, for any prime ℓ. Thus $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is the orthogonal complement to $(\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell)/\mathrm{tors}$ in $\mathrm{H}^2(\overline{X}, \mathbb{Z}_\ell(1))/\mathrm{tors}$. In particular, $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is naturally a Γ-module, so that the previous 4-term exact sequence is an exact sequence of Γ-modules.

Since $N(X_{\mathbb{C}})$ is the orthogonal complement to $T(X_{\mathbb{C}})$ in H^2, we obtain $T(X_{\mathbb{C}})^* = \mathrm{H}^2/N(X_{\mathbb{C}})$. Tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ we get

$$\mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \big(\mathrm{H}^2/N(X_{\mathbb{C}})\big) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell \cong \frac{\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))}{\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where we used $\mathrm{NS}(X^{\mathrm{s}})\{\ell\} \cong \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))\{\ell\}$. From the description of $\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}}$ in Proposition 5.2.9 (i) we now obtain a canonical isomorphism of Γ-modules

$$\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}} \cong \mathrm{Hom}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z})$$

and an exact sequence of Γ-modules

$$0 \longrightarrow N(X^{\mathrm{s}}) \longrightarrow N(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}} \longrightarrow 0. \quad (5.26)$$

The following proposition is a trimmed down version of Proposition 5.4.8 for surfaces, and is more suitable for concrete calculations.

Proposition 5.4.10 *Let X be a smooth, projective, geometrically integral surface over a field $k \subset \mathbb{C}$. Let k_{s} be the algebraic closure of k in $\mathbb{C}$. The composed map* (5.25) *coincides, up to sign, with the connecting map*

$$(\mathrm{Br}(X^{\mathrm{s}})_{\mathrm{div}})^{\Gamma} \longrightarrow \mathrm{H}^2(k, N(X^{\mathrm{s}}))$$

defined by the 2-extension of Γ-modules (5.26).

Proof. This is a special case of [CTS13b, Prop. 4.1]. □

Remark 5.4.11 This remark is a continuation of Remark 5.4.6 and uses the same notation. Let X be a smooth, projective, geometrically integral surface over a subfield of $\mathbb{C}$ such that $\mathrm{Pic}(X^{\mathrm{s}})$ is torsion-free. Then $\mathrm{NS}(X^{\mathrm{s}})$ is torsion-free, hence $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ is torsion-free for any prime ℓ (see (5.13)). Since X is a surface, this implies that $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell)$ is torsion-free. From Proposition 5.2.9 (i) we deduce that $\mathrm{Br}(X^{\mathrm{s}})$ is divisible. By [GvS, Prop. 1.2] the 2-term complex of Γ-modules

$$\mathrm{NS}(X^{\mathrm{s}})^* \longrightarrow T(X_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}, \quad (5.27)$$

which is the middle part of (5.26), is quasi-isomorphic to $\tau_{[1,2]}(\mathbf{R}p_*(\mathbb{G}_{m,X}))$. This explains the previous proposition, because the relevant differential in the spectral sequence coincides with the map attached to the exact triangle

$$(R^1p_*)(\mathbb{G}_{m,X})[-1] \longrightarrow \tau_{[1,2]}(\mathbf{R}p_*(\mathbb{G}_{m,X})) \longrightarrow (R^2p_*)(\mathbb{G}_{m,X})[-2].$$

5.4.3 Galois invariants of the geometric Brauer group

Theorem 5.4.12 *Let X be a smooth, projective, geometrically integral variety over a field k of characteristic exponent p. Then the cokernel of the natural map $\alpha\colon \mathrm{Br}(X)\to\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is the direct sum of a finite group of order coprime to p and a p-torsion group of finite exponent. In particular, if $\mathrm{char}(k)=0$, then the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(X^{\mathrm{s}})$ is finite if and only if $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is finite.*

This was proved in [CTS13b] when $\mathrm{char}(k)=0$. As pointed out by X. Yuan [Yua20], the same method works over an arbitrary ground field. Here we give the proof of [CTS13b] assuming $\mathrm{char}(k)=0$, and refer the reader to [Yua20] for the general case. See Theorem 16.1.4 for the case where k is a finite field.

We start with a lemma.

Lemma 5.4.13 *Let $L\subset k_{\mathrm{s}}$ be a finite separable extension of a field k of degree $[L:k]=n$. Write $\Gamma_k=\mathrm{Gal}(k_{\mathrm{s}}/k)$ and $\Gamma_L=\mathrm{Gal}(k_{\mathrm{s}}/L)$. Let X be a k-scheme and let $X_L=X\times_k L$. The following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{Br}(X) & \xrightarrow{\mathrm{res}_{L/k}} & \mathrm{Br}(X_L) & \xrightarrow{\mathrm{cores}_{L/k}} & \mathrm{Br}(X) \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_L} & & \downarrow{\scriptstyle \alpha} \\
\mathrm{Br}(X^{\mathrm{s}})^{\Gamma_k} & \hookrightarrow & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma_L} & \xrightarrow{\sigma} & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma_k}
\end{array}$$

Here $\sigma(x)=\sum\sigma_i(x)$, where $\sigma_i\in\Gamma_k$ are coset representatives of Γ_k/Γ_L. The composition of maps in each row of the diagram is the multiplication by n.

Proof. We have an isomorphism $L\otimes_k k_{\mathrm{s}}\xrightarrow{\sim}k_{\mathrm{s}}^{\oplus n}$ whose components correspond to the n distinct embeddings of L into k_{s}. By changing the base from X to X^{s} we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^p_{\text{ét}}(X,\mathbb{G}_m) & \xrightarrow{\mathrm{res}_{L/k}} & \mathrm{H}^p_{\text{ét}}(X_L,\mathbb{G}_m) & \xrightarrow{\mathrm{cores}_{L/k}} & \mathrm{H}^p_{\text{ét}}(X,\mathbb{G}_m) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}},\mathbb{G}_m) & \hookrightarrow & \mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}},\mathbb{G}_m)^{\oplus n} & \longrightarrow & \mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}},\mathbb{G}_m)
\end{array}$$

where the maps in the bottom row are the diagonal embedding and the sum. The representation of the Galois group Γ_k in $\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}},\mathbb{G}_m)^{\oplus n}$ is induced from the natural representation of Γ_L in $\mathrm{H}^p_{\text{ét}}(X^{\mathrm{s}},\mathbb{G}_m)$. Passing to Γ_k-invariant subgroups, and taking $p=2$, we obtain the statement of the lemma. □

Proof of Theorem 5.4.12. We assume that $\mathrm{char}(k)=0$. By Proposition 5.2.9 (ii) the group $\mathrm{Br}(X^{\mathrm{s}})[n]$ is finite for any positive integer n. Hence it is enough to show that $\mathrm{Coker}(\alpha)$ has finite exponent.

Suppose that $k \subset L \subset k_s$ is a finite extension of k such that $[L : k] = n$. By Lemma 5.4.13 restriction and corestriction induce the maps

$$\mathrm{Coker}(\alpha) \longrightarrow \mathrm{Coker}(\alpha_L) \longrightarrow \mathrm{Coker}(\alpha)$$

whose composition is the multiplication by n. Thus the kernel of the map $\mathrm{Coker}(\alpha) \to \mathrm{Coker}(\alpha_L)$ is annihilated by n, and to show that $\mathrm{Coker}(\alpha)$ has finite exponent it is enough to show that $\mathrm{Coker}(\alpha_L)$ has finite exponent.

Therefore without loss of generality we can replace k by any finite extension. In particular, we can assume that $X(k) \neq \emptyset$ and Γ_k acts trivially on the Néron–Severi group $\mathrm{NS}(X^s)$. Since $X(k) \neq \emptyset$, we have the exact sequence (5.24)

$$0 \longrightarrow \mathrm{Br}_1(X) \longrightarrow \mathrm{Br}(X) \xrightarrow{\alpha} \mathrm{Br}(X^s)^{\Gamma} \xrightarrow{\beta} \mathrm{H}^2(k, \mathrm{Pic}(X^s)).$$

Thus it is enough to show that $\mathrm{Im}(\beta)$ has finite exponent. We do this by considering finitely many curves on X and restricting our maps to each of these curves. This is a meaningful strategy because for a proper curve C over k we have $\mathrm{Br}(C^s) = 0$ by Theorem 5.6.1 (v) below.

More precisely, $\mathrm{NS}(X^s)/\mathrm{tors}$ is a finitely generated free abelian group, so we can choose finitely many, say m, curves in X^s such that the intersection pairing with the classes of these curves defines an injective group homomorphism $\iota\colon \mathrm{NS}(X^s)/\mathrm{tors} \hookrightarrow \mathbb{Z}^m$. By taking normalisation we obtain m morphisms from smooth projective curves defined over k_s to X^s. We replace k by a finite extension over which all these curves and morphisms are defined. We now have k-morphisms $C_i \to X$ for $i = 1, \ldots, m$.

By successively applying the Bertini theorem for hyperplane sections of smooth projective varieties [Jou84, Ch. I, Cor. 6.7] we find a smooth and connected curve in X^s. By replacing the field k by a finite extension we can assume that this curve is obtained by base change from k to k_s from a *smooth* and geometrically connected curve $C_0 \subset X$ defined over k. We now add C_0 to our finite family of curves equipped with finite morphisms to X.

A morphism $f\colon C \to X$, where C is a smooth, projective and geometrically integral curve over k gives rise to the commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(X^s)^{\Gamma} & \xrightarrow{\beta_X} & \mathrm{H}^2(k, \mathrm{Pic}(X^s)) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ 0 = \mathrm{Br}(C^s)^{\Gamma} & \xrightarrow{\beta_C} & \mathrm{H}^2(k, \mathrm{Pic}(C^s)) \end{array}$$

We have established the following claim.

Claim 1. *For any morphism $f\colon C \to X$ the group $\mathrm{Im}(\beta_X)$ is contained in the kernel of the right vertical map in the diagram.*

The exact sequence of Γ_k-modules

$$0 \longrightarrow \mathrm{Pic}^0(C^{\mathrm{s}}) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}}) \longrightarrow \mathrm{NS}(C^{\mathrm{s}}) \longrightarrow 0$$

gives rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} & & \mathrm{H}^2(k,\mathrm{Pic}^0(X^{\mathrm{s}})) & \to & \mathrm{H}^2(k,\mathrm{Pic}(X^{\mathrm{s}})) & \to & \mathrm{H}^2(k,\mathrm{NS}(X^{\mathrm{s}})) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{H}^2(k,\mathrm{Pic}^0(C^{\mathrm{s}})) & \to & \mathrm{H}^2(k,\mathrm{Pic}(C^{\mathrm{s}})) & \to & \mathrm{H}^2(k,\mathrm{NS}(C^{\mathrm{s}})) \end{array} \qquad (5.28)$$

The zero in the bottom row is due to the fact that $\mathrm{H}^1(k,\mathbb{Z})=0$.

A combination of the Bertini theorem and Zariski's connectedness theorem (see [SGA1, X, Cor. 2.11, p. 210]) implies that a connected finite étale cover of X^{s} restricts to a connected cover of C_0^{s}. In particular, writing A for the Picard variety of X, we obtain that the map of abelian varieties $A\to\mathbf{Pic}^0_{C_0/k}$ has trivial kernel. By the Poincaré reducibility theorem [MumAV, §19, Thm. 1] there exists an abelian subvariety $B\subset\mathbf{Pic}^0_{C_0/k}$ such that the natural map

$$A\times B\longrightarrow\mathbf{Pic}^0_{C_0/k}$$

is an isogeny of abelian varieties over k, that is, a surjective morphism with finite kernel. It follows that the kernel of $\mathrm{H}^2(k,\mathrm{Pic}^0(X^{\mathrm{s}}))\to\mathrm{H}^2(k,\mathrm{Pic}^0(C_0^{\mathrm{s}}))$ has finite exponent. From diagram (5.28) we now obtain the following statement.

Claim 2. *The kernel of the composite map*

$$\mathrm{H}^2(k,\mathrm{Pic}^0(X^{\mathrm{s}}))\longrightarrow\mathrm{H}^2(k,\mathrm{Pic}(X^{\mathrm{s}}))\longrightarrow\mathrm{H}^2(k,\mathrm{Pic}(C_0^{\mathrm{s}}))$$

has finite exponent.

In view of (5.28), Claims 1 and 2, to complete the proof it is enough to show that the map of Γ_k-modules

$$\mathrm{NS}(X^{\mathrm{s}})\longrightarrow\bigoplus_{i=1}^m\mathrm{NS}(C_i^{\mathrm{s}})=\mathbb{Z}^m$$

induces a map $\xi\colon\mathrm{H}^2(k,\mathrm{NS}(X^{\mathrm{s}}))\to\mathrm{H}^2(k,\mathbb{Z}^m)$ whose kernel has finite exponent. The map ξ is the composition of two maps:

$$\mathrm{H}^2(k,\mathrm{NS}(X^{\mathrm{s}}))\xrightarrow{\xi_1}\mathrm{H}^2(k,\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors})\xrightarrow{\xi_2}\mathrm{H}^2(k,\mathbb{Z}^m).$$

It is enough to show that the kernel of each of these has finite exponent.

From the exact sequence of Galois cohomology attached to the exact sequence of Γ_k-modules

$$0\longrightarrow\mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}\longrightarrow\mathrm{NS}(X^{\mathrm{s}})\longrightarrow\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}\longrightarrow 0$$

we deduce that $\mathrm{Ker}(\xi_1)$ is annihilated by the exponent of the finite group $\mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}$.

There exists a homomorphism of abelian groups $\sigma\colon \mathbb{Z}^m \to \mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$ such that the composition $\sigma \circ \iota$ is the multiplication by a positive integer on $\mathrm{NS}(X^{\mathrm{s}})/\mathrm{tors}$. This integer annihilates $\mathrm{Ker}(\xi_2)$. □

Remark 5.4.14 This proof can be used to produce an explicit upper bound for the size of the cokernel of $\alpha\colon \mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$, see [CTS13b, Thm. 2.2]. When $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ or k is a number field, Proposition 5.4.10 can also be used to give upper bounds for this cokernel, see [CTS13b, Thm. 4.2, 4.3]. In some cases, for example when X is a diagonal quartic surface in $\mathbb{P}^3_{\mathbb{Q}}$, with the help of Proposition 5.4.10 one can completely determine the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$, see Section 16.8.

5.5 Projective varieties with $\mathrm{H}^i(X, \mathcal{O}_X) = 0$

Theorem 5.5.1 *Let X be a smooth, projective and geometrically integral variety over a field k. Assume that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ and that $\mathrm{NS}(\overline{X})$ is torsion-free. Then $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ and $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ are finite groups.*

Proof. From the exact sequence (5.21) we see that the quotient $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is a subgroup of $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$. The result then follows from Corollary 5.1.3 and the finiteness of $\mathrm{H}^1(k, M)$ for any finitely generated torsion-free abelian group M. □

Theorem 5.5.2 *Let X be a smooth, projective and geometrically integral variety over a field k of characteristic zero. Assume that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ and that the Néron–Severi group $\mathrm{NS}(\overline{X})$ is torsion-free. Then we have the following properties.*

(i) *The groups $\mathrm{Br}(\overline{X})$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ are finite.*
(ii) $\mathrm{Br}(\overline{X}) = 0$ *if and only if* $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}} = 0$ *for every prime ℓ. In this case* $\mathrm{Br}(X) = \mathrm{Br}_1(X)$.
(iii) *If* $\dim X = 2$, *then* $\mathrm{Br}(\overline{X}) = 0$ *and* $\mathrm{Br}_1(X) = \mathrm{Br}(X)$.

Proof. By Corollary 5.1.3 the condition $\mathrm{H}^1_{\mathrm{zar}}(X, \mathcal{O}_X) = 0$ implies that $\mathrm{Pic}(\overline{X}) \cong \mathrm{NS}(\overline{X})$.

Let us first consider the case $k = \mathbb{C}$. Using the GAGA theorems, we get $\mathrm{H}^i_{\mathrm{an}}(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, and $\mathrm{Pic}(X) = \mathrm{H}^1_{\mathrm{zar}}(X, \mathcal{O}_X^*) \cong \mathrm{H}^1_{\mathrm{an}}(X, \mathcal{O}_X^*)$. The exponential sequence for the analytic topology

$$0 \longrightarrow \mathbb{Z}(2\pi\sqrt{-1}) \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0$$

then shows that the condition $\mathrm{H}^2_{\mathrm{zar}}(X, \mathcal{O}_X) = 0$ gives an isomorphism of finitely generated groups $\mathrm{Pic}(X) \cong \mathrm{H}^2_{\mathrm{an}}(X, \mathbb{Z})$, hence $\rho = b_2$.

If k is an arbitrary field of characteristic zero, we can find a subfield $k_0 \subset k$ finitely generated over $\mathbb{Q}$ and a variety X_0 over k_0 such that $X \cong X_0 \times_{k_0} k$. Choose an embedding $k_0 \hookrightarrow \mathbb{C}$. We proved that $\rho = b_2$ holds for $X_0 \times_{k_0} \mathbb{C}$. This implies that the same is true for $X_0 \times_{k_0} (k_0)_{\mathrm{s}}$ and thus also for $X^{\mathrm{s}} = X \times_k k_{\mathrm{s}}$.

Now Proposition 5.2.9 shows that $\mathrm{Br}(\overline{X})$ is isomorphic to the finite group $\oplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$. Statements (i) and (ii) then follow from Theorem 5.5.1. Statement (iii) follows from (ii) and Proposition 5.2.10. □

Corollary 5.5.3 *Let X be a smooth, projective, geometrically integral variety over a field k. In each of the following cases:*

(i) *X is a complete intersection of dimension at least 2 in projective space,*
(ii) *X is K3 surface,*

the groups $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$ and $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ are finite.

Proof. In both cases $\mathrm{Pic}(\overline{X})$ is torsion-free.

For case (i), see [SGA2, XII, Cor. 3.7] and [SGA7, XI, Thm. 1.8]. For case (ii), see [Huy16, Ch. 1, Prop. 2.4]. □

For smooth, projective, geometrically integral varieties over a field of characteristic zero, a similar statement is true for rationally connected varieties (see Definition 14.1.1), and in particular for unirational varieties.

Corollary 5.5.4 *Let $X \subset \mathbb{P}^n_k$ be a smooth complete intersection of dimension at least 3 over a field k. Then $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ is an isomorphism. If $\mathrm{char}(k)=0$, then $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism. If $\mathrm{char}(k) = p > 0$, then $\mathrm{Br}(k)\{\ell\} \to \mathrm{Br}(X)\{\ell\}$ is an isomorphism for any prime $\ell \neq p$.*

Proof. For such a variety X, the restriction map

$$\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_k) \cong \mathrm{Pic}(\mathbb{P}^n_{\bar{k}}) \longrightarrow \mathrm{Pic}(\overline{X})$$

is an isomorphism [SGA2, XII, Cor. 3.7]. It is compatible with the action of Γ, which thus acts trivially on $\mathrm{Pic}(\overline{X})$. Hence $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) = 0$. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(\overline{X})^\Gamma$ is surjective, since $\mathrm{Pic}(\mathbb{P}^n_k) \to \mathrm{Pic}(\mathbb{P}^n_{\bar{k}})^\Gamma$ is surjective. From the exact sequence (5.21) we conclude that $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ is an isomorphism.

For any prime $\ell \neq p$ we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & \mathrm{Pic}(\mathbb{P}^n_{\bar{k}})/\ell & \xrightarrow{\sim} & \mathrm{H}^2(\mathbb{P}^n_{\bar{k}}, \mathbb{Z}/\ell) & & & & \\ & & \downarrow \cong & & \downarrow \cong & & & & \\ 0 & \longrightarrow & \mathrm{Pic}(\overline{X})/\ell & \longrightarrow & \mathrm{H}^2(\overline{X}, \mathbb{Z}/\ell) & \longrightarrow & \mathrm{Br}(\overline{X})[\ell] & \longrightarrow & 0, \end{array}$$

here the middle vertical map is an isomorphism by [SGA7, XI, Thm. 1.6 (ii)]. By the exactness of the bottom row we have $\mathrm{Br}(\overline{X})[\ell] = 0$. □

5.6 The Picard and Brauer groups of curves

Let us first recall the structure of the Picard group of a smooth, projective, geometrically integral curve C over a field k. Let $g = \dim(\mathrm{H}^1(C, \mathcal{O}_C))$ be the genus of C. We have $\mathrm{NS}(C^{\mathrm{s}}) \cong \mathbb{Z}$, and the natural morphism $\mathbf{Pic}_{C/k} \to \mathbb{Z}$ is given by the degree map on divisors. For an integer n, let $\mathbf{Pic}^n_{C/k}$ be the component of $\mathbf{Pic}_{C/k}$ of degree n. The Picard variety $\mathbf{Pic}^0_{C/k}$ is an abelian variety of dimension g; it is the Jacobian J of the curve C. The Jacobian is principally polarised, hence isomorphic to its dual abelian variety, thus J is also the Albanese variety of C. The variety $\mathbf{Pic}^n_{C/k}$ is a k-torsor for J. For $g \geq 1$ there is a natural embedding $C \hookrightarrow \mathbf{Pic}^1_{C/k}$, so $\mathbf{Pic}^1_{C/k}$ is the Albanese torsor of C.

There is an exact sequence of Γ-modules

$$0 \longrightarrow J(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}}) \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{5.29}$$

The attached long exact sequence of Galois cohomology gives an exact sequence

$$0 \longrightarrow J(k) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}})^{\Gamma} \longrightarrow \mathbb{Z} \longrightarrow \mathrm{H}^1(k, J) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})) \longrightarrow 0.$$

The group $\mathrm{H}^1(k, J)$ classifies J-torsors over k. The map $\mathbb{Z}\to\mathrm{H}^1(k, J)$ sends $n \in \mathbb{Z}$ to the class of the torsor $\mathbf{Pic}^n_{C/k}$.

In the following theorem we collect results about the Brauer groups of curves.

Theorem 5.6.1 *Let C be a quasi-projective curve over a field k. Then the following statements hold.*

(i) *If $\alpha \in \mathrm{Br}(C)$ vanishes at each point of $C(K)$ for any field K containing k, then $\alpha = 0 \in \mathrm{Br}(C)$.*
(ii) *If k is algebraically closed, then $\mathrm{Br}(C) = 0$.*
(iii) *If k is separably closed of characteristic $p > 0$, then $\mathrm{Br}(C)$ is a p-primary torsion group.*
(iv) *If k is separably closed and C is proper over k, then $\mathrm{Br}(C) = 0$.*
(v) *If k is finite and C is proper over k, then $\mathrm{Br}(C) = 0$.*
(vi) *If k is not perfect, then $\mathrm{Br}(\mathbb{A}^1_k) \neq 0$. If k is separably closed, then $\mathrm{Br}(\mathbb{A}^1_k) = 0$ if and only if k is algebraically closed.*
(vii) *The natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism.*
(viii) *If k is perfect, then the natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{A}^1_k)$ is an isomorphism.*
(ix) *For any prime $\ell \neq \mathrm{char}(k)$, the map $\mathrm{Br}(k)\{\ell\}\to\mathrm{Br}(\mathbb{A}^1_k)\{\ell\}$ is an isomorphism.*

Proof. Let $C_{\mathrm{red}} \subset C$ be the underlying reduced curve. By Proposition 3.2.5 (ii), the map $\mathrm{Br}(C)\to\mathrm{Br}(C_{\mathrm{red}})$ is an isomorphism. Thus it is enough to prove the theorem when C is reduced. Then C is the union of connected purely

1-dimensional quasi-projective curves and of spectra of finite field extensions of k. Since a finite extension of a separably closed field is separably closed, and the Brauer groups of separably closed fields and of finite fields are trivial, to prove the theorem we may assume that C is purely 1-dimensional, reduced and connected.

Statement (i) is then a special case of Proposition 3.6.6.

By (i) and the triviality of the Brauer groups of separably closed fields and of finite fields, to prove (ii), (iii), (iv), (v) it is enough to consider the generic points of the irreducible components of C.

(ii) By Tsen's theorem (Theorem 1.2.14) the Brauer group of a function field in one variable over an algebraically closed field is trivial. Now (ii) follows from (i).

(iii) By a version of Tsen's theorem over a separably closed field (Proposition 3.8.2), the Brauer group of a function field in one variable over a separably closed field of characteristic $p > 0$ is p-primary. The result now follows from (i).

(iv) The normalisation of C is a finite disjoint union of proper, regular, integral curves over k. Hence, by (i), it is enough to prove that for a proper, regular, integral curve D over k we have $\mathrm{Br}(D) = 0$. Let K be the algebraic closure of k in the field $k(D)$. Since the local rings of D are normal, the elements of K are regular functions on D, that is, $K \subset \mathrm{H}^0(D, \mathcal{O}_D)$. Thus the structure morphism $D \to \mathrm{Spec}(k)$ factors through $D \to \mathrm{Spec}(K)$, and D is geometrically integral over K. Since D is a curve, we have $\mathrm{H}^2(D, \mathcal{O}_D) = 0$, thus (iv) follows from (ii) and Theorem 5.2.5.

(v) Arguing as in the proof of (iv) we reduce the statement to proving that $\mathrm{Br}(C) = 0$, where C is a regular, proper, geometrically integral curve over a finite field. The exact sequence (5.21) gives an isomorphism

$$\mathrm{Ker}[\mathrm{Br}(C) \longrightarrow \mathrm{Br}(C^{\mathrm{s}})] \xrightarrow{\sim} \mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})).$$

By (ii), we have $\mathrm{Br}(C^{\mathrm{s}}) = 0$. Now consider the exact sequence (5.29):

$$0 \longrightarrow J(k_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}}) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the Γ-module $J(k_{\mathrm{s}})$ is the group of k_{s}-points of the Jacobian J of C. By Lang's theorem on the triviality of the first Galois cohomology group of a finite field with coefficients in a connected algebraic group, we have $\mathrm{H}^1(k, J) = 0$. But $\mathrm{H}^1(k, \mathbb{Z}) = 0$, so we deduce $\mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})) = 0$. Hence $\mathrm{Br}(C) = 0$.

(vi) If k is algebraically closed, then (vi) is a particular case of (ii). Suppose that k has characteristic $p > 0$ and is not perfect. Then there is an element $c \in k \smallsetminus k^p$. It gives rise to a non-zero class in $\mathrm{H}^1_{\mathrm{fppf}}(k, \mu_p)$ and hence in $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_k, \mu_p)$. The étale Artin–Schreier covering of $\mathbb{A}^1_k = \mathrm{Spec}(k[x]) \to \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ given by $x^p - x = t$ gives a non-zero element of $\mathrm{H}^1_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/p) = \mathrm{H}^1_{\mathrm{fppf}}(\mathbb{A}^1_k, \mathbb{Z}/p)$. This finite étale cover extends to a finite cover $\mathbb{P}^1_k \to \mathbb{P}^1_k$ which is totally ramified of degree p above the point at infinity of $\mathbb{P}^1_k$. We claim that the cup-product of these two classes is a non-zero element of $\mathrm{H}^2_{\mathrm{fppf}}(\mathbb{A}^1_k,\mu_p)=\mathrm{Br}(\mathbb{A}^1_k)[p]$. For this it is enough to prove that the class of the corresponding cyclic algebra is non-zero in $\mathrm{Br}(k(t))$. By Proposition 1.3.8 this holds if and only if $c\in k\subset k(t)$ is not a norm of an element from $k(x)$. Consider the completion of $k(t)$ at the point at infinity. If c were a norm, then its image in the residue field, which is just k, would be a p-th power, see [SerCL, Ch. V, §3, Prop. 5 (i)].

(vii) For $C=\mathbb{P}^1_k$, we have an isomorphism of $\mathrm{Pic}(C^{\mathrm{s}})$ with the trivial Γ-module $\mathbb{Z}$ given by the degree map. The map $\mathrm{Pic}(C)\to\mathrm{Pic}(C^{\mathrm{s}})=\mathbb{Z}$ is an isomorphism. By (iv), $\mathrm{Br}(C^{\mathrm{s}})=0$. Since $\mathrm{H}^1(k,\mathbb{Z})=0$, the exact sequence (5.21) gives an isomorphism $\mathrm{Br}(k)\xrightarrow{\sim}\mathrm{Br}(\mathbb{P}^1_k)$.

(viii) Since the affine line has a k-point, we obtain from (5.21) that the natural map $\mathrm{Br}(k)\to\mathrm{Br}_1(\mathbb{A}^1_k)$ is an isomorphism. Since k is perfect, k_{s} is algebraically closed, hence $\mathrm{Br}(\mathbb{A}^1_{k_{\mathrm{s}}})=0$ by (ii). Thus $\mathrm{Br}(\mathbb{A}^1_k)=\mathrm{Br}_1(\mathbb{A}^1_k)$.

(ix) This follows from (iii) and (5.21). □

In connection with Theorem 5.6.1(vi), Knus, Ojanguren and Saltman [KOS76, Thm. 5.5] computed $\mathrm{Br}(\mathbb{A}^1_k)$ for non-perfect fields k of positive characteristic.

Remark 5.6.2 Let us return to the case of a smooth, projective and geometrically integral curve C over a field k. If C has a k-point or, more generally, a zero-cycle of degree 1, then (5.29) splits. In this case (5.21) gives an isomorphism $\mathrm{Pic}(C)=\mathrm{Pic}(C^{\mathrm{s}})^{\Gamma}$ and, in view of Theorem 5.6.1 (iv) below, a split exact sequence

$$0\longrightarrow\mathrm{Br}(k)\longrightarrow\mathrm{Br}(C)\longrightarrow\mathrm{H}^1(k,J)\longrightarrow 0.$$

5.7 The Picard and Brauer groups of a product

5.7.1 The Picard group of a product

Recall that if X is a smooth, projective, and geometrically integral variety over a field k, then the connected component of 0 of the Picard scheme $\mathbf{Pic}^0_{X/k}\subset\mathbf{Pic}_{X/k}$ is a projective and connected (but not necessarily reduced) group k-scheme, see Theorem 5.1.1.

Proposition 5.7.1 *Let X and Y be smooth, projective, geometrically integral varieties over a field k. Let $p_X\colon X\times_k Y\to X$ and $p_Y\colon X\times_k Y\to Y$ be the projection maps. Then the morphism of group k-schemes*

$$\mathbf{Pic}^0_{X/k}\times\mathbf{Pic}^0_{Y/k}\longrightarrow\mathbf{Pic}^0_{X\times_k Y/k}\qquad(5.30)$$

that sends (a,b) to $p_X^*(a)+p_Y^*(a)$, is an isomorphism.

Proof. By Galois descent, to prove that our map is an isomorphism we can assume that k is separably closed [EGA, IV$_2$, Prop. 2.7.1 (viii)]. Since X and Y are smooth and k is separably closed, we can choose base points $x_0 \in X(k)$ and $y_0 \in Y(k)$. The morphism (5.30) has a retraction $\big((\mathrm{id}, y_0)^*, (x_0, \mathrm{id})^*\big)$. In particular, it identifies $\mathbf{Pic}^0_{X/k} \times \mathbf{Pic}^0_{Y/k}$ with a direct factor of $\mathbf{Pic}^0_{X\times_k Y/k}$. Let $C \subset \mathbf{Pic}^0_{X\times_k Y/k}$ be the kernel of $\big((\mathrm{id}, y_0)^*, (x_0, \mathrm{id})^*\big)$. Then we have

$$\mathbf{Pic}^0_{X\times_k Y/k} \cong C \times \mathbf{Pic}^0_{X/k} \times \mathbf{Pic}^0_{Y/k}.$$

As a surjective image of a connected group k-scheme, C is connected. From the Künneth formula [Stacks, Lemma 0BED]

$$\mathrm{H}^1(X \times_k Y, \mathcal{O}) \cong \mathrm{H}^1(X, \mathcal{O}) \oplus \mathrm{H}^1(Y, \mathcal{O})$$

we see that (5.30) induces an isomorphism of tangent spaces at 0. Thus the tangent space to C at 0 is trivial, hence $C = 0$. □

We continue to assume that X and Y are smooth, projective, geometrically integral varieties over a field k. Let $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ and $B = \mathbf{Pic}^0_{Y/k,\mathrm{red}}$ be the Picard varieties of X and Y, respectively. The dual abelian variety $B^\vee$ of B is the Albanese variety of Y.

We choose $x_0 \in X(k_\mathrm{s})$ and $y_0 \in Y(k_\mathrm{s})$. By Corollary 2.5.8 the Picard scheme $\mathbf{Pic}_{X^\mathrm{s}/k_\mathrm{s}} \cong \mathbf{Pic}^\mathrm{s}_{X/k} = \mathbf{Pic}_{X/k} \times_k k_\mathrm{s}$ represents the relative Picard functor $\mathrm{Pic}_{X^\mathrm{s}/k_\mathrm{s}}$. Recall from Section 5.1 that there is a unique Poincaré invertible sheaf $\mathcal{P}$ on $X^\mathrm{s} \times \mathbf{Pic}^\mathrm{s}_{X/k}$ normalised so that it restricts trivially to $x_0 \times \mathbf{Pic}^\mathrm{s}_{X/k}$. An invertible sheaf $\mathcal{L}$ on $X^\mathrm{s} \times Y^\mathrm{s}$ that restricts trivially to $x_0 \times Y^\mathrm{s}$ and $X^\mathrm{s} \times y_0$ corresponds to a unique morphism $h\colon Y^\mathrm{s} \to \mathbf{Pic}^\mathrm{s}_{X/k}$ sending y_0 to 0 such that $\mathcal{L} \cong (\mathrm{id}, h)^*\mathcal{P}$. Since Y^s is reduced and connected, h factors through a morphism $h_0\colon Y^\mathrm{s} \to A^\mathrm{s}$. The morphism h_0 is zero if and only if $\mathcal{L}$ is trivial.

There is a canonical Albanese morphism $\mathrm{Alb}_{Y^\mathrm{s}, y_0}\colon Y^\mathrm{s} \to (B^\vee)^\mathrm{s}$ sending y_0 to 0, see Section 5.1. By the universal property of the Albanese variety, h_0 is uniquely written as the composition of $\mathrm{Alb}_{Y^\mathrm{s}, y_0}$ with a morphism of abelian varieties $(B^\vee)^\mathrm{s} \to A^\mathrm{s}$. This gives an isomorphism of abelian groups

$$\mathrm{Pic}(X^\mathrm{s} \times Y^\mathrm{s})_{x_0, y_0} \cong \mathrm{Hom}((B^\vee)^\mathrm{s}, A^\mathrm{s}),$$

where the subscript (x_0, y_0) denotes the kernel of $\big((\mathrm{id}, y_0)^*, (x_0, \mathrm{id})^*\big)$, and we write $\mathrm{Hom}((B^\vee)^\mathrm{s}, A^\mathrm{s})$ for the group of homomorphisms of abelian varieties $(B^\vee)^\mathrm{s} \to A^\mathrm{s}$. Thus we obtain a split exact sequence of abelian groups

$$0 \to \mathrm{Pic}(X^\mathrm{s}) \oplus \mathrm{Pic}(Y^\mathrm{s}) \longrightarrow \mathrm{Pic}(X^\mathrm{s} \times Y^\mathrm{s}) \longrightarrow \mathrm{Hom}((B^\vee)^\mathrm{s}, A^\mathrm{s}) \to 0. \quad (5.31)$$

The second map here is induced by the k-morphisms p_X and p_Y. The third map does not depend on the choice of x_0 and y_0. This implies that if $k_0 \subset k$ is a subfield and $\Gamma = \mathrm{Aut}(k/k_0)$, then (5.31) is an exact sequence of Γ-modules.

This exact sequence of Γ-modules is split when x_0 and y_0 are k_0-points, but not in general.

Proposition 5.7.2 *Let X and Y be smooth, projective and geometrically integral varieties over a field k. Assume that either $(X \times Y)(k) \neq \emptyset$ or $\mathrm{H}^3(k, k_s^*) = 0$, which holds when k is a number field. Then the natural map*

$$\mathrm{Br}_1(X) \oplus \mathrm{Br}_1(Y) \longrightarrow \mathrm{Br}_1(X \times Y)$$

has finite cokernel.

Proof. This is a consequence of exact sequences (5.31) and (5.21), and the fact that $\mathrm{Hom}((B^\vee)^s, A^s)$ is a finitely generated free abelian group. □

Proposition 5.7.3 *Let X and Y be smooth, projective and geometrically integral varieties over a field k. Let $A = \mathbf{Pic}^0_{X/k,\mathrm{red}}$ and $B = \mathbf{Pic}^0_{Y/k,\mathrm{red}}$ be the Picard varieties of X and Y, respectively. We have a commutative diagram of Γ-modules with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A(k_s) \oplus B(k_s) & = & A(k_s) \oplus B(k_s) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathrm{Pic}(X^s) \oplus \mathrm{Pic}(Y^s) & \to & \mathrm{Pic}(X^s \times Y^s) & \to & \mathrm{Hom}((B^\vee)^s, A^s) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{NS}(X^s) \oplus \mathrm{NS}(Y^s) & \to & \mathrm{NS}(X^s \times Y^s) & \to & \mathrm{Hom}((B^\vee)^s, A^s) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The bottom row is a split exact sequence of Γ-modules. If $(X \times Y)(k) \neq \emptyset$, then the middle row is also a split exact sequence of Γ-modules.

Proof. The vertical exact sequences are those of Theorem 5.1.1 (i). The upper row of the diagram comes from (5.30) and the middle row is (5.31). It remains to prove that the bottom row is split as a sequence of Γ-modules. This follows from the fact that for $\phi \in \mathrm{Hom}((B^\vee)^s, A^s)$ the class of the invertible sheaf $(\mathrm{Alb}_{X,x_0}, \phi \circ \mathrm{Alb}_{Y,y_0})^*\mathcal{P}$ in $\mathrm{NS}(X^s \times Y^s)$ does not depend on the choice of x_0 and y_0. The last statement of the proposition is clear: it is enough to choose $(x_0, y_0) \in (X \times Y)(k)$. □

Remark 5.7.4 (1) Let A_1 and A_2 be abelian varieties over an arbitrary field k. By a theorem of Chow, the natural map $\mathrm{Hom}(A_1^s, A_2^s) \to \mathrm{Hom}(\overline{A}_1, \overline{A}_2)$ is an isomorphism, see [Con06, Thm. 3.19].

(2) The bottom row of the previous diagram shows that

$$\mathrm{NS}(X^s \times Y^s)_{\mathrm{tors}} \cong \mathrm{NS}(X^s)_{\mathrm{tors}} \oplus \mathrm{NS}(Y^s)_{\mathrm{tors}},$$

because the abelian group $\mathrm{Hom}((B^\vee)^s, A^s)$ is torsion-free.

5.7.2 Topological Künneth formula in degrees 1 and 2

As a motivation for a version of the Künneth formula for étale cohomology in degrees 1 and 2, we give a topological analogue of this theorem for singular cohomology. The following proposition rectifies [SZ14, Prop. 2.2].

Proposition 5.7.5 *Let X and Y be non-empty path-connected CW-complexes such that $\mathrm{H}_1(X,\mathbb{Z})$ and $\mathrm{H}_1(Y,\mathbb{Z})$ are finitely generated abelian groups (which holds when X and Y are finite CW-complexes). Suppose $G=\mathbb{Z}$ or $G=\mathbb{Z}/n$, where n is a positive integer. Then we have canonical isomorphisms of abelian groups*

$$\mathrm{H}^1(X\times Y,G)\cong \mathrm{H}^1(X,G)\oplus \mathrm{H}^1(Y,G),$$

$$\mathrm{H}^2(X\times Y,G)\cong \mathrm{H}^2(X,G)\oplus \mathrm{H}^2(Y,G)\oplus \mathrm{Hom}\big(\mathrm{H}^1(X,G)^\vee,\mathrm{H}^1(Y,G)\big),$$

where for a G-module M we write $M^\vee=\mathrm{Hom}(M,G)$.

Proof. To simplify the notation we write $\mathrm{H}_n(X)$ for $\mathrm{H}_n(X,\mathbb{Z})$. Since X is non-empty and path-connected we have $\mathrm{H}_0(X)=\mathbb{Z}$, see [Hat02, Prop. 2.7]. The Künneth formula for homology [Hat02, Thm. 3.B.6] gives a split exact sequence of abelian groups

$$0\to\bigoplus_{i=0}^{n}\big(\mathrm{H}_i(X)\otimes \mathrm{H}_{n-i}(Y)\big)\to \mathrm{H}_n(X\times Y)\to\bigoplus_{i=0}^{n-1}\mathrm{Tor}(\mathrm{H}_i(X),\mathrm{H}_{n-1-i}(Y))\to 0.$$

Since $\mathrm{H}_0(X)=\mathbb{Z}$, in degrees 1 and 2 this gives canonical isomorphisms

$$\mathrm{H}_1(X\times Y)\cong \mathrm{H}_1(X)\oplus \mathrm{H}_1(Y) \tag{5.32}$$

and

$$\mathrm{H}_2(X\times Y)\cong \mathrm{H}_2(X)\oplus \mathrm{H}_2(Y)\oplus\big(\mathrm{H}_1(X)\otimes \mathrm{H}_1(Y)\big). \tag{5.33}$$

For any abelian group G, the universal coefficients theorem [Hat02, Thm. 3.2] gives the following (split) exact sequence of abelian groups

$$0\longrightarrow \mathrm{Ext}(\mathrm{H}_{n-1}(X),G)\longrightarrow \mathrm{H}^n(X,G)\longrightarrow \mathrm{Hom}(\mathrm{H}_n(X),G)\longrightarrow 0, \tag{5.34}$$

where the third map evaluates a cocycle on a cycle. This gives a canonical isomorphism

$$\mathrm{H}^1(X,G)\cong \mathrm{Hom}(\mathrm{H}_1(X),G). \tag{5.35}$$

The desired isomorphism for H^1 now follows from (5.32).

Using the functoriality of the universal coefficients formula (5.34) with respect to the projections of $X \times Y$ to X and Y, together with the isomorphisms (5.32) and (5.33), we obtain a split short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^2(X,G) \oplus \mathrm{H}^2(Y,G) \longrightarrow \mathrm{H}^2(X \times Y, G) \\ \longrightarrow \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G) \longrightarrow 0. \end{aligned} \tag{5.36}$$

The second map here has a retraction induced by the embedding of $X \times y_0$ and $x_0 \times Y$, for some base points x_0 and y_0. The third map in (5.36) is given by evaluating a cocycle on $X \times Y$ on the product of a cycle on X and a cycle on Y. A similar map with $G = G_1 \otimes G_2$ fits into the following commutative diagram with the natural right-hand vertical map:

$$\begin{array}{ccc} \mathrm{H}^2(X \times Y, G_1 \otimes G_2) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), G_1 \otimes G_2) \\ \cup \uparrow & & \uparrow \\ \mathrm{H}^1(X, G_1) \otimes \mathrm{H}^1(Y, G_2) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{H}_1(X), G_1) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), G_2) \end{array} \tag{5.37}$$

Let $G = \mathbb{Z}$. By assumption, $\mathrm{H}_1(X)$ and $\mathrm{H}_1(Y)$ are finitely generated abelian groups. Let M and N be their respective quotients by the torsion subgroups. The map induced by multiplication in $\mathbb{Z}$

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathbb{Z}) \otimes \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z})$$

coincides with $\mathrm{Hom}(M, \mathbb{Z}) \otimes \mathrm{Hom}(N, \mathbb{Z}) \to \mathrm{Hom}(M \otimes N, \mathbb{Z})$, which is clearly an isomorphism, so the displayed map is also an isomorphism. Using (5.35) we rewrite it as

$$\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z}) \cong \mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}).$$

Now (5.36) gives a canonical isomorphism

$$\mathrm{H}^2(X \times Y, \mathbb{Z}) \cong \mathrm{H}^2(X, \mathbb{Z}) \oplus \mathrm{H}^2(Y, \mathbb{Z}) \oplus \big(\mathrm{H}^1(X, \mathbb{Z}) \otimes \mathrm{H}^1(Y, \mathbb{Z})\big). \tag{5.38}$$

In view of the diagram (5.37) the last summand is embedded into $\mathrm{H}^2(X \times Y, \mathbb{Z})$ via the cup-product map. Since $\mathrm{H}^1(X, \mathbb{Z})$ is a free abelian group of finite rank, we can rewrite (5.38) and obtain the desired isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z})$.

Now let $G = \mathbb{Z}/n$. Then $\mathrm{Hom}(\mathrm{H}_1(X) \otimes \mathrm{H}_1(Y), \mathbb{Z}/n)$ is canonically isomorphic to

$$\mathrm{Hom}(\mathrm{H}_1(X), \mathrm{Hom}(\mathrm{H}_1(Y), \mathbb{Z}/n)) \cong \mathrm{Hom}(\mathrm{H}_1(X)/n, \mathrm{H}^1(Y, \mathbb{Z}/n)).$$

Since $\mathrm{Hom}(\mathrm{H}_1(X)/n, \mathbb{Z}/n) \cong \mathrm{H}^1(X, \mathbb{Z}/n)$, we have $\mathrm{H}^1(X, \mathbb{Z}/n)^\vee \cong \mathrm{H}_1(X)/n$. Now (5.36) produces the required isomorphism for $\mathrm{H}^2(X \times Y, \mathbb{Z}/n)$. □

Remark 5.7.6 (1) The naïve analogue of (5.38) is false for the third cohomology group. A counter-example is $X=\mathbb{RP}^2$.

(2) For $X=Y=\mathbb{RP}^2$ we have $\mathrm{H}_1(X)=\mathbb{Z}/2$, so in this case the map induced by multiplication in $\mathbb{Z}/n$ with $n=4$

$$\mathrm{Hom}(\mathrm{H}_1(X),\mathbb{Z}/n)\otimes\mathrm{Hom}(\mathrm{H}_1(Y),\mathbb{Z}/n)\longrightarrow\mathrm{Hom}(\mathrm{H}_1(X)\otimes\mathrm{H}_1(Y),\mathbb{Z}/n)$$

is zero. From diagram (5.37) we see that in this case the cup-product map

$$\mathrm{H}^1(X,\mathbb{Z}/n)\otimes\mathrm{H}^1(Y,\mathbb{Z}/n)\longrightarrow\mathrm{H}^2(X\times Y,\mathbb{Z}/n)$$

is zero.

5.7.3 Künneth formula for étale cohomology in degrees 1 and 2

Let k be a separably closed field. Let G be a finite commutative group k-scheme of order not divisible by char(k). The Cartier dual of G is defined as $\widehat{G}=\mathrm{Hom}(G,\mathbb{G}_{m,k})$ in the category of commutative group k-schemes.

For a proper and geometrically integral variety X over k, the natural pairing

$$\mathrm{H}^1_{\text{ét}}(X,G)\times\widehat{G}\longrightarrow\mathrm{H}^1_{\text{ét}}(X,\mathbb{G}_{m,X})=\mathrm{Pic}(X)$$

gives rise to a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X,G)\xrightarrow{\sim}\mathrm{Hom}(\widehat{G},\mathrm{Pic}(X)).\tag{5.39}$$

The map in (5.39) associates to a class of a G-torsor $\mathcal{T}\to X$ its 'type', see [Sko01, Thm. 2.3.6]. (To see that this map is an isomorphism, we can assume that $G=\mu_n$ so that $\widehat{G}=\mathbb{Z}/n$, and use the Kummer sequence.)

We consider an important particular case. Let n be a positive integer not divisible by char(k). Define S_X as the finite commutative group k-scheme whose Cartier dual is

$$\widehat{S}_X=\mathrm{H}^1_{\text{ét}}(X,\mu_n)\cong\mathrm{Pic}(X)[n].\tag{5.40}$$

For our purposes we shall often need to consider the twist $\widehat{S}_X(-1)$. So for a finite commutative group k-scheme G such that $nG=0$ we use the notation

$$G^\vee=\mathrm{Hom}(G,\mathbb{Z}/n).$$

In particular, we have $S_X^\vee=\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)$. The pairing $G\times G^\vee\to\mathbb{Z}/n$ gives rise to a canonical isomorphism $G\xrightarrow{\sim}(G^\vee)^\vee$.

Let $\mathcal{T}_X \to X$ be an S_X-torsor whose type is the identity in $\mathrm{End}(\mathrm{H}^1_{\text{ét}}(X, \mu_n)) \cong \mathrm{End}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n))$; it is unique up to isomorphism. The natural pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \times S_X^\vee \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$$

with the class $[\mathcal{T}_X] \in \mathrm{H}^1_{\text{ét}}(X, S_X)$ induces the identity map on $S_X^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$. In other words, the image of $[\mathcal{T}_X]$ with respect to the map induced by $a\colon S_X \to \mathbb{Z}/n$ equals $a \in S_X^\vee$.

Suppose that Y is also a proper and geometrically integral variety over k. The image of $[\mathcal{T}_X] \otimes [\mathcal{T}_Y]$ under the map

$$\mathrm{H}^1_{\text{ét}}(X, S_X) \otimes \mathrm{H}^1_{\text{ét}}(Y, S_Y) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)$$

induced by $a\colon S_X \to \mathbb{Z}/n$ and $b\colon S_Y \to \mathbb{Z}/n$, equals $a \otimes b \in S_X^\vee \otimes S_Y^\vee$.

We refer to [Mil80, Prop. V.1.16] for the existence and properties of the cup-product. Thus we can consider $[\mathcal{T}_X] \cup [\mathcal{T}_Y] \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y)$ and

$$a \cup b \in \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n \otimes \mathbb{Z}/n) \cong \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

The cup-product is functorial, so the image of $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ under the map induced by $a \otimes b$ is $a \cup b$. This can be rephrased by saying that the natural pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ S_X^\vee \otimes S_Y^\vee \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \tag{5.41}$$

with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ gives rise to the cup-product map

$$S_X^\vee \otimes S_Y^\vee = \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n).$$

It is important to note that (5.41) factors through the pairing

$$\mathrm{H}^2_{\text{ét}}(X \times_k Y, S_X \otimes S_Y) \ \times \ \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \tag{5.42}$$

The pairing (5.42) with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ induces a map

$$\varepsilon\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We thus have a commutative diagram, where the map ξ is induced by multiplication in $\mathbb{Z}/n$:

$$\begin{array}{ccc} S_X^\vee \otimes S_Y^\vee & \xrightarrow{\ \xi\ } & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \\ \cong \downarrow & & \downarrow \varepsilon \\ \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \cup\ } & \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \end{array} \tag{5.43}$$

The canonical isomorphism $\mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \cong \mathrm{Hom}(S_X, S_Y^\vee)$ allows us to rewrite ε as the map sending $\varphi \in \mathrm{Hom}(S_X, S_Y^\vee)$ to $\varepsilon(\varphi) = \varphi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y]$, where $\cup$ stands for the cup-product pairing

$$\mathrm{H}^1_{\text{ét}}(X, S_Y^\vee) \times \mathrm{H}^1_{\text{ét}}(Y, S_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times Y, S_Y^\vee \otimes S_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n).$$

We write $p_X : X \times_k Y \to X$ and $p_Y : X \times_k Y \to Y$ for the natural projections. Since X and Y are geometrically integral over the separably closed field k, we can choose base points $x_0 \in X(k)$ and $y_0 \in Y(k)$. We have the induced map

$$(\mathrm{id}_X, y_0)^* \colon \ \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n)$$

and a similar map for Y. Using these maps we see that

$$(p_X^*, p_Y^*) \colon \quad \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$$

is split injective, so we have an isomorphism

$$\mathrm{H}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) \cong \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^i_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \widetilde{\mathrm{H}}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n), \quad (5.44)$$

where $\widetilde{\mathrm{H}}^i_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$ is the intersection of kernels of $(\mathrm{id}_X, y_0)^*$ and $(x_0, \mathrm{id}_Y)^*$. Since k is separably closed, we have $\mathrm{H}^i(k, M) = 0$ for any abelian group M and any $i \geq 1$. Thus $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ goes to zero under the maps induced by the restrictions to $x_0 \times Y$ and to $X \times y_0$. This implies that $\mathrm{Im}(\varepsilon) \subset \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$.

Now we can give a Künneth formula that appeared in [SZ14, Thm. 2.6][1]. Part (iii) of the following result is the degree 2 case of [Mil80, Cor. VI.8.13].

Theorem 5.7.7 *Let X and Y be proper and geometrically integral varieties over a separably closed field k. Let n be a positive integer not divisible by* char(k)*. Then we have the following statements.*

(i) $\widetilde{\mathrm{H}}^1_{\text{ét}}(X \times_k Y, \mathbb{Z}/n) = 0$, *so there is a canonical isomorphism*

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X \times_k Y, \mathbb{Z}/n). \quad (5.45)$$

(ii) *Write* $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee = \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n), \mathbb{Z}/n)$ *and similarly for Y. The maps ε and ξ defined above fit into the following commutative diagram*

$$\begin{array}{ccc}
\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)^\vee, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \\
\downarrow{\scriptstyle \cup} & & \cong \downarrow{\scriptstyle \varepsilon} \\
\mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n) & \hookleftarrow & \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)
\end{array}$$

(5.46)

[1] The precise statement of [SZ14, Thm. 2.6] in the case of degree 2 requires an additional assumption. For example, it is enough to assume that $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module, see [Mil80, Cor. VI.8.13]. What the proof of [SZ14, Thm. 2.6] actually shows is Theorem 5.7.7 below.

Moreover, ε is an isomorphism.

(iii) *If $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module (which holds if $\mathrm{NS}(X)[n] = 0$), then all maps in (5.46) are isomorphisms, so we have*

$$\begin{aligned}&\mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/n)\\ &\qquad\cong \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mathbb{Z}/n) \oplus \left(\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)\right).\end{aligned}$$

Remark 5.7.8 Recall that for $\mathbb{Z}/n$-modules A and B the map ξ is the canonical map induced by multiplication in $\mathbb{Z}/n$:

$$\mathrm{Hom}(A, \mathbb{Z}/n) \otimes \mathrm{Hom}(B, \mathbb{Z}/n) \longrightarrow \mathrm{Hom}(A \otimes B, \mathbb{Z}/n) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, \mathbb{Z}/n)).$$

If $n = 4$ and $A \cong B \cong \mathbb{Z}/2$, then ξ is the zero map, cf. Remark 5.7.6 (2). Thus if X and Y are Enriques surfaces and $\mathrm{char}(k) \neq 2$, then commutativity of (5.46) implies that the cup-product map

$$\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/4) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/4) \longrightarrow \mathrm{H}^2_{\text{ét}}(X \times Y, \mathbb{Z}/4)$$

is identically zero. (We have $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/4) \cong \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/4) \cong \mathbb{Z}/2$.)

Remark 5.7.9 Suppose that k is a separable closure of a subfield $k_0 \subset k$ and $X = X_0 \times_{k_0} k$ for some k_0-variety X_0. Then the maps in (5.45) and (5.46) respect the action of the Galois group $\Gamma = \mathrm{Gal}(k/k_0)$. The torsor $\mathcal{T}_X$ is unique up to isomorphism, so the class $[\mathcal{T}_X] \in \mathrm{H}^1(X, S_X)$ is Γ-invariant (and similarly for $\mathcal{T}_Y$). Hence ε is an injective map of Γ-modules $\mathrm{Hom}(S_X, S_Y^\vee) \to \mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$, therefore its image $\widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$ is a Γ-submodule of $\mathrm{H}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$. After twisting, we obtain the following canonical direct sum decomposition of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n) \cong \mathrm{H}^2_{\text{ét}}(X, \mu_n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mu_n) \oplus \mathrm{Hom}(S_X, \widehat{S}_Y), \qquad (5.47)$$

for any n not divisible by $\mathrm{char}(k)$. Here $\mathrm{Hom}(S_X, \widehat{S}_Y)$ consists of the elements of $\mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n)$ that restrict trivially to $X \times y_0$ and $x_0 \times Y$ for any $x_0 \in X(k)$, $y_0 \in Y(k)$. Finally, we note that the decompositions (5.47) are compatible under the natural maps linking μ_n for different values of n.

Proof of Theorem 5.7.7. We have an obvious commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\;p_Y\;} & X \times_k Y \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle p_X} \\ \mathrm{Spec}(k) & \xleftarrow{\;\pi_X\;} & X \end{array}$$

Since X is connected, the map $\pi_X^* \colon \mathrm{H}^0_{\text{ét}}(k, \mathbb{Z}/n) = \mathbb{Z}/n \to \mathrm{H}^0_{\text{ét}}(X, \mathbb{Z}/n)$ is an isomorphism with section x_0^*. The k-variety Y is geometrically connected, hence p_X has connected fibres, therefore we have an isomorphism of étale X-sheaves $\mathbb{Z}/n \xrightarrow{\sim} p_{X*}(\mathbb{Z}/n)$. We also obtain that

$$p_X^*\colon \mathrm{H}^0_{\text{ét}}(X,\mathbb{Z}/n) \longrightarrow \mathrm{H}^0_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)$$

is an isomorphism with section $(\mathrm{id},y_0)^*$.

The proper base change theorem [Mil80, Cor. VI.2.3] implies that the constant étale X-sheaf $\pi_X^*\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)$ is canonically isomorphic to $R^ip_{X*}(\mathbb{Z}/n)$. Thus we have the Leray spectral sequence

$$E_2^{p,q}=\mathrm{H}^p_{\text{ét}}(X,\mathrm{H}^q_{\text{ét}}(Y,\mathbb{Z}/n))\Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X\times_k Y,\mathbb{Z}/n). \tag{5.48}$$

The standard properties of spectral sequences imply that the composition

$$\mathrm{H}^i_{\text{ét}}(X,\mathbb{Z}/n)\xrightarrow{\sim}\mathrm{H}^i_{\text{ét}}(X,\mathrm{H}^0_{\text{ét}}(Y,\mathbb{Z}/n))=E_2^{i,0}\longrightarrow \mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \tag{5.49}$$

coincides with p_X^*. The functoriality of the spectral sequence (5.48) in X gives rise to a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) & \longrightarrow & E_2^{0,i}=\mathrm{H}^0_{\text{ét}}(X,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)) \\
\downarrow{\scriptstyle (x_0,\mathrm{id})^*} & & {\scriptstyle x_0^*}\downarrow{\scriptstyle \cong} \\
\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n) & \xrightarrow{\cong} & E_2^{0,i}=\mathrm{H}^0_{\text{ét}}(k,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n))
\end{array}$$

Hence the composition

$$\mathrm{H}^i_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)\longrightarrow E_2^{0,i}=\mathrm{H}^0_{\text{ét}}(X,\mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n))\cong \mathrm{H}^i_{\text{ét}}(Y,\mathbb{Z}/n)$$

coincides with the pullback $(x_0,\mathrm{id})^*$.

For $i=1$ we deduce from the spectral sequence the split exact sequence

$$0\longrightarrow \mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)\xrightarrow{p_X^*}\mathrm{H}^1_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)\xrightarrow{(x_0,\mathrm{id})^*}\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n)\longrightarrow 0$$

with section p_Y^*. This gives (i).

Let us prove (ii). Diagram (5.46) is obtained from diagram (5.43) since $\mathrm{Im}(\varepsilon)$ is a subset of $\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)$, as explained before the statement of the theorem. It remains to show that ε is an isomorphism. We give two different proofs of this fact.

The first proof uses a counting argument based on the well-known Künneth formula with coefficients in a field.[2] Since the map (5.49) is injective, from the spectral sequence (5.48) we get an isomorphism

$$\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n)\cong \mathrm{H}^1_{\text{ét}}(X,\mathrm{H}^1_{\text{ét}}(Y,\mathbb{Z}/n)).$$

[2] The idea of this proof was communicated to us by Yang Cao.

As a particular case of (5.39) we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/n)) \cong \mathrm{Hom}(S_Y, S_X^\vee) \cong \mathrm{Hom}(S_X, S_Y^\vee).$$

Thus the source and the target of ε are isomorphic, so it is enough to show that

$$\varepsilon\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/n) \longrightarrow \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/n)$$

is *injective*. More generally, for an integer $m|n$ consider the map

$$\varepsilon_m\colon \mathrm{Hom}(S_X \otimes S_Y, \mathbb{Z}/m) \longrightarrow \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/m)$$

defined via pairing with $[\mathcal{T}_X] \cup [\mathcal{T}_Y]$. We prove that ε_m is injective by induction on $m|n$. If p is a prime, the usual Künneth formula [Mil80, Cor. VI.8.13] for the field $\mathbb{F}_p$ implies that the cup-product map

$$\cup\colon \mathrm{H}^1_{\text{ét}}(X, \mathbb{F}_p) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{F}_p) \longrightarrow \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{F}_p)$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(S_X, \mathbb{F}_p) \otimes \mathrm{Hom}(S_Y, \mathbb{F}_p) & \xrightarrow{\ \xi\ } & \mathrm{Hom}(S_X \otimes S_Y, \mathbb{F}_p) \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\varepsilon_p} \\
\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/p) \otimes \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/p) & \xrightarrow{\ \cup\ } & \widetilde{\mathrm{H}}^2_{\text{ét}}(X \times_k Y, \mathbb{Z}/p)
\end{array} \tag{5.50}$$

In this case ξ is an isomorphism, hence ε_p is also an isomorphism.

Now for $m|n$ assume that ε_a is injective for all $a|m$, $a \neq m$. Write $m = ab$. An injective map of abelian groups $A \hookrightarrow A'$ gives rise to an injective map $\mathrm{H}^1_{\text{ét}}(X, A) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X, A')$. Indeed, X is geometrically connected and so the induced map

$$A \cong \mathrm{H}^0_{\text{ét}}(X, A) \longrightarrow \mathrm{H}^0_{\text{ét}}(X, A'/A) \cong A'/A$$

is surjective. In particular, an injective map $\mathbb{Z}/a \hookrightarrow \mathbb{Z}/m$ gives rise to an injective map $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/a) \hookrightarrow \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/m)$. Let $C \subset \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/b)$ be its cokernel. Applying the same argument we get an embedding

$$\mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/a)) \hookrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/m)).$$

The cokernel of this map is contained in $\mathrm{H}^1_{\text{ét}}(X, C)$, which is contained in $\mathrm{H}^1_{\text{ét}}(X, \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/b))$.

This gives rise to the top row of the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \widetilde{\mathrm{H}}^2_{\text{ét}}(X\times Y,\mathbb{Z}/a) & \longrightarrow & \widetilde{\mathrm{H}}^2_{\text{ét}}(X\times Y,\mathbb{Z}/m) & \longrightarrow & \widetilde{\mathrm{H}}^2_{\text{ét}}(X\times Y,\mathbb{Z}/b) \\
& \uparrow \varepsilon_a & & \uparrow \varepsilon_m & & \uparrow \varepsilon_b \\
0 \longrightarrow & \mathrm{Hom}(S_X\otimes S_Y,\mathbb{Z}/a) & \longrightarrow & \mathrm{Hom}(S_X\otimes S_Y,\mathbb{Z}/m) & \longrightarrow & \mathrm{Hom}(S_X\otimes S_Y,\mathbb{Z}/b)
\end{array}$$

The diagram implies that the middle map is injective too. We conclude that $\varepsilon=\varepsilon_n$ is injective, hence an isomorphism.

The second proof uses a direct verification that ε is an isomorphism.

Let $\mathcal{D}(\mathrm{Ab})$ be the bounded derived category of the category of abelian groups Ab, and let $\mathcal{D}(Z)$ be the bounded derived category of étale sheaves of abelian groups on a scheme Z. Hom- and Ext-groups without subscript are taken in Ab or $\mathcal{D}(\mathrm{Ab})$. Each of $\mathcal{D}(\mathrm{Ab})$ and $\mathcal{D}(Z)$ has the canonical truncation functors $\tau_{\leq m}$. Let $\mathbf{R}\pi_{X*}\colon \mathcal{D}(X\times Y)\to\mathcal{D}(X)$ be the derived functor of π_X, and let $\mathbf{R}p_{Y*}\colon \mathcal{D}(Y)\to\mathcal{D}(\mathrm{Ab})$ be the derived functor of the structure morphism $p_Y\colon Y\to\mathrm{Spec}(k)$.

We need to recall the definition of the type of a torsor. The local-to-global spectral sequence of Ext-groups

$$E_2^{p,q}=\mathrm{H}^p_{\text{ét}}(Y,\mathcal{E}xt^q_Y(\widehat{G},\mathbb{G}_m))\Rightarrow \mathrm{Ext}^{p+q}_Y(\widehat{G},\mathbb{G}_m)$$

completely degenerates since $\mathcal{E}xt^q_Y(\widehat{G},\mathbb{G}_m)=0$ for $q\geq 1$, thus giving an isomorphism $\mathrm{H}^q_{\text{ét}}(Y,G)\xrightarrow{\sim}\mathrm{Ext}^q_Y(\widehat{G},\mathbb{G}_m)$, see [Sko01, Lemma 2.3.7]. Since

$$\mathbf{R}\mathrm{Hom}_Y(\widehat{G},\cdot)=\mathbf{R}\mathrm{Hom}(\hat{G},\mathbf{R}p_{Y*}(\cdot)),$$

we have canonical isomorphisms

$$\mathrm{Ext}^q_Y(\widehat{G},\mathbb{G}_m)\cong\mathrm{Ext}^q(\widehat{G},\mathbf{R}p_{Y*}\mathbb{G}_m)\cong\mathrm{Ext}^q(\widehat{G},\tau_{\leq q}\mathbf{R}p_{Y*}\mathbb{G}_m).$$

The type of a G-torsor on Y is defined as the composed map

$$\mathrm{H}^1_{\text{ét}}(Y,G)\xrightarrow{\sim}\mathrm{Ext}^1(\widehat{G},\tau_{\leq 1}\mathbf{R}p_{Y*}\mathbb{G}_m)\longrightarrow\mathrm{Hom}(\widehat{G},\mathrm{Pic}(Y)). \tag{5.51}$$

Here the second map is induced by the obvious exact triangle in $\mathcal{D}(\mathrm{Ab})$

$$\bar{k}^*\longrightarrow\tau_{\leq 1}\mathbf{R}p_{Y*}\mathbb{G}_m\longrightarrow\mathrm{Pic}(Y)[-1],$$

where $\mathrm{H}^0_{\text{ét}}(Y,\mathbb{G}_m)=k^*$, since Y is reduced and connected.

When $nG=0$, the type of a G-torsor on Y is an element of

$$\mathrm{Hom}(\widehat{G},\mathrm{Pic}(Y)[n])=\mathrm{Hom}(G^\vee,S_Y),$$

and the type map is the composition

$$\mathrm{H}^1_{\text{ét}}(Y,G) \longrightarrow \mathrm{Ext}^1(G^\vee, \tau_{\leq 1}\mathbf{R}p_{Y*}\mathbb{Z}/n) \longrightarrow \mathrm{Hom}(G^\vee, S_Y).$$

We claim that these maps fit into the following commutative diagram of pairings:

$$\begin{array}{cccccc}
\mathrm{H}^1_{\text{ét}}(X,G^\vee) & \times & \mathrm{H}^1_{\text{ét}}(Y,G) & \to & \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \\
|| & & \downarrow & & || \\
\mathrm{H}^1_{\text{ét}}(X,G^\vee) & \times & \mathrm{Ext}^1_Y(G^\vee,\mathbb{Z}/n) & \to & \mathrm{H}^2_{\text{ét}}(X\times_k Y,\mathbb{Z}/n) \\
|| & & || & & \uparrow \\
\mathrm{H}^1_{\text{ét}}(X,G^\vee) & \times & \mathrm{Ext}^1(\widehat{G},\tau_{\leq 1}\mathbf{R}p_{Y*}\mathbb{Z}/n) & \to & \mathrm{H}^2_{\text{ét}}(X,\tau_{\leq 1}\mathbf{R}\pi_{X*}\mathbb{Z}/n) \\
|| & & \downarrow & & \downarrow \\
\mathrm{H}^1_{\text{ét}}(X,G^\vee) & \times & \mathrm{Hom}(G^\vee,S_Y^\vee) & \to & \mathrm{H}^1_{\text{ét}}(X,S_Y^\vee)
\end{array}$$

The first two pairings are compatible by [Mil80, Prop. V.1.20]. The two lower pairings are natural, and the compatibility of the rest of the diagram is clear.

Now take $G = S_Y$, $\varphi \in \mathrm{Hom}(S_X, S_Y^\vee)$, then $\varepsilon(\varphi)$ is the value of the top pairing on $\varphi_*[\mathcal{T}_X] \in \mathrm{H}^1_{\text{ét}}(X, S_Y^\vee)$ and $[\mathcal{T}_Y] \in \mathrm{H}^1_{\text{ét}}(Y, S_Y)$. The type of $\mathcal{T}_Y$ is the identity in $\mathrm{Hom}(S_Y^\vee, S_Y^\vee)$, hence the commutativity of the diagram shows that $\varepsilon(\varphi)$, as an element of $\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y, \mathbb{Z}/n) \cong \mathrm{H}^1_{\text{ét}}(X, S_Y^\vee)$, equals $\varphi_*[\mathcal{T}_X]$. The type of this torsor is the precomposition of the type of $\mathcal{T}_X$, which is the identity map $S_X^\vee \to S_X^\vee$, with dual map $\varphi^\vee\colon S_Y \to S_X^\vee$. This implies that ε defines an isomorphism from $\mathrm{Hom}(S_X, S_Y^\vee)$ to $\widetilde{\mathrm{H}}^2_{\text{ét}}(X\times_k Y, \mathbb{Z}/n)$.

To prove (iii) we note that if $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n)$ is a free $\mathbb{Z}/n$-module, then ξ is an isomorphism. If $\mathrm{NS}(X)[n] = 0$, then the Kummer sequence gives $\mathrm{H}^1_{\text{ét}}(X, \mu_n) = A(k_{\rm s})[n]$, where A is the Picard variety of X, thus $\mathrm{H}^1_{\text{ét}}(X, \mu_n) \simeq (\mathbb{Z}/n)^{2g}$, where $g = \dim A$. □

Let X and Y be smooth, projective and geometrically integral varieties over a field k of characteristic exponent p. By Corollary 5.2.4 the natural map of Γ-modules

$$\mathrm{Br}(X^{\rm s})(p') \oplus \mathrm{Br}(Y^{\rm s})(p') \longrightarrow \mathrm{Br}(X^{\rm s} \times Y^{\rm s})(p')$$

is split injective. We can describe the cokernel of this map.

Let A and B be the Picard varieties of X and Y, respectively. Fix points $x_0 \in X(k^{\rm s})$ and $y_0 \in Y(k^{\rm s})$. By Proposition 5.7.3, the Γ-module $\mathrm{NS}(X^{\rm s}\times Y^{\rm s})$ is the direct sum of $\mathrm{NS}(X^{\rm s})$, $\mathrm{NS}(Y^{\rm s})$, and $\mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s})$ identified with the kernel of the pullback to $x_0 \times Y^{\rm s}$ and $X^{\rm s} \times y_0$. In view of Remark 5.7.9, the first Chern class defines a map $\mathrm{NS}(X^{\rm s} \times Y^{\rm s})/n \hookrightarrow \mathrm{H}^2_{\text{ét}}(X^{\rm s} \times Y^{\rm s}, \mu_n)$, which is the direct sum of similar maps for X and Y and a certain injective map of Γ-modules $\mu\colon \mathrm{Hom}((B^\vee)^{\rm s}, A^{\rm s})/n \to \mathrm{Hom}(S_X, \widehat{S}_Y)$. Then the Kummer sequence gives rise to a canonical direct sum decomposition of Γ-modules

$$\mathrm{Br}(X^{\rm s} \times Y^{\rm s})[n] \cong \mathrm{Br}(X^{\rm s})[n] \oplus \mathrm{Br}(Y^{\rm s})[n] \oplus \mathrm{Hom}(S_X, \widehat{S}_Y)/\mathrm{Im}(\mu). \quad (5.52)$$

By the last sentence of Remark 5.7.9, the decompositions (5.52) for various n are compatible.

Formula (5.52) for $\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})$ becomes simpler if we impose a condition on the torsion of the Néron–Severi groups of X^{s} and Y^{s}.

Corollary 5.7.10 *Let X and Y be smooth, projective and geometrically integral varieties over a field k of characteristic exponent p. Let A and B be the Picard varieties of X and Y, respectively. Let n be a positive integer coprime to* $\mathrm{char}(k)$. *If* $\mathrm{Pic}(X^{\mathrm{s}})[n]\neq 0$ *and* $\mathrm{Pic}(Y^{\mathrm{s}})[n]\neq 0$, *then assume also that n is coprime to* $|\mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}|\cdot|\mathrm{NS}(Y^{\mathrm{s}})_{\mathrm{tors}}|$. *Then we have a canonical isomorphism of Γ-modules*

$$\begin{aligned}&\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})[n]\\ &\qquad\cong \mathrm{Br}(X^{\mathrm{s}})[n]\oplus \mathrm{Br}(Y^{\mathrm{s}})[n]\oplus \mathrm{Hom}(B^\vee[n],A[n])/\big(\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})/n\big).\end{aligned}$$

Proof. From the isomorphism $\widehat{S}_X=\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}},\mu_n)=\mathrm{Pic}(X^{\mathrm{s}})[n]$ we see that this group is an extension of $\mathrm{NS}(X^{\mathrm{s}})[n]$ by $A[n]$, because $A(k_{\mathrm{s}})=nA(k_{\mathrm{s}})$. Thus in our assumptions $\widehat{S}_X\cong A[n]$ and $\widehat{S}_Y\cong B[n]$. Using the non-degeneracy of the Weil pairing $B[n]\times B^\vee[n]\to\mu_n$ we identify S_Y with $B^\vee[n]$, and hence obtain canonical isomorphisms

$$\mathrm{Hom}(S_X,\widehat{S}_Y)\cong\mathrm{Hom}(S_Y,\widehat{S}_X)\cong\mathrm{Hom}(B^\vee[n],A[n]).$$

The statement now follows from (5.52). (Note that since $A[n]$ and $B[n]$ are free $\mathbb{Z}/n$-modules, we can conclude using [Mil80, Cor. VI.8.13], so for this corollary we do not need the more sophisticated Theorem 5.7.7.) ☐

Remark 5.7.11 The map $\mu\colon \mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})\to\mathrm{Hom}(B^\vee[n],A[n])$ in Corollary 5.7.10 comes from the first Chern class map. Assume that $\mathrm{char}(k)=0$. Then this map is the *negative* of the natural map defined by the action of homomorphisms of abelian varieties on their n-torsion points. It is enough to consider the case when X and Y coincide with their respective Albanese varieties $A^\vee$ and $B^\vee$. For the verification in this case we refer the reader to [OSZ, Lemma 2.6] (based on the Appell–Humbert theorem), which should be applied to the abelian variety $A^\vee\times B^\vee$.

Corollary 5.7.10 can be used to compute the Brauer group of a product of two elliptic curves and the attached Kummer variety. Here we restrict ourselves to one example, referring to [SZ12] for further results.

Example 5.7.12 [SZ12, Prop. 4.1, Example A1] Let E be an elliptic curve over a number field k such that the representation of Γ in $E[\ell]$ is a surjective map $\Gamma\to\mathrm{GL}(E[\ell])$ for every prime ℓ. Let E' be an elliptic curve with complex multiplication over k, which has a k-point of order 6. Then for $A=E\times_k E'$ we have $\mathrm{Br}(\overline{A})^\Gamma=0$. For example, one can take $k=\mathbb{Q}$, the elliptic curve E with equation $y^2=x^3+6x+2$ of conductor 2^63^3, and the elliptic curve E' with equation $y^2=x^3+1$.

Chapter 6
Birational invariance

For a scheme X and a positive integer n the structure morphism $\mathbb{A}^n_X \to X$ induces an injective map $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{A}^n_X)$. Similarly, $\mathbb{P}^n_X \to X$ induces an injective map $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{P}^n_X)$. In Section 6.1 we give conditions on X under which these maps are isomorphisms.

In Section 6.2 we discuss the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}(K)$ of a field K over a subfield k. The definition of $\mathrm{Br}_{\mathrm{nr}}(K/k)$ only uses the discrete valuations of K that are trivial on k, so this group depends only on the extension of fields $k \subset K$. When K is the function field of an integral variety over k, the group $\mathrm{Br}_{\mathrm{nr}}(K/k)$ is a birational invariant that can be used even when one does not have an explicit smooth projective variety X over k with function field $K = k(X)$ at one's disposal. If we have such an X then there is an isomorphism $\mathrm{Br}(X) \simeq \mathrm{Br}_{\mathrm{nr}}(K/k)$. We also recall that the Galois module $\mathrm{Pic}(X^{\mathrm{s}})$ up to addition of a permutation module is a birational invariant. Another birational invariant of smooth projective varieties X is the Chow group $\mathrm{CH}_0(X)$ of zero-cycles. In Section 6.4 we define a natural pairing between $\mathrm{CH}_0(X)$ and $\mathrm{Br}(X)$ with values in $\mathrm{Br}(k)$. This is used to give a proof of Mumford's theorem that for a smooth complex surface X with $\mathrm{H}^2(X, \mathcal{O}_X) \neq 0$ the Chow group of 0-cycles of degree 0 is not "finite-dimensional" (see Theorem 6.4.6 (iii) and Remark 6.4.7).

6.1 Affine and projective spaces

Theorem 6.1.1 *Let X be an integral, regular, noetherian scheme with function field K. For any prime $\ell \neq \mathrm{char}(K)$ and any integer $n \geq 0$, the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\mathbb{A}^n_X)$ induces an isomorphism of ℓ-primary torsion subgroups.*

Proof. If X is integral and regular, then so is $\mathbb{A}^1_X$. Using induction we reduce the statement to the case $n = 1$. A section of $\mathbb{A}^1_X \to X$ gives rise to a

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commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{A}^1_X) & \hookrightarrow & \mathrm{Br}(\mathbb{A}^1_K) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(K) \end{array}$$

where the downwards pointing arrows are induced by the restriction to the section and the upwards pointing arrows are induced by structure morphisms. To prove the result, it is thus enough to prove that for a field K of characteristic different from ℓ, the map $\mathrm{Br}(K)\{\ell\}\to\mathrm{Br}(\mathbb{A}^1_K)\{\ell\}$ is an isomorphism: this is Theorem 5.6.1 (ix). □

Remark 6.1.2 We have already seen in Theorem 5.6.1 (vi) that when k is separably closed but not algebraically closed, then $\mathrm{Br}(\mathbb{A}^1_k) \neq 0$ and hence $\mathrm{Br}(\mathbb{A}^n_k) \neq 0$ for all $n \geq 1$. Moreover, if k is an algebraically closed field of characteristic $p > 0$ and $n \geq 2$ is an integer, then $\mathrm{Br}(\mathbb{A}^n_k) \neq 0$ [KOS76, Prop. 5.3], [Hür81, Thm. 4.4, Cor. 6.5]. These papers build upon earlier work of Zelinsky and Yuan (see [KO74b]).

Theorem 6.1.3 *For any field k the natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathbb{P}^n_k)$ is an isomorphism.*

Proof. We proceed by induction on n. In the case $n = 1$ this is Theorem 5.6.1 (vii). Suppose that $n \geq 2$ and we have the isomorphism $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}(\mathbb{P}^{n-1}_k)$.

Let $\psi\colon W\to\mathbb{P}^n_k$ be the blowing-up of $\mathbb{P}^n_k$ in a k-point P. The projection of $\mathbb{P}^n_k \smallsetminus P$ onto $\mathbb{P}^{n-1}_k$ extends to a morphism $\pi\colon W\to\mathbb{P}^{n-1}_k$ which is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}_k$ with a section. To see this we choose coordinates on $\mathbb{P}^n_k$ so that $P = (1 : 0 : \ldots : 0)$. The restriction of $\pi\colon W\to\mathbb{P}^{n-1}_k$ to the open set $\mathbb{P}^n_k \smallsetminus P$ sends $(x_0 : \ldots : x_n)$ to $(x_1 : \ldots : x_n)$. Then the morphism $\sigma\colon \mathbb{P}^{n-1}_k\to W$ defined by $\sigma(x_1 : \ldots : x_n) = (0 : x_1 : \ldots : x_n)$ is a section of π.

Let $K = k(\mathbb{P}^{n-1}_k)$ be the field of functions on $\mathbb{P}^{n-1}_k$. The section σ gives rise to a K-point s of the generic fibre of π, hence this generic fibre is isomorphic to the projective line $\mathbb{P}^1_K$. Theorem 3.5.5 implies that the restriction to the generic fibre of π defines an injective map $\mathrm{Br}(W) \hookrightarrow \mathrm{Br}(\mathbb{P}^1_K)$. The closed embedding of the section $\sigma(\mathbb{P}^{n-1}_k)$ into W defines a map $\mathrm{Br}(W)\to\mathrm{Br}(\mathbb{P}^{n-1}_k)$. Similarly, we have a restriction to the generic point $\mathrm{Br}(\mathbb{P}^{n-1}_k) \hookrightarrow \mathrm{Br}(K)$ and the map $\mathrm{Br}(\mathbb{P}^1_K)\to\mathrm{Br}(K)$ induced by the restriction to the K-point s of $\mathbb{P}^1_K$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(W) & \hookrightarrow & \mathrm{Br}(\mathbb{P}^1_K) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(\mathbb{P}^{n-1}_k) & \hookrightarrow & \mathrm{Br}(K) \end{array}$$

where the upwards pointing arrows are induced by π and the structure morphism $\mathbb{P}^1_K\to\mathrm{Spec}(K)$. By Theorem 5.6.1 (vii) we know that the vertical arrows in the right-hand part of the diagram are isomorphisms which are inverse to each other. The diagram shows that the map $\mathrm{Br}(\mathbb{P}^{n-1}_k)\to\mathrm{Br}(W)$ is an isomorphism. The induction assumption now implies that the natural map $\mathrm{Br}(k)\to\mathrm{Br}(W)$ is an isomorphism.

The contraction $\psi\colon W \to \mathbb{P}^n_k$ is a birational morphism of smooth varieties. The restriction of ψ to some non-empty open subset $U \subset W$ is an isomorphism. By Theorem 3.5.5 the restriction map $\mathrm{Br}(\mathbb{P}^n_k) \to \mathrm{Br}(U)$ is injective. Since it factors through $\psi^*\colon \mathrm{Br}(\mathbb{P}^n_k) \to \mathrm{Br}(W)$, we see that ψ^* is injective. It is clear that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{P}^n_k) & \xrightarrow{\psi^*} & \mathrm{Br}(W) \\ \uparrow & & \uparrow \\ \mathrm{Br}(k) & = & \mathrm{Br}(k) \end{array}$$

We know that the right-hand vertical map is an isomorphism. This implies that the left-hand vertical map is an isomorphism too. □

Corollary 6.1.4 *Let X be an integral, regular, noetherian scheme. For any positive integer n the canonical projection $\pi\colon \mathbb{P}^n_X \to X$ induces an isomorphism*

$$\pi^*\colon \mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Br}(\mathbb{P}^n_X).$$

Proof. Let K be the function field of X. Fix a section of $\mathbb{P}^n_X \to X$. As in the proof of the previous theorem we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(\mathbb{P}^n_X) & \hookrightarrow & \mathrm{Br}(\mathbb{P}^n_{k(X)}) \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(k(X)) \end{array}$$

where the downwards pointing arrows are induced by the restriction to the section and the upwards pointing arrows are induced by structure morphisms. By Theorem 6.1.3, the vertical arrows in the right-hand part of the diagram are mutually inverse isomorphisms. The corollary follows from the diagram. □

Remark 6.1.5 Theorem 6.1.3 and Corollary 6.1.4 are special cases of general results. A particular case of [Gab81, Thm. 2, p. 193] says that for any scheme X the map $\mathrm{Br}(X)_{\mathrm{tors}} \to \mathrm{Br}(\mathbb{P}^n_X)_{\mathrm{tors}}$ is an isomorphism. A proof of Theorem 6.1.3 via the unramified Brauer group was given by Saltman [Sal85].

6.2 The unramified Brauer group

The following definition goes back to D. Saltman.

Definition 6.2.1 *Let $k \subset K$ be an extension of fields. The* **unramified Brauer group** *of K over k is the subgroup $\mathrm{Br}_{\mathrm{nr}}(K/k) \subset \mathrm{Br}(K)$ defined as the intersection of images of the natural injective maps $\mathrm{Br}(A) \hookrightarrow \mathrm{Br}(K)$, for all discrete valuation rings A with field of fractions K such that $k \subset A$.*

It is clear that the image of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(K)$ is contained in $\mathrm{Br}_{\mathrm{nr}}(K/k)$.

Remark 6.2.2 If k is a separably closed field, then for any prime number $\ell \neq \mathrm{char}(k)$ the multiplicative group k^* is infinitely ℓ-divisible. Thus for any discrete valuation $v\colon K \to \mathbb{Z}$ and any element $a \in k^*$ we have $v(a) = 0$. It follows that any discrete valuation ring with field of fractions K contains k. Thus for a separably closed field k we can write $\mathrm{Br}_{\mathrm{nr}}(K)$ for $\mathrm{Br}_{\mathrm{nr}}(K/k)$.

The unramified Brauer group is functorial in the following sense.

Proposition 6.2.3 *Let $k \subset K \subset L$ be extensions of fields. The restriction map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ induces a map $\mathrm{Br}_{\mathrm{nr}}(K/k) \to \mathrm{Br}_{\mathrm{nr}}(L/k)$.*

Proof. Let $v\colon L \to \mathbb{Z}$ be a discrete valuation with valuation ring B such that $k \subset B$. The restriction of v to K can be trivial or non-trivial. In the first case $K \subset B$, hence $\mathrm{Br}(K) \to \mathrm{Br}(L)$ factors through $\mathrm{Br}(B)$. In the second case, $A = B \cap K$ is a discrete valuation ring with field of fractions K. The restriction to $\mathrm{Br}(L)$ of an element in the image of $\mathrm{Br}(A) \to \mathrm{Br}(K)$ is in the image of $\mathrm{Br}(B) \to \mathrm{Br}(L)$. □

Proposition 6.2.4 *Let $k \subset K \subset L$ be fields such that the extension L/K is finite and separable. The corestriction map $\mathrm{cores}_{K/L}\colon \mathrm{Br}(L) \to \mathrm{Br}(K)$ induces a map $\mathrm{Br}_{\mathrm{nr}}(L/k) \to \mathrm{Br}_{\mathrm{nr}}(K/k)$.*

Proof. Let $A \subset K$ be a discrete valuation ring that contains k. Let B be the integral closure of A in L. Then $k \subset B$. Due to the separability assumption, B is a semilocal Dedekind domain which is a finitely generated A-module. If $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$, then for any prime ideal $q \subset B$ we have $\beta \in \mathrm{Br}(B_q)$. By Corollary 3.5.6 this implies that $\beta \in \mathrm{Br}(B)$. The result now follows from Proposition 3.8.1. Over a field of characteristic zero, an alternative proof via residues can be given using Proposition 1.4.7. □

The following proposition, together with the explicit formulae from Section 1.3.4, is useful for computations.

Proposition 6.2.5 *Let $k \subset K$ be an extension of fields. Let A be a discrete valuation ring with field of fractions K and residue field κ such that $k \subset A$. Let A^{h} be the henselisation of A and let K^{h} be the fraction field of A^{h}. For an element $\alpha \in \mathrm{Br}(K)[n]$, where n is coprime to the characteristic exponent of k, the following properties are equivalent.*

(i) *We have $\alpha \in \mathrm{Br}(A) \subset \mathrm{Br}(K)$.*
(ii) *The image of α in $\mathrm{Br}(K^{\mathrm{h}})$ belongs to $\mathrm{Br}(A^{\mathrm{h}}) \subset \mathrm{Br}(K^{\mathrm{h}})$.*
(iii) *The Serre residue $r(\alpha) \in \mathrm{H}^1(\kappa, \mathbb{Z}/n)$ is zero.*
(iv) *The Witt residue $r_W(\alpha) \in \oplus_{\ell|n} \mathrm{H}^1(\kappa, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is zero.*
(v) *The Gysin residue $\partial(\alpha) \in \mathrm{H}^1(\kappa, \mathbb{Z}/n)$ is zero.*

Proof. This is an immediate consequence of Theorem 3.6.1. Note that $k \subset \kappa$, hence n is coprime to the characteristic exponent of κ. For the compatibility of residues see Theorems 1.4.14 and 2.3.5. □

Proposition 6.2.6 *Let $k \subset K$ be fields and let X be a smooth and geometrically integral variety over k. Let $K(X)$ be the function field of $X \times_k K$. Let $\alpha \in \mathrm{Br}(K)$ be an element of order not divisible by* $\mathrm{char}(k)$. *If the image of $\alpha \in \mathrm{Br}(K)$ in $\mathrm{Br}(K(X))$ belongs to $\mathrm{Br}_{\mathrm{nr}}(K(X)/k)$, then $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$.*

Proof. Let A be a discrete valuation ring with fraction field K and residue field κ_A, such that $k \subset A$. Clearly we have $k \subset \kappa_A$. The closed fibre of $X \times_k \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is $X_{\kappa_A} = X \times_k \kappa_A$; it is a smooth and geometrically integral variety over κ_A. Let $B \subset K(X)$ be the local ring of the generic point of X_{κ_A}. Thus B is a discrete valuation ring with fraction field $K(X)$ such that $k \subset B$. The residue field κ_B of B is the function field $\kappa_A(X_{\kappa_A})$. Since κ_A is integrally closed in κ_B, the restriction map $\mathrm{H}^1(\kappa_A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\kappa_B, \mathbb{Q}/\mathbb{Z})$ is injective (Lemma 11.1.2). By the functoriality of residues (Theorem 3.7.5), we conclude that $\partial_A(\alpha) = 0 \in \mathrm{H}^1(\kappa_A, \mathbb{Q}/\mathbb{Z})$. □

Proposition 6.2.7 *Let X be a proper, integral, regular variety over a field k with function field $k(X)$. The natural inclusion $\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$ induces an isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.*

Proof. This is a special case of Proposition 3.7.9. □

In spite of this proposition, there are good reasons for using the unramified Brauer group of function fields. Since its definition only involves discrete valuation rings, usually one can easily determine whether a given element of $\mathrm{Br}(K)$ belongs to $\mathrm{Br}_{\mathrm{nr}}(K/k)$. The group $\mathrm{Br}_{\mathrm{nr}}(K/k)$ is visibly a birational invariant of integral varieties over k. One may use it when no smooth projective model is available – or even known to exist (in positive characteristic).

Let us recall some basic definitions.

Definition 6.2.8 *Two integral varieties X and Y over a field k are k-***birationally equivalent** *if the following equivalent properties hold:*

(a) *There exist non-empty Zariski open sets $U \subset X$ and $V \subset Y$ such that U and V are isomorphic as varieties over k.*
(b) *There exists a k-isomorphism of fields $k(X) \simeq k(Y)$, where $k(X)$ and $k(Y)$ are function fields of X and Y, respectively.*

Two integral varieties X and Y over k are **stably k-birationally equivalent** *if there exist integers $n \geq 0$ and $m \geq 0$ such that $X \times_k \mathbb{P}^n_k$ is k-birationally equivalent to $Y \times_k \mathbb{P}^m_k$.*

*An integral variety X of dimension d is a k-***rational variety***, or is k-***rational***, if X is k-birationally equivalent to the projective space $\mathbb{P}^d_k$. (Equivalently, the function field $k(X)$ is a purely transcendental extension of k).*

An integral variety X is **stably k-rational** *if there exists an integer $n \geq 0$ such that $X \times_k \mathbb{P}^n_k$ is a k-rational variety.*

Proposition 6.2.9 *Let $k \subset K \subset L$ be fields, where $L = K(t_1, \ldots, t_n)$ is a purely transcendental extension of K. Then the restriction map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ induces an isomorphism $\mathrm{Br}_{\mathrm{nr}}(K/k) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{nr}}(L/k)$.*

Proof. It is enough to consider the case $L = K(\mathbb{P}^1_K) = K(t)$, where t is an independent variable. Let $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$. Then $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/K)$, but this group is equal to $\mathrm{Br}(\mathbb{P}^1_K)$ by Proposition 6.2.7. The map $\mathrm{Br}(K) \to \mathrm{Br}(\mathbb{P}^1_K)$ is an isomorphism (this is not completely obvious in positive characteristic), see Theorem 5.6.1 (vii). Thus β comes from a unique $\alpha \in \mathrm{Br}(K)$, so it is enough to show that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$.

Let us check that α belongs to the image of $\mathrm{Br}(A) \to \mathrm{Br}(K)$, where $A \subset K$ is a discrete valuation ring with fraction field K such that $k \subset A$. Let π be a uniformiser of A. Let $B \subset L$ be the 2-dimensional local ring at the closed point of $\mathrm{Spec}(A[t])$ defined by the ideal (π, t). By purity of the Brauer group for 2-dimensional regular noetherian rings [AG60, Prop. 7.4], [Gro68, II, Cor. 2.2, Prop. 2.3], $\beta \in \mathrm{Br}_{\mathrm{nr}}(L/k)$ is the image of an element $\gamma \in \mathrm{Br}(B) \subset \mathrm{Br}(L)$. The value of γ at $t = 0$ is an element of $\mathrm{Br}(A)$ whose image in $\mathrm{Br}(K)$ is α. Since this holds for any such A, we conclude that $\alpha \in \mathrm{Br}_{\mathrm{nr}}(K/k)$. □

Corollary 6.2.10 *Let k be a field and let X and Y be integral varieties over k. If X and Y are stably k-birationally equivalent, then*

$$\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \simeq \mathrm{Br}_{\mathrm{nr}}(k(Y)/k).$$

In particular, if X is stably k-rational, then $\mathrm{Br}(k) \cong \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$. □

Proposition 6.2.7 then gives

Corollary 6.2.11 *Let k be a field, and let X and Y be proper, integral, regular varieties over k. If X and Y are stably k-birationally equivalent, then $\mathrm{Br}(X) \simeq \mathrm{Br}(Y)$.* □

Galois action on the Picard group

Proposition 6.2.12 *Let X and Y be smooth, projective, geometrically integral varieties over a field k. If X and Y are stably k-birationally equivalent, then there exist finitely generated permutation Γ-modules P_1 and P_2 and an isomorphism of Γ-modules*

$$\mathrm{Pic}(X^{\mathrm{s}}) \oplus P_1 \simeq \mathrm{Pic}(Y^{\mathrm{s}}) \oplus P_2.$$

This gives an isomorphism $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) \simeq \mathrm{H}^1(k, \mathrm{Pic}(Y^{\mathrm{s}}))$.

If X is stably k-rational, then the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ is stably a permutation Γ-module: there are finitely generated permutation Γ-modules P_1 and P_2 such that $\mathrm{Pic}(X^{\mathrm{s}}) \oplus P_1 \simeq P_2$.

If there exists a smooth, projective, geometrically integral variety Z over k such that $X \times_k Z$ is k-rational, then the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ is a direct summand of a permutation module.

This proposition originates in works of Shafarevich, Manin, Voskresenskiĭ. For an elegant proof due to Moret-Bailly, see [CTS87a, Prop. 2.A.1].

Suppose that $X(k) \neq \emptyset$. In this case we have

$$\mathrm{Br}_1(X)/\mathrm{Br}(k) \cong \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$$

by Proposition 5.4.2 and Remark 5.4.3. Thus Proposition 6.2.12 is closely related to the birational invariance of $\mathrm{Br}_1(X)$. However, in some cases the birational invariant given by the Γ-module $\mathrm{Pic}(X^{\mathrm{s}})$ up to addition of a permutation module is finer than $\mathrm{Br}_1(X)/\mathrm{Br}(k)$, see [CTS77, §8].

Proposition 6.2.13 *Let k be a field. Let $\mathcal{C}$ be a class of smooth, projective, geometrically integral varieties X over K, where K ranges over arbitrary field extensions of k. Suppose that $\mathcal{C}$ is stable under field extensions, and that for each variety X in $\mathcal{C}$ one has $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. If one of the following statements holds for all varieties X/K in $\mathcal{C}$ which have the additional property $X(K) \neq \emptyset$, then it holds for all X/K in $\mathcal{C}$:*

(i) *the* $\mathrm{Gal}(K_{\mathrm{s}}/K)$*-module* $\mathrm{Pic}(X_{K_{\mathrm{s}}})$ *is a permutation module;*
(ii) *the* $\mathrm{Gal}(K_{\mathrm{s}}/K)$*-module* $\mathrm{Pic}(X_{K_{\mathrm{s}}})$ *is a direct summand of a permutation module;*
(iii) $\mathrm{H}^1(K, \mathrm{Pic}(X_{K_{\mathrm{s}}})) = 0$.

Proof. (Sketch) Let $F = K(X)$. The F-variety $X_F = X \times_K F$ has an F-point. The hypothesis $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ implies that $\mathbf{Pic}^0_{X/k} = 0$, so $\mathbf{Pic}_{X/k}$ is a twisted constant group k-scheme split by a separable closure k_{s} of k, see Theorem 5.1.1. This implies that the natural maps

$$\mathrm{Pic}(X_{k_{\mathrm{s}}}) \xrightarrow{\sim} \mathrm{Pic}(X_{k_{\mathrm{s}}(X)}) \xrightarrow{\sim} \mathrm{Pic}(X_{F_{\mathrm{s}}})$$

are isomorphisms. For more details, see [CTS87a, Thm. 2.B.1]. □

6.3 Examples of unramified classes

Here are three types of unramified Brauer classes which will be used to construct counter-examples to the Hasse principle in Section 13.3.3. We systematically use Proposition 6.2.5.

Example 6.3.1 Let k be a field, $\mathrm{char}(k) \neq 2$, and let $a \in k^*$. Let $P(x)$ be a separable polynomial in $k[x]$ such that $P(x) = Q(x)R(x)$, where $Q(x)$ is a polynomial of even degree. Let X be a smooth projective variety over k birationally equivalent to the smooth, affine, geometrically integral surface with equation

$$y^2 - az^2 = P(x).$$

Let $\alpha \in \mathrm{Br}(k(X))[2]$ be the class of the quaternion algebra $(a, Q(x))$. Let us show that α is *unramified*. By Proposition 6.2.7 this implies that $\alpha \in \mathrm{Br}(X)$.

Let $R \subset k(X)$ be a discrete valuation ring with fraction field $k(X)$ such that $k \subset R$. Let κ be the residue field of R and let $v\colon k(X)^* \to \mathbb{Z}$ be the valuation defined by R. As usual, we write

$$r\colon \mathrm{Br}(k(X)) \longrightarrow \mathrm{H}^1(\kappa, \mathbb{Z}/2) \cong \kappa^*/\kappa^{*2}$$

for the associated Serre residue with coefficients $\mathbb{Z}/2$ (Definition 1.4.3), cf. Proposition 6.2.5. By (1.18), $r(\alpha)$ is the class of $a^{v(Q(x))}$ in κ^*/κ^{*2}.

If a is a square in κ, then $r(\alpha) = 1$. If $v(x) < 0$, then $v(Q(x))$ is even, hence $r(\alpha) = 1$. Assume that $v(x) \geq 0$ and that a is not a square in κ. Then $v(y^2 - az^2)$ is even, hence the equality $y^2 - az^2 = Q(x)R(x) \in k(X)^*$ implies that $v(Q(x)) + v(R(x))$ is even. The polynomials $Q(x)$ and $R(x)$ are coprime, hence there exist polynomials $a(x)$ and $b(x)$ such that

$$a(x)Q(x) + b(x)R(x) = 1.$$

Since $v(x) \geq 0$, if $v(Q(x))$ is odd hence positive, then $v(R(x)) = 0$, but then $v(Q(x)) + v(R(x))$ is odd. Hence $v(Q(x))$ is even, so that $r(\alpha) = 1$. From Proposition 6.2.5 we see that α is unramified.

Let us show that if the following three conditions are satisfied, then α *is not in the image of the map* $\mathrm{Br}(k) \to \mathrm{Br}(k(X))$.

(1) a is not a square in k.
(2) $Q(x)$ has an irreducible factor $A(x)$ such that the image of a in the field $k[x]/(A(x))$ is not a square.
(3) $R(x)$ has an irreducible factor $B(x)$ such that the image of a in the field $k[x]/(B(x))$ is not a square.

Let $v_A\colon k(x)^* \to \mathbb{Z}$ and $v_B\colon k(x)^* \to \mathbb{Z}$ be the valuations defined by A and B, respectively. We have $v_A(Q(x)) = v_B(R(x)) = 1$. Since $P(x) = Q(x)R(x)$ is separable, we have $v_B(Q(x)) = v_A(R(x)) = 0$. Calculating the residues attached to v_A and v_B using (1.18) we obtain that $(a, cQ(x))$ and $(a, cR(x))$ are distinct non-zero elements of $\mathrm{Br}(k(x))$, for any $c \in k^*$.

Suppose that $\alpha = (a, Q(x)) \in \mathrm{Br}(k(X))$ is the image of a $\beta \in \mathrm{Br}(k)$. We have a commutative diagram of restriction maps

$$\begin{array}{ccc} \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}_{\mathrm{nr}}(k(X)/k) \\ \downarrow & & \downarrow \\ \mathrm{Br}(k(\sqrt{a})) & \xrightarrow{\sim} & \mathrm{Br}_{\mathrm{nr}}(k(\sqrt{a})(X)/k(\sqrt{a})) \end{array}$$

The isomorphism in the bottom row follows from Proposition 6.2.9, because the function field $k(\sqrt{a})(X)$ of $X \times_k k(\sqrt{a})$ is a purely transcendental extension of $k(\sqrt{a})$ of transcendence degree 2. It is clear that α is in the kernel of the right-hand vertical map, hence $\mathrm{res}_{k(\sqrt{a})/k}(\beta) = 0$ in $\mathrm{Br}(k(\sqrt{a}))$. By Proposition 1.1.9, we have $\beta = (a, c)$ for some $c \in k^*$. Then $(a, cQ(x)) \in \mathrm{Br}(k(x))$ goes to zero in $\mathrm{Br}(k(X))$. By Proposition 7.2.1, which is an easy consequence of Proposition 5.4.2, the kernel of the restriction map $\mathrm{Br}(k(x)) \to \mathrm{Br}(k(X))$ is of order at most 2 and is generated by the quaternion algebra $(a, P(x))$. It follows that the kernel of $\mathrm{Br}(k(x)) \to \mathrm{Br}(k(X))$ contains distinct non-zero elements $(a, cQ(x))$ and $(a, cR(x))$, which is impossible. This finishes the proof.

As we have seen, condition (2) in the above example implies that $(a, Q(x))$ is a ramified and hence non-zero element of $\mathrm{Br}(k(x))$. For the sake of completeness we give another equivalent condition for this.

Remark 6.3.2 Let k be a field, $\mathrm{char}(k) \neq 2$, and let $a \in k^* \smallsetminus k^{*2}$. For a separable polynomial $Q(x) \in k[x]$ the following conditions are equivalent.

(i) $(a, Q(x)) = 0 \in \mathrm{Br}(k(x))$.
(ii) $Q(x) = S(x)^2 - aT(x)^2$ where $S(x), T(x) \in k(x)$.
(iii) $Q(x) = S(x)^2 - aT(x)^2$ where $S(x), T(x) \in k[x]$.

The equivalence of (i) and (ii) follows from Proposition 1.1.8. Let $K = k(\sqrt{a})$. That (iii) implies (ii) is obvious. To prove that (ii) implies (iii), note that a monic irreducible polynomial in $k[x]$ either remains irreducible in $K[x]$ or is the norm of an irreducible polynomial in the principal ideal domain $K[x]$, and consider the decomposition of elements of $k[x]$ into a product of irreducible factors. That (ii) implies (iii) is a special case of the Cassels–Pfister theorem [Lam05, Ch. IX, Thm. 1.3], which states that for any quadratic form φ over k, if a polynomial in $k[x]$ is represented by φ in $k(x)$, then it is represented by φ in $k[x]$.

The following algebraic result will be used in Section 13.3.3.

Example 6.3.3 (Reichardt–Lind) Let k be a field, $\mathrm{char}(k) \neq 2$, and let $a, b \in k^*$. Let X be a smooth projective curve over k birationally equivalent to the affine curve

$$ay^2 = x^4 - b.$$

Let us show that *the class of the quaternion algebra* $\alpha = (y, b) \in \mathrm{Br}(k(X))$ *is unramified*. By Proposition 6.2.7 this implies that $\alpha \in \mathrm{Br}(X)$.

Let $R \subset k(X)$ be a discrete valuation ring such that $k \subset R$. Let κ be the residue field of R, let $v\colon k(X)^* \to \mathbb{Z}$ be the valuation and let $r\colon \mathrm{Br}(k(X)) \to \kappa^*/\kappa^{*2}$ be the associated Serre residue. By (1.18), the residue $r(\alpha)$ is the class of $b^{v(y)}$ in κ^*/κ^{*2}. If b is a square in κ or if $v(y)$ is even, the residue is 1. Assume that b is not a square in κ. If $v(x) < 0$ then $v(x^4 - b)$ is a multiple of 4, hence so is $v(ay^2)$, hence $v(y)$ is even. Assume that $v(x) \geq 0$. Since b is not a square in κ, we have $v(x^4 - b) = 0$. Thus $v(ay^2) = 0$ hence $v(y) = 0$.

When b is not a square in k, there does not seem to exist a simple criterion for $\alpha = (y, b)$ to be in the image of $\mathrm{Br}(k)$.

Example 6.3.4 [CT14] Let k be a field, $\mathrm{char}(k) \neq 2$, let $a, b, c \in k^*$. Let X be a smooth projective variety birationally equivalent to the affine variety with equation

$$(x^2 - ay^2)(z^2 - bt^2)(u^2 - abw^2) = c.$$

Computing residues for the valuations of $K(X)$ that are trivial on k, one checks that the class α of the quaternion algebra $(x^2 - ay^2, b)$ is unramified, hence is an element of $\mathrm{Br}(X)$.

Projecting to affine space $\mathbb{A}^4_k$ with coordinates (z, t, u, w), we represent $k(X)$ as the function field of a conic over $k(\mathbb{A}^4_k)$. Using this and Proposition 7.2.1, one shows that if none of a, b, ab is a square in k, then $\alpha \in \mathrm{Br}(k(X))$ is not in the image of $\mathrm{Br}(k)$. See [CT14, Thm. 4.1] for details.

Explicit examples of unramified classes in $\mathrm{Br}(\mathbb{C}(X))$, where X is a variety over $\mathbb{C}$, will be given in Section 11.3.3 (the Artin–Mumford example). See also Sections 12.1.2 and 12.2.1.

6.4 Zero-cycles and the Brauer group

In this section we collect some results about the relations between the Brauer group $\mathrm{Br}(X)$ of a variety X over a field k and another birational invariant of smooth and proper varieties, the Chow group of zero-cycles $\mathrm{CH}_0(X)$. See also [ABBB]. The basic reference for the Chow group is the first chapter of Fulton's book [Ful98].

Let $Z_0(X)$ be the free abelian group whose generators are the closed points of X. The elements of $Z_0(X)$ are called *zero-cycles*. In other words, a zero-cycle is a finite sum $\sum_P n_P P$, where P is a closed point and $n_P \in \mathbb{Z}$. A zero-cycle is called *effective* if $n_P \geq 0$ for all P. The degree map

$$\deg_k \colon Z_0(X) \longrightarrow \mathbb{Z}$$

sends a zero-cycle $\sum_i n_i P_i$ to $\sum_i n_i [k(P_i) : k]$.

A morphism of varieties $f \colon X \to Y$ gives rise to a natural homomorphism

$$f_* \colon Z_0(X) \longrightarrow Z_0(Y)$$

sending the closed point $P \in X$ to $[k(P) : k(f(P))] f(P)$. The degree map is compatible with morphisms of varieties over k.

A zero-cycle on a normal integral curve C is called *rationally equivalent to zero* if it is the divisor $\mathrm{div}_C(g)$ of a non-zero rational function $g \in k(C)^*$. The *Chow group* $\mathrm{CH}_0(X)$ of zero-cycles on X is defined as the quotient of $Z_0(X)$ by the group generated by the elements $\phi_*(\mathrm{div}_C(g))$, for all proper morphisms $\phi \colon C \to X$ where C is a normal integral curve over k and all $g \in k(C)^*$.

Let k be a field, let X be a variety over k and let $Y \subset X$ be a finite subscheme. Then $Y = \mathrm{Spec}(A)$, where $A = \prod_{i=1}^m A_i$ and each A_i is a local

k-algebra. For $i = 1, \ldots, m$, let k_i be the residue field of A_i and let $n_i = \dim_k(A_i)/[k_i : k]$. For each i, the composition

$$\mathrm{Spec}(k_i) \longrightarrow \mathrm{Spec}(A_i) \longrightarrow \mathrm{Spec}(A) \longrightarrow X$$

defines a closed point $P_i \in Y$ with residue field k_i. The *zero-cycle associated to* $Y \subset X$ is by definition the formal sum $\sum_{i=1}^{m} n_i P_i$.

If $f\colon X \to Y$ is a proper morphism, then $f_*\colon Z_0(X) \to Z_0(Y)$ induces a map

$$f_*\colon \mathrm{CH}_0(X) \longrightarrow \mathrm{CH}_0(Y).$$

In particular, if X is a proper variety over k, then the structure morphism $X \to \mathrm{Spec}(k)$ induces a degree map $\deg_k\colon \mathrm{CH}_0(X) \to \mathbb{Z}$. Define

$$A_0(X) = \mathrm{Ker}[\deg_k\colon \mathrm{CH}_0(X) \longrightarrow \mathbb{Z}].$$

By the functoriality of the Brauer group, an element $\alpha \in \mathrm{Br}(X)$ can be evaluated at a closed point $P\colon \mathrm{Spec}(k(P)) \to X$. We denote this value by $\alpha(P) \in \mathrm{Br}(k(P))$. Define

$$\langle \alpha, P \rangle = \mathrm{cores}_{k(P)/k}(\alpha(P)) \in \mathrm{Br}(k).$$

By linearity this extends to a pairing

$$\mathrm{Br}(X) \times Z_0(X) \longrightarrow \mathrm{Br}(k). \tag{6.1}$$

This pairing is functorial in X. Namely, let $f\colon X \to Y$ be a morphism of varieties over k, let $\alpha \in \mathrm{Br}(Y)$ and let $z \in Z_0(X)$. Using that the composition of restriction $\mathrm{res}_{k(P)/k(f(P))}\colon \mathrm{Br}(k(f(P))) \to \mathrm{Br}(k(P))$ with corestriction $\mathrm{cores}_{k(P)/k(f(P))}\colon \mathrm{Br}(k(P)) \to \mathrm{Br}(k(f(P)))$ is multiplication by the degree $[k(P) : k(f(P))]$, we obtain

$$\langle f^*(\alpha), z \rangle = \langle \alpha, f_*(z) \rangle. \tag{6.2}$$

Lemma 6.4.1 *Let k be a field, let X be a k-variety and let $Y = \mathrm{Spec}(A) \subset X$ be a finite subscheme. Let $[Y] \in Z_0(X)$ be the associated zero-cycle. For any $\alpha \in \mathrm{Br}(X)$ we have $\langle \alpha, [Y] \rangle = \mathrm{cores}_{A/k}(\alpha_Y) \in \mathrm{Br}(k)$.*

Proof. For the identity map $Y = X$, this is Lemma 3.8.6. The general case follows from the functoriality of the pairing. □

Proposition 6.4.2 *Let X be a proper variety over a field k. Then the pairing* (6.1) *induces a bilinear pairing*

$$\mathrm{Br}(X) \times \mathrm{CH}_0(X) \longrightarrow \mathrm{Br}(k). \tag{6.3}$$

This pairing is functorial with respect to proper morphisms.

Proof. Let $C \to X$ be a morphism from a proper, normal, integral curve C over k. Let $f\colon C \to \mathbb{P}^1_k$ be a dominant morphism. This is a finite locally free morphism of constant rank. Let $z_0 \in Z_0(C)$, respectively $z_1 \in Z_0(C)$, be the zero-cycle on C associated to the finite scheme $\mathrm{Spec}(A_0) = f^{-1}(p_0)$, respectively to $\mathrm{Spec}(A_1) = f^{-1}(p_1)$, where p_0 and p_1 are distinct k-points in $\mathbb{P}^1_k$. Let $\alpha \in \mathrm{Br}(X)$. By Lemma 6.4.1 we have

$$\langle \alpha, z_i \rangle = \mathrm{cores}_{A_i/k}(\alpha_{A_i}) \in \mathrm{Br}(k)$$

for $i = 0, 1$. By Proposition 3.8.1, we have $\mathrm{cores}_{A_i/k}(\alpha_{A_i}) = \langle \mathrm{cores}_{C/\mathbb{P}^1}(\alpha), p_i \rangle$. The map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism (Theorem 6.1.3). Thus $\mathrm{cores}_{C/\mathbb{P}^1}(\alpha) \in \mathrm{Br}(\mathbb{P}^1_k)$ is a constant class, hence $\langle \alpha, z_0 \rangle = \langle \alpha, z_1 \rangle$. Functoriality of the pairing follows from (6.2). □

Corollary 6.4.3 *Let $X \subset \mathbb{P}^n_k$, $n \geq 2$, be a hypersurface. A k-line $L \subset \mathbb{P}^n_k$, $L \not\subset X$, cuts out on X the zero-cycle $\sum_{i=1}^m n_i P_i$, where $n_i \in \mathbb{Z}$ and P_i is a closed point of $X \cap L$, for $i = 1, \ldots, m$. Let $\alpha \in \mathrm{Br}(X)$. Then*

$$\sum_{i=1}^m n_i \, \mathrm{cores}_{k(P_i)/k}(\alpha(P_i)) \in \mathrm{Br}(k)$$

does not depend on L.

Proof. We may assume that k is infinite. By Proposition 6.4.2 it suffices to check that for any k-lines L_1 and L_2 in $\mathbb{P}^n_k$ not contained in X, the zero-cycles cut out on X by L_1 and L_2 are rationally equivalent. Choose k-points $P \in L_1$ and $Q \in L_2$ both outside of X, and let L_3 be the line through P and Q. It is enough to prove that the zero-cycles cut out on X by L_1 and L_3 are rationally equivalent. Let $\Pi \simeq \mathbb{P}^2_k$ be the plane spanned by L_1 and L_3. Let C be the scheme-theoretic intersection $\Pi \cap X \subset \Pi$. Thus $C \subset \mathbb{P}^2_k$ is given by one equation, so C is Cohen–Macaulay. It is enough to prove that the zero-cycles cut out on C by L_1 and L_3 are rationally equivalent on C. Since $P \notin X$, the rational map from $\mathbb{P}^2_k$ to $\mathbb{P}^1_k$ given by l_1/l_3, where $l_i = 0$ is an equation of $L_i \subset \mathbb{P}^2$, defines a finite morphism $f\colon C \to \mathbb{P}^1_k$. Since C is Cohen–Macaulay, f is flat by [Mat86, Thm. 23.1]. Then [Ful98, Thm. 1.7] implies that the zero-cycles cut out on C by L_1 and L_3 are rationally equivalent. □

Proposition 6.4.4 *Let X and Y be smooth, proper, geometrically integral varieties over a field k.*

(i) *If X and Y are stably k-birationally equivalent, then* $\mathrm{CH}_0(X) \simeq \mathrm{CH}_0(Y)$ *and $A_0(X) \simeq A_0(Y)$.*
(ii) *If X is stably k-rational, then $A_0(X) = 0$.*

Proof. See [Ful98, Exercise 16.1.11]. □.

The following definition was given in [ACTP17]. See also [Mer08].

Definition 6.4.5 *A projective variety X over a field k is called* **universally CH_0-trivial** *if for any field extension $k \subset K$ the degree map*

$$\deg_K \colon \mathrm{CH}_0(X_K) \longrightarrow \mathbb{Z}$$

is an isomorphism.

For example, if X is smooth, projective and stably k-rational, then X is universally CH_0-trivial.

Theorem 6.4.6 *Let X be a smooth, projective, geometrically integral variety over a field k.*

(i) *Assume that X is universally CH_0-trivial. Then for every field extension K of k the natural map $\mathrm{Br}(K)\to\mathrm{Br}(X_K)$ is an isomorphism.*
(ii) *Assume that for every field extension $k \subset K$, the group $A_0(X_K)$ is a torsion group. Then there exists a positive integer N such that for every field extension $k \subset K$ the quotient $\mathrm{Br}(X_K)/\mathrm{Br}(K)$ is annihilated by N.*
(iii) *Let $k = \mathbb{C}$. Suppose that there exist a smooth, projective, integral curve Y over $\mathbb{C}$ and a morphism $f \colon Y\to X$ such that $f_* \colon \mathrm{CH}_0(Y)\to\mathrm{CH}_0(X)$ is surjective. Then $\mathrm{H}^2(X, \mathcal{O}_X) = 0$ and $\mathrm{Br}(X)$ is isomorphic to the finite group $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}}$.*

Proof. (i) It is enough to prove the statement over k. Since X is universally CH_0-trivial, it has a zero-cycle z of degree 1. The map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is injective because evaluating at z gives a section. Now let $\alpha \in \mathrm{Br}(X)$. Take $F = k(X)$ to be the function field of X. The pairing (6.3)

$$\mathrm{Br}(X_F) \times \mathrm{CH}_0(X_F) \longrightarrow \mathrm{Br}(F)$$

gives rise to the pairing

$$\mathrm{Br}(X) \times \mathrm{CH}_0(X_F) \longrightarrow \mathrm{Br}(F).$$

Let η be the generic point of X. It is clear that $\langle\alpha, \eta\rangle_F$ is the image of α under the natural map $\mathrm{Br}(X)\to\mathrm{Br}(F)$. Since X is smooth, this map is injective (Theorem 3.5.5). By hypothesis $z_F - \eta = 0$ in $\mathrm{CH}_0(X_F)$, hence $\langle\alpha, z\rangle_F = \langle\alpha, \eta\rangle_F \in \mathrm{Br}(F)$. Therefore, $\langle\alpha, \eta\rangle_F$ is the image of $\langle\alpha, z\rangle \in \mathrm{Br}(k)$ under the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(F)$, hence $\alpha \in \mathrm{Br}(X)$ is the image of $\langle\alpha, z\rangle \in \mathrm{Br}(k)$ under the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$. See also [ABBB].

(ii) Let P be a closed point of X. Let η be the generic point of X and let $F = k(X)$ be the field of fractions. By assumption there is a positive integer N such that $N(\deg_k(P)\eta - P_F) = 0 \in \mathrm{CH}_0(X_F)$. Arguing as above, we see that for any $\alpha \in \mathrm{Br}(X)$ we have $N(\deg_k(P)\alpha - \langle\alpha, P\rangle) = 0 \in \mathrm{Br}(X)$, hence $\mathrm{Br}(X)/\mathrm{Br}(k)$ is annihilated by $N\deg_k(P)$. The proof shows that the same statement, with the same factor $N\deg_k(P)$, holds for X_K over any field extension K of k.

(iii) As $\mathbb{C}$ is an algebraically closed field of infinite transcendence degree over $\mathbb{Q}$, there exists an algebraically closed field $F \subset \mathbb{C}$ of finite transcendence degree over $\mathbb{Q}$ such that Y and X descend to varieties Y_0 and X_0 over F, that is, $X \cong X_0 \otimes_F \mathbb{C}$ and $Y \cong Y_0 \otimes_F \mathbb{C}$, and $f\colon Y \to X$ descends to an F-morphism $f_0\colon Y_0 \to X_0$.

We first claim that for any such field F, the map $\mathrm{CH}_0(Y_0) \to \mathrm{CH}_0(X_0)$ is surjective. Let z_0 be a zero-cycle on X_0. By assumption, over $\mathbb{C}$ there exist a zero-cycle $\sum_i n_i w_i$ on Y, finitely many smooth, projective, integral curves C_j, morphisms $\theta_j\colon C_j \to X$, and rational functions $g_j \in \mathbb{C}(C_j)^*$ such that the equality

$$z_{0,\mathbb{C}} = f_*\big(\sum_i n_i w_i\big) + \sum_j \theta_{j,*}\big(\mathrm{div}_{C_j}(g_j)\big)$$

holds in the group of zero-cycles $Z_0(X)$. This equality involves only finitely many terms. One may thus realise all its constituents over a field $L \subset \mathbb{C}$ which is of finite type over F. This field L itself is the field of fractions of a regular F-algebra A of finite type. After suitable localisation, the displayed equality holds over such an A. Since F is algebraically closed, the F-rational points are Zariski dense on $\mathrm{Spec}(A)$, thus we can specialise the above equality to an equality over F. We obtain an equality of cycles on X_0. In this specialisation process, the zero-cycle $z \in Z_0(X_0)$ specialises to itself. This proves the claim.

Let us now consider $K = F(X_0)$, which we may embed into $\mathbb{C}$, and let η be the generic point of X_0 over F. By the previous claim applied to the algebraic closure of K in $\mathbb{C}$, there exists a finite extension L/K such that η_L is in the image of $\mathrm{CH}_0(Y_{0,L}) \to \mathrm{CH}_0(X_{0,L})$. A restriction-corestriction argument implies that there exists a positive integer N such that $N\eta \in \mathrm{CH}_0(X_{0,K})$ is in the image of $\mathrm{CH}_0(Y_{0,K})$, that is, we have $N\eta = f_{0,*}(z)$ for some $z \in \mathrm{CH}_0(Y_{0,K})$. From functoriality of the pairing between Chow groups of zero-cycles and Brauer groups (Proposition 6.4.2), for any $\alpha \in \mathrm{Br}(X_0)$ we obtain

$$\langle \alpha, N\eta \rangle = \langle \alpha, f_{0,*}(z) \rangle = \langle f_0^*(\alpha), z \rangle \in \mathrm{Br}(K).$$

We have $f_0^*(\alpha) \in \mathrm{Br}(Y_0)$ and $\mathrm{Br}(Y_0) = 0$ since Y_0 is a curve over the algebraically closed field F (Theorem 5.6.1). Thus $N\langle \alpha, \eta \rangle = 0 \in \mathrm{Br}(K)$. But

$$\langle \alpha, \eta \rangle \in \mathrm{Br}(K) = \mathrm{Br}(F(X_0))$$

is the image of $\alpha \in \mathrm{Br}(X_0)$ under the injective map $\mathrm{Br}(X_0) \to \mathrm{Br}(F(X_0))$. We thus have $N\,\mathrm{Br}(X_0) = 0$.

The map $\mathrm{Br}(X_0) \to \mathrm{Br}(X_0 \times_F \mathbb{C})$ is an isomorphism by Proposition 5.2.3. We thus conclude that $N\,\mathrm{Br}(X) = 0$ for the original X over $\mathbb{C}$. Proposition 5.2.9 now gives that $\mathrm{Br}(X)$ is isomorphic to the finite group $\mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$ and that $\rho(X) = b_2(X)$. By Hodge theory, this condition is equivalent to $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, see [Voi02, Ch. 6 and Thm. 11.30]. □

Remark 6.4.7 The proof of Theorem 6.4.6 (iii) given above is due to Salberger (unpublished). It is a Brauer group version of a theorem of Bloch

[Blo79], who, using the idea of the generic point, gave a radically new proof of a theorem of Mumford [Mum68]: the Chow group of zero-cycles of a smooth, complex, projective surface X with $p_g(X) \neq 0$ is not representable. Bloch's argument was much developed by Bloch and Srinivas [BS83] and then by many other authors. Under the assumptions of Theorem 6.4.6 (iii), A.A. Roitman showed that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for all $i \geq 2$.

Chapter 7
Severi–Brauer varieties and hypersurfaces

There is a natural bijection between the isomorphism classes of Severi–Brauer varieties over a field k and the isomorphism classes of central simple k-algebras. This leads to many intricate relations. In Section 7.1 we briefly recall the basic properties of Severi–Brauer varieties. Any such variety is birationally equivalent to a principal homogeneous space of a torus. We give a precise version of this statement. We then discuss morphisms from an arbitrary variety to a Severi–Brauer variety. In Section 7.2 we deal with another simple class of projective homogeneous varieties, namely smooth projective quadrics of dimension at least one. For a variety X of either type, the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is surjective and the kernel is a finite cyclic group with a natural generator. The knowledge of this kernel will be used to establish the non-vanishing of some classes in $\mathrm{Br}(\mathbb{C}(X))$, where $X\to\mathbb{P}^2_{\mathbb{C}}$ is a conic bundle (see Section 11.3.2). Recently the computation of the Brauer group of open varieties attracted attention in connection with arithmetic investigations of integral points. In Section 7.3 we give several examples of such computations.

7.1 Severi–Brauer varieties

The following definition is due to F. Châtelet.

Definition 7.1.1 *Let n be a positive integer. A Severi–Brauer variety of dimension $n-1$ over a field k is a twisted form of the projective space $\mathbb{P}^{n-1}_k$. Equivalently, this is a k-variety X such that there exist a field extension $k\subset K$ and an isomorphism of K-varieties $X\times_k K\simeq\mathbb{P}^{n-1}_K$.*

A twisted form of $\mathbb{P}^1_k$ is a smooth, projective, geometrically integral curve C of genus zero. The linear system defined by the anticanonical bundle embeds C as a smooth conic in $\mathbb{P}^2_k$. Conversely, any smooth plane conic is a twisted form of $\mathbb{P}^1_k$.

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The automorphism group of the projective space $\mathbb{P}^{n-1}_k$ is the algebraic group $\mathrm{PGL}_{n,k}$. Indeed, the line bundle $\mathcal{O}_{\mathbb{P}^n_k}(1)$ is the unique ample generator of $\mathrm{Pic}(\mathbb{P}^n_k)$, hence it is preserved by all k-automorphisms of the k-variety $\mathbb{P}^{n-1}_k$. By Proposition 1.2.3, the group $\mathrm{PGL}_n(k)$ is the automorphism group of the matrix algebra $M_n(k)$. Galois descent (see Section 1.3.2) then gives a bijection between the isomorphism classes of twisted forms of $\mathbb{P}^{n-1}_k$ and the isomorphism classes of twisted forms of $M_n(k)$, which are precisely the central simple algebras of degree n over k. Thus we obtain canonical bijections of pointed sets

$$\begin{aligned}\mathrm{H}^1(k,\mathrm{PGL}_{n,k}) &\cong \{\text{central simple algebras over } k \text{ of degree } n\}/\text{iso}\\ &\cong \{\text{Severi–Brauer varieties over } k \text{ of dimension } n-1\}/\text{iso}\end{aligned}$$

and a map of pointed sets $\mathrm{H}^1(k,\mathrm{PGL}_{n,k})\to\mathrm{Br}(k)$ which sends a central simple algebra A of degree n to its class $[A]\in\mathrm{Br}(k)$, as discussed in Section 1.3.3. For a Severi–Brauer variety X of dimension $n-1$ we denote by $[X]\in\mathrm{Br}(k)$ the image of the isomorphism class of X under this map.

For a central simple algebra A over k of degree n define $X(A)$ to be the k-scheme of right ideals of A of rank n. More precisely, for any commutative k-algebra R, the set $X(A)(R)$ is the set of right ideals of the k-algebra $A\otimes_k R$ which are projective R-modules of rank n and are direct summands of the R-module $A\otimes_k R$, see [KMRT, Ch. I, §1.C]. This is a closed subscheme of the Grassmannian of n-dimensional subspaces of the k-vector space A.

Theorem 7.1.2 *Let X be a variety over a field k. The following properties are equivalent.*

(i) *X is a Severi–Brauer variety of dimension $n-1$.*
(ii) *There is an isomorphism $\overline{X}\simeq\mathbb{P}^{n-1}_{\bar{k}}$.*
(iii) *There is an isomorphism $X^{\mathrm{s}}\simeq\mathbb{P}^{n-1}_{k_{\mathrm{s}}}$.*
(iv) *There is a central simple k-algebra A of degree n such that $X\simeq X(A)$.*

The central simple algebra A in (iv) is well defined up to isomorphism. If $X=X(A)$, then $[X]=[A]\in\mathrm{Br}(k)$.

For the proof of this theorem see [Lic68], [Art82], [KMRT, Ch. I, §1.C], [GS17, Ch. 5], [Kol], [Po18, §4.5.1].

For a Severi–Brauer variety X over k one can construct a central simple k-algebra A such that $X\simeq X(A)$ in the following direct manner (Quillen, Szabó, Kollár, see [Kol, Cor. 23]).

The exact sequence of locally free sheaves [Har77, Ch. II, Thm. 8.13]

$$0\longrightarrow \Omega^1_{\mathbb{P}^{n-1}_k}\longrightarrow \mathcal{O}(-1)^{\oplus n}_{\mathbb{P}^{n-1}_k}\longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}_k}\longrightarrow 0 \qquad (7.1)$$

gives an isomorphism $\mathrm{H}^1(\mathbb{P}^{n-1}_k,\Omega^1_{\mathbb{P}^{n-1}_k})\cong\mathrm{H}^0(\mathbb{P}^{n-1}_k,\mathcal{O}_{\mathbb{P}^{n-1}_k})\cong k$. Let T_X be the tangent bundle and let Ω^1_X be the cotangent bundle of X. The k-vector space $\mathrm{H}^1(X,\Omega^1_X)$ has dimension 1, as can be computed over $\bar{k}$, where we have $\overline{X}\simeq\mathbb{P}^{n-1}_{\bar{k}}$. We have canonical isomorphisms

$$\mathrm{H}^1(X,\Omega^1_X)\cong\mathrm{H}^1(X,\mathcal{H}om(T_X,\mathcal{O}_X))\cong\mathrm{Ext}^1_X(T_X,\mathcal{O}_X).$$

Hence, up to multiplying the maps by non-zero scalars in k, there is a unique non-split extension of vector bundles

$$0\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{F}_X\longrightarrow T_X\longrightarrow 0. \tag{7.2}$$

This is a twisted version of the dual sequence of (7.1). Now $A:=\mathrm{End}_X(\mathcal{F}_X)$ is a central simple k-algebra such that $X=X(A)$.

Note [Kol, Warning 24] that the functor $X(A)$ is defined using right ideals of rank n. If one defines this functor using left ideals of rank n, then A must be replaced by the opposite algebra A^{op}.

Let X be a Severi–Brauer variety of dimension $n-1$. The Picard group

$$\mathrm{Pic}(X^{\mathrm{s}})\cong\mathrm{Pic}(\mathbb{P}^{n-1}_{k_{\mathrm{s}}})\cong\mathbb{Z}$$

is generated by the class L_X of an ample line bundle of degree 1. The class of the canonical bundle $\omega_X\in\mathrm{Pic}(X)$ is $-nL_X$. The action of Γ on $\mathrm{Pic}(X^{\mathrm{s}})$ is trivial, so $L_X\in\mathrm{Pic}(X^{\mathrm{s}})^\Gamma$ and $\mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}))=0$. Next, $\mathrm{Br}(X^{\mathrm{s}})\simeq\mathrm{Br}(\mathbb{P}^{n-1}_{k_{\mathrm{s}}})=0$. Thus the exact sequence (5.21)

$$0\longrightarrow\mathrm{Pic}(X)\longrightarrow\mathrm{Pic}(X^{\mathrm{s}})^\Gamma\longrightarrow\mathrm{Br}(k)\longrightarrow\mathrm{Br}_1(X)\longrightarrow\ \mathrm{H}^1(k,\mathrm{Pic}(X^{\mathrm{s}}))$$

takes the following form:

$$0\longrightarrow\mathrm{Pic}(X)\longrightarrow\mathrm{Pic}(X^{\mathrm{s}})^\Gamma\xrightarrow{\partial_X}\mathrm{Br}(k)\longrightarrow\mathrm{Br}(X)\longrightarrow 0, \tag{7.3}$$

where $\mathrm{Pic}(X^{\mathrm{s}})\cong\mathbb{Z}$. The kernel of $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is equal to the kernel of $\mathrm{Br}(k)\to\mathrm{Br}(k(X))$. It is a finite cyclic group annihilated by n. Let us denote by $\alpha_X=\partial_X(L_X)$ the image of L_X in $\mathrm{Br}(k)$.

Proposition 7.1.3 *Let A be a central simple algebra over a field k and let $X=X(A)$ be the associated Severi–Brauer variety. Then $\alpha_X=[A]\in\mathrm{Br}(k)$.*

Proof. See [Lic68, p. 1217] and [GS17, Thm. 5.4.11]. □

Remark 7.1.4 (1) For $X=X(A)$ as above, Amitsur proved that the kernel of $\mathrm{Br}(k)\to\mathrm{Br}(k(X))$ is the finite cyclic subgroup generated by $[A]\in\mathrm{Br}(k)$.

(2) Recall that for a scheme X, an X-scheme Y which is locally for the étale topology isomorphic to $\mathbb{P}^{n-1}_X$, is called a *Severi–Brauer scheme*. For a generalisation of (1) to the torsion subgroup of the Brauer group of a Severi–Brauer scheme over an arbitrary base, see [Gab81, Ch. II, Thm. 2].

Proposition 7.1.5 (Châtelet) *Let A be a central simple algebra over a field k and let $X = X(A)$ be the associated Severi–Brauer variety. The following properties are equivalent:*

(i) $X(k) \neq \emptyset$;
(ii) $\alpha_X = 0$;
(iii) $X \simeq \mathbb{P}_k^{n-1}$;
(iv) *there is a k-algebra isomorphism $A \simeq M_n(k)$.*

Proof. Condition (i) implies that the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is injective, thus the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma}$ is surjective. This implies (ii), which itself implies that there is a line bundle L on X which over k_{s} is isomorphic to L_X. Such a line bundle L defines a k-morphism $X \to \mathbb{P}_k^n$ which becomes an isomorphism over k_{s}, hence is an isomorphism over k. This gives (iii), which trivially gives (i). The equivalence of (ii) and (iv) follows from Proposition 7.1.3. □

This proposition is a particular case of the following statement.

Proposition 7.1.6 *Let X_1 and X_2 be Severi–Brauer varieties over k of the same dimension. Then $X_1 \simeq X_2$ if and only if $\alpha_{X_1} = \alpha_{X_2} \in \mathrm{Br}(k)$.*

Proof. If $X_1 \simeq X_2$, then we have an isomorphism of sequences (7.3) for X_1 and X_2 sending L_{X_1} to L_{X_2}, hence $\alpha_{X_1} = \alpha_{X_2}$. Alternatively, an isomorphism $X_1 \simeq X_2$ gives rise to an isomorphism $A_1 \simeq A_2$, since $A_i = \mathrm{End}_{X_i}(\mathcal{F}_{X_i})$, for $i = 1, 2$, see (7.2). This implies $\alpha_{X_1} = \alpha_{X_2}$ by Proposition 7.1.3.

Let A_1 and A_2 be central simple k-algebras such that $X_i \cong X(A_i)$, for $i = 1, 2$. Since $\dim(X_1) = \dim(X_2)$, we have $\dim_k(A_1) = \dim_k(A_2)$. By Proposition 7.1.3 the condition $\alpha_{X_1} = \alpha_{X_2}$ is equivalent to the condition $[A_1] = [A_2]$. By Wedderburn's theorem (Theorem 1.2.4), central simple k-algebras of the same dimension and the same Brauer class, are isomorphic. Then $X_1 \cong X(A_1) \cong X(A_2) \cong X_2$. □

Proposition 7.1.7 *Let X_1 and X_2 be Severi–Brauer varieties over k. The following properties are equivalent.*

(i) *The classes α_{X_1} and α_{X_2} generate the same cyclic subgroup of* $\mathrm{Br}(k)$.
(ii) *X_1 and X_2 are stably k-birationally equivalent, i.e., there exist projective spaces $\mathbb{P}_k^r$ and $\mathbb{P}_k^s$ such that $X_1 \times_k \mathbb{P}_k^r$ and $X_2 \times_k \mathbb{P}_k^s$ are birationally equivalent.*

Proof. Since α_{X_1} is a multiple of α_{X_2}, the class α_{X_1} goes to zero in $\mathrm{Br}(k(X_2))$. Thus the generic fibre of the projection $X_1 \times X_2 \to X_2$ is a split Severi–Brauer variety, hence $X_1 \times X_2$ is birationally equivalent to $X_2 \times_k \mathbb{P}_k^s$ for some $s \geq 0$. Similarly, $X_1 \times X_2$ is birationally equivalent to $X_1 \times_k \mathbb{P}_k^r$ for some $r \geq 0$. □

It is an open question whether stably birationally equivalent Severi–Brauer varieties of the same dimension are birationally equivalent.

7.1.1 Two applications of Severi–Brauer varieties

We first justify the claim of Remark 1.2.17.

Proposition 7.1.8 *Let K be a henselian discretely valued field. Let $\widehat{K}$ be the completion of K. Then the natural map $\mathrm{Br}(K)\to\mathrm{Br}(\widehat{K})$ is an isomorphism.*

Proof. By [BLR90, III, §6, Cor. 10, p. 82], if X is a smooth variety over K, then $X(K)$ is a dense subset of $X(\widehat{K})$. Let α be an element of the kernel of $\mathrm{Br}(K)\to\mathrm{Br}(\widehat{K})$. Choose a Severi–Brauer variety X over K such that the class of X in $\mathrm{Br}(K)$ is α. Then $X(\widehat{K})\neq\emptyset$ by Proposition 7.1.5, hence $X(K)\neq\emptyset$ and this implies $\alpha=0$ by Proposition 7.1.5. Let now $\beta\in\mathrm{Br}(\widehat{K})$. There is a positive integer n such that β is the image of $\beta_1\in\mathrm{H}^1(\widehat{K},\mathrm{PGL}_n)$ under the map $\mathrm{H}^1(\widehat{K},\mathrm{PGL}_n)\to\mathrm{H}^2(\widehat{K},\mathbb{G}_m)$. Choose a closed embedding of algebraic K-groups $\mathrm{PGL}_n\hookrightarrow\mathrm{GL}_N$. Then $X=\mathrm{GL}_N/\mathrm{PGL}_n$ is a smooth variety over K. Applying [SerCG, Ch. I, §5.4, Prop. 36] and using Hilbert's theorem 90, we obtain the following commutative diagram of pointed sets with exact rows:

$$\begin{array}{ccccccc}
\mathrm{GL}_N(K) & \longrightarrow & X(K) & \longrightarrow & \mathrm{H}^1(K,\mathrm{PGL}_n) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{GL}_N(\widehat{K}) & \longrightarrow & X(\widehat{K}) & \longrightarrow & \mathrm{H}^1(\widehat{K},\mathrm{PGL}_n) & \longrightarrow & 0
\end{array}$$

Choose a lifting $\beta_2\in X(\widehat{K})$ of β_1. By the implicit function theorem (see Theorem 10.5.1), $\mathrm{GL}_N(\widehat{K})\beta_2$ is an open subset of $X(\widehat{K})$ in the topology induced by the topology of $\widehat{K}$. Since $X(K)$ is dense in $X(\widehat{K})$, we can find an $\alpha_2\in X(K)$ and a $g\in\mathrm{GL}_N(\widehat{K})$ such that $g\beta_2=\alpha_2$. Since $g\beta_2$ goes to $\beta_1\in\mathrm{H}^1(\widehat{K},\mathrm{PGL}_n)$ (see [SerCG, Ch. I, p. 55]), the image $\alpha_1\in\mathrm{H}^1(K,\mathrm{PGL}_n)$ of α_2 goes to β_1. This implies that the image $\alpha\in\mathrm{Br}(K)$ of α_1 goes to $\beta\in\mathrm{Br}(\widehat{K})$. □

The following proposition from [Duc98] is in the spirit of earlier results by Merkurjev and Suslin.

Proposition 7.1.9 (Ducros) *For any field k of characteristic zero there exists a field extension L of k of cohomological dimension $\mathrm{cd}(L)$ at most 1 such that k is algebraically closed in L.*

Proof. For k of characteristic zero, recall [SerCG, Ch. II, §3.1, Prop. 5] that $\mathrm{cd}(k)\leq 1$ if and only if $\mathrm{Br}(k')=0$ for every finite extension k'/k. By Proposition 7.1.5 this is also equivalent to the following condition: for any finite field extension k' of k, any Severi–Brauer variety over k' is trivial, equivalently, it has a k'-rational point.

If $\mathrm{cd}(k)\leq 1$, we take $L=k$. If $\mathrm{cd}(k)>1$, then there exist non-trivial Severi–Brauer varieties W over some finite extensions k'/k. Choose one

Severi–Brauer variety W in each k'-isomorphism class and consider the Weil restriction of scalars $R_{k'/k}(W)$. The finite products of these varieties form a filtering inductive system of geometrically integral varieties over k; their fields of functions are extensions $k \subset K$ such that k is algebraically closed in K. Passing to the inductive limit we obtain a field extension $k \subset k_1$ such that k is algebraically closed in k_1. Define $k_n := (k_{n-1})_1$ for $n \geq 2$. Let L be the inductive limit of k_n as $n \to \infty$. On the one hand, k is algebraically closed in L. On the other hand, any variety $R_{L'/L}(V)$, where V is a Severi–Brauer variety over a finite extension L' of L, is already defined over some k_n. Any integral variety has a rational point over its field of functions, so $R_{L'/L}(V)$ has a k_{n+1}-point which is also an L-point. Then V has an L'-point. This proves that $\mathrm{cd}(L) \leq 1$. □

Remark 7.1.10 As a consequence of a theorem of Kollár, one has the following stronger result [CT08b, Thm. 2.1]: *For any field k of characteristic zero there exists a field extension $k \subset L$ such that L is a C_1-field and k is algebraically closed in L.*

7.1.2 Torsors for tori as birational models of Severi–Brauer varieties

The following statement does not seem to be available in the literature.

Proposition 7.1.11 *Let A be a central simple algebra over a field k and let $X = X(A)$ be the associated Severi–Brauer variety. Let K be a maximal commutative étale k-subalgebra of A. The action of K on A by left multiplication defines a maximal k-torus $T \subset \mathrm{PGL}_A$ that fits into the exact sequence*

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow T \longrightarrow 1. \tag{7.4}$$

The natural action of PGL_A on X restricts to an action of T on X, which has a dense open orbit $E \subset X$ consisting of the points of X with trivial stabilisers in T. Then E is a k-torsor for T. Moreover, the connecting map defined by the exact sequence (7.4) sends the class $[E] \in \mathrm{H}^1(k,T)$ to the class $[A] \in \mathrm{Br}(k)$.

Proof. Let n be the degree of A. Let $c\colon \Gamma \to \mathrm{PGL}_{n,k}(k_{\mathrm{s}})$ be a 1-cocycle such that A is the twisted form of the matrix algebra $M_n(k)$ by c with respect to the action of $\mathrm{PGL}_{n,k}$ by conjugation. Twisting by c we obtain $X = (\mathbb{P}^{n-1}_k)_c$ and the inner form $\mathrm{PGL}_A = (\mathrm{PGL}_{n,k})_c$. After twisting, the left action of $\mathrm{PGL}_{n,k}$ on $\mathbb{P}^{n-1}_k$ becomes a left action of PGL_A on X.

We have a commutative diagram of algebraic groups over k:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,k} & \longrightarrow & R_{K/k}(\mathbb{G}_{m,K}) & \longrightarrow & T & \longrightarrow & 1 \\ & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_{m,k} & \longrightarrow & \mathrm{GL}_A & \longrightarrow & \mathrm{PGL}_A & \longrightarrow & 1 \end{array} \tag{7.5}$$

The choice of a cocycle c in its cohomology class is equivalent to the choice of an isomorphism $A \otimes_k k_{\mathrm{s}} \simeq M_n(k_{\mathrm{s}})$. Under this isomorphism, the k_{s}-subalgebra $K \otimes_k k_{\mathrm{s}} \subset A \otimes_k k_{\mathrm{s}}$ becomes a subalgebra of $M_n(k_{\mathrm{s}})$ conjugate to the subalgebra of diagonal matrices in $M_n(k_{\mathrm{s}})$. Hence the subset $E^{\mathrm{s}} \subset X^{\mathrm{s}} \simeq \mathbb{P}^{n-1}_{k_{\mathrm{s}}}$ is projectively isomorphic to the complement U to the union of coordinate hyperplanes in $\mathbb{P}^{n-1}_{k_{\mathrm{s}}}$. Moreover, the action of T^{s} on E^{s} is isomorphic to the action of the diagonal torus in $\mathrm{PGL}_{n,k_{\mathrm{s}}}$ on U. Hence E^{s} is a torsor for T^{s}. This implies that E is a k-torsor for T.

It is a general fact (and is immediate to check) that c^{-1} is a cocycle with coefficients in the inner form $\mathrm{PGL}_A = (\mathrm{PGL}_{n,k})_c$. Then $\mathrm{PGL}_{n,k} = (\mathrm{PGL}_A)_{c^{-1}}$. By a corollary of a theorem of Steinberg over perfect fields, extended in [BS68, §8.6] to arbitrary fields, the cocycle $c^{-1}\colon \Gamma \to \mathrm{PGL}_A(k_{\mathrm{s}})$ factors through a 1-cocycle $\mu\colon \Gamma \to T(k_{\mathrm{s}}) \subset \mathrm{PGL}_A(k_{\mathrm{s}})$, see [PR94, Prop. 6.18] and its proof.

Twisting (7.5) by μ with respect to the action of T by conjugation we obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_{m,k} & \longrightarrow & R_{K/k}(\mathbb{G}_{m,K}) & \longrightarrow & T & \longrightarrow & 1 \\ & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_{m,k} & \longrightarrow & \mathrm{GL}_{n,k} & \longrightarrow & \mathrm{PGL}_{n,k} & \longrightarrow & 1 \end{array} \tag{7.6}$$

Since $T \subset \mathrm{PGL}_A$ acts on X preserving E, twisting by μ turns X into $\mathbb{P}^{n-1}_k$, hence turns E into a trivial T-torsor. This shows that the class of E in $\mathrm{H}^1(k,T)$ is represented by the inverse of μ.

We note that $c\colon \Gamma \to \mathrm{PGL}_{n,k}(k_{\mathrm{s}})$ factors through $\mu^{-1}\colon \Gamma \to T(k_{\mathrm{s}})$. Since A is the twisted form of $M_n(k)$ by c, the connecting map $\mathrm{H}^1(k, \mathrm{PGL}_{n,k}) \to \mathrm{Br}(k)$ sends the class of c to $[A]$. By the commutativity of (7.6), the connecting map $\mathrm{H}^1(k,T) \to \mathrm{Br}(k)$ sends the class of μ^{-1}, which is equal to the class of the T-torsor E, to $[A]$. □

The following special case is better known, though it is sometimes stated in the weaker form of a stable birational equivalence.

Proposition 7.1.12 *Let X be the Severi–Brauer variety over a perfect field k attached to a cyclic algebra $D_k(\chi, a)$. Let $K \subset k_{\mathrm{s}}$ be the invariant subfield of* $\mathrm{Ker}(\chi) \subset \Gamma$. *Then X contains a dense open subset isomorphic to the k-torsor for the norm 1 torus $R^1_{K/k}(\mathbb{G}_{m,K})$ given by $\mathrm{N}_{K/k}(x) = a$.*

Proof. Write $A = D_k(\chi, a)$ and $n = [K : k]$. We note that $K \subset A$ is a maximal commutative étale k-subalgebra and $T = R_{K/k}(\mathbb{G}_{m,K})/\mathbb{G}_{m,k} \subset \mathrm{PGL}_A$ is the associated maximal k-torus. By Proposition 7.1.11, the Severi–Brauer variety X contains a dense open subset isomorphic to a k-torsor E for T such that the class $[E] \in \mathrm{H}^1(k, T)$ goes to

$$[A] = (\chi, a) \in \mathrm{Br}(K/k) = \mathrm{H}^2(G, K^*)$$

under the isomorphism $\mathrm{H}^1(k, T) \xrightarrow{\sim} \mathrm{Br}(K/k)$ provided by the connecting map of (7.4).

Let σ be the generator of $\mathrm{Gal}(K/k) \simeq \mathbb{Z}/n$ such that $\chi(\sigma) = 1 \in \mathbb{Z}/n$. We construct an isomorphism of k-tori $T \xrightarrow{\sim} R^1_{K/k}(\mathbb{G}_{m,K})$ as follows. The map $K^* \to K^*$ sending x to $\sigma(x)/x$ commutes with $\mathrm{Gal}(K/k)$ and hence induces an endomorphism ϕ of the k-torus $R_{K/k}(\mathbb{G}_{m,K})$. It is clear that $\mathrm{Ker}(\phi)$ is $\mathbb{G}_{m,k}$ naturally embedded in $R_{K/k}(\mathbb{G}_{m,K})$. By Hilbert's theorem 90 for a cyclic extension, $\mathrm{Coker}(\phi) = \mathbb{G}_{m,k}$ and the surjective map $R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k}$ is induced by the norm $\mathrm{N}_{K/k}$. We obtain an exact sequence of k-tori

$$1 \longrightarrow \mathbb{G}_{m,k} \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \xrightarrow{\phi} R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1. \quad (7.7)$$

Hence ϕ induces an isomorphism $\varphi \colon T \xrightarrow{\sim} R^1_{K/k}(\mathbb{G}_{m,K})$, which thus depends on the choice of the generator σ.

Recall that every k-torsor for $R^1_{K/k}(\mathbb{G}_{m,K})$ is isomorphic to the closed subset $Z_c \subset R_{K/k}(\mathbb{G}_{m,K})$ given by $\mathrm{N}_{K/k}(x) = c$ for some $c \in k^*$. Indeed, Shapiro's lemma and Hilbert's theorem 90 imply that $\mathrm{H}^1(k, R_{K/k}(\mathbb{G}_{m,K})) = \mathrm{H}^1(K, \mathbb{G}_{m,K}) = 0$. The exact sequence of tori

$$1 \longrightarrow R^1_{K/k}(\mathbb{G}_{m,K}) \longrightarrow R_{K/k}(\mathbb{G}_{m,K}) \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1 \quad (7.8)$$

gives an isomorphism

$$k^*/\mathrm{N}_{K/k}(K^*) = \widehat{\mathrm{H}}^0(G, K^*) \xrightarrow{\sim} \mathrm{H}^1(k, R^1_{K/k}(\mathbb{G}_{m,K})).$$

Every element of this group is represented by some $c \in k^*$. The exact sequence (7.8) shows that the inverse image of c in $R_{K/k}(\mathbb{G}_{m,K})$, which we called Z_c, is a k-torsor for $R^1_{K/k}(\mathbb{G}_{m,K})$ whose class is represented by c.

Let us show that the isomorphism φ induces an isomorphism $E \xrightarrow{\sim} Z_a$, which is equivalent to $\varphi_*[E] = [Z_a]$. We have $(\chi, a) = a \cup \partial(\chi)$, see (1.5). We obtain the following diagram of isomorphisms:

$$\begin{array}{ccccc}
[E] \in & \mathrm{H}^1(k,T) & \xrightarrow[\cong]{\varphi_*} & \mathrm{H}^1(k, R^1_{K/k}(\mathbb{G}_{m,K})) & \ni [Z_a] \\
 & \downarrow \cong & & \cong \uparrow & \\
(\chi, a) \in & \mathrm{H}^2(G, K^*) & \xleftarrow[\cong]{\cup \partial(\chi)} & \widehat{\mathrm{H}}^0(G, K^*) & \ni a
\end{array}$$

To prove that $\varphi_*[E] = [Z_a]$ it remains to show that this diagram commutes. It suffices to prove that the connecting map attached to (7.7) is the cup-product with the generator $\partial(\chi)$ of $\mathrm{H}^2(G,\mathbb{Z}) \simeq \mathbb{Z}/n$. This is the content of the following Lemma 7.1.13. □

Lemma 7.1.13 *The connecting map associated to the exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{\sigma-1} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{7.9}$$

sends $1 \in \mathbb{Z}$ *to* $\partial(\chi)$.

Proof. Recall that $\chi\colon G\to\mathbb{Z}/n$ sends σ to $1 \in \mathbb{Z}/n$. Then the map $\widetilde{\chi}\colon G\to\mathbb{Z}$ sending σ^i to $i \in \mathbb{Z}$ is a lifting of χ. Hence the cohomology class $\partial(\chi)$ is represented by the 2-cocycle $G \times G\to\mathbb{Z}$ sending a pair (σ^i, σ^j) to

$$\big(\widetilde{\chi}(\sigma^i) + \widetilde{\chi}(\sigma^j) - \widetilde{\chi}(\sigma^{i+j})\big)/n.$$

This equals 0 if $i + j < n$, and 1 otherwise.

Now we turn to the exact sequence (7.9). Let I be the kernel of $\mathbb{Z}[G]\to\mathbb{Z}$. The element $1 \in \mathbb{Z}[G]$ goes to $1 \in \mathbb{Z}$, hence the connecting map associated to the exact sequence of G-modules

$$0 \longrightarrow I \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

sends $1 \in \mathbb{Z}$ to the class in $\mathrm{H}^1(G, I)$ represented by the 1-cocycle $\sigma^i \mapsto \sigma^i - 1$. The function $G\to\mathbb{Z}[G]$ sending σ^i to $1 + \sigma + \ldots + \sigma^{i-1}$ is a lifting of this cocycle. Thus the image of $1 \in \mathbb{Z}$ under the connecting map defined by (7.9) is the class in $\mathrm{H}^2(G,\mathbb{Z})$ represented by the 2-cocycle sending a pair (σ^i, σ^j) to

$$(1 + \sigma + \ldots + \sigma^{i-1}) + \sigma^i(1 + \sigma + \ldots + \sigma^{j-1}) - (1 + \sigma + \ldots + \sigma^{k-1}),$$

where $k = i + j$ if $i + j < n$ and $k = i + j - n$ otherwise. Thus the value of this 2-cocycle on (σ^i, σ^j) is 0 if $i + j < n$, and the canonical generator $1 + \sigma + \ldots + \sigma^{n-1}$ of $\mathbb{Z} = \mathbb{Z}[G]^G$ otherwise. Thus the two 2-cocycles are the same. □

7.1.3 Morphisms to Severi–Brauer varieties

Let k be a field. Let Y be a Severi–Brauer variety and let X be an arbitrary k-scheme. A morphism $f\colon X\to Y$ gives rise to a map of Γ-modules $f^*\colon \mathrm{Pic}(Y^{\mathrm{s}})\to\mathrm{Pic}(X^{\mathrm{s}})$ and a distinguished class $f^*(L_Y) \in \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma}$. Moreover, we have a map of Γ-modules

$$\mathrm{H}^0(Y^{\mathrm{s}}, L_Y) \longrightarrow \mathrm{H}^0(X^{\mathrm{s}}, f^*(L_Y)).$$

The image of this map is a Γ-invariant finite-dimensional k_s-vector subspace V of $H^0(X^s, f^*(L_Y))$. Since f is a morphism, the line bundle $f^*(L_Y)$ on X^s is generated by the vector subspace of sections $V \subset H^0(X^s, f^*(L_Y))$, so that the natural map $V \otimes_{k_s} \mathcal{O}_{X^s} \to f^*(L_Y)$ is surjective.

There is a converse to this observation.

Proposition 7.1.14 *Let k be a field. Let X be a k-scheme. Let $L \in \mathrm{Pic}(X^s)^\Gamma$ and let $V \subset H^0(X^s, L)$ be a finite-dimensional Γ-invariant k_s-vector subspace such that the map $V \otimes_{k_s} \mathcal{O}_{X^s} \to L$ is surjective. Let $n = \dim(V)$. Then there is an $(n-1)$-dimensional Severi–Brauer variety Y over k and a k-morphism $f\colon X \to Y$ such that $f^*(L_Y) = L \in \mathrm{Pic}(X^s)$ and the map $f^*\colon H^0(Y^s, L_Y) \to H^0(X^s, L)$ is injective with image V.*

Proof. Under the assumption that X is proper over k, and $V = H^0(X^s, L)$, the above proposition is established in [Lie17, Thm. 3.4]. The proof by descent extends to the above more general statement. □

Let X be a smooth, quasi-projective, geometrically integral variety over a field k such that $k_s[X^s]^* = k_s^*$. By Proposition 5.4.2, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^s)^\Gamma \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(X),$$

which is functorial contravariant with respect to morphisms of such varieties. Let ∂_X denote the map $\mathrm{Pic}(X^s)^\Gamma \longrightarrow \mathrm{Br}(k)$. If $X(k) \neq \emptyset$, then $\mathrm{Br}(k) \to \mathrm{Br}_1(X)$ has a retraction, hence $\partial_X = 0$. More generally, if X has index d, so has a zero-cycle of degree d, then $d\,\partial_X(L) = 0$ for all $L \in \mathrm{Pic}(X^s)^\Gamma$.

We want to understand restrictions on the order of $\partial_X(L)$ in the general case. By abuse of notation, we use the same notation for a line bundle L on X^s and its class in $\mathrm{Pic}(X^s)$.

If Y is a Severi–Brauer variety of dimension $n-1$, then the image in $\mathrm{Pic}(Y^s)$ of the canonical bundle $\omega_Y \in \mathrm{Pic}(Y)$ is the opposite of $L_Y^{\otimes n}$. This implies that

$$n\,\partial_Y(L_Y) = 0.$$

Part (i) of the following proposition is stated in various degrees of generality by S. Lichtenbaum [Lic68, Lic69]. Part (ii) was recently suggested by A. Kuznetsov.

Proposition 7.1.15 *Let X be a smooth, projective, geometrically integral variety over a field k and let $L \in \mathrm{Pic}(X^s)^\Gamma$.*

(i) *If there exists a Γ-invariant vector subspace $V \subset H^0(X^s, L)$ of dimension $n \geq 1$, then $n\,\partial_X(L) = 0$.*
(ii) *Let $\chi(L)$ be the coherent Euler–Poincaré characteristic of L on X^s. Then*

$$\chi(L)\,\partial_X(L) = 0.$$

Proof. Let us prove (i). Suppose first that $n = 1$. Then there exists a unique effective Cartier divisor D on X^{s} with $\mathcal{O}_{X^{\mathrm{s}}}(D) \simeq L$ which is the zero set of a generator of the one-dimensional vector space V. This divisor is Γ-invariant. Hence it comes from $\mathrm{Div}(X)$, hence L comes from $\mathrm{Pic}(X)$, hence $\partial_X(L) = 0$.

Suppose that $n \geq 1$. If there exists a non-zero effective divisor $D \in \mathrm{Div}(X^{\mathrm{s}})$ such that for every element Δ of the linear system $V \subset \mathrm{H}^0(X^{\mathrm{s}}, L)$ the difference $\Delta - D$ is an effective divisor, then there is a maximal such D; it is called the fixed component of the linear system. Since V is stable under the action of Γ, the fixed component D is defined over k. Let $M\colon = L \otimes \mathcal{O}_{X^{\mathrm{s}}}(-D) \in \mathrm{Pic}(X^{\mathrm{s}})$. We may identify V with a Γ-invariant vector subspace of $\mathrm{H}^0(X^{\mathrm{s}}, M)$. Since there is now no fixed component in this linear system, by Proposition 7.1.14 there exists a k-morphism $g\colon U \to Y$, where $U \subset X$ is an open set which contains all codimension 1 points of the smooth variety X, so that $k_{\mathrm{s}}[U^{\mathrm{s}}]^* = k_{\mathrm{s}}^*$, and Y is a Severi–Brauer variety of dimension $n-1$, equipped with its natural line bundle $L_Y \in \mathrm{Pic}(Y^{\mathrm{s}})$. We have $n\,\partial_Y(L_Y) = 0$.

The inverse image $g^*(L_Y) \in \mathrm{Pic}(U^{\mathrm{s}})$ coincides with the restriction of the line bundle $M^{\mathrm{s}} \in \mathrm{Pic}(U^{\mathrm{s}})$. By functoriality we get $n\,\partial_U(M) = 0$. Since U contains all the codimension 1 points of X, the restriction map $\mathrm{Pic}(X^{\mathrm{s}}) \to \mathrm{Pic}(U^{\mathrm{s}})$ is an isomorphism. By functoriality again we have $n\,\partial_X(M) = 0$. Now we have $\partial(\mathcal{O}_{X^{\mathrm{s}}}(D)) = 0$ since D is defined over k. Since ∂ is additive, and we have $L = M \otimes \mathcal{O}_{X^{\mathrm{s}}}(D)$, we conclude $n\,\partial_X(L) = 0$. This proves (i).

Let us prove (ii). Let $\mathcal{O}(1) \in \mathrm{Pic}(X)$ be a very ample sheaf. By the Hirzebruch–Grothendieck Riemann–Roch theorem [Har77, Appendix A, Thm. 4.1] for any line bundle $L \in \mathrm{Pic}(X^{\mathrm{s}})$ there exists a polynomial $P(t) \in \mathbb{Q}[t]$ such that $\chi(L(m)) = P(m)$ for any integer m. Let a be a positive integer such that $a\,P(t) \in \mathbb{Z}[t]$.

Let $L \in \mathrm{Pic}(X^{\mathrm{s}})^{\Gamma}$. By a result of Serre (see [Har77, III, Thm. 5.2]), there exists an integer $m_0 = m_0(L)$ such that for any integer $m \geq m_0$ the line bundle $L(m)$ is very ample and satisfies $\mathrm{H}^i(X^{\mathrm{s}}, L(m)) = 0$ for $i > 0$, hence $\chi(L(m)) = h^0(X^{\mathrm{s}}, L(m))$. From (i) we deduce that $\chi(L(m))\,\partial_X(L(m)) = 0$. Since ∂ is additive and $\mathcal{O}(1) \in \mathrm{Pic}(X)$, this gives $\chi(L(m))\,\partial_X(L) = 0$.

We have $\chi(L(m)) - \chi(L) = a^{-1}R(m)$, where $R(t) \in \mathbb{Z}[t]$ is a polynomial with zero constant term. Let r be an integer such that $r\,\partial_X(L) = 0$. Choose $m \geq m_0$ to be a multiple of ra. Then $\chi(L(m)) - \chi(L)$ is an integer divisible by r. Thus $(\chi(L(m)) - \chi(L))\,\partial_X(L) = 0$, which implies $\chi(L)\,\partial_X(L) = 0$. □

7.2 Projective quadrics

Let C be a smooth, projective, geometrically integral curve of genus 0 over a field k. Since C is smooth, it has a k_{s}-point and hence $C^{\mathrm{s}} \cong \mathbb{P}^1_{k_{\mathrm{s}}}$, cf. Remark 1.1.12 (3). The anticanonical line bundle of C is very ample of degree 2, so it gives an embedding of C into $\mathbb{P}^2_k$ as a smooth conic. From the isomorphism

$C^{\mathrm{s}} \cong \mathbb{P}^1_{k_{\mathrm{s}}}$ we also see that the degree map gives an isomorphism of $\mathrm{Pic}(C^{\mathrm{s}})$ with the trivial Γ-module $\mathbb{Z}$, hence $\mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})) = 0$. Since $\mathrm{Br}(C^{\mathrm{s}}) = 0$ by Theorem 5.6.1 (iv), the exact sequence (5.21) can be written as

$$0 \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(C^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(C) \longrightarrow 0. \qquad (7.10)$$

Proposition 7.2.1 *Let k be a field, $\mathrm{char}(k) \neq 2$. Let C be a smooth conic over k. Let Q be the quaternion algebra over k associated to C as in Definition 1.1.11. Then the image of a generator of $\mathrm{Pic}(C^{\mathrm{s}})^{\Gamma} \cong \mathbb{Z}$ in $\mathrm{Br}(k)$ is the class of Q, so that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ is surjective with the kernel generated by the class of Q.*

Proof. By Remark 1.1.12 (3) or by the Riemann–Roch theorem, a smooth conic C has a k-point if and only if $C \cong \mathbb{P}^1_k$. In this case the natural map $\mathrm{Pic}(C) \to \mathrm{Pic}(C^{\mathrm{s}})$ is visibly an isomorphism. The natural map $\mathrm{Br}(k) \to \mathrm{Br}(\mathbb{P}^1_k)$ is an isomorphism by Theorem 5.6.1 (vii). On the other hand, Q is split over k by Proposition 1.1.8, so the class of Q in $\mathrm{Br}(k)$ is zero.

If C has no k-point, then Q is a division algebra by Proposition 1.1.8, so the class $[Q] \in \mathrm{Br}(k)$ is non-zero. By Exercise 1.1.13 (4), the class $[Q]$ lies in the kernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(k(C))$. This map factors through the natural map $\mathrm{Br}(C) \to \mathrm{Br}(k(C))$, which is injective by Theorem 3.5.5. We conclude that $[Q]$ is a non-zero element in the kernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$. To finish the proof it remains to show that the cokernel of $\mathrm{Pic}(C) \to \mathrm{Pic}(C^{\mathrm{s}})$ is annihilated by 2. This follows from the fact that the degree map identifies $\mathrm{Pic}(C^{\mathrm{s}})$ with $\mathbb{Z}$ and the canonical class of C is an element of $\mathrm{Pic}(C)$ of degree -2. □

Remark 7.2.2 This proposition goes back to Witt [Wit35, Satz, S. 465]. Since the smooth projective conics are precisely the twisted forms of the projective line, the above proof is a variant of Proposition 7.1.3, up to identification of the class of the quaternion algebra Q. There is a version of this proposition over a field of characteristic 2, with appropriate descriptions of smooth conics and quaternion algebras (see [GS17, Ch. I, Exercises 3, 4]).

Remark 7.2.3 Since $\mathrm{Br}(C^{\mathrm{s}}) = 0$ and $\mathrm{H}^1(k, \mathrm{Pic}(C^{\mathrm{s}})) = 0$, the Leray spectral sequence (5.19) shows that the homomorphism $\mathrm{H}^3(k, k_{\mathrm{s}}^*) \to \mathrm{H}^3(C, \mathbb{G}_m)$ is injective.

Proposition 7.2.4 *Let k be a field, $\mathrm{char}(k) \neq 2$. Let $X \subset \mathbb{P}^n_k$, $n \geq 2$, be a smooth projective quadric. The following properties hold.*

(a) *The map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is surjective.*
(b) *For $n = 2$, the conic X can be given by an equation*

$$x^2 - ay^2 - bt^2 = 0,$$

where $a, b \in k^*$. The map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism if and only if $X(k) \neq \emptyset$. If $X(k) = \emptyset$, then $\mathrm{Ker}[\mathrm{Br}(k) \to \mathrm{Br}(X)] \cong \mathbb{Z}/2$ is generated by the class of the quaternion algebra (a, b).

(c) *For $n = 3$, the quadric X can be given by an equation*

$$x^2 - ay^2 - bz^2 + dabt^2 = 0,$$

where $a, b, d \in k^*$. The class of d in k^*/k^{*2} is uniquely determined by X.

(c′) *If d is not a square, then $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism.*

(c″) *Let C be the conic $x^2 - ay^2 - bt^2 = 0$. If d is a square, then X is isomorphic to $C \times_k C$ and is k-birationally equivalent to $\mathbb{P}^1_k \times C$. In this case, the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism if and only if $X(k) \neq \emptyset$. If $X(k) = \emptyset$, then*

$$\mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(X)] = \mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(C)] \cong \mathbb{Z}/2$$

is generated by the class of the quaternion algebra (a, b).

(d) *For $n \geq 4$, the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is an isomorphism.*

Proof. Statement (b) was proved in Proposition 7.2.1.

By an argument going back to the Greeks, a smooth quadric $X \subset \mathbb{P}^n_k$ of dimension at least 1 with a k-point is birationally equivalent to the projective space: the stereographic projection from a k-point of X to a projective space $\mathbb{P}^{n-1}_k \subset \mathbb{P}^n_k$ is a birational equivalence.

By Theorem 6.2.11 we have $\mathrm{Br}(X^{\mathrm{s}}) = 0$, hence $\mathrm{Br}_1(X) = \mathrm{Br}(X)$. Thus statement (a) will follow from Proposition 5.4.2 once we prove that $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) = 0$ for all $n \geq 2$.

Let us prove (d). For $n \geq 4$ an easy direct proof shows that the restriction map $\mathrm{Pic}(\mathbb{P}^n_{k_{\mathrm{s}}})\to\mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism. Indeed, the homogeneous equation of X^{s} can be written as $x_0x_1 + q(x_2, \ldots, x_n) = 0$, where q is a non-degenerate quadratic form in $n - 1 \geq 3$ variables. The hyperplane $x_0 = 0$ cuts out the integral divisor D given by $x_0 = q(x_2, \ldots, x_n) = 0$ in $\mathbb{P}^n_{k_{\mathrm{s}}}$. The complement $X^{\mathrm{s}} \smallsetminus D$ is isomorphic to the affine space $\mathbb{A}^{n-1}_{k_{\mathrm{s}}}$. From the exact sequence

$$0 = k_{\mathrm{s}}[\mathbb{A}^{n-1}_{k_{\mathrm{s}}}]^*/k_{\mathrm{s}}^* \longrightarrow \mathbb{Z}[D] \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow \mathrm{Pic}(\mathbb{A}^{n-1}_{k_{\mathrm{s}}}) = 0$$

we conclude that $\mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^n_{k_{\mathrm{s}}})\to\mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism. Now the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\mathbb{P}^n_{k_{\mathrm{s}}}) & \xrightarrow{\cong} & \mathrm{Pic}(X^{\mathrm{s}}) \\ \cong\uparrow & & \uparrow \\ \mathrm{Pic}(\mathbb{P}^n_k) & \longrightarrow & \mathrm{Pic}(X) \end{array}$$

implies that $\mathrm{Pic}(X)\to\mathrm{Pic}(X^{\mathrm{s}})$ is an isomorphism. In particular, in this case we have $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}})) = 0$. Now the statement of (d) follows from the exact sequence (5.21).

Let us prove (c). Quadric surfaces were already discussed by F. Châtelet in the 1940s. We refer to [CTS93, Thm. 2.5] for detailed proofs of some of the following assertions. It is clear that any smooth quadric $X \subset \mathbb{P}^3_k$ can be given by an equation as in (c). We have $X^{\rm s} \simeq \mathbb{P}^1_{k_{\rm s}} \times \mathbb{P}^1_{k_{\rm s}}$, hence $\mathrm{Pic}(X^{\rm s}) \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, where e_i is the inverse image of a $k_{\rm s}$-point under the projection to the i-th factor, for $i = 1, 2$. These are the two rulings of the quadric surface $X^{\rm s}$. This implies that the $\mathbb{Z}$-basis $\{e_1, e_2\}$ of $\mathrm{Pic}(X^{\rm s})$ is stable under the action of Γ. The class of the hyperplane section is $e_1 + e_2$, which is thus in the image of $\mathrm{Pic}(X)$. Over $k(\sqrt{a}, \sqrt{d})$, the two rulings can be given by factorising the equation

$$x^2 - ay^2 = b(z^2 - adt^2). \tag{7.11}$$

The action of Γ on $\{e_1, e_2\}$ is trivial if d is a square. If d is not a square, then the action of Γ factors through its image $\mathrm{Gal}(k(\sqrt{d})/k)$; the generator of this group permutes e_1 and e_2. Using Shapiro's lemma we see that in all cases we have $\mathrm{H}^1(k, \mathrm{Pic}(X^{\rm s})) = 0$. The basic exact sequence (5.21) then becomes

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X^{\rm s})^{\Gamma} \xrightarrow{\partial_X} \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow 0.$$

Since X has a rational point over $k(\sqrt{a})$, the image of ∂_X lies in the group $\mathrm{Ker}[\mathrm{Br}(k) \to \mathrm{Br}(k(\sqrt{a})]$, hence is annihilated by 2.

If d is not a square, then $\mathrm{Pic}(X^{\rm s})^{\Gamma}$ is generated by $e_1 + e_2$, hence the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\rm s})^{\Gamma}$ is surjective in this case and thus the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism.

Let C be the smooth conic $x^2 - ay^2 - bt^2 = 0$. Suppose that d is a square. Then (7.11) gives that b is the ratio of norms from the quadratic extensions given by adjoining $\sqrt{a}$. From here it is easy to deduce that X is k-birationally equivalent to $C \times_k \mathbb{P}^1_k$. This implies that the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X) \subset \mathrm{Br}(k(X))$ coincides with the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(k(C))$, which by (b) is generated by the class of the quaternion algebra (a, b). We have $e_1, e_2 \in \mathrm{Pic}(X^{\rm s})^{\Gamma}$ and $e_1 + e_2$ comes from $\mathrm{Pic}(X)$. Since $2\partial_X(e_1) = 2\partial_X(e_2) = 0$ and $\partial_X(e_1 + e_2) = 0$, we have $\partial_X(e_1) = \partial_X(e_2)$ and this class generates the kernel of $\mathrm{Br}(k) \to \mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. Hence $\partial_X(e_1) = \partial_X(e_2) = (a, b)$. Using Proposition 7.1.14, one sees that each $e_i \in \mathrm{Pic}(X^{\rm s})^{\Gamma}$ gives rise to a morphism $X \to C$. Going over to $k_{\rm s}$, one checks that the morphism $X \to C \times_k C$ defined by (e_1, e_2) is an isomorphism.

Finally, statement (a) is now established for all $n \geq 2$. □

7.3 Some affine hypersurfaces

Proposition 7.3.1 *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface and let $Z \subset X$ be a smooth hyperplane section. If $n \geq 4$, then the natural map* $\mathrm{Br}(k) \to \mathrm{Br}(X \setminus Z)$ *is an isomorphism.*

Proof. As usual we write $\overline{X} = X \times_k \bar{k}$ and $\overline{Z} = Z \times_k \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Since $n \geq 4$ the restriction map $\mathrm{Pic}(\mathbb{P}^n_{\bar{k}}) \to \mathrm{Pic}(\overline{X})$ is an isomorphism [SGA2, XII, Cor. 3.7], so $\mathrm{Pic}(\overline{X}) = \mathbb{Z}[\overline{Z}]$. Thus every divisor class on $\overline{X}$ is a multiple of $[\overline{Z}]$, which implies that $\overline{Z}$ is integral. Let $U = X \smallsetminus Z$. If a rational function $f \in \bar{k}(X)^*$ is regular and invertible on $\overline{U}$, then $\mathrm{div}(f)$ is a multiple of an ample divisor $\overline{Z}$, hence $\mathrm{div}(f) = 0$. This shows that $\bar{k}[U]^* = \bar{k}^*$.

The natural restriction map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{U})$ is surjective because $\overline{X}$ is smooth. The kernel of this map is the cyclic subgroup generated by $[\overline{Z}]$, hence the exact sequence

$$0 \longrightarrow \mathbb{Z}[\overline{Z}] \longrightarrow \mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{U}) \longrightarrow 0$$

shows that $\mathrm{Pic}(\overline{U}) = 0$.

Since Z is a smooth complete intersection of dimension at least 2 in $\mathbb{P}^n_{\bar{k}}$ [SGA2, X, Thm. 3.10] gives $\pi_1(\overline{Z}) = 0$ hence $\mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z}) = 0$. Since $n \geq 4$, we have $\mathrm{Br}(\overline{X}) = 0$ by Corollary 5.5.4. From the exact sequence (3.15)

$$0 \longrightarrow \mathrm{Br}(\overline{X}) \longrightarrow \mathrm{Br}(\overline{U}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\overline{Z}, \mathbb{Q}/\mathbb{Z})$$

we conclude that $\mathrm{Br}(\overline{U}) = 0$. Now the exact sequence (5.21) gives the required statement. □

The following proposition is taken from [CTX09, §5.8].

Proposition 7.3.2 *Let k be a field,* $\mathrm{char}(k) \neq 2$. *Let $f(x,y,z)$ be a non-degenerate quadratic form and let $a \in k^*$. Let X be the affine quadric defined by the equation $f(x,y,z) = a$. Let d be the determinant of the quadratic form $f(x,y,z) - at^2$. If $X(k) \neq \emptyset$ and $d \notin k^{*2}$, then* $\mathrm{Br}(X)/\mathrm{Br}(k) \cong \mathbb{Z}/2$.

In [CTX09, §5.8] one proves that $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathbb{Z}/2$ and one gives an explicit algorithm which starting with a k-point on X produces an element in $\mathrm{Br}(X)$ whose image generates $\mathrm{Br}(X)/\mathrm{Br}(k)$. There is a misprint in the formulae in *loc. cit.*, so we give a corrected description of the algorithm here. Let $K = k(\sqrt{d})$. The algorithm generates a function $\rho \in k(X)^*$ whose divisor $\mathrm{div}_X(\rho)$ on the affine quadric X is the norm of a divisor on $X \times_k K$, which implies that the class of the quaternion algebra $(\rho, d) \in \mathrm{Br}(k(X))$ belongs to $\mathrm{Br}(X)$ and generates $\mathrm{Br}(X)/\mathrm{Br}(k)$.

Let $Y \subset \mathbb{P}^3_k$ be the smooth projective quadric given by the homogeneous equation

$$f(x,y,z) = at^2.$$

Let $M \in Y(k)$. Let $l_1(x,y,z,t)$ be a linear form with coefficients in k defining the tangent plane to Y at M. Then we have

$$f(x,y,z) - at^2 = l_1 l_2 + c(l_3^2 - d l_4^2),$$

where l_2, l_3, l_4 are linearly independent linear forms with coefficients in k and c is a constant in k^*. Conversely, if we have such an identity, then $l_1 = 0$ is an equation for the tangent plane at the k-point $l_1 = l_3 = l_4 = 0$. Define $\rho = l_1(x,y,z,t)/t \in k(X)$ and let $\alpha = (\rho, d) \in \mathrm{Br}(k(X))$. We have

$$\alpha = (l_1(x,y,z,t)/t, d) = (-c\,l_2(x,y,z,t)/t, d) \in \mathrm{Br}(k(X)).$$

Thus α is unramified on X away from the plane at infinity $t = 0$, and the finitely many closed points given by $l_1 = l_2 = 0$. By the purity theorem for the Brauer group of smooth varieties (Theorem 3.7.2 (i)) we see that α belongs to $\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. The complement to X in Y is the smooth projective conic C given by $f(x,y,z) = 0$. An easy computation shows that the residue of α at the generic point of this conic is the class of d in

$$k^*/k^{*2} = \mathrm{H}^1(k, \mathbb{Z}/2) \subset \mathrm{H}^1(k(C), \mathbb{Z}/2) \subset \mathrm{H}^1(k(C), \mathbb{Q}/\mathbb{Z})$$

(note that k is algebraically closed in $k(C)$). Since d is not a square in k, this class is not trivial. Thus $\alpha \in \mathrm{Br}(X)$ is not contained in the image of $\mathrm{Br}(k)$, and hence it generates $\mathrm{Br}(X)/\mathrm{Br}(k)$. Note that at any k-point of X, either l_1 or l_2 is not zero. Thus, taking into account the isomorphism (1.4) from Section 1.3.4, the map $X(k) \to \mathrm{Br}(k)$ defined by evaluation of α can be computed by means of the map $X(k) \to k^*/\mathrm{N}_{K/k}(K^*)$ given by either the function $\rho = l_1(x,y,z,t)/t$ or the function $-c\,l_2(x,y,z,t)/t$.

The following result was obtained by T. Uematsu [Uem16] by an explicit cocycle computation.

Proposition 7.3.3 *Let $K = \mathbb{C}(a,b,c)$ for independent variables a, b, c. Let $X \subset \mathbb{A}^3_K$ be the affine quadric*

$$x^2 + ay^2 + bz^2 + c = 0.$$

Then $\mathrm{Br}(X)/\mathrm{Br}(K) = 0$.

Exercise 7.3.4 Prove Proposition 7.3.3 without cocycle computations. *Hint*: Go over to the quadratic extension $K(\sqrt{b})/K$ where the quadric acquires a rational point. Then use [CTX09, §5.8].

The following propositions extend some of the computations in [Gun13].

Proposition 7.3.5 *Let k be a field. Let $Q(x) \in k[x]$ be a separable polynomial with $Q(0) \neq 0$. Let $X \subset \mathbb{A}^3_k$ be the affine surface $yz = xQ(x)$. Let $Z \subset X$ be the closed subset defined by $y = Q(x) = 0$, and let $U = X \setminus Z$. Then we have the following statements.*

(i) $U \simeq \mathbb{A}^2_k$, *hence* $\mathrm{Pic}(U) = 0$.
(ii) $k[X]^* = k[U]^* = k^*$.
(iii) $\mathrm{Pic}(X)$ *is a finitely generated torsion-free abelian group.*
(iv) *Assume that* $\mathrm{char}(k) = 0$. *Then* $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}(U)$ *and* $\mathrm{Br}(k) \xrightarrow{\sim} \mathrm{Br}(X)$.

Proof. The function $f(x,y,z) = x/y = z/Q(x)$ is defined everywhere on U. We have a morphism $U \to \mathbb{A}^2_k$ given by $(x,y,z) \mapsto (f(x,y,z), y)$. The image of the morphism $\mathbb{A}^2_k \to \mathbb{A}^3_k$, $(u,v) \mapsto (uv, v, uQ(uv))$, is contained in U. The two morphisms $U \to \mathbb{A}^2_k$ and $\mathbb{A}^2_k \to U$ are inverse to each other. This proves (i), from which the other statements easily follow. □

Corollary 7.3.6 *Let k be a field of characteristic zero. Let $a \in k^*$ and let $P(x) \in k[x]$ be a separable polynomial. Let $X \subset \mathbb{A}^3_k$ be the affine surface with equation $y^2 - az^2 = P(x)$. Then the quotient* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is a finite group.*

Proof. In view of Proposition 5.4.2, this follows from Proposition 7.3.5 applied over k_{s}. □

Remark 7.3.7 Note that the finiteness of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ for X as above is a general algebraic result. By contrast, if $Y \subset \mathbb{A}^3_{\mathbb{Q}}$ is the geometrically rational smooth surface given by $x^3 + y^3 + z^3 = a$, where $a \in \mathbb{Q}^*$, then the finiteness of $\mathrm{Br}(Y)/\mathrm{Br}_0(Y)$ is proved in [CTW12] using arithmetic arguments. The point here is that the 'curve at infinity' in this case is a curve of genus one.

Let X be as in Corollary 7.3.6. Write $P(x) = \prod_{i=1}^n P_i(x)$ as a product of irreducible polynomials. One easily checks that the classes of quaternion algebras $(a, P_i(x)) \in \mathrm{Br}(k(X))$ are contained in $\mathrm{Br}(X)$ (compare with Example 6.3.1). For an arbitrary polynomial $P(x)$, constructing a set of elements in $\mathrm{Br}(X)$ that generate the quotient of $\mathrm{Br}(X)$ modulo the image of $\mathrm{Br}(k)$ may require some work, see [Berg]. Here is one easy case.

Proposition 7.3.8 *Let k be a field of characteristic zero. Let $P(x) \in k[x]$ be a separable irreducible polynomial of degree d such that $K = k[x]/P(x)$ is a cyclic extension of k. Let $X \subset \mathbb{A}^3_k$ be the affine surface with equation $yz = P(x)$. Then* $\mathrm{Br}(X)/\mathrm{Br}(k) \simeq \mathbb{Z}/d$. *The class of the cyclic algebra $A = (K/k, \sigma, y)$ in* $\mathrm{Br}(k(X))$ *lies in* $\mathrm{Br}(X)$ *and generates* $\mathrm{Br}(X)/\mathrm{Br}(k)$. □

In connection with applications to the integral Brauer–Manin obstruction, the Brauer groups of many quasi-projective varieties have been computed in recent years. See [CTX09], [CTW12], [CTHa12], [JS17], [BK19], [BL19], [Harp17], [Harp19a], [Mit18], [Berg], [LM18], [CTWX].

Chapter 8
Singular schemes and varieties

This chapter collects and in some cases rectifies a number of results in the literature on the Brauer groups of singular schemes.

The Brauer group of a field is a torsion group, but this is not always so for schemes. Let X be an integral variety over a field k of characteristic zero and let $k(X)$ be the function field of X. If X is geometrically locally factorial, for example smooth, Theorem 3.5.5 says that the restriction map $\mathrm{Br}(X)\to\mathrm{Br}(k(X))$ is injective, in particular $\mathrm{Br}(X)$ is a torsion group. If, moreover, X is smooth over k, then, by Theorem 3.7.3, there is an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(k(X)) \xrightarrow{\{\partial_x\}} \oplus_{x\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z}).$$

Thus there is a purity theorem for $\mathrm{Br}(k(X))$: unramified classes in $\mathrm{Br}(k(X))$ lie in the subgroup $\mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. It is natural to ask whether and to what extent the above results fail for a singular variety over k.

In Section 8.1 we give elementary examples of quasi-projective varieties X, either non-reduced or reducible, such that the Brauer group $\mathrm{Br}(X)$ is not a torsion group. In Section 8.2 we study integral normal schemes with isolated singular points. Here the reader will find examples of affine integral normal surfaces X over $\mathbb{C}$ with one singularity and of projective hypersurfaces of dimension 3 with just one node over an algebraically closed field of any characteristic other than 2 such that $\mathrm{Br}(X)$ is not a torsion group. Brauer groups of possibly singular complete intersections and of projective cones are the subjects of Sections 8.3 and 8.4, respectively.

For some singular varieties X, the exact computation of $\mathrm{Br}(X)$, for example by comparison with the Brauer group of a desingularisation, turns out to be of interest in connection with arithmetic investigations [HS14], [BL20]. Section 8.5 treats singular curves over a field; these computations give more precise results than those in Section 3.6.2. The last section contains some more examples of the behaviour of the Brauer groups of singular schemes.

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8.1 The Brauer–Grothendieck group is not always a torsion group

In this section we give elementary examples of quasi-projective varieties X for which $\mathrm{Br}(X)$ is not a torsion group.

A non-reduced variety

Let Y be a smooth projective variety over a field k. Let $X = Y \times_k k[\varepsilon]$ where $\varepsilon^2 = 0$. Let $i\colon Y = X_{\mathrm{red}} \to X$ be the closed immersion. Since the k-algebra homomorphism $k[\varepsilon]\to k$ has a section, there is a morphism $s\colon X\to Y$ such that $s \circ i = \mathrm{id}$.

We have a split exact sequence of sheaves for the étale topology on X

$$0 \longrightarrow i_*\mathcal{O}_Y \longrightarrow \mathbb{G}_{m,X} \longrightarrow i_*\mathbb{G}_{m,Y} \longrightarrow 0,$$

where the first map sends x to $1+\epsilon x$. A closed immersion is a finite morphism, hence the functor i_* is exact for the étale topology [Mil80, Cor. II.3.6]. We obtain a split exact sequence

$$0 \longrightarrow \mathrm{H}^2(Y, \mathcal{O}_Y) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(Y) \longrightarrow 0.$$

If $\mathrm{H}^2(Y, \mathcal{O}_Y) \neq 0$, then the kernel of the reduction map $\mathrm{Br}(X)\to\mathrm{Br}(X_{\mathrm{red}})$ is a non-zero finite-dimensional vector space over k.

If $\mathrm{H}^2(Y, \mathcal{O}_Y) \neq 0$ and $\mathrm{char}(k) = 0$, then the kernel of the reduction map $\mathrm{Br}(X)\to\mathrm{Br}(X_{\mathrm{red}})$ is a positive-dimensional vector space over a field of characteristic zero, in particular $\mathrm{Br}(X)$ is not a torsion group. From the above exact sequence we also deduce $\mathrm{Br}(X)_{\mathrm{tors}} \cong \mathrm{Br}(X_{\mathrm{red}})_{\mathrm{tors}}$. By Theorem 4.2.1, this translates as an isomorphism $\mathrm{Br}(X)_{\mathrm{Az}} \cong \mathrm{Br}(X_{\mathrm{red}})_{\mathrm{Az}}$.

In characteristic $p > 0$, the kernel of $\mathrm{Br}(X)\to\mathrm{Br}(X_{\mathrm{red}})$ is a p-torsion group.

Remark 8.1.1 Let Y be a variety over a field k and let A is a local artinian k-algebra. The study of the kernel of $\mathrm{Br}(Y \times_k A)\to\mathrm{Br}(Y)$ led Artin and Mazur to define the formal Brauer group of Y, see [AM77, Ch. II, §4]. The group $\mathrm{H}^2(Y, \mathcal{O}_Y)$ is the tangent space to the formal Brauer group of Y (when it exists). This group is of importance in studying varieties over fields of positive characteristic. It is of particular interest in the case of K3 surfaces (e.g. smooth quartics in $\mathbb{P}^3_k$) over a finite field.

A reduced, reducible variety

Here is another type of example of non-torsion elements in the Brauer group, which works over fields of arbitrary characteristic.

Lemma 8.1.2 *Let k be a field. Let U be a non-empty open subset of a smooth projective curve C of genus at least 1 over k. For any integer r there exists a field K finitely generated over k such that the dimension of the $\mathbb{Q}$-vector space $\mathrm{Pic}(U_K)\otimes_{\mathbb{Z}}\mathbb{Q}$ is at least r. There exists a field extension L/k such that $\mathrm{Pic}(U_L)\otimes_{\mathbb{Z}}\mathbb{Q}$ is an infinite-dimensional $\mathbb{Q}$-vector space.*

Proof. One may assume that $C(k)\neq\emptyset$. It is enough to prove that if A is an abelian variety over k, then $\dim_{\mathbb{Q}}(A(K)\otimes_{\mathbb{Z}}\mathbb{Q})$, where K is finitely generated over k, can be made arbitrarily large, while $\dim_{\mathbb{Q}}(A(L)\otimes_{\mathbb{Z}}\mathbb{Q})$ can be made infinite for even larger field extension L/k. Indeed, the generic point of A is a point of $A(k(A))$ no multiple of which belongs to $A(k)$. Now extend the ground field from k to $k(A)$ and iterate the process. □

Let k be a prime field. Let $S\subset\mathbb{P}^3_k$ be a smooth cubic surface. Up to replacing k by a finite extension, we can find a plane $H\subset\mathbb{P}^3_k$ which intersects S transversally along a smooth cubic E with a rational point. Write $Y=S\cup H\subset\mathbb{P}^3_k$ and $X=S\sqcup H$. Let $p\colon X\to Y$ be the natural morphism and let $i\colon E\hookrightarrow Y$ be the natural inclusion. Both these morphisms are finite, thus i_* and π_* are exact functors for the étale topology [Mil80, Cor. II.3.6]. Hence $R^jp_*=0$ and $R^ji_*=0$ for any $j>0$. We have an exact sequence of sheaves for the étale topology on Y

$$1\longrightarrow\mathbb{G}_{m,Y}\longrightarrow p_*\mathbb{G}_{m,X}\longrightarrow i_*\mathbb{G}_{m,E}\longrightarrow 1.$$

The associated long exact cohomology sequence gives an exact sequence

$$\mathrm{Pic}(S)\oplus\mathrm{Pic}(H)\longrightarrow\mathrm{Pic}(E)\longrightarrow\mathrm{Br}(Y).$$

Now $\mathrm{Pic}(H)\cong\mathrm{Pic}(\mathbb{P}^2_k)\simeq\mathbb{Z}$ and we have $\mathrm{Pic}(S)\subset\mathrm{Pic}(X^{\mathrm{s}})$. As is well known [Man74, Thm. 24.4], any smooth cubic surface over k_{s} is the blow-up of $\mathbb{P}^2_{k_{\mathrm{s}}}$ in 6 points, hence $\mathrm{Pic}(S^{\mathrm{s}})\cong\mathbb{Z}^7$. Thus $\mathrm{Pic}(S)\oplus\mathrm{Pic}(H)$ is a finitely generated free abelian group of rank at most 8. The group $\mathrm{Pic}(E)$ contains $E(k)$ as a subgroup. The same statements hold after replacing k by any field extension K. Using Lemma 8.1.2 one finds a field K finitely generated over its prime subfield k such that $\mathrm{Br}(Y_K)$ contains non-torsion elements and $\dim_{\mathbb{Q}}(\mathrm{Br}(Y_K)\otimes_{\mathbb{Z}}\mathbb{Q})$ is arbitrarily large. One can also find a field extension L/k such that $\dim_{\mathbb{Q}}(\mathrm{Br}(Y_L)\otimes_{\mathbb{Z}}\mathbb{Q})=\infty$.

One may replace H and S by any two smooth surfaces in $\mathbb{P}^3$ transversally intersecting in a smooth curve of genus at least 1. The same argument also works for the Zariski topology, thus giving examples with non-torsion groups $\mathrm{H}^2_{\mathrm{zar}}(X_L,\mathbb{G}_m)$.

Replacing $S,H,E\subset\mathbb{P}^3_k$ by their respective intersections with any Zariski open set $W\subset\mathbb{P}^3_k$ such that $W\cap E\neq\emptyset$ produces examples where Y is affine and $\mathrm{Br}(Y_K)$ is non-torsion of rank as big as one wishes.

The above example implies the existence of an affine variety X over a finite field such that $\mathrm{Br}(X)$ is not a torsion group. Indeed, let us start with a field L of positive characteristic p and an affine variety Y over L with a non-torsion element $\beta \in \mathrm{Br}(Y)$. The field L is a filtered union of $\mathbb{F}_p$-algebras of finite type A_i, $i \in I$. There exists an $i \in I$ such that Y comes from an affine A_i-scheme of finite type Y_i, and β is the image of some $\beta_i \in \mathrm{Br}(Y_i)$. The element β_i of the Brauer group of the affine $\mathbb{F}_p$-variety Y_i has infinite order.

8.2 Isolated singularities

In this section we give examples of normal varieties for which the Brauer group is not torsion.

This section elaborates on [Gro68, Ch. II, §1, Rem. 11 (b)] and on further literature [Dan68, Dan72, Oja74], [Chi76, Thm. 1.1], [DF92, Ber05, Kol16]. It ends with an example due to B. Totaro.

Let X be a normal, integral, noetherian scheme with function field K. Assume that the singular locus $X_{\rm sing}$ is the union of finitely many closed points $P_1, \ldots, P_n$. Let k_i denote the residue field at P_i, let $k_{i,{\rm s}}$ be a separable closure of k_i and let $G_i = \mathrm{Gal}(k_{i,{\rm s}}/k_i)$, for $i = 1, \ldots, n$. We write R_i for the local ring $\mathcal{O}_{X,P_i}$ and $R_i^{\rm sh}$ for the strict henselisation of R_i. Let $\mathrm{Cl}(X)$ be the *class group* of X, defined as the cokernel of the divisor map

$$\mathrm{div}\colon K^* \longrightarrow \bigoplus_{x\in X^{(1)}} \mathbb{Z}.$$

We define the étale sheaf $\mathcal{D}iv_X$ by the condition that the following sequence is exact:

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow j_*\mathbb{G}_{m,K} \longrightarrow \mathcal{D}iv_X \longrightarrow 0. \tag{8.1}$$

Taking étale cohomology of (8.1) and using Lemma 2.4.1, we get an isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathcal{D}iv_X) \xrightarrow{\sim} \mathrm{Ker}[\mathrm{Br}(X) {\to} \mathrm{Br}(K)]. \tag{8.2}$$

Sending a Cartier divisor to the associated Weil divisor defines a natural injective map $\mathcal{D}iv_X {\to} \oplus_{x\in X^{(1)}} i_{x*}(\mathbb{Z}_{k(x)})$. This is an isomorphism when X is regular or, more generally, when X is geometrically locally factorial. Let $\mathcal{P}_X$ be the cokernel of this map. This gives an exact sequence

$$0 \longrightarrow \mathcal{D}iv_X \longrightarrow \bigoplus_{x\in X^{(1)}} i_{x*}(\mathbb{Z}_{k(x)}) \longrightarrow \mathcal{P}_X \longrightarrow 0. \tag{8.3}$$

It is clear that $\mathcal{P}_X$ is supported on $X_{\rm sing}$. Looking at the stalks of the terms of (8.1) and (8.3) at the points P_i we see that

$$\mathcal{P}_X = \bigoplus_{i=1}^{n} i_{P_i*}(\mathrm{Cl}(R_i^{\rm sh})),$$

where $i_{P_i}\colon \operatorname{Spec}(k_i)\to X$ is the natural map $P_i\to X$. Taking étale cohomology of (8.3) and using Lemma 2.4.1 together with (8.2), we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathcal{D}iv_X) \longrightarrow \bigoplus_{x\in X^{(1)}} \mathbb{Z} \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K).$$

Using the definition of $\mathrm{Cl}(X)$, we deduce the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K). \quad (8.4)$$

If X is the spectrum of a semilocal ring, one obtains the exact sequence

$$0 \longrightarrow \mathrm{Cl}(X) \longrightarrow \bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K). \quad (8.5)$$

If $X = \operatorname{Spec}(R)$ is the spectrum of a local ring R with field of fractions K and residue field k, and $G = \mathrm{Gal}(k_{\mathrm{s}}/k)$, then the exact sequence takes the form

$$0 \longrightarrow \mathrm{Cl}(R) \longrightarrow \mathrm{Cl}(R^{\mathrm{sh}})^{G} \longrightarrow \mathrm{Br}(R) \longrightarrow \mathrm{Br}(K).$$

Let $U\colon = X \setminus \{x\}$ and let $U^{\mathrm{sh}}\colon = \operatorname{Spec}(R^{\mathrm{sh}}) \setminus \{\overline{x}\}$. Since U is regular, we have $\mathrm{Pic}(U) \cong \mathrm{Cl}(U) \cong \mathrm{Cl}(R)$ and $\mathrm{Pic}(U^{\mathrm{sh}}) \cong \mathrm{Cl}(U^{\mathrm{sh}}) \cong \mathrm{Cl}(R^{\mathrm{sh}})$. The last displayed sequence then becomes the formula (7) in [Gro68, Ch. II, §1].

Remark 8.2.1 (Cf. [Ber05]) In [Gro68, Ch. II, §1, (7)] it is claimed that for any normal scheme X with isolated singular points $P_1, \ldots, P_n$ there is the following general formula:

$$\mathrm{H}^1_{\text{ét}}(X, \mathcal{D}iv_X) \cong \bigoplus_{i=1}^{n} [\mathrm{Pic}(\operatorname{Spec}(R_i^{\mathrm{sh}}) \setminus \overline{P_i})^{G_i} / \mathrm{Im}\big(\mathrm{Pic}(\operatorname{Spec}(R_i) \setminus P_i)\big)]. \quad (8.6)$$

(Here $\mathrm{Pic}(\operatorname{Spec}(R_i) \setminus P_i) \cong \mathrm{Cl}(R_i)$ and $\mathrm{Pic}(\operatorname{Spec}(R_i^{\mathrm{sh}}) \setminus \overline{P_i}) \cong \mathrm{Cl}(R_i^{\mathrm{sh}})$.) This is not correct. In view of (8.2) this would imply

$$\bigoplus_{i=1}^{n} [\mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} / \mathrm{Im}(\mathrm{Cl}(R_i))] \xrightarrow{\sim} \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(K)].$$

There is a natural surjective map

$$[\bigoplus_{i=1}^{n} \mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i}] / \mathrm{Im}(\mathrm{Cl}(X)) \longrightarrow \bigoplus_{i=1}^{n} [\mathrm{Cl}(R_i^{\mathrm{sh}})^{G_i} / \mathrm{Im}(\mathrm{Cl}(R_i))].$$

Formula (8.6) holds if and only if the map $\mathrm{Cl}(X)\to\oplus_{i=1}^{n}\mathrm{Cl}(R_i)$ is surjective. Ojanguren's Example (4) in Section 8.6 below is precisely built on an example where this map is not surjective [Oja74, §2, p. 511].

Example 8.2.2 Let R be the local ring of the vertex of the cone over a smooth projective plane curve $X\subset\mathbb{P}^2_{\mathbb{C}}$ of degree d. This is a 2-dimensional local normal domain. As explained in Childs [Chi76, Thm. 6.1], work of Danilov [Dan68, Dan72] gives an isomorphism $\mathrm{Cl}(R^{\mathrm{h}})/\mathrm{Cl}(R)\cong\mathrm{Cl}(\widehat{R})/\mathrm{Cl}(R)$. Moreover, this quotient is the finite-dimensional complex vector space $\oplus_{i\geq 1}\mathrm{H}^1(X,\mathcal{O}_X(i))$, which has positive dimension if $d\geq 4$. Hence for these values of d the kernel of the map $\mathrm{Br}(R)\to\mathrm{Br}(K)$ is a non-zero vector space over $\mathbb{C}$. In particular, there are non-torsion elements in this kernel. Note that this implies that the kernel of $\mathrm{Br}_{\mathrm{Az}}(R)\to\mathrm{Br}(K)$ is zero, because $\mathrm{Br}_{\mathrm{Az}}(R)$ is a torsion group (Theorem 3.3.1).

Let $\mathrm{Spec}(R_i)$ be affine Zariski open neighbourhoods of the vertex of the cone. We have $R=\varinjlim R_i$, in fact, R is the union of the rings R_i. By Section 2.2.2 we have $\mathrm{Br}(R)=\varinjlim\mathrm{Br}(R_i)$. Let $\alpha\in\mathrm{Br}(R)$ be a non-torsion element in the kernel of the map $\mathrm{Br}(R)\to\mathrm{Br}(K)$. There exist an i and an $\alpha_i\in\mathrm{Br}(R_i)$ such that the image of α_i in R is α. Thus α_i is a non-torsion element in the kernel of $\mathrm{Br}(R_i)\to\mathrm{Br}(K)$. Now $Y:=\mathrm{Spec}(R_i)$ is an affine, normal, integral surface over $\mathbb{C}$ such that $\mathrm{Br}(Y)$ is not a torsion group.

The following proposition and its proof were communicated to us by B. Totaro.

Proposition 8.2.3 *Let k be an algebraically closed field of characteristic different from 2. Let $X\subset\mathbb{P}^4_k$ be a hypersurface of degree $d\geq 3$. Assume that X is smooth outside one k-point, and that this point is a node. Then* $\mathrm{Br}(X)$ *is not a torsion group.*

Proof. Let P be the node of X. Let Y be the blow-up of X at P. Then Y is a smooth hypersurface in the blow-up W of $\mathbb{P}^4$ at P. The Picard group $\mathrm{Pic}(W)$ is the free abelian group generated by $H:=f^*\mathcal{O}(1)$ and E, where $f\colon W\to\mathbb{P}^4$ is the blow-up morphism and $E=f^{-1}(P)\subset W$ is the exceptional divisor. The hypersurface X has multiplicity 2 at the point P, hence the class of Y in $\mathrm{Pic}(W)$ is $dH-2E$.

The divisors H and $H-E$ are nef on W, because they are pullbacks of ample divisors by the contraction $f\colon W\to\mathbb{P}^4$ (for H) and the $\mathbb{P}^1$-bundle $W\to\mathbb{P}^3$ (for $H-E$). Therefore, any linear combination $aH+b(H-E)$ with $a,b>0$ is in the interior of the nef cone in $\mathrm{NS}(W)\otimes_{\mathbb{Z}}\mathbb{R}\cong\mathbb{R}^2$, hence is ample by Kleiman's criterion [Kle66, §2, Thm. 1]. Since $d\geq 3$, it follows that $[Y]=dH-2E$ is ample on W.

We want to apply the Grothendieck–Lefschetz theorem [SGA2, Cor. XII.3.6] to the smooth hypersurface $Y\subset W$ to deduce that the restriction map

$$\mathrm{Pic}(W)\longrightarrow\mathrm{Pic}(Y)$$

is an isomorphism. Besides ampleness of $\mathcal{O}_W(Y)$, the hypothesis of that theorem requires $\mathrm{H}^i(Y, \mathcal{O}_Y(-mY)) = 0$ for $i = 1, 2$ and all integers $m > 0$. By the long exact sequence of cohomology deduced from the exact sequence

$$0 \longrightarrow \mathcal{O}_W(-Y) \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

given by the divisor $Y \subset W$, and the twists of that sequence by the invertible bundles $\mathcal{O}_W(nY)$, it is enough to show that $\mathrm{H}^i(W, \mathcal{O}_W(-mY)) = 0$ for $1 \leq i \leq 3$ and all integers $m > 0$. Since each $\mathcal{O}_W(mY)$ for $m > 0$ is an ample sheaf on the 4-dimensional smooth projective variety Y, this follows by the Kodaira vanishing theorem combined with Serre's duality theorem. Indeed, W is the blow-up of $\mathbb{P}^4$ in a point, hence is a toric variety, and the Kodaira vanishing theorem holds for smooth projective toric varieties in any characteristic [Fuj07, Cor. 1.5].

Thus $\mathrm{Pic}(Y)$ is generated by $f^*\mathcal{O}(1)$ and the exceptional divisor $E_Y \subset Y$. Since $\mathrm{codim}_X(P) = 3$, by [Har77, Prop. 6.5 (b)] the class group $\mathrm{Cl}(X)$ is isomorphic to

$$\mathrm{Cl}(X \smallsetminus \{P\}) \cong \mathrm{Pic}(X \smallsetminus \{P\}) \cong \mathrm{Pic}(Y \smallsetminus E_Y) = \mathbb{Z}\mathcal{O}(1).$$

In particular X is factorial. Let R be the local ring of X at the node P. From the exact sequence (8.4)

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(R^{\mathrm{h}}) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(k(X))$$

we see that $\mathrm{Cl}(R^{\mathrm{h}})$ is the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$. Let A be the local ring of the quadratic cone $xy = zw$ in $\mathbb{A}^4_k$ at the vertex $(0,0,0,0)$. The class group $\mathrm{Cl}(A^{\mathrm{h}})$ is isomorphic to $\mathbb{Z}$ generated by the Weil divisor $x = z = 0$. The rings R and A are isomorphic étale-locally, hence there is an isomorphism of henselisations $R^{\mathrm{h}} \cong A^{\mathrm{h}}$. It follows that $\mathrm{Cl}(R^{\mathrm{h}}) \cong \mathbb{Z}$ is the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$, so that $\mathrm{Br}(X) \otimes \mathbb{Q} \simeq \mathbb{Q}$. □

Remark 8.2.4 Let $\bar{k}$ be an algebraically closed field of characteristic $p > 0$ which is not algebraic over a finite field. An example of a normal projective surface X over $\bar{k}$ such that $\mathrm{Br}(X)$ is not a torsion group was recently given in [Ess21, Thm. 2.3].

8.3 Intersections of hypersurfaces

Proposition 8.3.1 *Let k be an algebraically closed field of characteristic exponent p. Let $X \subset \mathbb{P}^N_k$ be a closed subscheme.*

(i) *If X is defined by the vanishing of at most $N - 3$ homogeneous forms, then* $\mathrm{Br}(X)$ *has no prime-to-p torsion. For example, this holds for any hypersurface $X \subset \mathbb{P}^N_k$ with $N \geq 4$.*

(ii) *If X is defined by the vanishing of at most $N-4$ homogeneous forms, then* $\mathrm{Br}(X)$ *is uniquely ℓ-divisible for any prime $\ell \neq p$. For example, this holds for any hypersurface $X \subset \mathbb{P}^N_k$ with $N \geq 5$.*

Proof. Let $\ell \neq p$ be a prime. By [Kat04, Cor. B.6] the restriction map

$$\mathrm{H}^2_{\text{ét}}(\mathbb{P}^N_k, \mathbb{Z}/\ell) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}/\ell)$$

is an isomorphism under hypothesis (i), and

$$\mathrm{H}^3_{\text{ét}}(\mathbb{P}^N_k, \mathbb{Z}/\ell) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/\ell)$$

is an isomorphism under hypothesis (ii). The Kummer sequence (3.2) then gives that the map $\mathrm{Br}(\mathbb{P}^N_k)[\ell] \to \mathrm{Br}(X)[\ell]$ is surjective. Since $\mathrm{Br}(\mathbb{P}^N_k) = 0$ by Theorem 6.1.3, this finishes the proof in case (i). In case (ii), from $\mathrm{H}^3_{\text{ét}}(\mathbb{P}^N_k, \mathbb{Z}/\ell) = 0$ we deduce $\mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/\ell) = 0$, and the Kummer sequence (3.2) gives $\mathrm{Br}(X)/\ell \hookrightarrow \mathrm{H}^3_{\text{ét}}(X, \mathbb{Z}/\ell) = 0$. □

Purity on some singular varieties

Corollary 5.5.4 can be extended to some singular complete intersections. K. Česnavičius showed us that the following theorem is essentially a consequence of results of Michèle Raynaud [MR62], a text which contains many more purity theorems in a possibly singular context. Some of the purity results for singular schemes have recently been extended by Česnavičius and Scholze [ČS19, Thm. 7.2.8].

Theorem 8.3.2 *Let k be a field of characteristic zero. Let $X \subset \mathbb{P}^N_k$ be a complete intersection of dimension $d \geq 3$. Assume that the codimension of the singular locus X_{sing} in X is at least 4. Let $U = X \setminus X_{\mathrm{sing}}$. Then the natural maps* $\mathrm{Br}(k) \to \mathrm{Br}(X) \to \mathrm{Br}(U)$ *are isomorphisms, and the natural map* $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ *is an isomorphism.*

Proof. The assumptions on X and on the codimension of X_{sing} imply [SGA2, XI, Cor. 3.14] that X is geometrically locally factorial. Theorem 3.5.5 then gives that the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. In particular $\mathrm{Br}(X)$ is a torsion group.

The restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(U)$ is surjective since X is locally factorial and is injective since the codimension of X_{sing} in X is at least 2, so it is an isomorphism.

Let us first assume that k is algebraically closed. We want to prove that

$$\mathrm{Br}(X) \cong \mathrm{Br}(U) = 0.$$

Quite generally, for any complete intersection $X \subset \mathbb{P}^N_k$ of dimension d over an algebraically closed field k, any $i < d$, and any integer $n > 0$ invertible in k, the restriction map $\mathrm{H}^i_{\text{ét}}(\mathbb{P}^N_k,\mu_n)\to\mathrm{H}^i_{\text{ét}}(X,\mu_n)$ is an isomorphism, see [Kat04]. In particular, $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^N_k,\mu_n)\cong\mathrm{H}^2_{\text{ét}}(X,\mu_n)$. Now, from the Kummer sequence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \mathrm{Pic}(\mathbb{P}^N_k)/n & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(\mathbb{P}^N_k,\mu_n) & & & & \\
 & & \downarrow\cong & & \downarrow\cong & & & & \\
0 & \longrightarrow & \mathrm{Pic}(X)/n & \xrightarrow{\cong} & \mathrm{H}^2_{\text{ét}}(X,\mu_n) & \longrightarrow & \mathrm{Br}(X)[n] & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(U)/n & \longrightarrow & \mathrm{H}^2_{\text{ét}}(U,\mu_n) & \longrightarrow & \mathrm{Br}(U)[n] & \longrightarrow & 0
\end{array}$$

Since $\mathrm{char}(k)=0$, this already gives $\mathrm{Br}(X)[n]=0$ for any integer n, hence $\mathrm{Br}(X)=0$. To prove that $\mathrm{Br}(U)=0$ it is then enough to show that for any prime ℓ the restriction map $\mathrm{H}^2_{\text{ét}}(X,\mu_\ell)\to\mathrm{H}^2_{\text{ét}}(U,\mu_\ell)$ is an isomorphism.

Let us describe the relevant results from [MR62]. Let X be a noetherian scheme, let $Y\subset X$ be a closed subscheme, and let $U=X\smallsetminus Y$. Assume that ℓ is invertible on X. The *étale depth* $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$ of X along Y is defined in [MR62, Déf. 1.2], which refers to [MR62, Prop. 1.1 (iii)]. If $n=\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$, then for any X' étale over X, the restriction map

$$\mathrm{H}^i_{\text{ét}}(X',\mathbb{Z}/\ell)\longrightarrow\mathrm{H}^i_{\text{ét}}(X'\times_X U,\mathbb{Z}/\ell)$$

is an isomorphism for $i<n-1$ and is injective for $i=n-1$.

One defines a similar notion locally at any point x of X, as follows. Let $\overline{X}_{\bar{x}}=\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{\bar{x}})$ be the strict henselisation of X at a geometric point $\bar{x}$ above x. Define $\mathrm{depth}_x(X)(\mathbb{Z}/\ell)=\mathrm{depth}_{\bar{x}}(\overline{X}_{\bar{x}})(\mathbb{Z}/\ell)$, which is the étale depth of the local scheme $\overline{X}_{\bar{x}}$ at its closed point $\bar{x}$. By [MR62, Thm. 1.8], $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)$ can be computed locally:

$$\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)=\inf_{y\in Y}\ \mathrm{depth}_y(X)(\mathbb{Z}/\ell),$$

where y ranges through the points of the scheme Y.

The *geometric depth* of an excellent local ring A is defined in [MR62, Déf. 5.3]. If A is a complete intersection, then the geometric depth of A coincides with the dimension of A [MR62, Prop. 5.4]. For an excellent local ring A of characteristic zero, the étale depth is greater than or equal to the geometric depth [MR62, Thm. 5.6].

We now resume the proof of the theorem. Let X be as in the statement of the theorem. Write $Y=X_{\mathrm{sing}}$ so that $U=X\smallsetminus Y$. Since X is a complete intersection, so is $\overline{X}_{\bar{y}}$, where y is a point of Y and $\bar{y}$ is a geometric point over y. Since $\mathrm{codim}_X(Y)\geq 4$, we have $\dim(\overline{X}_{\bar{y}})\geq 4$. We conclude that the étale depth at the local ring of X at y is at least 4. Thus $\mathrm{depth}_Y(X)(\mathbb{Z}/\ell)\geq 4$ for any prime number ℓ, hence the restriction map $\mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}/\ell)\to\mathrm{H}^2_{\text{ét}}(U,\mathbb{Z}/\ell)$ is an isomorphism. Thus $\mathrm{H}^2_{\text{ét}}(X,\mu_\ell)\cong\mathrm{H}^2_{\text{ét}}(U,\mu_\ell)$ for any prime number ℓ.

Finally, let k be an arbitrary field of characteristic zero. Let $\bar{k}$ be an algebraic closure of k. Since X is geometrically locally factorial, $\overline{X}$ is locally factorial, hence the restriction maps $\bar{k}[X]^*\to\bar{k}[U]^*$ and $\mathrm{Pic}(\overline{X})\to\mathrm{Pic}(\overline{U})$ are isomorphisms. As a complete intersection of dimension at least 1 in projective space, $\overline{X}$ satisfies $\bar{k}=\mathrm{H}^0(\overline{X},\mathcal{O}_{\overline{X}})$ hence $\bar{k}^*=\bar{k}[X]^*$. For a complete intersection X of dimension at least 3 in $\mathbb{P}^N_{\bar{k}}$, the restriction map $\mathbb{Z}=\mathrm{Pic}(\mathbb{P}^N_{\bar{k}})\to\mathrm{Pic}(\overline{X})$ is an isomorphism [SGA2, XII, Cor. 3.7], and both groups are generated by the hyperplane section class, which is defined over k. We already know that $\mathrm{Br}(\overline{X})\cong\mathrm{Br}(\overline{U})=0$. From the exact sequence (5.21) we then get isomorphisms $\mathrm{Br}(k)\cong\mathrm{Br}(X)\cong\mathrm{Br}(U)$.

By purity for the smooth k-variety U (Theorem 3.7.1(i)), the subgroup $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)\subset\mathrm{Br}(k(X))=\mathrm{Br}(k(U))$ is contained in $\mathrm{Br}(U)\subset\mathrm{Br}(k(U))$. Thus $\mathrm{Br}(k)\to\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ is an isomorphism. □

Remark 8.3.3 Proposition 8.2.3 implies that the codimension condition on the singular locus in Theorem 8.3.2 is best possible. See also the examples at the end of Section 8.4.

8.4 Projective cones

Proposition 8.4.1 *Let k be a field of characteristic zero. Let $Y\subset\mathbb{P}^n_k$, $n\geq 2$, be an integral closed subvariety. Let $X\subset\mathbb{P}^{n+1}_k$ be the projective cone over Y. Write $U=X\setminus X_{\mathrm{sing}}$.*

(i) *The restriction map* $\mathrm{Br}(X)\to\mathrm{Br}(U)$ *is the composition of the map* $\mathrm{Br}(X)\to\mathrm{Br}(k)$ *given by evaluation at P and the map* $\mathrm{Br}(k)\to\mathrm{Br}(U)$ *induced by the structure morphism* $U\to\mathrm{Spec}(k)$.
(ii) *If Y is smooth, then U is the complement to the vertex of the cone X and* $\mathrm{Br}(U)\cong\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.

Proof. Let $\alpha\in\mathrm{Br}(X)$. Let $K=k(X)$ be the function field of X. The K-variety $X_K=X\times_k K$ has two obvious K-points: the point P_K given by the vertex $P\in X(k)$ and the point given by the generic point $\eta\in X$. Any point $M\in X_K(K)$ distinct from P_K lies on the projective line $\mathbb{P}^1_K\subset X_K$ through M and P_K. Since $\mathrm{Br}(K)\to\mathrm{Br}(\mathbb{P}^1_K)$ is an isomorphism (Theorem 5.6.1 (vii)) we have

$$\alpha(\eta)=\alpha(P_K)=\mathrm{res}_{K/k}(\alpha(P))\in\mathrm{Br}(K).$$

But $\alpha(\eta)$ is just the image of α under the restriction map $\mathrm{Br}(X)\to\mathrm{Br}(k(X))$. The latter map is the composition $\mathrm{Br}(X)\to\mathrm{Br}(U)\to\mathrm{Br}(k(X))$, where the map $\mathrm{Br}(U)\to\mathrm{Br}(k(X))$ is injective since U is smooth over k (Theorem 3.5.5). Hence $\mathrm{Br}(X)\to\mathrm{Br}(U)$ factors as $\mathrm{Br}(X)\longrightarrow\mathrm{Br}(k)\longrightarrow\mathrm{Br}(U)$, where the first arrow is evaluation at P and the second arrow is induced by the structure map $U \to \mathrm{Spec}(k)$.

Assume that Y is smooth. Then $U = X \smallsetminus \{P\}$ is also a smooth integral variety. The projection map $p\colon U \to Y$ is a Zariski locally trivial $\mathbb{A}^1$-bundle, hence the generic fibre U_η is isomorphic to $\mathbb{A}^1_{k(U)}$. Since $\mathrm{char}(k) = 0$, the map $\mathrm{Br}(k(Y)) \to \mathrm{Br}(U_\eta)$ is an isomorphism by Theorem 6.1.1. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(U) & \hookrightarrow & \mathrm{Br}(U_\eta) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle \cong} \\ \mathrm{Br}(Y) & \hookrightarrow & \mathrm{Br}(k(Y)) \end{array}$$

This shows that the map $p^*\colon \mathrm{Br}(U) \to \mathrm{Br}(Y)$ is injective. It is also surjective, because an element of $\mathrm{Br}(k(Y))$ whose image in $\mathrm{Br}(U_\eta)$ is contained in $\mathrm{Br}(Y)$ must be unramified as $p\colon U \to Y$ is locally trivial for the Zariski topology on Y, hence this element is contained in $\mathrm{Br}(Y)$.

The k-varieties X and $Y \times_k \mathbb{P}^1_k$ are is k-birationally equivalent, hence $\mathrm{Br}_{\mathrm{nr}}(k(X)/k) \cong \mathrm{Br}_{\mathrm{nr}}(k(Y)/k)$ by Proposition 6.2.9. Since Y is smooth, $\mathrm{Br}(Y) = \mathrm{Br}_{\mathrm{nr}}(k(Y)/k) \subset \mathrm{Br}(k(Y))$ (Proposition 6.2.7). □

Let us discuss the case where $Y \subset \mathbb{P}^{N-1}_{\mathbb{C}}$, $N \geq 3$, is a smooth projective hypersurface. Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be the projective cone over Y. The vertex is the only singularity of X; it has codimension $N-1$ in X. Let $U \subset X$ be the complement to the vertex of X. By Proposition 8.4.1, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is zero. On the other hand, Proposition 8.3.1 says that $\mathrm{Br}(X)$ is torsion-free for $N \geq 4$ and $\mathrm{Br}(X)$ is uniquely divisible for $N \geq 5$.

For $N \geq 5$, we actually have $\mathrm{Br}(X) = \mathrm{Br}(U) = 0$. Indeed, X is geometrically locally factorial by [SGA2, XI, Cor. 3.14], hence by Theorem 3.5.5 the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(U)$ is injective. Since Y is a smooth hypersurface in $\mathbb{P}^{N-1}$ with $N - 1 \geq 4$, we have $\mathrm{Br}(Y) = 0$ by Corollary 5.5.4. But U is an $\mathbb{A}^1$-bundle over the smooth variety Y, so we have $\mathrm{Br}(U) = 0$ by the same argument as above.

If $N = 3$, then Y is a smooth curve of degree d in $\mathbb{P}^2_{\mathbb{C}}$. We then have $\mathrm{Br}(Y) = 0$ by Theorem 5.6.1 and then $\mathrm{Br}(U) = 0$ for the $\mathbb{A}^1$-bundle U over Y. Let R be the local ring of the vertex P of the cone. Assume $d \geq 4$. By Example 8.2.2, there exists an element of infinite order in $\mathrm{Br}(R)$. Thus one can find an element $\alpha \in \mathrm{Br}(V)$ of infinite order for some Zariski open set $V \subset X$ containing P. Since U is smooth, so is $U \cap V$, the restriction map $\mathrm{Pic}(U) \to \mathrm{Pic}(U \cap V)$ is surjective, and $\mathrm{Br}(U \cap V)$ is a torsion group. The Mayer–Vietoris sequence (Theorem 3.2.4) for the covering $X = U \cup V$ gives an exact sequence

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(U) \oplus \mathrm{Br}(V) \longrightarrow \mathrm{Br}(U \cap V).$$

A multiple of $(0,\alpha)\in \mathrm{Br}(U)\oplus \mathrm{Br}(V)$ goes to zero in $\mathrm{Br}(U\cap V)$ hence comes from a non-torsion element in $\mathrm{Br}(X)$. In particular, the map $\mathrm{Br}(X)\to\mathrm{Br}(U)$ is not injective.

If $N=4$, then $Y\subset\mathbb{P}^3_{\mathbb{C}}$ is a smooth surface, hence $\mathrm{NS}(Y)$ is torsion-free and $\mathrm{H}^3(Y,\mathbb{Z})=0$. By Proposition 8.3.1 (i) the group $\mathrm{Br}(X)$ is torsion-free. If the surface Y is of degree at least 4, then $\mathrm{H}^2(Y,\mathcal{O}_Y)\neq 0$, hence, by Hodge theory (see [Voi02, Ch. 6 and Thm. 11.30]), we have $b_2>\rho$ and then, from Proposition 5.2.9 we obtain $\mathrm{Br}(Y)\simeq(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}\neq 0$. For the $\mathbb{A}^1$-bundle U over the smooth variety Y we then have $\mathrm{Br}(U)\cong\mathrm{Br}(Y)\neq 0$. By Proposition 8.4.1, the map $\mathrm{Br}(X)\to\mathrm{Br}(U)$ factors through $\mathrm{Br}(\mathbb{C})=0$. This gives an example of a hypersurface X of dimension 3 with an isolated singularity P of codimension 3 such that the map $\mathrm{Br}(X)\to\mathrm{Br}(U)$, where $U=X\smallsetminus P$, is not an isomorphism. This shows that the condition on the codimension of the singular locus in Theorem 8.3.2 is necessary. Note that in this case we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(X))\cong\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(Y))\cong\mathrm{Br}(Y)\cong\mathrm{Br}(U)\neq 0$.

8.5 Singular curves and their desingularisation

Let k be a field of characteristic zero with an algebraic closure $\bar{k}$ and Galois group $\Gamma=\mathrm{Gal}(\bar{k}/k)$. In this section we give a complement to Proposition 3.6.6, as developed in [HS14].

Let C be a *reduced*, separated, purely 1-dimensional curve over k. We define the *normalisation* $\widetilde{C}$ as the disjoint union of normalisations of the irreducible components of C. The normalisation morphism $\nu\colon\widetilde{C}\to C$ factors as

$$\widetilde{C}\xrightarrow{\nu'}C'\xrightarrow{\nu''}C,$$

where C' is a maximal intermediate curve universally homeomorphic to C, see [BLR90, Section 9.2, p. 247] or [Liu10, Section 7.5, p. 308]. The curve C' is obtained from $\widetilde{C}$ by identifying the points which have the same image in C. This is the seminormalisation of C (see [Stacks, Section 0EUK]).

In particular, there is a canonical bijection $\nu''\colon C'(K)\xrightarrow{\sim}C(K)$ for any field extension K/k. The curve C' has relatively mild singularities: for each singular point $s\in C'(\bar{k})$ the branches of $\overline{C}'$ through s intersect like n coordinate axes at $0\in\mathbb{A}^n_k$.

We define three reduced 0-dimensional schemes naturally arising in this situation. Let Λ be the k-scheme of geometric irreducible components of C (or the geometric connected components of $\widetilde{C}$); it is the disjoint union of finite integral k-schemes $\lambda=\mathrm{Spec}(k(\lambda))$ such that $k(\lambda)$ is the algebraic closure of k in the function field of the corresponding irreducible component $k(C_\lambda)=k(\widetilde{C}_\lambda)$. Let

$$\Pi\colon=C_{\mathrm{sing}},\quad \Psi\colon=\left(\Pi\times_C\widetilde{C}\right)_{\mathrm{red}}. \tag{8.7}$$

Thus Ψ is the union of fibres of $\nu\colon \widetilde{C}\to C$ over the singular points of C with their reduced subscheme structures. The morphism ν'' induces an isomorphism $\left(\Pi\times_C C'\right)_{\rm red}\xrightarrow{\sim}\Pi$, so we can identify these schemes. Let $i\colon \Pi\to C$, $i'\colon \Pi\to C'$ and $j\colon \Psi\to\widetilde{C}$ be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ {\scriptstyle j}\uparrow & & {\scriptstyle i'}\uparrow & \nearrow{\scriptstyle i} & \\ \Psi & \xrightarrow{\nu'} & \Pi & & \end{array}$$

The restriction of ν to the smooth locus of C induces isomorphisms

$$\widetilde{C}\smallsetminus j(\Psi)\xrightarrow{\sim} C'\smallsetminus i'(\Pi)\xrightarrow{\sim} C\smallsetminus i(\Pi).$$

An algebraic group over Π is a product $G=\prod_\pi i_{\pi*}(G_\pi)$, where π ranges over the irreducible components of Π, the morphism $i_\pi\colon \mathrm{Spec}(k(\pi))\to\Pi$ is the natural closed immersion, and G_π is an algebraic group over the field $k(\pi)$.

Lemma 8.5.1 (i) *The maps $\mathbb{G}_{m,C'}\to\nu'_*(\mathbb{G}_{m,\widetilde{C}})$ and $\mathbb{G}_{m,C'}\to i'_*(\mathbb{G}_{m,\Pi})$ give rise to the exact sequence of étale sheaves on C'*

$$0\longrightarrow \mathbb{G}_{m,C'}\longrightarrow \nu'_*(\mathbb{G}_{m,\widetilde{C}})\oplus i'_*(\mathbb{G}_{m,\Pi})\longrightarrow i'_*(\nu'_*(\mathbb{G}_{m,\Psi}))\longrightarrow 0, \tag{8.8}$$

where $\nu'_*(\mathbb{G}_{m,\Psi})$ is a torus over Π and the third map sends (a,b) to $a-b$.

(ii) *The map $\mathbb{G}_{m,C}\to\nu''_*(\mathbb{G}_{m,C'})$ gives rise to the exact sequence of étale sheaves on C*

$$0\longrightarrow \mathbb{G}_{m,C}\longrightarrow \nu''_*(\mathbb{G}_{m,C'})\longrightarrow i_*\mathbb{U}\longrightarrow 0, \tag{8.9}$$

where $\mathbb{U}$ is a commutative unipotent group over Π.

Proof. See [BLR90], the proofs of Propositions 9.2.9 and 9.2.10, or [Liu10, Lemma 7.5.12]. By [Mil80, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (8.8) at each geometric point $\bar{x}$ of C'. If $\bar{x}\notin i'(\Pi)$, this is obvious since locally at $\bar{x}$ the morphism ν' is an isomorphism, and the stalks $(i'_*(\mathbb{G}_{m,\Pi}))_{\bar{x}}$ and $(i'_*(\nu'_*\mathbb{G}_{m,\Psi}))_{\bar{x}}$ are zero. Now let $\bar{x}\in i'(\Pi)$, and let $\mathcal{O}^{\rm sh}_{C',\bar{x}}$ be the strict henselisation of the local ring of $\bar{x}$ in C'. Each geometric point $\bar{y}$ of $\widetilde{C}$ belongs to exactly one geometric connected component of $\widetilde{C}$. Let $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}$ be the strict henselisation of the local ring of $\bar{y}$ in its geometric connected component. By the construction of C' we have an exact sequence

$$0\longrightarrow \mathcal{O}^{\rm sh}_{C',\bar{x}}\longrightarrow k(\bar{x})\times\prod_{\nu'(\bar{y})=\bar{x}}\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}\longrightarrow \prod_{\nu'(\bar{y})=\bar{x}}k(\bar{y})\longrightarrow 0,$$

where $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}\to k(\bar{y})$ is the reduction modulo the maximal ideal of $\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}}$, and $k(\bar{x})\to k(\bar{y})$ is the multiplication by -1. We obtain an exact sequence of abelian groups

$$1 \longrightarrow (\mathcal{O}^{\rm sh}_{C',\bar{x}})^* \longrightarrow k(\bar{x})^* \times \prod_{\nu'(\bar{y})=\bar{x}} (\mathcal{O}^{\rm sh}_{\widetilde{C},\bar{y}})^* \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y})^* \longrightarrow 1.$$

Using [Mil80, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (8.8) at $\bar{x}$, so that (i) is proved.

To prove (ii) consider the exact sequence

$$0 \longrightarrow \mathbb{G}_{m,C} \longrightarrow \nu''_*(\mathbb{G}_{m,C'}) \longrightarrow \nu''_*(\mathbb{G}_{m,C'})/\mathbb{G}_{m,C} \longrightarrow 0.$$

The morphism ν'' is an isomorphism away from $i(\Pi)$, so the restriction of the sheaf $\nu''_*(\mathbb{G}_{m,C'})/\mathbb{G}_{m,C}$ to $C \smallsetminus i(\Pi)$ is zero. Hence $\nu''_*(\mathbb{G}_{m,C'})/\mathbb{G}_{m,C} = i_*\mathbb{U}$ for some sheaf $\mathbb{U}$ on Π. To see that $\mathbb{U}$ is a unipotent group scheme it is enough to check the stalks at geometric points. Let $\bar{x}$ be a geometric point of $i(\Pi)$, and let $\bar{y}$ be the unique geometric point of C' such that $\nu''(\bar{y}) = \bar{x}$. Let $\mathcal{O}^{\rm sh}_{C,\bar{x}}$ and $\mathcal{O}^{\rm sh}_{C',\bar{y}}$ be the corresponding strictly henselian local rings. The stalk $(\nu''_*(\mathbb{G}_{m,C'})/\mathbb{G}_{m,C})_{\bar{x}}$ is $(\mathcal{O}^{\rm sh}_{C',\bar{y}})^*/(\mathcal{O}^{\rm sh}_{C,\bar{x}})^*$, and by [Liu10, Lemma 7.5.12 (c)], this is a unipotent group over the field $k(\bar{x})$. □

For fields $k_1, \ldots, k_n$, we have ${\rm Br}\big(\coprod_{i=1}^n {\rm Spec}(k_i)\big) = \oplus_{i=1}^n {\rm Br}(k_i)$.

Proposition 8.5.2 *Let k be a field of characteristic zero. Let C be a reduced, separated, purely 1-dimensional curve over k, and let Λ, Π and Ψ be the schemes defined in (8.7). Let $\Lambda = \coprod_\lambda {\rm Spec}(k(\lambda))$ be the decomposition into the disjoint union of connected components, so that $\widetilde{C} = \coprod_\lambda \widetilde{C}_\lambda$, where $\widetilde{C}_\lambda$ is a smooth geometrically integral curve over the field $k(\lambda)$. Then there is an exact sequence*

$$0 \longrightarrow {\rm Br}(C) \longrightarrow {\rm Br}(\Pi) \oplus \bigoplus_{\lambda\in\Lambda} {\rm Br}(\widetilde{C}_\lambda) \longrightarrow {\rm Br}(\Psi), \tag{8.10}$$

where the maps are the composition of canonical maps

$${\rm Br}(\widetilde{C}_\lambda) \longrightarrow {\rm Br}(\widetilde{C}_\lambda \cap \Psi) \longrightarrow {\rm Br}(\Psi),$$

and the opposite of the restriction map ${\rm Br}(\Pi)\to{\rm Br}(\Psi)$.

Proof. Let π range over the irreducible components of Π, so that $\mathbb{U}$ in Lemma 8.5.1 decomposes as $\mathbb{U} = \prod_\pi i_{\pi*}(U_\pi)$, where U_π is a commutative unipotent group over the field $k(\pi)$. Since i_* is an exact functor [Mil80, Cor. II.3.6], we have ${\rm H}^n_{\text{ét}}(C, i_*\mathbb{U}) = {\rm H}^n_{\text{ét}}(\Pi, \mathbb{U}) = \prod_\pi {\rm H}^n(k(\pi), U_\pi)$. The field k has characteristic 0, and it is well known that this implies that any commutative unipotent group has zero cohomology in degree $n > 0$. (Such a group has a composition series with factors $\mathbb{G}_a$, and ${\rm H}^n(k, \mathbb{G}_a) = 0$

for any $n > 0$, see [SerCL, Ch. X, Prop. 1].) Thus the long exact sequence of cohomology groups associated to (8.9) gives rise to an isomorphism $\mathrm{Br}(C) = \mathrm{H}^2_{\text{ét}}(C, \mathbb{G}_{m,C}) \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(C, \nu''_*(\mathbb{G}_{m,C'}))$. Since ν'' is finite, the functor ν''_* is exact [Mil80, Cor. II.3.6], so we obtain an isomorphism $\mathrm{Br}(C) \xrightarrow{\sim} \mathrm{Br}(C')$. We now apply similar arguments to (8.8). Hilbert's theorem 90 gives $\mathrm{H}^1_{\text{ét}}(\Pi, \nu'_*(\mathbb{G}_{m,\Psi})) = \mathrm{H}^1_{\text{ét}}(\Psi, \mathbb{G}_{m,\Psi}) = 0$, so we obtain the exact sequence (8.10). □

8.6 Some examples

(1) Let k be a field of characteristic different from 2 with $a, b \in k^*$ such that the quaternion algebra class $(a, b) \in \mathrm{Br}(k)$ is non-zero. (For example, $k = \mathbb{R}$ and $a = b = -1$.) Consider the singular affine curve over k defined by the equation

$$y^2 = x^2(x+b).$$

Let X be the open set given by $x + b \neq 0$. Consider the quaternion algebra

$$A = (a, x+b) \in \mathrm{Br}_{\mathrm{Az}}(X).$$

Over the function field $k(X)$ of X, we have

$$(a, x+b) = (a, (y/x)^2) = 0 \in \mathrm{Br}(k(X)).$$

But the evaluation of A at the singular point $(x, y) = (0, 0)$ is the non-zero element $(a, b) \in \mathrm{Br}(k)$, thus $A \neq 0$ lies in the kernel of $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(k(X))$. Compare with Proposition 3.6.6.

(2) Let k and $a, b \in k^*$ be the same as in (1). Consider the normal affine surface over k defined by the equation

$$y^2 - az^2 = x^2(x+b).$$

Let X be the open set given by $x + b \neq 0$. Consider the quaternion algebra

$$A = (a, x+b) \in \mathrm{Br}_{\mathrm{Az}}(X).$$

Over the function field $k(X)$ of X, we have

$$(a, x+b) = (a, (y^2 - az^2)/x^2) = 0 \in \mathrm{Br}(k(X)).$$

The evaluation of A at the singular point $(x, y, z) = (0, 0, 0)$ is the non-zero element $(a, b) \in \mathrm{Br}(k)$. Thus $A \neq 0$ lies in the kernel of $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$.

(3) Let k and $a, b \in k^*$ be the same as in (1). Consider the quadratic cone $X \subset \mathbb{A}^4_k$ defined by

$$x^2 - ay^2 - bz^2 + abt^2 = 0.$$

Its singular locus is the point $P = (0, 0, 0, 0)$, which has codimension 3 in X. The class $(a, b) \in \mathrm{Br}(k)$ gives rise to $\alpha = (a, b)_X \in \mathrm{Br}_{\mathrm{Az}}(X) \subset \mathrm{Br}(X)$. This class is non-zero, because its evaluation at P is $(a, b) \in \mathrm{Br}(k)$. But the image of α in $\mathrm{Br}(k(X))$ is zero, since

$$(a, b)_{k(X)} = (a, (x^2 - ay^2)/(z^2 - at^2)) = 0 \in \mathrm{Br}(k(X)).$$

In particular, the map $\mathrm{Br}(X)\to\mathrm{Br}(k(X))$ is not injective. The k-variety X is a complete intersection with one singular point of codimension 3. The variety X is not geometrically locally factorial, as one may see by factorisation of $x^2 - ay^2 = b(z^2 - at^2)$ over a field extension of k. This shows that in Theorem 3.5.5 one cannot remove the hypothesis that X is geometrically locally factorial. The same comments hold for the local ring of X at P. If one replaces X by its closure in $\mathbb{P}^4_k$, we see that in Theorem 8.3.2 one cannot weaken the hypothesis that the singular locus is of codimension at least 4.

(4) If X is a noetherian integral scheme with an isolated singularity $P \in X$, and R_P is the local ring of X at P, then the restriction map

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(R_P)$$

is injective. Indeed, one may write $X = U \cup V$ where U is regular and V contains P. By Theorem 3.5.7 this implies that the restriction map $\mathrm{Br}(X)\to\mathrm{Br}(V)$ is injective. Passing over to the limit over all V containing P gives the result.

The affine surface X over $\mathbb{C}$ given by $z^3 = (1 - x - y)xy$ is normal with exactly three singular points P_i, $i = 1, 2, 3$. Let R_i be the local ring of X at P_i. Ojanguren shows in [Oja74] that the natural map

$$\mathrm{Br}_{\mathrm{Az}}(X) \longrightarrow \prod_{i=1}^{3} \mathrm{Br}_{\mathrm{Az}}(R_i)$$

has a non-trivial kernel.

Chapter 9
Varieties with a group action

One often needs to study the Brauer group of a variety equipped with an action of an algebraic group. The Brauer groups of connected algebraic groups themselves as well as the Brauer groups of their homogeneous spaces can be explicitly computed in many cases. In Section 9.1 we deal with tori and in Section 9.2 with simply connected semisimple groups. We then turn our attention to the unramified Brauer group of function fields of homogeneous spaces; the challenge here is to compute these groups without having to construct an explicit smooth projective model. In Section 9.3 we discuss Bogomolov's theorems which compute the unramified Brauer group of the invariant field of a linear action of a finite group over an algebraically closed field, and a related theorem of Saltman. Finally, in Section 9.4 we give an overview of the unramified Brauer groups of homogeneous spaces over an arbitrary field (mostly without proofs).

9.1 Tori

The étale cohomology of split tori has been studied by many authors, e.g. [Mag78, GP08, GS14].

Lemma 9.1.1 *Let X be a smooth geometrically integral variety over a field k of characteristic zero. Let $\Gamma := \mathrm{Gal}(k_{\mathrm{s}}/k)$. There are split exact sequences*

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{m,X}, \mathbb{Q}/\mathbb{Z}) \longrightarrow \big((\mathbb{Q}/\mathbb{Z})(-1)\big)^{\Gamma} \longrightarrow 0,$$

where $(\mathbb{Q}/\mathbb{Z})(-1)$ is the direct limit of the Galois modules $(\mathbb{Z}/n)(-1)$ for $n\to\infty$, and

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

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Proof. Let Y be the closed subset of $\mathbb{A}^1_X$ which is the zero section of the structure morphism $\mathbb{A}^1_X \to X$. Then $X \cong Y$. The open subset $\mathbb{A}^1_X \setminus Y$ is isomorphic to $\mathbb{G}_{m,X}$. The unit section of the structure morphism $\mathbb{G}_{m,X} \to X$ is an embedding $X \hookrightarrow \mathbb{G}_{m,X}$ such that the composition $X \hookrightarrow \mathbb{G}_{m,X} \to \mathbb{A}^1_X \to X$ is an isomorphism.

For any integer $n > 0$ we have the Gysin exact sequence (2.16)

$$\ldots \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X}, \mathbb{Z}/n) \longrightarrow \mathrm{H}^{i-1}_{\text{ét}}(X, (\mathbb{Z}/n)(-1)) \longrightarrow \mathrm{H}^{i+1}_{\text{ét}}(\mathbb{A}^1_X, \mathbb{Z}/n) \longrightarrow \ldots$$

As $n > 0$ is invertible in X, the natural map $\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X, \mathbb{Z}/n)$ is an isomorphism for any integer i, see [Mil80, Cor. VI.4.20]. Specialisation at the unit section of $\mathbb{G}_{m,X} \to X$ shows that all maps $\mathrm{H}^i_{\text{ét}}(\mathbb{A}^1_X, \mathbb{Z}/n) \to \mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X}, \mathbb{Z}/n)$ are split injective. Putting everything together, we get split short exact sequences

$$0 \longrightarrow \mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/n) \longrightarrow \mathrm{H}^i_{\text{ét}}(\mathbb{G}_{m,X}, \mathbb{Z}/n) \longrightarrow \mathrm{H}^{i-1}_{\text{ét}}(X, (\mathbb{Z}/n)(-1)) \longrightarrow 0.$$

For $i = 1$, this gives the first exact sequence. For $i = 2$, this gives the exact sequence

$$0 \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_n) \longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,X}, \mu_n) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \longrightarrow 0.$$

One then uses the compatible exact sequences

$$0 \longrightarrow \mathrm{Pic}(X)/n \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_n) \longrightarrow \mathrm{Br}(X)[n] \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Pic}(\mathbb{G}_{m,X})/n \longrightarrow \mathrm{H}^2_{\text{ét}}(\mathbb{G}_{m,X}, \mu_n) \longrightarrow \mathrm{Br}(\mathbb{G}_{m,X})[n] \longrightarrow 0$$

given by the Kummer sequence. The map $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{G}_{m,X})$ is the composition $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{A}^1_X) \to \mathrm{Pic}(\mathbb{G}_{m,X})$. The first map is an isomorphism since X is regular and the second map is surjective since $\mathbb{A}^1_X$ is regular. Since $\mathbb{G}_{m,X}/X$ has the unit section, we conclude that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{G}_{m,X})$ is an isomorphism. We now get the exact sequence

$$0 \longrightarrow \mathrm{Br}(X)[n] \longrightarrow \mathrm{Br}(\mathbb{G}_{m,X})[n] \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/n) \longrightarrow 0.$$

Since X and $\mathbb{G}_{m,X}$ are regular, both $\mathrm{Br}(X)$ and $\mathrm{Br}(\mathbb{G}_{m,X})$ are torsion groups, so we obtain the second exact sequence of the lemma. □

Let k be a field with separable closure k_{s}. Let T be a k-torus. By definition there exists an isomorphism of k_{s}-algebraic groups $T^{\mathrm{s}} = T \times_k k_{\mathrm{s}} \cong \mathbb{G}^d_{m,k_{\mathrm{s}}}$ for some positive integer d. The group $k_{\mathrm{s}}[T]^*$ of invertible functions on T^{s} is the direct sum of k_{s}^* and the character group $\widehat{T} = \mathrm{Hom}_{k_{\mathrm{s}}\text{-groups}}(T^{\mathrm{s}}, \mathbb{G}_{m,k_{\mathrm{s}}})$.

In particular, for any integer n invertible in k, there is a natural isomorphism $\mathrm{H}^0_{\text{ét}}(T^{\mathrm{s}},\mathbb{G}_m)/n \cong \widehat{T}/n$.

Proposition 9.1.2 *Let k be a field of characteristic zero. Let T be a torus of dimension $d \geq 1$ over k with character group $\widehat{T}$.*

(i) *There is a functorial isomorphism of Γ-modules*

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \widehat{T} \otimes (\mathbb{Q}/\mathbb{Z})(-1)$$

and a non-canonical isomorphism of abelian groups

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mathbb{Q}/\mathbb{Z}) \simeq (\mathbb{Q}/\mathbb{Z})^d.$$

(ii) *There is a functorial isomorphism of Γ-modules*

$$\wedge^2(\widehat{T}) \otimes (\mathbb{Q}/\mathbb{Z})(-1) \xrightarrow{\sim} \mathrm{Br}(T^{\mathrm{s}})$$

and a non-canonical isomorphism of abelian groups

$$\mathrm{Br}(T^{\mathrm{s}}) \simeq (\mathbb{Q}/\mathbb{Z})^{d(d-1)/2}.$$

If k is algebraically closed, $T \simeq \mathrm{Spec}(k[x_1,x_1^{-1},\dots,x_d,x_d^{-1}])$ and ζ is a primitive n-th root of unity, the composition

$$\wedge^2(\widehat{T}) \otimes \mathbb{Z}/n \xrightarrow{\sim} \mathrm{Br}(T)[n] \otimes \mu_n \longrightarrow \mathrm{Br}(k(T))[n] \otimes \mu_n$$

sends $x_i \wedge x_j$ to $(x_i,x_j)_\zeta \otimes \zeta$, where $(x_i,x_j)_\zeta$ is defined at the end of Section 1.3.4.

(iii) *There is a split exact sequence of abelian groups*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}_1(T) \longrightarrow \mathrm{H}^2(k,\widehat{T}) \longrightarrow 0.$$

Proof. (i) Since $\mathrm{Pic}(T^{\mathrm{s}})=0$, for any integer n, the Kummer sequence gives a natural isomorphism

$$\mathrm{H}^0_{\text{ét}}(T^{\mathrm{s}},\mathbb{G}_m)/n \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mu_n),$$

hence $\widehat{T}/n \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mu_n)$. We thus obtain an isomorphism

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \widehat{T} \otimes (\mathbb{Q}/\mathbb{Z})(-1).$$

(ii) Using this isomorphism and the second (split) exact sequence of Lemma 9.1.1 for $X=\mathbb{G}_m^{d-1}$, we obtain by induction a non-canonical isomorphism $\mathrm{Br}(T^{\mathrm{s}}) \simeq (\mathbb{Q}/\mathbb{Z})^{d(d-1)/2}$. In particular, for each $n \geq 1$, the order of $\mathrm{Br}(T^{\mathrm{s}})[n]$ is $n^{d(d-1)/2}$.

Consider the cup-product pairing of étale cohomology groups

$$\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mu_n)\times \mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mu_n)\longrightarrow \mathrm{H}^2_{\text{ét}}(T^{\mathrm{s}},\mu_n^{\otimes 2})=\mathrm{Br}(T^{\mathrm{s}})[n]\otimes\mu_n, \qquad (9.1)$$

where the last equality follows from the Kummer sequence and the vanishing of $\mathrm{Pic}(T^{\mathrm{s}})$. This pairing is compatible with the cup-product pairing of Galois cohomology groups

$$\mathrm{H}^1(k_{\mathrm{s}}(T),\mu_n)\times \mathrm{H}^1(k_{\mathrm{s}}(T),\mu_n)\longrightarrow \mathrm{H}^2(k_{\mathrm{s}}(T),\mu_n^{\otimes 2}) \qquad (9.2)$$

via the injective map $\mathrm{H}^1_{\text{ét}}(T^{\mathrm{s}},\mu_n)\hookrightarrow \mathrm{H}^1(k_{\mathrm{s}}(T),\mu_n)$ induced by the inclusion of the generic point $\mathrm{Spec}(k_{\mathrm{s}}(T))\to T^{\mathrm{s}}$. Since $\mathrm{char}(k)=0$, the field k_{s} is algebraically closed. Thus $(a,a)=(a,-a)=0$ for any $a\in \mathrm{H}^1(k_{\mathrm{s}}(T),\mu_n)$. (For the last equality, see the end of Section 1.3.4.) Hence the pairings (9.2) and (9.1) are alternating. We thus have a map of Γ-modules

$$\xi\colon\ \wedge^2\,(\widehat{T})\otimes\mathbb{Z}/n\longrightarrow \mathrm{Br}(T^{\mathrm{s}})[n]\otimes\mu_n.$$

It is enough to prove that ξ is an isomorphism of abelian groups. We already know that the two groups have the same cardinality, so it remains to show that ξ is injective.

Let us fix an isomorphism of k_{s}-tori

$$T^{\mathrm{s}}\simeq \mathbb{G}^d_{m,k_{\mathrm{s}}}=\mathrm{Spec}(k_{\mathrm{s}}[x_1,x_1^{-1},\ldots,x_d,x_d^{-1}])\subset \mathbb{A}^d_{m,k_{\mathrm{s}}}=\mathrm{Spec}(k_{\mathrm{s}}[x_1,\ldots,x_d]).$$

The free $\mathbb{Z}/n$-module $\wedge^2(\widehat{T})\otimes\mathbb{Z}/n$ is generated by the elements $x_i\wedge x_j$ for $1\leq i<j\leq d$. Let $\alpha=\sum_{i<j}a_{ij}x_i\wedge x_j$ be a non-zero element of $\wedge^2(\widehat{T})\otimes\mathbb{Z}/n$, where each a_{ij} is a non-negative integer less than n. Write β for the image of α in $\mathrm{Br}(k_{\mathrm{s}}(T))[n]\otimes\mu_n$. Let r be the smallest value such that $a_{rt}\neq 0$ for some t. Let K_r be the field $k_{\mathrm{s}}(x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_d)$. By formula (1.18), the residue of β at the divisor $x_r=0$ of $\mathbb{A}^d_{m,k_{\mathrm{s}}}$ is the class of $\prod_{t>r}x_t^{a_{rt}}$ in K_r^*/K_r^{*n}. This class is non-trivial, hence $\beta\neq 0$. This shows that the composition of ξ with the natural map $\mathrm{Br}(T^{\mathrm{s}})[n]\otimes\mu_n\to\mathrm{Br}(k_{\mathrm{s}}(T))[n]\otimes\mu_n$ is injective, so ξ is injective. This proves (ii).

(iii) In view of $\mathrm{Pic}(T^{\mathrm{s}})=0$, the spectral sequence (5.19)

$$E_2^{p,q}=\mathrm{H}^p(k,\mathrm{H}^q_{\text{ét}}(T^{\mathrm{s}},\mathbb{G}_m))\Longrightarrow \mathrm{H}^{p+q}_{\text{ét}}(T,\mathbb{G}_m) \qquad (9.3)$$

gives rise to an isomorphism $\mathrm{H}^2(k,k_{\mathrm{s}}[T]^*)\xrightarrow{\sim}\mathrm{Br}_1(T)$, hence to an isomorphism $\mathrm{H}^2(k,k_{\mathrm{s}}^*)\oplus\mathrm{H}^2(k,\widehat{T})\xrightarrow{\sim}\mathrm{Br}_1(T)$ which gives (iii). □

Proposition 9.1.3 *Let T be a torus over a field k.*

(i) *If k is perfect and T is split, i.e., $T\cong\mathbb{G}^n_{m,k}$ for some $n\geq 1$, then the natural map $\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^{\Gamma}$ is surjective.*

(ii) *If* $\mathrm{char}(k) = 0$, *then the image of* $\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^{\Gamma}$ *contains the subgroup of* $\mathrm{Br}(T^{\mathrm{s}})^{\Gamma}$ *consisting of the elements of odd order. Hence the map* $\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^{\Gamma}$ *is surjective if* $(\wedge^2(\widehat{T})/2)^{\Gamma} = 0$.

Proof. We have $\mathrm{H}^0(T^{\mathrm{s}}, \mathbb{G}_m) \cong k_{\mathrm{s}}^* \oplus \widehat{T}$ and $\mathrm{H}^1(T^{\mathrm{s}}, \mathbb{G}_m) \cong \mathrm{Pic}(T^{\mathrm{s}}) = 0$. The spectral sequence (9.3) thus gives rise to an exact sequence

$$0 \longrightarrow \mathrm{H}^2(\Gamma, k_{\mathrm{s}}^* \oplus \widehat{T}) \longrightarrow \mathrm{Br}(T) \longrightarrow \mathrm{Br}(T^{\mathrm{s}})^{\Gamma} \longrightarrow \mathrm{H}^3(\Gamma, k_{\mathrm{s}}^* \oplus \widehat{T}) \longrightarrow \mathrm{H}^3(T, \mathbb{G}_m).$$

Write

$$\mathrm{Br}_e(T) = \mathrm{Ker}[\mathrm{Br}(T)\to\mathrm{Br}(k)], \quad \mathrm{H}^3_e(T, \mathbb{G}_m) = \mathrm{Ker}[\mathrm{H}^3(T, \mathbb{G}_m)\to\mathrm{H}^3(k, \mathbb{G}_m)]$$

for the kernels of the evaluation maps at the neutral element $e \in T(k)$. Then we get an exact sequence

$$0\to\mathrm{H}^2(\Gamma, \widehat{T}) \longrightarrow \mathrm{Br}_e(T) \longrightarrow \mathrm{Br}(T^{\mathrm{s}})^{\Gamma} \xrightarrow{\alpha} \mathrm{H}^3(\Gamma, \widehat{T}) \longrightarrow \mathrm{H}^3_e(T, \mathbb{G}_m). \quad (9.4)$$

Since (9.3) is functorial in the k-variety T, the exact sequence (9.4) is functorial with respect to homomorphisms of tori over k. Hence for any homomorphism of k-tori $R\to T$ we get a commutative diagram with exact rows

$$\begin{array}{ccccc} \mathrm{Br}_e(T) & \longrightarrow & \mathrm{Br}(T^{\mathrm{s}})^{\Gamma} & \xrightarrow{\alpha} & \mathrm{H}^3(\Gamma, \widehat{T}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Br}_e(R) & \longrightarrow & \mathrm{Br}(R^{\mathrm{s}})^{\Gamma} & \longrightarrow & \mathrm{H}^3(\Gamma, \widehat{R}) \end{array} \quad (9.5)$$

Let us prove (i). If $\dim(R) = 1$, then $\mathrm{Br}(R^{\mathrm{s}}) = 0$ (using that k is perfect; for k arbitrary we would only get a result up to the characteristic of k). This implies that the composition of maps

$$\mathrm{Br}(T^{\mathrm{s}})^{\Gamma} \xrightarrow{\alpha} \mathrm{H}^3(\Gamma, \widehat{T}) \longrightarrow \mathrm{H}^3(\Gamma, \widehat{R}) \quad (9.6)$$

is zero.

For $T \simeq \mathbb{G}_{m,k}^n$ we have an isomorphism of trivial Γ-modules $\widehat{T} \simeq \mathbb{Z}^n$, and we have $\mathrm{H}^3(\Gamma, \mathbb{Z}^n) \cong \mathrm{H}^3(\Gamma, \mathbb{Z})^{\oplus n}$. Thus by (9.6) the map α is zero, hence the map $\mathrm{Br}(T)\to\mathrm{Br}(T^{\mathrm{s}})^{\Gamma}$ is surjective.

Let us prove (ii). Let $[n]\colon T\to T$ be the multiplication by n map. The induced map $\widehat{T}\to\widehat{T}$ is multiplication by n. By the functoriality of the isomorphism in Proposition 9.1.2 (i), the map induced by $[n]$ on $\mathrm{Br}(T^{\mathrm{s}})$ is multiplication by n^2.

Consider the diagram (9.5) where $T\to R$ is the homomorphism $[n]\colon T\to T$. The commutativity of the diagram implies that the image of α is annihilated by $n^2 - n$, hence by 2. Now (ii) follows from the exactness of the top row of diagram (9.5). □

For an arbitrary torus T over a field k, it would be interesting to find some explicit description of the map $\alpha\colon \mathrm{Br}(T^{\mathrm{s}})^{\Gamma}\to\mathrm{H}^3(\Gamma,\widehat{T})$ from (9.4).

In the case when the field k is algebraically closed, Harari and Skorobogatov [HS03, Thm. 1.6] computed the Brauer group of a torsor for a k-torus.

Theorem 9.1.4 *Let k be an algebraically closed field of characteristic zero. Let $T=\mathbb{G}_{m,k}^n$ for some integer $n\geq 1$. Let X be a smooth integral variety over k such that $k[X]^*\cong k^*$ and $\mathrm{Pic}(X)$ is a finitely generated free abelian group. Let $f\colon Y\to X$ be a torsor for T over X. If $k[Y]^*\cong k^*$ and $\mathrm{Pic}(Y)$ is a finitely generated free abelian group, then $f^*\colon \mathrm{Br}(X)\to\mathrm{Br}(Y)$ is an isomorphism.*

9.2 Simply connected semisimple groups

Proposition 9.2.1 *Let k be a field of characteristic zero. Let G be a simply connected semisimple group over k. Let E be a G-torsor over k and let X be a smooth, projective, geometrically integral variety over k birationally equivalent to E. Then the following natural maps are isomorphisms:*

$$\text{(i)}\quad \mathrm{Br}(k)\xrightarrow{\sim}\mathrm{Br}(E);\qquad\qquad \text{(ii)}\quad \mathrm{Br}(k)\xrightarrow{\sim}\mathrm{Br}(X).$$

Proof. For G and E as above we have $k_{\mathrm{s}}[G]^*\cong k_{\mathrm{s}}^*$, hence $k_{\mathrm{s}}[E]^*\cong k_{\mathrm{s}}^*$. We also have $\mathrm{Pic}(G^{\mathrm{s}})=0$, see [San81, Lemme 6.9 (iv)], hence $\mathrm{Pic}(E^{\mathrm{s}})=0$. The natural map $\mathrm{Br}(k)\to\mathrm{Br}(G)$ is an isomorphism [Gil09], hence $\mathrm{Br}(E^{\mathrm{s}})=0$.

The exact sequence (5.20) then gives an isomorphism in (i). For X as in the proposition, there exists a non-empty open set $U\subset E$ and a birational morphism $U\to X$. Since X is projective and E is smooth, we may assume that U contains all codimension 1 points of E. By purity for the Brauer group (Theorem 3.7.1) the restriction map $\mathrm{Br}(E)\to\mathrm{Br}(U)$ is an isomorphism. Since X is smooth, the map $\mathrm{Br}(X)\to\mathrm{Br}(U)$ is injective. Now (ii) follows from (i). □

If G is not simply connected, the map $\mathrm{Br}(k)\to\mathrm{Br}(G)$ is not necessarily an isomorphism even when k is algebraically closed of characteristic zero, see [Ive76].

The following important theorem of Bruhat and Tits [BT87, §4.7] will be used in the proof of Proposition 10.1.15 leading to Corollary 11.2.3.

Theorem 9.2.2 (Bruhat–Tits) *Let K be a complete discretely valued field with perfect residue field of cohomological dimension 1. Let X be a torsor for a simply connected semisimple group over K. Then X has a K-point.*

9.3 Theorems of Bogomolov and Saltman

In this section we discuss theorems of Bogomolov and Saltman. We refer to [CTS07, §6] and to [GS17, Ch. 6, §6] for most of the proofs and for the history of the subject.

In view of Remark 6.2.2, for a separably closed field k and any field extension $k \subset K$ we can write $\mathrm{Br}_{\mathrm{nr}}(K)$ for $\mathrm{Br}_{\mathrm{nr}}(K/k)$. We shall follow this convention.

An abelian group generated by at most two elements will be called *bicyclic.*

Theorem 9.3.1 *Let L be a field finitely generated over an algebraically closed field k of characteristic zero. Let G be a finite group of automorphisms of L over k, and let $\mathcal{B}_G$ be the set of bicyclic subgroups of G. Then*

$$\mathrm{Br}_{\mathrm{nr}}(L^G) = \{\alpha \in \mathrm{Br}(L^G) \mid \alpha_H \in \mathrm{Br}_{\mathrm{nr}}(L^H) \text{ for all } H \in \mathcal{B}_G\},$$

where α_H is the restriction of $\alpha \in \mathrm{Br}(L^G)$ to $\mathrm{Br}(L^H)$.

Proof. (Cf. [CTS07, Thm. 6.1]) Let $K = L^G$ and let $\alpha \in \mathrm{Br}(K)$ be such that $\partial_A(\alpha) \neq 0$ for some discrete valuation ring $A \subset K$ with fraction field K. By Proposition 6.2.3 it is enough to show that there is a subgroup $H \in \mathcal{B}_G$ such that $\alpha_H \notin \mathrm{Br}_{\mathrm{nr}}(L^H)$.

The following facts can be found in [SerCL, I, §7]. Let $\mathfrak{p}$ be a prime ideal in the semilocal Dedekind ring $\widetilde{A}$ which is the integral closure of A in L, let $D \subset G$ be the associated decomposition group, and let $I \subset D$ be the inertia group, which is a normal subgroup of G. The localisation $B = \widetilde{A}_{\mathfrak{p}} \subset L$ is a discrete valuation ring. There is a tower of fields $K = L^G \subset L^D \subset L^I \subset L$ and a corresponding tower of discrete valuation rings $A = B^G \subset B^D \subset B^I$. The corresponding residue field extensions are $F = F \subset E = E$, and we have $D/I = \mathrm{Gal}(E/F) = \mathrm{Gal}(L^I/L^D)$. The Galois extension L^I/K is unramified, i.e., a uniformiser of A is also a uniformiser of B^I.

Moreover, since the residue characteristic is zero, the inertia group I is isomorphic to a cyclic group, namely, the subgroup μ of roots of unity in F [SerCL, IV, §2, Cor. 1 et 2]. Furthermore, the conjugacy action of D on the normal subgroup I is then trivial, since this action can be identified with the action of $D/I = \mathrm{Gal}(E/F)$ on $\mu \subset F$, and all the roots of unity are in $k \subset F$. Thus I is central in D.

If $\alpha_I \notin \mathrm{Br}_{\mathrm{nr}}(L^I)$, we are done, since I is a cyclic subgroup of G. Thus we may assume that $\alpha_I \in \mathrm{Br}_{\mathrm{nr}}(L^I)$. Since B^D/A is an unramified extension of discrete valuation rings which induces an isomorphism on the residue fields, the assumption $\partial_A(\alpha) \neq 0$ implies that $\partial_{B^D}(\alpha) \neq 0 \in \mathrm{H}^1(F, \mathbb{Q}/\mathbb{Z})$. On the other hand, $\partial_{B^I}(a) = 0 \in \mathrm{H}^1(E, \mathbb{Q}/\mathbb{Z})$.

Since B^I/B^D is unramified, the commutative diagram given by Proposition 1.4.7 (using Theorems 1.4.14 and 2.3.5)

$$\begin{array}{ccc} \mathrm{Br}(K^I) & \xrightarrow{\partial_{B^I}} & \mathrm{H}^1(E,\mathbb{Q}/\mathbb{Z}) \\ \uparrow & & \uparrow{\scriptstyle \mathrm{res}_{F/E}} \\ \mathrm{Br}(K^D) & \xrightarrow{\partial_{B^D}} & \mathrm{H}^1(F,\mathbb{Q}/\mathbb{Z}) \end{array}$$

implies that $\partial_{B^D}(\alpha)$ can be identified with a non-trivial character of $D/I = \mathrm{Gal}(E/F)$. Let $g \in D$ be an element of D whose class $\bar{g}$ in D/I satisfies $\partial_{B^D}(\alpha)(\bar{g}) \neq 0 \in \mathbb{Q}/\mathbb{Z}$, let $H = \langle I, g\rangle \subset D$ be the subgroup spanned by I and g, and let F_1 be the residue class field of B^H. Inserting $\mathrm{Br}(K^H)\to\mathrm{H}^1(F_1,\mathbb{Q}/\mathbb{Z})$ in the above diagram, one immediately sees that $\partial(\alpha_H) \neq 0$, since $\partial(\alpha_H)$ may be identified with a character of $\mathrm{Gal}(E/F_1) = H/I$ which does not vanish on $\bar{g}$. This is enough to conclude, since H is an extension of the cyclic group $\langle\bar{g}\rangle$ by the central cyclic subgroup I (see above), hence is an abelian group spanned by two elements. □

Let G be a finite group. Consider a faithful representation $G\to\mathrm{GL}(V)$, where V is a finite-dimensional complex vector space. Write $\mathbb{C}(V)$ for the purely transcendental extension of $\mathbb{C}$, which is the field of rational functions on V considered as an affine space over $\mathbb{C}$. Then the subfield of invariants $\mathbb{C}(V)^G$ is the function field of the quotient V/G. Speiser's lemma (see, e.g. [CTS07, Thm. 3.3]) states that the stably birational equivalence class of V/G does not depend on the choice of a faithful representation $G\to\mathrm{GL}(V)$. By Corollary 6.2.10, this implies that $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G)$ does not depend on the choice of V. In particular, considering the left action of $\mathrm{GL}(V)$ on $\mathrm{End}(V)$ gives a faithful representation of G in $\mathrm{End}(V)$, so we get an isomorphism $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) \simeq \mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(\mathrm{GL}(V)/G))$.

If $G = A$ is a finite abelian group, then V is a direct sum of 1-dimensional representations, i.e. characters of A. This implies that $\mathbb{C}(V)^A/\mathbb{C}$ is purely transcendental (Fischer's theorem, see [GS17, Thm. 6.6.8]). In this case, by Proposition 6.2.9, we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^A) = 0$. Combining this with Theorem 9.3.1, one gets the following result.

Theorem 9.3.2 (Bogomolov) *Let $G \subset \mathrm{GL}(V)$ be a finite group. Let $\mathcal{B}$ be the set of bicyclic subgroups of G. Then the unramified Brauer group of the field $\mathbb{C}(V)^G$ over $\mathbb{C}$ is given by the formula*

$$\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)^G) = \mathrm{Ker}[\mathrm{H}^2(G,\mathbb{C}^*) \longrightarrow \prod_{A\in\mathcal{B}} \mathrm{H}^2(A,\mathbb{C}^*)],$$

where G acts trivially on $\mathbb{C}^*$ and $\mathrm{H}^2(G,\mathbb{C}^*)\to\mathrm{H}^2(A,\mathbb{C}^*)$ is the restriction map.

See [Bog87], [CTS07, Thm. 7.1], [GS17, Thm. 6.6.12]. Fischer's theorem implies that the set $\mathcal{B}$ of bicyclic subgroups can be replaced by the larger set of all abelian subgroups. One may also write $\mathrm{H}^2(G,\mathbb{C}^*) \cong \mathrm{H}^3(G,\mathbb{Z})$ and similarly for each A. The same formula gives the value of $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(H/G))$, where G is a finite subgroup of $H = \mathrm{SL}_{n,\mathbb{C}}$ or any simply connected semisimple group over $\mathbb{C}$ (see [CT12b] and [LA17]).

This theorem led to numerous examples of finite p-groups G such that the quotient $\mathrm{GL}_{n,\mathbb{C}}/G$ is not rational (E. Noether's problem). D. Saltman (1984) was the first to use the unramified Brauer group to disprove the rationality of $\mathrm{GL}_{n,\mathbb{C}}/G$ for some finite groups G. Bogomolov [Bog87] developed a technique for computing $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(\mathrm{GL}_{n,\mathbb{C}}/G))$ when G is a central extension of abelian groups. See [CTS07, §7] and the references therein. Since [CTS07] was written, many papers have been devoted to the computation of the group $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(\mathrm{GL}_{n,\mathbb{C}}/G))$ in Theorem 9.3.2, which often goes under the name of 'Bogomolov multiplier'. (Recall that $\mathrm{H}^2(G,\mathbb{C}^*) \cong \mathrm{H}^3(G,\mathbb{Z})$ is the Schur multiplier of the finite group G.) Kunyavskiĭ [Kun10] proved that the Bogomolov multiplier vanishes for all finite simple groups.

Definition 9.3.3 *Let G be a finite group. A finitely generated free abelian group with an action of G is called a G-***lattice***.*

For a finitely generated free abelian group M we write $\mathbb{C}[M]$ for the group $\mathbb{C}$-algebra of M. Let $\mathbb{C}(M)$ be the field of fractions of $\mathbb{C}[M]$. In other words, $\mathbb{C}(M)$ is the function field $\mathbb{C}(T)$ of the complex torus $T = \mathrm{Spec}(\mathbb{C}[M])$. If M is a G-lattice, then $\mathbb{C}(M)^G$ is the function field of the quotient T/G.

Theorem 9.3.4 (Saltman [Sal90]) *Let G be a finite group and let M be a faithful G-lattice. Then*

$$\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(M)^G) = \mathrm{Ker}[\mathrm{H}^2(G,\mathbb{C}^*\oplus M) \longrightarrow \prod_{A\in\mathcal{B}} \mathrm{H}^2(A,\mathbb{C}^*\oplus M)],$$

where $\mathcal{B}$ is the set of bicyclic subgroups of G.

Further work along these lines has been done by D. Saltman, E. Peyre [P08], and by B. Kahn and Nguyen Thi Kim Ngan [KN16].

There is an extension of Theorem 9.3.1 to almost free actions of (not necessarily connected) reductive groups, see [Bog89, Thm. 2.1] and [CTS07, Thm. 6.4].

Theorem 9.3.5 (Bogomolov) *Let k be an algebraically closed field of characteristic zero, let G be a reductive group over k, and let X be an integral affine variety over k with an action of G such that all stabilisers are trivial. Write $\mathcal{B}_G$ for the set of finite bicyclic subgroups of $G(k)$. Then*

$$\mathrm{Br}_{\mathrm{nr}}(k(X)^G) = \{\alpha \in \mathrm{Br}(k(X)^G) \mid \alpha_A \in \mathrm{Br}_{\mathrm{nr}}(k(X)^A) \text{ for all } A \in \mathcal{B}_G\},$$

where α_A is the restriction of $\alpha \in \mathrm{Br}(k(X)^G)$ to $\mathrm{Br}(k(X)^A)$.

The following theorem was proved in several instalments.

Theorem 9.3.6 *Let k be an algebraically closed field of characteristic zero. Let G be a connected linear algebraic group over k and let $H \subset G$ be a connected algebraic subgroup. Let X_c be a smooth compactification of G/H. Then* $\mathrm{Br}(X_c) = 0$.

The case $H = \mathrm{PGL}_n \subset G = \mathrm{GL}_N$ is due to Saltman [Sal85].

For semisimple simply connected G, the result is a theorem of Bogomolov [Bog89, Thm. 2.4]. For a detailed account of his proof, see [CTS07, §9]. The proof given there builds upon Theorem 9.3.5.

The result in the general case was obtained by Borovoi, Demarche and Harari in [BDH13]. Their proof uses a long arithmetic detour. A direct reduction to the case when G is semisimple and simply connected was then given by Borovoi [Bor13].

In the special case when $G = \mathrm{GL}_n$ and H is a connected semisimple group, a proof in arbitrary characteristic is given by Blinstein and Merkurjev in [BM13, Thm. 5.10].

Over a separably closed field of characteristic $p > 0$, assuming that the connected groups G and H are smooth and reductive, Borovoi, Demarche and Harari [BDH13] prove that $\mathrm{Br}(X_c)$ is a p-primary torsion group.

Remark 9.3.7 (1) Let $k = \mathbb{C}$. There exists a subgroup $A \subset \mathrm{SL}_n$, where A is an extension of a finite abelian group by a torus, such that we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(\mathrm{SL}_n/A)) \neq 0$. Such examples can be constructed by a method suggested by C. Demarche. Suppose that a group H' is a central extension of a finite abelian group A by a finite abelian group Z. Let us embed Z into a torus T and define $H = (T \times H')/Z$. Then H is a central extension of A by T. Suppose that we are given an embedding $H \hookrightarrow G = \mathrm{SL}_n$. Since T commutes with H, there is a right action of T on G/H'. But H is generated by T and H', hence the natural morphism $G/H' \to G/H$ is a right torsor for the quotient torus T/Z. This torus is split, hence G/H' is stably birationally equivalent to G/H. Thus the natural map $\mathrm{Br}_{\mathrm{nr}}(G/H) \to \mathrm{Br}_{\mathrm{nr}}(G/H')$ is an isomorphism. Using Theorem 9.3.2, Bogomolov [Bog87] constructed examples with $\mathrm{Br}_{\mathrm{nr}}(G/H') \neq 0$. (See also [CTS07].)

(2) Let $k = \mathbb{C}$. For a subgroup $A \subset G$, where G is semisimple and simply connected, and A is an extension of a group of multiplicative type by a semisimple simply connected group, we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(G/A)) = 0$. This follows by combining Theorem 9.3.6 with [LA15, Prop. 26], itself an elaboration of Corollary 11.2.3.

9.4 Homogeneous spaces over an arbitrary field

Let k be a field with separable closure k_s and let $\Gamma = \mathrm{Gal}(k_s/k)$.

Definition 9.4.1 *For a continuous discrete Γ-module M and $i \geq 0$ define*

$$Ш^i_\omega(\Gamma, M) := \mathrm{Ker}[\mathrm{H}^i(\Gamma, M) \longrightarrow \prod_{g\in\Gamma} \mathrm{H}^i(\langle g\rangle, M)],$$

where $\langle g\rangle$ is the **closed** *subgroup of Γ generated by g.*

Using hypercohomology one extends this definition to bounded complexes of Galois modules. The following statements are proved using standard properties of Galois cohomology.

(1) If $K \subset k_s$ is a Galois extension of k such that $\mathrm{Gal}(k_s/K)$ acts trivially on M, then the inflation map $\mathrm{H}^1(\mathrm{Gal}(K/k), M^{\mathrm{Gal}(k_s/K)}) \to \mathrm{H}^1(\Gamma, M)$ induces an isomorphism

$$\mathrm{Ker}[\mathrm{H}^1(\mathrm{Gal}(K/k), M) \longrightarrow \prod_{g\in\mathrm{Gal}(K/k)} \mathrm{H}^1(\langle g\rangle, M)] \cong Ш^1_\omega(\Gamma, M).$$

(2) If, in addition, the abelian group M is finitely generated and torsion-free, then the inflation map $\mathrm{H}^2(\mathrm{Gal}(K/k), M^{\mathrm{Gal}(k_s/K)}) \to \mathrm{H}^2(\Gamma, M)$ induces an isomorphism

$$\mathrm{Ker}[\mathrm{H}^2(\mathrm{Gal}(K/k), M) \longrightarrow \prod_{g\in\mathrm{Gal}(K/k)} \mathrm{H}^2(\langle g\rangle, M)] \cong Ш^2_\omega(\Gamma, M).$$

Let G be a finite group and let M be a left G-lattice. The dual G-lattice M° is $\mathrm{Hom}_\mathbb{Z}(M, \mathbb{Z})$ with the action of G given by $(g\phi)(m) = \phi(g^{-1}m)$.

Definition 9.4.2 *A G-lattice M is* **flasque** *if it satisfies the following equivalent properties*

(i) *For any subgroup $H \subset G$, the Tate cohomology group $\widehat{\mathrm{H}}^{-1}(H, M)$ is zero.*
(ii) *For any subgroup $H \subset G$, we have $\mathrm{H}^1(H, M^\circ) = 0$.*
(iii) *For any subgroup $H \subset G$, we have $\mathrm{Ext}^1_H(M, \mathbb{Z}) = 0$.*

See [CTS87b, 0.5] for the equivalence of (i), (ii), and (iii). If Γ is a profinite group and M is a finitely generated free abelian group which is a continuous, discrete Γ-module, then Γ acts on M via a finite quotient by an open normal subgroup of Γ. Let us call this quotient G. The Γ-module M is called *flasque* if the G-lattice M is flasque. This definition does not depend on the choice of G.

Work of many authors [Vos98, CTS77, San81, Bog89, BDH13, CTK98, BK00, BKG04, CTK06, CTS07, CT08a, Bor13, BM13] has led to the following theorem.

Theorem 9.4.3 *Let k be a field of characteristic zero with an algebraic closure $\bar{k}$ and Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let X be a homogeneous space of a connected linear algebraic group such that the stabilisers of geometric points are connected. Let X_c be a smooth compactification of X. Then the group $\mathrm{Pic}(\overline{X}_c)$ is free and finitely generated, and the following properties hold.*

(i) $\mathrm{Br}(\overline{X}_c) = 0$, *hence* $\mathrm{Br}(X_c) = \mathrm{Br}_1(X_c)$.
(ii) $\mathrm{Pic}(\overline{X}_c)$ *is a flasque Γ-module.*
(iii) *For any procyclic subgroup $C \subset \Gamma$ we have* $\mathrm{H}^1(C, \mathrm{Pic}(\overline{X}_c)) = 0$.
(iv) *There is an exact sequence*

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_c) \longrightarrow Ш^1_\omega(\Gamma, \mathrm{Pic}(\overline{X}_c)) \longrightarrow \mathrm{H}^3(k, \bar{k}^*).$$

(v) *If $X(k) \neq \emptyset$, then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_c) \longrightarrow Ш^1_\omega(\Gamma, \mathrm{Pic}(\overline{X}_c)) \longrightarrow 0.$$

The group $\mathrm{Br}(X_c)$, up to isomorphism, does not depend on the choice of X_c. Indeed, by Proposition 6.2.7, $\mathrm{Br}(X_c)$ is isomorphic to the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k(X)/k)$.

A connected linear algebraic group over an algebraically closed field is a rational variety, see [Bor91, IV.14.14]. This implies that $\overline{X}_c$ is unirational. That the group $\mathrm{Pic}(\overline{X}_c)$ is free and finitely generated is a general property of smooth, projective, geometrically unirational varieties over a field of characteristic zero.

The key statements are (i) and (ii). For simplicity assume that $X(k) \neq \emptyset$. Then $X = G/H$, where G and H are connected linear algebraic groups over k. The vanishing of $\mathrm{Br}(\overline{X}_c) = 0$, which is a generalisation of Bogomolov's theorem for G semisimple and simply connected, is Theorem 9.3.6. In this generality it is due to Borovoi, Demarche and Harari [BDH13]. Once $\mathrm{Br}(\overline{X}_c) = 0$ has been established, the isomorphism $\mathrm{Br}(X_c)/\mathrm{Br}(k) \cong \mathrm{H}^1(\Gamma, \mathrm{Pic}(\overline{X}_c))$ follows from Proposition 5.4.2.

Statement (ii), which is a generalisation of results of Voskresenskiĭ [Vos98] and Colliot-Thélène and Sansuc [CTS77], was established by Colliot-Thélène and Kunyavskiĭ [CTK06, Thm. 5.1].

Statement (ii) implies (iii) for purely algebraic reasons (the duality for Tate cohomology of a finite group acting on a finitely generated free abelian group and the periodicity of cohomology of a finite cyclic group). From (i) and (iii) one immediately gets (iv), which implies (v).

Lemma 9.4.4 *Let k be a field of characteristic zero with an algebraic closure $\bar{k}$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let W be a smooth projective variety over k such that $\mathrm{Pic}(\overline{W})$ is a finitely generated free abelian group and $\mathrm{H}^1(C, \mathrm{Pic}(\overline{W})) = 0$ for all procyclic subgroups $C \subset \Gamma$. If there exists a cyclic field extension K/k such that $\mathrm{Pic}(W_K) = \mathrm{Pic}(\overline{W})$, then $\mathrm{H}^1(k, \mathrm{Pic}(\overline{W})) = 0$ and the Γ-module $\mathrm{Pic}(\overline{W})$ is a direct summand of a permutation Γ-module.*

Proof. Let $K \subset \bar{k}$ be a cyclic extension of k as in the statement, so that $G = \mathrm{Gal}(K/k)$ is cyclic. Let $M = \mathrm{Pic}(\overline{W})$. The group $\mathrm{Gal}(\bar{k}/K)$ acts trivially on the finitely generated torsion-free abelian group M, hence $\mathrm{H}^1(K, M) = 0$. The restriction-inflation sequence gives $\mathrm{H}^1(G, M) \cong \mathrm{H}^1(K/k, M) \cong \mathrm{H}^1(k, M)$. The map $\Gamma \to G$ is surjective, so we can find an element $g \in \Gamma$ whose image generates G. Let $E = \bar{k}^g$ be the fixed field of g. The field extensions K/k and E/k are linearly disjoint. In particular, $\mathrm{H}^1(K/k, M) \cong \mathrm{H}^1(KE/E, M)$. We have $\mathrm{H}^1(KE, M) = 0$, so the restriction-inflation sequence gives $\mathrm{H}^1(KE/E, M) \cong \mathrm{H}^1(E, M)$, and the latter group is trivial by assumption. We thus get $\mathrm{H}^1(K/k, M) = 0$ and then $\mathrm{H}^1(k, M) = 0$. This remains true if k is replaced by a finite field extension. The last part of the statement is a consequence of a theorem of Endo and Miyata (cf. [CTS77, Prop. 2, p. 184]): if G is a finite cyclic group acting on a finitely generated torsion-free abelian group M such that $\mathrm{H}^1(H, M) = 0$ for all subgroups $H \subset G$, then M is a direct summand of a permutation G-module. □

Combining Proposition 6.2.12, Theorem 9.4.3 and Lemma 9.4.4, one gets the following corollary.

Corollary 9.4.5 *Let k be a field of characteristic zero. Let X be a smooth, projective, geometrically integral variety over k with a k-point. Assume that X is stably k-birationally equivalent to a homogeneous space of a connected linear algebraic group such that the stabilisers of geometric points are connected. If there exists a finite cyclic extension K/k such that $\mathrm{Pic}(X_K) = \mathrm{Pic}(\overline{X})$, then the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is an isomorphism and the Γ-module $\mathrm{Pic}(\overline{X})$ is a direct summand of a permutation Γ-module.*

Example 9.4.6 A Châtelet surface Y given by the affine equation

$$y^2 - az^2 = (x - e_1)(x - e_2)(x - e_3),$$

where $a \in k \smallsetminus k^{*2}$ and $e_i \neq e_j$ for $i \neq j$, admits a smooth compactification Y_c such that $\mathrm{Pic}(Y_{c,K}) \cong \mathrm{Pic}(\overline{Y}_c)$, where $K = k(\sqrt{a})$. However, $\mathrm{Br}(Y_c)/\mathrm{Br}(k) \cong (\mathbb{Z}/2)^2$ (see Exercise 11.3.7). Corollary 9.4.5 then shows that such a Châtelet surface is not stably k-birationally equivalent to any homogeneous space of a connected linear group with connected geometric stabilisers.

One would like to have a formula for $Ш^1_\omega(\Gamma, \mathrm{Pic}(\overline{X}_c))$ in terms of the homogeneous space X and not in terms of its smooth compactification X_c. Let G be a connected linear algebraic group over a field k of characteristic zero and let X be a homogeneous space of G over k. Let $\overline{H} \subset \overline{G}$ be the stabiliser of a $\bar{k}$-point of X. *Assume that $\overline{H}$ is an extension of a group of multiplicative type $\overline{S}$ by a connected linear algebraic group with trivial group of characters.* Then there is a natural group k-scheme S of multiplicative type such that $\overline{S} = S \times_k \bar{k}$, see [Bor93, 1.7 and 6.1], [CTK06, p. 739], [BDH13, p. 673]. Let T be the torus over k which is the maximal toric quotient of G. Then there is an induced homomorphism $S \to T$ defined over k. Let $[\widehat{T} \to \widehat{S}]$

be the dual map of respective groups of characters, viewed as a complex of Γ-modules in degrees -1 and 0.

Theorem 9.4.7 ([BDH13, Thm. 8.1, Cor. 8.3]) *With notation as above assume that* $\mathrm{Pic}(\overline{G})=0$. *Then there is an exact sequence*

$$0\to\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)\longrightarrow Ш^1_\omega(k,[\widehat{T}\to\widehat{S}])\longrightarrow\mathrm{Ker}[\mathrm{H}^3(k,\bar{k}^*)\to\mathrm{H}^3_{\text{ét}}(X_c,\mathbb{G}_m)].$$

If $\overline{H}$ *is connected, then* S *is a torus and we have the same sequence with* $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)$ *replaced by* $\mathrm{Br}(X_c)/\mathrm{Br}_0(X_c)$.

Let us mention some special cases, some of which are used in the proof of the general result.

- $G=T$ *is a torus and* $\overline{H}=1$. Here $S=1$, and
$$Ш^1_\omega([\widehat{T}\to\widehat{S}])=Ш^1_\omega([\widehat{T}\to 0])=Ш^2_\omega(k,\widehat{T}).$$
Under the assumption $X(k)\neq\emptyset$, i.e., $X\simeq T$, the result in this case appeared in [CTS87b]. The proof uses the theorem of Endo and Miyata mentioned above: for any finite cyclic group G any H^1-trivial G-lattice is a direct summand of a permutation G-lattice (cf. [CTS77, Prop. 2]).
- G *is a simply connected semisimple group,* $\mu\subset G$ *is a finite central subgroup and* $X=G/\mu$. Here $T=1$, $S=\mu$, so
$$Ш^1_\omega([\widehat{T}\to\widehat{S}])=Ш^1_\omega([0\to\hat{\mu}])=Ш^1_\omega(k,\hat{\mu}),$$
where $\hat{\mu}=\mathrm{Hom}_{k\text{-groups}}(\mu,\mathbb{G}_{m,k})$. The result in this case was obtained in [CTK98]. The proof relies on a reduction to the case of a finite ground field k together with the above mentioned theorem on tori.
- G *is a simply connected semisimple group and* $\overline{H}$ *is connected.* Here $T=1$ and we have
$$Ш^1_\omega([\widehat{T}\to\widehat{S}])=Ш^1_\omega([0\to\widehat{S}])=Ш^1_\omega(k,\widehat{S}).$$
Under the assumption $X(k)\neq\emptyset$, the result in this case appeared in [CTK06] where Theorem 9.2.2 was used.
- $G=\mathrm{GL}_{n,k}$ *and* $H\subset G$ *is semisimple.* In this case
$$Ш^1_\omega([\widehat{T}\to\widehat{S}])=Ш^1_\omega([\mathbb{Z}\to 0])=Ш^2_\omega(k,\mathbb{Z})=0.$$

The proof of $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)=Ш^2_\omega(k,\widehat{T})$, where X_c is a smooth compactification of a torus T, is done directly at the level of the field k. The proofs of most other computations of $\mathrm{Br}_1(X_c)/\mathrm{Br}_0(X_c)=Ш^1_\omega(\Gamma,\mathrm{Pic}(\overline{X}_c))$ rely on various reductions involving change of the ground field k. Let us mention some of them, without going into details.

If X is a homogeneous space of a semisimple group G, it is helpful to reduce to the case when G is quasi-split, that is, G contains a Borel subgroup B.

Indeed, in this case the maximal torus of B is a quasi-trivial torus, that is, a product of tori of the form $R_{k'/k}(\mathbb{G}_{m,k'})$, where k' is finite separable extension of k. This implies that G is a rational variety over k. To reduce to this situation one extends the ground field k to the function field K of the variety of Borel subgroups of G. One then uses the fact that the map $\mathrm{Pic}(X_c \times_k k_{\mathrm{s}}) \to \mathrm{Pic}(X_c \times_k K_{\mathrm{s}})$ is an isomorphism, see Proposition 6.2.13.

Another way to reduce to the case when G is quasi-split is first to reduce to the case when k is the fraction field of a finitely generated $\mathbb{Z}$-algebra over which G, X, X_c can be extended, and then use Chebotarev's density theorem to reduce the whole situation to the case of a finite field where the Galois action is preserved. See [CTK98] for details.

One also uses algebraic and arithmetic results from the theory of connected linear algebraic groups: a semisimple algebraic group over a finite field is quasi-split; a quasi-split semisimple group over a field k is birationally equivalent to the product of an affine space and a torus. One also uses Theorem 9.2.2.

The above theorems do not cover the case of quotients $\mathrm{GL}_{n,k}/G$ where G is a non-commutative finite subgroup subscheme of $\mathrm{GL}_{n,k}$. Such an extension of Theorem 9.3.2 to more general ground fields is given in [CT12a] for constant G and in [LA17] for more general G. The case when G is constant and $k = \mathbb{Q}$ is of interest in connection with the inverse Galois problem [Har07a, Dem10, HW20]. For further work on unramified Brauer groups of quotients, see [Dem10] and [LA14, LA15, LA17].

Exercise 9.4.8 [CTS77, Prop. 7] Let K/k be a finite Galois extension of fields. Let $T = R^1_{K/k}(\mathbb{G}_{m,K})$ be the norm 1 torus, that is, the kernel of the norm map $R_{K/k}(\mathbb{G}_{m,K}) \to \mathbb{G}_{m,k}$. Show that $\mathrm{Br}_{\mathrm{nr}}(k(T)/k) \cong \mathrm{H}^3(\mathrm{Gal}(K/k), \mathbb{Z})$. If $\mathrm{Gal}(K/k) \cong (\mathbb{Z}/p)^2$, where p is a prime, show that $\mathrm{Br}_{\mathrm{nr}}(k(T)/k) \cong \mathbb{Z}/p$. Thus T is not k-rational. This example of a non-k-rational linear algebraic group was first given by C. Chevalley (with a different proof).

Chapter 10
Schemes over local rings and fields

The object of study in this chapter is a scheme over the spectrum of a local ring. A separately standing Section 10.1 is devoted to the concepts of a split variety and of a split fibre of a morphism of varieties; for arithmetic applications and for the calculation of the Brauer group, split fibres should be considered as 'good' or 'non-degenerate'. In Section 10.2 we look at the classical case of quadrics over a discrete valuation ring.

In the ensuing sections the local ring is henselian or complete. In Section 10.3 we consider regular, integral, proper schemes of relative dimension 1 over a henselian discrete valuation ring. The study of the Brauer group of such schemes goes back to Artin and Grothendieck [Gro68, III, §3]. We also discuss the parallel situation of proper regular desingularisations of a 2-dimensional henselian local ring, already considered in [Art87]. This leads to local-to-global theorems for the Brauer group of the function field. It also leads to comparison of index and exponent of a central simple algebra of the function field of such schemes under suitable assumptions on the residue field of the local ring, as initiated by Artin and by Saltman. In Section 10.4 we analyse the Brauer group of the generic fibre of a smooth proper scheme over a henselian discrete valuation ring. In Section 10.5 we discuss various properties of the Brauer group of a variety over a local field with respect to evaluation at rational and closed points.

10.1 Split varieties and split fibres

10.1.1 Split varieties

Recall that $\bar{k}$ is an algebraic closure of k, and $k_{\rm s}$ is a separable closure of k in $\bar{k}$. For a k-scheme X we write $X^{\rm s} = X \times_k k_{\rm s}$ and $\overline{X} = X \times_k \bar{k}$.

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We follow the convention that an integral scheme is by definition non-empty, see [Stacks, Def. 01OK]. For an integral scheme X over a field k we call the residue field of the generic point $\eta \in X$ the function field of X and denote it by $k(X)$. We write k_X for the algebraic closure of k in $k(X)$.

One can give criteria for an integral scheme X over a field k to be geometrically reduced or geometrically irreducible in terms of the extension of fields $k \subset k(X)$. We refer to Section 2.6 for the definition of a separable field extension. The integral k-scheme X is geometrically reduced if and only if $k(X)$ is a separable k-algebra [EGA, IV_2, Prop. 4.6.1 (e)]. Next, X is geometrically irreducible if and only if k is separably closed in $k(X)$, that is, the only separable algebraic field extension of k in $k(X)$ is k itself [EGA, IV_2, Prop. 4.5.9 (c)]. See also [Po18, §2.2].

Recall our standard convention that a variety over a field k is a separated scheme of finite type over k.

Definition 10.1.1 *Let X be an irreducible variety over a field k. The* **multiplicity** *of X is the length of the (artinian) local ring of X at the generic point η of X. The* **geometric multiplicity** *of X is the length of the (artinian) local ring of $\overline{X}$ at a point $\bar{\eta}$ of $\overline{X}$ over η.*

The definition of geometric multiplicity does not depend on the choice of $\bar{\eta}$ because such points are conjugate under the action of $\mathrm{Aut}(\bar{k}/k)$.

The multiplicity of an irreducible k-variety X is 1 if and only if X contains a non-empty open reduced k-subscheme. The geometric multiplicity of X is 1 if and only if X contains a non-empty open geometrically reduced k-subscheme. By the above criterion, this is equivalent to the property: X contains an integral open k-subscheme U such that $k(U)$ is a separable k-algebra. Equivalently, X contains a dense open smooth k-subscheme, see [Stacks, Lemma 056V]. The multiplicity divides the geometric multiplicity; the ratio is the geometric multiplicity of the reduced subscheme X_{red} [BLR90, §9.1, Lemma 4 (a)]. It is a power of the characteristic exponent of k [BLR90, §9.1, Lemma 4 (c)].

Lemma 10.1.2 *If Y is a normal integral scheme over a field k, then the structure morphism $Y \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k_Y)$. If $X \to Y$ is a morphism of integral schemes over k, and Y is normal, then there is a natural embedding $k_Y \subset k_X$.*

Proof. Let $y \in Y$ be a point and let $\mathcal{O}_{Y,y}$ be the local ring of Y at y. Since Y is normal, $\mathcal{O}_{Y,y}$ is integrally closed in the function field $k(Y)$. Thus the inclusions $k \subset k_Y \subset k(Y)$ induce inclusions $k \subset k_Y \subset \mathcal{O}_{Y,y}$. It follows that k_Y is contained in $\mathrm{H}^0(Y, \mathcal{O}_Y)$, so that the structure morphism $Y \to \mathrm{Spec}(k)$ factors through $\mathrm{Spec}(k_Y)$. The structure morphism $X \to \mathrm{Spec}(k)$ factors through $Y \to \mathrm{Spec}(k)$, hence also through $\mathrm{Spec}(k_Y)$, thus $k_Y \subset k(X)$. □

The following definition was introduced in [Sko96].

Definition 10.1.3 *A variety over a field k is* **split** *if it contains an open geometrically integral k-subscheme.*

Proposition 10.1.4 *Let X be a variety over a field k. The following properties are equivalent.*

(i) *X is split;*
(ii) *X contains an open integral k-subscheme U such that $k_U = k$ which is geometrically reduced;*
(iii) *X contains an open integral k-subscheme of geometric multiplicity 1 which is geometrically irreducible;*
(iv) *X contains an open k-subscheme which is smooth and geometrically irreducible.*

Proof. Let us show that (i) implies (ii). Let $U \subset X$ be an open geometrically integral subscheme. Then $k(U)$ is separable over k, so k_U is also separable over k. But k is separably closed in $k(U)$, thus $k_U = k$.

Conversely, $k_U = k$ implies that k is separably closed in $k(X)$, so X is geometrically irreducible. Thus (ii) implies (i).

An open integral subscheme $U \subset X$ has geometric multiplicity 1 if and only if it contains a dense open subscheme which is geometrically reduced, so (i) and (iii) are equivalent. This happens precisely when U contains a dense open smooth subscheme, so (iii) implies (iv). Any smooth scheme over k is geometrically reduced [Stacks, Lemma 056T], hence (iv) implies (i). □

Lemma 10.1.5 *A variety over a field k which contains a smooth k-point is split.*

Proof. Let X be a variety over k with a smooth k-point P. Then there exists a smooth irreducible Zariski open set $U \subset X$ which contains P. In particular, U is geometrically reduced. Lemma 10.1.2 applied to the morphism $P\colon \mathrm{Spec}(k)\to U$ gives $k_U = k$. We conclude by invoking Proposition 10.1.4. □

Definition 10.1.6 *A variety Z over a field k is* **geometrically split** *if the k_{s}-scheme $Z^{\mathrm{s}} = Z \times_k k_{\mathrm{s}}$ is split.*

Equivalently, a variety over a field is geometrically split if it contains a non-empty smooth open subscheme. In particular, a variety over a field is geometrically split if and only if it contains a smooth closed point.

Remark 10.1.7 The notions of a split variety has its origin in arithmetical considerations.

(1) Let $X \subset \mathbb{P}^n_{\mathbb{F}}$ be a quasi-projective, integral, split variety over a finite field $\mathbb{F}$. Then there is an integer N which depends only on the Hilbert polynomial of X such that for any finite field extension $\mathbb{F} \subset E$ with cardinality of E at least equal to N, the variety X has a smooth E-point. The key ingredient here is the Lang–Weil–Nisnevich inequality [Po18, Thm. 7.7.1] for the number of rational points of a variety over a finite field.

(2) Let k be a number field. If X is a split variety over k, then for almost all places v of k the variety X has a smooth k_v-point. This follows from a spreading out argument, Hensel's lemma and the previous result for finite fields.

10.1.2 Split fibres

Proposition 10.1.8 *Let R be a regular local ring with fraction field K, maximal ideal $\mathfrak{m}$ and residue field k. Let $f\colon X\to\mathrm{Spec}(R)$ be a flat R-scheme of finite type such that every point of the closed fibre is a regular point of X. Let $i\colon R\hookrightarrow R'$ be a flat extension of local rings such that $\mathfrak{m}$ generates the maximal ideal $\mathfrak{m}'\subset R'$ and the residue field k' of R' is a separable extension of k (not necessarily algebraic).*

Then for any morphism $\sigma\colon \mathrm{Spec}(R')\to X$ such that $f\sigma=\mathrm{Spec}(i)$ the point $\sigma(\mathrm{Spec}(k'))$ lies in the smooth locus of the closed fibre $X_k=X\times_R k$.

Proof. Let A be the local ring of X at $P=\sigma(\mathrm{Spec}(k'))$ and let $B=A/\mathfrak{m}A$ be the local ring of the closed fibre X_k at P. Let $\mathfrak{m}_A$ be the maximal ideal of A and let $\mathfrak{m}_B$ be the maximal ideal of B so that $A/\mathfrak{m}_A=B/\mathfrak{m}_B=k(P)$ is the residue field of P. The exact sequence

$$0\longrightarrow \mathfrak{m}A\longrightarrow A\longrightarrow B\longrightarrow 0$$

gives rise to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}A & \longrightarrow & \mathfrak{m}_A & \longrightarrow & \mathfrak{m}_B & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathfrak{m}A\cap\mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_B^2 & \longrightarrow & 0
\end{array}$$

Using $\mathfrak{m}^2A\subset\mathfrak{m}A\cap\mathfrak{m}_A^2$ we see that the upper row of the following diagram is exact:

$$\begin{array}{ccccccc}
\mathfrak{m}A/\mathfrak{m}^2A & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & 0\\
\downarrow{\scriptstyle\sigma^*} & & \downarrow{\scriptstyle\sigma^*} & & & & \\
\mathfrak{m}R'/\mathfrak{m}^2R' & \xrightarrow{\cong} & \mathfrak{m}'/\mathfrak{m}'^2 & & & &
\end{array}\qquad(10.1)$$

The isomorphism in the lower row is due to the assumption $\mathfrak{m}R'=\mathfrak{m}'$. The vertical maps are induced by the homomorphism of local rings $\sigma^*\colon A\to R'$, so that the diagram is commutative.

Since X is flat over R, the ring A a flat R-module. Tensoring $\mathfrak{m}^n\hookrightarrow R$ with A we obtain that for any $n\geq 1$ the natural map $A\otimes_R\mathfrak{m}^n\to\mathfrak{m}^nA$ is an isomorphism. Tensoring the exact sequence of R-modules

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow \mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0$$

with A we obtain isomorphisms

$$\mathfrak{m}A/\mathfrak{m}^2A \xrightarrow{\sim} A \otimes_R (\mathfrak{m}/\mathfrak{m}^2) \cong B \otimes_k (\mathfrak{m}/\mathfrak{m}^2). \tag{10.2}$$

The local ring homomorphism $i\colon R\to R'$ makes R' a flat R-module, so the same arguments give an isomorphism

$$\mathfrak{m}R'/\mathfrak{m}^2R' \xrightarrow{\sim} k' \otimes_k (\mathfrak{m}/\mathfrak{m}^2). \tag{10.3}$$

Using (10.2) and (10.3) we rewrite (10.1) as follows:

$$\begin{array}{ccccccc} B \otimes_k (\mathfrak{m}/\mathfrak{m}^2) & \longrightarrow & \mathfrak{m}_A/\mathfrak{m}_A^2 & \longrightarrow & \mathfrak{m}_B/\mathfrak{m}_B^2 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \sigma^*} & & \downarrow{\scriptstyle \sigma^*} & & & & \\ k' \otimes_k (\mathfrak{m}/\mathfrak{m}^2) & \xrightarrow{\cong} & \mathfrak{m}'/\mathfrak{m}'^2 & & & & \end{array} \tag{10.4}$$

In view of $\mathfrak{m}_A\mathfrak{m} \subset \mathfrak{m}_A^2$ the first arrow in the top row of (10.4) factors through $k(P)\otimes_k(\mathfrak{m}/\mathfrak{m}^2)$. The resulting map $k(P)\otimes_k(\mathfrak{m}/\mathfrak{m}^2)\to k'\otimes_k(\mathfrak{m}/\mathfrak{m}^2)$ is injective, because it is induced by the embedding $k(P) \hookrightarrow k'$. From commutativity of (10.4) we deduce that the upper row of that diagram gives rise to a short exact sequence of $k(P)$-vector spaces

$$0 \longrightarrow k(P) \otimes_k \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow 0. \tag{10.5}$$

Since R and A are regular local rings, we have $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(R)$ and $\dim_{k(P)}(\mathfrak{m}_A/\mathfrak{m}_A^2) = \dim(A)$. Since X is flat over R, by [Liu10, Prop. 4.3.12] we have $\dim(B) = \dim(A) - \dim(R)$. In view of (10.5), this equals $\dim_{k(P)}(\mathfrak{m}_B/\mathfrak{m}_B^2)$, hence B is regular.

Let V be a regular open neighbourhood of P in the closed fibre X_k. The residue field $k(P)$ is a subfield of k', which is separable over k, hence $k(P)$ is separable over k. It is finitely generated over k. By one of the equivalent properties of separable field extensions given in Section 2.6, there exist a smooth integral k-variety U and a birational k-morphism $\phi\colon U\to V$ sending the generic point of U to P. Because U is smooth, it contains a closed point M whose residue field is a finite separable extension of k. The residue field of the image $\phi(M)$ in the regular k-variety V has the same property. By [Liu10, Prop. 4.3.30], the point $\phi(M)$ is in the smooth locus of V. Since $\phi(M)$ is a specialisation of P, we conclude that P lies in the smooth locus of V. $\square$

Corollary 10.1.9 *Let R be a regular local ring with residue field k. Let X be a regular scheme which is a flat R-scheme of finite type. If the morphism $X\to\mathrm{Spec}(R)$ has a section, then this section meets the closed fibre X_k in a smooth k-point. Hence X_k is a split k-variety.*

Proof. Taking $R' = R$ in Proposition 10.1.8 we obtain a smooth k-point P in the closed fibre X_k. The last statement now follows from Lemma 10.1.5. □

Corollary 10.1.10 *Let $f\colon X\to Y$ be a flat morphism of regular varieties over a field k. Let P be a point of Y. The fibre X_P is geometrically split if and only if f has a section locally at P for the étale topology, i.e., the morphism $X\times_Y \mathrm{Spec}(R)\to\mathrm{Spec}(R)$ has a section, where R is the strict henselisation of the local ring of Y at P.*

Proof. Let $X' = X\times_Y R$ and let X'_0 be the closed fibre of X'/R. It is enough to show that X'_0 is split if and only if X'/R has a section.

The Y-scheme $\mathrm{Spec}(R)$ is the direct limit of étale schemes V/Y, thus X' is the limit of schemes $V\times_Y X$. But $V\times_Y X$ is étale over a regular scheme X, hence X' is regular. Now R is a regular local ring and X' is regular, so if X'/R has a section, then X'_0 is split by Corollary 10.1.9.

Conversely, since X'_0 is split over a separably closed field, Proposition 10.1.4 (iv) implies that X'_0 has a smooth rational point P. By assumption the morphism $X\to Y$ is flat, so X' is a flat R-scheme. Hence the morphism $X'\to\mathrm{Spec}(R)$ is smooth in a neighbourhood of P. Since R is henselian, P can be lifted to a section of X'/R [EGA, IV_4, Thm. 18.5.17]. □

In the case of a regular integral scheme over a discrete valuation ring, the multiplicity of an irreducible component of the closed fibre has a clear geometric meaning.

Lemma 10.1.11 *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$ and residue field $k = R/\mathfrak{m}$. Let X be a regular integral scheme with a faithfully flat morphism $f\colon X\to\mathrm{Spec}(R)$. Then the (non-empty) closed fibre X_t is the principal divisor*

$$(\pi) = \sum_{i=1}^{n} m_i C_i \ \in\ \mathrm{Div}(X),$$

where $C_1,\ldots,C_n$ are the (reduced) irreducible components of X_k, and m_i is the multiplicity of C_i, for $i = 1,\ldots,n$.

Proof. Since f is faithfully flat, X_k is non-empty, and each C_i is a Weil divisor on X. Since X is regular, each C_i is a Cartier divisor and the local ring $\mathcal{O}_{X,C_i}$ of X at the generic point of C_i is a discrete valuation ring. The local ring of X_k at the generic point of C_i is $\mathcal{O}_{X,C_i}/\pi\mathcal{O}_{X,C_i}$, which by assumption is a local Artinian ring of length m_i. Hence the valuation of π is m_i. Thus the Cartier divisors $X_k = (\pi)$ and $\sum_{i=1}^n m_i C_i$ coincide at codimension 1 points of X; this implies that they coincide as Cartier divisors on X. □

Proposition 10.1.12 *Let R be a discrete valuation ring with field of fractions K, maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Let Y and Y' be regular, integral and flat R-schemes of finite type, with smooth generic fibres Y_K and Y'_K. Assume that Y' is a proper R-scheme. If there is a rational map from Y_K to Y'_K, then for any irreducible component $C \subset Y_k$ of geometric multiplicity 1 there exist an irreducible component $C' \subset Y'_k$ of geometric multiplicity 1 and an embedding $k_{C'} \subset k_C$. In particular, if Y_k is split, then Y'_k is split too.*

Proof. The generic fibre Y_K is an integral variety over K. Write $F = K(Y_K)$. Let $\mathcal{O}_C$ be the local ring of Y at the generic point of C. Since Y is integral, the field of fractions of $\mathcal{O}_C$ is F; the residue field of $\mathcal{O}_C$ is $k(C)$. Since Y is regular, $\mathcal{O}_C$ is a discrete valuation ring. The geometric multiplicity of C is 1, hence $k(C)$ is a separable extension of k.

A rational map from Y_K to Y'_K can be thought of as an F-point of Y'_K. By the valuative criterion of properness, it uniquely extends to a morphism $\mathrm{Spec}(\mathcal{O}_C)\to Y'$. Let $P \in Y'_k$ be the image of the closed point of $\mathrm{Spec}(\mathcal{O}_C)$. Since $k(C)$ is separable over k, by Proposition 10.1.8, P is a smooth point of Y'_k, hence P belongs to (the smooth locus of) a unique irreducible component C' of Y'_k. Since C' contains a smooth point, the geometric multiplicity of C' is 1. A morphism of integral k-schemes $P\colon \mathrm{Spec}(k(C))\to C'_{\mathrm{smooth}}$ gives rise to an embedding $k_{C'} \subset k_C$ by Lemma 10.1.2. □

Corollary 10.1.13 *Let R be a discrete valuation ring with field of fractions K and residue field k. Let X be a regular, integral, proper and flat R-scheme with smooth generic fibre. Let Σ_X be the (possibly, empty) partially ordered set of irreducible components of geometric multiplicity 1 of X_k, where C dominates D if there exists an embedding of k_D into k_C. The set of isomorphism classes of finite separable field extensions $k \subset k_C$, where C is a minimal element of Σ_X, is a birational invariant of the generic fibre X_K as a smooth, integral, proper variety over K. In particular, the property of the closed fibre X_k to be split is a birational invariant of X_K.*

Proof. Suppose that X and Y are regular, integral, proper and flat R-schemes, with smooth generic fibres, such that $K(X_K) \cong K(Y_K)$. Define the partially ordered set Σ_Y in the same way as Σ_X. Let C be a minimal element of Σ_X. By Proposition 10.1.12 there exists a $C' \in \Sigma_Y$ such that $k_{C'}$ can be embedded into k_C. By the same proposition, there is a $C'' \in \Sigma_X$ such that $k_{C''}$ can be embedded into $k_{C'}$. By minimality of C we have $k_{C''} \simeq k_C$, hence $k_C \simeq k_{C'}$. Since C is minimal in Σ_X, then, by Proposition 10.1.12, C' is minimal in Σ_Y. The last statement then follows from the fact that X_k is split if and only if there is a $C \in \Sigma_X$ such that $k_C = k$, see Proposition 10.1.4. □

In some concrete cases, for example when the generic fibre is a quadric and the residue field is of characteristic different from 2, it is not difficult to determine this set of finite separable extensions.

One can give a criterion for the closed fibre to be split in terms of the generic fibre [Sko96, Lemma 1.1].

Theorem 10.1.14 *Let R be a discrete valuation ring with field of fractions K and residue field k. Let X be a regular, integral, proper and flat R-scheme with smooth generic fibre. Then the closed fibre X_k is split if and only if there exists a flat local homomorphism of discrete valuation rings $i\colon R\to R'$ satisfying the following properties, where k' is the residue field of R' and K' is the fraction field of R':*

(a) *k' is a separable k-algebra, and k is algebraically closed in k';*
(b) *the maximal ideal of R generates the maximal ideal of R';*
(c) *the generic fibre X_K has a K'-point.*

Following Bourbaki, an extension k'/k satisfying the conditions in (a) is called *regular* [BouV, §17, no. 4, Déf. 2].

Proof of Theorem 10.1.14. Assume that X_k is split, so that X_k contains an open geometrically integral subscheme U. Let R' be the local ring of X at the generic point of U. Since X is regular, R' is a discrete valuation ring. It is clear that the residue field of R' is $k(U)$ and the fraction field is $k(X)$. Since U is geometrically integral over k, the field $k(U)$ is a separable extension of k in which k is algebraically closed, so (a) is satisfied. The multiplicity of U is 1, so Lemma 10.1.11 shows that the maximal ideal of R' is generated by the maximal ideal of R, which is (b). Finally, the generic point of X_K is a K'-point, so (c) holds as well.

To prove the converse, let $i\colon R\to R'$ be as in the statement of the theorem. By the valuative criterion of properness, the given K'-point of X_K extends to an R-morphism $\phi\colon \mathrm{Spec}(R')\to X$. Since the field extension $k\subset k'$ is separable, by Proposition 10.1.8 the morphism ϕ factors through the smooth locus X_{smooth} of X/R. Let $P=\phi(\mathrm{Spec}(k'))$ be the image of the closed point of $\mathrm{Spec}(R')$ in $X_{\mathrm{smooth}}\cap X_k$. It follows that X_k contains an open irreducible smooth subscheme U such that $P\in U$.

Let us show that U is geometrically integral. Since U is smooth over k, it is geometrically reduced. By Lemma 10.1.2 applied to the morphism of k-schemes $\phi\colon \mathrm{Spec}(k')\to U$, the field k_U is a subfield of the algebraic closure of k in k'. But k is algebraically closed in k' by assumption, hence $k_U=k$, so U is geometrically irreducible. □

As an example of application of this theorem let us prove the following statement which is essentially [CTK06, Thm. 4.2].

Proposition 10.1.15 *Let k be a field of characteristic zero. Let $f\colon X\to Y$ be a proper dominant morphism of smooth and geometrically integral varieties over k. Assume that the generic fibre X_η is birationally equivalent to a $k(Y)$-torsor for a simply connected semisimple group over $k(Y)$. Then for any point $y\in Y$ of codimension 1, the fibre X_y is split.*

Proof. Write $\kappa=k(y)$. The local ring $\mathcal{O}_y$ of Y at y is a discrete valuation ring with residue field κ. Let $\pi\in\mathcal{O}_y$ be a uniformiser. Since k has characteristic zero, there is an isomorphism of discrete valuation rings and of k-algebras between the completion of $\mathcal{O}_y$ and the discrete valuation ring $\kappa[[t]]$, sending π to t. Let $\kappa \subset L$ be a field extension as in Lemma 7.1.9. As $\mathrm{cd}(L) \leq 1$, by Theorem 9.2.2 (Bruhat–Tits) any torsor for a simply connected semisimple group over $L((t))$ has an $L((t))$-point. The generic fibre X_η of the morphism $f\colon X \to Y$ is a proper variety over $k(\eta) = k(Y)$ birationally equivalent to such a torsor. By the lemma of Lang and Nishimura [CTCS80, Lemme 3.1.1], [Po18, Thm. 3.6.4], X_η has an $L((t))$-point. The local extension of discrete valuation rings $\mathcal{O}_y \subset L[[t]]$ satisfies the conditions of Theorem 10.1.14, so by this theorem the fibre X_y is split. □

Remark 10.1.16 Using the result mentioned in Remark 7.1.10, a similar argument gives the following statement [CT08b, Thm. 2.2]: *Let k be a field of characteristic zero. Let $f\colon X \to Y$ be a proper dominant morphism of smooth and geometrically integral varieties over k. Assume that the generic fibre X_η is $k(Y)$-birationally equivalent to a smooth hypersurface of degree d in $\mathbb{P}^n_{k(Y)}$, $n \geq d^2$. Then for every point $y \in Y$ of codimension 1, the fibre X_y is split.*

Let us give an example (taken from [LS18]) when one can determine if the closed fibre is split using only the information about the birational equivalence class of the generic fibre without constructing an explicit model.

Proposition 10.1.17 *Let k be a field of characteristic zero. Let $k_1, \ldots, k_n$ be finite field extensions of k, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\text{g.c.d.}\big(m_1, \ldots, m_n\big) = 1.$$

Let m be an integer and let X be the affine $k((t))$-variety defined by the equation

$$\prod_{i=1}^{n} \mathrm{N}_{k_i/k}(x_i)^{m_i} = t^m, \tag{10.6}$$

in the $k((t))$-torus $(\prod_{i=1}^n R_{k_i/k}\mathbb{G}_m) \times_k k((t))$. Let $\mathcal{X}$ be a regular connected scheme equipped with a proper morphism $\mathcal{X} \to \mathrm{Spec}(k[[t]])$ whose generic fibre is smooth, geometrically integral, and contains X as an open subscheme. Then the closed fibre $\mathcal{X}_k$ is split if and only if $r|m$, where

$$r = \text{g.c.d.}\big(m_1[k_1 : k], \ldots, m_n[k_n : k]\big).$$

Proof. Equation (10.6) with right-hand side replaced by 1 defines a k-torus. Hence X is a $k((t))$-torsor for this torus; in particular, it is geometrically integral.

If $r|m$ we can write $m = s_1 m_1[k_1 : k] + \ldots + s_n m_n[k_n : k]$ for some $s_i \in \mathbb{Z}$. Then $x_i = t^{s_i}$, for $i = 1, \ldots, n$, is a $k((t))$-point of X. By the valuative criterion of properness, it gives rise to a section of $\mathcal{X} \to \mathrm{Spec}(k[[t]])$. By Corollary 10.1.9 the closed fibre $\mathcal{X}_k$ is split.

Conversely, assume that $\mathcal{X}_k$ is split, so that $\mathcal{X}_k$ has a geometrically irreducible component C of multiplicity 1. Let $\mathcal{O}_C$ be the local ring of C in $\mathcal{X}$.

This a discrete valuation ring with field of fractions $k((t))(X)$ and residue field $k(C)$. Let $A = \widehat{\mathcal{O}}_C$ be the completion of $\mathcal{O}_C$. This is also a discrete valuation ring with residue field $k(C)$. Let K be the field of fractions of A and let $v\colon K^* \to \mathbb{Z}$ be the valuation. Then $k[[t]] \subset A$ is an unramified extension of complete discrete valuation rings, so $v(t) = 1$. In fact, A is isomorphic to $k(C)[[t]]$. Since C is geometrically irreducible, k is algebraically closed in $k(C)$, hence also in K.

The generic fibre X has a canonical $k((t))(X)$-point Q defined by the generic point of X. This point is contained in the affine open subset given by (10.6). Since $k((t))(X) \subset K$, we can think of Q as a K-point of X. Suppose that Q has coordinates (x_i), where $x_i \in K \otimes_k k_i$ for $i = 1, \ldots, n$. Since k is algebraically closed in K, the k-algebra $K_i = K \otimes_k k_i$ is a field, hence K_i is a complete discretely valued field which is an unramified extension of K of degree $[k_i : k]$. This implies that $v(\mathrm{N}_{K_i/K}(x_i)) = s_i[k_i : k]$ for some $s_i \in \mathbb{Z}$. But then (10.6) gives that $m = s_1 m_1 [k_1 : k] + \ldots + s_n m_n [k_n : k]$, so we are done. □

Split fibres naturally arise in applications to number-theoretic questions. The following general theorem [Den19] includes many classical results, in particular the Ax–Kochen–Ershov theorem.

Theorem 10.1.18 (J. Denef) *Let $f\colon X \to Y$ be a proper dominant morphism of smooth, projective, geometrically integral varieties over a number field k with geometrically integral generic fibre. Assume that for any discrete valuation ring R of the function field $k(Y)$ with $k \subset R$ the generic fibre of f has a regular proper model over* $\mathrm{Spec}(R)$ *with split closed fibre. Then for almost all places v of k the induced map $X(k_v) \to Y(k_v)$ is surjective.*

For motivation, proof and generalisations, see [CT08b] and [LSS20].

We refer to [CT11] for further discussion and applications of the type of results discussed in this section.

10.2 Quadrics over a discrete valuation ring

Varieties fibred into quadrics are a natural testing ground in at least two contexts: problems of rationality of complex varieties, and problems of existence of rational points on varieties over number fields. In both contexts, computation of the Brauer group of the total space of a family of quadrics plays an important rôle, as we shall see in Chapters 11 and 12 for varieties over $\mathbb{C}$ and in Chapter 14 for rational points. This is usually reduced to a local computation, namely the computation of the Brauer group of the total space of a regular proper scheme over a discrete valuation ring and the way it depends on the structure of the closed fibre.

In this section R is a discrete valuation ring with fraction field K. Let $\mathfrak{m} \subset R$ be the maximal ideal and let $k = R/\mathfrak{m}$ be the residue field. Any generator of $\mathfrak{m}$ is called a uniformiser. We assume that $\mathrm{char}(k) \neq 2$, hence also that $\mathrm{char}(K) \neq 2$. For $a \in R$ we denote by $\bar{a} \in k$ the reduction of a modulo $\mathfrak{m}$.

Let us recall a standard reduction. By a linear change of variables, any non-degenerate quadratic form ϕ in $n+1$ variables over the field K can be written in diagonal form $\sum_{i=0}^n a_i x_i^2$, where $a_i \in K^*$ for $i = 0, \ldots, n$. Multiplying the equation by an element of K^* and multiplying each x_i by an element of K^* does not change the isomorphism class of the smooth quadric $Q \subset \mathbb{P}^n_K$ defined by $\phi = 0$. Thus without loss of generality one can assume that ϕ is $\sum_{i=0}^n a_i x_i^2$, where $a_0 = 1$, and for $i = 1, \ldots, n$ the valuation of $a_i \in K^*$ is 0 or 1, and, moreover, at least a half of the coefficients a_i have valuation 0.

10.2.1 Conics

Lemma 10.2.1 *Any smooth conic over K has a regular model which is the closed subscheme of $\mathbb{P}^2_R$ given either by an equation*

$$x^2 - ay^2 - bz^2 = 0 \qquad \text{(type I)}$$

with $a, b \in R^*$, or by an equation

$$x^2 - ay^2 - \pi z^2 = 0, \qquad \text{(type II)}$$

where $a \in R^*$, π is a uniformiser, and $\bar{a}$ is not a square in k.

Proof. In view of the standard reduction described above, we only need to explain why a conic over K given by $x^2 - ay^2 - \pi z^2 = 0$, where $a \in R^*$ and $\bar{a}$ is a square in k^*, has a model of type (I). The equation can be rewritten as

$$x^2 - (b^2 + \pi c)y^2 - \pi z^2 = 0$$

with $b \in R^*$ and $c \in R$. This is easily transformed into

$$x^2 - (1 + \pi d)y^2 - \pi z^2 = 0$$

with $d \in R$. The linear change of variables $x = y + \pi u$ gives

$$\pi^2 u^2 + 2\pi uy - \pi dy^2 - \pi z^2 = 0.$$

Dividing by π we obtain

$$\pi u^2 + 2uy - dy^2 - z^2 = 0.$$

Since $1+\pi d \in R^*$, we can diagonalise this equation and obtain a diagonal quadratic form of type (I). □

Proposition 10.2.2 *Let* $W \to \mathrm{Spec}(R)$ *be a proper flat morphism such that* W *is regular and connected, and the generic fibre of* $W \to \mathrm{Spec}(R)$ *is a smooth conic over* K*. Then the natural map* $\mathrm{Br}(R) \to \mathrm{Br}(W)$ *is surjective.*

Proof. Let X be the generic fibre of $W \to \mathrm{Spec}(R)$ and let $\mathcal{X} \to \mathrm{Spec}(R)$ be the integral model of X given above. By a special case of Proposition 3.7.10 that only involves purity for regular 2-dimensional schemes (which has been known for some time, see [Gro68, II, Prop. 2.3]), there is an isomorphism $\mathrm{Br}(W) \simeq \mathrm{Br}(\mathcal{X})$ compatible with the maps $\mathrm{Br}(R) \to \mathrm{Br}(W)$ and $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$.

Thus we can assume that $W = \mathcal{X}$ as above. The conic X over K is a Severi–Brauer variety of dimension 1. The exact sequence (7.3) shows that the map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is surjective. Since $\mathrm{char}(K) \neq 2$, the kernel of this map is spanned by the class of the quaternion algebra $(a,b)_K$ in case (I) and $(a,\pi)_K$ in case (II).

Choose any $\beta \in \mathrm{Br}(\mathcal{X})$. Let β_X be the image of β under the injective map $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(X)$. Let $\alpha \in \mathrm{Br}(K)$ be any element mapping to β_X. Consider the exact sequence

$$0 \longrightarrow \mathrm{Br}(R) \longrightarrow \mathrm{Br}(K) \xrightarrow{\partial_R} \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$$

from Theorem 3.6.1 (i). Comparing residues on $\mathrm{Spec}(R)$ and on $\mathcal{X}$ using Theorem 3.7.5 one shows that either $\partial_R(\alpha) = 0$, or $\partial_R(\alpha)$ is the non-trivial class in $\mathrm{H}^1(k(\sqrt{\overline{a}})/k, \mathbb{Z}/2)$, and this last possibility can occur only in case (II). If $\partial_R(\alpha) = 0$, then $\alpha \in \mathrm{Br}(R)$, hence the images of α and β in $\mathrm{Br}(X)$ coincide, thus they also coincide in $\mathrm{Br}(\mathcal{X})$ since $\mathcal{X}$ is regular. If $\partial_R(\alpha) \neq 0$, we have

$$\partial_R(\alpha) = \partial_R((a,\pi)_K)$$

hence $\alpha = (a,\pi)_K + \gamma$, where $\gamma \in \mathrm{Br}(R)$. We then get

$$\beta = (a,\pi)_{K(X)} + \gamma_{K(X)} \in \mathrm{Br}(K(X)).$$

But $(a,\pi)_{K(X)} = 0$. Thus $\beta - \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(K(X))$ vanishes, hence $\beta = \gamma_{\mathcal{X}} \in \mathrm{Br}(\mathcal{X})$. The map $\mathrm{Br}(R) \to \mathrm{Br}(\mathcal{X})$ is thus surjective. This proves the statement for $\mathcal{X}$, and hence also for W. □

Corollary 10.2.3 *In the notation of Proposition 10.2.2 let* Z *be a closed subscheme of the closed fibre of* $W \to \mathrm{Spec}(R)$*. Then the image of the restriction map* $\mathrm{Br}(W) \to \mathrm{Br}(Z)$ *is contained in the image of* $\mathrm{Br}(k) \to \mathrm{Br}(Z)$*.*

10.2.2 Quadric surfaces

References for this section are [Sko90], [CTS93, §3], [CTS94, Thm. 2.3.1], [Pir18, Thm. 3.17] and [CT18].

We write a diagonal quadratic form $\sum_{i=0}^n a_i x_i^2$ over K as $\langle a_0, \ldots, a_n\rangle$.

Let q be a quadratic form of rank 4 over K. By a linear change of variables and multiplication of q by an element of K^* we can reduce q to one of the following forms:

(I) $q = \langle 1, -a, -b, abd\rangle$, where $a, b, d \in R^*$.
(II) $q = \langle 1, -a, -b, \pi\rangle$, where $a, b \in R^*$ and $\pi \in R$ is a uniformiser.
(III) $q = \langle 1, -a, -\pi, \pi b\rangle$, where $a, b \in R^*$ and $\pi \in R$ is a uniformiser.

Proposition 10.2.4 *Let $X \subset \mathbb{P}^3_K$ be a smooth quadric, defined by a diagonal quadratic form of rank 4 over K as in (I), (II) or (III) above. Let W be a proper, regular, integral scheme over R whose generic fibre is isomorphic to X. Then we have the following statements.*

In case (I) the map $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is surjective. If $d \in R$ is not a square, this map is an isomorphism. If d is a square, the kernel is spanned by the class $(a, b) \in \mathrm{Br}(R)$.

In case (II) the map $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is an isomorphism.

In case (III), if either $\bar{a}$ or $\bar{b}$ is a square in k, or if $\bar{a}\cdot\bar{b}$ is not a square in k, then $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is surjective. Any element of $\mathrm{Br}(K)$ whose image in $\mathrm{Br}(X)$ lies in $\mathrm{Br}(W)$ belongs to $\mathrm{Br}(R)$.

In case (III), if $\bar{a}\cdot\bar{b}$ is a square in k, then the image of $(a, \pi) \in \mathrm{Br}(K)$ in $\mathrm{Br}(X)$ belongs to $\mathrm{Br}(W)$ and spans the cokernel of the map $\mathrm{Br}(R)\to\mathrm{Br}(W)$.

The cokernel of $\mathrm{Br}(R)\to\mathrm{Br}(W)$ is non-zero if and only if ab is not a square in K, $\bar{a}\cdot\bar{b}$ is a square in k but neither $\bar{a}$ nor $\bar{b}$ is a square in k.

Proof. By [Har77, Prop. III.9.7], the morphism $f\colon W\to\mathrm{Spec}(R)$ is flat; since f is projective, it is also surjective. Each fibre of f has dimension 2 at every point. Let $W_0 = W\times_R k$ be the closed fibre of f. Each irreducible component D of W_0 of multiplicity $e = e_D$ gives rise to a commutative diagram of complexes

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Br}(W) & \longrightarrow & \mathrm{Br}(X) & \xrightarrow{\partial_D} & \mathrm{H}^1(k(D), \mathbb{Q}/\mathbb{Z}) \\
& \uparrow & & \uparrow & & \uparrow{\scriptstyle e\,\mathrm{res}_{k(D)/k}} \\
0 \longrightarrow & \mathrm{Br}(R) & \longrightarrow & \mathrm{Br}(K) & \xrightarrow{\partial} & \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})
\end{array}$$

In the top sequence the map $\mathrm{Br}(W)\to\mathrm{Br}(X)$ is injective since W is regular. The bottom sequence is the exact sequence given by Theorem 3.6.1 (ii). The right-hand vertical arrow is the restriction map followed by multiplication by e (by the functoriality of residues, see Theorem 3.7.5). Since X is a smooth quadric over K, the middle vertical map is surjective by Proposition 7.2.4 (a), and it is an isomorphism unless ab is a square in K, in which case the kernel is of order at most 2 spanned by the class of the quaternion algebra (a, π).

We can obtain information about the structure of W_0 by considering the closed subscheme $\mathcal{X} \subset \mathbb{P}^3_R$ given by $q = 0$ and examining its closed fibre $\mathcal{X}_0$. We note that $\mathcal{X}$ is integral and the morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is flat.

In case (I) the morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is smooth, and in case (II) the scheme $\mathcal{X}$ is regular and $\mathcal{X}_0$ is a cone over a smooth conic. In both cases $\mathcal{X}_0$ is split, hence W_0 is also split by Proposition 10.1.12. Thus we can find a geometrically irreducible component $D \subset W_0$ of multiplicity $e = 1$. Then the map $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k(D), \mathbb{Q}/\mathbb{Z})$ is injective (see Lemma 11.1.2). From the diagram we get surjectivity of $\mathrm{Br}(R) \to \mathrm{Br}(W)$ in these cases, and computation of the kernel follows from Proposition 7.2.4 (c).

Let us now consider case (III).

In case (III) the closed fibre $\mathcal{X}_0 \subset \mathbb{P}^3_k$ is given by the equation $x^2 - \bar{a}y^2 = 0$. If $\bar{a}$ is a square, this is the union of two planes intersecting along the line $x = y = 0$, in particular, $\mathcal{X}_0$ is split. If $\bar{a}$ is not a square, $\mathcal{X}_0$ is an integral scheme which splits up over $k(\sqrt{a})$ as a union of two planes. In each case the scheme $\mathcal{X}$ is singular at the points of $\mathcal{X}_0$ given by $x = y = z^2 - \bar{b}t^2 = 0$. (See [Sko90, §2].)

Let $\beta \in \mathrm{Br}(W)$. Its image under $\mathrm{Br}(W) \hookrightarrow \mathrm{Br}(X)$ is the image of some element $\alpha \in \mathrm{Br}(K)$, which is uniquely defined if ab is not a square, and is defined up to addition of (a, π) is ab is a square.

If $\bar{a}$ is a square, then intersecting one of the two irreducible components of $x^2 - \bar{a}y^2 = 0$ with the smooth locus $\mathcal{X}_{\mathrm{smooth}}$ of the morphism $\mathcal{X} \to \mathrm{Spec}(R)$ we obtain a geometrically irreducible component of multiplicity 1 of the closed fibre of $\mathcal{X}_{\mathrm{smooth}} \to \mathrm{Spec}(R)$. By Proposition 10.1.12, this implies that W_0 is split, so has a geometrically irreducible component D of multiplicity 1. By the above diagram and Lemma 11.1.2 we have $\partial(\alpha) = 0$, hence $\alpha \in \mathrm{Br}(R)$. If $\bar{b}$ is a square, we consider the quadratic form $q' = \langle 1, -b, -\pi, \pi a \rangle$ and the subscheme $\mathcal{X}' \subset \mathbb{P}^3_R$ given by $q' = 0$. Since $X \subset \mathbb{P}^3_K$ can also be given by $q' = 0$, the generic fibre of $\mathcal{X}' \to \mathrm{Spec}(R)$ is isomorphic to X. The same argument as above then gives that $\alpha \in \mathrm{Br}(R)$.

Now assume that neither $\bar{a}$ nor $\bar{b}$ is a square. Then $\mathcal{X}_0$ is the integral subscheme of $\mathbb{P}^3_k$ given by $x^2 - \bar{a}y^2 = 0$. Applying Proposition 10.1.12 we find an irreducible component $D \subset W_0$ of multiplicity 1 such that the integral closure of k in $k(D)$ is contained in $k(\sqrt{\bar{a}})$. Using Lemma 11.1.2 we see that $\partial(\alpha)$ goes to zero in $\mathrm{H}^1(k(\sqrt{\bar{a}}), \mathbb{Q}/\mathbb{Z})$, so $\partial(\alpha)$ is contained in the image of the $\mathbb{Z}/2$-module generated by the class of $\bar{a}$ in k^*/k^{*2} under the natural map $\mathrm{H}^1(k, \mathbb{Z}/2) \to \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$. Applying this argument to the model given by $q' = 0$ we obtain the same statement with $\bar{a}$ replaced by $\bar{b}$. If $\bar{a} \cdot \bar{b}$ is not a square, we conclude that $\partial(\alpha) = 0$, proving that $\alpha \in \mathrm{Br}(R)$. In all of these cases, we conclude that the map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

For the rest of the proof we assume that $\bar{a} \cdot \bar{b}$ is a square but neither $\bar{a}$ nor $\bar{b}$ is a square.

From the above argument we know that $\alpha \in \mathrm{Br}(R)$ or $\alpha + (a, \pi) \in \mathrm{Br}(R)$. Let us show that (a, π) has trivial residues on W. We actually prove the triviality of residues of (a, π) with respect to any discrete valuation v of the function field $K(X)$ of X. It is enough to consider only those v which extend the valuation of K defined by R. In $K(X)$ we have

$$x^2 - ay^2 = \pi(z^2 - b),$$

where both sides are non-zero. Thus in $\mathrm{Br}(K(X))$ we have the equality

$$(a, \pi) = (a, x^2 - ay^2) + (a, z^2 - b) = (a, z^2 - b),$$

since $(a, x^2 - ay^2) = 0$ by Proposition 1.1.8. To compute residues, we can go over to the field extension $\widehat{K} \subset \widehat{K(X)}$, where $\widehat{K}$ is the completion of K and $\widehat{K(X)}$ is the completion of $K(X)$ defined by v. We have $ab \in \widehat{K}^{*2}$, hence $ab \in \widehat{K(X)}^{*2}$. But then in $\mathrm{Br}(\widehat{K(X)})$ we have $(a, z^2 - b) = (b, z^2 - b) = 0$. Hence the residue of (a, π) at v is zero.

We now invoke purity for the Brauer group of the regular scheme W, see Theorem 3.7.6. (The earlier Theorem 3.7.2 is enough for our purposes here since we assume that 2 invertible in R and we are only concerned with the 2-torsion in the Brauer group.) We conclude that the image of $(a, \pi) \in \mathrm{Br}(K)$ in $\mathrm{Br}(X)$ belongs to $\mathrm{Br}(W) \subset \mathrm{Br}(X)$ and generates the quotient $\mathrm{Br}(W)/\mathrm{Br}(R)$.

If ab is a square in K, then the homogeneous equation of the quadric X is

$$x^2 - ay^2 - \pi z^2 + a\pi t^2 = 0.$$

Proposition 1.1.8 implies that the image of (a, π) in $\mathrm{Br}(X)$ is zero, hence the image of (a, π) in $\mathrm{Br}(W) \subset \mathrm{Br}(X)$ is zero. In this case $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is onto.

If ab is not a square in K, then $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is an isomorphism by Proposition 7.2.4 (c′). Suppose that the image of (a, π) comes from some $\alpha \in \mathrm{Br}(R)$. Then $\alpha = (a, \pi) \in \mathrm{Br}(K)$, which is absurd because $\partial((a, \pi)) \neq 0$ as $\bar{a}$ is not a square. Hence in this case $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is not surjective. □

The following statement is a stronger version of Corollary 1.4.9 in the situation considered here.

Corollary 10.2.5 *In the notation of Proposition* 10.2.4 *let Z be a closed subscheme of the closed fibre of $W \to \mathrm{Spec}(R)$. Then the image of the restriction map* $\mathrm{Br}(W) \to \mathrm{Br}(Z)$ *is contained in the image of* $\mathrm{Br}(k) \to \mathrm{Br}(Z)$.

Proof. To prove the result, we may assume that R is henselian. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(W)$ is surjective, as follows from Proposition 10.2.4. □

10.3 Schemes of dimension 2 over a henselian local ring

Let D be a central simple algebra over a field F. Let $\mathrm{ind}(D)$ be the *index* of D, that is, the square root of the dimension of the division algebra representing the class $[D] \in \mathrm{Br}(F)$. The index $\mathrm{ind}(D)$ can be also characterised as the smallest degree of a field extension of F that splits D. Let $\exp(D)$ be the *exponent* of D, that is, the order of $[D]$ in $\mathrm{Br}(F)$. The following facts were established in the 1930s by Brauer, Albert and others, see [Alb31].

- $\exp(D)$ divides $\mathrm{ind}(D)$; moreover, the primes which divide $\exp(D)$ are the same as the primes which divide $\mathrm{ind}(D)$.
- Let F be a number field or a p-adic field. Every central division algebra D over F of exponent $\exp(D) = n$ is cyclic of degree n, hence is split by a cyclic extension of F of degree n. In particular, $\mathrm{ind}(D) = \exp(D)$.
- If F is a number field and D splits over each completion of F, then D splits over F (the Albert–Brauer–Hasse–Noether theorem).
- Every central simple algebra over the function field of a curve over $\mathbb{C}$ is split (Tsen's theorem).

Such properties have applications to quadratic forms over F: the local-to-global principle for a quadratic form to be isotropic (i.e., to have the zero value on some non-zero vector) and the determination of the u-invariant of F (the maximum dimension of an anisotropic quadratic form over F).

One may wonder whether similar properties hold for other 'arithmetic fields'. Among the first examples one can think of are field extensions of $\mathbb{C}$ of transcendence degree 2. In this case the equality of index and exponent was established relatively recently by de Jong [deJ04]. One may also consider more local situations, such as function fields in one variable over $\mathbb{C}((t))$ or the purely local situation, that is, finite extensions of $\mathbb{C}((x,y))$. Further up the cohomological dimension are function fields of curves over a p-adic field. As early as 1970, Lichtenbaum [Lic69], using Tate's duality theorems for abelian varieties over a p-adic field, established a local-to-global principle in this context. Later, Saltman showed that over such a field the index divides the square of the exponent [Sal97].

We shall explain some of these results. Our starting point is the following theorem, which is a more general version of a theorem of Artin about families of curves over a henselian discrete valuation ring (written up by Grothendieck [Gro68, III, Thm. 3.1]).

Theorem 10.3.1 *Let R be a noetherian henselian local ring with residue field k. Let X be a regular scheme of dimension 2 equipped with a proper morphism $X\to\mathrm{Spec}(R)$ whose closed fibre X_0 has dimension 1. Then we have the following statements.*

(i) *The natural map $\mathrm{Br}(X)\to\mathrm{Br}(X_0)$ is an isomorphism.*
(ii) *If k is separably closed or finite, then $\mathrm{Br}(X) = 0$.*

Proof. For statement (i) see [CTOP02, Thm. 1.8]. Statement (ii) then follows from Theorem 5.6.1 (iv) and (v). □

Remark 10.3.2 (1) For ℓ invertible in k, the ℓ-primary part of this theorem is relatively easy to prove using the Kummer exact sequence [CTOP02, Thm. 1.3].

(2) When R is a discrete valuation ring, Theorem 10.3.1 removes the excellence hypothesis in Artin's theorem [Gro68, III, Thm. 3.1].

(3) The following two situations are of particular interest.

- The "semi-global" case: R is a henselian discrete valuation ring, X is integral, and the generic fibre of $X \to \mathrm{Spec}(R)$ is smooth and geometrically integral. This is the case considered in [Gro68, III, §3], with the additional hypothesis that the discrete valuation ring is excellent (used only to handle the p-torsion part of the theorem, where $p = \mathrm{char}(k)$).
- The "local" case: R is a 2-dimensional, noetherian, henselian local domain and the morphism $X \to \mathrm{Spec}(R)$ is birational. Then X is a resolution of singularities of $\mathrm{Spec}(R)$. If R is excellent, such a desingularisation always exists (Hironaka, Abhyankar, Lipman [Lip78], see [Stacks, Thm. 0BGP]).

In the 'semi-global' case, we have the following theorem.

Theorem 10.3.3 *Let R be an excellent henselian discrete valuation ring with residue field k and fraction field K. Let F be the function field of a smooth, projective, connected curve over K. Let D be a central division algebra over F.*

(i) *If k is algebraically closed of characteristic zero, then* $\mathrm{ind}(D) = \exp(D)$. *The algebra D is cyclic of degree $n = \exp(D)$. It is split by a field extension $F(\sqrt[n]{f})$ for some $f \in F^*$.*
(ii) *If k is a finite field, then* $\mathrm{ind}(D) | \exp(D)^2$.

Proof. Let us prove (i). This is a very slight variation on the proof of [CTOP02, Thm. 2.1], which is Theorem 10.3.5 below. We fix an isomorphism $\mathbb{Z}/n \simeq \mu_n$.

By [Lip78], [Stacks, Thm. 0BGP] there exists a regular, projective, integral model $X \to \mathrm{Spec}(R)$ of the smooth, projective curve over K with function field F. Then X is flat over R. The purity theorem gives an exact sequence (3.20):

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(F) \xrightarrow{\{\partial_x\}} \oplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

By Theorem 10.3.1 (ii), we have $\mathrm{Br}(X) = 0$. Thus the total Gysin residue is an injective map

$$\mathrm{Br}(F) \hookrightarrow \oplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Let $n = \exp(D)$ and let $\xi \in \mathrm{Br}(F)[n]$ be the class of D. Let Δ be the sum of the closures of codimension 1 points of X where ξ has a non-zero residue. By blowing up X we can assume that Δ is a strict normal crossing divisor [Liu10, Ch. 9, Thm. 2.26]. Since $\dim(X) = 2$, the singular locus Δ_{sing} is a union of closed points; these are the points where any two of the components meet. Replace X by its blow up in $\Delta_{\rm sing}$, write C for the strict transform of Δ and write E for the exceptional divisor. Thus D is unramified over $X\smallsetminus(C+E)$ and both C and E are (not necessarily connected) regular curves in X without common component, such that $C+E$ has normal crossings. If $C+E=0$, i.e., if ξ is unramified on X, then $\xi=0$ and the theorem is clear. We thus assume that $C+E\neq 0$.

Let S be a finite set of closed points of X including all points of intersection of C and E and at least one point of each component of $C+E$. Since X is projective over $\mathrm{Spec}(R)$, there exists an affine open subset $U\subset X$ containing S. The semilocalisation of U at S is a semilocal regular domain, hence a unique factorisation domain. Thus there exists an $f\in F^*$ such that the divisor of f on X is of the form $\mathrm{div}_X(f)=C+E+G$, where the support of G does not contain any point of S, hence in particular has no common component with $C+E$. Let L/F be the splitting field of the polynomial T^n-f. At the generic point of each component of $C+E$, the extension L/F is totally ramified of degree n. In particular, L/F is a field extension of degree n. Let ξ_L be the image of ξ under the restriction map $\mathrm{Br}(F)\to\mathrm{Br}(L)$. To prove (i) it suffices to show that $\xi_L=0$.

Let X' be the normalisation of X in L and let $\pi\colon Y\to X'$ be a projective birational morphism such that Y is regular. Let B be the integral closure of R in L. The ring B is a henselian discrete valuation ring with the same algebraically closed residue field k as R. By the universal property of normalisation, the composition $X'\to X\to\mathrm{Spec}(R)$ factors through a projective morphism $X'\to\mathrm{Spec}(B)$ [Stacks, Lemma 0C4Q], hence induces a projective morphism $Y\to\mathrm{Spec}(B)$. By Theorem 10.3.1 (ii), we have $\mathrm{Br}(Y)=0$. Just as above, the total Gysin residue map on Y defines an injective map

$$\mathrm{Br}(L)\hookrightarrow\bigoplus_{y\in Y^{(1)}}\mathrm{H}^1(\kappa(y),\mathbb{Q}/\mathbb{Z}).$$

It is thus enough to show that ξ_L is unramified on Y. Let $y\in Y$ be a codimension 1 point. We show that $\partial_y(\xi_L)=0$. Let $x\in X$ be the image of y under the map $Y\to X'\to X$.

Suppose first that $\mathrm{codim}(x)=1$. If $\overline{\{x\}}$ is not a component of $C+E$, then $\partial_x(\xi)=0$, hence, by Proposition 1.4.7, we have $\partial_y(\xi_L)=0$. Suppose that $D=\overline{\{x\}}$ is a component of $C+E$. Then f is a uniformiser of the discrete valuation ring $\mathcal{O}_{X,x}$. The extension L/F is totally ramified at x. The restriction map $\mathrm{Br}(F)\to\mathrm{Br}(L)$ induces multiplication by the ramification index on the character groups of the residue fields (Proposition 1.4.7). Hence ξ_L is unramified at y.

Suppose now that $\mathrm{codim}(x)=2$. Note that x is in the closed fibre, hence the residue field $\kappa(x)=k$, which is algebraically closed. If $x\notin C+E$, then $\xi\in\mathrm{Br}(\mathcal{O}_{X,x})$, hence ξ_L is unramified at y. If x is a regular point of $C+E$, then without loss of generality we can assume that x is in C but not in E (indeed, here the rôles of E and C are interchangeable). Let C_0 be the irreducible component of C that contains x, and let $V \subset C_0$ be the complement to the intersection of C_0 with E. Then $x \in V$. Let $\pi \in \mathcal{O}_{X,x}$ be a local equation of C at x. (This is also a local equation of V at x.) By the exact sequence (3.15) and Theorem 3.7.3, the residue $\partial_\pi(\xi) \in \kappa(C_0)^*/\kappa(C_0)^{*n}$ comes from an element of $\mathrm{H}^1(V, \mathbb{Z}/n)$. Since C is regular, we can choose a $\delta \in \mathcal{O}_{X,x}$ such that (π, δ) is a regular system of parameters of $\mathcal{O}_{X,x}$. As $\partial_\pi(\xi)$ comes from an element of $\mathrm{H}^1(V, \mathbb{Z}/n)$, it goes to zero under the map $\kappa(C_0)^*/\kappa(C_0)^{*n} \to \mathbb{Z}/n$ induced by the valuation defined by x on the field $\kappa(C_0)$, which is the fraction field of the discrete valuation ring $\mathcal{O}_{X,x}/(\pi)$. Thus $\partial_\pi(\xi)$ is the class of a unit of $\mathcal{O}_{X,x}/(\pi)$, and such a unit lifts to a unit $\mu \in \mathcal{O}^*_{X,x}$. Using (1.18) (via Theorems 1.4.14 and 2.3.5), we obtain that the residues of $\xi - (\mu, \pi)$ at all points of codimension one of $\mathcal{O}_{X,x}$ are trivial. Since $\mathcal{O}_{X,x}$ is a regular two-dimensional ring, this implies that $\xi - (\mu, \pi)$ is the class of an element $\eta \in \mathrm{Br}(\mathcal{O}_{X,x})$. Now, again using (1.18) (together with Theorems 1.4.14 and 2.3.5), we obtain

$$\partial_y(\xi_L) = \partial_L((\mu, \pi)) = \overline{\mu}^{-v_y(\pi)} \bmod \kappa(y)^{*n} ,$$

where $\kappa(y)$ is the residue field of y and $\overline{\mu}$ is the class of μ in $\kappa(y)$. This class comes from $\kappa(x) = k$, which is algebraically closed, therefore $\overline{\mu}$ is an n-th power and $\partial_y(\xi_L) = 0$.

Suppose now that x belongs to $C \cap E$. There exists a regular system of parameters (π, δ) defining (C, E) such that $f = u\pi\delta$, where $u \in \mathcal{O}^*_{X,x}$. Since the ramification of ξ on $\mathrm{Spec}(\mathcal{O}_{X,x})$ is only along π and δ, it can be shown that

$$\xi = \eta + (\pi, \mu_1) + (\delta, \mu_2) + r(\pi, \delta) ,$$

for some $\eta \in \mathrm{Br}(\mathcal{O}_{X,x})$, where $\mu_1, \mu_2 \in \mathcal{O}^*_{X,x}$ and $r \in \mathbb{Z}$. (This uses a Bloch–Ogus argument similar to the one used in the proof of Theorem 11.3.9, see [CTOP02] for details.) Since $f = u\pi\delta$, taking into account that $(\pi, -\pi) = 0$ (see the end of Section 1.3.4), we get

$$(\pi, \delta) = (\pi, fu^{-1}\pi^{-1}) = (\pi, f) + (\pi, -u^{-1}) .$$

The symbol (π, f) vanishes over L and the other symbols become unramified at y, since any unit in $\mathcal{O}^*_{X,x}$ induces an n-power in the algebraically closed residue field k.

For the proof of (ii) in the case when the index is coprime to the residual characteristic we refer to [Sal97] (see also the review Zentralblatt 0902.16021). This restriction was recently lifted by Parimala and Suresh [PS14]. □.

Remark 10.3.4 The technique used in the proof is essentially the technique of [FS89] and [Sal97]. For function fields of curves over the field of fractions K of a complete discrete valuation ring R with arbitrary residue field k, Harbater, Hartmann and Krashen introduced a new, patching technique which among other things gives bounds [HHK09, Thm. 5.5] for the index in terms of similar bounds for the field K and for the function fields of curves over the residue field k.

Here is a 'local' analogue of Theorem 10.3.3.

Theorem 10.3.5 *Let R be a 2-dimensional henselian local, normal, excellent domain with fraction field F and residue field k. Let D be a central division algebra over F. Then we have the following statements.*

(i) *If k is separably closed and $\exp(D)$ is not divisible by $\mathrm{char}(k)$, then $\mathrm{ind}(D) = \exp(D)$. Moreover, D is cyclic.*
(ii) *If k is a finite field and $\exp(D)$ is not divisible by $\mathrm{char}(k)$, then we have $\mathrm{ind}(D)|\exp(D)^2$.*

Proof. For (i) see [FS89, Thm. 1.6], [CTOP02, Thm. 2.1]. For (ii) see [Hu13, Thm. 3.4]. □

Lemma 10.3.6 *Let R be a discrete valuation ring with fraction field K. Let $\widehat{R}$ be the completion of R and let $\widehat{K}$ be the fraction field of $\widehat{R}$. If the image of $\alpha \in \mathrm{Br}(K)$ in $\mathrm{Br}(\widehat{K})$ lies in $\mathrm{Br}(\widehat{R}) \subset \mathrm{Br}(\widehat{K})$, then α belongs to $\mathrm{Br}(R) \subset \mathrm{Br}(K)$.*

Proof. There exists a positive integer n and elements $x_1 \in \mathrm{H}^1(K, \mathrm{PGL}_n)$ and $x_2 \in \mathrm{H}^1(\widehat{R}, \mathrm{PGL}_n)$, with the same image in $\mathrm{H}^1(\widehat{K}, \mathrm{PGL}_n)$, such that the injective map $\mathrm{H}^1(K, \mathrm{PGL}_n) \to \mathrm{Br}(K)$ sends x_1 to α. There is an embedding of reductive group R-schemes $\mathrm{PGL}_{n,R} \hookrightarrow \mathrm{GL}_{N,R}$ for some N. Then $E = \mathrm{GL}_{N,R}/\mathrm{PGL}_{n,R}$ is a smooth affine R-scheme. We have an exact sequence of pointed sets [SerCG, Ch. 1, §5, Prop. 36]

$$E(K) \longrightarrow \mathrm{H}^1(K, \mathrm{PGL}_n) \longrightarrow \mathrm{H}^1(K, \mathrm{GL}_N),$$

and a similar sequence for $\widehat{R}$ in place of K. By Hilbert's theorem 90 we have $\mathrm{H}^1(K, \mathrm{GL}_N) = 0$ (Theorem 1.3.1), so we can lift x_1 to a point $y_1 \in E(K)$. It is known that $\mathrm{H}^1(A, \mathrm{GL}_N) = 0$ for any local ring A, cf. [Mil80, Lemma III.4.10], hence we can lift x_2 to a point $y_2 \in E(\widehat{R})$. There exists an element $g \in \mathrm{GL}_N(\widehat{K})$ such that $gy_1 = y_2$. As GL_N is an open subset of an affine space, any element $g \in \mathrm{GL}_N(\widehat{K})$ can be written as a product $g_2 g_1$ where $g_1 \in \mathrm{GL}_N(K)$ and $g_2 \in \mathrm{GL}_N(\widehat{R})$. Then $g_1 y_1 = g_2^{-1} y_2$ is an element of $E(\widehat{K})$ contained in $E(K) \cap E(\widehat{R}) = E(R)$. This implies that $\alpha \in \mathrm{Br}(R)$. □

For a more general statement, see [CTPS12, Lemma 4.1].

Theorem 10.3.7 *Let R be a noetherian henselian local domain with residue field k. Let X be an integral regular scheme of dimension 2 equipped with a proper morphism $X \to \mathrm{Spec}(R)$ whose closed fibre X_0 is of dimension 1. Let F be the function field of X. Let Ω_X be the set of rank 1 valuations v on F associated to codimension 1 points on X. Let F_v denote the completion of F with respect to v. Then the natural restriction map $\mathrm{Br}(F) \to \prod_{v \in \Omega_X} \mathrm{Br}(F_v)$ is injective.*

Proof. Since $\alpha \in \mathrm{Br}(F)$ goes to zero in each $\mathrm{Br}(F_v)$ for v attached to the points of codimension 1 of the regular scheme X, by Lemma 10.3.6 (via a patching argument) α can be represented by an Azumaya algebra over an open set $U \subset X$ which contains all codimension 1 points of X. Since X is regular and 2-dimensional, by a theorem of Auslander, Goldman and Grothendieck [Gro68, II, §2, Thm. 2.1] there exists an Azumaya algebra over X whose class in $\mathrm{Br}(F)$ is α. We thus have $\alpha \in \mathrm{Br}(X) \subset \mathrm{Br}(F)$.

An irreducible component C of the curve X_0 defines a valuation $v \in \Omega_X$. The image of α in $\mathrm{Br}(F_v)$ belongs to the subgroup $\mathrm{Br}(\mathcal{O}_v) \subset \mathrm{Br}(F_v)$, where $\mathcal{O}_v$ is the ring of integers of the complete field F_v. By assumption, this image is zero. Thus the image of α in the Brauer group of the function field of C is zero.

Now let P be a closed point of X_0. Since X is regular, there exists a closed integral curve $D \subset X$ through P which is regular at P. Arguing as above, we see that the value of α at the generic point of D is zero. This implies that the restriction of α to the local ring of P on D is zero, hence $\alpha(P) = 0$. We now apply Proposition 5.6.1 (i) to conclude that the image of α in $\mathrm{Br}(X_0)$ is zero. Now Theorem 10.3.1 implies that $\alpha = 0$. □

Remark 10.3.8 (1) The above proof is essentially given by Y. Hu in [Hu12, §3]. It extends proofs in [CTOP02].

(2) For R complete, in the semi-global case, a different proof of Theorem 10.3.7 is given in [CTPS12, Thm. 4.3]. This proof relies on [HHK09].

(3) There exist examples of R, X and F as above such that the map

$$\mathrm{H}^1(F, \mathbb{Q}/\mathbb{Z}) \longrightarrow \prod_{v \in \Omega_X} \mathrm{H}^1(F_v, \mathbb{Q}/\mathbb{Z})$$

has a non-trivial kernel. See [CTPS12, §6].

(4) Let D be a central simple algebra over a field F. The relation between $\mathrm{ind}(D)$ and $\exp(D)$ over specific fields F has been the object of much study. Suppose $F = \mathbb{C}(X)$ is the function field of an integral algebraic variety X of dimension d over $\mathbb{C}$. It would be interesting to know if $\mathrm{ind}(D) | \exp(D)^{d-1}$ for any D over F, which is the best possible bound [CT02]. The case $d = 1$ is Tsen's theorem. The case $d = 2$ is a theorem of de Jong [deJ04], [CT06]. For more work on the comparison of index and exponent over various fields of geometric or arithmetic origin, see [Lie08, Lie11, Lie15], [KL08], [HHK09] and [AAI$^+$].

The following theorem combines [CTPS16, Prop. 2.10] and work of Izquierdo [Izq19].

Theorem 10.3.9 *Let R be a 2-dimensional, normal, excellent, henselian local domain with algebraically closed residue field k of characteristic zero. Let K be the fraction field of R. Let $X \to \mathrm{Spec}(R)$ be a resolution of singularities such that the reduced divisor associated to the closed fibre Y/k is a divisor* *on X with strict normal crossings. For each place v of K, let K_v be the completion of K at v. Then we have the following statements.*

(i) *There is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v\in X^{(1)}} \mathrm{H}^1(k(x),\mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{x\in X^{(2)}} (\mathbb{Q}/\mathbb{Z})(-1) \longrightarrow 0.$$

(ii) *For each $v \in R^{(1)}$ there are isomorphisms*

$$\mathrm{Br}(K_v) \xrightarrow{\sim} \mathrm{H}^1(k(v),\mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})(-1).$$

(iii) *The sum of these maps for all $v \in R^{(1)}$ fits into an exact sequence*

$$\mathrm{Br}(K) \longrightarrow \oplus_{v\in R^{(1)}}\mathrm{Br}(K_v) \longrightarrow (\mathbb{Q}/\mathbb{Z})(-1) \longrightarrow 0,$$

(iv) *If R is regular, then the map $\mathrm{Br}(K)\to \oplus_{v\in R^{(1)}} \mathrm{Br}(K_v)$ is injective. Assume that Y is a curve. Let Γ be the graph associated to the reduced divisor Y_{red} whose vertices correspond to the irreducible components of Y_{red} and the edges correspond to the intersection points of components. This graph is connected. Let n_v be the number of vertices and let n_e be the number of edges. Let $c = n_e - n_v + 1$ be the Betti number of Γ. Let $m_Y = c+2\sum_{y\in Y^{(1)}} g_y$, where g_y is the genus of the smooth, irreducible, projective curve defined by y. Then we have*

$$\mathrm{Ker}[\mathrm{Br}(K)\to \oplus_{v\in R^{(1)}} \mathrm{Br}(K_v)] \cong (\mathbb{Q}/\mathbb{Z})^{m_Y}.$$

(v) *We have $\mathrm{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{m_Y}$.*

Statements (iv) and (v) are due to Izquierdo. Statement (iv) is important, it is one of the building blocks for the Poitou–Tate duality theorems which Izquierdo establishes for finite commutative groups and for tori over K, with respect to just the completions at the points of codimension 1 of $\mathrm{Spec}(R)$.

10.4 Smooth proper schemes over a henselian discrete valuation ring

The content of the present section was developed in [CTS13a].

Let R be a henselian discrete valuation ring with field of fractions K and residue field k. We assume that $\mathrm{char}(K) = 0$ and that k is perfect. Let $\overline{K}$ be an algebraic closure of K, and let $K_{\mathrm{nr}} \subset \overline{K}$ be the maximal unramified extension of K. Let R_{nr} be the ring of integers of K_{nr}. Let

$$\mathfrak{g} = \mathrm{Gal}(\overline{K}/K), \quad G = \mathrm{Gal}(K_{\mathrm{nr}}/K), \quad I = \mathrm{Gal}(\overline{K}/K_{\mathrm{nr}}).$$

The valuation of K gives rise to a split exact sequence of G-modules

$$1 \longrightarrow R_{\rm nr}^* \longrightarrow K_{\rm nr}^* \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We have $\mathrm{Br}(K_{\rm nr}) = 0$ (Theorem 1.2.15), which implies $\mathrm{H}^2(G, K_{\rm nr}^*) \cong \mathrm{Br}(K)$.

Let $\pi\colon \mathcal{X} \to \mathrm{Spec}(R)$ be a faithfully flat proper morphism of integral schemes with geometrically integral generic fibre $X = \mathcal{X} \times_R K$. In particular, π is of finite type, but since R is noetherian, π is of finite presentation. If π is also assumed to be smooth, then by [Stacks, Lemma 0E0N] (a consequence of Stein factorisation) all fibres are geometrically connected, hence geometrically integral. Write

$$X_{\rm nr} = X \times_K K_{\rm nr}, \quad \mathcal{X}_{\rm nr} = \mathcal{X} \times_R R_{\rm nr}, \quad \overline{X} = X \times_K \overline{K}.$$

Lemma 10.4.1 *If the proper R-scheme $\mathcal{X}$ is smooth over R, then the following natural map is surjective:*

$$\mathrm{Br}(K) \oplus \mathrm{Ker}[\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_{\rm nr})] \longrightarrow \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\rm nr})].$$

Proof. The map is well defined since $\mathrm{Br}(K_{\rm nr}) = 0$, so that the composition $\mathrm{Br}(K) \to \mathrm{Br}(X) \to \mathrm{Br}(X_{\rm nr})$ is zero.

The restriction map $\mathrm{Pic}(\mathcal{X}_{\rm nr}) \to \mathrm{Pic}(X_{\rm nr})$ is surjective since $\mathcal{X}_{\rm nr}$ is regular. The kernel of this map is generated by the classes of components of the closed fibre of $\mathcal{X}_{\rm nr} \to \mathrm{Spec}(R)$. The closed fibre is a principal divisor in $\mathcal{X}_{\rm nr}$. Since we assume that it is integral, the restriction map gives an isomorphism of G-modules

$$\mathrm{Pic}(\mathcal{X}_{\rm nr}) \xrightarrow{\sim} \mathrm{Pic}(X_{\rm nr}). \tag{10.7}$$

There is a Hochschild–Serre spectral sequence attached to the morphism $\mathcal{X}_{\rm nr} \to \mathcal{X}$:

$$E_2^{pq} = \mathrm{H}^p(G, \mathrm{H}^q_{\text{ét}}(\mathcal{X}_{\rm nr}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathcal{X}, \mathbb{G}_m),$$

and a similar sequence attached to $X_{\rm nr} \to X$, see [Mil80, Thm. III.2.20, Remark III.2.21 (b)]. By functoriality the maps in these sequences are compatible with the inclusions of the generic fibres $X \hookrightarrow \mathcal{X}$ and $X_{\rm nr} \hookrightarrow \mathcal{X}_{\rm nr}$. We have $\mathrm{H}^0_{\text{ét}}(\mathcal{X}_{\rm nr}, \mathbb{G}_m) = R_{\rm nr}^*$ because the morphism $\pi\colon \mathcal{X} \to \mathrm{Spec}(R)$ is proper with geometrically integral fibres. The low degree terms of the two spectral sequences give rise to the following commutative diagram of exact sequences, where the equality is induced by (10.7):

$$\begin{array}{ccccccc}
\mathrm{H}^2(G, R_{\rm nr}^*) & \to & \mathrm{Ker}[\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(\mathcal{X}_{\rm nr})] & \to & \mathrm{H}^1(G, \mathrm{Pic}(\mathcal{X}_{\rm nr})) & \to & \mathrm{H}^3(G, R_{\rm nr}^*) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\mathrm{H}^2(G, K_{\rm nr}^*) & \to & \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\rm nr})] & \to & \mathrm{H}^1(G, \mathrm{Pic}(X_{\rm nr})) & \to & \mathrm{H}^3(G, K_{\rm nr}^*)
\end{array}$$

The inclusion of G-modules $R_{\rm nr}^* \hookrightarrow K_{\rm nr}^*$ has a G-module retraction, hence the map $\mathrm{H}^3(G, R_{\rm nr}^*) \to \mathrm{H}^3(G, K_{\rm nr}^*)$ is injective. Since $\mathrm{H}^2(G, K_{\rm nr}^*) = \mathrm{Br}(K)$, the statement follows from the above diagram. □

Proposition 10.4.2 *Assume that the proper R-scheme $\mathcal{X}$ is smooth over R with geometrically integral fibres. Assume also that* $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ *and that the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *is torsion-free. Then*

$$\mathrm{Br}_1(X) = \mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{nr}})].$$

Proof. For any prime $\ell \neq \mathrm{char}(k)$ the smooth base change theorem in étale cohomology for the smooth and proper morphism $\pi\colon \mathcal{X} \to \mathrm{Spec}(R)$ implies that the natural action of the inertia subgroup I on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$ is trivial. Indeed, by [Mil80, Cor. VI.4.2] the étale sheaf $R^2\pi_*(\mu_{\ell^m})$ is locally constant for every $m \geq 1$. Also, the fibre of $R^2\pi_*(\mu_{\ell^m})$ at the generic geometric point $\mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(R)$ is $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m})$. Now it follows from [Mil80, Remark V.1.2 (b)] that the action of $\mathfrak{g}$ on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m})$ factors through

$$\pi_1(\mathrm{Spec}(R), \mathrm{Spec}(\overline{K})) \cong \mathrm{Gal}(K_{\mathrm{nr}}/K) = G \cong \mathfrak{g}/I,$$

see [Mil80, Ex. I.5.2(b)]. Thus I acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^m})$ for every m, hence I acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$.

Since K has characteristic zero, for any prime ℓ the Kummer sequence gives a Galois equivariant embedding

$$\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)).$$

For any $\ell \neq \mathrm{char}(k)$ we conclude that I acts trivially on $\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell$, hence also on $\mathrm{Pic}(\overline{X}) \cong \mathrm{NS}(\overline{X}) \subset \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell$. Thus $\mathrm{H}^1(K_{\mathrm{nr}}, \mathrm{Pic}(\overline{X})) = \mathrm{H}^1(I, \mathrm{NS}(\overline{X})) = 0$. From the exact sequence

$$\mathrm{Br}(K_{\mathrm{nr}}) \longrightarrow \mathrm{Ker}[\mathrm{Br}(X_{\mathrm{nr}}) \to \mathrm{Br}(\overline{X})] \longrightarrow \mathrm{H}^1(K_{\mathrm{nr}}, \mathrm{Pic}(\overline{X}))$$

we conclude that the map $\mathrm{Br}(X_{\mathrm{nr}}) \to \mathrm{Br}(\overline{X})$ is injective. The result follows. □

We are also interested in the situation when $\mathrm{H}^2(X, \mathcal{O}_X)$ is not necessarily zero, so we must take into account the transcendental Brauer group as well.

Proposition 10.4.3 *Let ℓ be a prime, $\ell \neq \mathrm{char}(k)$. Let $\pi\colon \mathcal{X} \to \mathrm{Spec}(R)$ be a smooth proper morphism with geometrically integral fibres, such that the closed geometric fibre has no connected unramified cyclic covering of degree ℓ. Then the group* $\mathrm{Br}(X)\{\ell\}$ *is generated by the images of* $\mathrm{Br}(\mathcal{X})\{\ell\}$ *and* $\mathrm{Br}(K)\{\ell\}$.

Proof. Let $Y = \mathcal{X} \times_R k$ be the closed fibre of π. We note that Y is a regular subscheme of codimension 1 of the regular scheme $\mathcal{X}$. Thus we can apply the exact sequence (3.17):

$$0 \longrightarrow \mathrm{Br}(\mathcal{X})[\ell^m] \longrightarrow \mathrm{Br}(X)[\ell^m] \longrightarrow \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m). \qquad (10.8)$$

Let $\overline{Y} = Y \times_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k. As $\overline{Y}$ is connected, the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(k, \mathrm{H}^q_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(Y, \mathbb{Z}/\ell^n)$$

gives rise to the exact sequence

$$0 \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/\ell^n) \longrightarrow \mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^n) \longrightarrow \mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n).$$

By assumption $\overline{Y}$ has no connected unramified cyclic covering of degree ℓ, hence $\mathrm{H}^1_{\text{ét}}(\overline{Y}, \mathbb{Z}/\ell^n) = 0$.

Let $A \in \mathrm{Br}(X)\{\ell\}$. Take an m such that $A \in \mathrm{Br}(X)[\ell^m]$. The image of A in $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m)$ belongs to the image of $\mathrm{H}^1(k, \mathbb{Z}/\ell^m)$. We have the exact sequence (3.12)

$$0 \longrightarrow \mathrm{Br}(R)[\ell^m] \longrightarrow \mathrm{Br}(K)[\ell^m] \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/\ell^m) \longrightarrow 0,$$

with compatible maps to the terms of sequence (10.8) (see Lemma 2.3.6 and Theorem 3.7.5). Hence there exists an element $\alpha \in \mathrm{Br}(K)[\ell^m]$ such that $A - \alpha \in \mathrm{Br}(X)[\ell^m]$ goes to zero in $\mathrm{H}^1_{\text{ét}}(Y, \mathbb{Z}/\ell^m)$. By the exactness of (10.8) we have $A - \alpha \in \mathrm{Br}(\mathcal{X})[\ell^m]$. □

Remark 10.4.4 (1) Let k be a field of characteristic $p > 0$. For a smooth and proper morphism π, it is an interesting problem to decide when a similar statement holds for the p-primary torsion subgroup $\mathrm{Br}(X)\{p\}$. For the elements of $\mathrm{Br}(X)$ split by an unramified extension of K, including those of order divisible by p, such a statement follows from Lemma 10.4.1 (see also [Bri07, Prop. 6]). But the recent work of Bright and Newton [BN, Thm. C] shows that Proposition 10.4.3 does not in general extend to $\mathrm{Br}(X)\{p\}$. See Theorem 13.3.18.

(2) The hypotheses of Proposition 10.4.3 apply in particular when the fibres of $\pi\colon \mathcal{X} \to \mathrm{Spec}(R)$ are smooth complete intersections of dimension at least 2 in the projective space (an application of the weak Lefschetz theorem in étale cohomology, see [Kat04]). In particular they apply to smooth surfaces of arbitrary degree in $\mathbb{P}^3$.

10.5 Varieties over a local field

We start with the following statement, which is a generalisation of known results such as the implicit function theorem for varieties over a complete local field to the henselian case. Versions of this statement hold for the fields of fractions of much more general henselian valuation rings, see [Con12], [Mor12] and [GGM14, §3.1].

Theorem 10.5.1 *Let K be either the fraction field of a henselian discrete valuation ring, the field of complex numbers $\mathbb{C}$, or the field of real numbers $\mathbb{R}$.*

For any variety X over K there is a unique structure of a topological space on $X(K)$ which is functorial and compatible with fibred products, and such that open (respectively, closed) immersions in X give rise to open (respectively, closed) embeddings in $X(K)$, and étale morphisms give rise to local homeomorphisms. The topological space $X(K)$ is Hausdorff.

If $f\colon X\to Y$ is a smooth morphism of varieties over K, then the induced map $X(K)\to Y(K)$ is topologically open.

If X is a smooth K-variety and $U\subset X$ is a Zariski dense open set, then $U(K)$ is dense in $X(K)$.

Assume further that K is locally compact. Then $X(K)$ is locally compact. If $f\colon X\to Y$ is a proper morphism of varieties over K, then the induced map $X(K)\to Y(K)$ is topologically proper.

A discretely valued field is locally compact if and only if it is complete and has finite residue field, see [CF67, Ch. II, §7].

10.5.1 Evaluation at rational and closed points

Proposition 10.5.2 *Let A be a henselian discrete valuation ring with field of fractions K. Let X be a variety over K and let $\mathcal{A}\in \mathrm{Br}(X)$. The evaluation map $X(K)\to\mathrm{Br}(K)$ sending $M\in X(K)$ to $\mathcal{A}(M)\in\mathrm{Br}(K)$ is locally constant and its image is annihilated by some positive integer.*

Proof. Take any $P\in X(K)$. Then $\alpha=\mathcal{A}-\mathcal{A}(P)\in\mathrm{Br}(X)$ satisfies $\alpha(P)=0$. By Corollary 3.4.4 there exists an étale morphism $f\colon U\to X$ such that $f^*\alpha=0$ and P lifts to a point $M\in U(K)$. Then α vanishes on $f(U(K))\subset X(K)$. Since $P\in f(U(K))$, this is an open neighbourhood of $P\in X(K)$ by the implicit function theorem (Theorem 10.5.1). The last statement is a special case of Lemma 3.4.5. □

It is clear that the same result also holds for a variety X over the field of real numbers $\mathbb{R}$.

By a p-adic field we mean a finite extension of $\mathbb{Q}_p$.

Proposition 10.5.3 *Let k be a p-adic field and let R be the ring of integers of k. Let $\mathcal{X}$ be a proper R-scheme and let $X=\mathcal{X}\times_R k$. Let $\alpha\in\mathrm{Br}(\mathcal{X})$ and let $\alpha_X\in\mathrm{Br}(X)$ be the restriction of α to X. Then for any closed point $P\in X$ we have $\alpha_X(P)=0\in\mathrm{Br}(k(P))$.*

Proof. Let $L=k(P)$. Let $S\subset L$ be the ring of integers of the p-adic field L. Let $\mathbb{F}$ be the residue field of the complete discrete valuation ring S. Since $\mathcal{X}$ is proper over R, we have $X(L)=\mathcal{X}(S)$. By functoriality of the evaluation map, $\alpha_X(P)\in\mathrm{Br}(L)$ is the restriction of an element in $\mathrm{Br}(S)$. Since $\mathbb{F}$ is finite, we have $\mathrm{Br}(S) \cong \mathrm{Br}(\mathbb{F}) = 0$, where the first equality follows from Theorem 3.4.2 (i) and the second one from Theorem 1.2.13. □

Theorem 10.5.4 *Let k be a p-adic field and let R be the ring of integers of k. Let $\mathcal{X}$ be a regular, proper, integral, flat R-scheme with geometrically integral generic fibre X/k. If $\alpha \in \mathrm{Br}(X)$ vanishes at each closed point of a non-empty open set $U \subset X$, then α lies in $\mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(X)$.*

Proof. Here is a sketch of the proof for the prime to p-part of the statement [CTS96, Thm. 2.1]. Let ℓ be a prime, $\ell \neq p$. Using Chebotarev's theorem for varieties over a finite field, a suitable version of Hensel's lemma, and Theorem 3.7.5, one sees that the assumption implies that $\alpha \in \mathrm{Br}(X)\{\ell\}$ has trivial residues at the codimension 1 points of $\mathcal{X}$. Hence, by Gabber's purity theorem, α comes from an element of $\mathrm{Br}(\mathcal{X})$. For the general case, combine the result of Saito and Sato [SS14, Thm. 1.1.3], which is conditional on purity for the Brauer group for regular schemes, with this purity theorem, proved recently by Česnavičius using previous work of Gabber (Theorem 3.7.6). □

The case when X is a curve goes back to Lichtenbaum [Lic69].

Corollary 10.5.5 (Lichtenbaum) *Let X be a smooth, projective, geometrically integral curve over a p-adic field k. If $\alpha \in \mathrm{Br}(X)$ vanishes at each closed point of X, then $\alpha = 0$.*

Proof. Let R be the ring of integers of k. There exists a regular proper flat model $\mathcal{X}\to\mathrm{Spec}(R)$ (as proved independently by Lipman and Shafarevich). By the previous theorem, α lies in $\mathrm{Br}(\mathcal{X}) \subset \mathrm{Br}(X)$. By Theorem 10.3.1, we have $\mathrm{Br}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Br}(\mathcal{X}_0)$, where $\mathcal{X}_0$ is the closed fibre of $\mathcal{X}\to\mathrm{Spec}(R)$. But $\mathrm{Br}(\mathcal{X}_0) = 0$ by Theorem 5.6.1 (v), hence $\alpha = 0$. □

Proposition 10.5.6 *Let k be a p-adic field and let R be the ring of integers of k. Let ℓ be a prime number invertible in R. Let $\pi\colon \mathcal{X}\to\mathrm{Spec}(R)$ be a smooth proper morphism with geometrically integral fibres, such that the closed geometric fibre has no connected unramified cyclic covering of degree ℓ. Let $X = \mathcal{X} \times_R k$. Assume $X(K) \neq \emptyset$. Then for any $A \in \mathrm{Br}(X)\{\ell\}$ the evaluation map $\mathrm{ev}_A\colon X(K)\to\mathrm{Br}(K)$ has constant image.*

Proof. By Proposition 10.4.3 we can write $A = A_1 + A_2 \in \mathrm{Br}(X)$, where $A_1 \in \mathrm{Br}(\mathcal{X})$ and $A_2 \in \mathrm{Br}(K)$. By Proposition 10.5.3 the map ev_{A_1} sends $X(K)$ to 0. Hence $\mathrm{ev}_A(P) = A_2 \in \mathrm{Br}(K)$ for any $P \in X(K)$. □

Remark 10.5.7 (1) By Proposition 6.4.2, evaluation on closed points of a smooth, projective, geometrically integral curve over a p-adic field induces a bilinear pairing

$$\mathrm{Br}(X) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Br}(k) \cong \mathbb{Q}/\mathbb{Z}.$$

That the left kernel of this pairing is trivial (and, more precisely, the pairing induces a duality) was proved by Lichtenbaum as a consequence of the Tate duality theorems for abelian varieties over a p-adic field.

(2) Let X be a smooth and geometrically integral curve over a p-adic field k. Let U be a non-empty open subset of X. If $\alpha \in \mathrm{Br}(U)$ vanishes at each closed point of U, then α lies in $\mathrm{Br}(X)$ and, moreover, $\alpha = 0$.

Let us explain this. Let P be a closed point in $X \smallsetminus U$ and let $K = k(P)$ be the residue field of P. Write $X_K = X \times_k K$. The morphism $P\colon \mathrm{Spec}(K) \to X$ gives rise to the morphism $\mathrm{Spec}(K \otimes_k K) \to X_K$ that can be precomposed with the dual morphism of the multiplication map $K \otimes_k K \to K$ to define a K-point $\widetilde{P}\colon \mathrm{Spec}(K) \to X_K$ above P.

Suppose that α has a non-trivial residue $\chi \in \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ at P. Let $N > 1$ be the order of χ in $\mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$. Write α_K for the image of α in $\mathrm{Br}(U_K)$. The multiplicity of $\widetilde{P}$ in the fibre $\mathrm{Spec}(K \otimes_k K)$ of $X_K \to X$ above P is 1, so by the functoriality of residues (Theorem 3.7.5) the residue of $\alpha_K \in \mathrm{Br}(U_K)$ at $\widetilde{P}$ is $\chi \in \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$.

Let π be a local equation at $\widetilde{P} \in X_K$. By Proposition 3.6.4 and a limit argument from section 2.2.2, α_K differs from the cup-product (χ, π) by an element $\beta \in \mathrm{Br}(V)$, where $V \subset X_K$ is a Zariski neighbourhood of $\widetilde{P}$. By Proposition 10.5.2, there exists a p-adic neighbourhood $W \subset V(K)$ of $\widetilde{P}$ such that β is constant on W and π is invertible on $W \smallsetminus \widetilde{P}$. The assumption on α then implies that (χ, π) takes a constant value on $W \smallsetminus \widetilde{P}$. By the implicit function theorem (Theorem 10.5.1), after shrinking V, the local parameter π defines an isomorphism between $W \subset V(K)$ and a p-adic neighbourhood of $0 \in \mathbb{A}^1_K(K)$. Thus for points $M \neq \widetilde{P}$ in $W \subset U(K)$, the value $\pi(M) \in K$ takes all possible valuations. Thus $(\chi, \pi(M)) = v(\pi(M))\chi \in \mathrm{H}^1(K, \mathbb{Z}/N)$ is not constant, which is a contradiction. (For a similar and more detailed argument in a global context, we refer the reader to Theorem 13.4.1.) We conclude that α has zero residues on X, hence belongs to $\mathrm{Br}(X)$. Since α vanishes at all closed points of U, by the continuity of the evaluation map it vanishes at all closed points of X. One then applies Corollary 10.5.5.

(3) There exist smooth, projective, geometrically integral curves X over a p-adic field with non-zero elements in $\mathrm{H}^1_{\text{ét}}(X, \mathbb{Z}/\ell)$ which vanish at each closed point of X, see [Sai85] and [CTPS12, §6].

(4) Let X be a variety over the field of real numbers $\mathbb{R}$. The natural pairing

$$X(\mathbb{R}) \times \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2$$

is locally constant on $X(\mathbb{R})$ hence induces a map $\mathrm{Br}(X) \to (\mathbb{Z}/2)^S$, where S is the set of connected components of $X(\mathbb{R})$ for the real topology.

The real analogue of Tate's duality theorem for abelian varieties over a p-adic field and of Corollary 10.5.5 goes back to Witt (1934). For a smooth, projective, geometrically connected curve over $\mathbb{R}$, evaluating elements of $\mathrm{Br}(X)$ on the real points induces an isomorphism $\mathrm{Br}(X) \xrightarrow{\sim} (\mathbb{Z}/2)^S$. In particular, $\mathrm{Br}(X) = 0$ if $X(\mathbb{R}) = \emptyset$. If X is a quasi-projective but possibly singular curve over $\mathbb{R}$, then the map $\mathrm{Br}(X) \to (\mathbb{Z}/2)^S$ is injective [CTOP02, Prop. 1.13].

10.5.2 The index of a variety over a p-adic field

Let R be the ring of integers of a p-adic field K and let k be the (finite) residue field. Let $\mathcal{X}$ be a regular, connected, projective, flat R-scheme. Let X/K be the generic fibre of $\mathcal{X}$. We assume that X is geometrically integral. The closed fibre $\mathcal{X}_0/k$ is a divisor $\sum_i e_i D_i$, where e_i is a positive integer and D_i is an integral variety over k. Let f_i be the degree over k of the integral closure of k in the function field $k(D_i)$. In this context one defines the following positive integers.

(1) I_{Br} is the order of $\mathrm{Ker}[\mathrm{Br}(K)\to\mathrm{Br}(X)/\mathrm{Br}(\mathcal{X})]$.
(2) I is the g.c.d. of the degrees of the closed points of X.
(3) I_0 is the g.c.d. of the $e_i f_i$.

The positive integer I is called the *index* of X. Note, by the way, that the kernel in (1) is cyclic; by the purity theorem 3.7.6 it does not depend on the choice of $\mathcal{X}$.

Theorem 10.5.8 *We have* $I_{\mathrm{Br}} = I = I_0$.

Saito and Sato [SS14, Thm. 5.4.1] proved this theorem assuming purity for the Brauer group of regular schemes, a result which is now known in full generality (Theorem 3.7.6). Earlier results had been obtained by Lichtenbaum [Lic69] (in the case of curves), then in [CTS96, Thm. 3.1] (for the prime-to-p part, in arbitrary dimension) and in [GLL13, Cor. 9.1] which shows $I = I_0$. The paper [GLL13] studies the case of a henselian discrete valuation ring R with an arbitrary residue field k.

10.5.3 Finiteness results for the Brauer group

In the good reduction case, Section 10.4 gives some control on the Brauer group of a smooth proper variety over a p-adic field. Here are two general results under weaker assumptions.

Proposition 10.5.9 *Let X be a variety over a p-adic field K. Then for any positive integer n the group* $\mathrm{Br}(X)[n]$ *is finite.*

Proof. The Kummer exact sequence shows that $\mathrm{Br}(X)[n]$ is a quotient of $\mathrm{H}^2_{\text{ét}}(X,\mu_n)$. Consider the spectral sequence

$$E_2^{pq} = \mathrm{H}^p(K, \mathrm{H}^q_{\text{ét}}(\overline{X},\mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mu_n).$$

The groups $\mathrm{H}^q_{\text{ét}}(\overline{X},\mu_n)$ are finite for any $q \geq 0$ (see [Mil80, Cor. VI.4.5]). The Galois cohomology groups $\mathrm{H}^p(K,M)$, where K is a p-adic field and M is finite, are finite for all $p \geq 0$ [SerCG, Ch. 2, §5, Prop. 14]. □

Proposition 10.5.10 *Let X be a smooth, proper and geometrically integral variety over a p-adic field K. Let $X_{\mathrm{nr}} = X \times_K K_{\mathrm{nr}}$, where K_{nr} is the maximal unramified extension of K. Then* $\mathrm{Ker}[\mathrm{Br}(X) \to \mathrm{Br}(X_{\mathrm{nr}})]/\mathrm{Br}_0(X)$ *is finite.*

Proof. [CTS13a, Prop. 2.1] We assume that $K_{\mathrm{nr}} \subset \overline{K}$ and use the notation $\mathfrak{g} = \mathrm{Gal}(\overline{K}/K)$, $G = \mathrm{Gal}(K_{\mathrm{nr}}/K)$, $I = \mathrm{Gal}(\overline{K}/K_{\mathrm{nr}})$. Consider the Hochschild–Serre spectral sequence [Mil80, Thm. III.2.20, Remark III.2.21 (b)] attached to the morphism $X_{\mathrm{nr}} \to X$:

$$E_2^{pq} = \mathrm{H}^p(G, \mathrm{H}^q_{\text{ét}}(X_{\mathrm{nr}}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbb{G}_m). \tag{10.9}$$

Since $\mathrm{H}^2(G, K_{\mathrm{nr}}^*) = \mathrm{Br}(K)$, the exact sequence of low degree terms of (10.9) shows that the group under consideration is a subgroup of $\mathrm{H}^1(G, \mathrm{Pic}(X_{\mathrm{nr}}))$. There is an exact sequence of continuous discrete $\mathfrak{g}$-modules

$$0 \longrightarrow \mathrm{Pic}^0(\overline{X}) \longrightarrow \mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{NS}(\overline{X}) \longrightarrow 0.$$

By the representability of the Picard functor, and since $\mathrm{char}(K) = 0$, there exists an abelian variety A over K such that $A(\overline{K})$ is isomorphic to $\mathrm{Pic}^0(\overline{X})$ as a $\mathfrak{g}$-module (Theorem 5.1.1). Thus we rewrite the previous sequence as

$$0 \longrightarrow A(\overline{K}) \longrightarrow \mathrm{Pic}(\overline{X}) \longrightarrow \mathrm{NS}(\overline{X}) \longrightarrow 0. \tag{10.10}$$

The Hochschild–Serre spectral sequence attached to $\overline{X} \to X_{\mathrm{nr}}$ is

$$E_2^{pq} = \mathrm{H}^p(I, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X_{\mathrm{nr}}, \mathbb{G}_m).$$

By Hilbert's theorem 90 we have $\mathrm{H}^1(I, \overline{K}^*) = 0$. Since $\mathrm{Br}(K_{\mathrm{nr}}) = 0$ we obtain that the natural map $\mathrm{Pic}(X_{\mathrm{nr}}) \to \mathrm{Pic}(\overline{X})^I$ is an isomorphism. Now, by taking I-invariants in (10.10) we obtain the exact sequence of G-modules

$$0 \longrightarrow A(K_{\mathrm{nr}}) \longrightarrow \mathrm{Pic}(X_{\mathrm{nr}}) \longrightarrow \mathrm{NS}(\overline{X})^I.$$

The group $\mathrm{NS}(\overline{X})$ is finitely generated by the theorem of Néron and Severi [SGA6, XIII] hence so is $\mathrm{NS}(\overline{X})^I$. Thus there is a G-module N, finitely generated as an abelian group, that fits into the exact sequence of continuous discrete G-modules

$$0 \longrightarrow A(K_{\mathrm{nr}}) \longrightarrow \mathrm{Pic}(X_{\mathrm{nr}}) \longrightarrow N \longrightarrow 0.$$

The resulting exact sequence of cohomology groups gives us an exact sequence

$$\mathrm{H}^1(G, A(K_{\mathrm{nr}})) \longrightarrow \mathrm{H}^1(G, \mathrm{Pic}(X_{\mathrm{nr}})) \longrightarrow \mathrm{H}^1(G, N). \tag{10.11}$$

We note that G is canonically isomorphic to the profinite completion $\widehat{\mathbb{Z}}$, with the Frobenius as a topological generator. If M is a continuous discrete G-module which is finitely generated as an abelian group, then $\mathrm{H}^1(G, M)$ is finite. To see this, let G' be a finite index subgroup of G that acts trivially on M. The group $G' \simeq \widehat{\mathbb{Z}}$ has a dense subgroup $\mathbb{Z}$ generated by a power of the Frobenius. Now $\mathrm{H}^1(G', M)$ is the group of continuous homomorphisms

$$\mathrm{Hom}_{\mathrm{cont}}(G', M) = \mathrm{Hom}_{\mathrm{cont}}(G', M_{\mathrm{tors}}) = M_{\mathrm{tors}},$$

which is visibly finite. An application of the restriction-inflation sequence finishes the proof of the finiteness of $\mathrm{H}^1(G, M)$.

To complete the proof of the proposition it remains to prove the finiteness of $\mathrm{H}^1(G, A(K_{\mathrm{nr}}))$. By [Mil86, Prop. I.3.8] this group is isomorphic to $\mathrm{H}^1(G, \pi_0(A_0))$, where $\pi_0(A_0)$ is the group of connected components of the closed fibre A_0 of the Néron model of A over $\mathrm{Spec}(R)$. Since $\pi_0(A_0)$ is finite, we see that $\mathrm{H}^1(G, \pi_0(A_0))$ is finite. □

10.5.4 Unramified Brauer classes and evaluation at points

The following lemma and theorem are due to O. Wittenberg (private communication). A partial earlier result in this direction is [Mer02, Prop. 3.4].

Lemma 10.5.11 *Let X be a smooth geometrically integral variety over a field k. For any $\alpha \in \mathrm{Br}(X)$ and any point $P\colon \mathrm{Spec}(k((t)))\to X$ there exists a point $P'\colon \mathrm{Spec}(k((t)))\to X$ such that the latter map is dominant and we have $\alpha(P) = \alpha(P')$.*

Proof. A k-morphism $P'\colon \mathrm{Spec}(k((t)))\to X$ is dominant if it induces an inclusion of the fields of functions $k(X) \subset k((t))$. Let $x \in X$ be the image of the k-morphism $P\colon \mathrm{Spec}(k((t)))\to X$. Since X is smooth over k, there exist an open subset $U \subset X$ containing x and an étale morphism $f\colon U\to\mathbb{A}^d_k$. Let $Q = f(P) \in \mathbb{A}^d_k(k((t)))$. The field $k((t))$ is of infinite transcendence degree over k [MS39, §3, Lemma 1]. We can choose a $k((t))$-point Q' in $\mathbb{A}^d_k$ as close as we wish to Q in the topology of the field $k((t))$ such that the d coordinates of Q' are algebraically independent over k. Moreover, by the implicit function theorem (Theorem 10.5.1) over the field $k((t))$, we can choose Q' with the additional property that Q' lifts to a $k((t))$-point P' in U which is as close as we wish to P. Corollary 3.4.4 applied over $k((t))$ (see the proof of Proposition 10.5.2) then ensures the equality $\alpha(P') = \alpha(P)$ in $\mathrm{Br}(k((t)))$. Since the coordinates of Q' are algebraically independent over k, the morphism $Q'\colon \mathrm{Spec}(k((t)))\to\mathbb{A}^d_k$ induces a k-embedding of the fields of functions $k(x_1, \ldots, x_d) \subset k((t))$. But $Q' = f(P')$ and f is dominant, hence this embedding factors as $k(x_1, \ldots, x_d) \subset k(X) \subset k((t))$, which shows that $P'\colon \mathrm{Spec}(k((t)))\to X$ is dominant. □

Theorem 10.5.12 *Let X be a smooth geometrically integral variety over a field k of characteristic zero. Let $\alpha \in \mathrm{Br}(X) \subset \mathrm{Br}(k(X))$. The following conditions are equivalent:*

(i) $\alpha \in \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$;
(ii) *for any field extension L/k and any $P \in X(L((t)))$, the value $\alpha(P)$ is contained in the image of* $\mathrm{Br}(L) \to \mathrm{Br}(L((t)))$;
(iii) *for any field extension L/k with $\mathrm{cd}(L) \leq 1$ and any $P \in X(L((t)))$ we have $\alpha(P) = 0$ in $\mathrm{Br}(L((t)))$.*

Proof. It is clear that (ii) implies (iii). Let us prove that (iii) implies (ii). Choose an embedding $L \subset L'$ as in Proposition 7.1.9. We have a commutative diagram with exact rows (3.12)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(L') & \longrightarrow & \mathrm{Br}(L'((t))) & \longrightarrow & \mathrm{H}^1(L', \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Br}(L) & \longrightarrow & \mathrm{Br}(L((t))) & \longrightarrow & \mathrm{H}^1(L, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Here the third arrow in each row is the Gysin residue (equal to the Witt residue), and the vertical arrows are restriction maps. The right-hand vertical map is injective because L is algebraically closed in L'. This diagram implies the statement of (ii).

Let us prove that (ii) implies (i). Let $A \subset k(X)$ be a discrete valuation ring which contains k. Let κ be the residue field of A. By the Cohen structure theorem, the completion of A is isomorphic to $\kappa[[t]]$. We have $k \subset \kappa$, hence we have a k-embedding $k(X) \subset \kappa((t))$ such that $A = k(X) \cap \kappa[[t]]$. This gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(\kappa[[t]]) & \longrightarrow & \mathrm{Br}(\kappa((t))) & \longrightarrow & \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow = & & \\ 0 & \longrightarrow & \mathrm{Br}(A) & \longrightarrow & \mathrm{Br}(k(X)) & \longrightarrow & \mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Here the top row is (3.12), and the bottom row comes from Proposition 3.6.4. (Note that $\partial = r_W$.) The assumption of (ii) applied to $L = \kappa$ implies that the image of α in $\mathrm{Br}(\kappa((t)))$ goes to zero in $\mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z})$. By the diagram this implies $\alpha \in \mathrm{Br}(A)$. Thus (ii) implies (i).

Let us prove that (i) implies (ii). If X is projective, we have $X(L((t))) = X(L[[t]])$. Hence $P \in X(L[[t]])$, and thus $\alpha(P) \in \mathrm{Br}(L[[t]]) = \mathrm{Br}(L)$, proving (ii). Let us now drop the assumption that X be projective (and avoid the resolution of singularities). By Lemma 10.5.11 we may assume that the morphism $P\colon \mathrm{Spec}(L((t))) \to X$ is dominant while keeping the value of $\alpha(P) \in \mathrm{Br}(L((t)))$. Then we have a k-embedding $k(X) \subset L((t))$. By the functoriality of the unramified Brauer group we have $\alpha(P) \in \mathrm{Br}(L[[t]]) = \mathrm{Br}(L)$. □

Chapter 11
The Brauer group and families of varieties

In this section we are interested in the following question. Let $f\colon X\to Y$ be a dominant morphism of regular integral varieties over a field k. Can one compute the Brauer group $\mathrm{Br}(X)$ and its elements from the Brauer group of the base $\mathrm{Br}(Y)$ and the Brauer group of the generic fibre $\mathrm{Br}(X_\eta)$, in terms of the geometry of varieties X, Y and the morphism f? For example, when is the induced map $f^*\colon \mathrm{Br}(Y)\to\mathrm{Br}(X)$ surjective or injective? Recall that $\mathrm{Br}(X)$ is naturally a subgroup of $\mathrm{Br}(X_\eta)$, so when $\mathrm{Br}(X_\eta)$ is known, computing $\mathrm{Br}(X)$ involves determining the elements of $\mathrm{Br}(X_\eta)$ that are unramified on X. In general, this is a hard problem even if the generic fibre has very simple geometry, for instance, when X_η is finite.

The focus of Section 11.1 is the so-called *vertical* subgroup $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ of $\mathrm{Br}(X)$. It is defined as the set of elements of $\mathrm{Br}(X)$ whose restriction to $\mathrm{Br}(X_\eta)$ belongs to the image of $\mathrm{Br}(k(Y))$, where $k(Y)=k(\eta)$ is the function field of Y. There are several reasons to be interested in $\mathrm{Br}_{\mathrm{vert}}(X/Y)$.

- In some cases there are clean-cut algebraic formulae for $\mathrm{Br}_{\mathrm{vert}}(X/Y)$, whereas it may be difficult to give such formulae for the full Brauer group $\mathrm{Br}(X)$. For example, when $Y=\mathbb{P}^1_k$ and X_η is geometrically integral, generators of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ are explicitly computed in terms of the structure of the degenerate fibres of $f\colon X\to Y$.
- For certain types of morphisms, e.g. for families of quadrics of relative dimension at least 1 or for families of Severi–Brauer varieties, the full Brauer group is vertical, that is, the inclusion $\mathrm{Br}_{\mathrm{vert}}(X/Y)\subset\mathrm{Br}(X)$ is an equality.
- Over a number field k, the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ features in the definition of an obstruction to the existence of a k-point $P\in Y(k)$ such that the fibre X_P is smooth and has points in all completions of k.

In Section 11.2 we consider some dominant proper morphisms $f\colon X\to Y$ for which the map $f^*\colon \mathrm{Br}(Y)\to\mathrm{Br}(X)$ is surjective.

In Section 11.3.1 we illustrate the computations of Section 11.1 in the basic classical case (originally due to Iskovskikh) of conic bundles over $\mathbb{P}^1_k$,

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with singular fibres allowed. The main result of Section 11.3.2 is a formula for $\mathrm{Br}(X)$ in the case when X_η is a conic and Y is a rational surface over $\mathbb{C}$. This is used in Section 11.3.3 to recover the Artin–Mumford examples of unirational non-rational threefolds, along with several other examples.

Section 11.4 gives a description of the 2-torsion subgroup $\mathrm{Br}(X)[2]$, where $f\colon X\to Y$ is a double cover of a rational surface Y over an algebraically closed field k. In Section 11.5 we study a universal family of cyclic twists and the associated vertical Brauer group; this construction has useful arithmetic applications.

11.1 The vertical Brauer group

The following definition appeared for the first time in [Sko96, Def. 03].

Definition 11.1.1 *Let Y be an integral scheme with generic point $i\colon \eta\to Y$. Let $f\colon X\to Y$ be a dominant morphism, and let $X_\eta = X\times_Y \eta$ be the generic fibre of f. Write $j\colon X_\eta\to X$ for the natural inclusion, so that there is a cartesian square*

$$\begin{array}{ccc} X_\eta & \xrightarrow{j} & X \\ \downarrow & & \downarrow f \\ \eta & \xrightarrow{i} & Y \end{array} \tag{11.1}$$

The **vertical Brauer group** *of X/Y is*

$$\mathrm{Br}_{\mathrm{vert}}(X/Y) = \{A\in \mathrm{Br}(X) | j^*(A)\in \mathrm{Im}[\mathrm{Br}(\eta)\to\mathrm{Br}(X_\eta)]\}.$$

Let $\rho\colon \mathrm{Br}(\eta)\to\mathrm{Br}(X_\eta)$ be the map induced by the morphism $X_\eta\to\eta$ which is the left vertical arrow in (11.1). A formal consequence of the definition of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ is the exact sequence

$$0\longrightarrow \mathrm{Br}_{\mathrm{vert}}(X/Y)\longrightarrow \mathrm{Br}(X)\longrightarrow \mathrm{Br}(X_\eta)/\mathrm{Im}(\rho).$$

If X is regular and integral, then by Theorem 3.5.5 we have inclusions

$$\mathrm{Br}(X)\subset \mathrm{Br}(X_\eta)\subset \mathrm{Br}(\eta'),$$

where η' is the generic point of X and the first inclusion is induced by j. Then we have $\mathrm{Br}_{\mathrm{vert}}(X/Y) = \rho(\rho^{-1}(\mathrm{Br}(X)))$, so the following sequence is exact:

$$0\longrightarrow \mathrm{Ker}(\rho)\longrightarrow \rho^{-1}(\mathrm{Br}(X))\longrightarrow \mathrm{Br}_{\mathrm{vert}}(X/Y)\longrightarrow 0. \tag{11.2}$$

We shall mostly consider the case when X and Y are smooth, proper and geometrically integral varieties over a field k, so that $\eta = \mathrm{Spec}(k(Y))$ and $\eta' = \mathrm{Spec}(k(X))$. Then $\mathrm{Br}_{\mathrm{vert}}(X/Y) \subset \mathrm{Br}(X) \subset \mathrm{Br}(k(X))$ is the intersection of $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{nr}}(k(X)/k)$ (see Proposition 6.2.7) with the image of the restriction map $\mathrm{Br}(k(Y)) \to \mathrm{Br}(k(X))$. In other words, the elements of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ are the restrictions to $k(X)$ of the (possibly, ramified) classes in $\mathrm{Br}(k(Y))$ that become unramified in $k(X)$. It is clear that $\mathrm{Br}_0(X)$ is a subgroup of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$.

The following standard lemmas are frequently used.

Lemma 11.1.2 *Let $k \subset K$ be an extension of fields such that k is separably closed in K. Then the restriction map $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ is injective.*

Proof. Let K_{s} be a separable closure of K. Let k_{s} be the separable closure of k in K_{s}. The assumption implies that the natural map $\mathrm{Gal}(K_{\mathrm{s}}/K) \to \mathrm{Gal}(k_{\mathrm{s}}/k)$ is surjective. Thus a non-trivial character $\mathrm{Gal}(k_{\mathrm{s}}/k) \to \mathbb{Q}/\mathbb{Z}$ gives rise to a non-trivial character $\mathrm{Gal}(K_{\mathrm{s}}/K) \to \mathbb{Q}/\mathbb{Z}$. □

Lemma 11.1.3 *Let $k \subset K$ be a finite extension of fields. Then the kernel of the restriction map $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(K, \mathbb{Q}/\mathbb{Z})$ is finite.*

Proof. By Lemma 11.1.2 we can replace K by the separable closure of k in K. Thus we may assume that K is a finite separable extension of k. Increasing K we can assume that K/k is a finite Galois extension with Galois group G. In this case the kernel is $\mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$, which is finite. □

Lemma 11.1.4 *Let k be a field. Let k_i be a finite extension of k and let m_i be a positive integer, for $i = 1, \ldots, n$. Define $\mathcal{L} \subset \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ as*

$$\mathcal{L} = \bigcap_{i=1}^{n} \mathrm{Ker}[m_i \, \mathrm{res}_{k_i/k} \colon \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k_i, \mathbb{Q}/\mathbb{Z})].$$

Then $\mathcal{L}$ is an extension of a finite abelian group by $\mathrm{H}^1(k, \mathbb{Z}/m)$, where m is the g.c.d. of $m_1, \ldots, m_n$.

Proof. It is clear that $\mathcal{L}$ contains $\mathrm{H}^1(k, \mathbb{Z}/m) \subset \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$. Let us denote by F_i the kernel of $\mathrm{res}_{k_i/k} \colon \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(k_i, \mathbb{Q}/\mathbb{Z})$. Let F be the subgroup of $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ generated by the subgroups $F_1, \ldots, F_n$. By Lemma 11.1.3 each subgroup F_i is finite, hence F is finite too. Let us write $m = \sum_{i=1}^n a_i m_i$, where $a_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. If $x \in \mathcal{L}$, then $mx = \sum_{i=1}^n a_i m_i x \in F$, since $m_i x \in F_i$ for all i. To finish the proof, it remains to use the isomorphism $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})[m] \cong \mathrm{H}^1(k, \mathbb{Z}/m)$. □

Let k be a field of characteristic zero. Let X and Y be smooth integral varieties over k and let $f \colon X \to Y$ be a dominant morphism with generic fibre X_η, where $\eta = \mathrm{Spec}(k(Y))$ is the generic point of Y.

Then there is a commutative diagram of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X_\eta) & \xrightarrow{\{\partial_V\}} & \bigoplus_{P\in Y^{(1)}} \bigoplus_{V\subset X_P} \mathrm{H}^1(k(V),\mathbb{Q}/\mathbb{Z}) \\ & & \uparrow{\scriptstyle f^*} & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Br}(Y) & \longrightarrow & \mathrm{Br}(k(Y)) & \xrightarrow{\{\partial_P\}} & \bigoplus_{P\in Y^{(1)}} \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}) \end{array} \tag{11.3}$$

The bottom row is the exact sequence (3.20); here P are the codimension 1 points of Y. In the top sequence, $V \subset X_P$ ranges over the irreducible components of the fibre X_P. Then the map $\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(V),\mathbb{Q}/\mathbb{Z})$ is $m_V\mathrm{res}_{k(V)/k(P)}$, where m_V is the multiplicity of V_P in X_P. The diagram (11.3) commutes by the functoriality of residues (Theorem 3.7.5).

In the top complex, the map $\mathrm{Br}(X)\to\mathrm{Br}(X_\eta)$ is injective since X is regular. *If, moreover, $f\colon X\to Y$ is flat, then the top row of the diagram is exact.* Indeed, then the fibres of f have dimension $\dim(X)-\dim(Y)$ at every point, so the codimension 1 points of X that do not meet X_η are the generic points of the fibres at codimension 1 points of Y. Hence in this case the top row of (11.3) is obtained from (3.19) by taking the inductive limit over all the open subsets $f^{-1}(U)\subset X$, where U is a non-empty open subset of Y.

For an irreducible component V of X_P we define κ_V as the algebraic closure of $k(P)$ in $k(V)$. For $f\colon X\to Y$ as above (not necessarily flat), the fibre X_P at a codimension 1 point $P\in Y$ is a variety over $k(P)$ which is split if and only if it contains an irreducible component V of multiplicity $m_V=1$ such that $\kappa_V=k(P)$, see Proposition 10.1.4.

When X_η is geometrically integral, there is a non-empty Zariski open subset $U\subset Y$ such that the fibres of f at the points of U are geometrically integral [EGA, IV_3, Thm. 9.7.7]. Thus if X_η is geometrically integral, then all but finitely many fibres of f over the points of codimension 1 in Y are geometrically integral, hence split.

Proposition 11.1.5 *Let $f\colon X\to Y$ be a dominant morphism of smooth integral varieties over a field k of characteristic zero with geometrically integral generic fibre. Let S be the finite set of points $P\in Y$ of codimension 1 such that the fibre X_P is not split (for example, X_P can be empty). Then every element of $\mathrm{Br}_{\mathrm{vert}}(X/Y)$ can be written as $f^*(\alpha)$, where $\alpha\in\mathrm{Br}(k(Y))$ is such that if $P\notin S$, then $\partial_P(\alpha)=0$, and if $P\in S$, then*

$$\partial_P(\alpha)\in\bigcap_{V\subset X_P}\mathrm{Ker}[m_V\mathrm{res}_{\kappa_V/k(P)}\colon \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(\kappa_V,\mathbb{Q}/\mathbb{Z})]. \tag{11.4}$$

If f is flat, then for any $\alpha\in\mathrm{Br}(k(Y))$ which satisfies the above condition, the inverse image $f^*(\alpha)\in\mathrm{Br}(X_\eta)$ belongs to $\mathrm{Br}(X)$.

Proof. This follows from diagram (11.3) in view of Lemma 11.1.2. □

If X_P is empty, then the condition in (11.4) is vacuous.

This proposition shows, in particular, that split fibres can be disregarded, that is, counted as 'good' fibres, for the purpose of determining the vertical Brauer group attached to a morphism of varieties.

Corollary 11.1.6 *Let $f\colon X\to Y$ be a dominant morphism of smooth integral varieties over a field k of characteristic zero with geometrically integral generic fibre.*

(i) *Assume that for every point $P\in Y$ of codimension 1, the g.c.d. of the multiplicities m_V, where V is an irreducible component of X_P, is 1. (This condition is satisfied if the fibres of f over all points of Y of codimension 1 are geometrically split.) Then $\mathrm{Br}_{\mathrm{vert}}(X/Y)/f^*(\mathrm{Br}(Y))$ is finite.*

(ii) *Assume that for every point $P\in Y$ of codimension 1, the g.c.d. of the integers $m_V[\kappa_V:k(P)]$, where V is an irreducible component of X_P, is 1. (This condition is satisfied if the fibres of f over all points of Y of codimension 1 are split.) Then $\mathrm{Br}_{\mathrm{vert}}(X/Y)=f^*(\mathrm{Br}(Y))$.*

Proof. Proposition 11.1.5 implies that $\mathrm{Br}_{\mathrm{vert}}(X/Y)/f^*(\mathrm{Br}(Y))$ is a subquotient of

$$\bigoplus_{P\in Y^{(1)}}\mathrm{Ker}\,\big[\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\longrightarrow\bigoplus_{V\subset X_P}\mathrm{H}^1(\kappa_V,\mathbb{Q}/\mathbb{Z})\big],$$

where the map to $\mathrm{H}^1(\kappa_V,\mathbb{Q}/\mathbb{Z})$ is $m_V\mathrm{res}_{\kappa_V/k(P)}$. Now both statements follow from Lemmas 11.1.2 and 11.1.4. For (ii) we also use the fact that $\mathrm{cores}_{\kappa_V/k(P)}\circ\mathrm{res}_{\kappa_V/k(P)}$ is multiplication by $[\kappa_V:k(P)]$. This can be applied to the Galois module $\mathbb{Q}/\mathbb{Z}$ because $\mathrm{char}(k)=0$. □

Remark 11.1.7 The proof of Corollary 11.1.6 (i) actually shows that the subgroup of $\mathrm{Br}(k(Y))$ consisting of the classes α such that $f^*(\alpha)\in\mathrm{Br}(k(X))$ lies in the image of $\mathrm{Br}(X)$, is finite modulo the image of $\mathrm{Br}(Y)$.

Let k be a field of characteristic zero. Let X be a smooth, projective, geometrically integral variety over k. Let $f\colon X\to\mathbb{P}^1_k$ be a dominant morphism whose generic fibre X_η is geometrically integral over $K=k(\eta)=k(\mathbb{P}^1_k)$. For a closed point $P\in\mathbb{P}^1_k$, with residue field $k(P)$, let $X_P=\sum_{i=1}^n m_iV_i$ be the decomposition of the fibre X_P as a linear combination of integral divisors. Let k_i be the integral closure of $k(P)$ in the function field $k(V_i)$. Define

$$\mathcal{L}_P=\bigcap_{i=1}^n\mathrm{Ker}[m_i\mathrm{res}_{k_i/k(P)}\colon\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k_i,\mathbb{Q}/\mathbb{Z})]\subset\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}).$$

For almost all $P\in\mathbb{P}^1_k$ we have $\mathcal{L}_P=0$.

Recall that we write $\rho\colon\mathrm{Br}(K)\to\mathrm{Br}(X_\eta)$ for the restriction map.

Proposition 11.1.8 *With the hypotheses and notation as above, there are exact sequences*

$$\mathrm{Ker}(\rho) \longrightarrow \rho^{-1}(\mathrm{Br}(X))/\mathrm{Br}(k) \longrightarrow \mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \rho^{-1}(\mathrm{Br}(X)) \longrightarrow \bigoplus_{P\in\mathbb{P}^1_k} \mathcal{L}_P \longrightarrow \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z}),$$

where the last map is induced by corestriction $\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})$.

Proof. The first sequence follows from (11.2). Since X is integral and $\mathbb{P}^1_k$ is a regular integral curve, the dominant morphism $X\to\mathbb{P}^1_k$ is flat [Har77, Prop. III.9.7]. Thus the rows of the diagram (11.3) for $Y = \mathbb{P}^1_k$ are exact. Moreover, for $Y = \mathbb{P}^1_k$, the bottom exact sequence can be completed to the Faddeev exact sequence (1.34). (Recall that $\partial = r_W$.) We thus obtain the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X_\eta) & \xrightarrow{\{\partial_V\}} & \bigoplus_{P\in(\mathbb{P}^1_k)^{(1)}}\bigoplus_{V\subset X_P} \mathrm{H}^1(k(V),\mathbb{Q}/\mathbb{Z}) & & \\
 & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle \rho} & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{Br}(K) & \xrightarrow{\{\partial_P\}} & \bigoplus_{P\in(\mathbb{P}^1_k)^{(1)}} \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}) & \longrightarrow & \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z})\to 0
\end{array}$$

where each map $\mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(V),\mathbb{Q}/\mathbb{Z})$ is given by $m_V\mathrm{res}_{k(V)/k(P)}$, where $X_P = \sum m_V V$. The proposition now follows by diagram chase. □

When Corollary 11.1.6 can be applied, it gives sufficient conditions for triviality or at least finiteness of $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}_0(X)$. The above proposition can be used to give examples where this quotient is infinite.

Corollary 11.1.9 *Let k be a field finitely generated over $\mathbb{Q}$. Let X be a smooth, projective, geometrically integral variety over k. Let $f\colon X\to\mathbb{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. For a closed point $P\in\mathbb{P}^1_k$ let m_P be the multiplicity of the fibre X_P, that is, the g.c.d. of the multiplicities of the irreducible components of X_P.*

(i) *Consider the composed map, where the second arrow is the sum of corestriction maps* $\mathrm{cores}_{k(P)/k}$*:*

$$\bigoplus_{P\in\mathbb{P}^1_k} \mathrm{H}^1(k(P),\mathbb{Z}/m_P) \hookrightarrow \bigoplus_{P\in\mathbb{P}^1_k} \mathrm{H}^1(k(P),\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k,\mathbb{Q}/\mathbb{Z}).$$

The group $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X)$ is finite if and only if the kernel of this map is finite.

(ii) *If there exist an integer $m\geq 2$ and distinct k-points $P_1,P_2\in\mathbb{P}^1_k(k)$ such that the fibres X_{P_1} and X_{P_2} are divisors on X which are divisible by m, then $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X)$ is infinite.*

Proof. Since k is finitely generated over $\mathbb{Q}$, so is $K = k(\mathbb{P}^1_k)$. By Proposition 5.4.4 this implies that the kernel of $\rho\colon \mathrm{Br}(K) \to \mathrm{Br}(X_\eta)$ is finite. Statement (i) then follows from Proposition 11.1.8 and Lemma 11.1.4.

For any $\chi \in \mathrm{H}^1(k, \mathbb{Z}/m)$, consider the element of the first direct sum in (i) with the P_1-component χ, the P_2-component $-\chi$, and all the other components 0. This gives an embedding of $\mathrm{H}^1(k, \mathbb{Z}/m)$ into the kernel of the map in (i). It remains to observe that for any field k finitely generated over $\mathbb{Q}$ and any integer $m \geq 2$ the group $\mathrm{H}^1(k, \mathbb{Z}/m)$ is infinite. Indeed, the algebraic closure κ of $\mathbb{Q}$ in k is a finite extension of $\mathbb{Q}$. By Lemma 11.1.2, $\mathrm{H}^1(\kappa, \mathbb{Q}/\mathbb{Z})$ is a subgroup of $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$. But $\mathrm{H}^1(\kappa, \mathbb{Z}/m)$ is infinite for any $m \geq 2$ and any number field κ, see, e.g. [Har17, Exercise 18.4 (b)]. □

When the ground field k is a number field, one can give a necessary and sufficient condition for the finiteness of the vertical Brauer group modulo the image of $\mathrm{Br}(k)$.

Theorem 11.1.10 *Let k be a number field. Let X be a smooth, projective, geometrically integral variety over k. Let $f\colon X \to \mathbb{P}^1_k$ be a dominant morphism with geometrically integral generic fibre. Then* $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X)$ *is infinite if and only if there exist an integer $m \geq 2$ and two distinct points of $\mathbb{P}^1_k(k_\mathrm{s})$ such that the k_s-fibres of f at these points are divisors divisible by m.*

Proof. If there exist two such k_s-points that are k-points, then the group $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X)$ is infinite by Corollary 11.1.9 (ii). If this is not the case, there is a closed point $P \in \mathbb{P}^1_k$ such that $[k(P):k] \geq 2$ and the fibre X_P is a divisor on X divisible by m. By Corollary 11.1.9 (i) it is enough to prove that the kernel of

$$\mathrm{cores}_{k(P)/k}\colon \mathrm{H}^1(k(P), \mathbb{Z}/m) \longrightarrow \mathrm{H}^1(k, \mathbb{Z}/m)$$

is infinite. Let $\Gamma = \mathrm{Gal}(k_\mathrm{s}/k)$ and $\Gamma_P = \mathrm{Gal}(k_\mathrm{s}/k(P))$. Write $(\mathbb{Z}/m)[\Gamma/\Gamma_P]$ for the induced Γ-module $\mathbb{Z}/m \otimes_{\mathbb{Z}[\Gamma_P]} \mathbb{Z}[\Gamma]$. Consider the surjective map of Γ-modules $\sigma\colon (\mathbb{Z}/m)[\Gamma/\Gamma_P] \to \mathbb{Z}/m$ given by the sum of coordinates. Let $M = \mathrm{Ker}(\sigma)$. The isomorphism $\mathrm{H}^1(k(P), \mathbb{Z}/m) \cong \mathrm{H}^1(k, (\mathbb{Z}/m)[\Gamma/\Gamma_P])$ of Shapiro's lemma identifies $\mathrm{cores}_{k(P)/k}$ with the induced map σ_*. This map fits into the exact sequence

$$\mathbb{Z}/m \longrightarrow \mathrm{H}^1(k, M) \longrightarrow \mathrm{H}^1(k, (\mathbb{Z}/m)[\Gamma/\Gamma_P]) \xrightarrow{\sigma_*} \mathrm{H}^1(k, \mathbb{Z}/m).$$

Since M is a finite non-zero Γ-module, the group $\mathrm{H}^1(k, M)$ is infinite by [Har17, Exercise 18.4 (b)].

It remains to show that when all the multiple fibres of f are above k-points of $\mathbb{P}^1_k$, with pairwise coprime multiplicities, then $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)/\mathrm{Br}_0(X)$ is finite. This immediately follows from Corollary 11.1.9 (i). □

Exercise 11.1.11 Let t be a coordinate function on $\mathbb{A}^1_\mathbb{Q} \subset \mathbb{P}^1_\mathbb{Q}$. Let X be a smooth, projective, geometrically integral surface over $\mathbb{Q}$ with a morphism

$X \to \mathbb{P}^1_{\mathbb{Q}}$ whose generic fibre X_η is the smooth plane cubic curve over $\mathbb{Q}(t)$ defined by

$$u^3 + tv^3 + t^2w^3 = 0.$$

Prove that the group $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_{\mathbb{Q}})/\mathrm{Br}_0(X)$ is infinite, using a valuative argument to show that $\mathrm{div}_X(t) = 3D$ for some divisor D on X.

Remark 11.1.12 Vertical Brauer groups have been computed in various set-ups, including some cases where the generic fibre is given explicitly but no explicit smooth projective model has been constructed for the total space. Examples include families of quadrics over $\mathbb{P}^1_k$ and families of Severi–Brauer varieties over $\mathbb{P}^1_k$, see [Sko90], [CTS94] and the detailed discussion of conic bundles further below. In these cases one has $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k) = \mathrm{Br}(X)$.

In more general cases this equality may not hold, and there is no systematic way to compute the quotient $\mathrm{Br}(X)/\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)$.

For instance, one would like to compute $\mathrm{Br}(X)$ for a smooth, projective and geometrically integral variety X equipped with a morphism $X \to \mathbb{P}^1_k$ whose generic fibre is geometrically integral and birationally equivalent to a homogeneous space of a connected linear algebraic group G over $K = k(\mathbb{P}^1_k)$. Already in the case when the generic fibre X_K is birationally equivalent to a principal homogeneous space for $G = T \times_k K$, where T is a k-torus, it is difficult to compute $\mathrm{Br}(X)/\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)$. By [CTHS03, Lemme 2.1], in this case it is a subgroup of a known group, namely the unramified Brauer group of the k-torus T modulo $\mathrm{Br}(k)$, but in general one does not know which subgroup. A concrete case is when X_K is birationally equivalent to the affine K-variety with equation

$$\mathrm{N}_{L/k}(\Xi) = P(t),$$

where L/k is a finite separable extension and $P(t) \in k[t]$ is a non-zero polynomial. (The morphism to $\mathbb{P}^1_k$ is given by the coordinate t.) For some computations in this direction see [CTHS03] and [Wei12]; see also [VV12].

11.2 Families of split varieties

Using diagram (11.3) one can compute the Brauer group of a product of two varieties, under a simplifying assumption on the geometry of one of them. (For more general statements see Sections 5.7 and 16.3.)

Proposition 11.2.1 *Let k be a field of characteristic zero. Let X and Y be smooth, projective, geometrically integral varieties over k such that $X(k) \neq \emptyset$. If $\mathrm{Pic}(\overline{X})$ is torsion-free and $\mathrm{Br}(\overline{X}) = 0$, then $\mathrm{Br}(X \times_k Y)$ is generated by the images of $\mathrm{Br}(X)$ and $\mathrm{Br}(Y)$ with respect to the maps induced by projections. Moreover, if $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) = 0$, then the map $\mathrm{Br}(Y) \to \mathrm{Br}(X \times_k Y)$ is an isomorphism.*

Proof. The assumptions on X imply that $\mathrm{Br}(X) = \mathrm{Br}_1(X)$ and give a split exact sequence (see Section 5.4)

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \longrightarrow 0.$$

If one extends the ground field from k to the function field $K = k(Y)$ of Y, the assumptions on the geometric Picard group and on the geometric Brauer group of X_K over the algebraic closure $\overline{K}$ of K are preserved. For the Picard group, see Section 5.1. For the Brauer group, see Proposition 5.2.3. Thus one still has the analogous exact sequence for the K-variety X_K. Moreover, the map $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X_{\overline{K}})$ is an isomorphism and the absolute Galois group of $k(Y)$ acts on these finitely generated free abelian groups via its quotient Γ_k, which gives an isomorphism $\mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \xrightarrow{\sim} \mathrm{H}^1(K, \mathrm{Pic}(X_{\overline{K}}))$. Then the compatible split exact sequences

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{H}^1(k, \mathrm{Pic}(\overline{X})) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X_K) \longrightarrow \mathrm{H}^1(K, \mathrm{Pic}(X_{\overline{K}})) \longrightarrow 0$$

show that the natural map $\mathrm{Br}(X)/\mathrm{Br}(k) \to \mathrm{Br}(X_K)/\mathrm{Br}(K)$ is an isomorphism. All the fibres of the projection $X \times_k Y \to Y$ are geometrically integral. Corollary 11.1.6 (ii) applied to the projection $X \times_k Y \to Y$ then immediately gives that $\mathrm{Br}(X \times_k Y)$ is generated by the images of $\mathrm{Br}(X)$ and $\mathrm{Br}(Y)$ under the two projections. Compare with [Gon18]. □

Theorem 11.2.2 *Let k be a field of characteristic zero. Let $f \colon X \to Y$ be a dominant projective morphism of smooth geometrically integral varieties over k. Let $K = k(Y)$. Assume that the generic fibre X_K is birationally equivalent to a K-torsor for a simply connected semisimple group over K. Then the map $f^* \colon \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is an isomorphism.*

Proof. We have a commutative diagram of natural pullback maps

$$\begin{array}{ccc}
\mathrm{Br}(X) & \hookrightarrow & \mathrm{Br}(X_K) \\
\uparrow & & \cong\uparrow \\
\mathrm{Br}(Y) & \hookrightarrow & \mathrm{Br}(k(Y))
\end{array}$$

The injectivity of the horizontal arrows is due to the fact that X and Y are smooth and integral. The right-hand vertical arrow is an isomorphism by Proposition 9.2.1. Thus $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/Y)$. Now the result follows from Proposition 10.1.15 in view of Corollary 11.1.6 (ii). □

Corollary 11.2.3 *Let k be a field of characteristic zero. Let $H \hookrightarrow \mathrm{GL}_n$ be an arbitrary linear group over k. Let $H \hookrightarrow G$ be an embedding into a simply connected semisimple group G. Then $\mathrm{Br}_{\mathrm{nr}}(k(\mathrm{GL}_n/H)/k) \cong \mathrm{Br}_{\mathrm{nr}}(k(G/H)/k)$.*

Proof. (Cf. [LA15, Prop. 26]) Let $P = G \times_k \mathrm{GL}_n$. Consider the quotient P/H with respect to the diagonal action of H on the right. The projection of $P \to G$ induces a morphism $P/H \to G/H$ which is a left GL_n-torsor. Similarly, the morphism $P/H \to \mathrm{GL}_n/H$ induced by the projection $P \to \mathrm{GL}_n$ is a left G-torsor. Any GL_n-torsor is locally trivial for the Zariski topology, thus P/H is birationally equivalent to $G/H \times_k \mathrm{GL}_k$, hence P/H and G/H have isomorphic unramified Brauer groups (Corollary 6.2.10). Since G is simply connected and semisimple, Theorem 11.2.2 implies that the map $\mathrm{Br}_{\mathrm{nr}}(k(\mathrm{GL}_n/H)) \to \mathrm{Br}_{\mathrm{nr}}(k(P/H))$ is an isomorphism. □

Proposition 11.2.4 *Let $f\colon X \to Y$ be a proper surjective morphism of smooth geometrically integral varieties over a field k of characteristic zero such that the generic fibre X_K is a smooth quadric of dimension at least* 1. *Suppose that either all the fibres over points of codimension* 1 *in Y are split, or* $\dim(X_K) \geq 3$. *Then the map $f^*\colon \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is surjective.*

Proof. From Proposition 7.2.4 we see that $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/Y)$. By Corollary 11.1.6 (ii) we have $\mathrm{Br}_{\mathrm{vert}}(X/Y) = f^*\mathrm{Br}(Y)$, whenever all the fibres over the points of codimension 1 of Y are split. It remains to show that the splitness condition is satisfied when $\dim(X_K) \geq 3$. Recall that, for $P \in Y$ of codimension 1, the property that X_P is split does not depend on the choice of a smooth and proper model $X \to Y$ over the local ring $\mathcal{O}_{P,Y}$, see Corollary 10.1.13. If $\dim(X_K) \geq 3$, then the standard reduction procedure described in the introduction to Section 10.2 allows one to construct a model whose closed fibre is split. See [CTS93, §3] for more details. □

In the next two sections, we shall consider families $f\colon X \to Y$ for which the map $f^*\colon \mathrm{Br}(Y) \to \mathrm{Br}(X)$ is not necessarily surjective.

11.3 Conic bundles

In this section we assume that $\mathrm{char}(k) \neq 2$. With extra care, one could extend most results to an arbitrary ground field.

Recall that a conic over a field k is a closed subscheme $C \subset \mathbb{P}^2_k$ defined by an equation $F = 0$, where F is a non-zero quadratic form. Since $\mathrm{char}(k) \neq 2$, one can choose homogeneous coordinates so that $F(x,y,z) = ax^2 + by^2 + cz^2$ with a, b, c not all equal to zero. The conic C is smooth if and only if $abc \neq 0$. The conic C is reduced if and only if at least two of the coefficients are non-zero.

For any smooth conic C over k there is a quaternion k-algebra Q such that C is isomorphic to the conic $C(Q)$ attached to Q, see Definition 1.1.11. By Witt's theorem (Theorem 1.1.15) two conics are isomorphic if and only if they are associated to isomorphic quaternion algebras. (For a direct construction of a quaternion algebra from a conic, see Section 7.1.) By Proposition 1.1.8, we have $C(Q) \simeq \mathbb{P}^1_k$ if and only if Q is not a division algebra, see also Remark 1.1.12 (3).

Definition 11.3.1 *Let B be a smooth and geometrically integral variety over a field k. A* **conic bundle** *over B is a geometrically integral variety X over k equipped with a proper morphism $f\colon X \to B$ whose generic fibre is a smooth conic. A* **regular conic bundle** *is a conic bundle whose total space X is smooth over k.*

Let $K = k(B)$ be the function field of B and let X_K be the generic fibre of $f\colon X \to B$. Let $A \in \mathrm{Br}(K)$ be the class of the quaternion algebra Q such that $X_K \simeq C(Q)$. If $A = 0$, then $X_K \simeq \mathbb{P}^1_K$, hence $\mathrm{Br}(X_K) \cong \mathrm{Br}(K)$. If $A \neq 0$ or, equivalently, X_K has no K-point, then, by Proposition 7.2.1, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Br}(K) \longrightarrow \mathrm{Br}(X_K) \longrightarrow 0, \tag{11.5}$$

where $1 \in \mathbb{Z}/2$ is mapped to $A \in \mathrm{Br}(K)$. In particular, if $f\colon X \to B$ is a regular conic bundle, then $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/B)$.

11.3.1 Conic bundles over a curve

In the case when the base is a curve, any conic bundle has a regular model such that all fibres are irreducible conics.

Lemma 11.3.2 *Let B be a smooth and geometrically integral curve with function field $K = k(B)$. For any conic bundle $f\colon X \to B$ there is a regular conic bundle $X' \to B$ such that $X_K \simeq X'_K$ and all fibres of $X' \to B$ are reduced irreducible conics.*

Proof. There is a non-empty open subset $U \subset B$ such that all the fibres of $f^{-1}(U) \to U$ are smooth conics. Let P be a closed point of $\mathbb{P}^1_k$ not in U and let $\mathcal{O}_P$ be the local ring of P in B. By Lemma 10.2.1 the generic fibre X_K has a regular model over $\mathrm{Spec}(\mathcal{O}_P)$ which is the closed subscheme of $\mathbb{P}^2_{\mathcal{O}_P}$ given either by an equation $x^2 - ay^2 - bz^2 = 0$ with $a, b \in \mathcal{O}_P^*$, or by an equation $x^2 - ay^2 - \pi z^2 = 0$, where $a \in \mathcal{O}_P^*$, $\pi \in \mathcal{O}_P$ is a uniformiser, and $\bar{a} \in k(P)$ is not a square. We note that in both cases the closed fibre of this model is a reduced irreducible conic. There exists a Zariski open neighbourhood $U_P \subset B$ of P such that this regular model over $\mathrm{Spec}(\mathcal{O}_P)$ extends to a regular U_P-scheme $\mathcal{X}'_P$ and the restrictions of $\mathcal{X}'_P$ and X to $U_P \smallsetminus \{P\}$ are isomorphic. Thus we can glue $f^{-1}(U)$ with $\mathcal{X}'_P$ for all closed points $P \in B$ not in U to obtain a B-scheme X'. Each fibre of the morphism $X' \to \mathbb{P}^1_k$ is either a fibre of $f^{-1}(U) \to U$ or a fibre of one of the morphisms $\mathcal{X}'_P \to U_P$, and these schemes are separated, thus the k-scheme of finite type X' is separated and hence is a variety over k. We conclude that $X' \to B$ is a desired conic bundle. $\square$

Let $f\colon X\to B$ be a regular conic bundle over a smooth and geometrically integral curve B all of whose fibres are reduced irreducible conics. In view of Lemma 11.3.2 we may restrict our attention to such 'relatively minimal' conic bundles. As above, let $K=k(B)$ and let $A\in \mathrm{Br}(K)$ be the class of the quaternion algebra associated to the conic over K which is the generic fibre of f. Let S be the finite set of closed points $P\in B$ such that the fibre X_P is not smooth over $k(P)$. Then X_P is a singular, reduced and irreducible conic. Thus the restriction of X to $\mathrm{Spec}(\mathcal{O}_P)$ is isomorphic to the closed subscheme of $\mathbb{P}^2_{\mathcal{O}_P}$ given by $x^2-ay^2-\pi z^2=0$ for some $a\in\mathcal{O}_P^*$ such that the image of a in $k(P)$ is not a square. Let us denote this image by a_P. The closed fibre $X_P\subset \mathbb{P}^2_{k(P)}$ is given by $x^2-a_Py^2=0$. To a point $P\in S$ we associate the quadratic field extension $F_P=k(P)(\sqrt{a_P})$ of the residue field $k(P)$; it is the extension over which X_P decomposes as a pair of transversal lines, with a unique intersection point defined over $k(P)$. We have $A=(a,\pi)\in\mathrm{Br}(K)$, hence by (1.18) we get

$$\partial_P(A)=a_P\in k(P)^*/k(P)^{*2}=\mathrm{H}^1(k(P),\mathbb{Z}/2).$$

This shows that F_P depends only on the generic fibre X_K; equivalently, a_P depends only on X_K up to multiplication by a square in $k(P)^*$.

We now consider conic bundles over the projective line.

Lemma 11.3.3 *Let k be a field of characteristic zero. Let $f\colon X\to\mathbb{P}^1_k$ be a regular conic bundle all of whose fibres are reduced irreducible conics. Let $K=k(\mathbb{P}^1_k)$ and let $A\in\mathrm{Br}(K)$ be the class of the quaternion algebra associated to the conic X_K. Then the following properties are equivalent.*

(a) *The class A is in the image of* $\mathrm{Br}(k)\to\mathrm{Br}(K)$.
(b) *There exists a smooth conic C over k such that $X_K\simeq C\times_k K$.*
(c) *For every closed point $P\in\mathbb{P}^1_k$, the fibre X_P is smooth.*

If these properties do not hold, then the map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ *is injective.*

Proof. We shall use Proposition 1.1.8 without further mention.

It is clear that (b) implies (a). Let us show the reverse implication. Write $A=(u,v)\in\mathrm{Br}(K)$ with $u,v\in K^*$. Since k is infinite, there exists a k-point P in $\mathbb{P}^1_k$ where u and v are invertible. Let $u(P)\in k^*$ be the value of u at P and let $v(P)\in k^*$ be the value of v at P. Under the assumption of (a), the class $(u,v)\in\mathrm{Br}(K)$ is the image of $(u(P),v(P))\in\mathrm{Br}(k)$ under the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(K)$. Indeed, the composition of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(\mathcal{O}_P)$ and the evaluation map $\mathrm{Br}(\mathcal{O}_P)\to\mathrm{Br}(k)$ is the identity map on $\mathrm{Br}(k)$, and the natural map $\mathrm{Br}(\mathcal{O}_P)\to\mathrm{Br}(K)$ is injective. Thus (a) implies (b).

The equivalence of (a) and (c) follows from the Faddeev exact sequence (1.34) via the interpretation of the class of a_P in $k(P)^*/k(P)^{*2}$ as the residue of A at P.

The kernel of $\mathrm{Br}(k)\to\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k(X))$ is equal to the kernel of the composition

$$\mathrm{Br}(k) \hookrightarrow \mathrm{Br}(K)\to\mathrm{Br}(X_K) \hookrightarrow \mathrm{Br}(k(X)).$$

By (11.5), this map is injective unless $A \neq 0$ is in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$. This establishes the last statement. □

Proposition 11.3.4 *Let k be a field of characteristic zero. Let $f\colon X\to\mathbb{P}^1_k$ be a regular conic bundle all of whose fibres are reduced irreducible conics. Let S be the finite set of closed points $P \in \mathbb{P}^1_k$ such that the fibre X_P is not smooth over the residue field $k(P)$. If the class $A \in \mathrm{Br}(K)$ associated to the conic X_K is not in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$, then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \bigoplus_{P\in S}(\mathbb{Z}/2)_P/\langle\partial(A)\rangle \longrightarrow k^*/k^{*2},$$

where $\partial(A) \in \bigoplus_{P\in S}(\mathbb{Z}/2)_P = \bigoplus_{P\in S}\mathrm{H}^1(F_P/k(P),\mathbb{Z}/2)$ is the vector with coordinates $\partial_P(A)$. The last map sends $1 \in (\mathbb{Z}/2)_P$ to the class of the norm $\mathrm{N}_{k(P)/k}(a_P)$ in k^*/k^{*2}.

Proof. In our situation, the second exact sequence of Proposition 11.1.8 gives rise to the exact sequence

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \rho^{-1}(\mathrm{Br}(X)) \longrightarrow \bigoplus_{P\in S}\mathrm{H}^1(F_P/k(P),\mathbb{Z}/2) \longrightarrow \mathrm{H}^1(k,\mathbb{Z}/2).$$

By the last claim of Proposition 11.1.8, each map

$$\mathrm{H}^1(F_P/k(P),\mathbb{Z}/2) \longrightarrow \mathrm{H}^1(k,\mathbb{Z}/2)$$

sends the generator of $\mathrm{H}^1(F_P/k(P),\mathbb{Z}/2) \cong \mathbb{Z}/2$ to the class of $\mathrm{N}_{k(P)/k}(a_P)$ in $k^*/k^{*2} \cong \mathrm{H}^1(k,\mathbb{Z}/2)$. Recall that $\mathrm{Ker}(\rho)$ is the group of order 2 generated by A, see (11.5). The proposition now follows from the identification $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k) = \mathrm{Br}(X)$ established in Lemma 11.3.3. □

Corollary 11.3.5 *Let k be a field of characteristic zero. Let $f\colon X\to\mathbb{P}^1_k$ be a regular conic bundle all of whose fibres are reduced irreducible conics. Assume that the class associated to the conic X_K is not in the image of $\mathrm{Br}(k)\to\mathrm{Br}(K)$. Let S be the finite set of closed points $P \in \mathbb{P}^1_k$ such that the fibre X_P is not smooth over the residue field $k(P)$. Fix a k-point $M \in \mathbb{P}^1_k(k)$ such that the fibre X_M is smooth. Let $\mathbb{A}^1_k = \mathrm{Spec}(k[t]) = \mathbb{P}^1_k \smallsetminus \{M\}$. Then we have a direct sum decomposition*

$$\mathrm{Br}(X) = \mathrm{Br}(k) \oplus f^*B,$$

where $B \subset \mathrm{Br}(K) = \mathrm{Br}(k(t))$ is a finite subgroup whose elements have the following explicit description.

Let $P(t) \in k[t]$ be the monic irreducible polynomial such that P is the zero set of $P(t)$. Let $\tau_P \in k(P)$ be the image of t in $k(P) = k[t]/(P(t))$. Consider *the subgroup* $B \subset \mathbb{F}_2^{|S|}$ *of vectors* $\varepsilon = (\varepsilon_P)$ *such that*

$$\prod_{P\in S} \mathrm{N}_{k(P)/k}(a_P)^{\varepsilon_P} = 1 \in k^*/k^{*2}.$$

The injective map $B \to \mathrm{Br}(K)$ *sends* ε *to*

$$A_\varepsilon = \sum_{P\in S} \varepsilon_P \, \mathrm{cores}_{k(P)(t)/k(t)}(t-\tau_P, a_P),$$

where $(t-\tau_P, a_P)$ *is the class of the quaternion algebra* $Q(t-\tau_P, a_P)$ *in* $\mathrm{Br}(k(P)(t))$.

Proof. A calculation based on Proposition 1.4.7 and formula (1.18), gives $\partial_P(A_\varepsilon) = a_P^{\varepsilon_P}$. (Compare with the discussion at the end of Section 1.5.) This shows that the map $B \to \mathrm{Br}(K)$ is indeed injective. Then the statement follows from Proposition 11.3.4. □

Exercise 11.3.6 Show that for any $\varepsilon \in B$ the class A_ε is unramified at M and, moreover, $A_\varepsilon(M) = 0$.

Exercise 11.3.7 Let $P(x) \in k[x]$ be a separable polynomial and let $a \in k^*$, $a \notin k^{*2}$. Let $f\colon X \to \mathbb{P}^1_k$ be a regular conic bundle birationally equivalent to the Châtelet surface given by the affine equation

$$y^2 - az^2 = P(x)$$

over $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$. Prove the following statements.

(a) If $P(x)$ is irreducible, or is the product of two irreducible polynomials of odd degree, then $\mathrm{Br}(X)/\mathrm{Br}(k) = 0$.
(b) If $P(x)$ is the product of two non-constant irreducible polynomials of even degree, each of which is irreducible over $k(\sqrt{a})$, then $\mathrm{Br}(X)/\mathrm{Br}(k) = \mathbb{Z}/2$.
(c) Assume that the degree of $P(x)$ is even. (It is always possible to reduce to this case by choosing the point at infinity in $\mathbb{P}^1_k$ with smooth fibre.) Let n be the number of monic irreducible factors of $P(x)$ of even degree which remain irreducible over $k(\sqrt{a})$. Let m be the (even) number of monic irreducible factors of $P(x)$ of odd degree. Then $\mathrm{Br}(X)/\mathrm{Br}(k) = (\mathbb{Z}/2)^s$, where $s = n-1$ if $m = 0$ and $s = n+m-2$ if $m > 0$.

Remark 11.3.8 Let $X \to \mathbb{P}^1_k$ be a regular conic bundle all of whose fibres are reduced irreducible conics. We assumed $\mathrm{char}(k) = 0$, hence $k_{\mathrm{s}} = \overline{k}$. The generic fibre of $X^{\mathrm{s}} \to \mathbb{P}^1_{k_{\mathrm{s}}}$ is a smooth conic over the field $k_{\mathrm{s}}(\mathbb{P}^1_k)$. By Theorem 1.1.14, this conic has a rational point, hence is isomorphic to the projective line. Thus the function field $k_{\mathrm{s}}(X)$ is a purely transcendental extension of k_{s}. The smooth projective surface X^{s} is birationally equivalent to $\mathbb{P}^2_{k_{\mathrm{s}}}$, hence $\mathrm{Br}(X^{\mathrm{s}}) = 0$ and $\mathrm{Br}_1(X) = \mathrm{Br}(X)$. Besides the recipe of Proposition 11.3.4, one can determine $\mathrm{Br}(X)/\mathrm{Br}(k)$ by identifying the Galois action on $\mathrm{Pic}(X^{\mathrm{s}})$

and then computing $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$. By Remark 5.4.3 (2), the last group is $\mathrm{Br}_1(X)/\mathrm{Br}(k)$.

This method produces the finer birational invariant given by the Galois module $\mathrm{Pic}(X^{\mathrm{s}})$ up to addition of a permutation module (Proposition 6.2.12) but it is slightly less effective for producing explicit generators of the group $\mathrm{Br}(X)$. Further references are [CTSS87], [Sko96], [Sko01, §7.1].

11.3.2 Conic bundles over a complex surface

The following result is due to Artin and Mumford [AM72, §3, Thm. 1] in the case when S is a smooth, projective, rational surface. We sketch a proof based on the Bloch–Ogus theory and Kato complexes.

Theorem 11.3.9 *Let S be a smooth integral surface over $\mathbb{C}$. Let n be a positive integer.*

(i) *For any integer j, there is a natural complex*

$$0 \longrightarrow \mathrm{H}^2(\mathbb{C}(S), \mu_n^{\otimes j}) \xrightarrow{\{\partial_x\}} \bigoplus_{x \in S^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mu_n^{\otimes (j-1)}) \longrightarrow \bigoplus_{y \in S^{(2)}} \mu_n^{\otimes (j-2)} \longrightarrow 0.$$

(ii) *For each $x \in S^{(1)}$, the map ∂_x in* (i) *is the Gysin residue. When $j = 2$, the value of the residue map*

$$\partial_x \colon \mathrm{H}^2(\mathbb{C}(S), \mu_n^{\otimes 2}) \longrightarrow \mathbb{C}(x)^*/\mathbb{C}(x)^{*n}$$

on $a \cup b$, where $a, b \in \mathrm{H}^1(\mathbb{C}(S), \mu_n) \cong \mathbb{C}(S)^*/\mathbb{C}(S)^{*n}$, is the inverse of the value given by formula (1.18).

(iii) *For each $y \in S^{(2)}$ and each $x \in S^{(1)}$, the map*

$$\mathrm{H}^1(\mathbb{C}(x), \mu_n^{\otimes (j-1)}) \longrightarrow \mu_n^{\otimes (j-2)}$$

is zero when y is not in the closure of x, otherwise it is the sum of Gysin residues ∂_z computed at the points z above y on the normalisation of the closure of x. For $j = 1$, this is the map $\mathbb{C}(x)^*/\mathbb{C}(x)^{*n} \to \mathbb{Z}/n$ induced by the valuation at z.

(iv) *When $j = 1$, passing to the limit in n we obtain the complex*

$$0 \longrightarrow \mathrm{Br}(\mathbb{C}(S)) \longrightarrow \bigoplus_{x \in S^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z}) \longrightarrow \bigoplus_{y \in S^{(2)}} (\mathbb{Q}/\mathbb{Z})(-1) \longrightarrow 0.$$

The kernel of the map $\mathrm{Br}(\mathbb{C}(S)) \to \bigoplus_{x \in S^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z})$ is $\mathrm{Br}(S)$.

(v) *If S is projective and rational then the complexes in* (i) *are exact except at the last term, where the homology group is $\mu_n^{\otimes (j-2)}$, the map being the sum* *over all points $y \in S$. In this case we have an exact sequence*

$$0 \to \mathrm{Br}(\mathbb{C}(S)) \to \bigoplus_{x \in S^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{y \in S^{(2)}} (\mathbb{Q}/\mathbb{Z})(-1) \to (\mathbb{Q}/\mathbb{Z})(-1) \to 0.$$

Proof. Let X be a smooth integral variety over $\mathbb{C}$. We write η for the generic point $\mathrm{Spec}(\mathbb{C}(X))$. Let $q \geq 0$ be an integer. Let $\mathcal{H}^q(\mu_n^{\otimes j})$ be the Zariski sheaf on X associated to the presheaf $U \mapsto \mathrm{H}^q_{\text{ét}}(U, \mu_n^{\otimes j})$. Then there is the local-to-global spectral sequence

$$E_2^{pq} = \mathrm{H}^p_{\mathrm{zar}}(X, \mathcal{H}^q(\mu_n^{\otimes j})) \Rightarrow \mathrm{H}^n_{\text{ét}}(X, \mu_n^{\otimes j}). \tag{11.6}$$

By the Gersten conjecture for étale cohomology proved by Bloch and Ogus in 1974 (see [CT95a, CTKH97]) there is an exact sequence of Zariski sheaves

$$0 \to \mathcal{H}^q(\mu_n^{\otimes j}) \to i_{\eta *}\mathrm{H}^q(\mathbb{C}(X), \mu_n^{\otimes j}) \to \bigoplus_{x \in X^{(1)}} i_{x*}\mathrm{H}^{q-1}(\mathbb{C}(x), \mu_n^{\otimes (j-1)}) \to \ldots \tag{11.7}$$

which is a flasque resolution of the sheaf $\mathcal{H}^q(\mu_n^{\otimes j})$. Here $i_{\eta *}$ is the map induced by the natural map $i\colon \eta \to X$, and similarly for i_{x*}. This construction uses Gysin maps, so the maps in the exact sequence (11.7) are induced by Gysin residues. The second statement of (ii) follows from Theorems 1.4.14 and 2.3.5.

A priori different, explicit complexes with explicitly defined maps were later introduced by K. Kato in [Kat86], where they are called "arithmetical Bloch–Ogus complexes". Kato's complexes are compatible with maps from Milnor K-theory to Galois cohomology, thus the values of residues on symbols are easy to identify. See also [Kerz09] and [GS17, §8.1, §8.2]. That the maps in the Bloch–Ogus complex and in the Kato complex coincide up to precise changes of signs was established by Jannsen, Saito and Sato [JSS14, §3.5]. The above description of the maps in the complexes follows from the results of this paper. In the main case of interest further down in this section, the coefficients of the cohomology groups are $\mathbb{Z}/2$, hence there are no problems with signs.

Since flasque sheaves are acyclic, taking global sections of the flasque resolution gives the Zariski cohomology groups of the sheaves $\mathcal{H}^q(\mu_n^{\otimes j})$. In particular, we obtain

$$\mathrm{H}^p_{\mathrm{zar}}(X, \mathcal{H}^q(\mu_n^{\otimes j})) = 0, \quad p > q.$$

Together with the spectral sequence (11.6) this gives an injective map

$$\mathrm{H}^1_{\mathrm{zar}}(X, \mathcal{H}^2(\mu_n^{\otimes j})) \hookrightarrow \mathrm{H}^3_{\text{ét}}(X, \mu_n^{\otimes j}). \tag{11.8}$$

Now set $q = 2$ and $j = 2$. Taking global sections of the flasque resolution (11.7) we obtain a complex

$$0 \longrightarrow \mathrm{H}^2(\mathbb{C}(X), \mu_n^{\otimes 2}) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(\mathbb{C}(x), \mu_n) \longrightarrow \bigoplus_{x \in X^{(2)}} \mathbb{Z}/n \longrightarrow 0. \tag{11.9}$$

By the purity theorem for the Brauer group we have an exact sequence

$$0 \longrightarrow \mathrm{Br}(X)[n] \longrightarrow \mathrm{Br}(\mathbb{C}(X))[n] \longrightarrow \bigoplus_{x\in X^{(1)}} \mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/n).$$

It shows that the cohomology group of (11.9) at $\mathrm{H}^2(\mathbb{C}(X),\mu_n^{\otimes 2})$ is canonically isomorphic to $\mathrm{Br}(X)[n]\otimes\mu_n$. The cohomology group at the middle term is $\mathrm{H}^1_{\mathrm{zar}}(X,\mathcal{H}^2(\mu_n^{\otimes j}))$. Finally, the cohomology group at the right term is the cokernel of the map

$$\bigoplus_{x\in X^{(1)}} \mathbb{C}(x)^*/\mathbb{C}(x)^{*n} \longrightarrow \bigoplus_{x\in X^{(2)}} \mathbb{Z}/n$$

induced by the divisor map on the normalisation of the closure of x in X. This group is $\mathrm{CH}^2(X)/n$, the mod n quotient of the codimension 2 Chow group $\mathrm{CH}^2(X)$.

If X is a smooth, projective, integral, rational variety over $\mathbb{C}$, then we have $\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)=0$. Indeed, $\mathrm{H}^1_{\text{ét}}(X,\mathbb{Z}/n)$ is a birational invariant and it vanishes for $X=\mathbb{P}^m_{\mathbb{C}}$.

Let us specialise to the case when $X=S$ is a smooth, projective, integral rational surface. From $\mathrm{H}^1_{\text{ét}}(S,\mathbb{Z}/n)=0$ and Poincaré duality for étale cohomology we get $\mathrm{H}^3_{\text{ét}}(S,\mathbb{Z}/n)=0$. Now the inclusion (11.8) gives $\mathrm{H}^1_{\mathrm{zar}}(S,\mathcal{H}^2(\mu_n^{\otimes j}))=0$. The Chow group of zero-cycles of *degree zero* on any smooth projective connected variety is divisible, as one sees by reducing to the case of curves. Hence for a surface the degree map $\mathrm{CH}^2(S)\to\mathbb{Z}$ induces an isomorphism $\mathrm{CH}^2(S)/n \xrightarrow{\sim} \mathbb{Z}/n$.

Since S is a smooth and projective rational variety, we have $\mathrm{Br}(S)=0$ by Corollary 6.2.11. The complex (11.9) then gives the exact sequence

$$0\to\mathrm{Br}(\mathbb{C}(S))[n]\otimes\mu_n\to \oplus_{x\in X^{(1)}}\mathbb{C}(x)^*/\mathbb{C}(x)^{*n}\to \oplus_{x\in X^{(2)}}\mathbb{Z}/n\to\mathbb{Z}/n\to 0.$$

Twisting by $\mu_n^{\otimes(-1)}$ and passing to the direct limit over all integers n gives the exact sequence in (v), which is [AM72, §3, Thm. 1]. □

Theorem 11.3.10 *Let S be an integral surface over $\mathbb{C}$. All elements of* $\mathrm{Br}(\mathbb{C}(S))[2]$ *are the classes of quaternion algebras.*

Proof. Any element of order 2 in the Brauer group of a field of characteristic not equal to 2 is the class of a tensor product of quaternion algebras. This is a special case of Merkurjev's theorem [Mer81], [GS17, Thm. 1.5.8], which is itself a special case of the Merkurjev–Suslin theorem [MS82], [GS17, Thm. 8.6.5]. In the special case when the field is the field of rational functions on a surface over $\mathbb{C}$, this theorem was proved earlier by S. Bloch [Blo80, Thm. (5.7)]. The tensor product of two quaternion algebras over $\mathbb{C}(S)$ is similar to a quaternion algebra. This follows from Albert's criterion [GS17, Thm. 1.5.5] and the fact that a quadratic form in at least five variables over $\mathbb{C}(S)$ has a nontrivial zero [Lang52]. □

For an elaborate proof using more elementary tools, see [Art82, Thm. 6.2].

Corollary 11.3.11 *Let S be a smooth, projective, integral rational surface over $\mathbb{C}$. Suppose that $\{\gamma_x\} \in \oplus_{x\in S^{(1)}}\mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/2)$ has trivial image in $\oplus_{y\in S^{(2)}}\mathbb{Z}/2$. Then there exists a quaternion algebra α over $\mathbb{C}(S)$ whose class in $\mathrm{Br}(\mathbb{C}(S))$ has residue $\gamma_x \in \mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/2)$ at each $x \in S^{(1)}$. The class of α in $\mathrm{Br}(\mathbb{C}(S))$ is uniquely defined.*

Proof. This follows from Theorems 11.3.9 (v) and 11.3.10. □

Note that the above proof is far from constructive: it is not clear how to find rational functions f and g in $\mathbb{C}(S)^*$ such that $\alpha = (f, g) \in \mathrm{Br}(\mathbb{C}(S))[2]$.

Proposition 11.3.12 *Let S be a smooth integral surface over $\mathbb{C}$ and let $\pi\colon X\to S$ be a proper morphism. If X is smooth and all the fibres of π are conics, then the morphism π is flat and the locus $C \subset S$ of points P whose fibre $f^{-1}(P)$ is not smooth is a curve with at most ordinary quadratic singularities.*

Proof. See [Bea77, Ch. I, Prop. 1.2]. There the surface S is $\mathbb{P}^2_{\mathbb{C}}$ but the arguments are purely local for the étale topology on S. □

Lemma 11.3.13 *Let S be a smooth, projective, integral surface over $\mathbb{C}$ and let $\pi\colon X\to S$ be a regular conic bundle. Let X_η be the generic fibre of π. Let $\alpha \in \mathrm{Br}(\mathbb{C}(S))[2]$ be the associated quaternion algebra class. Let Σ be the finite set of codimension 1 points $x \in S$ such that $\partial_x(\alpha) \neq 0$. Then there exists a regular conic bundle $\pi'\colon X'\to S$ with generic fibre isomorphic to X_η, such that all fibres of π' at codimension 1 points not in Σ are smooth conics, and such that the fibre of π' at a point $x \in \Sigma$ is the union of two conjugate lines over $k(x)$ defined over the quadratic extension given by $\partial_x(\alpha) \in k(x)^*/k(x)^{*2}$.*

Proof. This follows by patching from a similar statement for regular conic bundles over discrete valuations rings, see Section 10.2. □

Theorem 11.3.14 *Let S be a smooth projective rational surface over $\mathbb{C}$ and let $\pi\colon X\to S$ be a regular conic bundle. Let $\alpha \in \mathrm{Br}(\mathbb{C}(S))[2]$ be the associated quaternion algebra class. Assume that $\alpha \neq 0$. Let $C_1, \ldots, C_n$ be the integral curves in S such that the residue of α at the generic point of C_i is non-zero:*

$$0 \neq \partial_{C_i}(\alpha) \in \mathrm{H}^1(\mathbb{C}(C_i),\mathbb{Z}/2) \cong \mathbb{C}(C_i)^*/\mathbb{C}(C_i)^{*2}.$$

Assume that each C_i is smooth and that $C = \cup_{i=1}^n C_i$ is a curve with at most ordinary quadratic singularities. Let $H \subset (\mathbb{Z}/2)^n$ be the subgroup consisting of the elements $(r_1, \ldots, r_n)$ such that for $i \neq j$ we have $r_i = r_j$ if there is a point $P \in C_i \cap C_j$ with the property that $\partial_P(\partial_{C_i}(\alpha)) = \partial_P(\partial_{C_j}(\alpha)) \in \mathbb{Z}/2$ is non-zero. Then $\mathrm{Br}(X)$ is the quotient of H by the diagonal element $(1, \ldots, 1)$ which is the image of α.

Proof. By the birational invariance of $\mathrm{Br}(X)$ we can assume without loss of generality that $\pi\colon X\to S$ satisfies the conclusion of Lemma 11.3.13. The generic fibre X_η of π is a smooth conic over the function field $\mathbb{C}(S)$. The natural map $\mathrm{Br}(\mathbb{C}(S))\to\mathrm{Br}(X_\eta)$ is surjective with kernel $\mathbb{Z}/2$ spanned by $\alpha\neq 0$ (Proposition 7.2.1). Choose any $\beta\in\mathrm{Br}(X)$. The image of β in $\mathrm{Br}(X_\eta)$ is the image of some $\rho\in\mathrm{Br}(\mathbb{C}(S))$. For $x\in S^{(1)}$ write $\gamma_x=\partial_x(\alpha)$. Comparing the residues of ρ on S and on X, see diagram (11.3), we note that for any x in $S^{(1)}$ the residue of ρ in $\mathrm{H}^1(\mathbb{C}(x),\mathbb{Q}/\mathbb{Z})$ lies in the subgroup of $\mathrm{H}^1(\mathbb{C}(x),\mathbb{Z}/2)$ generated by γ_x. Since S is rational, we have $\mathrm{Br}(S)=0$. Thus the total residue map $\mathrm{Br}(\mathbb{C}(S))\to\oplus_{s\in S^{(1)}}\mathrm{H}^1(\mathbb{C}(x),\mathbb{Q}/\mathbb{Z})$ is injective, hence $2\rho=0$, so that $\mathrm{Br}(X)$ is of exponent 2. Moreover, the injective image of $\mathrm{Br}(X)$ in $\mathrm{Br}(\mathbb{C}(X))$ coincides with the image of a certain subgroup of $\mathrm{Br}(\mathbb{C}(S))[2]$ under the natural map $\mathrm{Br}(\mathbb{C}(S))[2]\to\mathrm{Br}(\mathbb{C}(X))[2]$ (whose kernel $\mathbb{Z}/2$ is generated by α).

Let us prove that this subgroup consists of the classes $\rho\in\mathrm{Br}(\mathbb{C}(S))[2]$ unramified outside C and with the property that

$$(\partial_{C_1}(\rho),\ldots,\partial_{C_n}(\rho))=(r_1\gamma_1,\ldots,r_n\gamma_n)\in\bigoplus_{i=1}^n\mathrm{H}^1(\mathbb{C}(C_i),\mathbb{Z}/2)$$

is in the kernel of the map

$$\bigoplus_{i=1}^n\mathrm{H}^1(\mathbb{C}(C_i),\mathbb{Z}/2)\longrightarrow\bigoplus_{y\in S^{(2)}}\mathbb{Z}/2.$$

Indeed, let $v\colon\mathbb{C}(X)\to\mathbb{Z}$ be a discrete valuation of the function field $\mathbb{C}(X)$ of X. It restricts to a trivial valuation of $\mathbb{C}$. Without loss of generality we can assume that v restricts to a non-trivial valuation on $\mathbb{C}(S)$. Let F_v be the residue field of v. Since S is projective, the valuation v is centred at a point M of the scheme S. If $M\notin C$, then clearly $\pi^*(\rho)$ is unramified at v. If M is the generic point of one of the C_i's, then the residue of ρ at v is a multiple of the residue of α at v, hence is zero since $\alpha=0$ in $\mathrm{Br}(\mathbb{C}(X))$.

In the following arguments, we shall systematically use formula (1.18).

Assume that M is a closed point which lies on exactly one C_i. Since (11.9) is a complex, the residue γ_i can be represented by the class of a rational function invertible at M. One may lift this function to a rational function h on S invertible at M. If u is a local equation for $C_i\subset S$ at M, the difference $\alpha-(h,u)$ is in the Brauer group of the local ring of S at M, because it has trivial residues at the curves passing through M. Thus the image of (h,u) in $\mathrm{Br}(\mathbb{C}(X))$ is unramified at v. Similarly, the difference $\rho-r_i(h,u)$ is in the Brauer group of the local ring of S at M. Hence the image of ρ in $\mathrm{Br}(\mathbb{C}(X))$ is unramified at v.

Let us now consider the case when M lies at the intersection of two curves C_1 and C_2.

Suppose first that $\partial_M(\partial_{C_1}(\alpha)) = \partial_M(\partial_{C_2}(\alpha)) = 0 \in \mathbb{Z}/2$. Let u_1, respectively u_2, be a local equation at M for $C_1 \subset S$, respectively for $C_2 \subset S$. There are rational functions h_1 and h_2 on S invertible at M with the property that $\rho - r_1(h_1, u_1) - r_2(h_2, u_2)$ is in the Brauer group of the local ring of S at M. The residue of the image of ρ in $\mathrm{Br}(\mathbb{C}(X))$ at v is then the class of a product of powers of $h_1(M)$ and $h_2(M)$ in F_v^*/F_v^{*2}, and this is 1, since $h_1(M), h_2(M) \in \mathbb{C}^*$.

Suppose now that $\partial_M(\partial_{C_1}(\alpha)) = \partial_M(\partial_{C_2}(\alpha)) = 1 \in \mathbb{Z}/2$. By assumption, we then have $r_1 = r_2$. Thus locally around M, the residue of ρ is a multiple of the residue of α, hence there exists an integer s (equal to 0 or 1) such that $\rho - s\alpha$ is in the Brauer group of the local ring of S at M. Since α vanishes in $\mathrm{Br}(\mathbb{C}(X))$, we conclude that the image of ρ in $\mathrm{Br}(\mathbb{C}(X))$ is unramified at v. □

Remark 11.3.15 By a definition common in the literature on complex algebraic geometry, a "standard conic bundle" over a surface S is a proper flat morphism $f\colon X \to S$ of smooth, projective, geometrically integral varieties such that each fibre is a conic, the locus $C \subset S$ where f is not smooth is a simple normal crossings divisor with smooth components, and f is relatively minimal. (Relative minimality means that any birational S-morphism $X \to X'$, where X' is a smooth and projective variety with a morphism to S, is an isomorphism.) Assume that $X \to S$ is a standard conic bundle – this is a stronger assumption than the hypothesis of Theorem 11.3.14. At a point $P \in S$ where two irreducible components C_1 and C_2 of C meet, the conic bundle can be given by an equation

$$X^2 - uY^2 - vT^2 = 0$$

over the completion $\mathbb{C}[[u, v]]$, see [Bea77, Lemme 1.5.2]. Thus the associated quaternion algebra α is (u, v). Then

$$\partial_P(\partial_{C_1}(\alpha)) = 1 = \partial_P(\partial_{C_2}(\alpha)).$$

Hence for all C_i in a given connected component of C, the integers r_i are equal. Then Theorem 11.3.14 gives the formula $\mathrm{Br}(X) \simeq (\mathbb{Z}/2)^{c-1}$, where c is the number of connected components of C. This result is mentioned by V.A. Iskovskikh in [Isk97, Teorema, p. 206]; it can also be extracted from the paper [Zag77].

Remark 11.3.16 A. Pirutka [Pir18, Thm. 3.17] obtained an analogue of Theorem 11.3.14 for the total space of a family of 2-dimensional quadrics over a rational surface.

11.3.3 Variations on the Artin–Mumford example

Now let us take $S = \mathbb{P}^2_{\mathbb{C}}$. Let E_1 and E_2 be two transversal smooth cubic curves in S. Let $\gamma_i \in \mathrm{H}^1_{\text{ét}}(E_i, \mathbb{Z}/2)$, $\gamma_i \neq 0$, for $i = 1, 2$. By Corollary 11.3.11 there exists a unique quaternion algebra class $\alpha \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))[2]$ unramified outside of $E_1 \cup E_2$, with residues γ_1 on E_1 and γ_2 on E_2. Let $\pi\colon X \to \mathbb{P}^2_{\mathbb{C}}$ be a regular conic bundle whose generic fibre is a conic corresponding to $\alpha \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$ as in Definition 1.1.11. By Proposition 7.2.1 (Witt's theorem) the kernel of the map $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2)) \to \mathrm{Br}(\mathbb{C}(X))$ is $\mathbb{Z}/2$ generated by α. Theorem 11.3.14 gives $\mathrm{Br}(X) = \mathbb{Z}/2$.

Artin and Mumford [AM72] provided a concrete example of such a situation and proved that $\mathrm{Br}(X) \neq 0$ by computing $\mathrm{H}^3(X(\mathbb{C}), \mathbb{Z})_{\mathrm{tors}}$ on an explicit smooth projective model. In [AM72, §2] they construct a singular variety V which is a double cover of $\mathbb{P}^3_{\mathbb{C}}$ ramified along a special quartic surface with 10 nodes. They compute an explicit resolution of singularities $\widetilde{V} \to V$ and determine $\mathrm{H}^3(\widetilde{V}, \mathbb{Z})_{\mathrm{tors}}$. In [AM72, §3], they study general conic bundles over rational complex surfaces. At the end of [AM72, §4], they come back to the variety V of [AM72, §2] and show that it is birationally equivalent to a conic bundle, and look at it from this point of view. Here are some details (cf. [CTO89]).

Artin and Mumford start with the following data: a smooth conic $C \subset \mathbb{P}^2_{\mathbb{C}}$ and two smooth cubic curves E_1 and E_2 which are each tritangent to C in distinct points P_1, Q_1, R_1 and P_2, Q_2, R_2 and such that E_1 and E_2 intersect transversally. Such configurations exist. Indeed, let us start with a smooth conic C given by an equation $q(x, y, t) = 0$. Fix three distinct points P_1, Q_1, R_1 on C. Let $l_1 = 0$, $m_1 = 0$, $n_1 = 0$ be equations of the lines through two of these points. Let d_1 be the equation of a line which is transversal to $l_1 m_1 n_1 = 0$, in particular, d_1 does not pass through any of the points P_1, Q_1, R_1. The linear system of cubics given by $\lambda q d_1 + \mu l_1 m_1 n_1 = 0$ has 6 base points: P_1, Q_1, R_1 and the points A_1, B_1, C_1 given by $l_1 = d_1 = 0$, $m_1 = d_1 = 0$, $n_1 = d_1 = 0$, respectively. For each of the 6 points there is a cubic curve in the linear system which is not singular at that point (the cubic $l_1 m_1 n_1$ is non-singular at A_1, B_1, C_1, whereas $q d_1$ is non-singular at P_1, Q_1, R_1). By one of the Bertini theorems over a field of characteristic zero [Jou84, Ch. I, Thm. 6.3.2], there exist smooth curves in this pencil. Any such curve intersects $q = 0$ exactly in the points P_1, Q_1, R_1, each time with multiplicity 2, hence is tangent to $q = 0$ at these points. Fix such a curve, with equation $h_1 = 0$. Then we choose three other points P_2, Q_2, R_2 on C; they give rise to l_2, m_2, n_2 as above. Choose a line $d_2 = 0$ transversal to $l_2 m_2 n_2 = 0$ and such that $d_2 = 0$ does contain any of the common points of $h_1 = 0$ and $l_2 m_2 n_2 = 0$. The linear system of cubics $\lambda q d_2 + \mu l_2 m_2 n_2 = 0$ has no base point contained in the curve $h_1 = 0$. As above, the Bertini theorem ensures that the general member of this system is a smooth cubic curve $h_2 = 0$ tangent to C at P_2,

Q_2, R_2. Moreover, the same Bertini theorem ensures that one can find h_2 such that $h_2 = 0$ is transversal to $h_1 = 0$.

Let $l = 0$ be a general tangent line to C. Then it is not hard to check (see [CTO89]) that the quaternion algebra $(q/l^2, h_1h_2/l^6)$ is unramified outside $E_1 \cup E_2$ and its residue at E_i is a non-zero element $\gamma_i \in \mathrm{H}^1_{\text{ét}}(E_i, \mathbb{Z}/2)$, for $i = 1, 2$. Similarly, the unique non-trivial residue of the quaternion algebra $(q/l^2, h_1/l^3)$ is $\gamma_1 \in \mathrm{H}^1_{\text{ét}}(E_1, \mathbb{Z}/2)$. By Theorem 11.3.14, for any regular conic bundle $\pi\colon X \to \mathbb{P}^2_{\mathbb{C}}$ whose generic fibre is a conic with associated quaternion algebra $(q/l^2, h_1h_2/l^6)$, the image of the class $(q/l^2, h_1/l^3) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$ in $\mathrm{Br}(\mathbb{C}(X))$ is a non-trivial element of $\mathrm{Br}(X)$.

One advantage of this concrete representation is that it leads to a proof of the unirationality of this particular variety X. Indeed, the conic bundle acquires a rational section after the base change from $\mathbb{P}^2_{\mathbb{C}}$ to the double cover $z^2 = q(x, y, t)$. This equation defines a smooth quadric in $\mathbb{P}^3_{\mathbb{C}}$ which is a rational variety.

In Section 12.1.2 we shall use this very special example for a deformation argument.

Similar examples are given in [CTO89]. The ramification locus in [CTO89, Example 2.4] is a union of eight lines.

Exercise 11.3.17 Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be a regular conic bundle. If the ramification locus $C = \cup_{i=1}^n C_i$ is a union of $n \leq 5$ lines without triple intersections, then $\mathrm{Br}(X) = 0$.

In fact, one can drop the assumption about triple intersections. For this, blow up $\mathbb{P}^2_{\mathbb{C}}$ in the points where more than two lines meet. We obtain a surface S, where the reduced total transform of the five lines (including the exceptional curves produced in the process) is a divisor C with normal crossings. We also obtain a regular conic bundle $X' \to S$ unramified outside C. Check that for any initial configuration of 5 lines, we have $\mathrm{Br}(X) = 0$.

Exercise 11.3.18 Construct regular conic bundles $X \to \mathbb{P}^2_{\mathbb{C}}$ with $\mathrm{Br}(X) \neq 0$ ramified exactly in the union of six lines.

It is enough to take six lines in general position and partition them into two triples, say L_1, L_2, L_3 and M_1, M_2, M_3. Choose $\gamma_{L_1} \in \mathbb{C}(L_1)^*/\mathbb{C}(L_1)^{*2}$ to be the class of a rational function whose divisor on L_1 is $(L_1 \cap L_2) - (L_1 \cap L_3)$, and similarly for the other lines. One immediately checks that the assumptions of Corollary 11.3.11 are fulfilled for the family γ_x with $\gamma_x = \gamma_{L_1}$ at $x = L_1$, similarly at the other five lines, and $1 \in \mathbb{C}(x)^*/\mathbb{C}(x)^{*2}$ at other codimension 1 points. There thus exists a quaternion algebra (a, b) over $\mathbb{C}(S)$ which has exactly these residues. Thus one constructs a regular conic bundle $X \to S = \mathbb{P}^2_{\mathbb{C}}$ whose ramification locus is the union of these six lines in $\mathbb{P}^2_{\mathbb{C}}$.

Choosing six lines tangent to a given smooth conic, one produces a degenerate version of the Artin–Mumford example.

Exercise 11.3.19 Let (u, v) be the coordinates in $\mathbb{A}^2_{\mathbb{C}}$. Let $X \to \mathbb{A}^2_{\mathbb{C}}$ be the conic bundle given in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by the equation

$$S^2 v(v^2-1) - T^2 u(u^2-1) + uv(u^2-v^2)W^2 = 0.$$

Let $Y \to \mathbb{A}^2_{\mathbb{C}}$ be the conic bundle given in $\mathbb{A}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by the equation

$$S^2 - uT^2 - vR^2 = 0.$$

By computing residues on $\mathbb{A}^2_{\mathbb{C}}$ show that X and Y are birationally equivalent over $\mathbb{A}^2_{\mathbb{C}}$. *Hint*: Use the fact that if two quaternions algebras have the same class in the Brauer group, then the associated conics are isomorphic. Conclude that X is rational over $\mathbb{C}$. For background and a detailed proof, see [CT15].

Remark 11.3.20 *A construction of a unirational but not stably rational variety fibred in Severi–Brauer varieties over* $\mathbb{P}^2_{\mathbb{C}}$. In [CTO89, Exemple 2.4] one constructs a non-trivial unramified Brauer class in the function field of a conic bundle over $\mathbb{P}^2_{\mathbb{C}}$ without actually producing a nice explicit model. This example can be generalised.

Let p be a prime. Let L_1, respectively L_2, be the line in $\mathbb{P}^2_{\mathbb{C}}$ given by the affine equation $u = 0$, respectively by $v = 0$. Choose p distinct points on each of these affine lines. Join each of these p points on L_1 to all the p points on L_2. Let g_1 be an equation of the union of these p^2 lines. Do this construction again using disjoint sets of points. Let g_2 be an equation of the union of the second family of p^2 lines. Let ζ be a primitive p-th root of unity. Let $X \to \mathbb{P}^2_{\mathbb{C}}$ be a proper morphism such that X is smooth and the generic fibre X_η is the Severi–Brauer variety over $\mathbb{C}(\mathbb{P}^2) = \mathbb{C}(u, v)$ attached to the cyclic algebra $(g_1 g_2, u/v)_\zeta$.

By Amitsur's theorem ([GS17, Thm. 5.4.1], see also Section 7.1), the kernel of the restriction map $\mathrm{Br}(\mathbb{C}(\mathbb{P}^2)) \to \mathrm{Br}(\mathbb{C}(X))$ is the $\mathbb{Z}/p$-module generated by the class $(g_1 g_2, u/v)_\zeta$. Comparing the residues of $\alpha = (g_1 g_2, u/v)_\zeta$ and $\beta = (g_1, u/v)_\zeta$ at codimension 1 points of $\mathbb{P}^2_{\mathbb{C}}$, one sees that β is not a multiple of α, hence its image $\beta_{\mathbb{C}(X)} \in \mathrm{Br}(\mathbb{C}(X))$ does not vanish. One then shows that the residue of $\beta_{\mathbb{C}(X)}$ is trivial at any point x of codimension 1 of X by studying the behaviour of β at the point $y \in \mathbb{P}^2$ which is the image of x. (Note that y can have dimension 0, 1 or 2.) Thus $\mathrm{Br}(X) \neq 0$. This implies that X is not stably rational.

Let $K = \mathbb{C}(u, v) = \mathbb{C}(\mathbb{P}^2)$. Let $L = K(\sqrt[p]{g_1 g_2})$. By Proposition 7.1.12 the generic fibre X_η is birationally equivalent to the affine K-variety with equation $\mathrm{N}_{L/K}(\Xi) = u/v$. Let $E = K(\sqrt[p]{u/v})$. We have $E = \mathbb{C}(u, z)$, where $z^p = u/v$, so E is a purely transcendental extension of $\mathbb{C}$. The variety $X_E = X_\eta \times_K E$ is then birationally equivalent to the affine variety over E with equation $\mathrm{N}_{EL/E}(\Xi) = 1$. As is well-known (Hilbert's theorem 90 for a cyclic extension, see the proof of Proposition 7.1.11), the latter variety is an E-torus isomorphic to the cokernel of the diagonal embedding $\mathbb{G}_{m,E} \to R_{EL/E}(\mathbb{G}_m)$. But this is an open set of a projective space over E, hence the function field of X_E is purely transcendental over E, hence over $\mathbb{C}$. Thus the function field $\mathbb{C}(X)$ is contained in a purely transcendental extension of $\mathbb{C}$, hence X is unirational.

For some recent computations of unramified Brauer groups of conic bundles over threefolds, see [ABBP].

11.4 Double covers

The following theorem is a special case of [Sko17, Thm. 1.1]. We refer to [Sko17] for the proof of this theorem and more general results.

Theorem 11.4.1 *Let k be an algebraically closed field,* $\mathrm{char}(k) \neq 2$*. Let S be a smooth, projective, integral surface over k such that* $\mathrm{Pic}(S)[2] = 0$ *and* $\mathrm{Br}(S)[2] = 0$*, for instance a rational surface. Let X be a smooth, projective, integral surface over k with a morphism $\pi\colon X \to S$ which makes X a double cover ramified exactly along a smooth irreducible curve C. Let $j\colon C \hookrightarrow X$ be the natural closed embedding. There is a natural map $\Phi\colon \mathrm{Pic}(C)[2] \to \mathrm{Br}(X)[2]$, which gives rise to an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(C)[2]/j^*(\mathrm{Pic}(X)[\pi_*]) \longrightarrow \mathrm{Br}(X)[2] \longrightarrow \mathrm{Pic}(S)/\pi_*(\mathrm{Pic}(X)) \longrightarrow 0.$$

Here $\mathrm{Pic}(X)[\pi_*]$ denotes the kernel of $\pi_*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(S)$.

In the special case when $S = \mathbb{P}^2_k$ we have $\mathrm{Pic}(S) = \mathrm{Pic}(\mathbb{P}^2_k) = \mathbb{Z}$, hence $\mathrm{Pic}(S)/\pi_*(\mathrm{Pic}(X))$ is 0 or $\mathbb{Z}/2$.

Here we content ourselves with giving the definition of the map Φ. It comes from the comparison of the Gysin sequences for étale cohomology groups of S and X with coefficients $\mu_2 = \mathbb{Z}/2$:

$$\begin{array}{ccccccc}
\mathrm{H}^2_{\text{ét}}(X,\mu_2) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(X\smallsetminus C,\mu_2) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(X,\mu_2) \\
\uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle [0]} & & \uparrow{\scriptstyle \pi^*} \\
\mathrm{H}^2_{\text{ét}}(S,\mu_2) & \longrightarrow & \mathrm{H}^2_{\text{ét}}(S\smallsetminus C,\mu_2) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(S,\mu_2)
\end{array}$$

The morphism $\pi\colon X \to S$ is ramified along C with ramification index 2, hence the induced map $\mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) \to \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2)$ is zero.

Since S and X are smooth, the restriction maps

$$\mathrm{Pic}(S) \longrightarrow \mathrm{Pic}(S\smallsetminus C), \quad \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X\smallsetminus C)$$

are surjective, and the restriction maps

$$\mathrm{Br}(S) \longrightarrow \mathrm{Br}(S\smallsetminus C), \quad \mathrm{Br}(X) \longrightarrow \mathrm{Br}(X\smallsetminus C)$$

are injective. Using the Kummer sequences with coefficients μ_2, one obtains a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}(X)[2] & \longrightarrow & \mathrm{Br}(X \smallsetminus C)[2] & \longrightarrow & \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(X,\mu_2) \\ & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle [0]} & & \uparrow{\scriptstyle \pi^*} \\ 0 & \longrightarrow & \mathrm{Br}(S)[2] & \longrightarrow & \mathrm{Br}(S \smallsetminus C)[2] & \longrightarrow & \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^3_{\text{ét}}(S,\mu_2) \end{array}$$

We thus get a map

$$\varPhi\colon \mathrm{Ker}[\mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) \to \mathrm{H}^3_{\text{ét}}(S,\mu_2)] \longrightarrow \mathrm{Br}(X)[2]/\pi^*(\mathrm{Br}(S)[2]).$$

Assuming $\mathrm{Pic}(S)[2] = 0$, we have $\mathrm{H}^1_{\text{ét}}(S,\mu_2) = 0$ and thus by Poincaré duality $\mathrm{H}^3_{\text{ét}}(S,\mu_2) = 0$. If, moreover, $\mathrm{Br}(S)[2] = 0$, then we get a map

$$\varPhi\colon \mathrm{Pic}(C)[2] \cong \mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) \longrightarrow \mathrm{Br}(X)[2].$$

Remark 11.4.2 (1) We have a natural inclusion

$$\mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2) \hookrightarrow \mathrm{H}^1(k(C),\mathbb{Z}/2) \cong k(C)^*/k(C)^{*2},$$

where injectivity is due to the fact that C is normal and the isomorphism comes from the Kummer sequence. The image of $\mathrm{H}^1_{\text{ét}}(C,\mathbb{Z}/2)$ consists of the classes of rational functions $f \in k(C)^*$ such that $\mathrm{div}(f) = 2D$ for a divisor D on C. Via the map $\varPhi$ such a rational function gives rise to an element of $\mathrm{Br}(X)[2] \subset \mathrm{Br}(k(X))[2]$. By Merkurjev's theorem [Mer81], [GS17, Theorems 1.5.8 and 8.6.5] every such class is a sum of quaternion algebras. When k is algebraically closed, $k(X)$ is a C_2-field by a theorem of Lang [Lang52]. By Albert's criterion [GS17, Thm. 1.5.5] the class of a sum of quaternion algebras in $\mathrm{Br}(k(X))$ is equal to the class of a single quaternion algebra. It seems quite a challenge to construct such a quaternion algebra explicitly starting from a function f as above.

(2) Other papers have been concerned with double and more generally cyclic covers [F92, vG05, CV15, IOOV17]. In [IOOV17] for a double cover of $S = \mathbb{P}^2$ as above, one constructs an exact sequence

$$0 \to \mathrm{Pic}(X)/(\mathbb{Z}\pi^*(\mathcal{O}(1)) + 2\mathrm{Pic}(X)) \to (\mathrm{Pic}(C)/\mathbb{Z}K_C)[2] \to \mathrm{Br}(X)[2] \to 0,$$

where $K_C \in \mathrm{Pic}(C)$ is the canonical class. The map

$$(\mathrm{Pic}(C)/\mathbb{Z}K_C)[2] \longrightarrow \mathrm{Br}(X)[2]$$

has a description in terms of a geometric construction of Azumaya algebras on X. See also [CV15].

Remark 11.4.3 In a different direction, one can ask the following question. Suppose that $X \to S$ is a double cover of smooth, projective, complex surfaces. Can one compute the kernel of the restriction map $\mathrm{Br}(S)\to\mathrm{Br}(X)$? A restriction-corestriction argument shows that this kernel is contained in $\mathrm{Br}(S)[2]$. An interesting case is that of an Enriques surface S and its unramified double K3 covering $X\to S$ over an algebraically closed field of characteristic zero. Here $\mathrm{Pic}(S)\simeq\mathbb{Z}^{10}\oplus\mathbb{Z}/2$, $\mathrm{Br}(S)\cong\mathbb{Z}/2$, $\mathrm{Pic}(X)$ is torsion-free, and $\mathrm{Br}(X)\simeq(\mathbb{Q}/\mathbb{Z})^s$ for some integer s with $2\leq s\leq 21$. Beauville [Bea09] showed that the kernel of the map $\mathbb{Z}/2=\mathrm{Br}(S)\to\mathrm{Br}(X)\simeq(\mathbb{Q}/\mathbb{Z})^s$ depends on the Enriques surface S. He proved that in the (coarse) moduli space of Enriques surfaces, the surfaces S for which the kernel is non-zero, hence is equal to $\mathbb{Z}/2$, form a countable, infinite union of non-empty algebraic hypersurfaces. In [HS05] one finds an example definable over $\mathbb{Q}$ for which the map $\mathrm{Br}(S)\to\mathrm{Br}(X)$ is injective.

One step in Beauville's proof is the following general result [Bea09, Prop. 4.1]. Let $\pi\colon X\to S$ be a cyclic étale covering of smooth projective varieties over an algebraically closed field k. Let σ be a generator of the Galois group G of $\pi\colon X\to S$, and let $\mathrm{N}=\pi_*\colon \mathrm{Pic}(X)\to\mathrm{Pic}(S)$ be the natural norm homomorphism. Then the kernel of $\pi^*\colon\mathrm{Br}(S)\to\mathrm{Br}(X)$ is isomorphic to $\mathrm{Ker}(\mathrm{N})/(1-\sigma^*)\mathrm{Pic}(X)$.

11.5 The universal family of cyclic twists

Let k be a field of characteristic zero and let $n\geq 2$ be an integer. Let X and Y be smooth, projective, geometrically integral varieties over k such that X has a generically free action of μ_n and Y is birationally equivalent to the quotient of X by this action. The associated field extension $k(Y)\subset k(X)$ can be written as $k(X)=k(Y)[t]/(t^n-f)$ for some $f\in k(Y)^*$. For any $a\in k^*$ one may consider k-varieties X_a such that $k(X_a)\simeq k(Y)[t]/(t^n-af)$. We shall call them *cyclic twists* of X. We shall define a 'universal family of cyclic twists'. This is a smooth, projective, geometrically integral k-variety $\widetilde{\mathcal{X}}$ equipped with a proper morphism $\widetilde{\mathcal{X}}\to\mathbb{P}^1_k$ that is smooth over the open set $\mathbb{G}_{m,k}\subset\mathbb{P}^1_k$ and such that the fibre at $a\in k^*$ is the cyclic twist X_a as above.

In this section we calculate the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k)$ attached to the universal family of cyclic twists $\widetilde{\mathcal{X}}\to\mathbb{P}^1_k$. We shall identify $\mathrm{Br}_{\mathrm{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k)$ with a specific subgroup of $\mathrm{Br}(Y)$.

This seemingly very special example of a vertical Brauer group is important for arithmetic applications to 'ramified descent' in Section 14.2.5. Suppose that k is a number field and the variety Y discussed above is everywhere locally solvable. The vertical Brauer group of the universal family of cyclic twists is a subgroup of $\mathrm{Br}(Y)$; if it gives no Brauer–Manin obstruction to the Hasse principle on Y, for example, if it is equal to $\mathrm{Br}(k)$, then, under an appropriate assumption on the ramification, there is an $a\in k^*$ such that X_a is everywhere locally solvable, see Theorem 14.2.25.

Let X be a smooth, projective and geometrically integral variety over k equipped with a generically free action of μ_n. Thus there is a dense open subset $U \subset X$ on which μ_n acts freely. By [MumAV, Ch. II, §7, Thm. 1] there exists a variety V over k and a finite étale morphism $\pi\colon U \to V$ such that V is a geometric quotient of U. Then U/V is a μ_n-torsor [GIT, Prop. 0.9]. This implies that V is smooth and geometrically integral. Let Y be a smooth, projective and geometrically integral variety which contains V as a dense open set.

For $a \in k^*$ the cyclic twist X_a of X by a is the quotient of $X \times_k T_a$ by the diagonal action of μ_n, where T_a is the μ_n-torsor over k given by $x^n = a$. The twists are naturally parameterised by the points of $\mathbb{G}_{m,k}$ and there is a universal family of cyclic twists $\mathcal{X} \to \mathbb{G}_{m,k}$. More precisely, one defines $\mathcal{X}$ as the quotient of $X \times_k \mathbb{G}_{m,k}$ by the diagonal action of μ_n, where $\mu_n \subset \mathbb{G}_{m,k}$ acts on $\mathbb{G}_{m,k}$ by multiplication. (The quotient exists as a variety because $X \times_k \mathbb{G}_{m,k}$ is quasi-projective, see [MumAV, Ch. II, §7, Thm. 1].) Then $\mathcal{V} = (U \times_k \mathbb{G}_{m,k})/\mu_n$ is Zariski open in $\mathcal{X}$. The projection $U \times_k \mathbb{G}_{m,k} \to U$ gives rise to a map $\mathcal{V} \to V$ which is a $\mathbb{G}_{m,k}$-torsor. We have the following commutative diagram, where the vertical arrows are quotients by μ_n and the arrows pointing left are $\mathbb{G}_{m,k}$-torsors:

$$\begin{array}{ccccc} U & \longleftarrow & U \times_k \mathbb{G}_{m,k} & \longrightarrow & \mathbb{G}_{m,k} \\ \downarrow & & \downarrow & & \downarrow \\ V & \longleftarrow & \mathcal{V} & \longrightarrow & \mathbb{G}_{m,k} \end{array}$$

By Hilbert's theorem 90 any $\mathbb{G}_m$-torsor is trivial over the generic point. Hence $\mathcal{V}$, and thus $\mathcal{X}$, is stably birationally equivalent to Y.

Using Hironaka's theorem, we can compactify $\mathcal{X}$ to a smooth, projective, geometrically integral variety $\widetilde{\mathcal{X}}$ equipped with a morphism $f\colon \widetilde{\mathcal{X}} \to \mathbb{P}^1_k$ so that $\mathcal{X} = f^{-1}(\mathbb{G}_{m,k})$. In particular, the restriction of $\widetilde{\mathcal{X}} \to \mathbb{P}^1_k$ to $\mathbb{G}_{m,k} \subset \mathbb{P}^1_k$ is smooth with geometrically integral fibres. These fibres are twists of X, so that the fibre over $a \in k^*$ is X_a. Since $\widetilde{\mathcal{X}}$ is stably birationally equivalent to Y, we have an isomorphism (Propositions 6.2.7 and 6.2.9)

$$\mathrm{Br}(\widetilde{\mathcal{X}}) \cong \mathrm{Br}(Y).$$

Let $[U/V]$ be the class of the torsor $\pi\colon U \to V$ in $\mathrm{H}^1_{\text{ét}}(V, \mu_n)$. Let $F \in k(Y)^*$ be a non-zero rational function such that the generic fibre of π is given by the equation $x^n = F$. Since U is geometrically integral, F is not a constant function. In $k(\widetilde{\mathcal{X}})$ we have the relation $tu^n = F$, for some $u \in k(\widetilde{\mathcal{X}})^*$, where $t \in k(\widetilde{\mathcal{X}})^*$ is the coordinate on $\mathbb{G}_{m,k}$. Write $\mathrm{div}_Y(F) = \sum_D m_D D$, where each D is an integral divisor in Y and m_D is a non-zero integer. Let k_D be the integral closure of k in $k(D)$.

Recall from Lemma 11.1.4 the notation

$$\mathcal{L}(F) = \bigcap_D \mathrm{Ker}[m_D \,\mathrm{res}_{k_D/k} \colon \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{H}^1(k_D, \mathbb{Q}/\mathbb{Z})]. \tag{11.10}$$

For $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n)$ we denote by $[U/V] \cup \chi \in \mathrm{H}^2_{\text{ét}}(V, \mu_n)$ the element obtained via the cup-product

$$\mathrm{H}^1_{\text{ét}}(V, \mu_n) \times \mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow \mathrm{H}^1_{\text{ét}}(V, \mu_n) \times \mathrm{H}^1_{\text{ét}}(V, \mathbb{Z}/n) \longrightarrow \mathrm{H}^2_{\text{ét}}(V, \mu_n).$$

Let us denote by $A_\chi \in \mathrm{Br}(V)$ the image of this element under the map $\mathrm{H}^2_{\text{ét}}(V, \mu_n) \to \mathrm{Br}(V)$ defined by the Kummer sequence. The restriction of A_χ to $\mathrm{Br}(k(Y))$ is the class of the cyclic algebra (χ, F). For each irreducible divisor $D \subset Y$ supported in $\mathrm{div}_Y(F)$ we have

$$\partial_D(A_\chi) = m_D \mathrm{res}_{k(D)/k}(\chi) \in \mathrm{H}^1(k(D), \mathbb{Q}/\mathbb{Z}),$$

which is zero if and only if $m_D \mathrm{res}_{k_D/k}(\chi) = 0 \in \mathrm{H}^1(k_D, \mathbb{Q}/\mathbb{Z})$, by Lemma 10.1.2. We have $\partial_D(A_\chi) = 0$ if D is not contained in $Y \smallsetminus V$. Thus $A_\chi \in \mathrm{Br}(Y)$ if and only if $\chi \in \mathcal{L}(F)[n]$.

Proposition 11.5.1 *In the above notation and assumptions we have the following statements.*

(i) *The group* $\mathrm{Br}_{\text{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k) \subset \mathrm{Br}(\widetilde{\mathcal{X}}) \cong \mathrm{Br}(Y)$ *is generated by* $\mathrm{Br}(k)$ *and the classes* $A_\chi = (\chi, F) = f^*((\chi, t))$, *where* $\chi \in \mathcal{L}(F)[n]$.

(ii) *Let m be the g.c.d. of the integers m_D, for all integral divisors D in the support of* $\mathrm{div}_Y(F)$. *If $(m, n) = 1$, then* $\mathrm{Br}_{\text{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k)$ *is finite modulo* $\mathrm{Br}(k)$.

(iii) *If $(m_D, n) = 1$ for some integral divisor D in the support of* $\mathrm{div}_Y(F)$, *then each fibre of* $f \colon \widetilde{\mathcal{X}} \to \mathbb{P}^1_k$ *is geometrically split.*

Proof. (i) Let $\varphi \colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the finite morphism given by $t = z^n$. By the definitions of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ the base change of $\widetilde{\mathcal{X}}/\mathbb{P}^1_k$ along φ is a variety birationally equivalent to $X \times_k \mathbb{P}^1_k$ over $\mathbb{P}^1_k$. We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Br}(k(z)) & \longrightarrow & \mathrm{Br}(k(X \times_k \mathbb{P}^1_k)) & \hookleftarrow & \mathrm{Br}(X \times_k \mathbb{P}^1_k) \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{Br}(k(t)) & \longrightarrow & \mathrm{Br}(k(\widetilde{\mathcal{X}})) & \hookleftarrow & \mathrm{Br}(\widetilde{\mathcal{X}})
\end{array}$$

where the Brauer groups in the right-hand column are identified with the unramified (over k) subgroups of their ambient groups.

By definition, any element of $\mathrm{Br}_{\text{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k)$ comes from some $A \in \mathrm{Br}(k(t))$ whose image in $\mathrm{Br}(k(\widetilde{\mathcal{X}}))$ lies in $\mathrm{Br}(\widetilde{\mathcal{X}})$. The fibres of $\mathcal{X} \to \mathbb{G}_{m,k}$ are geometrically integral, thus A can be ramified only at 0 and ∞ (Proposition 11.1.5). Let $\chi \in \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z})$ be the residue of A at ∞. By the diagram, $\varphi^*A \in \mathrm{Br}(k(z))$ gives an element of $\mathrm{Br}(k(X \times_k \mathbb{P}^1_k))$ that lies in $\mathrm{Br}(X \times_k \mathbb{P}^1_k)$. However, all fibres of the projection $X \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$ are geometrically integral, which implies that already φ^*A is unramified over k, so that $\varphi^*A \in \mathrm{Br}(\mathbb{P}^1_k) = \mathrm{Br}(k)$. The covering $\varphi\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ is ramified at ∞ with ramification index n, hence $n\chi = 0$. Thus $\chi \in \mathrm{H}^1(k, \mathbb{Z}/n)$. The Faddeev exact sequence (Theorem 1.5.2) and formula (1.19) imply that up to addition of an element of $\mathrm{Br}(k)$, the class A is represented by the cyclic algebra (χ, t). In $k(\widetilde{\mathcal{X}})$ we have the relation $tu^n = F$, so the image of (χ, t) in $\mathrm{Br}(k(\widetilde{\mathcal{X}})) = \mathrm{Br}(k(Y \times_k \mathbb{P}^1_k))$ is the image of $(\chi, F) \in \mathrm{Br}(k(Y))$, which is exactly $A_\chi \in \mathrm{Br}(V) \subset \mathrm{Br}(k(Y))$. Thus $(\chi, t) \in \mathrm{Br}(\widetilde{\mathcal{X}})$ if and only if $A_\chi \in \mathrm{Br}(Y)$. We have seen that $A_\chi \in \mathrm{Br}(Y)$ if and only if $\chi \in \mathcal{L}(F)$. This proves (i).

(ii) Lemma 11.1.4 implies that $\mathcal{L}(F)[n]$ is finite in this case.

(iii) Since the fibres over the points other than $t = 0$ and $t = \infty$ are geometrically integral, it is enough to consider the fibre above 0 (the fibre above ∞ is treated similarly). This geometric fibre has an integral component of multiplicity 1 if and only if the morphism $\widetilde{\mathcal{X}} \to \mathbb{P}^1_k$ has a section over $\bar{k}[[t]]$ (see Corollary 10.1.10). By the valuative criterion of properness, it suffices to show that the generic fibre of $\widetilde{\mathcal{X}} \to \mathbb{P}^1_k$ has a $\bar{k}((t))$-point. The generic fibre is the cyclic cover of $Y \times_k \bar{k}((t))$ given by $x^n = t^{-1}F$. By assumption, there is an irreducible divisor $D \subset Y$ with $\mathrm{val}_D(F) = m$ such that $(m, n) = 1$. Take $a, b \in \mathbb{Z}$ such that $am - bn = 1$ and $a > 0$. Consider the 'constant' $\bar{k}[[t]]$-scheme $\mathcal{Y} = Y \times_k \bar{k}[[t]]$ and let $\mathcal{D} = D \times_k \bar{k}[[t]] \subset \mathcal{Y}$. Since Y is smooth over k and D is generically smooth, using Hensel's lemma, we can find a section s of $\mathcal{Y} \to \mathrm{Spec}(\bar{k}[[t]])$ such that the value of s at the generic point $\mathrm{Spec}(\bar{k}((t)))$ is outside the support of $\mathrm{div}_Y(F)$ and the value of s at the closed point $\mathrm{Spec}(\bar{k})$ is contained in D but not in any other irreducible component of $\mathrm{div}(F)$; moreover, we can arrange that the intersection index of $\mathcal{D}$ and s in $\mathcal{Y}$ is a. Let v be the valuation of the discrete valuation ring $\bar{k}[[t]]$. By the construction of s we have $v(F(s)) = am$, so $v(t^{-1}F(s)) = am - 1 = bn$. Thus s lifts to a $\bar{k}((t))$-point on the cyclic cover of $Y \times_k \bar{k}((t))$ given by $x^n = t^{-1}F$. This means that the generic fibre of $W \to \mathbb{P}^1_k$ has a $\bar{k}((t))$-point. □

We compute the group in Proposition 11.5.1 in two concrete situations. Let $p(x)$ and $q(y)$ be separable non-constant polynomials with coefficients in k, and let $n \geq 2$ be a positive integer. Let C_1 and C_2 be smooth, projective curves with affine equations $u^n = p(x)$ and $v^n = q(y)$, respectively.

Example A Let $n = 2$. Consider the affine surface $z^2 = p(x)q(y)$. It is birationally equivalent to the quotient of $C_1 \times_k C_2$ by the diagonal action of μ_2 on u and v. Indeed, $z = uv$ is invariant and satisfies $z^2 = p(x)q(y)$. For example, if $p(x)$ and $q(y)$ are of degree 3 or 4, we obtain a K3 surface. If $\deg p(x) = \deg q(y) = 3$, we obtain the Kummer surface associated to the product of elliptic curves C_1 and C_2. If $\deg p(x) = \deg q(y) = 4$, we obtain the Kummer surface associated to a 2-covering of the product of Jacobians of C_1 and C_2. Such a situation occurs in [SkS05].

Example B Here we assume that $n = \deg p(x) = \deg q(x)$. Let $P(x, y)$ and $Q(z, w)$ be homogeneous forms of degree n such that $p(x) = P(x, 1)$ and $q(x) = Q(x, 1)$. The smooth surface $S \subset \mathbb{P}^3_k$ of degree n given by $P(x, y) = Q(z, w)$ is birationally equivalent to the quotient of $C_1 \times_k C_2$ by the diagonal action of μ_n on u and v. Indeed, $z = u/v$ is invariant under this action of μ_n and satisfies $p(x) = q(y)z^n$. If $n = 3$, then S is a smooth cubic surface; such a situation occurs in Swinnerton-Dyer's paper [SwD01]. If $n = 4$, then S is a quartic K3 surface, cf. [GvS].

Let us consider both examples at the same time. In Example A, to fix ideas, we assume that the degrees of $p(x)$ and $q(y)$ are even. The ramification locus of the projection $C_1 \to \mathbb{P}^1_k$ given by x is exactly the zero set of $p(x)$, and similarly for C_2. Define

$$L_1 = k[x]/(p(x)), \quad L_2 = k[y]/(q(y)), \quad L = L_1 \otimes_k L_2.$$

Let $Z = \mathrm{Spec}(L) \subset C_1 \times_k C_2$ be the closed subset given by $p(x) = q(y) = 0$; this is the fixed locus of the action of μ_n. Let U be the complement to Z in $C_1 \times_k C_2$. It is clear that U is the largest open subset of $C_1 \times_k C_2$ on which the diagonal action of μ_n is free. The singular locus of the quotient $(C_1 \times_k C_2)/\mu_n$ is $Z/\mu_n \cong Z$. We define Y as the minimal resolution of this quotient. Each singular $\bar{k}$-point of $(C_1 \times_k C_2)/\mu_n$ is an isolated quotient singularity with a well known resolution. Over the completion of its local ring, it is isomorphic to the vertex of the affine cone over the rational normal curve of degree n. The exceptional divisor of the resolution is a smooth irreducible rational curve E with $(E^2) = -n$.

Let X be the blow-up of Z in $C_1 \times_k C_2$. Then we have a finite morphism $\pi\colon X \to Y$ of smooth projective varieties whose restriction to U is a torsor $\pi\colon U \to V$ with structure group μ_n. We have the following commutative diagram where the vertical arrows are quotient morphisms by the action of μ_n and the horizontal arrows are birational morphisms:

$$\begin{array}{ccccc} C_1 \times_k C_2 & \longleftarrow & X & \longrightarrow & S' \\ \downarrow & & \downarrow & & \downarrow \\ (C_1 \times_k C_2)/\mu_n & \longleftarrow & Y & \longrightarrow & S \end{array}$$

The surfaces S and S' feature only in Example B: here $S' \subset \mathbb{P}^4_k$ is given by $t^n = P(x, y) = Q(z, w)$ and the action of μ_n on S' is by multiplication on the coordinate t. The natural projection $S' \to S$ is a torsor for μ_n away from its ramification divisor D which is given by $P(x, y) = Q(z, w) = 0$. (Geometrically this is the union of n^2 lines joining two sets of n points each. So $D_{\mathrm{sing}}(\bar{k})$ consists of $2n$ points.) Note that S'_{sing} is the union of closed subsets $x = y = 0$ and $z = w = 0$; the image of S'_{sing} in S is D_{sing}. The morphism $X \to S'$ is obtained by blowing-up S'_{sing}, and the morphism $Y \to S$ is obtained by blowing-up D_{sing}.

With notation as before we can take $F = P(x, y)/x^n$, then

$$\mathcal{L}(F)[n] = \mathrm{H}^1(L/k, \mathbb{Z}/n) = \mathrm{Ker}[\mathrm{res}_{L/k} \colon \mathrm{H}^1(k, \mathbb{Z}/n) \to \mathrm{H}^1(L, \mathbb{Z}/n)].$$

Proposition 11.5.2 *Assume that we are either in the situation of Example A, with* $n = 2$ *and* $\deg(p(x))$, $\deg(q(x))$ *even, or Example B, with* $n = \deg(p(x)) = \deg(q(x))$.

(i) *If* $\mathcal{L}$ *is generated by the subgroups* $\mathrm{H}^1(L_1/k, \mathbb{Z}/n)$ *and* $\mathrm{H}^1(L_2/k, \mathbb{Z}/n)$, *then* $\mathrm{Br}_{\mathrm{vert}}(\widetilde{\mathcal{X}}/\mathbb{P}^1_k) = \mathrm{Br}_0(Y)$.

(ii) *For* $n = 2$ *the condition of* (i) *is satisfied when each of* $p(x)$ *and* $q(y)$ *is irreducible with a pluriquadratic splitting field.*

(iii) *If* n *is a prime number, the condition of* (i) *is satisfied when*

$$p(x) = a_1 x^n + a_2, \quad q(y) = a_3 y^n + a_4, \quad \textit{where} \quad a_1, a_2, a_3, a_4 \in k^*.$$

Proof. (i) Recall that C_1 and C_2 are curves with affine equations $u^n = p(x)$ and $v^n = q(y)$, respectively. We have two natural morphisms $Y \to C_i \to \mathbb{P}^1_k$ given by the projections to the coordinates x and y, respectively. The rational function F on Y can be represented by either $p(x)$ or $q(y)$ modulo n-th powers. Thus, if $\chi \in \mathrm{H}^1(L_1/k, \mathbb{Z}/n)$, then $A_\chi = (p(x), \chi) \in \mathrm{Br}(Y)$ belongs to the image of $\mathrm{Br}(k(x))$ in $\mathrm{Br}(k(Y))$. As an element of $\mathrm{Br}(k(x))$, the class $(p(x), \chi)$ is unramified away from the closed points of $\mathbb{A}^1_k$ given by the monic irreducible factors $r(x)$ of $p(x)$. The residue at the closed point $r \in \mathbb{A}^1_k$ given by $r(x) = 0$ is the restriction $\mathrm{res}_{k_r/k}(\chi) \in \mathrm{H}^1(k_r, \mathbb{Z}/n)$, where $k_r = k[x]/(r(x))$. Since $L_1 = \prod_r k_r$, where the sum is over all monic irreducible $r(x)$ dividing $p(x)$, we have $\mathrm{res}_{k_r/k}(\chi) = 0$. Hence $(p(x), \chi)$ is unramified everywhere on $\mathbb{A}^1_k$. This implies that $(p(x), \chi) \in \mathrm{Br}(k)$. Similar considerations apply to the case $\chi \in \mathrm{H}^1(L_2/k, \mathbb{Z}/n)$. This proves (i).

(ii) In this case L is the direct sum of copies of $L_1 L_2$, the compositum of L_1 and L_2. All these fields are pluriquadratic extensions of k, and the statement follows at once.

(iii) In this case n is coprime to $[k(\zeta) : k] = n - 1$, where ζ is a primitive n-th root of unity. A restriction-corestriction argument then shows that it is enough to establish (i) for $k = k(\zeta)$, but this is straightforward. □

If $p(x)$ and $q(y)$ are very general, then the map $k^*/k^{*2} \to L^*/L^{*2}$ is injective. Such is the case if $p(x)$ and $q(y)$ are both irreducible of degree 4, the Galois closure of each of the extensions $k[x]/(p(x))$ and $k[y]/(q(y))$ is an extension of k whose Galois group is the symmetric group S_4, and these Galois extensions are linearly disjoint. See [HS16, Prop. 3.1, Lemma 2.1].

For a proof of (iii) in terms of valuations which avoids discussing the geometry of underlying varieties, see [CT03, Prop. 3.5].

Chapter 12
Rationality in a family

The specialisation method allows one to prove that a smooth and projective complex variety is not stably rational if it can be deformed into a mildly singular variety Z whose desingularisation has a non-zero Brauer group. The original idea is due to C. Voisin [Voi15], who stated it in terms of the decomposition of the diagonal. In this chapter we present this method in the set-up proposed by Colliot-Thélène and Pirutka [CTP16] and later simplified by S. Schreieder. In this form the method can be applied under very mild additional assumptions. As an example of application, we construct a conic bundle over $\mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth sextic curve which is not stably rational.

In Section 12.2 we consider smooth projective fourfolds X with a dominant morphism $X \to \mathbb{P}^2_{\mathbb{C}}$ such that the generic fibre is a quadric. Using a calculation of $\mathrm{Br}(X)$ in this case, we present the striking recent example of Hassett, Pirutka and Tschinkel of an algebraic family of smooth projective fourfolds some of whose elements are rational, whereas others are not even stably rational.

Most of the material in this chapter follows the exposition in [CT18].

12.1 The specialisation method

12.1.1 Main theorem

The following theorem is Schreieder's improvement [Sch18, Prop. 26] of the specialisation method. The assumptions in [Sch18, Prop. 26] are weaker than in this section. The same proof also works in the more general setting of higher unramified cohomology with torsion coefficients in place of the Brauer group.

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Schreieder's proof is cast in the geometric language of the decomposition of the diagonal. We give here a more 'field-theoretic' proof. It is known that both points of view are equivalent, cf. [ACTP17, CTP16].

Theorem 12.1.1 *Let R be a discrete valuation ring with field of fractions K and algebraically closed residue field κ of characteristic zero. Let $\mathcal{X}$ be an integral projective scheme over R, whose generic fibre $X = \mathcal{X}_K$ is smooth and geometrically integral and whose closed fibre Z/κ is geometrically integral. Assume that*

(i) *there exist a non-empty open set $U \subset Z$ and a projective, birational desingularisation $f\colon \widetilde{Z} \to Z$ such that $V = f^{-1}(U) \to U$ is an isomorphism and such that $\widetilde{Z} \smallsetminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\widetilde{Z}$;*
(ii) *$X_{\overline{K}}$ is stably rational, where $\overline{K}$ is an algebraic closure of K.*

Then the restriction map $\mathrm{Br}(\widetilde{Z}) \to \oplus_i \mathrm{Br}(\kappa(Y_i))$ is injective. In particular, if each $\mathrm{Br}(Y_i) = 0$, then $\mathrm{Br}(\widetilde{Z}) = 0$.

Proof. The morphism $\mathcal{X} \to \mathrm{Spec}(R)$ is flat. We can replace R by its completion and thus assume that $R = \kappa[[t]]$ and $K = \kappa((t))$. Since $X_{\overline{K}}$ is stably rational, there exists a finite extension $K' = \kappa((t^{1/n}))$ of K such that $X_{K'}$ is stably rational over K'. Write $R' = \kappa[[t^{1/n}]]$ and $\mathcal{X}_{R'} = \mathcal{X} \times_R R'$. Since $\mathcal{X}_{R'}/R'$ is flat, each of the irreducible components of $\mathcal{X}_{R'}$ dominates $\mathrm{Spec}(R')$, by [EGA, IV$_2$, Prop. 2.3.4 (iii)]. Since X is geometrically integral, the generic fibre $X_{K'}$ of $\mathcal{X}_{R'}/R'$ is integral, hence $\mathcal{X}_{R'}$ is an integral scheme. Thus we can replace $\mathcal{X}/R$ by $\mathcal{X}_{R'}/R'$. This operation does not affect the closed fibre.

Now $\mathcal{X}/R$ is an integral projective scheme whose generic fibre X/K is stably rational over K and whose closed fibre Z/κ satisfies (i). Since X is stably rational over K, by Proposition 6.4.4, for any field extension $K \subset F$ the degree map $\mathrm{CH}_0(X_F) \to \mathbb{Z}$ is an isomorphism.

Let $L = \kappa(Z)$. We have a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \bigoplus_i \mathrm{CH}_0(Y_{i,L}) & \longrightarrow & \mathrm{CH}_0(\widetilde{Z}_L) & \longrightarrow & \mathrm{CH}_0(V_L) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \cong} & & \\ & & \mathrm{CH}_0(Z_L) & \longrightarrow & \mathrm{CH}_0(U_L) & \longrightarrow & 0 \end{array}$$

Let us explain how this diagram is constructed. For each i, the closed embedding $\rho_i\colon Y_i \to \widetilde{Z}$ induces a map $\rho_{i*}\colon \mathrm{CH}_0(Y_{i,L}) \to \mathrm{CH}_0(\widetilde{Z}_L)$. The top exact sequence is the classical localisation sequence for the Chow group [Ful98, Prop. 1.8]. The map $f_*\colon \mathrm{CH}_0(\widetilde{Z}_L) \to \mathrm{CH}_0(Z_L)$ is induced by the proper morphism $f\colon \widetilde{Z} \to Z$. The map $\mathrm{CH}_0(V_L) \xrightarrow{\sim} \mathrm{CH}_0(U_L)$ is the isomorphism induced by the isomorphism[1] $f\colon V \xrightarrow{\sim} U$. Finally, $\mathrm{CH}_0(Z_L) \to \mathrm{CH}_0(U_L)$ is the restriction map.

[1] Instead of assuming that $f^{-1}(U) \to U$ is an isomorphism, it would be enough, as in [Sch19a], to assume that this morphism is a universal CH_0-isomorphism.

Let ξ be the generic point of $\widetilde{Z}$ and let η be the generic point of Z. Choose $m \in V(\kappa)$ and let $n = f(m) \in U(\kappa)$. Thus η and n_L are smooth L-points of Z_L.

Let $S = L[[t]]$ and let F be the field of fractions of S. The extension $R \subset S$ of complete discrete valuation rings is compatible with the extension $\kappa \subset L$ of their residue fields. By Hensel's lemma, the points η and n_L lift to F-points of the generic fibre X_F of $\mathcal{X}_S/S$. Since the degree map $\mathrm{CH}_0(X_F)\to\mathbb{Z}$ is an isomorphism, these two points are rationally equivalent in X_F. By Fulton's specialisation theorem for the Chow group of a proper scheme over a discrete valuation ring [Ful98, Ch. 20, §3], we obtain $\eta = n_L \in \mathrm{CH}_0(Z_L)$. Then from the above diagram we conclude that

$$\xi = m_L + \sum_i \rho_{i*}(z_i) \in \mathrm{CH}_0(\widetilde{Z}_L),$$

where $z_i \in \mathrm{CH}_0(Y_{i,L})$. There is a natural bilinear pairing (6.3)

$$\mathrm{CH}_0(\widetilde{Z}_L) \times \mathrm{Br}(\widetilde{Z}) \longrightarrow \mathrm{Br}(L).$$

Suppose that $\alpha \in \mathrm{Br}(\widetilde{Z})$ goes to zero in $\mathrm{Br}(\kappa(Y_i))$, for each i. Since Y_i is smooth and integral, by Theorem 3.5.5 already the image of α in $\mathrm{Br}(Y_i)$ is zero. The value $\alpha(m_L) \in \mathrm{Br}(L)$ is just the image of $\alpha(m) \in \mathrm{Br}(\kappa) = 0$. Now the above equality implies $\alpha(\xi) = 0 \in \mathrm{Br}(L)$. But since $\widetilde{Z}$ is smooth and integral, the pairing of $\mathrm{Br}(\widetilde{Z})$ with the generic point $\xi \in \widetilde{Z}_L(L)$ induces the embedding $\mathrm{Br}(\widetilde{Z}) \hookrightarrow \mathrm{Br}(\kappa(Z)) = \mathrm{Br}(L)$. Thus $\alpha = 0 \in \mathrm{Br}(\widetilde{Z})$. □

Remark 12.1.2 (1) One may replace condition (ii) in the above theorem by the weaker condition that $X_{\overline{K}}$ is universally CH_0-trivial. The same proof works.

(2) In the proof of the theorem, under the assumption of (ii), instead of specialisation of the Chow group one can use specialisation of R-equivalence on rational points. See [CTP16, Remarque 1.19], [Pir18, §2.4] and [CT18, §6].

(3) In [CTP16, Thm. 1.14], condition (i) of Theorem 12.1.1 is replaced by the condition that the desingularisation $f\colon \widetilde{Z}\to Z$ is universally CH_0-trivial, i.e., for any field extension $\kappa \subset L$ the map $f_*\colon \mathrm{CH}_0(\widetilde{Z}_L)\to\mathrm{CH}_0(Z_L)$ is an isomorphism. Under condition (ii), this implies that $\mathrm{Br}(\widetilde{Z}) = 0$.

12.1.2 Irrational conic bundles with smooth ramification

The Artin–Mumford example was used by Voisin [Voi15] to prove that very general double coverings of $\mathbb{P}^3_{\mathbb{C}}$ ramified in a smooth quartic hypersurface are not stably rational. It was used by Colliot-Thélène and Pirutka [CTP16] to prove that very general quartic hypersurfaces in $\mathbb{P}^4_{\mathbb{C}}$ are not stably rational. The specialisation method was applied in [HKT16] and [BB18] to prove that for $d \geq 6$ very general conic bundles over $\mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth curve of degree d are not stably rational. Let us show how the Artin–Mumford example can be used to establish the following special case of this result.

Proposition 12.1.3 *There exists a standard conic bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ ramified in a smooth curve of degree 6 such that X is not stably rational.*

Proof. In Deligne's Bourbaki talk [Del71] we find the following presentation of the Artin–Mumford example. As in Section 11.3.3 we are given two transversal smooth cubic curves with homogeneous equations $h_1 = 0$ and $h_2 = 0$ and a smooth conic $q = 0$ which is tangent to the cubic $h_i = 0$ in three points P_i, Q_i, R_i, where $i = 1, 2$. Moreover, the points $P_1, Q_1, R_1, P_2, Q_2, R_2$ are distinct and disjoint from the intersection points of the two cubics. Let $g = 0$ be a cubic curve that meets the conic in the divisor $P_1 + Q_1 + R_1 + P_2 + Q_2 + R_2$. Multiplying g by a non-zero number we arrange that the curve $h_1 h_2 - g^2 = 0$ contains the conic as an irreducible component, so that

$$h_1 h_2 = g^2 + qc$$

for some homogeneous polynomial c of degree 4. Consider the vector bundle $\mathcal{V} = \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ on $\mathbb{P}^2_{\mathbb{C}}$ and the quadratic form $\Phi \colon \mathcal{V} \to \mathcal{O}$ given by

$$\Phi(x, y, z) = cx^2 + 2gxy - qy^2 - z^2.$$

The vanishing of Φ defines a flat conic bundle $X \subset \mathbb{P}(\mathcal{V}^*)$ over $\mathbb{P}^2_{\mathbb{C}}$ whose total space has nine singular points, which are ordinary quadratic singularities. Resolving the singularities gives a birational map $X' \to X$. There are many ways to prove that $\mathrm{Br}(X') \neq 0$, see Section 11.3.3.

One then considers the family of all quadratic forms $\Phi \colon \mathcal{V} \to \mathcal{O}$ given by

$$\Phi(x, y, z) = Cx^2 + 2Gxy - Qy^2 - z^2,$$

where C, G, Q are homogeneous forms of degrees $4, 3, 2$, respectively. We claim that for a very general triple of such forms, the vanishing of the discriminant $G^2 + QC = 0$ defines a smooth curve in $\mathbb{P}^2_{\mathbb{C}}$. (Then the total space X is smooth.) More precisely, suppose that $C = 0$, $G = 0$, $Q = 0$ are smooth curves such that the closed set $C = G = Q = 0$ is empty. We claim that for almost all $\lambda \in \mathbb{C}$, the curve $G^2 + \lambda QC = 0$ is smooth. By one of the Bertini theorems ([Jou84, Ch. I, Cor. 6.7], [GH78, Ch. I, §1, p. 137]) since G^2 and QC have no common factor, it is enough to show that for $\lambda \neq 0$, the curve $G^2 + \lambda QC = 0$ has no singular point with $G^2 = 0$ and $QC = 0$. Any such point would satisfy $2GG'_x + \lambda Q'_x C + \lambda QC'_x = 0$ and similar equations with respect to the variables y and z. If the point lies on $G = C = 0$ it then satisfies $QC'_x = QC'_y = QC'_z = 0$, hence $Q = 0$ by the non-singularity of the curve

$C = 0$. However, the set $G = C = Q = 0$ is empty, so we have a contradiction. A similar argument shows that the point cannot lie on $G = Q = 0$.

Voisin's deformation argument in its original form [Voi15] can now be applied: by specialising to the Artin–Mumford example in the version recalled above, we see that the very general conic bundle in the family defined by C, G, Q is not stably rational. Alternatively, one can use Theorem 12.1.1 or [CTP16, Thm. 1.14] to establish the result. □

12.2 Quadric bundles over the complex plane

Hassett, Pirutka and Tschinkel [HPT18] used the specialisation method to give the first examples of families $X \to B$ of smooth, projective, integral complex varieties with some fibres rational and some other fibres not even stably rational. A simplified version of the specialisation method, as proposed by Schreieder [Sch18, Sch19b], gives a streamlined proof of the main result of [HPT18] which avoids explicit resolution of singularities. This simplified specialisation method was described in Section 12.1. In this section, following [CT18], we give examples from [HPT18] in their simplest form.

12.2.1 A special quadric bundle

The references for this section are [HPT18], [Pir18], [CT18].

Let x, y, z be homogeneous coordinates in $\mathbb{P}^2_k$ and let U, V, W, T be homogeneous coordinates in $\mathbb{P}^3_k$. Let

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx).$$

Let $X \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ be the hypersurface given by the bihomogeneous equation

$$yzU^2 + zxV^2 + xyW^2 + F(x, y, z)T^2 = 0.$$

Let $p\colon X \to \mathbb{P}^2_{\mathbb{C}}$ be the morphism given by the projection $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$. Since X is a hypersurface in a regular scheme, its local rings are Cohen–Macaulay. The fibres of $p\colon X \to \mathbb{P}^2_{\mathbb{C}}$ are 2-dimensional quadrics; in particular, p is a flat morphism. The morphism p is smooth over the complement to the plane octic curve defined by the vanishing of the determinant

$$x^2y^2z^2F(x, y, z) = 0.$$

This equation describes the union of the smooth conic $F = 0$ and three tangents to this conic taken with multiplicity 2. The variety X has singular points over the singular points of the curve $xyzF(x, y, z) = 0$.

Part (a) of the following proposition is a result of Hassett, Pirutka, and Tschinkel [HPT18, Prop. 11]. Part (b) is a special case of the general statement [Sch18, Prop. 7], the proof of which builds on results of Pirutka ([Pir18, Thm. 3.17], [Sch18, Thm. 4]). As we shall now see, the proof of (a) can be modified to simultaneously give a proof of (b).

Proposition 12.2.1 *Let $\widetilde{X}\to X$ be a projective birational desingularisation of X. Let*

$$\alpha = (x/z, y/z) \in \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))$$

and let β be the image of α under the map $p^*\colon \mathrm{Br}(\mathbb{C}(\mathbb{P}^2))\to\mathrm{Br}(\mathbb{C}(X))$.

(a) *We have $\beta \in \mathrm{Br}(\widetilde{X})$ and $\beta \neq 0$.*
(b) *For each irreducible divisor $Y \subset \widetilde{X}$ the restriction of β to $\mathrm{Br}(\mathbb{C}(Y))$ is 0.*

Proof. The equation of X is symmetric in (x, y, z). In view of this symmetry, it is enough to consider the open set $z = 1$ with affine coordinates x and y. In the rest of the proof we consider only this open set. Then $\alpha = (x, y)$ has non-trivial residues precisely at $x = 0$ and $y = 0$. In particular, $\alpha \neq 0$.

Let $K = \mathbb{C}(\mathbb{P}^2) = \mathbb{C}(x, y)$, let $L = \mathbb{C}(X)$, and let X_η/K be the generic fibre of $p\colon X\to\mathbb{P}^2_{\mathbb{C}}$. The discriminant of the quadratic form $\langle y, x, xy, F(x, y, 1)\rangle$ is not a square in K, thus the map $\mathrm{Br}(K)\to\mathrm{Br}(X_\eta)$ is an isomorphism by Proposition 7.2.4 (c), so that the composition $\mathrm{Br}(K) \xrightarrow{\sim} \mathrm{Br}(X_\eta) \hookrightarrow \mathrm{Br}(L)$ is injective. Thus $\beta = p^*(\alpha) \in \mathrm{Br}(L)$ is non-zero.

Let v be a discrete valuation $L^*\to\mathbb{Z}$, let S be the valuation ring of v and let κ_v be the residue field. If $K \subset S$, then $v(x) = v(y) = 0$, hence (x, y) is unramified. If $K \not\subset S$, then $S \cap K = R$ is a discrete valuation ring with field of fractions K. The image of the closed point of $\mathrm{Spec}(R)$ in $\mathbb{P}^2_{\mathbb{C}}$ is then either a point m of codimension 1 or a (complex) closed point m of $\mathbb{P}^2_{\mathbb{C}}$.

Consider the first case. If the codimension 1 point m does not belong to $xy = 0$, then $\alpha = (x, y) \in \mathrm{Br}(K)$ is unramified at m, hence $\beta \in \mathrm{Br}(L)$ is unramified at v. Moreover, the evaluation of β in $\mathrm{Br}(\kappa_v)$ is just the image under $\mathrm{Br}(\mathbb{C}(m))\to\mathrm{Br}(\kappa_v)$ of the evaluation of α in $\mathrm{Br}(\mathbb{C}(m))$. By Tsen's theorem $\mathrm{Br}(\mathbb{C}(m)) = 0$, hence the image of β in $\mathrm{Br}(\kappa_v)$ is zero.

Suppose that m is a generic point of a component of $xy = 0$, say m is the generic point of $x = 0$. In $L = \mathbb{C}(X)$ we have an identity

$$yU^2 + xV^2 + xyW^2 + F(x, y, 1) = 0$$

with $yU^2 + xV^2 \neq 0$. In the completion of K at the generic point of $x = 0$, $F(x, y, 1)$ is a square, because $F(x, y, 1)$ modulo x is equal to $(y-1)^2$, a non-zero square. Thus, in the completion L_v, the quadratic form $\langle y, x, xy, 1\rangle$ has a non-trivial zero, hence (x, y) goes to zero in $\mathrm{Br}(L_v)$. Hence β is unramified at v, thus $\beta \in \mathrm{Br}(S)$ and the image of β in $\mathrm{Br}(\kappa_v)$ is zero.

Now consider the second case, i.e., m is a closed point of $\mathbb{P}^2_{\mathbb{C}}$. There is a local homomorphism of local rings $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},m}\to S$ which induces an embedding $\mathbb{C}\to\kappa_v$ of residue fields. If $x(m) \neq 0$, then x becomes a non-zero square in the residue field $\mathbb{C}$ hence in κ_v. This implies that the residue of $\beta = (x, y) \in \mathrm{Br}(L)$ at v is trivial. The analogous argument holds if $y(m) \neq 0$. It remains to discuss the case $x(m) = y(m) = 0$. We have $F(0, 0, 1) = 1 \in \mathbb{C}^*$. Thus $F(x, y, 1)$ reduces to 1 in κ_v, hence is a square in the completion L_v. As above, in the completion L_v, the quadratic form $\langle y, x, xy, 1\rangle$ has a non-trivial zero, hence (x, y) goes to zero in $\mathrm{Br}(L_v)$. Hence β is unramified at v, thus $\beta \in \mathrm{Br}(S)$ and the image of β in $\mathrm{Br}(\kappa_v)$ is zero. □

As in the reinterpretation [CTO89] of the Artin–Mumford examples, the intuitive idea behind the above result is that the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$ is ramified along $xyzF(x, y, z) = 0$ and the ramification of the symbol $(x/z, y/z)$, which is "contained" in the ramification of the quadric bundle $X \to \mathbb{P}^2_{\mathbb{C}}$, disappears when one pulls back $(x/z, y/z)$ to a smooth projective model of X: ramification kills ramification (Abhyankar's lemma). Here one also uses the fact that the smooth conic defined by $F(x, y, z) = 0$ is tangent to each of the lines $x = 0$, $y = 0$, $z = 0$, and does not vanish at the intersection point of any two of these three lines.

12.2.2 Rationality is not deformation invariant

In this section we complete the simplified proof of the theorem of Hassett, Pirutka and Tschinkel [HPT18].

Theorem 12.2.2 *There exist a smooth projective family of complex fourfolds $X \to T$, where T is an open subset of the affine line $\mathbb{A}^1_{\mathbb{C}}$, and points m and n in $T(\mathbb{C})$ such that the fibre X_n is rational whereas the fibre X_m is not stably rational.*

Proof. Consider the universal family of quadric bundles over $\mathbb{P}^2_{\mathbb{C}}$ given in $\mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}}$ by a bihomogeneous form of bidegree $(2, 2)$. It is given by a non-zero symmetric (4×4)-matrix whose entries $a_{ij}(x, y, z)$ $(1 \leq i, j \leq 4)$ are homogeneous quadratic forms in x, y, z. The parameter space is $B = \mathbb{P}^{59}_{\mathbb{C}}$ (the corresponding vector space is given by the coefficients of ten quadratic forms in three variables). The determinant defines a hypersurface of degree 8 in B. We have the map $X \to B$ whose fibres X_m are quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$, where $X_m \subset \mathbb{P}^3_{\mathbb{C}} \times \mathbb{P}^2_{C}$ is the zero set of a non-zero complex bihomogeneous form of bidegree $(2, 2)$.

Using Bertini's theorem on smoothness for general members of a base point free linear system, one shows [Sch19a, Lemma 19] that there exists a non-empty open set $B_0 \subset B$ such that the fibres of $X \to B$ over the points $m \in B_0$ are flat quadric bundles $X_m \to \mathbb{P}^2_{\mathbb{C}}$ which are smooth as complex varieties.

Using Bertini's theorem on smoothness for general members of a base point free linear system, and the fact that the fibres are 2-dimensional quadrics over a smooth surface (for dimension reasons the argument does not work for a family of conics over a smooth surface) one shows [Sch19a, Lemma 20] that there exist points $m \in B_0$ with the property that the corresponding quadric bundle $X_m \to \mathbb{P}^2_{\mathbb{C}}$ has a smooth total space X_m and satisfies $a_{1,1} = 0$. This implies that the morphism $X_m \to \mathbb{P}^2_{\mathbb{C}}$ has a rational section given by the point $(1, 0, 0, 0)$, hence the generic fibre of $X_m \to \mathbb{P}^2_{\mathbb{C}}$ is rational over $\mathbb{C}(\mathbb{P}^2)$, so that the complex variety X_m is rational over $\mathbb{C}$.

The special example in Section 12.2.1 defines a point $P_0 \in B(\mathbb{C})$. Let $Z = X_{P_0}$. Using Proposition 12.2.1, one finds a projective birational desingularisation $f\colon \widetilde{Z} \to Z$ and a non-empty open set $U \subset Z$ such that

- the induced map $V = f^{-1}(U) \to U$ is an isomorphism;
- $\widetilde{Z} \setminus V$ is a union $\cup_i Y_i$ of smooth irreducible divisors of $\widetilde{Z}$;
- there is a non-trivial element in $\mathrm{Br}(\widetilde{Z})$ which vanishes on each Y_i.

Theorem 12.1.1 then implies that the generic fibre of $X \to B$ is not geometrically stably rational.

The following has been known for some time.

Proposition 12.2.3 [dFF13, Prop. 2.3] *Let T be a smooth connected variety over an algebraically closed field k. Let $f\colon X \to T$ be a proper and smooth morphism with connected, projective fibres of relative dimension d. The set of points $m \in T$ such that X_m is rational, respectively, stably rational, is a countable union of locally closed subsets of T.*

In particular, if the geometric generic fibre of f is not (stably) rational, and $k = \mathbb{C}$ hence is uncountable, then there are uncountably many points $t \in T(\mathbb{C})$ such that the fibre X_t is not (stably) rational.

In the above family of quadric bundles we thus find points $n \in B_0(\mathbb{C})$ such that X_n is not stably rational and $m \in B_0(\mathbb{C})$ such that X_m is rational. Over an open set of the line joining m and n we get a projective family of smooth varieties with one fibre rational and another fibre not stably rational. □

Remark 12.2.4 In [HPT18] it is shown that the set of points $m \in B_0(\mathbb{C})$ such that X_m is rational is a countably *infinite* union of subvarieties of B_0. So rationality of smooth complex projective varieties is neither an open nor a closed condition in the classical topology.

Remark 12.2.5 The proof by Hassett, Pirutka and Tschinkel [HPT18] uses an explicit desingularisation of the variety Z in Section 12.2.1, with a description of the exceptional divisors appearing in the process. Schreieder's improvement of the specialisation method enables one to bypass this explicit desingularisation. Papers [HPT18] and [Sch18] contain many other results about families of quadric surfaces over the projective plane. For further developments the reader is referred to [ABBP], which gives a different approach to [HPT18] as well as some generalisations, to [ABP] and [Sch19a, Sch19b].

Let us summarise the current state of knowledge about the behaviour of rationality and stable rationality in the fibres of an algebraic family of proper and smooth varieties.

Recall that a property P of varieties over algebraically closed fields, which is stable under extensions of such fields, *extends by generisation* if for any smooth projective scheme X over $\mathrm{Spec}(\mathbb{C}[[t]])$, if P holds for the closed fibre, then P holds for the geometric generic fibre, that is, the fibre over an algebraic closure of $\mathbb{C}((t))$. Over fields of characteristic zero, rational connectedness, whose definition will be recalled in Section 14.1, enjoys this property (Kollár, Miyaoka, Mori [Kol99, Thm. IV.3.11]).

Let T be a smooth connected variety over $\mathbb{C}$ and let $X\to T$ be a proper and smooth morphism with connected, projective fibres of relative dimension d.

- For $d \leq 2$, stable rationality is equivalent to rationality, and either all fibres of $X\to T$ are rational or no fibre is rational.
- For arbitrary d, stable rationality specialises. Thus the set of points t such that X_t is stably rational is a countable union of *closed* subsets of T (Nicaise–Shinder [NSh19]).
- For arbitrary d, rationality specialises. Thus the set of points t such that X_t is rational is a countable union of *closed* subsets of T (Kontsevich–Tschinkel [KT19]).
- By the examples discussed in this section, for $d \geq 4$, neither rationality nor stable rationality extends by generisation (Hassett, Pirutka and Tschinkel [HPT18]).
- For $d = 3$ stable rationality does not extend by generisation (Hassett, Kresch and Tschinkel [HKT]).
- For $d = 3$ it is not known if rationality extends by generisation.

Chapter 13
The Brauer–Manin set and the formal lemma

This is the first of three chapters which deal with applications of the Brauer group to the arithmetic of varieties over a number field k. Section 13.1 is a collection of preliminary results from algebraic number theory and class field theory. In Section 13.2 we discuss the Hasse principle, weak and strong approximation. Section 13.3 contains the definition and basic properties of the Brauer–Manin obstruction, which is the fundamental reason why the knowledge of the Brauer group is necessary for the study of local-to-global principles for rational points. When the cokernel of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ is finite, the Brauer–Manin obstruction on X involves only finitely many primes; the set of these primes is studied in Section 13.3.2. Explicit examples of calculation of the Brauer–Manin obstruction to the Hasse principle and weak approximation are presented in Section 13.3.3. In Section 13.4 we state and prove Harari's formal lemma, which is a fundamental tool to study the Brauer–Manin obstruction for the total space of a family of varieties.

13.1 Number fields

Throughout this chapter, unless otherwise stated, k is a number field. We write Ω for the set of places of k. The completion of k at a place v is denoted by k_v. For a finite (=non-archimedean) place v we denote by $v\colon k_v^*\to\mathbb{Z}$ the discrete valuation of k_v. For example, $v_p\colon \mathbb{Q}_p^*\to\mathbb{Z}$ is the usual p-adic valuation. For $x\in\mathbb{Q}_p^*$ we have $|x|_p=p^{-v_p(y)}$. The unique extension of this norm to k_v is given by $|x|_v=|\mathrm{N}_{k_v/\mathbb{Q}_p}(x)|_p^{1/[k_v:\mathbb{Q}_p]}$, see [CF67, Ch. II, §10]. When $x\in\mathbb{R}$ or $x\in\mathbb{C}$ we write $|x|$ for the euclidean norm of x.

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13.1.1 Primes and approximation

Let S be a finite set of places of k. The image of the diagonal map $k \to \prod_{v\in S} k_v$ is dense. This property is called *weak approximation*, see [CF67, Ch. II, §6].

Let v_0 be a place of k such that $v_0 \notin S$. For any $\lambda_v \in k_v$ for $v \in S$, there exists an element $\lambda \in k$ with $|\lambda|_v \leq 1$ for every $v \in \Omega \setminus (S \cup \{v_0\})$ such that λ is arbitrarily close to λ_v in the topology of k_v for each $v \in S$. This property is called *strong approximation*, see [CF67, Ch. II, §15]. It is a generalisation of the Chinese remainder theorem.

Dirichlet's theorem on primes in an arithmetic progression can be extended to number fields in the following form [Has26, §8, Satz 13], [Lang70, Ch. VIII, §4, p. 167].

Theorem 13.1.1 (Dirichlet, Hasse) *Let $S \subset \Omega$ be a finite set of* **finite** *places of k and let $\lambda_v \in k_v$ for each $v \in S$. For any $\varepsilon > 0$ there exist $\lambda \in k^*$ and a finite place $v_0 \notin S$ of absolute degree 1 such that*

(i) $|\lambda - \lambda_v|_v < \varepsilon$ *for each place $v \in S$;*
(ii) $\lambda > 0$ *in each real completion of k;*
(iii) λ *is a unit at any place $v \notin S \cup \{v_0\}$, whereas $v_0(\lambda) = 1$.*

Here v_0 may not be fixed from the outset.

The next statement, whose proof combines Theorem 13.1.1 and a theorem of Waldschmidt in transcendental number theory, enables one to approximate also at the archimedean places, if one accepts to lose control over an infinite set of places of k that can be chosen from the outset. Typically, this will be the set of places split in a given finite extension of k, up to finitely many of them.

Theorem 13.1.2 (Dirichlet, Hasse, Waldschmidt, Sansuc) *Let $S \subset \Omega$ be a finite set and let $\lambda_v \in k_v$ for each $v \in S$. Let V be an infinite set of places of k. For any $\varepsilon > 0$ there exist $\lambda \in k^*$ and a finite place $v_0 \notin S$ of absolute degree 1 such that*

(i) $|\lambda - \lambda_v|_v < \varepsilon$ *for each $v \in S$,*
(ii) λ *is a unit at each finite place $v \notin S \cup \{v_0\} \cup V$ and $v_0(\lambda) = 1$.*

Proof. See [San82, Cor. 4.4, p. 264]. □

Here again v_0 may not be fixed from the outset.

We recall a corollary of the Chebotarev density theorem [Lang70, Ch. VIII, §4, Thm. 10], [CF67, Ch. VIII, §3]. This special case has an elementary proof (cf. [MumAV, App. I, p. 250]).

Theorem 13.1.3 *Let K/k be a finite extension of number fields. There exists an infinite set of places v of k which are completely split in K, i.e., such that the k_v-algebra $K \otimes_k k_v$ is isomorphic to $k_v^{[K:k]}$.*

Theorem 13.1.2 can be compared to the following proposition.

Proposition 13.1.4 [HW16, Lemma 5.2] *Let K/k be an extension of number fields. Let $S \subset \Omega$ be a finite set. Let $\xi_v \in \mathrm{N}_{K/k}(K \otimes_k k_v^*) \subset k_v^*$ for each $v \in S$. Then there exists an element $\xi \in k^*$ arbitrarily close to ξ_v for $v \in S$ and such that ξ is a unit outside S except possibly at the places above which K has a place of degree 1. Moreover, if v_0 is a place of k not in S, over which K has a place of degree 1, one can ensure that ξ is integral outside $S \cup \{v_0\}$.*

Chebotarev's theorem is used to prove the existence of such a place v_0, but the rest of the proof requires only the strong approximation theorem.

Here is another corollary of the Chebotarev density theorem.

Theorem 13.1.5 *Let K/k be a non-trivial extension of number fields. There exist infinitely many places v of k such that the k_v-algebra $K \otimes_k k_v$ has no direct summand isomorphic to k_v. In particular, given an irreducible polynomial $P(t)$ of degree at least 2, there exist infinitely many places v such that $P(t)$ has no root in k_v.*

It is well known that the second statement does not hold for reducible polynomials. A classical example is $P(t) = (t^2 - 13)(t^2 - 17)(t^2 - 221) \in \mathbb{Q}[t]$.

Here is another variation on the same theme [Har94, Prop. 2.2.1].

Theorem 13.1.6 *Let $k \subset K \subset L$ be number fields, where L/K is cyclic. There exist infinitely many places w of K of degree 1 over k which are inert in the extension L/K.*

13.1.2 Class field theory and the Brauer group

There is a vast literature on class field theory [Has26, AT09, SerCL, CF67, Lang70, NSW08]. We refer to Harari's recent book [Har17] both for proofs and for a list of further references to classical literature. For a historical perspective, see Hasse's contribution to [CF67] and Roquette's book [Roq05].

The Witt residue was introduced in Definition 1.4.11.

Definition 13.1.7 *For each place v of k define*

$$\mathrm{inv}_v \colon \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

as follows. If v is finite, let inv_v be the Witt residue $\mathrm{Br}(k_v) \to \mathrm{H}^1(\mathbb{F}_v, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v), \mathbb{Q}/\mathbb{Z})$ followed by evaluation at the Frobenius element. For a real place v define $\mathrm{inv}_v : \mathrm{Br}(k_v) \cong \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$. For a complex place v set $\mathrm{inv}_v = 0$.

The definition of $\mathrm{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$ given here is used in [SerCL, Ch. XIII, §2], [CF67, Ch. VI], [NSW08, Ch. VII, Cor. (7.1.4)], and [Har17, §8.2].

Theorem 13.1.8 (i) *For each finite place v of k, the map* inv_v *is an isomorphism. For each real place v, the map* inv_v *is the injective map* $\mathrm{Br}(k_v) = \mathbb{Z}/2 \hookrightarrow \mathbb{Q}/\mathbb{Z}$*. For each complex place v we have* $\mathrm{Br}(k_v) = 0$*.*

(ii) *The diagonal map* $\mathrm{Br}(k) \to \prod_{v\in\Omega} \mathrm{Br}(k_v)$ *factors through the direct sum* $\oplus_{v\in\Omega}\mathrm{Br}(k_v)$*.*

(iii) *The maps* inv_v *fit into an exact sequence*

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v\in\Omega} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{13.1}$$

where the map to $\mathbb{Q}/\mathbb{Z}$ is the sum of inv_v *for all $v \in \Omega$.*

See [Har17, Thm. 8.9; Thm. 14.11]. The fact that (13.1) is a complex is a generalisation of the Gauss quadratic reciprocity law. Injectivity of the second arrow is a celebrated theorem of A.A. Albert, R. Brauer, H. Hasse, and E. Noether, generalising results of Legendre and Hilbert.

Theorem 13.1.9 *Let K/k be an abelian extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. For each place $v \in \Omega$, let $G_v \subset G$ be the decomposition group of v. There is a well-defined isomorphism*

$$j_v\colon k_v^*/\mathrm{N}_{K/k}((K \otimes_k k_v)^*) \xrightarrow{\sim} G_v$$

called the norm residue homomorphism, or the local Artin map [SerCL, Ch. XIII, §4], [Har17, Ch. 9]*. For a valuation v of k unramified in K, this map sends an element $c \in k^*$ to $\mathrm{Frob}_v^{v(c)} \in G$ (ibid.). These maps fit into an exact sequence*

$$k^*/\mathrm{N}_{K/k}(K^*) \longrightarrow \bigoplus_{v\in\Omega} k_v^*/\mathrm{N}_{K/k}((K \otimes_k k_v)^*) \longrightarrow G \longrightarrow 1. \tag{13.2}$$

If K/k is cyclic, we have an exact sequence

$$1 \longrightarrow k^*/\mathrm{N}_{K/k}(K^*) \longrightarrow \bigoplus_{v\in\Omega} k_v^*/\mathrm{N}_{K/k}((K \otimes_k k_v)^*) \longrightarrow G \longrightarrow 1. \tag{13.3}$$

Corollary 13.1.10 [AT09, Ch. 7, §3, Cor. 1 of Thm. 9] *Let K/k be an abelian extension of number fields. The following properties hold.*

(i) *If $c \in k^*$ is a local norm for K/k at all places of k except possibly one place v_0, then c is also a local norm at v_0.*
(ii) (Hasse) *If K/k is cyclic, and $c \in k^*$ is a local norm for K/k at all places of k except possibly one place v_0, then it is a global norm for K/k.*

Suppose K/k is a finite Galois extension of fields with Galois group G. Let $T = R^1_{K/k}\mathbb{G}_{m,K}$ be the norm 1 torus. It fits into an exact sequence of k-tori

$$1 \longrightarrow T \longrightarrow R_{K/k}\mathbb{G}_{m,K} \longrightarrow \mathbb{G}_{m,k} \longrightarrow 1.$$

The dual exact sequence of character groups is the exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow \widehat{T} \longrightarrow 0.$$

From the first exact sequence and Hilbert's theorem 90 we obtain an isomorphism $F^*/\mathrm{N}_{K/k}(K \otimes_k F)^* \cong \mathrm{H}^1(F, T)$ for any field extension F/k. From the second exact sequence we get $\mathrm{H}^i(G, \widehat{T}) \cong \mathrm{H}^{i+1}(G, \mathbb{Z})$ for any integer $i \geq 1$. In particular, $\mathrm{H}^1(G, \widehat{T}) \cong \mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$. If k is a number field, almost all decomposition groups are cyclic, and all cyclic subgroups of G are decomposition groups. Thus Theorem 13.1.9 and Corollary 13.1.10 are special cases of the following theorem, which is a consequence – and a special case – of many results due to Tate, Nakayama, Poitou, Takahashi and other authors. A proof can be found in each of the following references: [Mil86, Ch. I, Thm. 4.20], [NSW08, Theorems (8.6.7), (8.6.9), (8.6.14)], [Har17, Exercise 17.8]. See also [Tate66a] and [SerCG, Ch. II, §5.8, Thm. 6] (local duality).

Let M be a continuous discrete Γ-module, where $\Gamma = \mathrm{Gal}(k_{\mathrm{s}}/k)$. Define

$$Ш^i(M) := \mathrm{Ker}[\mathrm{H}^i(k, M) \longrightarrow \prod_{v \in \Omega} \mathrm{H}^1(k_v, M)].$$

Let us explain the construction of the map $\mathrm{H}^i(k, M) \to \mathrm{H}^1(k_v, M)$ following [SerCG, Ch. 2, §6.1]. We can extend the valuation v to a valuation w of k_{s}. Let $D_w = \{g \in \Gamma | g(w) = w\}$ be the decomposition group of w. Let $\bar{k}_v$ be the union of completions of all finite subextensions of k_{s} at w. The field $\bar{k}_v$ is a separable closure of k_v with Galois group $\mathrm{Gal}(\bar{k}_v/k_v) \cong D_w$. The homomorphism $\mathrm{H}^i(k, M) \to \mathrm{H}^1(k_v, M)$ is defined as the restriction map to the subgroup $D_w \subset \Gamma$.

For a commutative algebraic group G locally of finite type over a field k we write $\mathrm{H}^i(k, G) = \mathrm{H}^i(k, G(k_{\mathrm{s}}))$, for $i \geq 0$.

Theorem 13.1.11 *Let k be a number field and let T be an arbitrary k-torus. Let $\widehat{T}$ be the character group of T considered as a Galois module. There are perfect dualities of finite abelian groups*

$$Ш^2(k, T) \times Ш^1(k, \widehat{T}) \longrightarrow \mathbb{Q}/\mathbb{Z}, \qquad Ш^1(k, T) \times Ш^2(k, \widehat{T}) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

and a natural exact sequence of abelian groups

$$\begin{aligned} \mathrm{H}^1(k, T) \longrightarrow \textstyle\bigoplus_{v \in \Omega} \mathrm{H}^1(k_v, T) \longrightarrow \mathrm{Hom}(\mathrm{H}^1(k, \widehat{T}), \mathbb{Q}/\mathbb{Z}) \\ \longrightarrow \mathrm{H}^2(k, T) \longrightarrow \textstyle\bigoplus_{v \in \Omega} \mathrm{H}^2(k_v, T). \end{aligned} \tag{13.4}$$

The map $\mathrm{H}^1(k_v, T) \to \mathrm{Hom}(\mathrm{H}^1(k, \widehat{T}), \mathbb{Q}/\mathbb{Z})$ is induced by the perfect pairing

$$\mathrm{H}^1(k_v, T) \times \mathrm{H}^1(k_v, \widehat{T}) \longrightarrow \mathrm{H}^2(k_v, \mathbb{G}_m) = \mathrm{Br}(k_v) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

given by the cup-product. The long exact sequence (13.4) is obtained as follows. Consider a finite Galois extension K/k which splits T, i.e., such that $T \times_k K \simeq \mathbb{G}_{m,K}^n$ for some n. Let $G = \mathrm{Gal}(K/k)$. Let $\mathbf{A}_K$ be the ring of adèles of K, see Section 13.1.3. Define $C_K(T) := T(\mathbf{A}_K)/T(K)$ and consider the exact sequence of the cohomology groups of G attached to the exact sequence of G-modules

$$1 \longrightarrow T(K) \longrightarrow T(\mathbf{A}_K) \longrightarrow C_K(T) \longrightarrow 1.$$

We identify $\mathrm{H}^1(G, C_K(T))$ with $\mathrm{Hom}(\mathrm{H}^1(k, \widehat{T}), \mathbb{Q}/\mathbb{Z})$ using the global duality theorem [NSW08, Thm. 8.4.4], whose proof relies on the duality theorem [NSW08, Thm. 3.1.11].

Remark 13.1.12 Note that *a priori* there are two possible definitions of the map $\mathrm{Br}(k_v)\to\mathbb{Q}/\mathbb{Z}$ (see Theorem 1.4.14). It is crucial for the discussion of global problems that we define the local invariants

$$\mathrm{inv}_v \colon \mathrm{Br}(k_v)\to\mathbb{Q}/\mathbb{Z} \quad \text{and} \quad j_v \colon k_v^*/\mathrm{N}_{K/k}((K \otimes_k k_v)^*)\to G$$

in a uniform way. It is not enough to define these maps up to sign, except obviously in the case of real places. Formulae for invariants of cup-products with values in $\mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}$ are called explicit reciprocity laws [SerCL, Ch. XIV], [Iwa68], [Har17, Ch. 9]. When applying formulae for residues of cup-products from Section 1.4.1, one should remember that they compute the Serre residue introduced in Definition 1.4.3 (and not the Witt residue introduced later in Definition 1.4.11). By Theorem 1.4.14, the Witt residue is the negative of the Serre residue. See [CTKS87] for a concrete example where one handles 3-torsion elements.

Remark 13.1.13 Let k be a number field and let T be a k-torus. Let K/k be a finite Galois field extension that splits T. Let $G = \mathrm{Gal}(K/k)$. As already mentioned, the decomposition groups G_v are cyclic for almost all places v, and any cyclic subgroup of G appears as a decomposition group at infinitely many places v. From Theorem 13.1.11 we see that $\mathrm{H}^1(k, T)$ is finite if and only if $\mathrm{H}^1(k_v, \widehat{T}) = 0$ for almost all places v, hence if and only if $\mathrm{H}^1(H, \widehat{T}) = 0$ for each cyclic subgroup $H \subset G$. If M is a free finitely generated abelian group with an action of G, then there is a perfect duality of (finite) Tate cohomology groups

$$\mathrm{H}^1(G, M) \times \widehat{\mathrm{H}}^{-1}(G, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})) \longrightarrow \mathbb{Q}/\mathbb{Z},$$

see, e.g., [Bro82, Ch. VI, §7, Exercise 3]. Using periodicity of cohomology of cyclic groups we see that $\mathrm{H}^1(k, T)$ is finite if and only if $\mathrm{H}^1(k, T^\circ)$ is finite, where T° is the torus with character group $\mathrm{Hom}_{\mathbb{Z}}(\widehat{T}, \mathbb{Z})$.

Now let L/k be an arbitrary finite field extension and let again T be the attached norm 1 torus $R^1_{L/k}\mathbb{G}_{m,L}$. Then we have $T^\circ \cong (R_{L/k}\mathbb{G}_{m,L})/\mathbb{G}_{m,k}$ where $\mathbb{G}_{m,k}\to R_{L/k}\mathbb{G}_{m,L}$ is the natural map. Using Hilbert's theorem 90, we obtain canonical isomorphisms

$$\mathrm{H}^1(k,T) \cong k^*/\mathrm{N}_{L/k}(L^*), \qquad \mathrm{H}^1(k,T^\circ) \cong \mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(L)].$$

Thus $k^*/\mathrm{N}_{L/k}(L^*)$ *is finite if and only if the relative Brauer group* $\mathrm{Br}(L/k) = \mathrm{Ker}[\mathrm{Br}(k)\to\mathrm{Br}(L)]$ *is finite* (cf. [Ser16, Ch. 6, Thm. 6.5 and Thm. 6.6]).

In fact, $\mathrm{Br}(L/k)$ and hence also $k^*/\mathrm{N}_{L/k}(L^*)$ are infinite if $L \neq k$. This is easy to prove using Chebotarev's theorem when the extension L/k is Galois. The only known proof of this statement for an arbitrary finite extension of number fields L/k, due to Fein, Kantor, and Schacher [FKS81], uses the classification of finite simple groups (see [Ser16, Ch. 6, Thm. 6.4] and [Har17, Exercise 18.8]).

13.1.3 Adèles and adelic points

In this section we use a very helpful article of B. Conrad [Con12] to which we refer for many carefully worked out details.

If v is a non-archimedean place of k, we denote by $\mathcal{O}_v$ the ring of integers of the completion k_v. We shall write S for a finite set of places of k containing all the archimedean places. Let $\mathcal{O}$ be the ring of integers of k and let $\mathcal{O}_S$ be the ring of S-integers, i.e. the elements of k that belong to $\mathcal{O}_v$ for $v \notin S$.

The product $\prod_{v\in\Omega} k_v$ is a topological ring equipped with the product topology, where each k_v carries its natural archimedean or non-archimedean topology. The ring of adèles $\mathbf{A}_k$ is defined as a subring of $\prod_{v\in\Omega} k_v$ given by the condition that all but finitely many components are in $\mathcal{O}_v$. The topology of $\mathbf{A}_k$ induced by the topology of $\prod_{v\in\Omega} k_v$ is such that a base is given by the open sets $\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{O}_v$, where U_v is open in k_v. We put

$$\mathbf{A}_{k,S} = \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v.$$

Then $\mathbf{A}_k$ is the direct limit of the open subrings $\mathbf{A}_{k,S}$ over all finite $S \subset \Omega$ containing the archimedean places. The diagonal image of k in $\mathbf{A}_k$ is discrete with compact quotient [CF67, Ch. II, Thm. 14.1], hence the diagonal image of $\mathcal{O}_S$ is discrete in $\mathbf{A}_{k,S}$.

If $X \subset \mathbb{A}^n_k$ is a closed affine subvariety, then the set $X(\mathbf{A}_k)$ is identified with a closed subset of $\mathbf{A}_k^n$ and so acquires a locally compact Hausdorff subspace topology. This topology does not depend on the closed immersion $X \hookrightarrow \mathbb{A}^n_k$, see [Con12, Prop. 2.1]. Since k is discrete in $\mathbf{A}_k$, the set $X(k)$ is discrete in $X(\mathbf{A}_k)$ if X is affine. Although a closed immersion $X \hookrightarrow X'$ gives rise to a closed embedding $X(\mathbf{A}_k) \hookrightarrow X'(\mathbf{A}_k)$ of topological spaces, this is not true for open immersions. The standard example is $\mathbb{G}_{m,k} \subset \mathbb{A}^1_k$. Indeed, the topology on the group of idèles $\mathbf{A}_k^*$ coming from the closed immersion $\mathbb{G}_{m,k} \subset \mathbb{A}^2_k$ given by $xy = 1$, is not the topology induced from $\mathbf{A}_k$. (The elements $a, b \in \mathbf{A}_k^*$

are close when not only a and b are close, but a^{-1} and b^{-1} are close too.) This shows that to equip the set $X(\mathbf{A}_k)$ with the structure of a topological space when X is not affine one cannot proceed by gluing over the affine open subsets. Following Weil and Grothendieck, this goal is achieved by working with integral models.

Nevertheless, the approach via gluing works for a local topological ring R such that R^* is open in R and has continuous inversion, e.g. if $R = k_v$ or $R = \mathcal{O}_v$. This crucially uses the fact that if $\{U_i\}$ is an open covering of X, then $X(R)$ is the union of the sets $U_i(R)$. See [Con12, Prop. 3.1, Prop. 5.4] and Theorem 10.5.1.

Let X be a variety over k (that is, a separated scheme of finite type over k). By [EGA, IV$_3$, §8.8] for some finite set T of places there exists a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_T$ with generic fibre X. Let $S \subset \Omega$ be a finite set containing T. It is clear that an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ gives rise to an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k$. Since $\mathcal{O}_T \subset k \subset \mathbf{A}_k$, we have $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k = X \times_k \mathbf{A}_k$, so an $\mathbf{A}_k$-valued point of $\mathcal{X} \times_{\mathcal{O}_T} \mathbf{A}_k$ is identified with an $\mathbf{A}_k$-valued point of X. This gives rise to a map of sets

$$\varinjlim \mathcal{X}(\mathbf{A}_{k,S}) \longrightarrow \mathcal{X}(\mathbf{A}_k) = X(\mathbf{A}_k). \tag{13.5}$$

Here the limit is over S, and it does not depend on T. An $\mathbf{A}_k$-valued point of X comes from an $\mathbf{A}_{k,S}$-valued point of $\mathcal{X}$ for some S, so this map is bijective.

The natural map of sets

$$\mathcal{X}(\mathbf{A}_{k,S}) \xrightarrow{\sim} \prod_{v \in S} X(k_v) \times \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v)$$

is a bijection [Con12, Thm. 3.6]. This implies that $X(\mathbf{A}_k)$ is the restricted topological product of the sets $X(k_v)$, for $v \in \Omega$, with respect to their subsets $\mathcal{X}(\mathcal{O}_v)$ for $v \notin S$. Here $X(k_v)$ and $\mathcal{X}(\mathcal{O}_v)$ are topologised via gluing, as explained above. This makes $\mathcal{X}(\mathbf{A}_{k,S})$ a locally compact Hausdorff topological space. If $S \subset S'$, then $\mathcal{X}(\mathbf{A}_{k,S}) \to \mathcal{X}(\mathbf{A}_{k,S'})$ is an open embedding. Using (13.5) we make $X(\mathbf{A}_k)$ a topological space in such a way that a subset of $X(\mathbf{A}_k)$ is open if its intersection with each $\mathcal{X}(\mathbf{A}_{k,S})$ is open. Then $X(\mathbf{A}_k)$ is a locally compact Hausdorff topological space with a countable basis of open sets. We note that the sets $\mathcal{X}(\mathbf{A}_{k,S})$ form an open covering of $X(\mathbf{A}_k)$. A morphism $f\colon X \to Y$ of varieties over k gives rise to a continuous map $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$.

We refer to $X(\mathbf{A}_k)$ as the *adelic space* of X and call its elements the *adelic points* of X. If X is an affine variety over k, the topology of the adelic space $X(\mathbf{A}_k)$ is the natural topology defined earlier in the affine case.

If X is proper, we can take $\mathcal{X}$ to be proper over $\mathcal{O}_T$. For $v \notin S$, by the valuative criterion of properness, we have $X(k_v) = \mathcal{X}(\mathcal{O}_v)$, hence $X(\mathbf{A}_k)$ coincides with the product topological space $\prod_{v \in \Omega} X(k_v)$, and so is compact by Tychonoff's theorem. More generally, if $X \to Y$ is a proper morphism of varieties over k, then the continuous map of topological spaces $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$

is topologically proper: the inverse image of a compact set is compact. If $X \to Y$ is a smooth *surjective* morphism of varieties over k with geometrically integral fibres, then $X(\mathbf{A}_k) \to Y(\mathbf{A}_k)$ is open [Con12, Thm. 4.5]. (The proof uses the Lang–Weil–Nisnevich estimates [LW54], [Po18, Thm. 7.7.1].)

For a finite set of places $T \subset \Omega$ let $\mathbf{A}_k^T$ be the ring of T-adèles of k, i.e. the adèles without the components at the places in T. We have the topological space $X(\mathbf{A}_k^T)$ of T-adelic points.

13.2 The Hasse principle and approximation

A variety X over a number field k is called *everywhere locally soluble* if

$$\prod_{v \in \Omega} X(k_v) \neq \emptyset.$$

Definition 13.2.1 *A variety X over a number field k fails* **the Hasse principle** *if X is everywhere locally soluble but has no k-points, that is, $\prod_{v\in\Omega} X(k_v) \neq \emptyset$ whereas $X(k) = \emptyset$. A class of varieties over k satisfies the Hasse principle if no variety in this class fails the Hasse principle.*

If two smooth, proper, integral varieties X and Y over a field F are birationally equivalent, then $X(F) \neq \emptyset$ if and only if $Y(F) \neq \emptyset$. This is a special case of the well known Lang–Nishimura lemma [Po18, Thm. 3.6.11]. As a consequence, for smooth, projective, geometrically integral varieties over a number field k the Hasse principle is a property which is k-birational invariant.

Corollary 13.1.10 for a quadratic extension K/k implies that smooth conics satisfy the Hasse principle. This can be generalised in several directions. The Hasse principle holds for smooth quadrics of dimension at least 1, this is Hasse's original result. The Hasse principle holds for Severi–Brauer varieties, as proved by F. Châtelet. This last result follows from the combination of Proposition 7.1.5 and Theorem 13.1.8 (iii). More generally, projective homogeneous spaces of connected linear algebraic groups satisfy the Hasse principle. This was proved by G. Harder as a consequence of the Hasse principle for principal homogeneous spaces of semisimple, simply connected linear algebraic groups (itself proved by Eichler, Kneser, Harder, Chernousov).

For any subset $S \subset \Omega$ we consider $\prod_{v\in S} X(k_v)$ as the topological space with respect to the product topology, where the topology on each $X(k_v)$ is inherited from the topology of k_v.

Definition 13.2.2 Weak approximation *holds for a variety X over a number field k if the image of the following diagonal map is dense:*

$$X(k) \longrightarrow \prod_{v \in \Omega} X(k_v).$$

This is implied by the condition that for any finite set $S \subset \Omega$, the image of $X(k)$ under the diagonal embedding

$$X(k) \longrightarrow \prod_{v \in S} X(k_v)$$

is dense. If $X(k) \neq \emptyset$, the two conditions are equivalent.

Weak approximation holds for semisimple simply connected linear algebraic groups (Kneser, Harder, Platonov) see [San81, Thm. 3.1]. It also holds if G is a semisimple adjoint group [San81, Prop. 9.8]

If an everywhere locally soluble variety over k satisfies weak approximation, then it has a k-point. Thus, *according to our definition*, if we have a class of varieties over k such that weak approximation holds for each variety of this class, then this class satisfies the Hasse principle. One should however be aware that for some classes of varieties it can be easy to prove weak approximation assuming the existence of a k-point, while it can be hard to prove the Hasse principle. The simplest example is the class of quadrics. Indeed, a smooth quadric of dimension at least 1 over a field k is birationally equivalent to projective space if and only if it has a k-point.

Proposition 13.2.3 (Kneser) *Let X and Y be smooth integral varieties over a number field k such that X is everywhere locally soluble. Suppose X and Y are birationally equivalent. If weak approximation holds for X, then it holds for Y.*

Proof. There exist non-empty Zariski open sets $U \subset X$ and $V \subset Y$ which are isomorphic. Let X be a finite set of places of k. By Theorem 10.5.1, the open subset $\prod_{v \in S} U(k_v) \subset \prod_{v \in S} X(k_v)$ is dense, and similarly the open subset $\prod_{v \in S} V(k_v) \subset \prod_{v \in S} Y(k_v)$ is dense. Weak approximation for X implies that $U(k)$ is dense in $\prod_{v \in S} U(k_v)$, thus $V(k)$ is dense in $\prod_{v \in S} V(k_v)$, and hence $V(k) \subset Y(k)$ is dense in $\prod_{v \in S} Y(k_v)$. □

In particular, to prove weak approximation for a smooth integral variety, it is enough to prove it for a non-empty Zariski open subset.

Definition 13.2.4 *A variety X over a number field k satisfies* **weak weak approximation** *if there exists a finite set $T \subset \Omega$ such that the image of the following diagonal map is dense:*

$$X(k) \longrightarrow \prod_{v \in \Omega \smallsetminus T} X(k_v).$$

Equivalently, for any finite set $S \subset \Omega$ with $S \cap T = \emptyset$, the image of the following diagonal map is dense:

$$X(k) \longrightarrow \prod_{v \in S} X(k_v).$$

This property holds for connected linear algebraic groups, see [PR94, Ch. VII, §7.3, Thm. 7].

Let X be an integral variety over a number field k. A subset $H \subset X(k)$ is called a *Hilbert set* if there exists an integral variety Z over k and a dominant quasi-finite morphism $Z \to X$ such that H is the set of k-points P with connected fibre $Z_P = Z \times_X P$. For basic properties of Hilbert sets, we refer to [Lang83b, Ch. 9] and [SerMW, Ch. 9]. The intersection of two Hilbert sets in $X(k)$ contains a Hilbert set.

Definition 13.2.5 *A variety X over a number field k satisfies* **hilbertian weak approximation** *if the image of any Hilbert set $H \subset X(k)$ under the following diagonal map is dense:*

$$X(k) \longrightarrow \prod_{v \in \Omega} X(k_v).$$

Assume that $X_{\text{smooth}}(k) \neq \emptyset$. If X satisfies hilbertian weak approximation, then any Hilbert subset of $X(k)$ is Zariski dense in X, so is not empty. The following result shows that hilbertian weak approximation holds for any non-empty Zariski open subset of the projective line.

Theorem 13.2.6 [Eke90] *Let $H \subset \mathbb{A}^1(k) = k$ be a Hilbert set. Let $S \subset \Omega$ be a finite set of places and let $\lambda_v \in k_v$ for each $v \in S$. Then for any $\varepsilon > 0$ there exists a $\lambda \in H$ such that $|\lambda - \lambda_v|_v < \varepsilon$ for each $v \in S$.*

Let $T \subset \Omega$ be a finite non-empty subset. Recall that $\mathbf{A}_k^T$ is the ring of T-adèles of k, i.e. the adèles without the components at the places of T.

Definition 13.2.7 *A variety X over a number field k satisfies* **strong approximation** *off a finite set $T \subset \Omega$ if the image of the diagonal map $X(k) \to X(\mathbf{A}_k^T)$ is dense.*

The classical example is provided by the affine space $\mathbb{A}_k^n$ for any $n \geq 1$, a special case of which is the Chinese remainder theorem. A less classical example is given by the complement of a closed subset of codimension at least 2 in $\mathbb{A}_k^n$. This was observed independently by D. Wei and by Y. Cao and F. Xu, see [HW16, Lemma 1.8].

Strong approximation off T holds for semisimple simply connected linear algebraic groups under a suitable non-compactness condition on the set of local points at the places in T (Kneser, Harder, Platonov [PR94, §7.4, Thm. 7.12]).

Definition 13.2.8 *A variety X over a number field k satisfies* **hilbertian strong approximation** *off a finite set $T \subset \Omega$ if the image of any Hilbert set $H \subset X(k)$ under the diagonal map $X(k) \to X(\mathbf{A}_k^T)$ is dense.*

If X is *proper*, then $X(\mathbf{A}_k^T) = \prod_{v \in \Omega \smallsetminus T} X(k_v)$. Thus if weak approximation holds for X, then strong approximation holds for X off any finite set $T \subset \Omega$, in particular, one can take $T = \emptyset$. The same is true in the hilbertian case.

The following theorem may be viewed as a further extension of the Chinese remainder theorem.

Theorem 13.2.9 [Eke90] *Let $H \subset \mathbb{A}^1(k) = k$ be a Hilbert set. Let $S \subset \Omega$ be a finite set of places and let $\lambda_v \in k_v$ for each $v \in S$. Let v_0 be a place of k not in S. Then for any $\varepsilon > 0$ there exists an element $\lambda \in H$ such that*

(i) $|\lambda - \lambda_v|_v < \varepsilon$ *for each* $v \in S$, *and*
(ii) $v(\lambda) \geq 0$ *at each finite place* $v \notin S \cup \{v_0\}$.

Note that v_0 can be chosen to be any place outside of S. Thus the affine line $\mathbb{A}^1_k$ satisfies hilbertian strong approximation off any *non-empty* finite set $T \subset \Omega$.

Ekedahl's theorem [Eke90, Thm. 1.3] is actually more general.

Theorem 13.2.10 *Let R be the ring of integers of a number field k. Let $\pi\colon X \to \mathrm{Spec}(R)$ be a morphism of finite type and let $\rho\colon Y \to X$ be an étale cover such that the generic fibre of the composed morphism $\pi\,\rho$ is geometrically irreducible. Let $T \subset \Omega$ be a finite set. If weak approximation (respectively, strong approximation off T) holds for $X \times_R k$, then weak approximation (respectively, strong approximation off T) holds for the set of points $x \in X(k)$ with connected fibres $\rho^{-1}(x)$.*

13.3 The Brauer–Manin obstruction

13.3.1 The Brauer–Manin set

Proposition 13.3.1 *Let k be a number field, let X be a variety over k and let $A \in \mathrm{Br}(X)$.*

(i) *There exist a finite set of places $T \subset \Omega$ containing all the archimedean places, a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_T$ with generic fibre X, and an element $\mathcal{A} \in \mathrm{Br}(\mathcal{X})$ such that $A \in \mathrm{Br}(X)$ is the restriction of $\mathcal{A}$ to X.*
(ii) *For $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_T)$ as in (i), for any finite place $v \notin T$ and for any point $M_v \in \mathcal{X}(\mathcal{O}_v) \subset X(k_v)$ we have $A(M_v) = \mathcal{A}(M_v) = 0$.*
(iii) *If X is proper, there exists a finite set of places $T \subset \Omega$ such that for all $v \notin T$ and for any $M_v \in X(k_v)$ we have $A(M_v) = 0$.*
(iv) *The map*
$$\mathrm{ev}_A\colon X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$$
which sends an adelic point (M_v) to $\sum_{v\in\Omega} \mathrm{inv}_v A(M_v) \in \mathbb{Q}/\mathbb{Z}$ is a well-defined continuous map whose image is annihilated by a positive integer.

Proof. (i) We have $X = \varprojlim \mathcal{X}_S$, where the limit is over separated $\mathcal{O}_S$-schemes $\mathcal{X}_S$ of finite type with generic fibre X such that $S \subset \Omega$ is finite and contains all the archimedean places. The transition maps are just base changes to smaller open sets of $\mathrm{Spec}(\mathcal{O}_S)$. By Section 2.2.2 we have $\mathrm{Br}(X) = \varinjlim \mathrm{Br}(\mathcal{X}_S)$, which implies (i).

(ii) This follows from $\mathrm{Br}(\mathcal{O}_v) = 0$ (Theorem 3.4.2 (i) and Theorem 1.2.13).

(iii) If X is proper, then in (i) we can take $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_T)$ to be proper. Then for any finite place $v \notin T$ we have $\mathcal{X}(\mathcal{O}_v) = X(k_v)$. Now (iii) follows from (ii) (cf. Proposition 10.5.3).

(iv) Let $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_T)$ be as in (i). By Section 13.1.3 the sets

$$\prod_{v\in S} U_v \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v),$$

where $S \subset \Omega$ is a finite set containing T and $U_v \subset X(k_v)$ is an open set for $v \in S$, form a basis of open sets of $X(\mathbf{A}_k)$. By (ii), the adelic evaluation map ev_A is well defined on such open sets. It is continuous on each of these open sets. Indeed, the local evaluation map $\mathrm{ev}_A\colon \mathcal{X}(\mathcal{O}_v)\to\mathrm{Br}(k_v)$ is zero for $v \notin T$ and $\mathrm{ev}_A\colon X(k_v)\to\mathrm{Br}(k_v)$ is continuous for any place v (Corollary 10.5.2). Lemma 3.4.5 implies that the image of the adelic evaluation map $\mathrm{ev}_A\colon X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is annihilated by a positive integer. □

Write $\mathbf{A}_k^{\mathbb{C}}$ for the ring of $\Omega_{\mathbb{C}}$-adèles $\mathbf{A}_k^{\Omega_{\mathbb{C}}}$, where $\Omega_{\mathbb{C}}$ is the set of complex places of k. In particular, $\mathbf{A}_k^{\mathbb{C}} = \mathbf{A}_k$ if k is totally real, e.g., if $k = \mathbb{Q}$. Since $\mathrm{Br}(\mathbb{C}) = 0$, the evaluation map $\mathrm{ev}_A\colon X(\mathbf{A}_k) \longrightarrow \mathbb{Q}/\mathbb{Z}$ factors through the evaluation map

$$\mathrm{ev}_A^{\mathbb{C}}\colon X(\mathbf{A}_k^{\mathbb{C}}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

The Brauer–Manin pairing

Let k be a number field and let X be a variety over k. By definition, *the Brauer–Manin pairing*

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

sends $(M_v) \in X(\mathbf{A}_k)$ and $A \in \mathrm{Br}(X)$ to

$$\mathrm{ev}_A\big((M_v)\big) = \sum_{v\in\Omega} \mathrm{inv}_v(A(M_v)) \in \mathbb{Q}/\mathbb{Z}.$$

If X is proper over k, then $X(\mathbf{A}_k) = \prod_v X(k_v)$. In this case the pairing becomes

$$\prod_{v\in\Omega} X(k_v) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

For any subset $B \subset \mathrm{Br}(X)$, we denote by $X(\mathbf{A}_k)^B \subset X(\mathbf{A}_k)$ the set of adelic points orthogonal to B with respect to the Brauer–Manin pairing, that is, the intersection of $\mathrm{ev}_A^{-1}(0)$ for $A \in B$. By the continuity of the evaluation map (Proposition 10.5.2), it is a closed subset of $X(\mathbf{A}_k)$. When B is finite, Proposition 13.3.1 (iv) shows that the map $X(\mathbf{A}_k)\to\mathrm{Maps}(B,\mathbb{Q}/\mathbb{Z})$ factors through $\mathrm{Maps}(B,\mathbb{Z}/n)$ for some n, hence $X(\mathbf{A}_k)^B$ is closed and open in $X(\mathbf{A}_k)$. The set $X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ is called *the Brauer–Manin set* of X. We abbreviate this notation by $X(\mathbf{A}_k)^{\mathrm{Br}}$.

Similarly, the evaluation map without complex components gives rise to the Brauer–Manin set $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.

If $X(\mathbf{A}_k)^{\mathrm{Br}}$ is empty, the set $X(\mathbf{A}_k)$ has a covering by open subsets $X(\mathbf{A}_k)\smallsetminus X(\mathbf{A}_k)^b$, for all $b\in\mathrm{Br}(X)$. If X is proper, the topological space $X(\mathbf{A}_k)$ is compact, hence there is a *finite* subset $B\subset\mathrm{Br}(X)$ such that

$$X(\mathbf{A}_k)=\bigcup_{b\in B}(X(\mathbf{A}_k)\smallsetminus X(\mathbf{A}_k)^b),$$

and therefore $X(\mathbf{A}_k)^B=\emptyset$.

Let X be a variety over k. For any $A\in\mathrm{Br}(X)$ we have the basic commutative diagram

$$\begin{array}{ccccc}
X(k) & \hookrightarrow & X(\mathbf{A}_k) & & \\
\downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow & \searrow^{\mathrm{ev}_A} & \\
\mathrm{Br}(k) & \longrightarrow & \bigoplus_{v\in\Omega}\mathrm{Br}(k_v) & \xrightarrow{\mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z}
\end{array}$$

where the bottom line is the complex given by the exact sequence of class field theory (13.1).

Theorem 13.3.2 (Manin) [Man71] *Let k be a number field and let X be a variety over k. The Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ contains the closure of the image of the diagonal map $X(k)\to X(\mathbf{A}_k)$.*

Proof. The inclusion $X(k)\subset X(\mathbf{A}_k)^{\mathrm{Br}}$ follows immediately from the above diagram. Since $X(\mathbf{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbf{A}_k)$, it contains the closure of $X(k)$. □

Manin's observation is that this simple theorem accounted for most counter-examples to the Hasse principle known at the time (the justification for the Cassels–Guy example came later [CTKS87]). In these examples, the rôle of (13.1) is played by some explicit form of the reciprocity law, mostly the quadratic reciprocity law. We review some of these examples in Section 13.3.3.

One commonly uses the following terminology.

If X is a variety over k such that $X(\mathbf{A}_k)\neq\emptyset$ but $X(\mathbf{A}_k)^{\mathrm{Br}}=\emptyset$, then one says that *there is a Brauer–Manin obstruction to the Hasse principle for X.*

If X is proper over k, then $X(\mathbf{A}_k) = \prod_{v\in\Omega} X(k_v)$ with the product topology. In this case, if the inclusion $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is not an equality, one says that *there is a Brauer–Manin obstruction to weak approximation for X.*

The space $X(\mathbf{A}_k)$ is the union of subsets

$$\prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v),$$

where $S \subset \Omega$ is a finite set containing all infinite places, and $\mathcal{X}\to\mathrm{Spec}(\mathcal{O}_S)$ is a separated scheme of finite type with generic fibre X. For each subset $B \subset \mathrm{Br}(X)$ there is an inclusion

$$\mathcal{X}(\mathcal{O}_S) \subset \left(\prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)\right)^B .$$

The Brauer–Manin pairing can be used to show the failure of strong approximation outside a finite set of places, or even to give counter-examples to the integral Hasse principle (proving the emptiness of the set $\mathcal{X}(\mathcal{O}_S)$ of $\mathcal{O}_S$-integral points).

As we have seen in Proposition 13.3.1, for a given element $A \in \mathrm{Br}(\mathcal{X})$, the image of ev_A on the set $\prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$ is computed using only the places in S.

Remark 13.3.3 More generally, let F be a contravariant functor from the category of k-schemes to the category of sets. For a variety X over k any element $\xi \in F(X)$ gives rise to a commutative diagram

$$\begin{array}{ccc} X(k) & \to & X(\mathbf{A}_k) \\ \downarrow & & \downarrow \\ F(k) & \to & \prod_{v\in\Omega} F(k_v) \end{array}$$

where the vertical arrows are given by 'evaluation' of ξ on k-points of X and on local points of X. This puts a constraint on the image of $X(k)$ in $X(\mathbf{A}_k)$. The Brauer–Manin obstruction corresponds to the functor $F(X) = \mathrm{Br}(X)$. Another useful example is the functor that associates to X the étale cohomology set $\mathrm{H}^1_{\text{ét}}(X, G)$, where G is a fixed algebraic group over k.

General properties of the Brauer–Manin obstruction

We can make some simple observations about the Brauer–Manin obstruction.

Remark 13.3.4 Recall that $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ is the image of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(X)$ induced by the structure morphism $X\to\mathrm{Spec}(k)$. If $X(k) \neq \emptyset$, then the homomorphism $\mathrm{Br}(k)\to\mathrm{Br}_0(X)$ has a section and so is an isomorphism. Using the injective map of the exact sequence (13.1) one shows that $\mathrm{Br}(k)\to\mathrm{Br}_0(X)$ is an isomorphism when $X(\mathbf{A}_k) \neq \emptyset$.

Remark 13.3.5 Here is another simple remark [CTS13a, Lemma 1.2]. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. For any $A \in \mathrm{Br}(X)$ we have $\sum_v \mathrm{inv}_v A(M_v) = 0 \in \mathbb{Q}/\mathbb{Z}$. By exactness of the sequence (13.1), there is a well-defined homomorphism $\rho\colon \mathrm{Br}(X)\to\mathrm{Br}(k)$ such that the image of $\rho(A)$ in each $\mathrm{Br}(k_v)$ equals $\mathrm{inv}_v A(M_v)$. Thus any element of $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ gives rise to a splitting of the natural map $\mathrm{Br}(k)\to\mathrm{Br}(X)$. In particular, if $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then

$$\mathrm{Br}(X) \simeq \big(\mathrm{Br}(X)/\mathrm{Br}_0(X)\big) \oplus \mathrm{Br}(k).$$

Remark 13.3.6 Let $B \subset \mathrm{Br}(X)$. The set $X(\mathbf{A}_k)^B$ depends only on the image of B in the quotient $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

Remark 13.3.7 Let us write X_v for $X \times_k k_v$. Let $\text{Б}(X) \subset \mathrm{Br}(X)$ be the subgroup consisting of the elements $A \in \mathrm{Br}(X)$ such that for each place $v \in \Omega$ there exists an $\alpha_v \in \mathrm{Br}(k_v)$ whose image in $\mathrm{Br}(X_v)$ is the same as the image of A. Assume that $X(\mathbf{A}_k) \neq \emptyset$. Then $\mathrm{Br}(k_v)\to\mathrm{Br}(X_v)$ is injective for each v, so that α_v is well defined and equal to the value of A at any k_v-point of X. By Proposition 13.3.1, we have $\alpha_v = 0$ for almost all v. For each adelic point $(M_v) \in X(\mathbf{A}_k)$ one then has

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(M_v)) = \sum_{v\in\Omega} \mathrm{inv}_v(\alpha_v) \in \mathbb{Q}/\mathbb{Z}.$$

The value of this sum does not depend on (M_v). Thus we have a homomorphism $\text{Б}(X)\to\mathbb{Q}/\mathbb{Z}$. If this homomorphism is not zero, then $X(k) = \emptyset$. The Brauer–Manin obstruction attached to the "small" subgroup $\text{Б}(X) \subset \mathrm{Br}(X)$ plays a great rôle in the study of the Hasse principle for homogeneous spaces of connected algebraic groups [Bor96, BCS08, Witt08] – but this subgroup is too small to control weak approximation.

To state a finiteness property of $\text{Б}(X)$, we recall that the Tate–Shafarevich group of an abelian variety A over a number field k is defined as

$$Ш(A) = \mathrm{Ker}[\mathrm{H}^1(k, A)\to \prod_{v\in\Omega} \mathrm{H}^1(k_v, A)].$$

A conjecture of Tate and Shafarevich asserts that $Ш(A)$ is finite for any abelian variety A over any number field.

Proposition 13.3.8 [BCS08, Prop. 2.14] *Let X be a smooth, projective, geometrically integral variety over a number field k such that $X(\mathbf{A}_k) \neq \emptyset$. Let A be the Picard variety of X. If $Ш(A \times_k K)$ is finite for any finite extension K/k, then $\text{Б}(X)/\mathrm{Br}_0(X)$ is finite.*

Remark 13.3.9 Let X be a smooth, projective, geometrically integral variety over a number field k. Suppose that there is a finite field extension K/k such that $X_K(\mathbf{A}_K)^{\mathrm{Br}} = \emptyset$. Then $X(K) = \emptyset$, hence $X(k) = \emptyset$. But can one conclude that $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$? This question is open in general. The answer is positive if $\mathrm{Pic}(\overline{X})$ is a finitely generated free abelian group and $\mathrm{Br}(\overline{X}) = 0$. The proof of this statement uses the theory of universal torsors as developed in [CTS87a]. Since $\mathrm{Br}(\overline{X}) = 0$, for any finite field extension K/k we have $\mathrm{Br}(X_K) = \mathrm{Br}_1(X_K)$. By [Sko01, Cor. 6.1.3 (1)] under our assumptions the condition $X(\mathbf{A}_k)^{\mathrm{Br}_1(X)} \neq \emptyset$ is equivalent to the existence of a universal torsor $\mathcal{T}\to X$ such that $\mathcal{T}(\mathbf{A}_k) \neq \emptyset$. The base change to K is a universal torsor $\mathcal{T}_K\to X_K$ such that $\mathcal{T}_K(\mathbf{A}_K) \neq \emptyset$, hence $X_K(\mathbf{A}_K)^{\mathrm{Br}} = X_K(\mathbf{A}_K)^{\mathrm{Br}_1(X_K)} \neq \emptyset$.

The formation of the Brauer–Manin set is functorial.

Proposition 13.3.10 *A morphism $f\colon X\to Y$ of varieties over a number field k induces a continuous map of their Brauer–Manin sets $X(\mathbf{A}_k)^{\mathrm{Br}}\to Y(\mathbf{A}_k)^{\mathrm{Br}}$.*

Proof. We have a continuous map of topological spaces $f\colon X(\mathbf{A}_k)\to Y(\mathbf{A}_k)$, see Section 13.1.3, and a map of Brauer groups $f^*\colon \mathrm{Br}(Y)\to\mathrm{Br}(X)$, see Section 3.2. For an adelic point $(P_v) \in X(\mathbf{A}_k)$ and $A \in \mathrm{Br}(Y)$ we have $(f^*A)(P_v) = A(f(P_v))$, hence f sends $X(\mathbf{A}_k)^{f^*A}$ to $Y(\mathbf{A}_k)^A$. Thus f sends $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{f^*\mathrm{Br}(Y)}$ to $Y(\mathbf{A}_k)^{\mathrm{Br}(Y)} = Y(\mathbf{A}_k)^{\mathrm{Br}}$. □

The following proposition deals with the behaviour of the Brauer–Manin set under birational equivalence.

Proposition 13.3.11 [CTPS16, Prop. 6.1] *Let k be a number field and let X and Y be birationally equivalent smooth, projective, and geometrically integral varieties over k.*

(i) *If $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then $Y(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$.*
(ii) *Assume, moreover, that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$ if and only if $Y(k)$ is dense in $Y(\mathbf{A}_k)^{\mathrm{Br}}$.*

We refer to [CTPS16, §6] for the proof of this proposition and for some additional remarks.

Failure of weak approximation

It is delicate to exhibit counter-examples to the Hasse principle or to prove that for a given place v the set $X(k)$ is not dense in $X(k_v)$. It is much easier to give counter-examples to weak approximation at a finite set of places.

Proposition 13.3.12 *Let k be a number field and let X be a projective variety over k. Assume that $X(k) \neq \emptyset$ and that there exist an element $\alpha \in \mathrm{Br}(X)$ and a place w of k such that α takes at least two different values on $X(k_w)$. Then there exists a finite set S of places of k containing w such that $X(k)$ is not dense in $\prod_{v\in S} X(k_v)$, so that weak approximation fails for X.*

Proof. By Proposition 13.3.1, there exists a finite set S of places of k such that α identically vanishes on each $X(k_v)$ for $v \notin S$. We thus have $w \in S$. Let $P \in X(k)$ be a rational point. For $v \in S$, $v \neq w$, let $N_v \in X(k_v)$ be the image of $P \in X(k) \subset X(k_v)$. Let $N_w \in X(k_w)$ be a point such that $\alpha(N_w) \in \mathrm{Br}(k_w)$ is not equal to $\mathrm{res}_{k_w/k}(\alpha(P)) \in \mathrm{Br}(k_w)$. By reciprocity (Theorem 13.1.8) and vanishing of α on $X(k_v)$ for $v \notin S$, we have

$$0 = \sum_{v\in S} \mathrm{inv}_v(\mathrm{res}_{k_v/k}(\alpha(P))).$$

Then we obtain

$$\sum_{v\in S} \mathrm{inv}_v(\alpha(N_v)) = \mathrm{inv}_w(\alpha(N_w)) - \mathrm{inv}_w(\mathrm{res}_{k_w/k}(\alpha(P))) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

This implies that for any choice of $N_v \in X(k_v)$ for $v \notin S$ we have

$$\sum_{v\in \Omega} \mathrm{inv}_v(\alpha(N_v)) \neq 0 \in \mathbb{Q}/\mathbb{Z}.$$

By Theorem 13.3.2, $(N_v) \in \prod_{v\in S} X(k_v)$ is not in the closure of $X(k)$. □

13.3.2 The structure of the Brauer–Manin set

When $\mathrm{Br}(X)$ is finite modulo $\mathrm{Br}_0(X)$, the Brauer–Manin set of X is an open and closed subset of $X(\mathbf{A}_k)$. More precisely, we have the following

Lemma 13.3.13 *Let X be a proper variety over a number field k. Assume that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Then there exists a finite set S of places of k such that*

$$X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v\notin S} X(k_v)$$

for an open and closed set $Z \subset \prod_{v\in S} X(k_v)$.

Proof. There is a finite set $B \subset \mathrm{Br}(X)$ that generates $\mathrm{Br}(X)$ modulo $\mathrm{Br}_0(X)$. By Proposition 13.3.1 (iii) there is a finite set S of places such that $A(M_v) = 0$ for each $A \in B$ and any $M_v \in X(k_v)$, where $v \notin S$. Thus for each $A \in B$ the evaluation map $\mathrm{ev}_A\colon X(\mathbf{A}_k)\to\mathbb{Q}/\mathbb{Z}$ is the composition of the projection $X(\mathbf{A}_k)\to \prod_{v\in S} X(k_v)$ and a continuous map $\prod_{v\in S} X(k_v)\to\mathbb{Q}/\mathbb{Z}$. The resulting map $\prod_{v\in S} X(k_v)\to(\mathbb{Q}/\mathbb{Z})^B$ is continuous with finite image (Proposition 13.3.1 (iv)) thus its kernel Z is an open and closed subset of $\prod_{v\in S} X(k_v)$. □

In this section we discuss how small the set S can be. We essentially follow the paper [CTS13a].

The following question was originally asked by Peter Swinnerton-Dyer.

Question 13.3.14 *Let X be a smooth, projective and geometrically integral variety over a number field k. Assume that $\mathrm{Pic}(\overline{X})$ is a finitely generated torsion-free abelian group. Can one choose S in Lemma 13.3.13 to be the union of the archimedean places of k and the places of bad reduction for X?*

The following result gives sufficient conditions under which the answer is positive.

Theorem 13.3.15 *Let k be a number field. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_S$ be the ring of S-integers of k. Let $\pi\colon \mathcal{X}\to\mathrm{Spec}(\mathcal{O}_S)$ be a smooth and proper $\mathcal{O}_S$-scheme with geometrically integral generic fibre X/k. Assume that*

(i) $\mathrm{H}^1(X,\mathcal{O}_X)=0$*;*
(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*
(iii) *the transcendental Brauer group* $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ *is a finite abelian group of order invertible in* $\mathcal{O}_S$*.*

Then $X(\mathbf{A}_k)^{\mathrm{Br}} = Z\times\prod_{v\notin S} X(k_v)$, where $Z\subset\prod_{v\in S} X(k_v)$ is an open and closed subset.

Proof. A proper morphism is of finite type, but since $\mathcal{O}_S$ is noetherian, π is of finite presentation. By Stein factorisation [Stacks, Lemma 0E0N], the closed fibres of π (which are smooth and proper) are geometrically connected, hence geometrically integral.

Let $X_v = X\times_k k_v$ for any v and $\mathcal{X}_v := \mathcal{X}\times_{\mathcal{O}_S} O_v$ for $v\notin S$. We claim that for any place $v\notin S$, the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(X_v)$ is contained in the sum of the images of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$. It is enough to prove this statement for the ℓ-primary component, for each prime ℓ.

Let p be the residual characteristic of v. The combination of assumptions (i) and (ii), Proposition 10.4.2 and Lemma 10.4.1 gives that the image of $\mathrm{Br}_1(X)$ in $\mathrm{Br}(X_v)$ is contained in the subgroup generated by the images of $\mathrm{Br}(k_v)$ and $\mathrm{Br}(\mathcal{X}_v)$. Assumption (iii) implies that $\mathrm{Br}(X)\{p\}\subset\mathrm{Br}_1(X)$. Thus the image of $\mathrm{Br}(X)\{p\}$ in $\mathrm{Br}(X_v)$ is contained in the subgroup generated by the images of $\mathrm{Br}(k_v)\{p\}$ and $\mathrm{Br}(\mathcal{X}_v)\{p\}$.

Let us prove the analogous statement for any prime $\ell\neq p$. By Proposition 10.4.3 we only need to check that $\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0,\mathbb{Z}/\ell)=0$, where $\overline{\mathcal{X}}_0$ is the closed geometric fibre of $\pi\colon \mathcal{X}_v\to\mathrm{Spec}(\mathcal{O}_v)$. By the smooth base change theorem for étale cohomology [Mil80, Cor. VI.4.2] the group $\mathrm{H}^1_{\text{ét}}(\overline{\mathcal{X}}_0,\mathbb{Z}/\ell)$ is isomorphic to $\mathrm{H}^1_{\text{ét}}(\overline{X}_v,\mathbb{Z}/\ell)$, which in turn is isomorphic to $\mathrm{H}^1_{\text{ét}}(\overline{X},\mathbb{Z}/\ell)$ by [Mil80, Cor. VI.4.3]. The Kummer exact sequence gives an isomorphism

$\mathrm{H}^1_{\text{ét}}(\overline{X},\mu_\ell) \xrightarrow{\sim} \mathrm{Pic}(\overline{X})[\ell]$, and the vanishing of the latter group follows from conditions (i) and (ii).

We can assume that $X(\mathbf{A}_k)^{\mathrm{Br}(X)} \neq \emptyset$. A point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}(X)}$ gives rise to a retraction $\rho\colon \mathrm{Br}(X) \to \mathrm{Br}(k)$ of the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ such that for each place v we have $\mathrm{inv}_v(\rho(A)) = A(M_v) \in \mathrm{Br}(k_v)$, see Remark 13.3.6. In particular, $\mathrm{Br}(k)$ is isomorphic to its image $\mathrm{Br}_0(X)$ in $\mathrm{Br}(X)$.

Assumptions (i) and (ii) imply that the quotient $\mathrm{Br}_1(X)/\mathrm{Br}(k)$ is finite (Theorem 5.5.1). Since we also assume (iii), the group $\mathrm{Br}(X)/\mathrm{Br}(k)$ is also finite. Hence $\mathrm{Br}(X)/\mathrm{Br}(k)$ is generated by the images of finitely many elements $A_i \in \mathrm{Br}(X)$ that can be assumed to satisfy $\rho(A_i) = 0$. For $v \notin S$, we have an equality

$$A_i \otimes_k k_v = \beta_{i,v} + \gamma_{i,v} \in \mathrm{Br}(X_v),$$

where $\beta_{i,v} \in \mathrm{Br}(\mathcal{X}_v)$ and $\gamma_{i,v} \in \mathrm{Br}(k_v)$. We have $\beta_{i,v}(M_v) = 0$ since the k_v-point M_v of X_v extends to an $\mathcal{O}_v$-point of $\mathcal{X}_v$ by the properness of $\mathcal{X}/\mathcal{O}$, and $\mathrm{Br}(\mathcal{O}_v) = 0$ (Theorem 3.4.2 (ii) and Theorem 1.2.13). It follows that $\gamma_{i,v} = 0 \in \mathrm{Br}(k_v)$. Hence $A_i \otimes_k k_v$ belongs to $\mathrm{Br}(\mathcal{X}_v)$, and so A_i vanishes at every point of $X(k_v) = \mathcal{X}_v(\mathcal{O}_v)$.

Let $B \subset \mathrm{Br}(X)$ be the finite group generated by the elements $A_i \in \mathrm{Br}(X)$. We conclude that the Brauer–Manin pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is induced by the pairing

$$\prod_{v\in S} X(k_v) \times B \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

This finishes the proof. □

Remark 13.3.16 Conditions (i) and (ii) together are equivalent to the assumption that $\mathrm{Pic}(\overline{X})$ is a finitely generated torsion-free abelian group. In general, condition (iii) cannot be dropped, see the discussion after Theorem 13.3.18 below. The finiteness of the transcendental Brauer group is closely related to the Tate conjecture for divisors, see Theorem 16.1.1.

One can give purely geometric conditions under which the assumptions of Theorem 13.3.15 are satisfied.

Corollary 13.3.17 *Let $\pi\colon \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a smooth proper $\mathcal{O}_S$-scheme with geometrically integral generic fibre X/k. Assume that*

(i) $\mathrm{H}^i(X,\mathcal{O}_X) = 0$ *for* $i = 1, 2$;
(ii) *the Néron–Severi group* $\mathrm{NS}(\overline{X})$ *has no torsion;*
(iii) *either* $\dim X = 2$, *or* $\mathrm{H}^3_{\text{ét}}(\overline{X},\mathbb{Z}_\ell)$ *is torsion-free for every prime* ℓ *invertible in* $\mathcal{O}_S$.

Then we have the same conclusion as in Theorem 13.3.15.

Proof. As already mentioned, the closed fibres are automatically geometrically integral. We only need to verify condition (iii) of Theorem 13.3.15. By Theorem 5.2.9 and Theorem 5.5.2, if $\mathrm{H}^2(X, \mathcal{O}_X) = 0$, then $\mathrm{Br}(\overline{X})$ is finite and isomorphic to the direct sum $\oplus_\ell \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}}$. In the surface case, Proposition 5.2.10 gives a (non-canonical) isomorphism $\mathrm{NS}(\overline{X})\{\ell\} \simeq \mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}}$. Thus under assumptions (ii) and (iii), we have $\mathrm{Br}(\overline{X})\{\ell\} = 0$ for $\ell \notin S$. The group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is a subgroup of $\mathrm{Br}(\overline{X})$. Thus hypothesis (iii) in Theorem 13.3.15 is satisfied. □

This corollary can be applied to rationally connected varieties. Indeed, over a field of characteristic zero such a variety X is $\mathcal{O}_X$-acyclic, that is, $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, and is algebraically simply connected [Deb01, Cor. 4.18], hence $\mathrm{Pic}(\overline{X})_{\text{tors}} = \mathrm{NS}(\overline{X})_{\text{tors}} = 0$.

In their very recent work, M. Bright and R. Newton consider the evaluation map for a smooth, projective and geometrically integral variety X over a p-adic field and $A \in \mathrm{Br}(X)\{p\}$. They link the filtration on $\mathrm{Br}(X)$ given by the radius of p-adic disks on which ev_A is constant to a filtration defined earlier by K. Kato, and give an explicit formula for ev_A in terms of this filtration.

Theorem 13.3.18 [BN, Thm. C] *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{H}^2(X, \mathcal{O}_X) \neq 0$. Let v be a finite place of k at which X has good ordinary reduction, with residue characteristic p. Then there exist a finite extension K/k, a place w of K over v, and an element $A \in \mathrm{Br}(X_K)\{p\}$ such that the evaluation map $\mathrm{ev}_A\colon X(K_w) \to \mathrm{Br}(K_w)$ is non-constant. In particular, if $X(\mathbf{A}_K) \neq \emptyset$, then A obstructs weak approximation on X_K.*

Thus, without assumption (iii) of Theorem 13.3.15 (or the stronger assumptions of Corollary 13.3.17), Question 13.3.14 has a negative answer.

Bright and Newton also show how to bound the set S of critical places in Lemma 13.3.13. The issue is the behaviour of p-primary elements of the Brauer group at a place of good reduction above a prime p. For a prime number ℓ invertible in the residue field of a place of good reduction and $A \in \mathrm{Br}(X)\{\ell\}$, the statement follows from Theorem 10.5.6.

Theorem 13.3.19 [BN, Thm. D] *Let X be a smooth, projective and geometrically integral variety over a number field k such that $\mathrm{Pic}(\overline{X})$ is torsion-free. Let S be the union of the following finite sets of places of k:*

(1) *archimedean places;*
(2) *places of bad reduction for X;*
(3) *places v satisfying $e_v \geq p-1$, where e_v is the absolute ramification index of k_v and p is the residue characteristic of k_v;*
(4) *places v for which, for any smooth proper model $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_v)$ of X, we have $\mathrm{H}^0(\mathcal{X}_0, \Omega^1_{\mathcal{X}_0}) \neq 0$, where $\mathcal{X}_0$ is the closed fibre of $\mathcal{X}$.*

Then the evaluation map $\mathrm{ev}_A\colon X(k_v) \to \mathrm{Br}(k_v)$ is constant for all $A \in \mathrm{Br}(X)$ and all places $v \notin S$.

In contrast to Theorem 13.3.15, this theorem makes no assumption about the finiteness of the transcendental Brauer group of X (although in the situation of the theorem it is natural to expect that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite, see Section 16.1). Thus Theorem 13.3.19 gives an explicit finite set S of places of k such that

$$X(\mathbf{A}_k)^{\mathrm{Br}} = Z \times \prod_{v \notin S} X(k_v)$$

for a closed subset $Z \subset \prod_{v \in S} X(k_v)$.

In the important particular case when X is a K3 surface, Theorem 16.7.3 says that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. Moreover, the set in (4) is empty, as follows from a theorem of Rudakov and Shafarevich. Thus for any K3 surface X over $\mathbb{Q}$ we have

$$X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}} = Z \times \prod_{p \notin S} X(\mathbb{Q}_p),$$

where Z is open and closed in $\prod_{v \in S} X(\mathbb{Q}_v)$ for the set S consisting of the real place, the primes of bad reduction, and the prime 2, see [BN, Remark 7.5]. M. Pagano showed that in general the prime 2 cannot be removed from S, see Example 13.3.26 below.

13.3.3 Examples of Brauer–Manin obstruction

Reducible varieties

The following statements are Brauer group versions of the results of Stoll [Sto07, Lemma 5.10, Prop. 5.11, Prop. 5.12].

Lemma 13.3.20 *Let $X = \mathrm{Spec}(k) \sqcup \mathrm{Spec}(k)$. Then $X(k) = X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.*

Proof. Write $X = P_1 \sqcup P_2$, where $P_1 \cong P_2 \cong \mathrm{Spec}(k)$. Take any (Q_v) in $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$. Let $S_1 \subset \Omega \smallsetminus \Omega_\mathbb{C}$ consist of the places v such that $Q_v = P_1$. The complement $S_2 = (\Omega \smallsetminus \Omega_\mathbb{C}) \smallsetminus S_1$ consists of the places v such that $Q_v = P_2$. If $S_1 = \Omega \smallsetminus \Omega_\mathbb{C}$, then $(Q_v) = P_1$, and if $S_2 = \Omega \smallsetminus \Omega_\mathbb{C}$, then $(Q_v) = P_2$. We now suppose that we are not in one of these cases and deduce a contradiction. Choose $v_1 \in S_1$ and $v_2 \in S_2$. Since neither v_1 nor v_2 is a complex place, there is an $\alpha \in \mathrm{Br}(k)$ such that $\mathrm{inv}_{v_1}(\alpha_{v_1}) = 1/2$, $\mathrm{inv}_{v_2}(\alpha_{v_2}) = 1/2$ and $\alpha_v = 0$ for $v \neq v_1, v_2$. Consider the element $\beta = (\alpha, 0) \in \mathrm{Br}(X) = \mathrm{Br}(k) \oplus \mathrm{Br}(k)$. Then $\sum_{v \in \Omega \smallsetminus \Omega_\mathbb{C}} \mathrm{inv}_v(\beta(Q_v)) = \mathrm{inv}_{v_1}(\alpha_{v_1}) \neq 0$. □

Proposition 13.3.21 *Let $X = X_1 \sqcup \cdots \sqcup X_n$ be a disjoint union of varieties over k. Then*

$$X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} = X_1(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} \sqcup \cdots \sqcup X_n(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}.$$

Proof. It is enough to prove the statement for $n = 2$. By functoriality of the Brauer–Manin set (Proposition 13.3.10), the right-hand side is included in the left-hand side. Let $Y = \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k)$. Consider the projection $p\colon X_1 \sqcup X_2 \to \operatorname{Spec}(k) \sqcup \operatorname{Spec}(k)$. By functoriality of the Brauer–Manin set, we have $p(X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}) \subset Y(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$. Lemma 13.3.20 says that $Y(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} = Y(k)$. Thus $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} \subset X_1(\mathbf{A}_k^{\mathbb{C}}) \sqcup X_2(\mathbf{A}_k^{\mathbb{C}})$. Since $\mathrm{Br}(X) = \mathrm{Br}(X_1) \oplus \mathrm{Br}(X_2)$, we have $X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} \subset X_1(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}} \sqcup X_2(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$. □

As a special example, if $X = X_1 \sqcup X_2$, $X_1(\mathbf{A}_k) = \emptyset$ and $X_2(\mathbf{A}_k) = \emptyset$ then $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset$ but we might have $X(\mathbf{A}_k) \neq \emptyset$.

Corollary 13.3.22 *Let X be a finite k-scheme. Then $X(k) = X(\mathbf{A}_k^{\mathbb{C}})^{\mathrm{Br}}$.*

Proof. For X of dimension zero we have $\mathrm{Br}(X_{\mathrm{red}}) = \mathrm{Br}(X)$ (Proposition 3.2.5), so one may assume that X is reduced. By Proposition 13.3.21 it is enough to prove the statement when $X = \operatorname{Spec}(K)$, where K is a non-trivial field extension of k. In this case, by Theorem 13.1.5, there are infinitely many places v such that k_v is not a direct summand of $K \otimes_k k_v$. For such places v we have $X(k_v) = \emptyset$, in particular $X(\mathbf{A}_k^{\mathbb{C}}) = \emptyset$. □

Let us discuss the famous counter-example to the Hasse principle over $\mathbb{Q}$

$$(x^2 - 13)(x^2 - 17)(x^2 - 221) = 0$$

from another point of view. Let us think of x as a coordinate on $\mathbb{A}^1_{\mathbb{Q}}$ and let $Z \subset \mathbb{G}_{m,\mathbb{Q}} = \mathbb{A}^1_{\mathbb{Q}} \smallsetminus \{0\}$ be the closed subset given by this polynomial. Consider

$$A = (x, 13) \in \mathrm{Br}(\mathbb{G}_{m,\mathbb{Q}}).$$

Take any $(x_v) \in Z(\mathbf{A}_{\mathbb{Q}})$, where v is a prime p or the infinite place. It is clear that $A(x_\infty) = 0$. If $p \neq 2$ and $p \neq 13$, then 13 and x_p are both in $\mathbb{Z}_p^*$, hence we have $(x_p, 13)_p = 1$ and thus $A(x_p) = 0 \in \mathrm{Br}(\mathbb{Q}_p)$. If $p = 13$, then $x_{13} \in \mathbb{Q}_{13}$ satisfies $x_{13}^2 = 17$ hence $x_{13} = \pm 2$ up to a square in $\mathbb{Q}_{13}^*$. Since $(\pm 2, 13)_{13} = -1$ we have $A(x_{13}) \neq 0$. Finally, let $p = 2$. Then $x_2^2 = 17$ in $\mathbb{Q}_2$, hence $x_2 = \pm 1$ up to a square in $\mathbb{Q}_2^*$. Then $(\pm 1, 13)_2 = 1$, as follows from the reciprocity law since $(\pm 1, 13)_{13} = 1$. We conclude that $A(x_2) = 0$. This easy calculation shows that there is a Brauer–Manin obstruction attached to the restriction of A to $\mathrm{Br}(Z)$.

This example is a baby case of the situation investigated by Q. Liu and F. Xu [LX15, Thm. 5.7] (finite closed subschemes of a torus), after the work of Stoll [Sto07, Thm. 8.2] (finite closed subschemes of a smooth projective curve of positive genus).

Let us now describe some counter-examples to the Hasse principle on geometrically irreducible varieties.

Iskovskikh's counter-example to the Hasse principle

The following example was explored in [CTCS80, Exemple 5.4]. In a different guise, the case $c = 3$ is due to Iskovskikh [Isk71].

Let $K = \mathbb{Q}(\sqrt{-1})$. We shall use the following properties, easily established using Hensel's lemma, and which are special cases of local class field theory. For a place v of $\mathbb{Q}$ let $K_v = K \otimes_{\mathbb{Q}} \mathbb{Q}_v$. If v is a place defined by a prime $p \equiv 1 \bmod 4$, then $K/\mathbb{Q}$ is split at v, that is, K_v is isomorphic to the product of two copies of $\mathbb{Q}_p$. Then $\mathbb{Q}_p = \mathrm{N}_{K/\mathbb{Q}}(K_v)$. For the infinite place v, an element of $\mathbb{R}^*$ is a norm from $K_v^* \simeq \mathbb{C}^*$ if and only if it is positive. If v is a place defined by a prime $p \equiv 3 \bmod 4$, then $K/\mathbb{Q}$ is inert at v, that is, K_v is an unramified quadratic extension of $\mathbb{Q}_p$. Then an element of $\mathbb{Q}_p^*$ is a norm of an element of K_p^* if and only if its p-adic valuation is even. An element of $\mathbb{Q}_2^*$ is a norm of an element of K_2^* if and only if it equals $2^r u$, where $r \in \mathbb{Z}$ and $u \in \mathbb{Z}_2^*$ is congruent to 1 modulo 4. Quite generally, an element $a \in \mathbb{Q}_v^*$ is a norm from K_v^* if and only if it is a sum of two squares in $\mathbb{Q}_v$, that is, if and only if the class of the quaternion algebra $(-1, a) = 0 \in \mathrm{Br}(\mathbb{Q}_v)$ (see Proposition 1.1.8).

Let $U = U_c$ be the smooth, affine, geometrically integral variety over $\mathbb{Q}$ defined by the equation

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1) \neq 0, \tag{13.6}$$

where $c \in \mathbb{N}$ is congruent to 3 modulo 4. Then U has points in all completions of $\mathbb{Q}$. This is clear for the real place and the p-adic places for $p \neq 2$: just take $x_p = 1/p$. For $p = 2$, take $x = 0$ if c is congruent to 3 mod 8 and $x = 1$ if c is congruent to 7 mod 8.

Consider the Azumaya algebra on U defined by the quaternion algebra $A = (c - x^2, -1)$. Let $X = X_c$ be a smooth projective compactification of U_c. As proved in Example 6.3.1 the class of A comes from a class in $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. By Proposition 10.5.2, for any place v of $\mathbb{Q}$, finite or infinite, the image of the evaluation map $\mathrm{ev}_A\colon X(\mathbb{Q}_v) \to \mathrm{Br}(\mathbb{Q}_v)$ coincides with the image of $\mathrm{ev}_A\colon U(\mathbb{Q}_v) \to \mathrm{Br}(\mathbb{Q}_v)$. Thus we need to compute the images of the maps

$$\phi_v\colon U(\mathbb{Q}_v) \longrightarrow \mathbb{Q}_v^*/\mathrm{N}_{K/\mathbb{Q}}(K_v^*) \subset \mathbb{Z}/2,$$

where ϕ_v sends $M_v = (x_v, y_v, z_v) \in U(\mathbb{Q}_v)$ to the class of $c - x_v^2$.

If v splits in K, the target group of ϕ_v is zero. For $v = v_\infty$, the equation

$$y_\infty^2 + z_\infty^2 = (c - x_\infty^2)(x_\infty^2 - c + 1) \in \mathbb{R}^*$$

forces $c - x_\infty^2 > 0$, hence the image of ϕ_v is zero.

Suppose that $v = p$ is a prime which is inert in K. If $v(x_v) < 0$, then $v(c - x_v^2)$ is even and thus $c - x_v^2$ is a norm. Suppose that $v(x_v) \geq 0$. From

$$(c - x_v^2) + (x_v^2 - c + 1) = 1$$

we deduce that at least one of $v(c - x_v^2)$ and $v(x_v^2 - c + 1)$ is zero. From

$$y_v^2 + z_v^2 = (c - x_v^2)(x_v^2 - c + 1) \in \mathbb{Q}_v^*,$$

we deduce that the sum of the valuations of $c - x_v^2$ and $x_v^2 - c + 1$ is even. Thus $v(c - x_v^2)$ is even, and the image of ϕ_v is zero.

For the unique ramified prime $v = 2$, as recalled above, an element of $\mathbb{Q}_2^*$ is a sum of two squares if and only if it is the product of a power of 2 and a unit in $\mathbb{Z}_2^*$ which is congruent to 1 modulo 4. Write $x_2 = u/v$ with u and v in $\mathbb{Z}_2$, not both divisible by 2. Up to multiplication by a square, $c - x_2^2$ is equal to $cv^2 - u^2$, which by the hypothesis on c is congruent to $3v^2 - u^2$ modulo 4. Up to multiplication by a square, $x_2^2 - c + 1$ is equal to $u^2 - (c-1)v^2$ which by the hypothesis on c is congruent to $u^2 - 2v^2$ modulo 4. The possible values for (u^2, v^2) modulo 4 are $(0, 1), (1, 0), (1, 1)$. In the first and second cases, $3v^2 - u^2$ is congruent to 3 modulo 4 hence is not a norm for $K_2/\mathbb{Q}_2$. In the third case, $u^2 - 2v^2$ is congruent to 3 modulo 4, hence is not a norm for $K_2/\mathbb{Q}_2$. Since $\mathbb{Q}_2^*/\mathrm{N}_{K/\mathbb{Q}}(K_2^*) \cong \mathbb{Z}/2$, and the product $(c - x_2^2)(x_2^2 - c + 1) = y_2^2 + z_2^2$ is a norm, we conclude that $c - x_2^2$ is never a norm for the extension $K_2/\mathbb{Q}_2$. Thus the image of ϕ_2 is $1 \in \mathbb{Z}/2$.

For any $(M_v) \in X(\mathbf{A}_\mathbb{Q})$, we thus have

$$\sum_{v \in \Omega} \mathrm{inv}_v(A(M_v)) = \frac{1}{2},$$

hence $X(\mathbb{A}_\mathbb{Q})^A = \emptyset$ implying $X(\mathbb{Q}) = \emptyset$.

Exercise 13.3.23 [CTCS80, Exemple 5.5], [San82, §2] Let $c \geq 2$ be an integer. Let X_c be a smooth projective variety over $\mathbb{Q}$ birationally equivalent to the affine surface with the equation

$$y^2 + 3z^2 = (c - x^2)(x^2 - c + 1).$$

Consider the unramified quaternion algebra $A = (c - x^2, -3)$ and prove that $X_c(\mathbf{A}_\mathbb{Q})^A = \emptyset$ if $c = 3^{2s+1}(3n - 1)$ for some integers $s \geq 0$ and $n \geq 1$.

Exercise 13.3.24 [CTCS80, Exemple 5.6] Let $c \geq 3$ be an integer. Let X_c be a smooth projective variety over $\mathbb{Q}$ birationally equivalent to the affine surface with the equation

$$y^2 + z^2 = (c - x^2)(x^2 - c + 2).$$

Consider the unramified quaternion algebra $A = (c - x^2, -1)$ and prove that $X_c(\mathbf{A}_\mathbb{Q})^A = \emptyset$ if $c = 4^n(8m + 7)$ for some integers $n \geq 0$ and $m \geq 0$.

Swinnerton-Dyer's counter-example to weak approximation

The reference is [SwD62]. Let U be the affine surface over $\mathbb{Q}$ defined by

$$y^2 + z^2 = (4x - 7)(x^2 - 2) \neq 0.$$

Let

$$A = (4x - 7, -1) \in \mathrm{Br}(U).$$

Let X be a smooth compactification of U. Using the equality

$$(4x - 7)(4x + 7) - 16(x^2 - 2) = -17,$$

and proceeding as in Example 6.3.1 to compute residues at an arbitrary discrete valuation of the field $k(X)$ trivial on k, one shows that A is in $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. For any prime $p \neq 2$, a similar valuation argument based on the same equality shows that $A \in \mathrm{Br}(X)$ vanishes identically on $U(\mathbb{Q}_p)$ and hence also, by continuity (Proposition 10.5.2), on $X(\mathbb{Q}_p)$. For $p = 2$ a more intricate computation in the same spirit as the computation in the previous example shows that A also vanishes identically on $U(\mathbb{Q}_2)$, hence on $X(\mathbb{Q}_2)$ by continuity. The set of real points $U(\mathbb{R})$ has two connected components: the first one given by $-\sqrt{2} < x < \sqrt{2}$ and the second one given by $x > 7/4$. The connected components of $X(\mathbb{R})$ are obtained by taking the closure of the connected components of $U(\mathbb{R})$ in $X(\mathbb{R})$. It is clear that A takes the non-zero value in $\mathrm{Br}(\mathbb{R})$ on any point of the first component, and the zero value on any point of the second component. The reciprocity law then implies that all $\mathbb{Q}$-points of X are contained in the second component of $X(\mathbb{R})$.

Principal homogeneous spaces of a particular torus

The following example is discussed in more detail in [CT14].

Let k be a number field, $a, b, c \in k^*$, and let U be the smooth, geometrically integral, affine variety over k defined by the equation

$$(x^2 - ay^2)(z^2 - bt^2)(u^2 - abw^2) = c.$$

It is clear that U is a principal homogeneous space for the 5–dimensional torus defined by the same equation with $c = 1$. Let X be a smooth compactification of U. Computing residues, one checks that the class of the quaternion algebra $A = (x^2 - ay^2, b) \in \mathrm{Br}(U)$ is contained in the subgroup $\mathrm{Br}(X) \subset \mathrm{Br}(U)$, see Example 6.3.4.

The subgroup $\text{Б}(X) \subset \mathrm{Br}(X)$ was defined in Remark 13.3.7.

Proposition 13.3.25 *Assume that $a, b, c \in k^*$ are such that for each place v of k the field extension $k_v(\sqrt{a}, \sqrt{b})/k_v$ is cyclic, hence of degree at most 2. Then we have the following statements.*

(i) *The class A is contained in the subgroup $Б(X) \subset \mathrm{Br}(X)$.*
(ii) *For each $(M_v) \in X(\mathbf{A}_k)$ one has*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(M_v)) = \sum_{v\,|\,a\in k_v^{*2}} (c,b)_v = \sum_{v\,|\,a\notin k_v^{*2}} (c,b)_v \in \mathbb{Z}/2.$$

Proof. (i) Let F be a field extension of k. If a, b or ab is a square in F, then the left-hand side of the equation of U has a linear factor. This easily implies that U_F is rational over F. For instance, if a is square in F, the equation may be rewritten

$$xy(z^2 - bt^2)(u^2 - abw^2) = c,$$

or with a further change of variables

$$x = cy(z^2 - bt^2)(u^2 - abw^2) \neq 0,$$

so that U_F is isomorphic to the open set of $\mathbb{A}^5_F$ defined by

$$y(z^2 - bt^2)(u^2 - abw^2) \neq 0.$$

Then the natural map $\mathrm{Br}(F) \to \mathrm{Br}(X_F)$ is an isomorphism by Corollary 6.2.11 and Theorem 6.1.3. This proves (i).

(ii) Let $v \in \Omega$ and let $M_v = (x_v, y_v, z_v, t_v, u_v, w_v)$ be a point of $U(k_v)$. Let us compute $(x_v^2 - ay_v^2, b)_v \in \mathrm{Br}(k_v)$. Assume that a is not a square in k_v. Then either b or ab is a square in k_v. In the first case $(x_v^2 - ay_v^2, b)_v = 0$, whereas in the second case $(x_v^2 - ay_v^2, b)_v = (x_v^2 - ay_v^2, a)_v = 0$ (Proposition 1.1.8). Now assume that a is a square in k_v. From the equation of U we obtain

$$(x_v^2 - ay_v^2, b)_v = (z_v^2 - bt_v^2, b)_v + (u_v^2 - abw_v^2, b)_v + (c, b)_v.$$

Since $a \in k_v^{*2}$ we see that $(u_v^2 - abw_v^2, b)_v = (u_v^2 - abw_v^2, ab)_v = 0$. The first term of the right-hand side is zero, hence $(x_v^2 - ay_v^2, b)_v = (c, b)_v$. By the continuity of ev_A this extends to any point of $X(k_v)$. □

Starting from this explicit formula, one easily produces counter-examples to the Hasse principle. For $k = \mathbb{Q}$ take $a = 17$, $b = 13$, $c = 5$.

The Reichardt–Lind counter-example to the Hasse principle

Let X be the smooth compactification of the smooth affine curve U over $\mathbb{Q}$ defined by

$$2y^2 = x^4 - 17 \neq 0.$$

One checks that $X(\mathbf{A}_\mathbb{Q}) \neq \emptyset$. (For the primes of good reduction this follows from Hensel's lemma and the fact that every smooth projective curve of genus 1 over a finite field has a rational point.) In Example 6.3.3 we proved that the quaternion algebra $A = (y, 17)$ defines an element of $\mathrm{Br}(X) \subset \mathrm{Br}(U)$. It is obvious that $A(U(\mathbb{R})) = 0$, hence by the continuity of ev_A we have $A(X(\mathbb{R})) = 0$. If p is a prime such that 17 is a square in $\mathbb{Q}_p$, then we have $A(U(\mathbb{Q}_p)) = 0$ and hence $A(X(\mathbb{Q}_p)) = 0$. Suppose that $p \neq 17$ is a prime such that 17 is not a square in $\mathbb{Q}_p$, in particular $p \neq 2$. Write v_p for the valuation of $\mathbb{Q}_p$. Let $x_p, y_p \in \mathbb{Q}_p$ be such that $2y_p^2 = x_p^4 - 17 \neq 0$. Suppose that $v_p(y_p) < 0$. Then from the equation we get $2v_p(y_p) = 4v_p(x)$, hence $v_p(y_p)$ is even. Suppose that $v_p(y_p) > 0$. From the equation we deduce that 17 is a square modulo p, which we have excluded. Thus $v_p(y_p)$ is even. This implies $(y_p, 17) = 0$. Thus $A(U(\mathbb{Q}_p)) = 0$, hence $A(X(\mathbb{Q}_p)) = 0$ for any prime $p \neq 17$. For $p = 17$, an *ad hoc* local computation shows that we have $\mathrm{inv}_{17}(A(U(\mathbb{Q}_{17}))) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$. This implies $\mathrm{inv}_{17}(A(X(\mathbb{Q}_{17}))) = \frac{1}{2}$. Thus the adelic evaluation map ev_A sends $X(\mathbf{A}_\mathbb{Q})$ to $\frac{1}{2} \in \mathbb{Q}/\mathbb{Z}$. We conclude that $X(\mathbb{Q}) = \emptyset$. (In fact, A is contained in the subgroup $\text{Б}(X) \subset \mathrm{Br}(X)$ defined in Remark 13.3.7, as can be deduced from Corollary 10.5.5.)

For the history of this example we quote from Cassels [Cas66, p. 284]: "... Lind [Lin40] in his dissertation gave examples of curves of genus 1 with points everywhere locally but not globally, including the example later given by Reichardt. We reproduce Lind's elegant argument, which has recently been rediscovered by Mordell, and which does not fall readily into the paradigm proposed in this paper. One has to prove that there are no solutions of

$$u^4 - 17v^4 = 2w^2 \qquad (*)$$

in coprime integers u, v, w. We first show that w is a quadratic residue of 17. For if p is an odd prime divisor of w, it follows from $(*)$ that 17 is a quadratic residue of p, so p is a quadratic residue of 17 by the law of quadratic reciprocity and 2 is in any case a quadratic residue of 17. Hence u^4 and w^2 are both quartic residues of 17. Then $(*)$ implies that 2 is a quartic residue of 17, which is not the case."

Reichardt [Rei42] considered this curve over $\mathbb{Q}(\sqrt{2})$, computed its non-empty set of $\mathbb{Q}(\sqrt{2})$-points, then showed that none of them is fixed by $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$.

Other examples

Counter-examples to the Hasse principle and weak approximation which are accounted for by the Brauer–Manin obstruction have been constructed for the following classes of varieties.

- Smooth projective curves of arbitrary genus $g \geq 1$.
- Smooth, projective, geometrically rational varieties of dimension at least 2, including smooth del Pezzo surfaces of degree d with $2 \leq d \leq 4$, in particular, smooth cubic surfaces.
- Smooth compactifications of homogeneous spaces of connected linear algebraic groups.
- Surfaces with a pencil of curves of genus one.
- K3 surfaces, such as smooth quartics in $\mathbb{P}^3$.
- Enriques surfaces.

We refer to surveys [VA13, VA17, Witt18] for precise references. Here we give only the example mentioned at the end of Section 13.3.2.

Exercise 13.3.26 (M. Pagano) The K3 surface $X \subset \mathbb{P}^3_{\mathbb{Q}}$ given by

$$x^3y + y^3z + z^3w + w^3x + xyzw = 0$$

has good reduction at the prime 2. The class of the quaternion algebra

$$A := \left(\frac{z^3 + xw^2 + xyz}{x^3}, -\frac{z}{x}\right) \in \mathrm{Br}(\mathbb{Q}(X))$$

is contained in $\mathrm{Br}(X)$. Moreover, $\mathrm{ev}_A\colon X(\mathbb{Q}_2) \to \mathrm{Br}(\mathbb{Q}_2)$ is a non-constant function, so A gives an obstruction to weak approximation on X. See [Pag, Thm. 1] for details and the relation to the ideas and constructions of [BN].

See Section 14.3 for counter-examples to the Hasse principle and weak approximation which are not explained by the Brauer–Manin obstruction.

Remark 13.3.27 Diagonal cubic surfaces $X \subset \mathbb{P}^3_{\mathbb{Q}}$ given by an equation

$$ax^3 + by^3 + cz^3 + dt^3 = 0,$$

where $a, b, c, d \in \mathbb{Q}^*$, were an early testing ground for the validity of the Hasse principle and the study of the Brauer–Manin obstruction. Mordell conjectured that they satisfy the Hasse principle. Cassels and Guy (1966) showed that the cubic surface given by

$$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0$$

is a counter-example. In 1987, in the paper [CTKS87], all non-singular diagonal surfaces with positive $a, b, c, d \in \mathbb{Z}$ such that $\max\{a, b, c, d\} \leq 100$ were numerically analysed, and the Brauer–Manin set was determined. It turns out that for such a surface X, we have either $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$ or a search for rational points of small height shows that $X(\mathbb{Q}) \neq \emptyset$. This provides computational support for Conjecture 14.1.2 discussed in the next chapter.

Without loss of generality we assume that the integers a, b, c, d have no common prime factor. Absorbing cubes by changing variables, we can assume that for any prime p the highest power of p dividing any coefficient is at most 2. An interesting aspect of the investigation is that all surfaces in the list with $X(\mathbf{A}_{\mathbb{Q}}) \neq \emptyset$ and $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} = \emptyset$ have the property that no prime p divides exactly one coefficient. A specific family of counter-examples to the Hasse principle is provided by surfaces with equation

$$x^3 + p^2y^3 + pqz^3 + q^2t^3 = 0,$$

where $p \equiv 2$ and $q \equiv 5$ modulo 9 are prime numbers.

This is another instance of a phenomenon taught to us by experience: in a given algebraic family of varieties over a number field, finding a counter-example to the Hasse principle is akin to finding a needle in a haystack. This is in sharp contrast to Proposition 13.3.12, which says that it is easy to find counter-examples to weak approximation.

The two phenomena have now been studied from a quantitative point of view. One tries to give estimates, or at least bounds, for the number of points in the parameter space (in the above example, the number of points $(a, b, c, d) \in \mathbb{P}^3(\mathbb{Q})$ with height at most H, as $H \to \infty$) for which the Hasse principle, or weak approximation fails. This is related to the geometry and arithmetic of the locus of the parameter space where the fibres need not be split. In the above example, this is the union of lines in $\mathbb{P}^3_{\mathbb{Q}}$ with homogeneous coordinates (a, b, c, d) given by the simultaneous vanishing of two coordinates. See the papers [BBL16, LS18, Bri18, Bro18].

13.3.4 The Brauer–Manin set of a product

Proposition 13.3.10 shows that for the varieties X and Y the Brauer–Manin set of $X \times_k Y$ is contained in $X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}$. In the following basic case this is an equality.

Theorem 13.3.28 *Let X and Y be smooth and geometrically integral varieties over a number field k. Then we have*

$$(X \times Y)(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}.$$

Proof. Below we give the proof in the case when X and Y are projective [SZ14, Thm. C]. The general case is due to Chang Lv, see [Lv20, Thm. 3.1].

It is clear that the left-hand side is contained in the right-hand side, so it remains to prove that $X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}$ is a subset of $(X \times Y)(\mathbf{A}_k)^{\mathrm{Br}}$.

We use the same notation as in Section 5.7.3, in particular, we denote by $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ the natural projections.

We can assume that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $Y(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. Since the Brauer group of a smooth projective variety is a torsion group, is enough to show that for any positive integer n the group $\mathrm{Br}(X \times Y)[n]$ is generated by $p_X^*\mathrm{Br}(X)[n]$,

$p_Y^*\mathrm{Br}(Y)[n]$ and the elements that pair trivially with $X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}$ with respect to the Brauer–Manin pairing. Using the Kummer sequence, we see that it suffices to show that $\mathrm{H}^2_{\text{ét}}(X\times Y, \mu_n)$ is generated by $p_X^*\mathrm{H}^2_{\text{ét}}(X,\mu_n)$, $p_Y^*\mathrm{H}^2_{\text{ét}}(Y,\mu_n)$ and the elements that pair trivially with $X(\mathbf{A}_k)^{\mathrm{Br}} \times Y(\mathbf{A}_k)^{\mathrm{Br}}$.

If $Z\to X$ is a torsor for a k-group of multiplicative type G annihilated by n, then the *type* of $Z\to X$ is the image of the class $[Z/X]$ under the composed map

$$\mathrm{H}^1_{\text{ét}}(X,G) \longrightarrow \mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}},G)^{\Gamma} \longrightarrow \mathrm{Hom}_k(\widehat{G},\mathrm{Pic}(X^{\mathrm{s}})) = \mathrm{Hom}_k(\widehat{G},\mathrm{Pic}(X^{\mathrm{s}})[n]),$$

see [Sko01, Def. 2.3.2]. As in Section 5.7.3 we denote by S_X and S_Y the finite commutative k-group schemes whose Cartier duals are

$$\widehat{S}_X = \mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}},\mu_n) \cong \mathrm{Pic}(X^{\mathrm{s}})[n], \qquad \widehat{S}_Y = \mathrm{H}^1_{\text{ét}}(Y^{\mathrm{s}},\mu_n) \cong \mathrm{Pic}(Y^{\mathrm{s}})[n].$$

The assumption $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ allows us to 'descend' various data defined in Section 5.7.3 over k_{s}, to k. Indeed, by the descent theory of Colliot-Thélène and Sansuc [Sko01, Cor. 6.1.3 (1)], $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies that there exists an S_X-torsor on X whose type is the identity in $\mathrm{End}(\widehat{S}_X)$. Let us choose one such torsor and call it $\mathcal{T}_X$. (The class of this torsor in $\mathrm{H}^1_{\text{ét}}(X,S_X)$ is well defined up to an element of $\mathrm{H}^1(k,S_X)$.) Then $\mathcal{T}_X^{\mathrm{s}}$ is isomorphic to the $S_{X^{\mathrm{s}}}$-torsor $\mathcal{T}_{X^{\mathrm{s}}}$ used in Section 5.7.3. Thus we have a map

$$\varepsilon\colon \mathrm{Hom}_k(S_X,\widehat{S}_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X\times Y,\mu_n)$$

that sends $\phi\in\mathrm{Hom}_k(S_X,\widehat{S}_Y)$ to $\varepsilon(\phi)=\phi_*[\mathcal{T}_X]\cup[\mathcal{T}_Y]$, where $\cup$ stands for the cup-product pairing

$$\mathrm{H}^1_{\text{ét}}(X,\widehat{S}_Y)\times\mathrm{H}^1_{\text{ét}}(Y,S_Y) \longrightarrow \mathrm{H}^2_{\text{ét}}(X\times Y,\mu_n).$$

Note that ε is injective, as follows from Theorem 5.7.7 (ii).

The theorem is a consequence of Claims 1 and 2:

Claim 1: $\mathrm{H}^2_{\text{ét}}(X\times Y,\mu_n) = p_X^*\mathrm{H}^2_{\text{ét}}(X,\mu_n)+p_Y^*\mathrm{H}^2_{\text{ét}}(Y,\mu_n)+\mathrm{Im}(\varepsilon)$.

Claim 2: $X(\mathbf{A}_k)^{\mathrm{Br}_1(X)[n]}\times Y(\mathbf{A}_k)^{\mathrm{Br}_1(Y)[n]} \subset (X\times Y)(\mathbf{A}_k)^{\mathrm{Im}(\varepsilon)}$.

Proof of Claim 1. We use the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(k,\mathrm{H}^q_{\text{ét}}(X^{\mathrm{s}},\mu_n)) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X,\mu_n). \tag{13.7}$$

Let us point out some consequences of the assumption $X(\mathbf{A}_k)\neq\emptyset$. The canonical maps

$$E_2^{p,0} = \mathrm{H}^p(k,\mu_n) \longrightarrow \mathrm{H}^p_{\text{ét}}(X,\mu_n)$$

are injective for $p\geq 3$. Indeed, for such p the natural map

$$\mathrm{H}^p(k,M) \longrightarrow \bigoplus_{k_v\simeq\mathbb{R}} \mathrm{H}^p(k_v,M)$$

is a bijection for any finite Galois module M, see [Mil86, Thm. I.4.10 (c)]. Next, the natural map $\mathrm{H}^p(k_v,M)\to\mathrm{H}^p_{\acute{e}t}(X\times_k k_v,M)$ is injective for any p since any k_v-point of X defines a section of this map. It follows that the composite map

$$\mathrm{H}^p(k,M)\longrightarrow \mathrm{H}^p_{\acute{e}t}(X,M)\longrightarrow \bigoplus_{k_v\simeq\mathbb{R}}\mathrm{H}^p_{\acute{e}t}(X\times_k k_v,M)$$

is injective, and this implies our claim. We note that

$$E_2^{2,0}=\mathrm{H}^2(k,\mu_n)\longrightarrow \mathrm{H}^2_{\acute{e}t}(X,\mu_n)$$

is also injective. The argument is similar; it uses the embedding of $\mathrm{Br}(k)$ into the direct sum of the Brauer groups $\mathrm{Br}(k_v)$, for all $v\in\Omega$, provided by global class field theory, together with the existence of k_v-points on X for every place v. This implies the triviality of all the canonical maps in the spectral sequence with target $E_2^{p,0}=\mathrm{H}^p(k,\mu_n)$ for $p\geq 2$.

Let us denote by $\mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)'$ the quotient of $\mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)$ by the (injective) image of $\mathrm{H}^2(k,\mu_n)$. Using the above remarks we obtain from (13.7) the following exact sequence:

$$\begin{aligned}0\longrightarrow \mathrm{H}^1(k,\mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n))\longrightarrow \mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)'\\ \longrightarrow \mathrm{H}^2_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n)^{\Gamma}\longrightarrow \mathrm{H}^2(k,\mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n)).\end{aligned}\tag{13.8}$$

There are similar sequences for Y and $X\times Y$ linked by the maps p_X^* and p_Y^*.

Let us define

$$\mathcal{H}=\pi_X^*\mathrm{H}^2_{\acute{e}t}(X,\mu_n)+\pi_Y^*\mathrm{H}^2_{\acute{e}t}(Y,\mu_n)+\mathrm{Im}(\varepsilon)\subset \mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n).$$

It is clear that the (injective) image of $\mathrm{H}^2(k,\mu_n)$ in $\mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)$ is contained in $\mathcal{H}$, so in order to prove Claim 1 it is enough to show that the natural map $\mathcal{H}\to\mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)'$ is surjective.

By (5.45) we have an isomorphism of Γ-modules induced by p_X and p_Y:

$$\mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n)\cong \mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}},\mu_n)\oplus \mathrm{H}^1_{\acute{e}t}(Y^{\mathrm{s}},\mu_n).\tag{13.9}$$

This implies that the image of $\mathrm{H}^1(k,\mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n))$ in $\mathrm{H}^2_{\acute{e}t}(X\times Y,\mu_n)'$ is contained in $\mathcal{H}$. In view of (13.8) it remains to show that every element of the kernel of the map

$$\alpha_{X\times Y}\colon \mathrm{H}^2_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n)^{\Gamma}\longrightarrow \mathrm{H}^2(k,\mathrm{H}^1_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n))$$

comes from $\mathcal{H}$. The isomorphism of Γ-modules (5.47) gives

$$\mathrm{H}^2_{\acute{e}t}(X^{\mathrm{s}}\times Y^{\mathrm{s}},\mu_n)^{\Gamma}\cong \mathrm{H}^2_{\acute{e}t}(X^{\mathrm{s}},\mu_n)^{\Gamma}\oplus \mathrm{H}^2_{\acute{e}t}(Y^{\mathrm{s}},\mu_n)^{\Gamma}\oplus \mathrm{Hom}_k(S_X,\widehat{S}_Y).$$

By Theorem 5.7.7 and (5.47), any $\varphi \in \mathrm{Hom}_k(S_X, \widehat{S}_Y)$, considered as an element of $\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}} \times Y^{\mathrm{s}}, \mu_n)^{\Gamma}$, lifts to $\phi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y] \in \mathrm{H}^2_{\text{ét}}(X \times Y, \mu_n)$. Hence $\alpha_{X\times Y}$ is zero on $\mathrm{Hom}_k(S_X, \widehat{S}_Y)$, so that $\alpha_{X\times Y}$ is the direct sum of

$$\alpha_X \colon \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mu_n)^{\Gamma} \longrightarrow \mathrm{H}^2(k, \mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mu_n))$$

and a similar map α_Y for Y. Thus $\mathrm{Ker}(\alpha_{X\times Y})$ is a surjective image of

$$\mathrm{H}^2_{\text{ét}}(X, \mu_n) \oplus \mathrm{H}^2_{\text{ét}}(Y, \mu_n) \oplus \mathrm{Im}(\varepsilon).$$

This finishes the proof of Claim 1.

Proof of Claim 2. Let M be a finite commutative group k-scheme such that $nM = 0$. If v is a non-archimedean place of k, we write $\mathrm{H}^1_{\mathrm{nr}}(k_v, M)$ for the unramified subgroup of $\mathrm{H}^1(k_v, M)$. By definition, it consists of the classes that are annihilated by the restriction to the maximal unramified extension of k_v. We write $P^1(k, M)$ for the restricted product of the abelian groups $\mathrm{H}^1(k_v, M)$, for $v \in \Omega$, relative to the subgroups $\mathrm{H}^1_{\mathrm{nr}}(k_v, M)$, where v is a non-archimedean place of k. By [Mil86, Lemma I.4.8] the image of the diagonal map

$$\mathrm{H}^1(k, M) \longrightarrow \prod_{v\in\Omega} \mathrm{H}^1(k_v, M)$$

is contained in $P^1(k, M)$. Let us denote this image by $U^1(k, M)$.

The local cup-product pairings for $v \in \Omega$

$$\cup_v \colon \mathrm{H}^1(k_v, M) \times \mathrm{H}^1(k_v, \widehat{M}) \longrightarrow \mathrm{H}^2(k_v, \mu_n)$$

give rise to the global Poitou–Tate pairing

$$(\ ,\ )\colon \quad P^1(k, M) \times P^1(k, \widehat{M}) \ \longrightarrow\ \mathbb{Z}/n.$$

It is a perfect duality of locally compact abelian groups, moreover, $U^1(k, M)$ and $U^1(k, \widehat{M})$ are exact annihilators of each other (the Poitou–Tate theorem, see [Mil86, Thm. I.4.10 (b)] or [Har17, Thm. 17.13]). We shall use this pairing with $M = \widehat{S}_Y$.

Let $(P_v) \in X(\mathbf{A}_k)$ be an adelic point that is Brauer–Manin orthogonal to $\mathrm{Br}_1(X)[n]$. Let $(Q_v) \in Y(\mathbf{A}_k)$ be an adelic point orthogonal to $\mathrm{Br}_1(Y)[n]$. For $\varphi \in \mathrm{Hom}_k(S_X, \widehat{S}_Y)$ the Brauer–Manin pairing of the adelic point $(P_v \times Q_v)$ of $X \times Y$ with the image of $\varepsilon(\varphi) = \varphi_*[\mathcal{T}_X] \cup [\mathcal{T}_Y]$ in $\mathrm{Br}(X \times Y)$ is given by the Poitou–Tate pairing of $\varphi_*[\mathcal{T}_X](P_v)$ with $[\mathcal{T}_Y](Q_v)$. Thus to prove Claim 2 we need to show that

$$(\varphi_*[\mathcal{T}_X](P_v), [\mathcal{T}_Y](Q_v)) = \sum_{v\in\Omega} \mathrm{inv}_v\big(\varphi_*[\mathcal{T}_X](P_v) \cup_v\ [\mathcal{T}_Y](Q_v)\big) = 0. \quad (13.10)$$

For any $a \in \mathrm{H}^1(k, \widehat{S}_Y)$ the image of $a \cup [\mathcal{T}_Y] \in \mathrm{H}^2_{\text{ét}}(Y, \mu_n)$ in $\mathrm{Br}(Y)[n]$ belongs to $\mathrm{Br}_1(Y)[n]$, and hence $(a, [\mathcal{T}_Y](Q_v)) = 0$. If an element of $P^1(k, S_Y)$ is orthogonal to $U^1(k, \widehat{S}_Y)$, then it is contained in $U^1(k, S_Y)$. Therefore, we must have

$$[\mathcal{T}_Y](Q_v) \in U^1(k, S_Y). \tag{13.11}$$

Similarly, for any $b \in \mathrm{H}^1(k, S_Y)$ the image of $\varphi_*[\mathcal{T}_X] \cup b \in \mathrm{H}^2_{\text{ét}}(X, \mu_n)$ in $\mathrm{Br}(X)[n]$ is contained in $\mathrm{Br}_1(X)[n]$, and hence $(\varphi_*[\mathcal{T}_X](P_v), b) = 0$. Since every element of $P^1(k, \widehat{S}_Y)$ orthogonal to $U^1(k, S_Y)$ belongs to $U^1(k, \widehat{S}_Y)$, this implies

$$\varphi_*[\mathcal{T}_X](P_v) \in U^1(k, \widehat{S}_Y). \tag{13.12}$$

Since (13.11) and (13.12) imply (13.10), this finishes the proof of Claim 2. □

13.4 Harari's formal lemma

Let k be a number field and let $a \in k^* \setminus k^{*2}$. Let $X = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$ and let $U = \mathbb{G}_{m,k} \subset \mathbb{A}^1_k$ be the open subset given by $t \neq 0$. The quaternion algebra class $(a, t) \in \mathrm{Br}(U)[2]$ has a non-trivial residue $a \in k^*/k^{*2}$ at $t = 0$ and $t = \infty$. There exist infinitely many places v of k such that a is a unit and not a square in the completion k_v. If v is such a finite, odd place, and $\pi_v \in k_v$ is a uniformiser, then $(a, \pi_v) \in \mathrm{Br}(k_v)$ is non-zero. Thus there are infinitely many places v such that α is not identically zero on $U(k_v)$. A conceptual reason for this is that (a, t) is ramified on X.

This is the easiest case of the following general result [Har94, Thm. 2.1.1, p. 226]. The presentation in this section follows [CT03].

Theorem 13.4.1 (Harari) *Let X be a smooth integral variety over a number field k. Let S be a finite set of places of k and let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_S)$ be a morphism of finite type with generic fibre X. Let $U \subset X$ be a non-empty open subset of X and let $\alpha \in \mathrm{Br}(U) \setminus \mathrm{Br}(X)$. There exist infinitely many places v of k for which there is a point $M_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ with $\alpha(M_v) \neq 0$.*

Proof. The first part of the proof is a reduction to the case when X is a curve.

There is an irreducible divisor $Z \subset X$ such that the residue of α at the generic point of Z is a non-zero element $\partial_Z(\alpha) \in \mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$. Let n be the order of $\partial_Z(\alpha)$. Then $\partial_Z(\alpha)$ is an element of the subgroup $\mathrm{H}^1(k(Z), \mathbb{Z}/n)$ of $\mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$. After replacing X by an open subset we can assume that Z is smooth, $U = X \setminus Z$ and $\alpha \in \mathrm{Br}(U)$. Using exact sequence (3.15) in Theorem 3.7.1, we see that $\partial_Z(\alpha)$ comes from a non-zero element

$$\rho \in \mathrm{H}^1_{\text{ét}}(Z, \mathbb{Z}/n) \subset \mathrm{H}^1(k(Z), \mathbb{Z}/n).$$

Let $Z_1 \to Z$ be the finite cyclic cover defined by ρ, or, equivalently, the Z-torsor for $\mathbb{Z}/n$ with class ρ. The scheme Z_1 is connected. Indeed, the invariant subfield K of the kernel of the homomorphism $\mathrm{Gal}(\overline{k(Z)}/k(Z)) \to \mathbb{Q}/\mathbb{Z}$ corresponding to $\partial_Z(\alpha)$ is a field extension of $k(Z)$ of degree n, so $\mathrm{Spec}(K)$ must be the generic point of Z_1.

Replacing X by an open subset, we may assume that there is a generically finite dominant morphism $Z \to \mathbb{A}^d_k$, where $d = \dim(Z)$. Since Z_1 is connected, Hilbert's irreducibility theorem [SerMW, Ch. 9] shows the existence of infinitely many k-points M of $\mathbb{A}^d_k$ such that the fibre of the composite morphism $Z_1 \to Z \to \mathbb{A}^d_k$ at M is integral. Let us choose one such point M. The inverse image of M under the morphism $Z \to \mathbb{A}^d_k$ is a closed point $P \in Z$ such that $\rho(P) \neq 0 \in \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z})$.

A local equation of $Z \subset X$ at P can be extended to a regular system of parameters of the regular local ring $\mathcal{O}_{X,P}$. One thus finds a closed integral curve $C \subset X$ containing the closed point P such that C is a smooth at P and transversal to Z at P. Shrinking X even more, we may assume that C is smooth and $Z \cap C = P$. Let $\alpha_C \in \mathrm{Br}(C \smallsetminus P)$ be the restriction of $\alpha \in \mathrm{Br}(X \smallsetminus Z)$. Since C and Z are transversal at P, by Theorem 3.7.5 the residue of α_C at P is

$$\partial_P(\alpha_C) = \rho(P) \in \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z}),$$

thus $\partial_P(\alpha_C) \neq 0$. The embedding $C \subset X$ extends to an embedding of integral models over a suitable open subset of $\mathrm{Spec}(\mathcal{O}_S)$. Therefore, it is enough to prove the statement of the theorem for the smooth connected curve C.

So let C be a connected integral curve over k with a closed point P. Write $U = C \smallsetminus \{P\}$. Let $\alpha \in \mathrm{Br}(U)$ be an element with a non-zero residue

$$\chi = \partial_P(\alpha) \in \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z})$$

of order n. Thus $\chi \in \mathrm{H}^1(k(P), \mathbb{Z}/n) \subset \mathrm{H}^1(k(P), \mathbb{Q}/\mathbb{Z})$. Replacing C by an open set, we may assume that C is affine, $C = \mathrm{Spec}(A)$, and P is defined be the vanishing of some $f \in A$. Let A^{h} be the henselisation of A at P. The natural restriction map $\mathrm{H}^1(A^{\mathrm{h}}, \mathbb{Z}/n) \to \mathrm{H}^1(k(P), \mathbb{Z}/n)$ is an isomorphism (Section 2.3.3, formula (2.18)). Thus there exists a connected affine curve $D = \mathrm{Spec}(B)$ over k and an étale morphism $q\colon D \to C$ such that q induces an isomorphism $Q = q^{-1}(P) \xrightarrow{\sim} P$ and, moreover, χ is the restriction of some $\xi \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n)$.

Let $V = D \smallsetminus Q$. The restriction of q to V is a morphism $q\colon V \to U$. Write $\alpha_V = q^*(\alpha) \in \mathrm{Br}(V)$. Define $g := f \circ q \in k[D]$ and consider the cup-product $(\xi, g) \in \mathrm{Br}(V)$ of $\xi \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n)$ with the class of g in $k[V]^*/k[V]^{*n} \subset \mathrm{H}^1_{\text{ét}}(V, \mu_n)$. By formula (1.20) for the Serre residue r and Theorem 2.3.5 (which gives $r = -\partial$) the difference $\beta = \alpha_V - (\xi, g)$ is an element of $\mathrm{Br}(V)$ with trivial residue $\partial_Q(\beta)$ at Q, hence $\beta \in \mathrm{Br}(D)$.

Replacing S by a larger finite set of places we can assume the existence of affine curves $\mathcal{C}$ and $\mathcal{D}$, each of them smooth over $\mathrm{Spec}(\mathcal{O}_S)$, such that $q\colon D\to C$ extends to an étale $\mathcal{O}_S$-morphism $q\colon \mathcal{D}\to\mathcal{C}$. Let $\mathcal{P}$ be the Zariski closure of P in $\mathcal{C}$. By increasing S further we can ensure the following properties:

$f \in A$ comes from an element $f \in \mathcal{O}_S[\mathcal{C}]$;
$\xi \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n)$ is the restriction of an element $\xi \in \mathrm{H}^1_{\text{ét}}(\mathcal{D}, \mathbb{Z}/n)$;
$\beta \in \mathrm{Br}(D)$ is the restriction of an element $\beta \in \mathrm{Br}(\mathcal{D})$;
the natural morphism $\mathcal{P}\to\mathrm{Spec}(\mathcal{O}_S)$ is finite and étale;
the inverse image $\mathcal{Q}$ of $\mathcal{P}$ in $\mathcal{D}$ is integral and maps isomorphically onto its image in $\mathcal{C}$.

By a version of Chebotarev's theorem (Theorem 13.1.6), there exist infinitely many places v of k for which there is a place w of $k(P)$ over v with $k_v \cong k(P)_w$ (i.e., w has degree 1 over v) and w is inert in the cyclic extension $k(P)(\chi)/k(P)$ defined by $\chi \in \mathrm{H}^1(k(P), \mathbb{Z}/n)$.

For such a place v the $k(P)_w$-points of D and C given by Q and P extend to isomorphic $\mathcal{O}_v$-points N_v^0 of $\mathcal{D}$ and M_v^0 of $\mathcal{C}$, respectively.

Let $N_v \in \mathcal{D}(\mathcal{O}_v)$ be such that $g(N_v) \neq 0$ and let $M_v = q(N_v) \in \mathcal{C}(\mathcal{O}_v)$ be the image of N_v. Then by viewing N_v and M_v as k_v-points, one has

$$\alpha(M_v) = \alpha_V(N_v) = \beta(N_v) + (\xi(N_v), g(N_v)) \in \mathrm{Br}(k_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

We have $\beta(N_v) \in \mathrm{Br}(\mathcal{O}_v) = 0$. The place w of $k(P)$ is inert in the cyclic extension $k(P)(\chi)$, so if N_v is close enough to N_v^0 for the v-adic topology on $\mathcal{D}(\mathcal{O}_v)$, which is an open set of $D(k_v)$, the class

$$\xi(N_v) \in \mathrm{H}^1(k(P)_w, \mathbb{Z}/n) = \mathrm{H}^1(k_v, \mathbb{Z}/n)$$

has order n. Indeed, for any k_v-variety W and any $\chi \in \mathrm{H}^1_{\text{ét}}(W, \mathbb{Z}/n)$, an application of Theorem 10.5.1 gives that the evaluation map $W(k_v)\to\mathrm{H}^1(k_v, \mathbb{Z}/n)$, given by $M_v \mapsto \chi(M_v)$ with values in the finite group $\mathrm{H}^1(k_v, \mathbb{Z}/n)$, is locally constant.

Using formula (1.20) for the Serre residue, Theorem 1.4.14 (for $r = -r_W$) and the fact that the isomorphism $\mathrm{Br}(k_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ is given by r_W (Definition 13.1.7), we find that $\alpha(M_v) = (\xi(N_v), g(N_v)) \in \mathbb{Z}/n \subset \mathbb{Q}/\mathbb{Z}$ is equal to $v(g(N_v))$ modulo n, where $v\colon k_v^*\to\mathbb{Z}$ is the valuation. The closed set

$$\mathcal{Q} \times_{\mathcal{O}_S} \mathcal{O}_v \subset \mathcal{D} \times_{\mathcal{O}_S} \mathcal{O}_v$$

contains the $\mathcal{O}_v$-section of $\mathcal{D} \times_{\mathcal{O}_S} \mathcal{O}_v\to\mathrm{Spec}(\mathcal{O}_v)$ defined by N_v^0, and is finite and étale over $\mathcal{O}_v$. The map g induces an étale map $D \times_k k_v\to\mathbb{A}^1_{k_v}$ sending N_v^0 to $0 \in \mathbb{A}^1(k_v)$, hence by Theorem 10.5.1 induces a local isomorphism from $D(k_v)$ to $\mathbb{A}^1(k_v)$. Thus on any neighbourhood of N_v^0 in $D(k_v)$, the function g takes all possible valuations. The set $\mathcal{D}(\mathcal{O}_v)$ is open in $D(k_v)$. We conclude that there exists a point $N_v \in \mathcal{D}(\mathcal{O}_v)$ arbitrarily close to N_v^0 such that $v(g(N_v)) \equiv 1 \mod n$, hence its image $M_v \in \mathcal{C}(\mathcal{O}_v) \subset C(k_v)$ satisfies $\alpha(M_v) \neq 0$. □

Remark 13.4.2 In the simple example before the theorem, there also exist infinitely many places v of k such that α identically vanishes on $U(k_v)$, namely those where a becomes a square in k_v. Thus one may ask the following question.

Let k be a number field and let $U \subset X$ be a non-empty open subset of a smooth geometrically integral variety over k. Let $\alpha \in \mathrm{Br}(U) \setminus \mathrm{Br}(X)$. Does there exist infinitely many places v such that α vanishes identically on $U(k_v)$?

This holds if X is a curve, but can fail in higher dimension. Suppose that $\dim(X) \geq 2$ and there exists a geometrically integral subvariety $Z \subset X \setminus U$ of codimension 1 in X such that $\partial(\alpha) \in \mathrm{H}^1(k(Z), \mathbb{Q}/\mathbb{Z})$ defines a cyclic extension $L/k(Z)$ such that k *is algebraically closed in* L. Then, for almost all places v of k, the class α takes at least one non-zero value on $U(k_v)$.

Starting from Theorem 13.4.1, a combinatorial argument leads to the following extremely useful result. This version of D. Harari's "formal lemma" [Har94, Cor. 2.6.1, p. 233] was given in [CTS00].

Theorem 13.4.3 (Harari) *Let X be a smooth geometrically integral variety over a number field k. Let $U \subset X$ be a non-empty open set and let $B \subset \mathrm{Br}(U)$ be a finite subgroup. Let $(P_v) \in U(\mathbf{A}_k)^{B\cap \mathrm{Br}(X)}$. For any finite set S of places of k there exists an adelic point $(M_v) \in U(\mathbf{A}_k)$, where $M_v = P_v$ for $v \in S$, such that for any $\beta \in B$ we have*

$$\sum_{v\in\Omega} \mathrm{inv}_v\,(\beta(M_v)) = 0.$$

Proof. Replacing S by a larger finite set of places which contains all the archimedean places of k, we can find $\mathcal{O}_S$-schemes $\mathcal{X}$ and $\mathcal{U}$ of finite type, together with an open immersion of $\mathcal{O}_S$-schemes $\mathcal{U}\to\mathcal{X}$ which gives the open immersion $U\to X$ after restricting to the generic point $\mathrm{Spec}(k)$ of $\mathrm{Spec}(\mathcal{O}_S)$. In doing so we can ensure that $P_v \in \mathcal{U}(\mathcal{O}_v)$ for $v \notin S$. Since B is finite, by increasing S further, we may assume that $B \subset \mathrm{Br}(\mathcal{U})$ and $B\cap\mathrm{Br}(X) \subset \mathrm{Br}(\mathcal{X})$. Since $\mathrm{Br}(\mathcal{O}_v) = 0$, this implies that $\beta(P_v) = 0$ for any $\beta \in B$ and any $v \notin S$. Likewise, $\beta(M_v) = 0$ for any $\beta \in B \cap \mathrm{Br}(X)$ and *any* point $M_v \in \mathcal{X}(\mathcal{O}_v)$, where $v \notin S$.

Let $\alpha \in B$, $\alpha \notin \mathrm{Br}(X)$. According to Theorem 13.4.1, there exist an infinite set T_α of places of k disjoint from S and a family of points $(N_v)_{v\in T_\alpha}$ with each $N_v \in U(k_v) \cap \mathcal{X}(\mathcal{O}_v)$ such that for $\alpha(N_v) \neq 0$ for each $v \in T_\alpha$. The elements of $B \cap \mathrm{Br}(X) \subset \mathrm{Br}(\mathcal{X})$ take the zero value at a point N_v since we have $N_v \in \mathcal{X}(\mathcal{O}_v)$. Thus for each $v \in T_\alpha$, the evaluation of the elements of B at the point N_v defines a homomorphism

$$\varphi_{\alpha,v}\colon B/(B\cap \mathrm{Br}(X)) \longrightarrow \mathrm{Br}(k_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

such that $\varphi_{\alpha,v}(\alpha) \neq 0$. Since $B/(B \cap \mathrm{Br}(X))$ is a finite group, the group $\mathrm{Hom}(B/(B\cap\mathrm{Br}(X)), \mathbb{Q}/\mathbb{Z})$ is finite too. Thus there exists an infinite subset of T_α such that the attached homomorphisms $\varphi_{\alpha,v}$ are all equal. Replacing T_α by this subset, we may thus assume that there exists a homomorphism

$$\varphi_\alpha \colon B/(B\cap \mathrm{Br}(X)) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

with the property $\varphi_\alpha(\alpha)\neq 0$, such that for any $\beta\in B$ and any $v\in T_\alpha$ we have

$$\varphi_\alpha(\beta)=\mathrm{inv}_v(\beta(N_v))\in \mathbb{Q}/\mathbb{Z}. \tag{13.13}$$

Let C be the subgroup of $\mathrm{Hom}(B/(B\cap\mathrm{Br}(X)),\mathbb{Q}/\mathbb{Z})$ generated by the φ_α for $\alpha\in B$. Consider the natural bilinear pairing

$$B/(B\cap \mathrm{Br}(X))\times C \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Since $\varphi_\alpha(\alpha)\neq 0$, the left kernel of this pairing is zero. We thus obtain an injective map of $B/(B\cap\mathrm{Br}(X))$ into $\mathrm{Hom}(C,\mathbb{Q}/\mathbb{Z})$. Comparing the orders of these finite groups we conclude that $C\cong \mathrm{Hom}(B/(B\cap\mathrm{Br}(X)),\mathbb{Q}/\mathbb{Z})$.

From the assumption on the given $(P_v)\in U(\mathbf{A}_k)^{B\cap\mathrm{Br}(X)}$ and the conditions imposed on S at the beginning of the proof, we deduce that the linear map $B{\to}\mathbb{Q}/\mathbb{Z}$ that sends β to $-\sum_{v\in S}\beta(P_v)$ descends to a linear map $B/(B\cap\mathrm{Br}(X)){\to}\mathbb{Q}/\mathbb{Z}$. We have just seen that such a map can be written as a sum of maps φ_α (possibly with repetitions). By (13.13) each of the φ_α involved in this sum can be written as $\beta\mapsto\mathrm{inv}_v(\beta(N_v))$, this time without repeating v – since for each α we have an infinite set of places T_α at our disposal. We have thus found a finite set T of places $v\notin S$ and points $N_v\in U(k_v)\cap\mathcal{X}(\mathcal{O}_v)$ for $v\in T$ such that

$$\sum_{v\in S}\mathrm{inv}_v(\beta(P_v))+\sum_{v\in T}\mathrm{inv}_v(\beta(N_v))=0$$

for each $\beta\in B$. We then have

$$\sum_{v\in S}\mathrm{inv}_v(\beta(P_v))+\sum_{v\in T}\mathrm{inv}_v(\beta(N_v))+\sum_{v\notin S\cup T}\mathrm{inv}_v(\beta(P_v))=0$$

for each $\beta\in B$. This completes the proof once we take $M_v=N_v$ for $v\in T$ and $M_v=P_v$ for $v\notin T$. □

Remark 13.4.4 In [Witt07, p. 17] one finds the following variant of the formal lemma. Let k be a number field and let X be a smooth, proper, geometrically integral variety over k. For any non-empty open subset U of X, any finite subgroup $B\subset\mathrm{Br}(U)$, any finite set $S\subset\Omega$ and any family $(P_v)\in X(\mathbb{A}_k)^{\mathrm{Br}(X)\cap B}$ such that $P_v\in U(k_v)$ for $v\in S$, there exists a finite set $S_1\subset\Omega$ containing S and points $Q_v\in U(k_v)$ for $v\in S_1$ such that $Q_v=P_v$ for $v\in S$ and

$$\sum_{v\in S_1}\mathrm{inv}_v\left(\beta(Q_v)\right)=0\quad\text{for any}\quad\beta\in B.$$

Remark 13.4.5 Let $X \subset X_c$ be a smooth compactification. Theorem 13.4.3 for $U \subset X_c$ implies the same theorem for $U \subset X$ since $\mathrm{Br}(X_c) \subset \mathrm{Br}(X)$. Since $\mathrm{Br}(X_c) = \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$, in the condition of Theorem 13.4.3 we can replace $B \cap \mathrm{Br}(X)$ by the smaller subgroup $B \cap \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$, with the same conclusion.

Let us illustrate the idea and the use of the formal lemma on a simple case. Let k be a number field, let $a \in k^* \setminus k^{*2}$ and let $P(x)$ and $Q(x)$ be two coprime irreducible polynomials of odd degree in $k[x]$. Let U be the smooth, affine, geometrically integral variety over k defined by the equation

$$y^2 - az^2 = P(x)Q(x) \neq 0$$

and let $U \subset X$ be a smooth compactification. Consider the quaternion class $\beta = (a, P(x)) \in \mathrm{Br}(U)$. Suppose X, and hence U, has k_v-points for all completions of k. For almost all places v, the set $U(k_v)$ contains a point (x_v, y_v, z_v) with $v(x_v) = -1$ and also a point with $v(P(x_v)) = 0$. If v is inert in $k(\sqrt{a})$ and $v(x_v) = -1$, then $(a, P(x_v)) = 1/2 \in \mathbb{Q}/\mathbb{Z}$. If v is unramified in $k(\sqrt{a})$ and $v(P(x_v)) = 0$, then $(a, P(x_v)) = 0 \in \mathbb{Q}/\mathbb{Z}$.

Suppose we are given a finite set S of places of k and points $M_v \in U(k_v)$ for $v \in S$. It is then easy to extend the family $(M_v)_{v \in S}$ to an adelic point $(M_v) \in U(\mathbb{A}_k)$ such that

$$\sum_{v \in \Omega} \beta(M_v) = 0.$$

One can check that the map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ is surjective (Exercise 11.3.7 (i)) and that $\beta \in \mathrm{Br}(U)$ does not belong to $\mathrm{Br}(X)$. From the above equality and the exact sequence (13.2) it follows that there exists an element $c \in k^*$ such that the family $(M_v) \in U(\mathbb{A}_k)$ is the image of an adelic point of the k-variety V_c given by the system of equations

$$\begin{aligned} u_1^2 - av_1^2 &= cP(x) \\ u_2^2 - av_2^2 &= c^{-1}Q(x) \end{aligned}$$

via the projection map defined by the formal identity

$$x + \sqrt{a}y = (u_1 + \sqrt{a}v_1)(u_2 + \sqrt{a}v_2).$$

One thus reduces the question of the Hasse principle for

$$y^2 - az^2 = P(x)Q(x) \neq 0$$

to the Hasse principle for the varieties V_c.

A formal lemma for torsors

The following statement and its proof are taken from [BMS14, Prop. 3.1].

Theorem 13.4.6 *Let U be a smooth and geometrically integral variety over a number field k. Let T be a k-torus. Let $Y \to U$ be a T-torsor over U, and let $\theta \in \mathrm{H}^1_{\text{ét}}(U,T)$ be its class. Let $B \subset \mathrm{Br}(U)$ be the finite subgroup consisting of cup-products $\theta \cup \gamma$, where γ is an element of the finite group $\mathrm{H}^1(k,\widehat{T})$. Let $(M_v) \in U(\mathbf{A}_k)$ be a point orthogonal to $B \cap \mathrm{Br}_{\mathrm{nr}}(k(U)/k)$. Let $S \subset \Omega$ be a finite set of places. Then there exists an $\alpha \in \mathrm{H}^1(k,T)$ such that the twisted torsor Y^α has points in all completions of k and such that for each $v \in S$, the point M_v lies in the image of $Y^\alpha(k_v) \to U(k_v)$.*

Proof. By Theorem 13.4.3 and Remark 13.4.5 there exists an adelic point $(P_v) \in U(\mathbf{A}_k)$ with $M_v = P_v$ for $v \in S$ such that

$$\sum_{v\in\Omega} \mathrm{inv}_v\ [\theta(P_v) \cup \gamma] = 0 \in \mathbb{Q}/\mathbb{Z} \quad \text{for all} \quad \gamma \in \mathrm{H}^1(k,\widehat{T}).$$

Thus $(\theta(P_v)) \in \oplus_{v\in\Omega}\mathrm{H}^1(k_v,T)$ is orthogonal to $\mathrm{H}^1(k,\widehat{T})$. From the exact sequence (13.1.11) we obtain that $(\theta(P_v))$ is the image of some $\alpha \in \mathrm{H}^1(k,T)$ under the diagonal map $\mathrm{H}^1(k,T) \to \oplus_{v\in\Omega} \mathrm{H}^1(k_v,T)$. Twisting Y by α gives a torsor over Y under T which has a k_v-point over P_v for each $v \in \Omega$. □

Remark 13.4.7 In the proof of [CTHS03, Thm. 3.1] there is a similar argument with a stronger hypothesis and a stronger conclusion. There we have an additional condition $\bar{k}[U]^* = \bar{k}^*$. Starting from an element of $X(\mathbf{A}_k)^{\mathrm{Br}}$, in the situation described in *loc. cit.* one produces an adelic point on a suitable twist Y^α with the additional property that this adelic point is orthogonal to (a suitable subgroup) of the unramified Brauer group of Y^α.

Chapter 14
Are rational points dense in the Brauer–Manin set?

Let X be a smooth, projective and geometrically integral variety over a number field k. When $X(k)$ is dense in $X(\mathbf{A}_k)$, weak approximation holds for X. In the previous chapter we have seen that this is impossible if the Brauer–Manin set $X(\mathbf{A}_k)^{\mathrm{Br}}$ is smaller than $X(\mathbf{A}_k)$. Thus a natural question is this: is $X(k)$ a dense subset of $X(\mathbf{A}_k)^{\mathrm{Br}}$? Write $X(k)^{cl}$ for the closure of $X(k)$ in $X(\mathbf{A}_k)$. If $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ we shall say that *weak approximation holds for the Brauer–Manin set of* X. Informally speaking, the question is whether the Brauer–Manin obstruction is the only obstruction to weak approximation – and, in particular, to the Hasse principle, – in the sense that weak approximation holds for the adelic points which are not obstructed by the Brauer group. One would like to determine geometric classes of varieties such that weak approximation holds for their Brauer–Manin sets. It is not reasonable to expect this for all varieties, and it is interesting to explore further obstructions.

In Section 14.1 we discuss Colliot-Thélène's conjecture that weak approximation holds for the Brauer–Manin set of rationally connected varieties. This is a rather strong conjecture. Indeed, it implies that rational points are Zariski dense on any smooth, projective, and geometrically rational variety over a number field with at least one rational point. This is not known already for general conic bundles over the projective line and for general del Pezzo surfaces of degree 1. Any progress in this direction would be significant.

In Section 14.2 we look at Schinzel's Hypothesis (H), its consequences for rational points, and we report on results of Green, Tao and Ziegler from additive combinatorics that in some cases can be used instead of Schinzel's Hypothesis.

We then state a conjecture of Harpaz and Wittenberg which allows one to deduce Colliot-Thélène's conjecture for a variety fibred over $\mathbb{P}^1_k$ from the same conjecture for the smooth k-fibres. We explain the idea of the proof in an important particular case.

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In a number of concrete cases, the conjecture of Harpaz and Wittenberg was proved by analytic methods, thus leading to new results on existence and density of rational points on certain types of varieties.

In Section 14.3 we give an overview of the theory of obstructions to the local-to-global principle, in other words, various canonically defined subsets of $X(\mathbf{A}_k)$ that contain $X(k)$. We discuss relations between them and give several examples to demonstrate insufficiency of these obstructions for several types of varieties which are not rationally connected.

14.1 Rationally connected varieties: a conjecture

Rationally connected varieties

Let k be a field *of characteristic zero*. As usual, we denote an algebraic closure of k by $\bar{k}$. Unless otherwise mentioned, a *geometrically rational* variety X over k is a smooth, projective, geometrically integral variety such that $\overline{X} = X \times_k \bar{k}$ is birationally equivalent to the projective space of the same dimension.

Concrete examples of such varieties whose geometry is simple but arithmetic is difficult are smooth projective birational models of affine hypersurfaces in $\mathbb{A}_k^{d+1}$ with a simple looking defining equation of the following type. Let $x_1, \ldots, x_n, t$ be coordinates on $\mathbb{A}_k^{d+1}$. Let K/k be a finite field extension of degree d with a basis $e_1, \ldots, e_d$ of K as a vector space over k. Write $\Xi := \sum_{i=1}^d x_i e_i$. One sometimes says that Ξ is a variable with values in K. Let $\mathrm{N}_{K/k}\colon K \to k$ be the norm map. Let $P(t) \in k[t]$ be a non-zero polynomial in t. Consider the hypersurface in $\mathbb{A}_k^{d+1}$ given by the equation

$$\mathrm{N}_{K/k}(\Xi) = P(t).$$

This is a geometrically rational variety. Indeed, over $\bar{k}$ the equation can be rewritten as $x_1 \ldots x_n = P(t)$. The dense open set defined by $\prod_{i=1}^n x_i \neq 0$ is isomorphic to an open set of the affine space $\mathbb{A}_k^n$ with coordinates $x_2, \ldots, x_n$.

We refer to [Kol99, Ch. IV.3], [Deb01, Ch. 4] and [AK01, Def.-Thm. 29] for the following definitions and statements.

Definition 14.1.1 *A* **rationally connected** *variety over a field k of characteristic zero is a smooth, proper and geometrically integral variety X over k such that for any algebraically closed field K containing k, any two K-points of X are connected by a rational curve, i.e., belong to the image of a morphism $\mathbb{P}^1_K \to X_K$.*

Rationally connected varieties have been studied by Kollár, Miyaoka and Mori, and by Campana. In characteristic zero, they can be characterised by many equivalent properties. In particular, in the above definition one may simply assume that any two points are connected by a chain of rational curves, or even that two 'general' points are connected by such a chain.

In arbitrary characteristic, there is a notion of separably rationally connected varieties [Kol99, Ch. IV, Def. (3.2.3)], [AK01, Def. 35]. In characteristic zero, as assumed here, both notions coincide.

(1) A rationally connected variety of dimension 1 is a smooth conic.

(2) A separably rationally connected variety of dimension 2 is a geometrically rational surface [Kol99, Ch. IV, 3.3.5].

(3) Any smooth, projective, geometrically unirational variety is a rationally connected variety. (The converse is an open question.) For example, any smooth compactification of a homogeneous space of a connected linear algebraic group is a rationally connected variety.

(4) By a theorem of Campana and Kollár–Miyaoka–Mori [Kol99, Thm. V.2.13], any Fano variety (that is, a smooth, projective, geometrically integral variety with ample anticanonical bundle) is rationally connected. In particular, smooth hypersurfaces in $\mathbb{P}^n_k$ of degree $d \leq n$ are rationally connected.

(5) If $\mathrm{char}(k) = 0$ and $f\colon X \to C$ is a dominant morphism of smooth, projective, geometrically integral k-varieties such that C is a curve and the generic geometric fibre of f is rationally connected, then f has a section over $\bar{k}$. This is a deep theorem, proved by Graber, Harris and Starr [GHS03, Thm. 1.1] (with an extension by de Jong and Starr to separably rationally connected varieties in arbitrary characteristic [deJS03]). It implies that every closed fibre of f has an irreducible component of multiplicity 1. As a consequence of the existence of sections over curves one obtains that if $X \to Y$ is a dominant k-morphism of smooth, projective, geometrically integral k-varieties such that Y and the generic geometric fibre are rationally connected, then X is rationally connected [GHS03, Cor. 1.3].

(6) If X is a rationally connected variety over a field k of characteristic zero, then $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ and $\mathrm{Pic}(\overline{X})$ is a free abelian group of finite type [Deb01, Cor. 4.18]. In particular, $\mathrm{Br}(\overline{X})$ is finite and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Theorem 5.5.2).

(7) By a theorem of Enriques, Manin, Iskovskikh [Isk79] and Mori [Kol99, III.2], any geometrically rational smooth projective surface over a field k is k-birationally equivalent to a surface of at least one of the following families:

(i) A smooth del Pezzo surface of degree d, where $1 \leq d \leq 9$.
(ii) A conic bundle over a conic (possibly with degenerate fibres).

A *del Pezzo surface* is a smooth, projective, geometrically integral surface X such that the anticanonical bundle ω_X^{-1} is ample. The integer $d = (\omega_X.\omega_X)$ is called the degree of X; it satisfies $1 \leq d \leq 9$. Del Pezzo surfaces of degree 4 are smooth complete intersections of two quadrics in $\mathbb{P}^4_k$, and del Pezzo surfaces of degree 3 are smooth cubic surfaces in $\mathbb{P}^3_k$, see [Man74] and [Kol99].

The arithmetic theory of geometrically rational, smooth, projective surfaces X of degree $d = (\omega_X.\omega_X) \geq 5$ is not difficult. If such a surface X has a k-point, then it is rational over k, that is, X is k-birationally equivalent to $\mathbb{P}^2_k$. (If $d = 5$ or $d = 7$, then $X(k) \neq \emptyset$.) If k is a number field, the property $X(k)^{cl} = X(\mathbf{A}_k)$ holds for such surfaces. In particular, they satisfy the Hasse principle.

The reference for del Pezzo surfaces is [Man74, IV.29.4; IV.30.1]; for del Pezzo surfaces of degree 5 see also [Sko01, Cor. 3.1.5]. For conic bundles over a conic, see [Isk79, Teorema 5]. More recent references are [VA13, Ch. 2] and [Po18, §9.4.2].

Colliot-Thélène's conjecture

In the case of surfaces, the following conjecture was put forward as an open question by Colliot-Thélène and Sansuc in 1979, see [CTS80]. The general question was raised in Colliot-Thélène's lectures at the Institut Henri Poincaré in 1999 and mentioned again in [CT03].

Conjecture 14.1.2 *If X is a rationally connected variety over a number field k, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

In other words, the Brauer–Manin obstruction to the Hasse principle for a rationally connected variety should be the only obstruction, and the Brauer–Manin set of a rationally connected variety is conjectured to satisfy weak approximation.

Since $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite when X is rationally connected, the closed set $X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)$ is open and the conjecture is birationally invariant [CTPS16, Prop. 6.1].

Also, if the conjecture holds, and $X(k) \neq \emptyset$, then *weak weak approximation* (Definition 13.2.4) holds for X.

A partial converse is due to Harari: if a smooth, projective and geometrically integral variety X over a number field k satisfies weak weak approximation over every finite extension of k, then the geometric fundamental group of X is trivial, see [Har00, Cor. 2.4] and the remark after it. This implies $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. See Theorem 13.3.18 for a connection between weak approximation and the condition $\mathrm{H}^2(X, \mathcal{O}_X) = 0$.

Here are some of the consequences of conjectural weak weak approximation for rationally connected varieties. In particular, these are consequences of Conjecture 14.1.2.

(1) For any rationally connected variety X over a number field k with a k-point, the set $X(k)$ is Zariski dense in X. Already in dimension 2, i.e., for geometrically rational surfaces, this is not known.
(2) Over any rationally connected variety X over a number field k with a k-point, Hilbert's irreducibility theorem holds [Ser92, Ch. 3, §5].
(3) Let G be a finite group. For any number field k there exists a finite Galois field extension K/k with $\mathrm{Gal}(K/k) \simeq G$ ([Ser92, Thm. 3.5.9],

[Har07a]). The case $k = \mathbb{Q}$ is the inverse Galois problem, a well-known open problem. For recent progress, see [HW20].

(4) Let X be a geometrically rational surface over a number field k such that the Brauer–Manin obstruction is the only obstruction to the existence of a rational point on X over any finite extension of k. Then X contains a point defined over some abelian extension of k [Kan87, Thm. 3, Remark].

There is theoretical evidence for Conjecture 14.1.2 for geometrically rational conic bundle surfaces. Indeed, in this case it follows from Schinzel's Hypothesis (H), see Section 14.2.1.

Conjecture 14.1.2 is known for conic bundles over the projective line with $r \leq 5$ geometric degenerate fibres. The case $r \leq 3$ is easy: in this case the Hasse principle and weak approximation hold. Châtelet surfaces are the simplest non-trivial family of conic bundles with $r = 4$. They are given by an affine equation

$$y^2 - az^2 = P(x),$$

where $a \in k^*$ and $P(x) \in k[x]$ is a separable polynomial of degree 3 or 4. The conjecture was proved for such surfaces by Colliot-Thélène, Sansuc and Swinnerton-Dyer [CTSS87], after an earlier result of Colliot-Thélène, Coray and Sansuc [CTCS80]. In particular, it allows one to describe the values of $c \in \mathbb{Z}$ for which the surface of Iskovskikh's type (13.6) over $\mathbb{Q}$

$$y^2 + z^2 = (c - x^2)(x^2 - c + 1)$$

has a rational point, and similarly for the surfaces in Examples 13.3.23 and 13.3.24.

For general conic bundles with $r = 4$ the conjecture was proved by Salberger (unpublished) and by Colliot-Thélène [CT90]. The case $r = 5$ is due to Salberger and Skorobogatov [SSk91]. Swinnerton-Dyer discusses some special cases with $r = 6$. For short proofs of his results see [Sko01, Ch. 7].

Conjecture 14.1.2 is known for del Pezzo surfaces of degree 4 with a k-point [SSk91]. This is one case where theorems about zero-cycles ultimately lead to results on rational points. For general del Pezzo surfaces of degree 4, Wittenberg in his thesis [Witt07] develops a method of Swinnerton-Dyer [SwD95, CTSS98b] to produce strong evidence – conditional on Schinzel's Hypothesis (H) and finiteness of Tate–Shafarevich groups of elliptic curves.

In higher dimensions, the case of intersections of two quadrics has been much discussed (Mordell; Swinnerton-Dyer; Colliot-Thélène, Sansuc and Swinnerton-Dyer [CTSS87]; Heath-Brown [HB18]). Let us quote the results for arbitrary smooth complete intersections of two quadrics in $\mathbb{P}^n_k$. For $n \geq 5$, if there is a k-point, then weak approximation holds. For $n \geq 7$, the Hasse principle is known. For $n \geq 5$, this is also conjectured to hold, and is proved conditionally on Schinzel's Hypothesis (H) and the finiteness of Tate–Shafarevich groups of elliptic curves in [Witt07].

For diagonal cubic surfaces X over $\mathbb{Q}$, there is numerical evidence [CTKS87] that $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$ implies $X(\mathbb{Q}) \neq \emptyset$. For diagonal cubic hypersurfaces of dimension at least 3 over $\mathbb{Q}$, Swinnerton-Dyer [SwD01] proves the Hasse principle conditionally on the finiteness of Tate–Shafarevich groups of elliptic curves over number fields.

Further evidence is given by unconditional theorems for some classes of singular cubic hypersurfaces [CTS89].

When the number of variables is large with respect to the degree, the circle method can be applied. A celebrated general result is due to Birch [Bir61], and a more general result for smooth projective varieties $X \subset \mathbb{P}^n_{\mathbb{Q}}$ is due to Browning and Heath-Brown [BH17]. In particular, such varieties satisfy the Hasse principle and weak approximation if $\dim(X) \geq (\deg(X)-1)2^{\deg(X)}-1$.

This method also gives good results in relatively low dimension for cubic hypersurfaces: smooth cubic hypersurfaces in $\mathbb{P}^n_{\mathbb{Q}}$ have rational points when $n \geq 9$ (Heath-Brown [HB83]) and satisfy the Hasse principle for $n = 8$ (Hooley [Hoo88]). See [Hoo14] for a discussion of the case $n = 7$.

If X is a smooth, projective, integral variety birationally equivalent to a homogeneous space of a connected linear algebraic group with connected geometric stabilisers, then $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$. The case when the stabilisers are trivial is a theorem of Sansuc [San81, Cor. 8.7]. Using [San81, Lemme 6.1] (a special case of Harari's formal lemma) as in the proof of [San81, Cor. 8.7], one immediately reduces the case of connected stabilisers to a theorem of Borovoi [Bor96, Cor. 2.5]. For such a variety X, one has the refined statement that $X(k)$ is not empty if $X(\mathbb{A}_k)^{\text{Б}(X)} \neq \emptyset$, where $\text{Б}(X) \subset \mathrm{Br}(X)$ is defined in Remark 13.3.7.

It is worth mentioning that in the function field case, that is, when k is the function field of a curve over a finite field, geometric methods may give sharper, unconditional results. Zhiyu Tian [Tian17] proved the Hasse principle for smooth complete intersections of two quadrics in $\mathbb{P}^n_k$ for $n \geq 5$ in odd characteristic, and the Hasse principle for smooth cubic hypersurfaces in $\mathbb{P}^n_k$ for $n \geq 5$ in characteristic $p \geq 7$.

14.2 Schinzel's Hypothesis (H) and additive number theory

14.2.1 Applications of Schinzel's hypothesis

Let us state Schinzel's Hypothesis (H) (1958) [SS58], which is a generalisation of conjectures of Bouniakowsky (1854) [Bou57, p. 328] and Dickson (1904). Its quantitative version is the conjecture of Bateman and Horn (1962), a more general version of conjectures of Hardy and Littlewood (1922).

Conjecture 14.2.1 Schinzel's Hypothesis (H) *Let $P_i(x) \in \mathbb{Z}[x]$, for $i = 1, \ldots, r$, be non-constant irreducible polynomials with positive leading coefficients. Assume that no prime divides all the numbers $\prod_{i=1}^{r} P_i(m)$, where $m \in \mathbb{Z}$. Then there exist infinitely many positive integers n such that each $P_i(n)$ is a prime number, for $i = 1, \ldots, r$.*

After stating his conjecture Bouniakowsky adds this remark: "*Il est à présumer que la démonstration rigoureuse du théorème énoncé sur les* progressions arithmétiques des ordres supérieurs *conduirait, dans l'état actuel de la théorie des nombres, à des difficultés insurmontables ; néanmoins, sa réalité ne peut pas être révoquée en doute*". This is still the case today.

Note that only primes p with $p \leq \sum_i \deg(P_i)$ may divide all the numbers $\prod_{i=1}^{n} P_i(m)$. The only known case of this conjecture is the case of one polynomial of degree one: this is Dirichlet's theorem on primes in an arithmetic progression. That theorem was used by Hasse (1924) to prove the Hasse principle for zeros of quadratic forms in 4 variables once the case of 3 variables is known. In 1979, it was noticed [CTS82] that Hypothesis (H) can be used to give conditional proofs of the Hasse principle for other diophantine equations. Here is one of the simplest cases, taken directly from [CTS82, §5].

Theorem 14.2.2 (Colliot-Thélène–Sansuc) *Let $P(x) \in \mathbb{Q}[x]$ be an irreducible polynomial and let $a \in \mathbb{Q}^*$. Assume Schinzel's Hypothesis (H). Then the Hasse principle and weak approximation hold for any smooth model of the affine variety*

$$y^2 - az^2 = P(x) \neq 0.$$

Proof. Let us denote this affine variety by U. By the implicit function theorem (Theorem 10.5.1), it is enough to prove the theorem for U. Here, we shall make two simplifying hypotheses. We shall assume $a > 0$ and shall prove weak approximation only at the finite places. We refer the reader to [CTS82, §5] for the technical arguments required to handle the real place. Such extra efforts are often needed when handling the archimedean places.

Assume that we are given points

$$(y_p, z_p, x_p) \in U(\mathbb{Q}_p)$$

for all primes p. Let S be a finite set of primes containing $p = 2$, the primes p such that $v_p(a) \neq 0$, the primes p such that $P(x) \notin \mathbb{Z}_p[x]$, the primes for which the reduction of $P(x)$ modulo p has degree less than $\deg P(x)$ or is not separable, and the primes $p \leq \deg(P)$.

Using the irreducibility of $P(x)$, Hensel's lemma and Schinzel's Hypothesis (H), one finds $\lambda \in \mathbb{Q}$ very close to each $x_p \in \mathbb{Q}_p$ for $p \in S$ and such that

$$P(\lambda) = q \prod_{p \in S} p^{n_p} \in \mathbb{Q},$$

where $n_p \in \mathbb{Z}$ and q is a prime not in S ("the Schinzel prime").

Then the rational number $P(\lambda) \neq 0$ is represented by the quadratic form $y^2 - az^2$ in each completion of $\mathbb{Q}$ (including the reals, since we assumed $a > 0$), except possibly in $\mathbb{Q}_q$. By Corollary 13.1.10, $P(\lambda)$ is represented by this form over $\mathbb{Q}_q$ and over $\mathbb{Q}$. Using weak approximation on the affine conic $y^2 - az^2 = P(\lambda)$ and the implicit function theorem (Theorem 10.5.1), one concludes that weak approximation away from the reals holds for U. □

In the above theorem, the condition that $P(t)$ is irreducible implies that for any smooth projective model X of U we have $\mathrm{Br}(X)/\mathrm{Br}_0(X) = 0$ (see Proposition 11.3.4, the exercises after it, and Remark 11.3.8). If one allows the separable polynomial $P(x)$ to be reducible, then the finite group $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ can be non-zero, and there can be counter-examples to the Hasse principle, e.g. Iskovskikh's counter-example (Section 13.3.3). Using a descent argument [CTS82] or Harari's formal lemma for elements of the Brauer group (Theorem 13.4.3) or for U-torsors for suitable tori (Theorem 13.4.6), one can prove that Schinzel's hypothesis implies that $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.

To prove general theorems along these lines, it is convenient to use the following Hypothesis (H_1). As noted by Serre, this statement follows from Hypothesis (H). The proof of this implication is given in [CTS94, Prop. 4.1].

Conjecture 14.2.3 Hypothesis (H_1) *Let k be a number field and let $P_i(t)$, for $i = 1, \cdots, n$, be non-constant irreducible polynomials in $k[t]$. Let S be a finite set of places of k containing the infinite places, the finite places v where the coefficients of some $P_i(t)$ are either all contained in the maximal ideal of $\mathcal{O}_v$ or one of the coefficients is not in $\mathcal{O}_v$, and the finite places above the primes $p \leq [k:\mathbb{Q}] \sum_{i=1}^n \deg(P_i)$. Given $\lambda_v \in k_v$ for $v \in S$, one can find $\lambda \in k$, integral away from S, arbitrarily close to each λ_v in the v-adic topology for finite $v \in S$, arbitrarily large in the archimedean completions k_v, and such that for each $i = 1, \cdots, n$, $P_i(\lambda) \in k$ is a unit in the ring of integers of k_w for all places $w \notin S$ except perhaps one place w_i, where it is a uniformiser.*

Using Theorem 13.4.1 (Harari's formal lemma), one proves the following general result. For the definition of the vertical Brauer group attached to a morphism $X \to Y$, see Section 11.1. When a morphism $X \to Y$ of varieties over a number field k is given, we write $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$ for $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}(X/Y)}$.

Theorem 14.2.4 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre $X_m/k(m)$ contains a component Y of multiplicity 1 such that the integral closure of $k(m)$ in the function field of Y is an* **abelian** *extension of $k(m)$. Assuming Hypothesis (H), if $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then there exists a point $c \in \mathbb{P}^1(k)$ such that X_c is smooth and $X_c(\mathbf{A}_k) \neq \emptyset$. Moreover, given a finite set S of places of k and an adelic point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can choose this point $c \in \mathbb{P}^1(k)$ so that X_c contains a k_v-point close to M_v for each $v \in S$.*

Proof. This is [CTSS98, Thm. 1.1]. □

Here the same reciprocity argument as in Hasse's proof is used: the abelian extensions mentioned in the theorem give rise to a cyclic extension L/K of number fields together with an element in K^* which is a local norm at all places of K except possibly one; then one concludes that the element is a global norm (Corollary 13.1.10).

The following special case was proved in [CTS94].

Theorem 14.2.5 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is a smooth quadric of dimension* 1 *or* 2*. Then* $\mathrm{Br}(X) = \mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)$*. Assuming Schinzel's hypothesis* (H)*, we have* $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$*.*

In relative dimension at least 3, this theorem holds unconditionally and is easy to prove.

14.2.2 Additive combinatorics enters

Over the field of rational numbers, a breakthrough happened in 2010. Work of B. Green and T. Tao, and further work with T. Ziegler (2012) established a statement which implies a two-variable version of Schinzel's Hypothesis (H), when restricted to a system of polynomials with integral coefficients each of total degree one. We refer the reader to the report [Zie14].

The initial results of Green and Tao, together with further work by L. Matthiesen on additive combinatorics, first led to unconditional results in the spirit of 'Schinzel implies Hasse'. This is the work of Browning, Matthiesen and Skorobogatov [BMS14]. A typical result is the unconditional proof of weak approximation for the Brauer–Manin set of a conic bundle over $\mathbb{P}^1_{\mathbb{Q}}$ when all the singular fibres are above $\mathbb{Q}$-rational points of $\mathbb{P}^1_{\mathbb{Q}}$. They also prove a similar result for the total space of quadric bundles of relative dimension 2 over $\mathbb{P}^1_{\mathbb{Q}}$. Until then, for most such $\mathbb{Q}$-varieties, we did not know that existence of one rational point implies that rational points are Zariski dense – unless one was willing to accept Schinzel's Hypothesis (H).

The work of Green, Tao and Ziegler led to further progress. Here is the exact result used, reproduced from [HSW14].

Theorem 14.2.6 (Green–Tao–Ziegler) *Let $L_1(x,y), \ldots, L_r(x,y)$ be pairwise non-proportional linear forms with coefficients in $\mathbb{Z}$, and let $c_1, \ldots, c_r$ be integers. Assume that for each prime p there exists a pair $(m,n) \in \mathbb{Z}^2$ such that p does not divide $L_i(m,n) + c_i$ for any $i = 1, \ldots, r$. Let $K \subset \mathbb{R}^2$ be an open convex cone containing a point $(m,n) \in \mathbb{Z}^2$ such that $L_i(m,n) > 0$ for $i = 1, \ldots, r$. Then there exist infinitely many pairs $(m,n) \in K \cap \mathbb{Z}^2$ such that each $L_i(m,n) + c_i$ is a prime.*

From this theorem, Harpaz, Skorobogatov and Wittenberg [HSW14] deduce a number of results on weak approximation for the Brauer–Manin set. Let us describe the argument in a simple case. We start with an easy consequence of Theorem 14.2.6, where for simplicity we do not consider approximation at the real place.

Proposition 14.2.7 [HSW14, Prop. 1.2] *Let e_i, for $i = 1, \ldots, m$, be distinct integers. Let S be a finite set of primes containing all the primes that divide some $e_i - e_j$ for $i \neq j$. Suppose that we are given $(u_p, v_p) \in \mathbb{Q}_p^2$ for each $p \in S$. Then there exist a pair $(u_0, v_0) \in \mathbb{Q}^2$ which is arbitrarily close to each (u_p, v_p) for the p-adic topology, and distinct primes p_i outside of S such that for each i,*

$$u_0 - e_i v_0 = p_i q_i \in \mathbb{Q}^*,$$

where $q_i \in \mathbb{Q}^*$ is a unit outside of S. □

Theorem 14.2.8 *Let $k = \mathbb{Q}$. Let U be the surface*

$$y^2 - az^2 = b \prod_{i=1}^{2n} (t - e_i) \neq 0,$$

where $a, b \in \mathbb{Q}^*$ and $e_1, \ldots, e_{2n}$ are distinct elements of $\mathbb{Q}$. Assume that $a > 0$. Let X be a smooth projective variety containing U as a dense open subset and let $(M_p) \in X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$. Then there are $\mathbb{Q}$-points of U arbitrarily close to (M_p) at the finite primes. In particular, $\mathbb{Q}$-points are Zariski dense in U and weak weak approximation holds.

Proof. A linear change of variables allows us to assume $e_i \in \mathbb{Z}$ for each i. The argument in the proof of Theorem 14.2.4 would require the use of Schinzel's Hypothesis in the (H_1) version for the system of polynomials $t - e_i$, but this is a difficult open conjecture even over $\mathbb{Q}$.

Instead, we shall use a simple but slightly mysterious trick and replace the variable t by two variables (u, v) such that $t = u/v$. Let V be the variety

$$Y^2 - aZ^2 = b \prod_{i=1}^{2n} (u - e_i v) \neq 0, \quad v \neq 0.$$

The formulae $y = Y/v^n$, $z = Z/v^n$, $t = u/v$ give an isomorphism $V \cong U \times \mathbb{G}_m$, where the coordinate on $\mathbb{G}_m$ is v. Let $V \subset V_c$ be a smooth compactification. Then V_c is birationally equivalent to $X \times \mathbb{P}^1_\mathbb{Q}$. By Corollary 6.2.11 the Brauer group is a stable birational invariant, so $\mathrm{Br}(X) \cong \mathrm{Br}(V_c)$.

Since X is geometrically rational, $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. By Lemma 13.3.13, moving (M_p) in a small adelic neighbourhood we can arrange that $(M_p) \in U(\mathbf{A}_\mathbb{Q}) \cap X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$ and find an adelic point $(N_p) \in V(\mathbf{A}_\mathbb{Q})$ orthogonal to $\mathrm{Br}(V_c)$ which projects to $(M_p) \in U(\mathbf{A}_\mathbb{Q})$.

We are given a finite set S of places and a neighbourhood of $M_p \in U(\mathbb{Q}_p)$ for each $p \in S$. We fix neighbourhoods of the points N_p that project into the given neighbourhoods of M_p. Now we apply Theorem 13.4.3 (Harari's formal lemma) to $(N_p) \in V(\mathbf{A}_\mathbb{Q})$ and the finite family of quaternion algebras $\alpha_i = (a, u - e_i v) \in \mathrm{Br}(V)$; these classes need not belong to $\mathrm{Br}(V_c)$. This gives a new element $(P_p) \in V(\mathbf{A}_\mathbb{Q})$ with $P_p = N_p$ for $p \in S$ (the points with coordinates Y_p, Z_p, u_p, v_p) and such that $\sum_p \mathrm{inv}_v(\alpha_i(P_p)) = 0$ for each i. Let $K = \mathbb{Q}(\sqrt{a})$. From the exact sequence of class field theory (13.3)

$$1 \longrightarrow \mathbb{Q}^*/\mathrm{N}(K^*) \longrightarrow \bigoplus_{p \leq \infty} \mathbb{Q}_p^*/\mathrm{N}(K_p)^* \longrightarrow \mathbb{Z}/2,$$

we conclude that for each i there exists a $c_i \in \mathbb{Q}^*$ such that for each place p, the map $\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(\mathbb{Q}_p)$ sends the quaternion class (a, c_i) to $(a, u_p - e_i v_p)$. We thus have elements $c_i \in \mathbb{Q}^*$ and an adelic point (R_p) on the variety given by the system

$$\begin{cases} Y^2 - aZ^2 & = & b\prod_{i=1}^{2n}(u - e_i v) \neq 0, \\ y_i^2 - az_i^2 & = & c_i(u - e_i v) \neq 0, \quad i = 1, \ldots, 2n, \end{cases}$$

such that R_p projects to P_p. The variety given by this system is isomorphic to the product of the conic $Y^2 - aZ^2 = b\prod_{i=1}^{2n} c_i$ and the variety W given by

$$y_i^2 - az_i^2 = c_i(u - e_i v) \neq 0, \quad i = 1, \ldots, 2n.$$

The conic given by $Y^2 - aZ^2 = b\prod_{i=1}^{2n} c_i$ satisfies Hasse principle and weak approximation. Now we apply Proposition 14.2.7 to a finite set S of primes containing the primes dividing some $e_i - e_j$, $i \neq j$, the prime 2 and the primes p such that a or some c_i is not a unit in $\mathbb{Z}_p$. This produces a pair $(u_0, v_0) \in \mathbb{Q}^2$ close to each (u_p, v_p) at each place $p \in S$ such that each equation

$$y_i^2 - az_i^2 = c_i(u_0 - e_i v_0) \neq 0$$

has solutions in all completions of $\mathbb{Q}$, except possibly in $\mathbb{Q}_{p_i}$. Since each of these equations is the equation of a conic, it has a solution over $\mathbb{Q}$, and it satisfies weak approximation. □

Theorem 14.2.8 is a special case of the following result.

Theorem 14.2.9 [HSW14, Cor. 3.2] *Let X be a smooth, proper, geometrically integral variety over $\mathbb{Q}$ with a dominant morphism $f\colon X \to \mathbb{P}^1_\mathbb{Q}$ satisfying the following conditions.*

(i) *The generic fibre of f is geometrically integral.*
(ii) *The non-split fibres of f are above $\mathbb{Q}$-rational points of $\mathbb{P}^1_\mathbb{Q}$ and each such fibre contains a component Y of multiplicity 1 such that the integral closure of $\mathbb{Q}$ in $\mathbb{Q}(Y)$ is an abelian extension of $\mathbb{Q}$.*
(iii) *The Hasse principle and weak approximation hold for the smooth fibres.*

Then $X(\mathbb{Q})^{cl} = X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$.

See Definition 10.1.3 for the definition of a split scheme over a field. The fibre X_m over the residue field $k(m)$ at a closed point m is split if and only if it contains a multiplicity 1 component which is geometrically integral.

Here are some concrete examples.

Corollary 14.2.10 [HSW14, Cor. 4.1] *Let $K_i/\mathbb{Q}$, for $i = 1, \ldots, r$, be cyclic extensions. Let $P_i(t)$, for $i = 1, \ldots, r$, be non-zero separable polynomials that are products of linear factors over $\mathbb{Q}$. Let X be a smooth projective variety over $\mathbb{Q}$ that contains the variety given by the system of equations*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = P_i(t) \neq 0, \quad i = 1, \ldots, r,$$

as a dense open subset. Then $X(\mathbb{Q})^{cl} = X(\mathbf{A}_\mathbb{Q})^{\mathrm{Br}}$.

Corollary 14.2.11 [HSW14, Cor. 4.2] *Let $K_i/\mathbb{Q}$, for $i = 1, \ldots, r$, be cyclic extensions. Let $b_i \in \mathbb{Q}^*$ and $e_i \in \mathbb{Q}$, for $i = 1, \ldots, r$. Then the variety over $\mathbb{Q}$ given by the system of equations*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = b_i(t - e_i) \neq 0, \quad i = 1, \ldots, r,$$

satisfies the Hasse principle and weak approximation.

To put this last result in perspective, here is what has been proved without the input of additive combinatorics. Corollary 14.2.11 is obvious when $r = 1$. The case $r = 2$ and K_1 and K_2 both of degree 2 is easy, as it reduces to quadrics. An old result of Birch, Davenport and Lewis obtained by the circle method gives the statement for $r = 2$ and $K_1 = K_2$ of arbitrary degree over $\mathbb{Q}$. The case $r = 3$ and $K_1 = K_2 = K_3$ of degree 2 over $\mathbb{Q}$ was covered by Colliot-Thélène, Sansuc and Swinnerton-Dyer in [CTSS87]. Not much else was known.

The results above concern the total space of a 1-parameter family $X \to \mathbb{P}^1_k$ with the following properties:

(i) The smooth fibres satisfy *weak approximation* (in particular, they satisfy the Hasse principle).
(ii) Each non-split fibre X_m over a closed point m contains a component Y of multiplicity 1 such that the algebraic closure of $k(m)$ in $k(Y)$ is *abelian*.
(iii) $k = \mathbb{Q}$ and the non-split fibres are over $\mathbb{Q}$-*rational* points (this last hypothesis is needed to use the results of Green, Tao and Ziegler).

It took some time to obtain results without assuming (i) or (ii). Unconditional results without these assumptions were obtained under the following strong condition: all fibres of the morphism $X \to \mathbb{P}^1_k$, except possibly one fibre above a k-point, contain a geometrically integral component of multiplicity 1 (Harari [Har94, Har97]).

Several unconditional results were obtained for varieties given by an equation

$$\mathrm{N}_{K/k}(\Xi) = P(t) \tag{14.1}$$

for an arbitrary finite field extension K/k, assuming that the polynomial $P(t)$ has at most two roots over $\bar{k}$ [HBS02, CTHS03, BH12, DSW12].

The first interesting results where the abelianity condition was relaxed, while allowing an arbitrary number of bad fibres, were obtained by D. Wei [Wei12]. Assuming Schinzel's hypothesis (H), he considered (14.1) where K/k is an arbitrary field extension of degree 3.

The case when $k = \mathbb{Q}$, K/k is an arbitrary field extension, and $P(t) \in \mathbb{Q}[t]$ is a polynomial of arbitrary degree all of whose roots are in $\mathbb{Q}$, was dealt with in [BM17].

14.2.3 Hypothesis of Harpaz and Wittenberg

In this section we discuss a new approach of Harpaz and Wittenberg that *does not require conditions (i) or (ii)*. The following statement is [HW16, Conjecture 9.1]; we shall refer to it as Hypothesis (HW). As explained in [HW16, §9.1], this is a replacement for a homogeneous version of Hypothesis (H_1), itself a consequence of Schinzel's Hypothesis (H). A closely related, more geometric version of (HW) is given below (Proposition 14.2.13).

Conjecture 14.2.12 Hypothesis (HW) *Let k be a number field. Let $n \geq 1$ be an integer and let $P_1(t), \ldots, P_n(t) \in k[t]$ be pairwise distinct irreducible monic polynomials. Write $k_i = k[t]/(P_i(t))$ and let $a_i \in k_i$ denote the class of t. Suppose that for each $i = 1, \ldots, n$ we are given a finite extension L_i of k_i and an element $b_i \in k_i^*$. Let S be a finite set of places of k containing the real places of k and the finite places above which, for some i, b_i is not a local unit or L_i/k_i is ramified. Finally, for each $v \in S$, fix an element $t_v \in k_v$. Assume that for each $i = 1, \ldots, n$ and each $v \in S$, there exists an $x_{i,v} \in (L_i \otimes_k k_v)^*$ such that*

$$t_v - a_i = b_i \mathrm{N}_{L_i \otimes_k k_v / k_i \otimes_k k_v}(x_{i,v}) \quad \in k_i \otimes_k k_v.$$

Then there exists a $t_0 \in k$ satisfying the following conditions:

(1) *t_0 is arbitrarily close to t_v for $v \in S$;*
(2) *for every $i = 1, ..., n$ and every finite place w of k_i with $w(t_0 - a_i) > 0$, either w lies above a place of S or the field L_i has a place of degree 1 over w.*

Note that the above conjecture for a given family $\{a_i, b_i, k_i, L_i\}_{i \in I}$ and any finite family S of places as above is equivalent to the same conjecture where all the $i \in I$ with $k_i = L_i$ are removed.

Techniques of analytic number theory have already established Hypothesis (HW) in a number of significant cases, which we now list.

It is convenient to define

$$\varepsilon = \sum_{i=1}^{n} [k_i : k].$$

For an arbitrary number field k, Hypothesis (HW) is known in the following cases.

(i) $\varepsilon \leq 2$ (see [HW16, Thm. 9.11 (i)]). The essential ingredient is strong approximation for the complement of a closed subset of codimension at least 2 in $\mathbb{A}^m_k$. In the case $k_1 = k_2 = k$, one may use Dirichlet's theorem to prove the result. This is the proof of [CTSS98, Thm. 2.2.1]. To handle approximation at the non-archimedean places, that proof makes use of Theorem 13.1.2.

(ii) $\varepsilon = 3$ and $[L_i : k_i] = 2$ for each i (see [HW16, Thm. 9.11 (ii)]).

When $k = \mathbb{Q}$, Hypothesis (HW) is also known in these cases:

(iii) Any $\varepsilon = n \geq 1$, $k_i = \mathbb{Q}$ for each $i = 1, \ldots, n$, and arbitrary number fields $L_1, \ldots, L_n$. This important case is due to Matthiesen [Mat18], who used the results by Green, Tao and Ziegler, as well as her joint work with Browning [BM17]. See [HW16, Thm. 9.14].

(iv) $\varepsilon = 3$, $n = 2$, $k_1 = \mathbb{Q}$ and $[k_2 : \mathbb{Q}] = 2$. This case was established by Browning and Schindler [BS19] who used [Mat18] to strengthen the sieve method approach of Browning and Heath-Brown in [BH12].

(v) $\varepsilon = 3$, $n = 1$, $[k_1 : \mathbb{Q}] = 3$ and the extension L_1/k_1 is abelian, or is of degree 3, or is of the shape $L_1 = k_1(c^{1/p})$ for some $c \in k_1$ and p a prime number – these are all special cases of "almost abelian" extensions as defined in [HW16, Def. 9.4]. As noticed in [HW16, Remark 9.7], this follows from the work of Heath-Brown and Moroz on primes represented by cubic forms in two variables.

In [HW16, Prop. 9.9, Cor. 9.10] we find more geometric versions of Hypothesis (HW).

Let k be a number field. Let $k \subset k_i \subset L_i$ be finite field extensions and let $a_i \in k_i$ and $b_i \in k_i^*$, for $i = 1, \ldots, r$. To this set of data, one attaches the affine k-variety defined in $\mathbb{A}^2_k \times \prod_i R_{L_i/k}(\mathbb{A}^1_{L_i})$ by the system of equations

$$u - a_i v = b_i \mathrm{N}_{L_i/k_i}(\Xi_i), \quad i = 1 \ldots, r, \tag{14.2}$$

where u, v are variables and Ξ_i is a variable with values in the k-vector space L_i. Here we write

$$\mathrm{N}_{L_i/k_i} \colon R_{L_i/k}(\mathbb{A}^1_{L_i}) \longrightarrow R_{k_i/k}(\mathbb{A}^1_{k_i})$$

for the map induced by the norm map from L_i to k_i. Let W be the smooth open subset of the variety (14.2) defined by the conditions

$$(u,v) \neq (0,0), \qquad \Xi_i \notin [R_{L_i/k}(\mathbb{A}^1_{L_i}) \setminus R_{L_i/k}(\mathbb{G}_{m,L_i})]_{\text{sing}}.$$

There is a natural projection map $W \to \mathbb{A}^2_k \setminus (0,0) \to \mathbb{P}^1_k$.

A key insight is that $W \to \mathbb{P}^1_k$ is a kind of universal model for varieties fibred over $\mathbb{P}^1_k$ with bad, irreducible fibres of multiplicity 1 over the closed points with residue field k_i defined by $(u,v) = (a_i, 1)$ and split by the extension L_i/k_i.

Proposition 14.2.13 [HW16, Cor. 9.10] *If strong approximation off any finite place v_0 of k holds for every k-variety W as above, then Hypothesis* (HW) *holds.*

Note that the assumption of strong approximation implies that the varieties W satisfy the Hasse principle.

Harpaz and Wittenberg prove the following theorem [HW16, Cor. 9.25], the main ideas of which will be discussed in Section 14.2.4 below.

Theorem 14.2.14 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is a rationally connected variety. Assuming Hypothesis* (HW)*, if $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, then there exists a point $t_0 \in \mathbb{P}^1(k)$ with smooth fibre X_{t_0} such that $X_{t_0}(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$. Moreover, given a finite set S of places of k and an adelic point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$, one can choose t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Remark 14.2.15 For $X \to \mathbb{P}^1_k$ as above, with rationally connected generic fibre, the total space X is a rationally connected variety by the theorem of Graber, Harris and Starr, hence the quotient $\mathrm{Br}(X)/\mathrm{Br}(k)$ is finite. The general results of Harpaz and Wittenberg [HW16, Thm. 9.17, Cor. 9.23] hold for $X \to \mathbb{P}^1_k$ with weaker assumptions on the generic fibre.

Using Borovoi's theorem [Bor96] mentioned at the end of Section 14.1 we obtain the following statement.

Corollary 14.2.16 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is birationally equivalent to a homogeneous space of a connected linear algebraic group over $k(\mathbb{P}^1)$ with connected geometric stabilisers. Assuming Hypothesis* (HW)*, we have $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$.*

This result applies in particular to any smooth projective model X of a variety given by a system of equations

$$\mathrm{N}_{K_i/k}(\Xi_i) = P_i(t), \quad i = 1, \ldots, n.$$

Such systems have been considered in many special situations.

Remark 14.2.17 It is a non-trivial algebraic problem to decide when such a variety X satisfies $\mathrm{Br}(X) = \mathrm{Br}_0(X)$. For instance, if the polynomials $P_i(t)$ are all of degree 1 and no two of them are proportional, do we have $\mathrm{Br}(X) = \mathrm{Br}_0(X)$?

14.2.4 Main steps of the proof of Theorem 14.2.14

For a finite place v of k we denote the ring of integers of k_v by $\mathcal{O}_v$, the maximal ideal of $\mathcal{O}_v$ by $\mathfrak{m}_v$, and the residue field $\mathcal{O}_v/\mathfrak{m}_v$ by $\kappa(v)$.

As mentioned in [HW16, Remark 9.18 (i)], if one assumes Hypothesis (HW) for arbitrary irreducible monic polynomials $P_1(t), \ldots, P_n(t)$, one can give a proof of Theorem 14.2.14 which is shorter than the proof of [HW16, Theorem 9.17] (where a minimal set of such polynomials is chosen for each situation under consideration). This is what we do here, under some further simplifying assumptions. Unlike the proof in [HW16], the proof below uses Severi–Brauer schemes.

Two important steps are Theorems 14.2.18 and 14.2.21, leading to the proof of Theorem 14.2.14. The first of them addresses the following question: given a dominant morphism $X \to \mathbb{P}^1_k$ for which there is no vertical Brauer–Manin obstruction, is there a k-point in $\mathbb{P}^1_k$ such that the fibre over this point is smooth and everywhere locally soluble?

Theorem 14.2.18 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre contains a component of multiplicity* 1. *Assume $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$. Then, assuming Hypothesis* (HW)*, there exists a point $t_0 \in k = \mathbb{A}^1_k(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and an adelic point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can choose t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. We can choose $\mathbb{A}^1_k = \mathrm{Spec}(k[t]) \subset \mathbb{P}^1_k$ so that the fibre X_∞ at the point at infinity is smooth and geometrically integral. For simplicity of notation *let us only consider the case where all the singular fibres are above k-points of* $\mathbb{A}^1_k$. Let $a_1 \ldots, a_n \in k$ be the coordinates of these points.

Thus all the other fibres, including the fibre at infinity, are smooth and geometrically integral. We concentrate on the existence of a k-point with everywhere locally soluble fibre and omit the proof of the last claim of the theorem.

Let E_i be an irreducible component of multiplicity 1 in the fibre above a_i and let L_i be the integral closure of k in the function field $k(E_i)$. Let $U \subset X$ be the complement to the union of the fibre at infinity and the fibres above the points a_i. Let $T := \prod_{i=1}^n R^1_{L_i/k}\mathbb{G}_{m,L_i}$ be the product of norm 1 tori.

Consider the T-torsor over U given by the equations

$$t - a_i = \mathrm{N}_{L_i/k}(\Xi_i) \neq 0, \quad i = 1, \ldots, n.$$

This torsor over $U \subset X$ is the inverse image of a T-torsor over the complement to $\{a_1, \ldots, a_n\}$ in $\mathbb{P}^1_k$. Thus in this particular case the group B introduced in the proof of the formal lemma for torsors (Theorem 13.4.6) consists of elements coming from $\mathrm{Br}(k(\mathbb{P}^1))$, which are therefore vertical elements. Applying this theorem, under the hypothesis $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, we find elements $b_i \in k^*$ and a family $(M_v) \in U(\mathbf{A}_k)$, with projections $t_v \in k_v$ for $v \in \Omega$, such that for each $v \in \Omega$ the system

$$0 \neq t_v - a_i = b_i \mathrm{N}_{L_i/k}(\Xi_i), \quad i = 1, \ldots, n,$$

has solutions over k_v.

Choose a *finite* set S of places of k, large enough for various purposes. Firstly, we include into S all archimedean places and all non-archimedean places v for which $v(a_i) < 0$ for some i. Next, we require that the following conditions hold.

(i) Each L_i/k is unramified at any $v \notin S$.
(ii) Each b_i is a unit at any $v \notin S$.
(iii) The fibre at infinity X_∞ has good reduction at any $v \notin S$ and has points in all k_v for $v \notin S$ (this is possible by the Lang–Weil–Nisnevich estimates as this fibre is smooth and geometrically integral).
(iv) Each $E_{i,\mathrm{smooth}}$ (which is geometrically integral over L_i) has points in all completions $L_{i,w}$, where w is a place of L_i not lying above a place of S.
(v) There exists a connected, regular proper model $\mathcal{X}/\mathbb{P}^1_{\mathcal{O}_S}$ of $X/\mathbb{P}^1_k$.

Given a place $v \notin S$ and a point $t_v \in k_v$ one can consider the reduction of X_{t_v} at v. Namely, t_v extends to a unique point of $\mathbb{P}^1(\mathcal{O}_v)$, we consider the restriction of $\mathcal{X}/\mathbb{P}^1_{\mathcal{O}_S}$ to $\mathcal{O}_v$, and then the reduction modulo $\mathfrak{m}_v$.

Now we appeal to Hypothesis (HW). It produces a point $t_0 \in k$ very close to t_v for $v \in S$, and such that for any i and any $v \notin S$ either $v(t_0 - a_i) \leq 0$ or there exists a place of L_i of degree 1 over v.

Claim. $X_{t_0}(\mathbf{A}_k) \neq \emptyset$.

For $v \in S$ this follows from the implicit function theorem (Theorem 10.5.1).

If $v \notin S$ and $v(t_0 - a_i) < 0$ for some i, then $v(t_0) < 0$ and so the fibre X_{t_0} reduces modulo $\mathfrak{m}_v$ to the same smooth $\kappa(v)$-variety as the fibre X_∞, hence has k_v-points.

If $v \notin S$ and $v(t_0 - a_i) = 0$ for each i, then t_0 does not reduce to the same point as any of the a_i. Thus, provided the set S was chosen large enough, the fibre X_{t_0} reduces to a smooth and geometrically integral variety over the finite field $\kappa(v)$ with a fixed Hilbert polynomial. This allows one to apply the Lang–Weil–Nisnevich estimates which guarantee that the reduction of X_{t_0} modulo $\mathfrak{m}_v$ has a $\kappa(v)$-point. By Hensel's lemma, X_{t_0} has a k_v-point.

Finally, suppose that $v \notin S$ is such that $v(t_0 - a_i) > 0$ for some i. On the one hand, this implies that X_{t_0} reduces to the same variety over $\kappa(v)$ as X_{a_i}. On the other hand, by Hypothesis (HW), this implies that there is a place w of L_i of degree 1 over v such that $E_i \times_{L_i} L_{i,w}$ is geometrically integral over $L_{i,w}$. Again, provided the fixed set S was chosen large enough, the reduction of $E_i \times_{L_i} L_{i,w}$ over the field $\kappa(w) = \kappa(v)$ is geometrically integral and by the Lang–Weil–Nisnevich estimates has a smooth $\kappa(v)$-point. Using Hensel's lemma, we conclude that X_{t_0} contains a k_v-point. □

Assume that the smooth fibres of $X \to \mathbb{P}^1_k$ satisfy the Hasse principle and weak approximation. Then, assuming Hypothesis (HW), the proof of Theorem 14.2.18 easily implies that $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, hence also in $X(\mathbf{A}_k)^{\mathrm{Br}}$ – which then coincides with $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$. Such a general result was out of reach of the theorems based on Hypothesis (H).

The following (unconditional) corollary was originally obtained by the descent method, see [CTS00, Thm. A] which is an improvement of an earlier result [CTSS98, §2.2].

Corollary 14.2.19 [HW16, Thm. 9.11 (i)] *Let k be a number field and let X be a smooth, projective and geometrically integral variety over k. Let $f\colon X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre contains an irreducible component of multiplicity* 1. *Assume that the sum of the degrees of the closed points of $\mathbb{P}^1_k$ with a non-split fibre is at most* 2. *If $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}} \neq \emptyset$, then there exists a point $t_0 \in k = \mathbb{A}^1(k)$ such that X_{t_0} is smooth and has points in all completions of k. Moreover, given a finite set S of places of k, and an adelic point $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{vert}}}$, one can choose t_0 such that X_{t_0} contains k_v-points close to M_v for $v \in S$.*

Proof. The case when the non-split fibres are above two k-points corresponds to the case $n \leq 2$ in the proof of Theorem 14.2.18. In general, we have $\varepsilon \leq 2$, where ε is defined after the statement of Hypothesis (HW) in Section 14.2.3. As recalled there, Hypothesis (HW) is known for $\varepsilon \leq 2$. □

Lemma 14.2.20 *Let K be a field extension of k. Let $\rho_1, \ldots, \rho_n \in \mathrm{Br}(k)$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Br}(K)$ be elements of order not divisible by* $\mathrm{char}(k)$. *Let F be the function field of the product of Severi–Brauer varieties over K with classes $\alpha_1, \ldots, \alpha_n$. Let F' be the function field of the product of Severi–Brauer varieties over K with classes $\alpha_1 - \mathrm{res}_{K/k}(\rho_1), \ldots, \alpha_n - \mathrm{res}_{K/k}(\rho_n)$. For $\gamma \in \mathrm{Br}(K)$ we have $\mathrm{res}_{F/K}(\gamma) \in \mathrm{Br}_{\mathrm{nr}}(F/k)$ if and only if $\mathrm{res}_{F'/K}(\gamma) \in \mathrm{Br}_{\mathrm{nr}}(F'/k)$.*

Proof. For $i = 1, \ldots, n$ let P_i be a Severi–Brauer variety over k with class ρ_i. Let $P = P_1 \times_k \ldots \times_k P_n$. Each ρ_i goes to zero in $\mathrm{Br}(k(P))$, hence the images of α_i and $\alpha_i - \mathrm{res}_{K/k}(\rho_i)$ in $\mathrm{Br}(K(P))$ are the same for each i. By Lemma 7.1.7 this implies the existence of an isomorphism of field extensions of K:

$$F(P)(x_1, \ldots, x_r) \simeq F'(P)(y_1, \ldots, y_s),$$

where $x_1, \ldots, x_r, y_1, \ldots, y_s$ are independent variables, for some r and s. Now the lemma follows from Propositions 6.2.3, 6.2.6 and 6.2.9. □

Theorem 14.2.21 *Let X be a smooth, projective and geometrically integral variety over a number field k and let $f\colon X\to\mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is geometrically integral and that each closed fibre contains a component of multiplicity* 1. *Let $U\subset\mathbb{P}^1_k$ be a non-empty open set such that $X_U=f^{-1}(U)\to U$ is a smooth morphism. Let $B\subset\mathrm{Br}(X_U)$ be a finite subgroup and let $(M_v)\in X(\mathbf{A}_k)$ be an adelic point orthogonal to the intersection of* $\mathrm{Br}(X)$ *with* $B+f^*(\mathrm{Br}(k(\mathbb{P}^1)))\subset\mathrm{Br}(k(X))$. *Then, assuming Hypothesis* (HW), *for any finite set of places S there exists a point $t_0\in U(k)$ such that $X_{t_0}(\mathbf{A}_k)^B$ contains an adelic point (P_v), where P_v is close to M_v for each $v\in S$.*

Proof. Let k be an arbitrary field of characteristic zero. By Gabber's theorem (Theorem 4.2.1) the cohomological Brauer–Grothendieck group of a smooth variety coincides with its Brauer–Azumaya group. Thus we can assume that $B\subset\mathrm{Br}(X_U)$ is generated by the classes of Azumaya algebras A_i over X_U, for $i=1,\ldots,n$. Let $Y_U\to X_U$ be the fibred product of the corresponding Severi–Brauer schemes. Using resolution of singularities, the morphism $Y_U\to X_U$ can be extended to a morphism $g\colon Y\to X$, where Y is a smooth, projective, geometrically integral variety over k.

Let us prove that *each closed fibre of the composition $h\colon Y\to X\to\mathbb{P}^1_k$ contains an irreducible component of multiplicity* 1. Indeed, this condition is equivalent to the condition that h is locally split for the étale topology on $\mathbb{P}^1_k$. To check it we can assume $k=\bar{k}$. Since the morphism $X\to\mathbb{P}^1_k$ is locally split by assumption, for any closed point $m\in\mathbb{P}^1_k$ there exists a connected étale neighbourhood $V\to\mathbb{P}^1_k$ whose image contains m and such that $X_V\to V$ has a section $V\to X_V$. The image of this section is an integral curve $W\subset X_V$ which is étale over $\mathbb{P}^1_k$. The morphism $Y\to X$ gives rise to a morphism $Y_V\to X_V$. Let $Y_W\to W$ be the restriction of $Y_V\to X_V$ to the curve W. The generic fibre of $Y_W\to W$ is a product of Severi–Brauer varieties. Applying Tsen's theorem (Theorem 1.2.14 (i)) to $k(W)$ shows that $Y_W\to W$ has a rational section, which must be a morphism since W is a regular curve and $Y_W\to W$ is proper. This proves that h is locally split for the étale topology.

For each closed point $M\in\mathbb{P}^1\setminus U$, let $E_M\subset Y_P$ be an irreducible component of multiplicity 1. Let $\alpha\in\mathrm{Br}(U)$. Assume $h^*(\alpha)\in\mathrm{Br}(Y_U)$ belongs to the subgroup $\mathrm{Br}(Y)\subset\mathrm{Br}(Y_U)$. By Theorem 3.7.5, the residue of α at P lies in the kernel of $\mathrm{H}^1(k(M),\mathbb{Q}/\mathbb{Z})\to\mathrm{H}^1(k(E_M),\mathbb{Q}/\mathbb{Z})$, which is a finite group (Lemma 11.1.3). From the exact sequence

$$0\longrightarrow\mathrm{Br}(\mathbb{P}^1_k)\longrightarrow\mathrm{Br}(U)\longrightarrow\bigoplus_{M\in\mathbb{P}^1_k\setminus U}\mathrm{H}^1(k(M),\mathbb{Q}/\mathbb{Z})$$

(see Sections 1.5 and 3.6) and the isomorphism $\mathrm{Br}(k)\cong\mathrm{Br}(\mathbb{P}^1_k)$ (see Section 6.1) we conclude that the subgroup of $\mathrm{Br}(U)$, which consists of the elements α

such that $h^*(\alpha) \in \mathrm{Br}(Y_U)$ belongs to $\mathrm{Br}(Y) \subset \mathrm{Br}(Y_U)$, is finite modulo $\mathrm{Br}(k)$. Let $\gamma_1, \ldots, \gamma_m \in \mathrm{Br}(U)$ be elements generating this group modulo $\mathrm{Br}(k)$.

Write B' for the intersection of $\mathrm{Br}(X)$ with $B + f^*(\mathrm{Br}(k(\mathbb{P}^1_k))) \subset \mathrm{Br}(k(X))$. Let $\beta \in B \subset \mathrm{Br}(X_U)$ and let $\xi \in \mathrm{Br}(k(\mathbb{P}^1))$. Suppose $\beta + f^*(\xi) \in \mathrm{Br}(k(X))$ belongs to $\mathrm{Br}(X)$. Then $f^*(\xi)$ belongs to $\mathrm{Br}(X_U)$. Just as above, by Theorem 3.7.5, the residues of ξ at the closed points $M \in U$ are trivial, and the residues at the closed points $M \in \mathbb{P}^1_k$, $M \notin U$, belong to finitely many classes. By the same exact sequence as above, this implies that the class of ξ lies in a finite subgroup of $\mathrm{Br}(U)/\mathrm{Br}(k)$, and we conclude that $B' \subset \mathrm{Br}(k(X))$ is finite modulo the image of $\mathrm{Br}(k)$.

From now on, let k be a number field. By assumption, we have an adelic point $(M_v) \in X(\mathbf{A}_k)^{B'}$. By Proposition 10.5.2, each element of $\mathrm{Br}(X)$ is locally constant on $X(k_v)$. Since X is proper over k, by Proposition 13.3.1 (iii), each element of $\mathrm{Br}(X)$ vanishes on $X(k_v)$ for almost all places v. By the implicit function theorem for the smooth k_v-variety $X \times_k k_v$, the set $X_U(k_v)$ is dense in $X(k_v)$ for the v-topology. Since $B' \subset \mathrm{Br}(X)$ is finite modulo $\mathrm{Br}(k)$, this implies that there is an adelic point $(M'_v) \in X_U(\mathbf{A}_k)^{B'}$ such that M'_v is very close to M_v for each $v \in S$. We now rename M'_v and call it M_v.

By Harari's formal lemma (Theorem 13.4.3) we may assume that

$$\sum_{v\in\Omega} \mathrm{inv}_v A_i(M_v) = \sum_{v\in\Omega} \mathrm{inv}_v \gamma_j(M_v) = 0,$$

for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By class field theory (Theorem 13.1.8 (iii)) it follows that there exists a $\rho_i \in \mathrm{Br}(k)$ whose image in $\mathrm{Br}(k_v)$ is $A_i(M_v)$ for each place $v \in \Omega$.

Let $A'_i = A_i - \rho_i \in \mathrm{Br}(X_U)$, for $i = 1, \ldots, n$. We can choose Azumaya algebras over X_U representing these classes and consider the associated Severi–Brauer schemes. Let Y'_U be the fibred product of these schemes over X_U. As above, we extend the smooth, projective morphism $Y'_U \to X_U$ to a morphism $Y' \to X$, where Y' is a smooth, projective and geometrically integral variety over k.

Since $A'_i(M_v) = 0$, there is a k_v-point N_v in the fibre of $Y'_U \to X_U$ above M_v. Thus we have an adelic point $(N_v) \in Y'_U(\mathbf{A}_k)$ above $(M_v) \in X_U(\mathbf{A}_k)$.

We claim that $(N_v) \in Y'_U(\mathbf{A}_k)$ is orthogonal to $\mathrm{Br}_{\mathrm{vert}}(Y'/\mathbb{P}^1_k)$, where the vertical part of the Brauer group is taken with respect to the morphism $Y' \to \mathbb{P}^1_k$. Indeed, $\mathrm{Br}_{\mathrm{vert}}(Y'/\mathbb{P}^1_k)$, consists of the images of the elements of $\mathrm{Br}(U)$ which become unramified on Y'. By Lemma 14.2.20, these elements of $\mathrm{Br}(U)$ are exactly those which become unramified on Y. Modulo $\mathrm{Br}(k)$, this group is spanned by the classes γ_j, for $j = 1, \ldots, m$, and we have

$$\sum_{v\in\Omega} \mathrm{inv}_v\, \gamma_j(N_v) = 0,$$

since N_v is over M_v.

If we now assume Hypothesis (HW) and apply Theorem 14.2.18 to $Y'\to\mathbb{P}^1_k$, we find that there exists a $t_0 \in U(k)$ such that the fibre Y'_{t_0} has an adelic point (R_v), with R_v close to N_v for $v \in S$. Let $Q_v \in X_{t_0}(k_v)$ be the image of R_v under the morphism $Y'_{t_0}\to X_{t_0}$. For each $i = 1, \ldots, n$ and each $v \in \Omega$ we have $A'_i(Q_v) = 0$, hence $A_i(Q_v)$ is the image of ρ_i in $\mathrm{Br}(k_v)$. Thus

$$\sum_{v\in\Omega} \mathrm{inv}_v A_i(Q_v) = 0,$$

with Q_v close to M_v for $v \in S$. □

Remark 14.2.22 In the special case when $\mathbb{P}^1_k \setminus U$ is a union of k-points, the morphism $Y'\to\mathbb{P}^1_k$ is smooth over U, hence the simplifying assumption made in the above proof of Theorem 14.2.18 applies to $Y'\to\mathbb{P}^1_k$.

We are now ready to sketch the proof of Theorem 14.2.14.

Proof of Theorem 14.2.14. (Sketch) The generic fibre X_η of the morphism $X\to\mathbb{P}^1_k$ is a rationally connected variety. As mentioned in Section 14.1, this implies that $\mathrm{Br}(X_\eta)$ is finite modulo the image of $\mathrm{Br}(k(t))$. Thus we can choose an open subset $U \subset \mathbb{P}^1_k$ such that $X_U\to U$ is smooth and there is a finite group $B \subset \mathrm{Br}(X_U)$ that generates $\mathrm{Br}(X_\eta)$ modulo the image of $\mathrm{Br}(k(t))$. By the theorem of Graber–Harris–Starr [GHS03] the rational connectedness of X_η also implies that each closed fibre of $f\colon X\to\mathbb{P}^1_k$ contains an irreducible component of multiplicity 1. Thus Theorem 14.2.21 can be applied. Then one looks for a t_0 as in that theorem with the additional condition that the image of B generates the finite group $\mathrm{Br}(X_{t_0})/\mathrm{Br}(k)$. By Harari's specialisation result ([Har94, §3] and [Har97, Thm. 2.3.1], see also [HW16, Prop. 4.1]), the set of k-points such that the last condition is fulfilled is a Hilbert set. Thus we need to show that in Theorem 14.2.21 we can require t_0 to be an element of a Hilbert set. For this we refer to [HW16, Thm. 9.22] (see also [Sme15, Prop. 6.1]). □

Building on the results of additive combinatorics one obtains the following *unconditional* statement, first proved by Skorobogatov [Sko13]. His proof (of a slightly more general statement) uses the result of Browning and Matthiesen [BM17] on systems of equations

$$u - a_i v = b_i \mathrm{N}_{L_i/k}(\Xi_i), \quad i = 1\ldots, r,$$

obtained using additive combinatorics, but his argument looks somewhat different as it uses descent and universal torsors [CTS87a, Sko01]. In the proof we give here, descent has been replaced by the use of the formal lemma for torsors in the proof of Theorem 14.2.18.

Theorem 14.2.23 (Skorobogatov) *Let $U \subset \mathbb{A}^1_{\mathbb{Q}}$ be the open subset given by $P_1(t)\dots P_n(t) \neq 0$, where each polynomial $P_i(t)$ is a product of linear factors over $\mathbb{Q}$. Let X_0 be the smooth quasi-affine variety over $\mathbb{Q}$ defined by*

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = P_i(t) \neq 0, \quad i = 1, \dots, n,$$

where $K_1, \dots, K_n$ are number fields, and let $g\colon X_0 \to U$ be the projection to the coordinate t. Let $X_0 \subset X$ be an open embedding into a smooth, projective and geometrically integral variety over $\mathbb{Q}$ equipped with a dominant morphism $f\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ extending the map $X_0 \to U$. Then $X(\mathbb{Q})^{cl} = X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. In particular, if $X(\mathbb{Q})$ is not empty, then $X(\mathbb{Q})$ is Zariski dense in X and weak weak approximation holds for X.

Proof. The statement of the theorem does not depend on the choice of X. A convenient way to construct X is as follows. Let T be the product of norm 1 tori given by

$$\mathrm{N}_{K_i/\mathbb{Q}}(\Xi_i) = 1, \quad i = 1, \dots, n.$$

Choose a smooth T-equivariant compactification $T \subset Y$, which exists by [CTHS03]. The contracted product $X_0 \times^T Y$ has a natural proper morphism to U such that all fibres are smooth compactifications of T-torsors, in particular, they are geometrically integral. Extending $X_0 \times^T Y \to U$ we produce a smooth, projective and geometrically integral variety X over $\mathbb{Q}$ together with a proper morphism $f\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ such that $X_U \to U$ is smooth and $X_0 \subset X_U$ is an open subset.

Let $m \in U$ be a closed point and let X_m be the closed fibre at m. We have arranged that X_m is smooth and geometrically integral; moreover, X_m is geometrically rational. Any element of $\mathrm{Br}(X_\eta)$ is a restriction of some $\beta \in \mathrm{Br}(X \times_{\mathbb{P}^1_{\mathbb{Q}}} V)$, where V is a non-empty open subset of U. By the Gysin sequence (2.16), the residue of β at the generic point of X_m lies in

$$\mathrm{H}^1_{\text{ét}}(X_m, \mathbb{Q}/\mathbb{Z}) \subset \mathrm{H}^1(k(m)(X_m), \mathbb{Q}/\mathbb{Z}).$$

Using the fact that smooth, projective, rational varieties over an algebraically closed field of characteristic zero have no non-trivial finite étale covers, one shows that the natural map

$$\mathrm{H}^1(k(m), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_m, \mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Thus the above residue is an element of $\mathrm{H}^1(k(m), \mathbb{Q}/\mathbb{Z})$.

Using the Faddeev exact sequence (Theorem 1.33) one sees that

$$\mathrm{Br}(X_\eta) \subset f^*\mathrm{Br}(\mathbb{Q}(\mathbb{P}^1)) + \mathrm{Br}(X_U).$$

From this one deduces that there exists a finite subgroup $B \subset \mathrm{Br}(X_U)$ which surjects onto the (finite) group $\mathrm{Br}(X_\eta)/f^*\mathrm{Br}(\mathbb{Q}(\mathbb{P}^1))$. Then one proceeds as in the proofs of Theorems 14.2.21 and 14.2.14. The composition $Y'_U \to X_U \to U$ is smooth; the complement of U consists of $\mathbb{Q}$-points. Since $k = \mathbb{Q}$, Matthiesen's theorem (see Section 14.2.3) guarantees the validity of Hypothesis (HW) in the present situation. Thus in our case the proof of Theorem 14.2.14, via Theorem 14.2.21, gives an unconditional result. □

Again over the rationals, using Matthiesen's results [Mat18], Harpaz and Wittenberg [HW16] prove the more general, unconditional result:

Theorem 14.2.24 *Let X be a smooth, projective, geometrically integral variety over $\mathbb{Q}$ and let $X \to \mathbb{P}^1_{\mathbb{Q}}$ be a morphism with rationally connected generic fibre. Assume that all non-split fibres are above $\mathbb{Q}$-points of $\mathbb{P}^1_{\mathbb{Q}}$. If $X_P(\mathbb{Q})$ is dense in $X_P(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}(X_P)}$ for smooth fibres X_P over rational points of $\mathbb{P}^1_{\mathbb{Q}}$, then $X(\mathbb{Q})$ is dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$.*

The theorem also holds if $X \to \mathbb{P}^1_{\mathbb{Q}}$ has exactly two non-split fibres, one above a $\mathbb{Q}$-point and another one above a closed point of degree 2. Indeed, in this case Hypothesis (HW) was proved by Browning and Schindler in [BS19]. This subsumes the earlier result of Derenthal–Smeets–Wei [DSW12] for the equation (14.2) where $P(t)$ is irreducible of degree 2 and K is an arbitrary number field. Both proofs are based on the work of Browning and Heath-Brown [BH12], who used sieve methods.

14.2.5 Fibrations with two non-split fibres and ramified descent

A very special example for Corollary 14.2.19 is the Hasse principle for quadratic forms in four variables. Consider the system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2) = b_2(x_2^2 - a_2 y_2^2), \tag{14.3}$$

where a_1, a_2, b_1, b_2 are non-zero elements of a number field k, and assume that it has solutions everywhere locally. Let S be the set of places of k containing the infinite places, the places above 2, and the primes where at least one of a_1, a_2, b_1, b_2 is not a unit. Hasse's method is to apply Dirichlet's theorem on primes in an arithmetic progression to find a $t_0 \in k$ which is a unit away from $S \cup \{v_0\}$, where v_0 is a finite place where t_0 has valuation 1, and such that t_0 is close to the t-coordinate of given k_v-points for $v \in S$. Then each conic $t_0 = b_i(x_i^2 - a_i y_i^2)$, $i = 1, 2$, has points in all completions of k except possibly in k_{v_0}. The reciprocity law (Corollary 13.1.10) then implies that it has a solution also in k_{v_0} and in k.

We now give a *different* proof of this result (which does not use Theorem 14.2.18). The argument based on the Tate–Nakayama duality and the formal lemma for torsors directly produces a point t_0 such that each of the two equations $t_0 = b_i(x_i^2 - a_i y_i^2)$ has solutions in *all* completions. (Note that the proof of Theorem 14.2.18 given above is arranged in such a way that the fibre at infinity is smooth, but for the equation (14.3) the fibre at infinity is not smooth.)

Let U be the quasi-affine variety given by (14.3). Assume that U is everywhere locally soluble. Let $L = k(\sqrt{a_1}, \sqrt{a_2})$. The equation

$$0 \neq t = \mathrm{N}_{L/k}(\Xi)$$

defines a torsor over $\mathbb{G}_{m,k}$ whose structure group is the norm 1 torus $T = R^1_{L/k}(\mathbb{G}_{m,L})$. Let $Y \to U$ be the torsor obtained by pulling it back to U via the projection $U \to \mathbb{G}_{m,k}$ given by the coordinate t. We have $\mathrm{Br}_{\mathrm{nr}}(k(U)/k) = \mathrm{Im}(\mathrm{Br}(k))$, since U is birationally equivalent to the product of $\mathbb{P}^1_k$ and a quadric, hence the Brauer group of a smooth projective model of U is reduced to the image of $\mathrm{Br}(k)$. The formal lemma for torsors (Theorem 13.4.6) now gives an element $c \in k^*$ such that the system

$$0 \neq t = b_1(x_1^2 - a_1 y_1^2) = b_2(x_2^2 - a_2 y_2^2) = c\mathrm{N}_{L/k}(\Xi)$$

is everywhere locally soluble. This implies that the system

$$b_1(x_1^2 - a_1 y_1^2) = c = b_2(x_2^2 - a_2 y_2^2)$$

is everywhere locally soluble too. In other words, the fibre of $U \to \mathbb{G}_{m,k}$ over $c \in k^*$ is everywhere locally soluble. It is the product of two conics, so we use the Hasse principle for conics to conclude that $U(k) \neq \emptyset$.

If one considers a system of equations

$$0 \neq t = b_i \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, r,$$

with arbitrary number fields $k_1, \ldots, k_r$, and with no vertical Brauer–Manin obstruction to the existence of a rational point, the same argument will produce an element $c \in k^*$ such that the system

$$c = b_i \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, r,$$

is everywhere locally soluble. However, in the case of arbitrary number fields $k_1, \ldots, k_r$ one cannot ensure that this system has solutions in k: here the obstruction coming from the vertical Brauer group is not enough. As in Theorem 14.2.14, the whole Brauer group of (a smooth projective model) of the variety must be taken into account.

Note that in Theorem 14.2.18 there is no geometric assumption on the generic fibre of $X \to \mathbb{P}^1_k$ other than it is geometrically integral. This theorem can be applied to the problem of lifting an adelic point to some twist of a ramified cyclic cover, as discussed in Section 11.5.

Theorem 14.2.25 *Let k be a number field. Let X and Y be smooth, projective and geometrically integral varieties over k. Assume that μ_n acts on X, and Y is birationally equivalent to X/μ_n. Let $F \in k(Y)^*$ be a rational function such that the generic fibre of $X \to X/\mu_n$ is given by $t^n = F$. Write $\mathrm{div}(F) = \sum m_D D$, where D are irreducible divisors in Y. Let k_D be the algebraic closure of k in the function field $k(D)$. Assume that $(n, m_D) = 1$ for some D. Let $B \subset \mathrm{Br}(Y)$ be the subgroup consisting of the classes (χ, F), where χ is an element of the finite group*

$$\bigcap_D \mathrm{Ker}[m_D \, \mathrm{res}_{k_D/k} \colon \mathrm{H}^1(k, \mathbb{Z}/n) \to \mathrm{H}^1(k_D, \mathbb{Z}/n)].$$

If $Y(\mathbf{A}_k)^B \neq \emptyset$, then there exists an element $c \in k^*$ such that the twisted cover X_c with generic fibre $ct^n = F$ has points in all completions of k. Moreover, given a finite set S of places of k and an adelic point $(M_v) \in Y(\mathbf{A}_k)^B$, where $M_v \notin \mathrm{Supp}(\mathrm{div}(F))$ for $v \in S$, one can choose c close to $F(M_v)$ for $v \in S$.

Proof. In the notation of Section 11.5 consider the morphism $W \to \mathbb{P}^1_k$. Recall that W is stably birationally equivalent to Y and that at most two closed fibres of $W \to \mathbb{P}^1_k$, namely the k-fibres above 0 and ∞, are non-split. The k-fibres of $W \to \mathbb{P}^1_k$ other than the fibres at 0 and ∞ are cyclic twists of X. By Proposition 11.5.1 (i) the vertical Brauer group $\mathrm{Br}_{\mathrm{vert}}(W/\mathbb{P}^1_k)$ is generated by B modulo the image of $\mathrm{Br}(k)$. By Proposition 11.5.1 (iii), the assumption $(n, m_D) = 1$ implies that each closed fibre of $W \to \mathbb{P}^1_k$ contains an irreducible component of multiplicity 1. It remains to apply Corollary 14.2.19. □

Let n be a positive integer. Let $a, b, c, d \in k^*$ and let $S \subset \mathbb{P}^3_k$ be the smooth surface given by

$$ax^n + by^n = cz^n + dw^n.$$

We assume that S is everywhere locally soluble and there is no Brauer–Manin obstruction with respect to the finite subgroup $B \subset \mathrm{Br}(S)$ consisting of the classes $(\chi, a(x/y)^n + b)$. Here χ belongs to the kernel of the restriction map

$$\mathrm{H}^1(k, \mathbb{Z}/n) \longrightarrow \mathrm{H}^1(L, \mathbb{Z}/n),$$

where L is the étale k-algebra

$$L = \big(k[t]/(t^n + b/a)\big) \otimes_k \big(k[t]/(t^n + d/c)\big).$$

Then, by Theorem 14.2.25, there exists an element $\rho \in k^*$ such that each of the smooth plane curves

$$ax^n + by^n = \rho v^n, \quad cz^n + dw^n = \rho v^n \tag{14.4}$$

is everywhere locally soluble. When $n = p$ is a prime, we have $B = 0$, hence the relevant vertical Brauer group is reduced to the image of $\mathrm{Br}(k)$ (Proposition 11.5.2).

For $n = 3$ this statement is a starting point in Swinnerton-Dyer's paper [SwD01, Lemma 2, p. 901], see also a similar situation in [SkS05] and [HS16]. The challenge here, assuming $S(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, is to produce a $\rho \in k^*$ such that for each of the two curves (14.4) there is no Brauer–Manin obstruction to the existence of a k-point, or at least to the existence of a zero-cycle of degree 1.

14.3 Beyond the Brauer–Manin obstruction

14.3.1 Insufficiency of the Brauer–Manin obstruction

Let k be a number field. For $n \geq 4$ any smooth hypersurface $X \subset \mathbb{P}^n_k$ satisfies $\mathrm{Br}(k) = \mathrm{Br}(X)$ (Corollary 5.5.4). Thus $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$. If $d > n$, where d is the degree of X, then the canonical bundle on X is ample, and the Bombieri–Lang conjecture ([Lang91], [Po18, 9.5.3]) states that $X(k)$ is not Zariski dense in X. A refinement of this conjecture, also discussed in [Lang91], predicts that a 'hyperbolic' hypersurface has finitely many rational points. If these conjectures are true, then for any such X with $X(k) \neq \emptyset$, the set $X(k)$ cannot be dense in $X(\mathbf{A}_k)^{\mathrm{Br}} = X(\mathbf{A}_k)$. It is also very unlikely that the Hasse principle holds for rational points of smooth projective hypersurfaces of dimension at least 3 of large degree. Conditional examples with $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ and $X(k) = \emptyset$ can be found in [SW95, Po01].

In 1999 Skorobogatov [Sko99] gave the first unconditional example of a smooth, projective, geometrically integral variety X over a number field k such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ but nevertheless $X(k) = \emptyset$. In this example $k = \mathbb{Q}$ and the variety X is a surface of Kodaira dimension zero, which geometrically is a bielliptic surface. See also [Sko01, Ch. 8].

14.3.2 Quadric bundles over a curve, I

The following, more elementary example was constructed in [CTPS16]. The idea to use a curve with a unique rational point is due to Poonen [Po10].

Let C be a smooth, projective, geometrically integral curve over a number field $k \subset \mathbb{R}$ such that $C(k)$ consists of just one point, $C(k) = \{P\}$. Poonen [Po10b, Thm. 1.1] showed that such a curve C exists for any number field k. Mazur and Rubin [MR10, Thm. 1.1] proved that C can be chosen to be an elliptic curve. Let us write v_0 for the given real place $k \subset \mathbb{R}$. Let $\Pi \subset C(\mathbb{R})$ be an open interval containing P. Let $f\colon C \to \mathbb{P}^1_k$ be a surjective morphism unramified at P. Choose a coordinate function t on $\mathbb{A}^1_k = \mathbb{P}^1_k \smallsetminus f(P)$ such that f is unramified above $t = 0$. We have $f(P) = \infty$. Take any $a > 0$ in

k such that a is an interior point of the interval $f(\Pi)$ and f is unramified above $t = a$.

Let $v \neq v_0$ be a place of k. From the exact reciprocity sequence 13.3 from class field theory we see that there exists a quadratic homogeneous form $Q(x_0, x_1, x_2)$ over k of rank 3 that represents zero in all completions of k other than k_v and k_{v_0}, but not in k_v or k_{v_0}, and we may take Q positive definite at v_0. Choose an $n \in k$ with $n > 0$ and $-nQ(1,0,0) \in k_v^{*2}$. Let $Y_1 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(x_0, x_1, x_2) + nt(t-a)x_3^2 = 0$, and let $Y_2 \subset \mathbb{P}^3_k \times \mathbb{A}^1_k$ be given by $Q(X_0, X_1, X_2) + n(1-aT)X_3^2 = 0$. We glue Y_1 and Y_2 by identifying $T = t^{-1}$, $X_3 = tx_3$, and $X_i = x_i$ for $i = 0, 1, 2$. This produces a quadric bundle $Y \to \mathbb{P}^1_k$ with exactly two degenerate fibres (over $t = a$ and $t = 0$), each given by the quadratic form $Q(x_0, x_1, x_2)$ of rank 3. It is straightforward to check that Y is smooth over k.

Define $X = Y \times_{\mathbb{P}^1_k} C$. This is a flat surjective proper morphism $X \to C$ whose fibres are geometrically integral quadrics. The assumption that f is unramified at $t = 0$ and $t = a$ implies that X is also smooth.

For example, we can take $k = \mathbb{Q}$, $\mathbb{Q}_v = \mathbb{Q}_2$ and consider Y defined by

$$x_0^2 + x_1^2 + x_2^2 + 7t(t-a)x_3^2 = 0.$$

Proposition 14.3.1 *In the above notation we have $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ whereas $X(k) = \emptyset$.*

Proof. Since $C(k) = \{P\}$ we have $X(k) \subset X_P$. The fibre X_P is the smooth quadric $Q(x_0, x_1, x_2) + nx_3^2 = 0$. This quadratic form is positive definite thus X_P has no points in $k_{v_0} = \mathbb{R}$ and so $X(k) = \emptyset$. By assumption X_P has local points in all completions of k other than k_v and k_{v_0}. The condition $-nQ(1,0,0) \in k_v^{*2}$ implies that X_P contains k_v-points, so X_P has local points in all completions of k but one. Choose $N_u \in X_P(k_u)$ for each place $u \neq v_0$. Let $M \in \Pi$ be such that $f(M) = a$. Then the singular point of the real fibre X_M (the vertex of the quadratic cone) is a smooth real point of X. Take it as the v_0-component of the adelic point (N_u) of X.

We claim that $(N_u) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Indeed, the fibres of $X \to C$ are geometrically integral. Thus, by Proposition 11.2.4 the natural map $\mathrm{Br}(C) \to \mathrm{Br}(X)$ is surjective. Thus it is enough to show that the adelic point on C such that its components at all places other than v_0 are equal to P and its component at v_0 is M, is orthogonal to $\mathrm{Br}(C)$. The real point M is path-connected to P, so this adelic point is in the connected component of the diagonal image of the k-point P in $C(\mathbf{A}_k)$. By the continuity of the real evaluation map it is contained in $C(\mathbf{A}_k)^{\mathrm{Br}}$, so the proposition follows. □

14.3.3 Distinguished subsets of the adelic space

Let k be a number field. Let G be a linear algebraic group over k. Let X and Y be varieties over k and let $f\colon Y\to X$ be a left G-torsor. Up to isomorphism, such torsors are classified by the pointed set $\mathrm{H}^1_{\text{ét}}(X,G)$. For a number field k, we have a natural map

$$\theta\colon \mathrm{H}^1(k,G) \longrightarrow \prod_{v\in\Omega} \mathrm{H}^1(k_v,G).$$

By a theorem of Borel and Serre, the fibres of θ are finite, see [SerCG, III.4.6].

For any ring R containing k, the pullback of the torsor $f\colon Y\to X$ to an R-point of X induces a map $X(R)\longrightarrow \mathrm{H}^1(R,G)$. When $R=k$ we obtain a map $X(k)\to\mathrm{H}^1(k,G)$. When R is the ring of adèles $\mathbf{A}_k$ we obtain a map

$$X(\mathbf{A}_k) \longrightarrow \prod_v \mathrm{H}^1(k_v,G).$$

Define $X(\mathbf{A}_k)^f \subset X(\mathbf{A}_k)$ as the inverse image of $\theta(\mathrm{H}^1(k,G))$ under this map. It is clear that the diagonal map $X(k)\hookrightarrow X(\mathbf{A}_k)$ gives an embedding $X(k)\subset X(\mathbf{A}_k)^f$, cf. Remark 13.3.3.

To a continuous 1-cocycle $\sigma\colon \Gamma=\mathrm{Gal}(\bar{k}/k)\to G(\bar{k})$ one associates an inner form G_σ of G, which is a twisted $(\bar{k}/k)$-form of G defined with respect to the action of G on itself by conjugations. (See Section 1.3.2.) The isomorphism class of G_σ depends only on the class $[\sigma]\in\mathrm{H}^1(k,G)$. Twisting the left G-torsor $f\colon Y\to X$ we obtain a left G_σ-torsor $f_\sigma\colon Y^\sigma\to X$; its isomorphism class depends only on $[\sigma]$. See [Sko01, Ch. 2] for more details.

The class $[\sigma]$ is the image of a k-point $P\in X(k)$ under the map $X(k)\to\mathrm{H}^1(k,G)$ if and only if there exists a k-point $M\in Y^\sigma(k)$ such that $f_\sigma(M)=P$. This implies

$$X(k)=\coprod_{[\sigma]\in\mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(k)).$$

Similarly, by the definition of $X(\mathbf{A}_k)^f$ we have

$$X(\mathbf{A}_k)^f=\bigcup_{[\sigma]\in\mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)).$$

Proposition 14.3.2 *Let X be a variety over a number field k. Let G be a linear algebraic group over k. Let $f\colon Y\to X$ be a G-torsor. Then $X(\mathbf{A}_k)^f$ is a closed subset of $X(\mathbf{A}_k)$.*

Proof. See [Sko09a, Cor. 2.7] in the case when X is proper, and [CDX, Prop. 6.4] in general. The proof combines existence of good models over an open set of the ring of integers of k, the implicit function theorem over local fields, finiteness of $\mathrm{H}^1(k_v, G)$ for a linear algebraic group G over a local field k_v, the theorem of Borel and Serre mentioned above, and Lang's theorem that torsors under a connected linear algebraic group over a finite field have a rational point. □

Let us define

$$X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,G)} = \bigcap_f X(\mathbf{A}_k)^f,$$

where f ranges over all G-torsors $f\colon Y \to X$.

For a smooth, projective variety over k, a close relation between the intersection of $X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,T)}$ for all k-tori T, and $X(\mathbf{A}_k)^{\mathrm{Br}}$ was established by Colliot-Thélène and Sansuc [CTS87a]. This was extended by Skorobogatov [Sko99] to groups of multiplicative type, see [Sko01, §6.1]. The set $X(\mathbf{A}_k)^f$ attached to a torsor $f\colon Y \to X$ for a finite, non-commutative group k-scheme was used by Harari [Har00] to construct examples where weak approximation fails and this failure is not accounted for by the Brauer–Manin obstruction. The general definition of $X(\mathbf{A}_k)^f$ was spelled out in [HS02, §4], see also [Sko01, §5.3].

For various classes of linear groups we define closed subsets of $X(\mathbf{A}_k)$ containing $X(k)$, as follows:

$$X(\mathbf{A}_k)^{\mathrm{desc}} = \bigcap_{\text{linear } G} X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,G)},$$

$$X(\mathbf{A}_k)^{\text{ét}} = \bigcap_{\text{finite } G} X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,G)},$$

$$X(\mathbf{A}_k)^{\mathrm{conn}} = \bigcap_{\text{connected linear } G} X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,G)}.$$

If X is a smooth quasi-projective variety over a field k, then $\mathrm{Br}_{\mathrm{Az}}(X) \cong \mathrm{Br}(X)$ by Gabber's theorem (Theorem 4.2.1). So, if X is smooth and quasi-projective over a number field k, the connection between torsors for PGL_n and Azumaya algebras gives $X(\mathbf{A}_k)^{\mathrm{Br}} = \bigcap_n X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,\mathrm{PGL}_n)}$, see [HS02, Thm. 4.10]. One concludes that

$$X(\mathbf{A}_k)^{\mathrm{desc}} \subset X(\mathbf{A}_k)^{\mathrm{Br}}. \tag{14.5}$$

A theorem of Harari [Har02] gives

$$X(\mathbf{A}_k)^{\mathrm{Br}} \subset X(\mathbf{A}_k)^{\mathrm{conn}}$$

for any geometrically integral X.

Allowing $f\colon Y\to X$ to be a torsor for any *finite* group k-scheme G, we define more subsets of $X(\mathbf{A}_k)$ containing $X(k)$:

$$X(\mathbf{A}_k)^{\text{ét,Br}} = \bigcap_{f} \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}});$$

$$X(\mathbf{A}_k)^{\text{ét,desc}} = \bigcap_{f} \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}).$$

Next, allowing $f\colon Y\to X$ to be a torsor for any linear group k-scheme G, define

$$X(\mathbf{A}_k)^{\text{desc,desc}} = \bigcap_{f} \bigcup_{[\sigma]\in \mathrm{H}^1(k,G)} f_\sigma(Y^\sigma(\mathbf{A}_k)^{\mathrm{desc}}).$$

Skorobogatov's example was first interpreted in [Sko99] as an example of a smooth projective variety such that $X(\mathbf{A}_k)^{\mathrm{Br}}\neq\emptyset$ but $X(\mathbf{A}_k)^{\text{ét,Br}}=\emptyset$. It was then interpreted in [HS02] as an example with $X(\mathbf{A}_k)^{\mathrm{H}^1_{\text{ét}}(X,G)}=\emptyset$ for a suitable finite k-group G, hence with $X(\mathbf{A}_k)^{\mathrm{desc}}=\emptyset$. This raised the question of the relation between these various obstructions.

Theorem 14.3.3 *Let k be a number field. Let X be a smooth, quasi-projective, geometrically integral variety over k. Then*

$$X(\mathbf{A}_k)^{\mathrm{desc}} = X(\mathbf{A}_k)^{\text{ét,desc}} = X(\mathbf{A}_k)^{\text{ét,Br}}.$$

In particular, the following two conditions are equivalent:

(i) *any left G-torsor $Y\to X$, where G is a linear group k-scheme, has a twisted form $Y^\sigma\to X$ such that $Y^\sigma(\mathbf{A}_k)\neq\emptyset$;*
(ii) *any left G-torsor $Y\to X$, where G is a finite group k-scheme, has a twisted form $Y^\sigma\to X$ such that $Y^\sigma(\mathbf{A}_k)^{\mathrm{Br}}\neq\emptyset$.*

This theorem is a consequence of the following inclusions

$$X(\mathbf{A}_k)^{\mathrm{desc}} \subset X(\mathbf{A}_k)^{\text{ét,desc}} \subset X(\mathbf{A}_k)^{\text{ét,Br}} \subset X(\mathbf{A}_k)^{\mathrm{desc}}.$$

The second inclusion is (14.5). In the case when X is projective, the first inclusion is a theorem of Skorobogatov [Sko09a, Thm. 1.1] built on results of Stoll [Sto07], and the third inclusion is a theorem of Demarche [Dem09] built on results of Harari [Har02]. The general case of quasi-projective varieties is due to Cao, Demarche, and Xu, see [CDX, Thm. 7.5]. Among the ingredients of their proof is the result that with suitable modifications and isotropy conditions at the archimedean places, the Brauer–Manin obstruction to strong approximation for homogeneous spaces of connected linear algebraic groups with connected stabilisers is the only obstruction (Borovoi and Demarche [BD13], after work of Colliot-Thélène and Xu [CTX09], Harari [Har08], and Demarche [Dem11]).

This theorem is complemented by the following result of Y. Cao [Cao20], which answers a question that was asked by Poonen in the case when X is projective.

Theorem 14.3.4 (Y. Cao) *Let X be a smooth, quasi-projective, geometrically integral variety over a number field k. Then*

$$X(\mathbf{A}_k)^{\text{desc},\text{desc}} = X(\mathbf{A}_k)^{\text{desc}},$$

hence

$$X(\mathbf{A}_k)^{\text{desc},\text{desc}} = X(\mathbf{A}_k)^{\text{desc}} = X(\mathbf{A}_k)^{\text{ét},\text{desc}} = X(\mathbf{A}_k)^{\text{ét},\text{Br}}.$$

As an immediate corollary, we obtain $X(\mathbf{A}_k)^{\text{desc},\ldots,\text{desc}} = X(\mathbf{A}_k)^{\text{desc}}$.

Y. Harpaz and T. Schlank [HSc13] used étale homotopy theory of Artin and Mazur to produce a subset $X(\mathbf{A}_k)^h \subset X(\mathbf{A}_k)$ which contains $X(k)$. For any smooth and geometrically integral variety X (not necessarily proper) they prove that $X(\mathbf{A}_k)^h = X(\mathbf{A}_k)^{\text{ét},\text{Br}}$. See also [Stix].

The above results raise the question: is the étale Brauer–Manin obstruction the only obstruction to the existence of rational points, that is, if the set $X(k)$ is empty, then is the set $X(\mathbf{A}_k)^{\text{ét},\text{Br}}$ empty too? This seems to be unlikely already in the case of smooth hypersurfaces of dimension at least 3 and of arbitrary degree in projective space, which have trivial geometric fundamental group [SGA2, X, Thm. 3.10] and trivial Brauer group $\text{Br}(X) = \text{Br}_0(X)$ (Corollary 5.5.4). Hence the question in that case reduces to the very unlikely Hasse principle for such hypersurfaces.

Unconditional examples of smooth, projective, geometrically integral varieties X over a number field k with $X(\mathbf{A}_k)^{\text{ét},\text{Br}} \neq \emptyset$ but $X(k) = \emptyset$ have been found. Poonen [Po10] uses a threefold with a dominant morphism to a curve with finitely many rational points such that the generic fibre is a Châtelet surface. Over $\bar{k}$ such a variety becomes birationally equivalent to the product of a curve (of genus at least one) and a projective space. Harpaz and Skorobogatov [HS14] construct surfaces with a dominant morphism to a curve (of genus at least one) with finitely many rational points such that the fibres over some rational points are singular unions of curves of genus 0.

14.3.4 Quadric bundles over a curve, II

In this section we discuss simpler examples of varieties over a number field k such that $X(\mathbf{A}_k)^{\text{ét},\text{Br}} \neq \emptyset$ but $X(k) = \emptyset$ that were constructed in [CTPS16].

To control the étale Brauer–Manin obstruction, one uses the following proposition.

Proposition 14.3.5 *Let $f\colon X\to B$ be a surjective flat morphism of smooth, proper, geometrically integral varieties over a field k of characteristic zero. If the geometric generic fibre is simply connected and all geometric fibres are reduced, then for any torsor $X'\to X$ for a finite k-group scheme G there exists a G-torsor $B'\to B$ such that there is an isomorphism $X'\cong X\times_B B'$ of G-torsors over X.*

Proof. By [SGA1, X, Cor. 2.4] the hypotheses imply that each geometric fibre is simply connected. The result then follows from [SGA1, IX, Cor. 6.8]. □

It is easy to construct examples over $k=\mathbb{Q}$ that are varieties fibred into quadrics of dimension $d\geq 2$ over a curve: in fact, the threefold from Section 14.3.4 is such an example (also reproduced in [Po18, §8.6.2]).

Proposition 14.3.6 *The threefold X over $k=\mathbb{Q}$ considered in Proposition 14.3.1 satisfies $X(\mathbf{A}_k)^{\text{ét,Br}}\neq\emptyset$ and $X(k)=\emptyset$.*

Proof. We keep the notation of the proof of Proposition 14.3.1. There we constructed an adelic point $(N_u)\in X(\mathbf{A}_k)^{\rm Br}$. Let us show that in fact

$$(N_u)\in X(\mathbf{A}_k)^{\text{ét,Br}}.$$

Let G be a finite k-group scheme. Proposition 14.3.5 implies that any G-torsor X'/X is isomorphic to $X\times_C C'\to X$ for some G-torsor C'/C. Let $\sigma\in Z^1(k,G)$ be a 1-cocycle defining the k-torsor which is the fibre of $C'\to C$ at P. Twisting X'/X and C'/C by σ and replacing the group G by the twisted group G^σ and changing notation, we can assume that C' contains a k-point P' that maps to P in C. The irreducible component C'' of C' that contains P' is a geometrically integral curve over k. Let $X''\subset X'$ denote the inverse image of C'' in X'. The fibres of the morphism $X\to C$ are geometrically integral, hence such are also the fibres of $X'\to C'$ and $X''\to C''$. Thus X'' is a geometrically integral variety over k.

There are natural isomorphisms $X''_{P'}\cong X'_{P'}\cong X_P$, so we can define $N'_u\in X''(k_u)$ as the point that maps to $N_u\in X(k_u)$ for each $u\neq v$. The map $C''\to C$ is finite and étale. The image of $C''(\mathbb{R})$ in $C(\mathbb{R})$ is thus closed and open. The image of the connected component of $P'\in C''(\mathbb{R})$ is the whole connected component of $P\in C(\mathbb{R})$, hence contains Π. The inverse image in $C''(\mathbb{R})$ of the interval Π is a disjoint union of intervals, one of which contains P' and maps bijectively onto Π. Let us call this interval Π'. Let M' be the unique point of Π' over M. Let $N'_v\in X''_{M'}(\mathbb{R})$ be the point that maps to $N_v\in X_M(\mathbb{R})$. Thus the adelic point $(N'_u)\in X''(\mathbf{A}_k)\subset X'(\mathbf{A}_k)$ projects to the adelic point $(N_u)\in X(\mathbf{A}_k)$. By the definition of the étale Brauer–Manin obstruction, to prove that $(N_u)\in X(\mathbf{A}_k)^{\text{ét,Br}}$ it suffices to show that $(N'_u)\in X'(\mathbf{A}_k)^{\rm Br}$. This follows by the argument in the last paragraph of Proposition 14.3.1. □

It is more delicate to give examples with $d=1$, that is, conic bundles.

Theorem 14.3.7 *There exist a real quadratic field k, an elliptic curve E and a smooth, projective and geometrically integral surface X over k with a surjective morphism $f\colon X\to E$ satisfying the following properties:*

(i) *the fibres of $f\colon X\to E$ are conics;*
(ii) *there exists a closed point $P\in E$ such that the field $k(P)$ is a totally real biquadratic extension of $\mathbb{Q}$ and the restriction $X\setminus f^{-1}(P)\to E\setminus P$ is a smooth morphism;*
(iii) *$X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}\neq\emptyset$ and $X(k)=\emptyset$.*

Here one can take $k=\mathbb{Q}(\sqrt{10})$ and take E to be the elliptic curve

$$y^2+y=x^3+x^2-12x-21$$

of conductor 67 and discriminant -67. We refer to [CTPS16, §5] for the construction of the conic bundle $f\colon X\to E$ and the proof of Theorem 14.3.7. Note that in this theorem we cannot take the ground field to be $\mathbb{Q}$, see Proposition 14.3.11. The construction given in [CTPS16] works over a number field with at least two real places, and requires good control of the Galois representation on the torsion points of E over k.

In all these unconditional examples with $X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}\neq\emptyset$ and $X(k)=\emptyset$, the varieties X have a non-constant map to a curve of genus at least one, hence have a non-trivial Albanese variety. A. Smeets [Sme17] has given examples with trivial Albanese varieties. Under the *abc* conjecture, he even produces examples with trivial geometric fundamental group. In the function field case, where k is a function field in one variable over a finite field, unconditional examples with trivial geometric fundamental group have been found by Kebekus, Pereira and Smeets [KPS19].

Definition 14.3.8 *Let X be a variety over a number field k. Define the topological space $X(\mathbf{A}_k)_\bullet$ by replacing $X(k_v)$ in the definition of $X(\mathbf{A}_k)$ by the set of connected components $\pi_0(X(k_v))$, for each archimedean place v.*

This definition is due to Poonen and Stoll, see [Sto07]. The evaluation map defined by a class in the Brauer group of a variety X over $\mathbb{R}$ is constant on each connected component of $X(\mathbb{R})$, see Remark 10.5.7 (4). Thus $X(\mathbf{A}_k)^{\mathrm{Br}}$ is the inverse image of a well-defined subset $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}\subset X(\mathbf{A}_k)_\bullet$ under the natural map $X(\mathbf{A}_k)\to X(\mathbf{A}_k)_\bullet$.

Here are some complements to Proposition 14.3.6 and Theorem 14.3.7.

Proposition 14.3.9 *Let E be an elliptic curve over a number field k such that the Tate–Shafarevich group $\text{Ш}(E)$ is finite. Let $f\colon X\to E$ be a Severi–Brauer scheme over E. Then $X(\mathbf{A}_k)^{\mathrm{Br}}\neq\emptyset$ implies $X(k)\neq\emptyset$. Moreover, $X(k)$ is dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$.*

Proof. Since $f\colon X\to E$ is a projective morphism of projective varieties with smooth geometrically integral fibres, by a spreading argument, the Lang–Weil–Nisnevich estimates and Hensel's lemma, one sees that there exists a finite set of places Σ such that $E(k_v) = f(X(k_v))$ for $v \notin \Sigma$. We may assume that Σ contains the archimedean places of k. At an arbitrary place v the set $f(X(k_v))$ is open and closed in $E(k_v)$. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. By functoriality (Proposition 13.3.10) we then have $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. The finiteness of $Ш(E)$ implies [Sko01, Prop. 6.2.4] the exactness of the Cassels–Tate dual sequence

$$0 \longrightarrow E(k) \otimes \widehat{\mathbb{Z}} \longrightarrow \prod_v E(k_v)_\bullet \longrightarrow \mathrm{Hom}(\mathrm{Br}(E), \mathbb{Q}/\mathbb{Z}), \tag{14.6}$$

where $E(k_v)_\bullet = E(k_v)$ if v is a finite place of k, and $E(k_v)_\bullet = \pi_0(E(k_v))$ if v is an archimedean place. By a theorem of Serre [Ser64, II, Thm. 3], the image of $E(k) \otimes \widehat{\mathbb{Z}}$ in that product coincides with the topological closure of $E(k)$. Approximating at the places of Σ, we find a k-point $M \in E(k)$ such that the fibre $X_M = f^{-1}(M)$ is a Severi–Brauer variety with points in all k_v for $v \in \Sigma$, hence also for all places v. Since X_M is a Severi–Brauer variety over k, it contains a k-point (see Section 13.2), hence $X(k) \neq \emptyset$. For the last statement of the theorem we include into Σ the places where we want to approximate. If $k_v \simeq \mathbb{R}$, each connected component $X(k_v)$ maps surjectively onto a connected component of $E(k_v)$. By Proposition 7.1.5 Severi–Brauer varieties with a k-point are isomorphic to the projective space, hence satisfy weak approximation. It remains to apply the implicit function theorem. □

Remark 14.3.10 The same argument works more generally for any projective morphism $f\colon X \to E$ with split fibres, provided that the smooth k-fibres satisfy the Hasse principle. For the last statement to hold, the smooth k-fibres also need to satisfy weak approximation.

The following proposition explains why a counter-example similar to that of Theorem 14.3.7 cannot be constructed over $\mathbb{Q}$.

Proposition 14.3.11 *Let E be an elliptic curve over a number field k such that both $E(k)$ and $Ш(E)$ are finite. Let $f\colon X \to E$ be a conic bundle. Suppose that there exists a real place v_0 of k such that for all real places $v \neq v_0$ all k_v-fibres of $f\colon X \to E$ are smooth. Then $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$.*

Proof. If a k-fibre of f is not smooth, then this fibre contains a k-point. Thus we can assume that the fibres above $E(k)$ are smooth. Let $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$. Then $(f(M_v)) \in E(\mathbf{A}_k)^{\mathrm{Br}}$. Set $N_v = f(M_v)$ for each place v. The finiteness of $Ш(E)$ implies the exactness of (14.6). Hence there exists an $N \in E(k)$ such that $N = N_v$ for each finite place v and such that N lies in the same connected component as N_v for v archimedean. The fibre X_N is a smooth conic with points in all finite completions of k. For an archimedean place $v \neq v_0$, the map $X(k_v) \to E(k_v)$ sends each connected component of $X(k_v)$ onto a connected component of $E(k_v)$. Since N and N_v are in the same connected component of $E(k_v)$, this implies that $X_N(k_v) \neq \emptyset$. Thus the conic X_N has points in all completions of k except possibly k_{v_0}. By the reciprocity law (Corollary 13.1.10) it has points in all completions of k and hence in k. □

14.3.5 Curves, K3 surfaces, Enriques surfaces

One may wonder whether there are classes beyond rationally connected varieties for which one could hope that the Brauer–Manin obstruction or one of its substitutes discussed in this chapter controls the existence of rational points.

Let X be a smooth, projective, geometrically integral variety over a number field k. Here are some open problems concerning the closure $X(k)^{cl}$ of $X(k)$ in the topological space $X(\mathbf{A}_k)_\bullet$ introduced in Definition 14.3.8.

Curves

In [Sko01, p. 133] Skorobogatov asked if $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$ for every smooth and projective curve X over a number field k. Computational evidence (the Mordell–Weil sieve [Fly04, Po06, BSt08, BSt10]) seems to suggest that the answer to this question could be in the affirmative. When X is a curve, it is an open question whether the image of $X(k)$ is dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$. If the genus of X is 1, this is the case if the Tate–Shafarevich group of the Jacobian of X is finite [Sko01, Thm. 6.2.3, Cor. 6.2.4]. More generally, if X is a torsor for an abelian variety A with finite Tate–Shafarevich group, then $X(k)$ is dense in $X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$. If X is a curve of higher genus with Jacobian J such that $\text{Ш}(J)$ is finite (which is expected to be always true) and also $J(k)$ is finite, it is a theorem of Scharaschkin and (independently) Skorobogatov [Sko01, Cor. 6.2.6] that $X(k) = X(\mathbf{A}_k)_\bullet^{\mathrm{Br}}$. Stoll has shown that the same statement remains true under the weaker assumption that J is isogenous to an abelian variety which has a direct factor A of positive dimension such that $\text{Ш}(A)$ and $A(k)$ are both finite [Sto07].

The question whether $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$ has interesting connections with Grothendieck's section conjecture [Stix, Ch. 11, Thm. 161].

Although the case of number fields remains wide open, more is known in the function field case. A theorem of Poonen and Voloch [PV10] states that under certain conditions on a smooth projective curve X over a global field k of finite characteristic, one has $X(k) = X(\mathbf{A}_k)^{\mathrm{Br}}$.

K3 surfaces

That some form of the Hasse principle could hold for classes of varieties beyond curves and rationally connected varieties seems to have been mentioned for the first time in the introduction to [CTSS98b]. This paper is concerned with certain families of surfaces X with a pencil of curves of genus 1, among which one finds some classes of diagonal quartics in $\mathbb{P}^3_k$. The result is that conditionally on two hard conjectures (Schinzel's hypothesis and finiteness of Tate–Shafarevich groups of elliptic curves), under suitable hypotheses,

$X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ implies $X(k) \neq \emptyset$. (To quote from [CTSS98b]: "Until we embarked on the current research, we would not have supposed that this was a sensible question to ask".) Further work along these lines was done by Swinnerton-Dyer [SwD00]. The treatment of geometrically Kummer surfaces in [SkS05, HS16, Harp19b] does not rely on Schinzel's hypothesis. More precisely, just like Swinnerton-Dyer's paper [SwD01], it uses the only known case of Schinzel's hypothesis: the Dirichlet theorem on primes in an arithmetic progression.

The following conjecture was proposed in [Sko09b], see also the introduction to [SZ08].

Conjecture 14.3.12 (Skorobogatov) *Let X be a K3 surface over a number field k. Then $X(k)$ is dense in $X(\mathbf{A}_k)^{\mathrm{Br}}$.*

There are conditional results in this direction, particularly, but not exclusively, for surfaces which are geometrically Kummer, see [CTSS98b, SkS05, HS16, Harp19b]. See also [EJ] for some numerical experiments with rational points on Kummer surfaces.

Enriques surfaces

Enriques surfaces with interesting $X(\mathbf{A}_k)^{\text{ét,Br}}$ were studied by Harari and Skorobogatov in [HS05], where the following example was constructed. Let Y be the Kummer surface over $\mathbb{Q}$ with affine equation

$$z^2 = (x^2 - a)(x^2 - ab^2)(y^2 - a)(y^2 - ac^2),$$

where $a = 5$, $b = 13$, $c = 2$. Let X be the quotient of Y by the involution that changes the signs of all three coordinates. Then X is an Enriques surface such that $X(\mathbb{Q})$ is not dense in $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}}$. Following work of Várilly-Alvarado and Viray [VV11], an Enriques surface X such that $X(\mathbf{A}_k)^{\text{ét,Br}} = \emptyset$, hence $X(k) = \emptyset$, whereas $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$, was constructed in [BBMPV].

One may ask if $X(k)^{cl} = X(\mathbf{A}_k)^{\text{ét,Br}}$ for any Enriques surface X. See [Sko09a, §3] for a discussion of this question for arbitrary surfaces of Kodaira dimension zero.

Chapter 15
The Brauer–Manin obstruction for zero-cycles

The Brauer–Manin obstruction for rational points has an analogue for zero-cycles, which conjecturally governs the local-to-global principle for zero-cycles on an arbitrary smooth projective variety X – unlike the original version for rational points! For example, one expects that if X has a family of local zero-cycles of degree 1 for each completion of k, which is orthogonal to $\mathrm{Br}(X)$ with respect to the Brauer–Manin pairing, then X has a global zero-cycle of degree 1. The precise conjectures are stated in Section 15.1.

If one knows that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points (as is conjectured for rationally connected varieties), then in some cases one can conclude that the Brauer–Manin obstruction controls the existence of zero-cycles of degree 1 as well. This work of Y. Liang is presented in Section 15.2.

In 1988 P. Salberger proved the conjectures of Section 15.1 for arbitrary conic bundles over the projective line. His method may be viewed as an accessible analogue of Schinzel's Hypothesis (H). We describe the method in a simple case in Section 15.3.

Finally, in Section 15.4 we explain the general fibration theorem of Harpaz and Wittenberg, which roughly says that the Brauer–Manin obstruction controls the local-to-global properties of zero-cycles of degree 1 on a variety fibred over $\mathbb{P}^1_k$ with rationally connected generic fibre, if the same property holds for the smooth fibres.

15.1 Local-to-global principles for zero-cycles

The Brauer–Manin pairing for zero-cycles

Let k be a number field. As in the previous chapter, we denote by Ω the set of places of k. For a place $v \in \Omega$ we always identify $\mathrm{Br}(k_v)$ with a subgroup of $\mathbb{Q}/\mathbb{Z}$ via the local invariant inv_v, see Definition 13.1.7.

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Let X be a smooth, projective, geometrically integral variety over k. For each place $v \in \Omega$ we have the pairing from Section 6.4:

$$\mathrm{CH}_0(X_{k_v}) \times \mathrm{Br}(X_{k_v}) \longrightarrow \mathrm{Br}(k_v) \subset \mathbb{Q}/\mathbb{Z}. \tag{15.1}$$

If v is an archimedean place, this pairing vanishes on $\mathrm{N}_v(\mathrm{CH}_0(X_{k'_v}))$, where k'_v is an algebraic closure of k_v and N_v is the natural norm map

$$\mathrm{CH}_0(X_{k'_v}) \to \mathrm{CH}_0(X_{k_v}).$$

Define

$$\mathrm{CH}'_0(X_{k_v}) = \mathrm{CH}_0(X_{k_v})/\mathrm{N}_v(\mathrm{CH}_0(X_{k'_v}))$$

if v is archimedean, and $\mathrm{CH}'_0(X_{k_v}) = \mathrm{CH}_0(X_{k_v})$ otherwise.

For any $\alpha \in \mathrm{Br}(X)$ there exists a finite set $S \subset \Omega$ and a smooth projective model $\mathcal{X}$ over the ring of S-integers $\mathcal{O}_S \subset k$ such that α belongs to the image of $\mathrm{Br}(\mathcal{X}) \to \mathrm{Br}(X)$, see Proposition 13.3.1. If $v \notin S$ is a non-archimedean place, then by Proposition 10.5.3 we have $\alpha(M_v) = 0$ for any closed point M_v of X_{k_v}. Thus each $\alpha \in \mathrm{Br}(X)$ pairs trivially with the local Chow groups $\mathrm{CH}_0(X_{k_v})$ for almost all places $v \in \Omega$, as well as with $\mathrm{N}_v(\mathrm{CH}_0(X_{k'_v}))$ if v is archimedean. Therefore we have a well-defined map

$$\prod_{v \in \Omega} \mathrm{CH}'_0(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Class field theory gives an exact sequence (13.1)

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow \bigoplus_{v \in \Omega} \mathrm{Br}(k_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The mere fact that this is a complex at the middle term implies that the following sequence is a complex too:

$$\mathrm{CH}_0(X) \longrightarrow \prod_{v \in \Omega} \mathrm{CH}'_0(X_{k_v}) \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Here the first map is the diagonal map and the second map is induced by the local pairings (15.1).

For an abelian group A, write $\widehat{A} = \varprojlim A/n$, where the projective limit is over the set of natural numbers ordered by divisibility. Since $\mathrm{Br}(X_{k_v})$ is a torsion group, the local pairing (15.1) gives rise to a pairing

$$\widehat{\mathrm{CH}_0(X_{k_v})} \times \mathrm{Br}(X_{k_v}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

From this we obtain a complex

$$\widehat{\mathrm{CH}_0(X)} \longrightarrow \prod_{v \in \Omega} \widehat{\mathrm{CH}'_0(X_{k_v})} \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{15.2}$$

Recall from Section 6.4 that $A_0(X)$ is the degree 0 subgroup of $\mathrm{CH}_0(X)$. By [CT95b, Thm. 1.3 (b)], if v is an archimedean place, then $\mathrm{N}_v(A_0(X_{k'_v}))$ is the divisible subgroup of $A_0(X_{k_v})$. Thus restricting (15.2) to the degree 0 subgroups we obtain a complex

$$\widehat{A_0(X)} \longrightarrow \prod_{v\in\Omega} \widehat{A_0(X_{k_v})} \longrightarrow \mathrm{Hom}(\mathrm{Br}(X), \mathbb{Q}/\mathbb{Z}). \tag{15.3}$$

Conjectures

Work of Cassels and Tate on elliptic curves (the Cassels–Tate dual exact sequence), of Colliot-Thélène and Sansuc on geometrically rational surfaces [CTS81], and of Kato and Saito [KS86] on higher class field theory has led to the following general conjecture, which encompasses a number of earlier conjectures.

Conjecture 15.1.1 (E) *For any smooth, projective, geometrically integral variety X over a number field k, the complex* (15.2) *is exact.*

Conjecture (E) subsumes the following two conjectures.

Conjecture 15.1.2 ($\mathbf{E}_0$) *For any smooth, projective, geometrically integral variety X over a number field k, the complex* (15.3) *is exact.*

Conjecture 15.1.3 ($\mathbf{E}_1$) *For any smooth, projective, geometrically integral variety X over a number field k, if there exists a family (z_v) of local zero-cycles of degree 1 on X such that, for all $A \in \mathrm{Br}(X)$, we have*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z},$$

then there exists a zero-cycle of degree 1 on X.

For the history of these conjectures, see [CTS81], [KS86, p. 303], [Sai89, §8], [CT95b], [CT99], [vHa03], and the introduction to [Witt12]. Note that these conjectures are about *all* smooth, projective, geometrically integral varieties over number fields. The groups involved are rather mysterious. Indeed, it is a conjecture of Bloch and Beilinson that for a smooth and projective variety X over a number field k, the group $\mathrm{CH}_0(X)$ is finitely generated. Important work on the groups $\mathrm{CH}_0(X_{k_v})$ was done by Kollár [Kol04] and by S. Saito and K. Sato [SS10, SS14].

In the case $X = \mathrm{Spec}(k)$, conjecture (E) follows from the exact sequence (13.1) of class field theory.

For curves, classical results of Cassels and Tate imply the conjecture – modulo finiteness of Tate–Shafarevich groups. See [Man66] (for curves of genus 1), [Sai89], [CT99], [Witt12, Remark 1.1 (iv), p. 2121].

For Châtelet surfaces, i.e. smooth projective models X of surfaces given by an affine equation $y^2 - az^2 = P(x)$, where $a \in k^*$ and $P(x) \in k[x]$ is a separable polynomial of degree 3 or 4, conjectures (E_0) and (E_1) were proved in [CTSS87] by reduction to the theorem $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$, also proved there. Indeed, these surfaces have the very special property that any zero-cycle of degree 1 is rationally equivalent to a k-point. Then Salberger [Salb88], by an innovative method, to be discussed in Section 15.3, proved the conjectures for arbitrary conic bundles over $\mathbb{P}^1_k$. For varieties fibred over $\mathbb{P}^1_k$, with generic fibre a Severi–Brauer variety, further progress was achieved in papers by Colliot-Thélène, Swinnerton-Dyer, Skorobogatov [CTS94, CTSS98], and Salberger [Salb03].

A series of papers by Colliot-Thélène [CT00, CT10], Frossard [Fro03], van Hamel [vHa03], Wittenberg [Witt12] and Y. Liang [Lia12, Lia13a, Lia13b, Lia14, Lia15] established cases of conjecture (E) for varieties fibred over an arbitrary curve C, when the Tate–Shafarevich group of the Jacobian of C is finite, and the generic fibre is birationally equivalent to a Severi–Brauer variety or to a homogeneous space of a connected linear algebraic group, with various restrictions. Most of these results are now covered by the work of Harpaz and Wittenberg [HW16].

The smooth projective surface X over $\mathbb{Q}$ with the property $X(\mathbf{A}_{\mathbb{Q}})^{\mathrm{Br}} \neq \emptyset$ but $X(\mathbb{Q}) = \emptyset$ discovered by Skorobogatov in [Sko99] does not belong to the class of varieties handled in [HW16] (it is not geometrically uniruled). B. Creutz has recently shown that it contains a zero-cycle of degree one [Cre17], as predicted by the conjecture.

15.2 From rational points to zero-cycles

Let k be a number field. There are classes of smooth projective varieties X over k for which one can prove that if $X(K)^{cl} = X(\mathbb{A}_K)^{\mathrm{Br}}$ holds for all finite field extensions K/k then the conjectures for zero-cycles on X mentioned above are true. This insight is due to Y. Liang [Lia13b].

Here is a baby case.

Proposition 15.2.1 [Lia13b, Prop. 3.2.3] *Let k be a number field and let X be a smooth, proper, geometrically integral variety over k. Assume that for any finite field extension K/k, the Hasse principle holds for rational points of X_K. Then the Hasse principle holds for zero-cycles of degree 1 on X.*

Proof. By the Lang–Weil–Nisnevich estimates and Hensel's lemma, there exists a finite set S of places of k containing all the archimedean places, such that for any place $v \notin S$, one has $X(k_v) \neq \emptyset$. Fix a closed point m of some degree N on X. For each $v \in S$, let $z_v = z_v^+ - z_v^-$ be a local zero-cycle of degree 1 where z_v^+ and z_v^- are effective zero-cycles. Let $z_v^1 = z_v^+ + (N-1)z_v^-$. This is an effective zero-cycle on X_{k_v} of degree congruent to 1 modulo N.

Since S is finite, we can add to each z_v^1 a suitable positive multiple $n_v m$ of the closed point m and ensure that all the effective local cycles $z_v^2 = z_v^1 + n_v m$, for $v \in S$, have the same common degree d congruent to 1 modulo N.

Here comes the basic argument. Let $Y = X \times_k \mathbb{P}^1_k$ and let $f\colon Y \to \mathbb{P}^1_k$ be the natural projection. Fix a rational point $q \in \mathbb{P}^1(k)$. On Y_{k_v} we have the effective zero-cycle $z_v^2 \times q$ of degree d, for $v \in S$.

A moving lemma [CTS94, Lemma 6.2.1] based on the implicit function theorem (Theorem 10.5.1) ensures that there exists an effective zero-cycle z_v^3 on Y very close to $z_v^2 \times q$ and such that z_v^3 and $f_*(z_v^3)$ are "reduced". This means that $z_v^3 = \sum_j R_j$ with distinct closed points R_j on Y_{k_v}, mapping to distinct closed points $f(R_j)$ and such that $f\colon R_j \to f(R_j)$ is an isomorphism for each j. We may assume that for each $v \in S$, the support of the zero-cycle $f_*(z_v^3)$ lies in $\mathrm{Spec}(k[t]) = \mathbb{A}^1_k \subset \mathbb{P}^1_k$. Each $f_*(z_v^3)$ is defined by a separable monic polynomial $P_v(t) \in k_v[t]$ of degree d. We choose a finite place v_0 outside of S and a monic *irreducible* polynomial $P_{v_0}(t) \in k_{v_0}[t]$ of degree d. By weak approximation on the coefficients, we then approximate the $P_v(t)$, for $v \in S \cup \{v_0\}$, by a monic polynomial $P(t) \in k[t]$ of degree d. The polynomial $P(t)$ is irreducible, hence it defines a closed point $M \in \mathbb{A}^1_k$ of degree d.

If the approximation is close enough, Krasner's lemma [Po18, Prop. 3.5.74] and the implicit function theorem (Theorem 10.5.1, [Po18, Prop. 3.5.73]) imply that the fibre $f^{-1}(M) = X \times_k k(M)$ has points in all completions of $k(M)$ at the places above $v \in S$. By the choice of S, it also has points in all the other completions. By assumption, $X \times_k k(M)$ satisfies the Hasse principle over $k(M)$, hence it has a $k(M)$-point. Thus X has a point in an extension of degree d. As d is congruent to 1 mod N, and the closed point m has degree N, we conclude that the k-variety X has a zero-cycle of degree 1. □

Theorem 15.2.2 (Y. Liang) [Lia13b, Thm. 3.2.1] *Let k be a number field and let X be a smooth, projective, geometrically integral variety over k. Assume that $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$ and that the Néron–Severi group $\mathrm{NS}(\overline{X})$ is torsion-free. For any finite field extension K of k, assume that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for rational points of X_K. Then the Brauer–Manin obstruction to the existence of a zero-cycle of degree 1 on X is the only obstruction: conjecture (E_1) holds for X.*

Proof. Over any field k of characteristic zero, the assumptions on the geometry of X imply that $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite (Theorem 5.5.2). Let $A_1, \ldots, A_n$ be elements of $\mathrm{Br}(X)$ whose images generate $\mathrm{Br}(X)/\mathrm{Br}_0(X)$.

Let S be a finite set of places containing the archimedean places such that X has a smooth projective model $\mathcal{X}$ over the ring $\mathcal{O}_S$ of S-integers, the elements A_i extend to elements of $\mathrm{Br}(\mathcal{X})$, and $X(k_v) \neq \emptyset$ for $v \notin S$. By Proposition 10.5.3, each A_i vanishes when evaluated on any zero-cycle of X_{k_v}.

Suppose that we have a family of zero-cycles z_v of degree 1 on X_{k_v}, for $v \in \Omega$, which is orthogonal to $\mathrm{Br}(X)$ with respect to the Brauer–Manin pairing. This is equivalent to the condition

$$\sum_{v\in\Omega} \mathrm{inv}_v(A_i(z_v)) = 0 \in \mathbb{Q}/\mathbb{Z}, \quad \text{for all} \quad i = 1, \ldots, n.$$

Let N be an integer which is a multiple of the degree of a closed point of X and also annihilates each $A_i \in \mathrm{Br}(X)$.

Let $Y = X \times_k \mathbb{P}^1_k$ and let $f\colon Y \to \mathbb{P}^1_k$ be the projection. Proceeding as in the previous proof, we replace the original zero-cycles z_v, $v \in S$, by reduced effective zero-cycles z'_v on Y each of the same degree d congruent to 1 modulo N, with the property that $f_*(z'_v)$ is reduced. We may choose coordinates so that the support of $f_*(z'_v)$ lies in $\mathrm{Spec}(k[t]) = \mathbb{A}^1_k \subset \mathbb{P}^1_k$. The cycles $f_*(z'_v)$ are then defined by the vanishing of separable, monic polynomials $P_v(t)$ of degree d. One then approximates the $P_v(t)$ for $v \in S$ and an irreducible monic polynomial $P_{v_0} \in k_{v_0}[t]$ of degree d at a place $v_0 \notin S$ by a monic polynomial $P(t) \in k[t]$ of degree d. Just as before, $P(t)$ defines a closed point $M \in \mathbb{P}^1_k$.

For each place $v \in S$, we have the effective zero-cycle z'_v close to z_v on $X_M \otimes_k k_v$. This gives rise to $k(M)_w$-rational points R_w of the $k(M)$-variety $X_{k(M)}$ over the completions of $k(M)$ at the places w above the places in S.

At each place w of $k(M)$ above a place $v \notin S$, we take an arbitrary $k(M)_w$-point, for instance, a point coming from a k_v-point on X. Then we have

$$\sum_{w\in\Omega_{k(M)}} \mathrm{inv}_w(A_i(R_w)) = 0 \in \mathbb{Q}/\mathbb{Z}, \quad \text{for all} \quad i = 1, \ldots, n.$$

This is enough to ensure that the adelic point $(R_w) \in X_{k(M)}(\mathbf{A}_{k(M)})$ is orthogonal to $\mathrm{Br}(X_{k(M)})$ provided we can choose the point M, i.e. the polynomial $P(t)$, in such a way that the map

$$\mathrm{Br}(X)/\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X_{k(M)})/\mathrm{Br}(k(M))$$

is surjective. By [Lia13b, Prop. 3.1.1] (an easy special case of a more general theorem of Harari [Har97, Thm. 2.3.1]), the geometric assumptions on X imply that there exists a finite Galois extension L of k such that the above surjectivity holds for any closed point M as long as $L \otimes_k k(M)$ is a field. But this last condition is easy to ensure. Indeed, it is enough to require from the very beginning that N is also a multiple of $[L : k]$. Then $d = [k(M) : k]$, being congruent to 1 modulo N, is prime to $[L : k]$. □

Liang [Lia13b, Thm. A and Thm. B] proves the following general result.

Theorem 15.2.3 *Let k be a number field. Let X be a smooth, projective, geometrically integral, rationally connected variety over k. Assume that for any finite field extension K of k, the set $X(K)$ is dense in $X(\mathbf{A}_K)^{\mathrm{Br}}$. Then conjecture* (E) *holds for X.*

To prove this, Liang first proves a version of Theorem 15.2.2 for zero-cycles of degree 1, keeping track of "approximation" modulo a positive integer. Here z is said to be close to z_v modulo n if z and z_v have the same image in $\mathrm{CH}_0(X_{k_v})/n$. This uses work of Wittenberg [Witt12]. Under the hypothesis that X is a rationally connected variety, one then proceeds from this statement to the exact sequence (E). This uses results of Kato–Saito [KS83], Kollár–Szabó [KS03] and Saito–Sato [SS10, SS14] on the Chow groups of zero-cycles of rationally connected varieties over finite fields and local fields.

This gives the following result [Lia13b, Cor. of Thm. B].

Corollary 15.2.4 *If a smooth projective variety X over a number field is birationally equivalent to a homogeneous space of a connected linear algebraic group with connected geometric stabilisers, then conjecture* (E) *holds for X.*

Indeed, the property $X(k)^{cl} = X(\mathbf{A}_k)^{\mathrm{Br}}$ is known for such varieties (Sansuc [San81] when the stabilisers are trivial, Borovoi [Bor96] in general).

Until [Lia13b] the validity of (E) was unknown even for smooth compactifications of 3-dimensional tori.

Further work along these lines was done by Ieronymou [Ier19] who extended Theorem 15.2.2 to K3 surfaces. His proof uses Theorems 16.7.2 and 16.7.5. See also the paper of Balestrieri and Newton [BN19] who extended Theorem 15.2.2 to generalised Kummer varieties and their products.

15.3 Salberger's method

In the paper [Salb88] Salberger devised a method which gives unconditional proofs of Conjecture (E) for large classes of varieties for which the rational point analogue is still conjectural. Salberger's method was streamlined in [CTS94, CTSS98], where it is interpreted as an appropriate substitute to Schinzel's Hypothesis (H).

In this section we display this argument in a simple case.

Theorem 15.3.1 (Salberger) *Let k be a number field, let $a, c \in k^*$, and let $P(t) \in k[t]$ be a monic irreducible polynomial of degree d. Assume that the equation*

$$y^2 - az^2 = c\,P(t) \neq 0 \tag{15.4}$$

is solvable in k_v for all places v of k. Then we have the following statements.

(i) *For any integer $N \geq d$ equation* (15.4) *has a solution in a field extension of k of degree N.*

(ii) *There exists a zero-cycle of degree* 1 *on the affine surface* (15.4).

Proof. Statement (ii) follows from (i) by considering N and $N+1$. Of course, (15.4) is solvable in $k(\sqrt{a})$, so the task is to prove that (15.4) has a solution in an odd degree extension of k.

Let us prove (i). Let U be the affine surface defined by (15.4). There is a finite set of places S containing all the archimedean and dyadic places of k, all the finite places v such that $v(a) \neq 0$, and such that at any place $w \notin S$ we have the following properties: $c \in \mathcal{O}_w^*$, the coefficients of $P(t)$ are in $\mathcal{O}_w$, the reduction of P modulo the maximal ideal of $\mathcal{O}_w$ is a separable polynomial.

For each $v \in S$, choose a monic separable polynomial $G_v(t) \in k_v[t]$ of degree N with all its roots in k_v, each of them corresponding to the image of a point of $U(k_v)$. In particular, $G_v(t)$ is coprime to $P(t)$.

Choose a place $v_0 \notin S$ such that a is a square in k_{v_0}. Then choose a monic irreducible polynomial $G_{v_0}(t) \in k_{v_0}[t]$ of degree N over k_{v_0}.

For each $v \in S \cup \{v_0\}$ Euclid's algorithm gives polynomials $Q_v(t)$ and $R_v(t)$ in $k_v[t]$, with $\deg(R_v(t)) < d \leq N$ such that

$$G_v(t) = P(t)Q_v(t) + R_v(t).$$

Hence each $Q_v(t)$ is monic and $\deg(Q_v(t)) = N - d$. Each $R_v(t)$ is coprime to $P(t)$ since $G_v(t)$ is coprime to $P(t)$.

Let K be the field $k[t]/P(t)$. Let $\xi_v \in (K \otimes_k k_v)^*$ be the image of $R_v(t)$.

Let V be the set of places of k outside of S for which a is a square in k_v. By Chebotarev's theorem 13.1.3, this set V is infinite.

Theorem 13.1.2 (a modified version of Dirichlet's theorem) for the field K implies that there is an element $\xi \in K^*$ which is arbitrarily close to each ξ_v for $v \in S \cup \{v_0\}$ (thus also at the archimedean places) and such that its prime decomposition in K involves only primes split in $k(\sqrt{a})/k$, primes above the primes in $S \cup \{v_0\}$, and a prime w such that $w(\xi) = 1$ and w has degree 1 over k. The element $\xi \in k[t]/P(t)$ lifts to a unique polynomial $R(t)$ of $k[t]$ such that $\deg(R(t)) < d$.

Choose a place $v_1 \notin S \cup \{v_0\}$ such that a is a square at v_1. If $N > d$ we use strong approximation (13.2.7) in k away from v_1 and the primes in V to produce a monic polynomial $Q(t) \in k[t]$ of degree $N - d$ whose coefficients are integral away from $V \cup \{v_1\}$ and very close to respective coefficients of $Q_v(t)$ for $v \in S \cup \{v_0\}$. If $N = d$, then take $Q(t) = 1$.

One then defines

$$G(t) := P(t)Q(t) + R(t).$$

By Krasner's lemma [Po18, Prop. 3.5.74] this polynomial is irreducible, since it is close to the irreducible polynomial $G_{v_0}(t)$. It is monic and has integral coefficients away from $V \cup S \cup \{v_0, v_1\}$.

Let $L = k[t]/G(t)$. This is a field extension of degree N of k. Let $\theta \in L$ be the class of t. The element θ is integral outside $V \cup S \cup \{v_0, v_1\}$. The theorem will follow from the claim:

Claim. *The conic over L with equation $y^2 - az^2 = c\,P(\theta)$ has an L-point.*

If w is a place of L above a place in S, then the conic has an L_w-point because $G(t)$ is very close to $G_v(t)$. If w is a place above v_0, v_1 or a place in V, then the conic has an L_w-point because a is a square in k_v.

The formula for the resultant of two polynomials shows that the product of the conjugates of $P(\theta)$ is an element of k^* which is equal, up to sign, to the product of the conjugates of $G(\alpha)$, where α is the class of t in $K = k[t]/P(t)$. Since $P(\alpha) = 0$, the definition of $G(t)$ implies that $G(\alpha) = R(\alpha) = \xi$. The degree 1 condition on the prime w implies that $\mathrm{N}_{K/k}(\xi) \in k^*$, away from $S \cup \{v_0\} \cup V$, has in its factorisation only one prime, and that its valuation at this prime is 1. Since $P(\theta) \in L$ is integral away from $S \cup \{v_0, v_1\} \cup V$, this implies that the prime decomposition of $P(\theta) \in L$ involves only one prime w' of L not dividing a prime of $S \cup \{v_0, v_1\} \cup V$. Thus the conic has points in all completions of L except possibly at the prime w'. Corollary 13.1.10 then implies that it has an L-point. □

Remark 15.3.2 In the above proof, we appealed to Theorem 13.1.2, which is a modified version of Dirichlet's theorem 13.1.1, because we needed to approximate at the real places of k. If $a \in k$ is totally positive, to get the above existence result, there is no need to introduce the infinite set V of places, Dirichlet's theorem is enough.

Remark 15.3.3 In the above theorem we assumed that $U(\mathbf{A}_k) \neq \emptyset$ but we did not assume the existence of an adelic point orthogonal to the unramified Brauer group of U. But this is automatic. Indeed, the hypothesis that $P(t)$ is irreducible implies that the Brauer group of a smooth projective model of $y^2 - az^2 = P(t)$ is reduced to the image of $\mathrm{Br}(k)$, see Exercise 11.3.7 (a).

Remark 15.3.4 Salberger's method may be interpreted as a successful substitute for Schinzel's hypothesis (H). Given an irreducible polynomial $P(t)$ over a number field k, it is hard to find an almost integral element $\alpha \in k$ such that $P(\alpha)$ is almost a prime. However, for any $N \geq \deg(P(t))$ one may produce a field extension L of k of degree N and an almost integral element $\beta \in L$ such that $P(\beta)$ is almost a prime in L. There is a similar comparison in the case of a finite set of polynomials (see [CT98, Prop. 17] for the example of twin primes). The above proof then becomes parallel to the proof of Theorem 14.2.2.

15.4 A fibration theorem for zero-cycles

The papers mentioned at the end of Section 15.1 were inspired by Salberger's paper [Salb88] and its reformulation in [CTS94, CTSS98]. In a manner parallel to the (conditional) case of rational points (Theorem 14.2.4), Salberger's method allows one to obtain unconditional results for zero-cycles on varieties X fibred over $\mathbb{P}^1_k$ when the following two assumptions are fulfilled:

- for any closed point $m \in \mathbb{P}^1_k$, the fibre X_m contains an irreducible component Y of multiplicity 1 such that the integral closure of $k(m)$ in $k(Y)$ is *abelian*;
- the Hasse principle and weak approximation hold for the smooth closed fibres of $X \to \mathbb{P}^1_k$.

If all fibres over closed points of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ are split, the second assumption can be weakened [CT10, Lia12] using arguments similar to those of Harari [Har94, Har07b] in the case of rational points.

These restrictions on the algebra and arithmetic of fibres have now been removed. In fact, we have the following unconditional results.

Theorem 15.4.1 (Harpaz and Wittenberg) [HW16, Thm. 8.3] *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f\colon X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is rationally connected. If the smooth fibres satisfy Conjecture* (E)*, then X satisfies Conjecture* (E)*.*

Combined with Corollary 15.2.4, this implies

Theorem 15.4.2 [HW16, Thm. 1.4] *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f\colon X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that the generic fibre is birationally equivalent to a homogeneous space of a connected linear algebraic group over $k(\mathbb{P}^1)$ with connected geometric stabilisers. Then Conjecture* (E) *holds for X.*

This last result can be applied to smooth projective models of varieties given by a system of equations

$$\mathrm{N}_{K_i/k}(\Xi_i) = P_i(t), \quad i = 1, \dots, n,$$

where K_i is a finite étale k-algebra (for example, a finite field extension) and $P_i(t) \in k[t]$, for each $i = 1, \dots, n$.

Harpaz and Wittenberg actually prove their result for varieties fibred over a smooth projective curve C of arbitrary genus, under the assumption that Conjecture (E) holds for C, for instance when the Tate–Shafarevich group of the Jacobian of C is finite.

We shall only describe one idea in the proof of Theorem 15.4.1. This is a zero-cycle analogue of Theorem 14.2.18.

Theorem 15.4.3 *Let X be a smooth, projective, geometrically integral variety over a number field k, and let $f\colon X \to \mathbb{P}^1_k$ be a dominant morphism. Assume that all non-split fibres of f are above k-points of $\mathbb{A}^1_k$, say given by $t = e_i \in k$, where $i = 1, \dots, n$ and t is a coordinate in $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$. Assume that each non-split fibre contains an irreducible component of multiplicity* 1*. If there exists an adelic point $(P_v) \in X(\mathbf{A}_k)$ which is orthogonal to $\mathrm{Br}_{\mathrm{vert}}(X/\mathbb{P}^1_k)$, then for any integer $N \geq n$ there exists a closed point $m \in \mathbb{P}^1_k$ of degree N such that the fibre X_m has points in all the completions of $k(m)$.*

Proof. Write $P(t) = \prod_{i=1}^{n}(t - e_i)$. Let $U \subset \mathbb{A}^1_k$ be the open set given by $P(t) \neq 0$, and let $V = f^{-1}(U)$. Fix an irreducible component $E_i \subset X_{e_i}$ of multiplicity 1, and let k_i be the integral closure of k in $k(E_i)$. The smooth locus $E_{i,\mathrm{smooth}}$ is a geometrically integral variety over k_i.

Since $X(\mathbf{A}_k) \neq \emptyset$, the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{vert}}(X) \subset \mathrm{Br}(X)$ is injective. By Corollary 11.1.6, the assumption on the multiplicity of E_i implies that $\mathrm{Br}_{\mathrm{vert}}(X)/\mathrm{Br}(k)$ is a finite group. Using this, by a small deformation argument we can assume that $(P_v) \in V(\mathbf{A}_k)$.

Let T be the product of norm 1 k-tori attached to the extensions k_i/k, for $i = 1, \ldots, n$. Consider the T-torsor over U given by the system of equations

$$t - e_i = \mathrm{N}_{k_i/k}(\Xi_i), \quad i = 1, \ldots, n.$$

Its pullback to V is a T-torsor over V. Since there is no vertical Brauer–Manin obstruction for rational points, by the formal lemma for torsors and rational points (Theorem 13.4.6) applied to this torsor over V, there exist $b_i \in k^*$, for $i = 1, \ldots, n$, and for each $v \in \Omega$, $\alpha_v \in f(V(k_v)) \subset U(k_v) \subset k_v$, such that the system of equations

$$\alpha_v - e_i = b_i \mathrm{N}_{k_i/k}(\Xi_i) \neq 0, \quad i = 1, \ldots, n,$$

has solutions with $\Xi_i \in (k_i \otimes_k k_v)^*$. Let $S \subset \Omega$ be a finite set of places containing the infinite places, the primes where at least one of the extensions k_i/k is ramified, the primes of bad reduction for X, the primes where at least one e_i is not integral, then the primes v dividing some $e_i - e_j$, where $i \neq j$, and the primes where b_i is not a unit. Then $f\colon X \to \mathbb{P}^1_k$ extends to a dominant morphism $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_S}$, where $\mathcal{X}$ is proper over $\mathcal{O}_S$. Using the Lang–Weil–Nisnevich estimates, we arrange that for any closed point $s \in \mathbb{P}^1_{\mathcal{O}_S}$ such that the fibre $\mathcal{X}_s$ is split, $\mathcal{X}_s$ has a smooth rational point over the residue field of s. This is achieved by including in S enough places with small residue characteristic.

By Chebotarev's theorem (Theorem 13.1.3), there are infinitely many primes outside of S that completely split in each of the extensions k_i/k. Let v_0 and v_1 be such primes.

By the implicit function theorem (Theorem 10.5.1, [Po18, Prop. 3.5.73]) for each $v \in S$ we can find pairwise distinct $\alpha_r^v \in f(V(k_v)) \subset U(k_v) \subset k_v$, for $r = 1, \ldots, N$, that are close to α_v. In particular, we can arrange that for any $v \in S$ and $i = 1, \ldots, n$ we have

$$b_i^{-1}(\alpha_r^v - e_i) \in \mathrm{N}_{k_i/k}((k_i \otimes_k k_v)^*) \subset k_v^*.$$

For each place $v \in S$, we define

$$G_v(t) = \prod_{r=1}^{N}(t - \alpha_r^v).$$

We note that $G_v(e_i)$ is the product of a global element $(-1)^N b_i^N \in k$ and an element of $\mathrm{N}_{k_i/k}((k_i \otimes_k k_v)^*) \subset k_v^*$. Let $G_{v_1}(t) \in \mathcal{O}_{v_1}[t]$ be a monic irreducible polynomial of degree N with integral coefficients. Dividing $G_v(t)$ by $P(t)$ in $k_v[t]$, for $v \in S \cup \{v_1\}$, and using Lagrange interpolation, we obtain

$$G_v(t) = P(t)Q_v(t) + \sum_{i=1}^{n} G_v(e_i) \frac{\prod_{j \neq i}(t - e_j)}{\prod_{j \neq i}(e_i - e_j)},$$

where the polynomials $Q_v(t)$ are monic of degree $N - n$.

Applying Proposition 13.1.4, for each $i = 1, \dots, n$ we find an element $c_i \in k$ close to $G_v(e_i)$ for $v \in S \cup \{v_1\}$ and such that for any $v \notin S \cup \{v_1\}$ either c_i is a unit at v, or k_i has a place of degree 1 over v. Moreover, we choose c_i integral away from $S \cup \{v_0, v_1\}$.

Using strong approximation in k away from v_0 for the coefficients of polynomials, we find $Q(t) \in k[t]$ with coefficients integral away from $S \cup \{v_0, v_1\}$ and close to each $Q_v(t)$ for $v \in S \cup \{v_1\}$ coefficient-wise. Consider the polynomial

$$G(t) = P(t)Q(t) + \sum_{i=1}^{n} c_i \frac{\prod_{j \neq i}(t - e_j)}{\prod_{j \neq i}(e_i - e_j)}.$$

By construction, $G(e_i) = c_i$ for $i = 1, \dots, n$. Also, the coefficients of $G(t)$ are integral away from $S \cup \{v_0, v_1\}$. Moreover, in the v-adic topology, where $v \in S \cup \{v_1\}$, the element c_i is close to $G_v(e_i)$ and $Q(t)$ is close to $Q_v(t)$, hence $G(t)$ is close to $G_v(t)$. Since $G_{v_1}(t)$ is irreducible in $k_{v_1}[t]$, we see that $G(t)$ is irreducible in $k[t]$.

Write $F = k[t]/(G(t))$, so that $m = \mathrm{Spec}(F)$ is the closed point of $U \subset \mathbb{A}^1_k$ defined by $G(t) = 0$. We claim that X_m has points in all completions of F.

If w is a place of F over $v \in S$, then $G(t)$ is v-adically close to $G_v(t) = \prod_{r=1}^{N}(t - \alpha_r^v)$. But each $\alpha_r^v \in k_v$ lifts to $V(k_v)$, proving the claim for such w.

The primes of F not above S are closed points of $\mathbb{P}^1_{\mathcal{O}_S}$. We only need to consider the finitely many closed points in $\mathbb{P}^1_{\mathcal{O}_S}$ where the closure of m in $\mathbb{P}^1_{\mathcal{O}_S}$ meets the closure of one of the e_i's. Indeed, S was chosen big enough so that the fibre of $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_S}$ above any other closed point is split, and then by the Lang–Weil–Nisnevich estimates has a smooth rational point over the residue field.

Let w be a closed point of $\mathbb{P}^1_{\mathcal{O}_S}$ contained in the closure of m and in the closure of e_i. The degree of w is 1 since the degree of e_i is 1. This closed point w lies above a prime $v \notin S$ dividing $c_i = G(e_i)$. Let us first consider the case $v \neq v_0, v_1$. By the construction of c_i, the field k_i has a place of degree 1 over v. This implies that the fibre $\mathcal{X}_w$ is split. By the choice of S, we have that $\mathcal{X}_w$ has a smooth rational point over the residue field. As $\mathcal{X}_w$ is the reduction of X_m at the place w of $F = k(m)$, we can apply Hensel's lemma to deduce that X_m has a point in the completion F_w.

It remains to deal with v_0 and v_1. Recall that these primes are split in all extensions k_i/k. This implies that all the fibres of $X \times_k k_{v_0} \to \mathbb{P}^1_{v_0}$ are split. The same argument works for v_1. □

Remark 15.4.4 The argument in Theorem 15.3.1 only shows the existence of a fibre X_M over a closed point M which has points in all completions of $k(M)$ except, possibly, one. In the above theorem, whose proof uses Proposition 13.1.4, we get a fibre with points in *all completions* of $k(M)$.

Using their work [HW16], Harpaz and Wittenberg have recently removed the assumption of connectedness of stabilisers in Corollary 15.2.4.

Theorem 15.4.5 [HW20, Thm. A] *If a smooth projective variety X over a number field is birationally equivalent to a homogeneous space of a connected linear algebraic group over k, then conjecture* (E) *holds for X.*

Chapter 16
The Tate conjecture, abelian varieties and K3 surfaces

M. Artin and J. Tate conjectured that the Brauer group of a smooth and projective variety over a finite field is a finite group. In his 1966 Bourbaki talk [Tate66b], Tate explains why this is analogous to the conjectured finiteness of the Tate–Shafarevich group of an abelian variety over a number field. In [Tate66b, Tate91] he also discusses how this is closely related to his conjectures [Tate63, Tate66b] on the image of the cycle map from the Picard group to the second ℓ-adic étale cohomology group. The precise interplay between these statements has been discussed in many papers, see [Tate91].

The ℓ-adic Tate conjecture for divisors on abelian varieties over a finite field was proved by Tate in 1966. Work of several people over a long period of time established the conjecture for K3 surfaces over any finite field, see [MP15], [Ben15] and [Tot17].

Already in [Tate63], Tate extended his conjectures over finite fields to conjectures over fields finitely generated over the prime subfield. Here again, the question for divisors has various incarnations, see Section 16.1 below.

One may ask the following questions about the Brauer group:

Question 1. Let X be a smooth, proper and geometrically integral variety over a field k that is *finitely generated* over its prime subfield. Is $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ finite?

Question 2. Let X be a smooth, proper and geometrically integral variety over a field k that is *finitely generated* over its prime subfield. Is the image of $\mathrm{Br}(X)$ in $\mathrm{Br}(X^{\mathrm{s}})$ finite?

These questions are closely related, see Theorem 5.4.12.

The ℓ-adic Tate conjecture for divisors on an abelian variety A over a field finitely generated over its prime field was proved in positive characteristic by Zarhin (and Mori) and in characteristic zero by Faltings. This leads to the finiteness of $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^{\Gamma}$ for each $\ell \neq \mathrm{char}(k)$. A similar result for K3 surfaces over a field finitely generated over $\mathbb{Q}$ was proved by S.G. Tankeev [Tan88] and Y. André [And96], see the brief sketch by D. Ramakrishnan in [Tate91, Thm. 5.6].

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The first main result of this chapter says that if A is an abelian variety over a field finitely generated over its prime field, then $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^{\Gamma}=0$ for almost all ℓ. This is proved in Section 16.2, using the material developed in Sections 5.3 and 16.1. In Section 16.3 we discuss the Brauer group of a product of varieties over a field k finitely generated over $\mathbb{Q}$ and deduce finiteness results for the Brauer group of a variety over k dominated by a product of curves.

The rest of the chapter is devoted to a positive answer to Questions 1 and 2 for K3 surfaces over fields which are finitely generated over $\mathbb{Q}$. In Section 16.4 we recall the basic properties of K3 surfaces such as their Hodge structure and the period map. Section 16.5 introduces the original Kuga–Satake construction [KS67] and its interpretation by Deligne [Del72]. A uniform version of this construction in terms of Shimura varieties carrying universal families of abelian varieties is given in Section 16.6. Section 16.7 contains the proofs of the main results in the case of K3 surfaces over a field finitely generated over $\mathbb{Q}$: the Tate conjecture for divisors and the finiteness of $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$. The exposition in these sections is based on the papers [SZ08, OS18, OSZ] and previous work of André [And96], Rizov [Riz10] and Madapusi Pera [MP15, MP16]. Finally, in the last section we discuss the example of diagonal quartic surfaces over $\mathbb{Q}$, where a classification of all possible Brauer groups has been recently completed.

16.1 Tate conjecture for divisors

The Tate conjecture for divisors asserts that the equivalent properties in the following theorem hold when the ground field k is finitely generated over its prime field.

Theorem 16.1.1 *Let X be a smooth, projective, geometrically integral variety over a field k. Let $\Gamma=\mathrm{Gal}(k_{\mathrm{s}}/k)$. Let $\ell\neq\mathrm{char}(k)$ be a prime. The following conditions are equivalent.*

(i) *The natural injective map* $(\mathrm{NS}(X^{\mathrm{s}})\otimes\mathbb{Z}_\ell)^{\Gamma}\to\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))^{\Gamma}$ *is an isomorphism.*

(ii) *The natural injective map* $(\mathrm{NS}(X^{\mathrm{s}})\otimes\mathbb{Q}_\ell)^{\Gamma}\to\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Q}_\ell(1))^{\Gamma}$ *is an isomorphism.*

(iii) $(T_\ell(\mathrm{Br}(X^{\mathrm{s}})))^{\Gamma}=0$.

(iv) $(V_\ell(\mathrm{Br}(X^{\mathrm{s}})))^{\Gamma}=0$.

(v) $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$ *is finite.*

(vi) *The image of the map* $\mathrm{Br}(X)\{\ell\}\to\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ *is finite.*

Proof. As explained in Section 5.3, the Kummer exact sequence gives rise to an exact sequence of finitely generated $\mathbb{Z}_\ell$-modules (5.12)

$$0\longrightarrow\mathrm{NS}(X^{\mathrm{s}})\otimes\mathbb{Z}_\ell\longrightarrow\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}},\mathbb{Z}_\ell(1))\longrightarrow T_\ell(\mathrm{Br}(X^{\mathrm{s}}))\longrightarrow 0,$$

with a continuous action of Γ. When tensored with $\mathbb{Q}_\ell$, it gives the exact sequence

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell(1)) \longrightarrow V_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0. \qquad (16.1)$$

By Theorem 5.3.1 (ii) this sequence is split as a sequence of Γ-modules. Thus (ii) is equivalent to (iv). The group $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ is torsion-free. Thus (iii) is equivalent to (iv). It is clear that (iii) implies (i). That (i) implies (ii) follows from the simple observation that for any finitely generated $\mathbb{Z}_\ell$-module M with an action of Γ, the map $M^\Gamma \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \to (M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)^\Gamma$ is surjective.

For any abelian group A with an action of a group Γ, one has a natural isomorphism $T_\ell(A)^\Gamma \cong T_\ell(A^\Gamma)$. The group $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ is of cofinite type (Proposition 5.2.9), hence so is $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$, that is, $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m \oplus B$, where B is a finite abelian group. Then $T_\ell(\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma) \simeq \mathbb{Z}_\ell^m$. It follows that (iii) is equivalent to (v), because both statements are equivalent to $m = 0$.

The equivalence of (v) and (vi) is a consequence of Theorem 5.4.12. □

Corollary 16.1.2 [CTS13b, Thm. 6.2] *Let U be a smooth, quasi-projective and geometrically integral variety over a field k that is finitely generated over $\mathbb{Q}$. Then the cokernel of the natural map $\mathrm{Br}(U) \to \mathrm{Br}(U^{\mathrm{s}})^\Gamma$ is finite. If ℓ is a prime such that the ℓ-adic Tate conjecture for divisors holds for a smooth compactification of U, then $\mathrm{Br}(U^{\mathrm{s}})\{\ell\}^\Gamma$ is finite.*

Proof. (Sketch) One uses Theorem 3.7.2 over k_{s}. The first statement is a consequence of Theorem 5.4.12 and finiteness of $\mathrm{H}^1_{\text{ét}}(Y^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z})^\Gamma$ for any smooth quasi-projective variety Y over a field k that is finitely generated over $\mathbb{Q}$ (a theorem of Katz and Lang, see [CTS13b, Prop. 6.1]). One proves the second statement by combining the same finiteness theorem with Theorem 16.1.1. □

Corollary 16.1.3 *Let A be an abelian variety over a field k that is finitely generated over its prime subfield. Then $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^\Gamma$ is finite for all primes ℓ not equal to* $\mathrm{char}(k)$.

Proof. This follows from Theorem 16.1.1 because statement (ii) in that theorem (the original Tate conjecture for divisors) is known for abelian varieties (which are projective): over a finite field, this was proved by Tate, over a field finitely generated over the prime field, it was proved by Zarhin in characteristic $p > 2$ [Zar75, Zar76], by Faltings in characteristic zero [Fal83, Fal86], and by Mori in characteristic 2, see [Mor85]. □

Finite fields

Let $\mathbb{F}$ be a finite field of cardinality $|\mathbb{F}| = q$. The Galois group $\Gamma = \mathrm{Gal}(\mathbb{F}_{\mathrm{s}}/\mathbb{F})$ is isomorphic to the procyclic group $\widehat{\mathbb{Z}}$ generated by the Frobenius automorphism $\mathrm{Frob}\colon x \mapsto x^q$.

Theorem 16.1.4 *Let X be a smooth, projective, geometrically integral variety over a finite field $\mathbb{F}$ of characteristic p. The natural map $\mathrm{Br}(X)\to\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ has finite kernel, and its cokernel is the product of a finite group and a p-primary torsion group. In particular, for any prime $\ell \neq p$, $\mathrm{Br}(X)\{\ell\}$ is finite if and only if $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$ is finite, and these two groups are equal for almost all primes ℓ.*

Proof. Let A be the Picard variety of X, and let $T := \mathrm{NS}(X^{\mathrm{s}})_{\mathrm{tors}}$. The groups $\mathrm{Br}(\mathbb{F}) = \mathrm{H}^2(\mathbb{F}, \mathbb{F}_{\mathrm{s}}^*)$, $\mathrm{H}^3(\mathbb{F}, \mathbb{F}_{\mathrm{s}}^*)$, $\mathrm{H}^2(\mathbb{F}, A(\mathbb{F}_{\mathrm{s}}))$ and $\mathrm{H}^2(\mathbb{F}, T)$ all vanish for the same reason: the cohomological dimension of $\Gamma \cong \widehat{\mathbb{Z}}$ is 1 and the coefficient module is a torsion group, see [SerCG, Ch. I, §3.1]. By a theorem of Lang, for the connected algebraic group A over the finite field $\mathbb{F}$, we have $\mathrm{H}^1(\mathbb{F}, A(\mathbb{F}_{\mathrm{s}})) = 0$.

For any continuous, discrete Γ-module M that is finitely generated as an abelian group, the group $\mathrm{H}^1(\Gamma, M)$ is finite. To see this, use the restriction-inflation exact sequence to reduce to the case when M is a trivial Γ-module. Then $\mathrm{H}^1(\mathbb{F}, M) = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, M)$ is the group of continuous homomorphisms from the compact group Γ to the discrete, finitely generated abelian group M, hence $\mathrm{H}^1(\mathbb{F}, M) = \mathrm{Hom}(\Gamma, M_{\mathrm{tors}})$. Since $\Gamma \cong \widehat{\mathbb{Z}}$, the group $\mathrm{Hom}(\Gamma, M_{\mathrm{tors}}) \cong M_{\mathrm{tors}}$ is finite. The Néron–Severi group $\mathrm{NS}(X^{\mathrm{s}})$ is a finitely generated abelian group, so we conclude that $\mathrm{H}^1(\mathbb{F}, \mathrm{NS}(X^{\mathrm{s}}))$ is finite.

By Theorem 5.1.1 we have the exact sequence of Γ-modules

$$0 \longrightarrow A(\mathbb{F}_{\mathrm{s}}) \longrightarrow \mathrm{Pic}(X^{\mathrm{s}}) \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \longrightarrow 0.$$

From the long exact sequence of Galois cohomology groups we deduce that $\mathrm{H}^1(\mathbb{F}, \mathrm{Pic}(X^{\mathrm{s}})) \cong \mathrm{H}^1(\mathbb{F}, \mathrm{NS}(X^{\mathrm{s}}))$ is finite. For a finite field, this group is the kernel of $\mathrm{Br}(X)\to\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$, which is thus finite.

Let ℓ be a prime, $\ell \neq p$. Theorem 5.3.1 (i) shows that there exists a positive integer N not depending on ℓ, and for each $\ell \neq p$ there is a Γ-submodule $M_\ell \subset T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ such that $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))/M_\ell$ is annihilated by N and the exact sequence of continuous Γ-modules (5.12)

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1)) \longrightarrow T_\ell(\mathrm{Br}(X^{\mathrm{s}})) \longrightarrow 0$$

pulled back with respect to the map $M_\ell \to T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ is split. Tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ we obtain the exact sequence

$$0 \longrightarrow \mathrm{NS}(X^{\mathrm{s}}) \otimes_{\mathbb{Z}} (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{\mathrm{div}} \longrightarrow \mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}} \longrightarrow 0,$$

whose pullback with respect to the map $M_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}}$ is split. An easy diagram chase shows that the cokernel of the map

$$[\mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{\mathrm{div}}]^{\Gamma} \longrightarrow [\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}}]^{\Gamma}$$

is killed by N. Since $\mathrm{H}^3_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Z}_\ell(1))_{\mathrm{tors}}$ is finite for all $\ell \neq p$, and zero for almost all ℓ (see Proposition 5.2.9), by (5.4) this implies that the cokernel of the lower horizontal map in the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{Br}(X)\{\ell\} \\ \downarrow & & \downarrow \\ \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{\Gamma} & \longrightarrow & \mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma} \end{array} \tag{16.2}$$

is killed by some positive integer which does not depend on ℓ.

Since $\mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ is a torsion group, $\mathrm{H}^2(\mathbb{F}, \mathrm{H}^1_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))) = 0$ because $\mathbb{F}$ has cohomological dimension 1. Thus the Hochschild–Serre spectral sequence for Galois cohomology gives the surjectivity of the map

$$\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow \mathrm{H}^2_{\text{ét}}(X^{\mathrm{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{\Gamma}.$$

The commutativity of (16.2) implies that the prime-to-p subgroup of the cokernel of $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is killed by a positive integer. The group $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$, hence also any subquotient, is an ℓ-primary torsion group of cofinite type (Proposition 5.2.9). Hence the cokernel of $\mathrm{Br}(X)\{\ell\} \to \mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$ is finite for any $\ell \neq p$ and is zero for almost all ℓ. □

Remark 16.1.5 The above proposition and its proof should be compared to Theorem 5.4.12. For a variety X over a finite field as above, the cokernel of $\mathrm{Br}(X) \to \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is actually finite, so the restriction $\ell \neq p$ in the above proposition can be dropped, see [Yua20, Thm. 1.3 (4)].

The following theorem is proved for surfaces in [Tate66b, Thm. 5.2] and in arbitrary dimension in [Zar82], [Lic83]. It is a consequence of Deligne's work on the Weil conjectures.

Theorem 16.1.6 *Let X be a smooth, projective, geometrically integral variety over a finite field $\mathbb{F}$ of characteristic p. If there exists a prime $\ell \neq p$ such that $\mathrm{Br}(X)\{\ell\}$ is finite, then $\mathrm{Br}(X)\{\ell\}$ is finite for all primes $\ell \neq p$, and is zero for almost all ℓ.*

Proof. By Theorem 16.1.4, the finiteness of $\mathrm{Br}(X)\{\ell\}$ implies the finiteness of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^{\Gamma}$. By Theorem 16.1.1, this implies $(V_\ell(\mathrm{Br}(X^{\mathrm{s}})))^{\Gamma} = 0$.

Let us write (16.1), which is an exact sequence of continuous Γ-modules and of finite-dimensional $\mathbb{Q}_\ell$-vector spaces, as

$$0 \longrightarrow U_\ell \longrightarrow W_\ell \longrightarrow V_\ell \longrightarrow 0.$$

Thus we have an equality of characteristic polynomials of $\mathrm{Frob} \in \Gamma$:

$$\det(t\,\mathrm{Id} - \mathrm{Frob}^*, W_\ell) = \det(t\,\mathrm{Id} - \mathrm{Frob}^*, U_\ell) \cdot \det(t\,\mathrm{Id} - \mathrm{Frob}^*, V_\ell) \in \mathbb{Q}_\ell[t].$$

It is clear that $\det(t\,\mathrm{Id} - \mathrm{Frob}^*, U_\ell)$ is the characteristic polynomial of Frob acting on the $\mathbb{Q}$-vector space $\mathrm{NS}(X^{\mathrm{s}}) \otimes \mathbb{Q}$, so it belongs to $\mathbb{Q}[t]$ and does not depend on ℓ.

The polynomial $\det(t\,\mathrm{Id}-\mathrm{Frob}^*, W_\ell)$ is in $\mathbb{Q}[t]$ and does not depend on ℓ. For surfaces this was already known in 1964. In general this is a special case of Deligne's result [Del74, Thm. 1.6], [Del80, Cor. 3.3.9].

Thus $\det(t\,\mathrm{Id}-\mathrm{Frob}^*, V_\ell)\in\mathbb{Q}[t]$ is independent of ℓ. This polynomial has a root equal to 1 if and only if $(V_\ell(\mathrm{Br}(X^{\mathrm{s}})))^\Gamma\neq 0$. Thus if $(V_\ell(\mathrm{Br}(X^{\mathrm{s}})))^\Gamma=0$ for one $\ell\neq p$, then $\det(\mathrm{Id}-\mathrm{Frob}^*, V_\ell)\in\mathbb{Q}^*$. This implies that the endomorphism $\mathrm{Id}-\mathrm{Frob}^*$ of the torsion-free $\mathbb{Z}_\ell$-module $T_\ell(\mathrm{Br}(X^{\mathrm{s}}))$ is injective for all $\ell\neq p$, with finite cokernel, and is an isomorphism for all but finitely many $\ell\neq p$. Since

$$T_\ell(\mathrm{Br}(X^{\mathrm{s}}))\otimes_{\mathbb{Z}_\ell}(\mathbb{Q}_\ell/\mathbb{Z}_\ell)=\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}},$$

we get that $[\mathrm{Br}(X^{\mathrm{s}})\{\ell\}_{\mathrm{div}}]^\Gamma$ is finite for all $\ell\neq p$ and zero for almost all ℓ.

By Proposition 5.2.9, for each $\ell\neq p$, the quotient of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}$ by its maximal divisible subgroup is a finite group which is zero for almost all ℓ. We conclude that $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ is finite for any $\ell\neq p$ and zero for almost all ℓ. By Theorem 16.1.4, the natural map $\mathrm{Br}(X)\to\mathrm{Br}(X^{\mathrm{s}})^\Gamma$ has finite kernel. Thus the prime-to-p subgroup of $\mathrm{Br}(X)$ is finite. □

Remark 16.1.7 For arbitrary fields finitely generated over a finite field, Y. Qin [Qin, Thm. 1.2] recently proved the following version of Theorem 16.1.6. *Let k be a field finitely generated over the finite field $\mathbb{F}_p$. For a smooth, projective, and geometrically integral variety X over k, finiteness of $\mathrm{Br}(X^{\mathrm{s}})\{\ell\}^\Gamma$ for one prime $\ell\neq p$ implies finiteness of the subgroup of $\mathrm{Br}(X^{\mathrm{s}})^\Gamma$ consisting of the elements of order not divisible by p.*

Corollary 16.1.8 *Let X be a smooth, projective, geometrically integral variety over a finite field of characteristic p. For any prime $\ell\neq p$, the equivalent conditions in Theorem 16.1.1 are equivalent to finiteness of $\mathrm{Br}(X)\{\ell\}$. They are also equivalent to the statement: the natural map*

$$\mathrm{Pic}(X)\otimes\mathbb{Z}_\ell\longrightarrow \mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell(1))$$

is an isomorphism. If they hold for one ℓ they hold for all ℓ.

Proof. The first claim follows from Theorem 16.1.1 and Theorem 16.1.4. To prove the second claim, note that the Kummer sequence gives rise to compatible short exact sequences of finite abelian groups

$$0\longrightarrow \mathrm{Pic}(X)/\ell^n\longrightarrow \mathrm{H}^2_{\text{ét}}(X,\mu_{\ell^n})\longrightarrow \mathrm{Br}(X)[\ell^n]\longrightarrow 0.$$

Taking projective limits over n gives the exact sequence

$$0\longrightarrow \mathrm{Pic}(X)\otimes\mathbb{Z}_\ell\longrightarrow \mathrm{H}^2_{\text{ét}}(X,\mathbb{Z}_\ell(1))\longrightarrow T_\ell(\mathrm{Br}(X))\longrightarrow 0.$$

If $\mathrm{Br}(X)\{\ell\}$ is finite, then we have $T_\ell(\mathrm{Br}(X))=0$. The converse holds since $\mathrm{Br}(X)\{\ell\}$ is a torsion group of cofinite type. (Indeed, it is an extension of a subgroup of $\mathrm{Br}(X^{\mathrm{s}})$ by a finite group.) The last claim follows from Theorem 16.1.6. □

Remark 16.1.9 The ℓ-adic Tate conjecture for divisors on all smooth, projective, connected surfaces over a finite field implies the same conjecture for all smooth, projective varieties, see [Mor19, Thm. 4.3]. It also implies the ℓ-adic Tate conjecture for divisors on all smooth, proper, connected varieties over fields finitely generated over a finite field [Amb20, Thm. 1.1.2].

Remark 16.1.10 Let X be a smooth, projective, geometrically integral *surface* over a finite field $\mathbb{F}$ of characteristic $p \neq 2$. Let $\ell \neq p$ be a prime. The abelian group $\big(\mathrm{Br}(X)/\mathrm{Br}(X)_{\mathrm{div}}\big)\{\ell\}$ carries a natural pairing with values in $\mathbb{Q}/\mathbb{Z}$ defined by Artin and Tate, see [Tate66b]. (The Tate conjecture predicts that $\mathrm{Br}(X)_{\mathrm{div}} = 0$.) This pairing can be thought of as an analogue of the Cassels–Tate pairing on the Tate–Shafarevich group of an elliptic curve over a number field. The Artin–Tate pairing is skew-symmetric. In his thesis, using ideas from algebraic topology (Steenrod squares, Stiefel–Whitney classes), Tony Feng proved that this pairing is alternating [Fen20, Thm. 1.2]; this is non-trivial for $\ell = 2$. This implies that the cardinality of the finite group $\big(\mathrm{Br}(X)/\mathrm{Br}(X)_{\mathrm{div}}\big)\{\ell\}$ is a square. See [Fen20] for the history of this question and references to the literature.

16.2 Abelian varieties

For an abelian variety A over a field k and a prime $\ell \neq \mathrm{char}(k)$ the cohomology group $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)$ has a nice interpretation in terms of the torsion subgroup $A[\ell] := A(k_{\mathrm{s}})[\ell]$. Namely, for each $n \geq 1$ we have a canonical isomorphism

$$\mathrm{H}^n_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/\ell) = \wedge^n_{\mathbb{Z}/\ell} \mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mathbb{Z}/\ell).$$

The Kummer sequence gives a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell) = \mathrm{Pic}(A^{\mathrm{s}})[\ell] = A^\vee[\ell],$$

where $A^\vee$ is the dual abelian variety of A. We have $(A^\vee)^\vee = A$, see [Lang83a, Ch. V, §2, Prop. 9], [MumAV, p. 132]. The ℓ-torsion subgroups of A and $A^\vee$ are related by the Weil pairing

$$e_{\ell,A} \colon A[\ell] \times A^\vee[\ell] \longrightarrow \mu_\ell,$$

which is a perfect Γ-invariant pairing. Thus we obtain a canonical isomorphism of Γ-modules

$$\mathrm{H}^1_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell) = \mathrm{Hom}(A[\ell], \mu_\ell),$$

which gives a canonical isomorphism and an injective map of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell) = \mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell} A[\ell], \mu_\ell) \hookrightarrow \mathrm{Hom}(A[\ell], A^\vee[\ell]).$$

By definition, $\wedge^2_{\mathbb{Z}/\ell}A[\ell]$ is the quotient of $A[\ell]\otimes_{\mathbb{Z}/\ell}A[\ell]$ by the $\mathbb{Z}/\ell$-submodule generated by $x\otimes x$ for $x\in A[\ell]$. Hence a homomorphism $\phi\colon A[\ell]\to A^\vee[\ell]$ comes from an element of $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell}A[\ell],\mu_\ell)$ if and only if $e_{\ell,A}(x,\phi x)=0$ for all $x\in A[\ell]$.

Definition 16.2.1 *A homomorphism $\phi\colon A[\ell]\to A^\vee[\ell]$ is called* **symmetric** *if we have $e_{\ell,A}(x,\phi y)=e_{\ell,A^\vee}(\phi x,y)$ for any $x,y\in A[\ell]$.*

Lemma 16.2.2 *Let $\ell\neq 2$. Then the injective image of $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)$ in $\mathrm{Hom}(A[\ell],A^\vee[\ell])$ is the subgroup $\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}$ of symmetric homomorphisms.*

Proof. This crucially uses the subtle fact that the Weil pairings for A and $A^\vee$ differ by sign, see [Lang83a, Ch. VII, §2, Thm. 5 (iii), p. 193] or [Oda69, Cor. 1.3 (ii)]. That is, we have

$$e_{\ell,A^\vee}(y,x)=-e_{\ell,A}(x,y)$$

for all $x\in A[\ell]$, $y\in A^\vee[\ell]$. Thus $\phi\in\mathrm{Hom}(A[\ell],A^\vee[\ell])_{\mathrm{sym}}$ if and only if

$$e_{\ell,A}(x,\phi y)=-e_{\ell,A^\vee}(\phi y,x)=-e_{\ell,A}(y,\phi x).$$

Equivalently, ϕ is symmetric if and only if the bilinear form $e_{\ell,A}(x,\phi y)$ is skew-symmetric:

$$e_{\ell,A}(x,\phi y)=-e_{\ell,A}(y,\phi x),\quad x,y\in A[\ell].$$

When $\ell\neq 2$, the group $\wedge^2_{\mathbb{Z}/\ell}A[\ell]$ is the quotient of $A[\ell]\otimes_{\mathbb{Z}/\ell}A[\ell]$ by the $\mathbb{Z}/\ell$-submodule generated by the elements of the form $x\otimes y+y\otimes x$ for $x,y\in A[\ell]$. Hence $\phi\in\mathrm{Hom}(A[\ell],A^\vee[\ell])$ is contained in the image of $\mathrm{Hom}(\wedge^2_{\mathbb{Z}/\ell}A[\ell],\mu_\ell)$ if and only if $e_{\ell,A}(x,\phi y)+e_{\ell,A}(y,\phi x)=0$ for all $x,y\in A[\ell]$, which says that ϕ is symmetric. □

For abelian varieties A and B we write $\mathrm{Hom}(A,B)$ for the group of homomorphisms $A\to B$ (defined over k). A divisor D on A^{s} defines the homomorphism $A^{\mathrm{s}}\to(A^\vee)^{\mathrm{s}}$ sending $a\in A(k_{\mathrm{s}})$ to the linear equivalence class of $T_a^*(D)-D$ in $\mathrm{Pic}^0(A^{\mathrm{s}})$, where T_a is the translation by a in A^{s}. If L is the class of D in $\mathrm{NS}(A^{\mathrm{s}})$, then this map depends only on L, and is denoted by $\varphi_L\colon A^{\mathrm{s}}\to(A^\vee)^{\mathrm{s}}$ [MumAV, §8]. For $\alpha\in\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})$ we denote by $\alpha^\vee\in\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})$ the dual map of α, see [MumAV, §13]. Then $\varphi_L^\vee=\varphi_L$. Moreover, if we define

$$\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}}=\{u\in\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})\mid u=u^\vee\},$$

then the group homomorphism

$$\mathrm{NS}(A^{\mathrm{s}})\longrightarrow\mathrm{Hom}(A^{\mathrm{s}},(A^\vee)^{\mathrm{s}})_{\mathrm{sym}},\quad L\mapsto\phi_L,$$

is an isomorphism [Lang83a], [MumAV, §20, formula (I) and Thm. 1 on p. 186, Thm. 2 on p. 188 and Remark on p. 189]. For any $\alpha \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ we have $(\alpha^\vee)^\vee = \alpha$, and thus

$$\alpha + \alpha^\vee \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}. \tag{16.3}$$

We have $\mathrm{Hom}(A, B) = \mathrm{Hom}_\Gamma(A^{\mathrm{s}}, B^{\mathrm{s}}) = \mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})^\Gamma$. Since $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})$ has no torsion, the group $\mathrm{Hom}(A, B)/\ell$ is a subgroup of $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})/\ell$.

The action of homomorphisms on points of order ℓ defines a natural map of Γ-modules

$$\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}}) \longrightarrow \mathrm{Hom}(A[\ell], B[\ell]).$$

A homomorphism $A^{\mathrm{s}} \to B^{\mathrm{s}}$ annihilates $A[\ell]$ if and only if it factors through the multiplication by ℓ map, hence the image of $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})$ in $\mathrm{Hom}(A[\ell], B[\ell])$ is $\mathrm{Hom}(A^{\mathrm{s}}, B^{\mathrm{s}})/\ell$. We thus obtain an embedding

$$\mathrm{Hom}(A, B)/\ell \subset \mathrm{Hom}_\Gamma(A[\ell], B[\ell]).$$

Now let $B = A^\vee$. Then for any $\alpha \in \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})$ and any $x, y \in A[\ell]$ we have

$$e_{\ell, A^\vee}(\alpha x, y) = e_{\ell, A}(x, \alpha^\vee y),$$

see [Lang83a, Ch. VII, §2, Thm. 4], [MumAV, p. 186], or [Oda69, Cor. 1.3 (ii)]. Thus $\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell$ is a subgroup of $\mathrm{Hom}(A[\ell], A^\vee[\ell])_{\mathrm{sym}}$. Note that if $\ell \neq 2$, then using (16.3) we see that this subgroup consists precisely of the elements of $\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})/\ell$ that define symmetric homomorphisms on ℓ-torsion subgroups:

$$\mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})_{\mathrm{sym}}/\ell = \mathrm{Hom}(A^{\mathrm{s}}, (A^\vee)^{\mathrm{s}})/\ell \,\cap\, \mathrm{Hom}(A[\ell], A^\vee[\ell])_{\mathrm{sym}}. \tag{16.4}$$

(To see that the natural inclusion of the left-hand side into the right-hand side is an isomorphism, note that any α in the right-hand side lifts to $\frac{\ell+1}{2}(\alpha + \alpha^\vee)$.)

Now we are ready to prove the main theorem of this section [SZ08, Thm. 1.1]. Assume that the field k is finitely generated over its prime subfield. By Corollary 16.1.3 we know that $\mathrm{Br}(A^{\mathrm{s}})\{\ell\}^\Gamma$ is finite for all prime numbers $\ell \neq \mathrm{char}(k)$. Our aim now is to prove that this group is actually zero for almost all ℓ. For this it is enough to prove that $\mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma = 0$ for almost all ℓ. By Theorem 5.3.1 for almost all ℓ we have an exact sequence of Γ-modules (5.17):

$$0 \longrightarrow (\mathrm{NS}(A^{\mathrm{s}})/\ell)^\Gamma \longrightarrow \mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)^\Gamma \longrightarrow \mathrm{Br}(A^{\mathrm{s}})[\ell]^\Gamma \longrightarrow 0,$$

so our task is to prove that for almost all ℓ each Γ-invariant class in $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}}, \mu_\ell)$ comes from a divisor on A.

Theorem 16.2.3 *Let A be an abelian variety over a field k that is finitely generated over its prime subfield. Then* $\mathrm{Br}(A^{\mathrm{s}})[\ell]^{\Gamma}=0$ *for almost all primes ℓ. The subgroup of* $\mathrm{Br}(A^{\mathrm{s}})^{\Gamma}$ *that consists of the elements of order not divisible by* $\mathrm{char}(k)$ *is finite.*

Proof. The second statement follows from the first statement and Corollary 16.1.3. The first statement is a consequence of the following variant of the Tate conjecture on homomorphisms first stated by Zarhin in [Zar77]: for abelian varieties A and B over k the natural injective map

$$\mathrm{Hom}(A,B)/\ell \hookrightarrow \mathrm{Hom}_{\Gamma}(A[\ell],B[\ell]) \tag{16.5}$$

is an isomorphism for almost all ℓ. In the finite characteristic case this is due to Zarhin [Zar77, Thm. 1.1]. When k is a number field, [Zar85, Cor. 5.4.5] based on the results of Faltings [Fal83] says that for almost all ℓ we have

$$\mathrm{End}(A)/\ell \cong \mathrm{End}_{\Gamma}(A[\ell]). \tag{16.6}$$

The same proof works over arbitrary fields that are finitely generated over $\mathbb{Q}$, if one replaces the reference to [Zar85, Prop. 3.1] by the reference to the corollary on p. 211 of [Fal86]. Applying (16.6) to the abelian variety $A\times B$, one deduces that (16.5) is a bijection.

Since the Néron–Severi group of an abelian variety is torsion-free, $\mathrm{NS}(A^{\mathrm{s}})^{\Gamma}/\ell$ maps injectively into $(\mathrm{NS}(A^{\mathrm{s}})/\ell)^{\Gamma}$. By the Kummer sequence, $(\mathrm{NS}(A^{\mathrm{s}})/\ell)^{\Gamma}$ is a subgroup of $\mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)^{\Gamma}$, so we have

$$\mathrm{NS}(A^{\mathrm{s}})^{\Gamma}/\ell \subset (\mathrm{NS}(A^{\mathrm{s}})/\ell)^{\Gamma} \subset \mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)^{\Gamma}. \tag{16.7}$$

In view of (16.4), taking $B=A^{\vee}$ in (16.5) gives an isomorphism of

$$\mathrm{NS}(A^{\mathrm{s}})^{\Gamma}/\ell \cong \mathrm{Hom}_{\Gamma}(A^{\mathrm{s}},(A^{\vee})^{\mathrm{s}})_{\mathrm{sym}}/\ell \cong \mathrm{Hom}(A,A^{\vee})_{\mathrm{sym}}/\ell$$

with $\mathrm{Hom}_{\Gamma}(A[\ell],A^{\vee}[\ell])_{\mathrm{sym}} \cong \mathrm{H}^2_{\text{ét}}(A^{\mathrm{s}},\mu_\ell)^{\Gamma}$ for almost all ℓ. Hence all three groups in (16.7) coincide. By Theorem 5.3.1 (iv), this implies that for almost all ℓ we have $\mathrm{Br}(A^{\mathrm{s}})[\ell]^{\Gamma}=0$. □

16.3 Varieties dominated by products

The following results appeared as [SZ14, Theorems A, B].

Theorem 16.3.1 *Let k be a field finitely generated over $\mathbb{Q}$. Let X and Y be smooth, projective and geometrically integral varieties over k. Then*

$$\big(\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})/(\mathrm{Br}(X^{\mathrm{s}})\oplus \mathrm{Br}(Y^{\mathrm{s}}))\big)^{\Gamma}$$

is a finite group.

Proof. By Corollary 5.2.4 the Γ-module $\mathrm{Br}(X^{\mathrm{s}})\oplus\mathrm{Br}(Y^{\mathrm{s}})$ is a direct summand of $\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})$. Since $\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})$ is a torsion group such that $\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})[n]$ is finite for every positive integer n, the same is true for the group in the statement of the theorem. Thus it is enough to prove the following statements.

(a) *For every prime* ℓ *we have* $V_\ell\big((\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})/(\mathrm{Br}(X^{\mathrm{s}})\oplus\mathrm{Br}(Y^{\mathrm{s}})))^\Gamma\big)=0$.

(b) $\big(\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})[\ell]/(\mathrm{Br}(X^{\mathrm{s}})[\ell]\oplus\mathrm{Br}(Y^{\mathrm{s}})[\ell])\big)^\Gamma=0$ *for almost all primes* ℓ.

Let us prove (a). We can pass to the limit in the isomorphism of Corollary 5.7.10, taking into account what was said in Remark 5.7.11. This produces an isomorphism of Γ-modules between

$$V_\ell(\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}}))/(V_\ell(\mathrm{Br}(X^{\mathrm{s}}))\oplus V_\ell(\mathrm{Br}(Y^{\mathrm{s}})))$$
$$\cong V_\ell\big(\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})/(\mathrm{Br}(X^{\mathrm{s}})\oplus\mathrm{Br}(Y^{\mathrm{s}}))\big)$$

and the quotient of $\mathrm{Hom}_{\mathbb{Q}_\ell}\big(V_\ell(B^\vee),V_\ell(A)\big)$ by $\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})\otimes\mathbb{Q}_\ell$ (embedded via the map given by the action on torsion points). Hence we obtain

$$\begin{aligned}&V_\ell\big((\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})/(\mathrm{Br}(X^{\mathrm{s}})\oplus\mathrm{Br}(Y^{\mathrm{s}})))^\Gamma\big)\\ \cong\ & V_\ell\big(\mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})/(\mathrm{Br}(X^{\mathrm{s}})\oplus\mathrm{Br}(Y^{\mathrm{s}}))\big)^\Gamma\\ \cong\ & \big(\mathrm{Hom}_{\mathbb{Q}_\ell}\big(V_\ell(B^\vee),V_\ell(A)\big)/\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})\otimes\mathbb{Q}_\ell\big)^\Gamma.\end{aligned}$$

By the fundamental results of Faltings [Fal83, Fal86], the Γ-modules $V_\ell(B^\vee)$ and $V_\ell(A)$ are semisimple and

$$\mathrm{Hom}_\Gamma(V_\ell(B^\vee),V_\ell(A))\cong\mathrm{Hom}(B^\vee,A)\otimes\mathbb{Q}_\ell.$$

By a theorem of Chevalley [Che54, p. 88], the semisimplicity of Γ-modules $V_\ell(B^\vee)$ and $V_\ell(A)$ implies the semisimplicity of $\mathrm{Hom}_{\mathbb{Q}_\ell}(V_\ell(B^\vee),V_\ell(A))$. From this we deduce (a).

Let us prove (b). By Corollary 5.7.10 and Remark 5.7.11 it is enough to show that

$$\big(\mathrm{Hom}(B^\vee[\ell],A[\ell])/(\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})/\ell)\big)^\Gamma=0$$

for almost all primes ℓ. Since $\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})^\Gamma=\mathrm{Hom}(B^\vee,A)$, the exact sequence

$$0\to\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})^\Gamma/\ell\to\big(\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})/\ell\big)^\Gamma\to\mathrm{H}^1(k,\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}}))$$

implies, in view of the finiteness of $\mathrm{H}^1(k,\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}}))$, that for all but finitely many primes ℓ we have

$$\big(\mathrm{Hom}((B^\vee)^{\mathrm{s}},A^{\mathrm{s}})/\ell\big)^\Gamma\cong\mathrm{Hom}(B^\vee,A)/\ell.$$

If we further assume that $\ell>2\dim(A)+2\dim(B)-2$, then, by a theorem of Serre [Ser94], the semisimplicity of the Γ-modules $B^\vee[\ell]$ and $A[\ell]$ implies the semisimplicity of $\mathrm{Hom}(B^\vee[\ell], A[\ell])$. Hence we obtain

$$\begin{aligned} &\big(\mathrm{Hom}(B^\vee[\ell], A[\ell])/(\mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}})/\ell)\big)^{\Gamma} \\ = \;& \mathrm{Hom}(B^\vee[\ell], A[\ell])^{\Gamma}/(\mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}})/\ell)^{\Gamma} \\ = \;& \mathrm{Hom}_{\Gamma}(B^\vee[\ell], A[\ell])/(\mathrm{Hom}(B^\vee, A)/\ell) \;=\; 0. \end{aligned}$$

This is zero for almost all ℓ, since (16.5) is a bijection for almost all ℓ. □

Corollary 16.3.2 *Let k be a field finitely generated over $\mathbb{Q}$. Let X and Y be smooth, projective and geometrically integral varieties over k. Assume that either $\mathrm{H}^3(k, k_{\mathrm{s}}^*) = 0$ (for example, k is a number field) or $(X \times Y)(k) \neq \emptyset$. Then the cokernel of the natural map*

$$\mathrm{Br}(X) \oplus \mathrm{Br}(Y) \longrightarrow \mathrm{Br}(X \times Y)$$

is finite.

Proof. From the functoriality of the spectral sequence (5.19), in view of our assumption, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Br}(X\times Y) & \to & \mathrm{Br}(X^{\mathrm{s}}\times Y^{\mathrm{s}})^{\Gamma} & \to & \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}}\times Y^{\mathrm{s}})) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Br}(X)\oplus \mathrm{Br}(Y) & \to & \mathrm{Br}(X^{\mathrm{s}})^{\Gamma}\oplus \mathrm{Br}(Y^{\mathrm{s}})^{\Gamma} & \to & \mathrm{H}^2(k, \mathrm{Pic}(X^{\mathrm{s}}))\oplus \mathrm{H}^2(k, \mathrm{Pic}(Y^{\mathrm{s}})) \end{array}$$

The middle vertical map is injective by Corollary 5.2.4. Next, the kernel of the right-hand vertical map is finite. Indeed, in view of the exact sequence (5.31) it is enough to remark that the abelian group $\mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}})$ is free and finitely generated, hence $\mathrm{H}^1(k, \mathrm{Hom}((B^\vee)^{\mathrm{s}}, A^{\mathrm{s}}))$ is finite.

By Theorem 16.3.1 this diagram shows that the subgroup of $\mathrm{Br}(X \times Y)$ generated by $\mathrm{Br}_1(X \times Y)$ and the images of $\mathrm{Br}(X)$ and $\mathrm{Br}(Y)$, has finite index. By Proposition 5.7.2, $\mathrm{Br}_1(X \times Y)$ is finite modulo $\mathrm{Br}_1(X) \oplus \mathrm{Br}_1(Y)$, so we are done. □.

Varieties dominated by products of curves

A smooth, projective and geometrically integral variety X over a field k is called a variety *dominated by a product of curves* if there is a dominant rational map from a product of geometrically integral k_{s}-curves to X^{s}.

Theorem 16.3.3 *Let k be a field finitely generated over $\mathbb{Q}$. Let X be a variety dominated by a product of curves. Then $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is finite.*

Proof. To prove the finiteness of $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ we can replace k by a finite field extension.

If V and W are smooth, projective and geometrically integral varieties over a field k that is finitely generated over $\mathbb{Q}$, then the cokernel of the natural map

$\mathrm{Br}(V^{\mathrm{s}})^{\Gamma} \oplus \mathrm{Br}(W^{\mathrm{s}})^{\Gamma} \to \mathrm{Br}(V^{\mathrm{s}} \times W^{\mathrm{s}})^{\Gamma}$ is finite by Theorem 16.3.1. The Brauer group of a smooth, projective, integral curve over an algebraically closed field is zero (Theorem 5.6.1). Thus if Z is a product of smooth, projective and geometrically integral curves over k, then $\mathrm{Br}(Z^{\mathrm{s}})^{\Gamma}$ is finite.

Replacing k by a finite extension we obtain geometrically integral k-curves $C_1, \ldots, C_n$ such that there is a dominant rational map f from $S = \prod_{i=1}^{n} C_i$ to X defined over k. Let $d = \dim(X)$. There is a dense open subset $U \subset S$ such that the restriction of f to U is a smooth morphism $U \to X$. After another finite extension of k we can find a k-point $P \in U(k)$. The induced map of tangent spaces $f_*\colon T_{S,P} \to T_{X,f(P)}$ is surjective, so we can choose a d-element subset $I \subset \{1, \ldots, n\}$ such that $f_*(T_{S_I,P}) = T_{X,f(P)}$, where $S_I \cong \prod_{i\in I} C_i$ is the k-fibre of the projection $S \to \prod_{i\notin I} C_i$ which contains P. Thus f restricts to a dominant, generically finite rational map from a product of d geometrically integral k-curves to X.

Since $\mathrm{char}(k) = 0$, we can assume that there is a smooth, projective and geometrically integral variety Y over k, a birational morphism $Y \to Z$, where Z is a product of d smooth, projective and geometrically integral curves, and a dominant, generically finite morphism $f\colon Y \to X$. By the birational invariance of the Brauer group (Corollary 6.2.11) we have $\mathrm{Br}(Y^{\mathrm{s}})^{\Gamma} \cong \mathrm{Br}(Z^{\mathrm{s}})^{\Gamma}$. By Theorem 3.5.5 the natural map $\mathrm{Br}(X^{\mathrm{s}}) \hookrightarrow \mathrm{Br}(k_{\mathrm{s}}(X))$ is injective. The standard restriction-corestriction argument then gives that the kernel of $f^*\colon \mathrm{Br}(X^{\mathrm{s}}) \to \mathrm{Br}(Y^{\mathrm{s}})$ is killed by the degree $[k_{\mathrm{s}}(Y) : k_{\mathrm{s}}(X)]$. Since $\mathrm{Br}(X^{\mathrm{s}})$ is a torsion group of cofinite type, this kernel is finite. Hence $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ is finite. □

The following statement can be applied, for example, to smooth surfaces in $\mathbb{P}^3_k$ given by a diagonal equation.

Corollary 16.3.4 *Let k be a field finitely generated over $\mathbb{Q}$. Let $f(t)$ and $g(t)$ be separable polynomials of degree $d \geq 2$. Let $F(x,y)$ and $G(x,y)$ be homogeneous forms of degree d such that $f(t) = F(t,1)$ and $g(t) = G(t,1)$. Let $X \subset \mathbb{P}^3_k$ be the surface with equation $F(x,y) = G(z,w)$. Then* $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ *is finite.*

Proof. An immediate verification shows that X is smooth. The surface X is dominated by the product of smooth plane curves of degree d, namely, the curves $z^d = F(x,y)$ and $z^d = G(x,y)$. Since $\mathrm{Pic}(X^{\mathrm{s}})$ is torsion-free [SGA2, XII, Cor. 3.7], the group $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is finite, see the exact sequence (5.21). The finiteness of $\mathrm{Br}(X^{\mathrm{s}})^{\Gamma}$ follows from Theorem 16.3.3. □

16.4 K3 surfaces

Preliminaries on K3 surfaces

For a detailed introduction to the geometry of K3 surfaces we refer the reader to Huybrechts' book [Huy16], see also [Voi02, §7.2]. Here we briefly recall the definition and the basic geometric properties of K3 surfaces.

In this chapter we use the term *lattice* for a finitely generated free abelian group together with a non-degenerate integral symmetric bilinear form.

A smooth, projective and geometrically integral surface X over a field k is called a *K3 surface* if $\Omega_X^2 \cong \mathcal{O}_X$ and $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. Standard examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}_k^3$ and double covers of $\mathbb{P}_k^2$ ramified in a smooth sextic curve.

Let $k = \mathbb{C}$. Using Serre duality and the Riemann–Roch theorem one finds that the classical (Betti) cohomology group $\mathrm{H}^2(X, \mathbb{Z})$ is a free abelian group of rank 22. We have the cup-product

$$\cup\colon \mathrm{H}^2(X, \mathbb{Z}) \times \mathrm{H}^2(X, \mathbb{Z}) \longrightarrow \mathrm{H}^4(X, \mathbb{Z}) \cong \mathbb{Z},$$

where the last isomorphism is due to the fact that $\dim(X) = 2$. This is a symmetric bilinear pairing. Poincaré duality implies that this pairing is a perfect duality, that is, it induces an isomorphism

$$\mathrm{H}^2(X, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}^2(X, \mathbb{Z}), \mathbb{Z}).$$

Thus the determinant of the matrix of this bilinear form with respect to a $\mathbb{Z}$-basis of $\mathrm{H}^2(X, \mathbb{Z})$ lies in $\mathbb{Z}^* = \{\pm 1\}$. Topological arguments (Wu's formula, Thom–Hirzebruch index theorem) give that the associated integral quadratic form is even, i.e., $x \cup x \in 2\mathbb{Z}$ for any $x \in \mathrm{H}^2(X, \mathbb{Z})$, and of signature $(3, 19)$. By the classification of even integral quadratic forms [Ser70, Ch. V, §2, Thm. 5] this implies that $\mathrm{H}^2(X, \mathbb{Z})$ can be written as the orthogonal direct sum

$$L \cong \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 3}. \tag{16.8}$$

Here E_8 is the (positive definite) root lattice of the root system E_8; the lattice $\mathrm{E}_8(-1)$ is obtained by multiplication of the form on E_8 by -1, and U is the hyperbolic lattice of rank 2.

Hodge structures of complex tori

Let M be a finitely generated free abelian group. Following Deligne, an *integral Hodge structure* on M is a representation of the 2-dimensional real torus $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ in $\mathrm{GL}(M_\mathbb{R})$. Then we have a Hodge decomposition $M_\mathbb{C} = \oplus_{p,q} M^{p,q}$ such that $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ acts on $M^{p,q}$ by $z^p \bar{z}^q$. The space

$M^{q,p}$ is the complex conjugate of $M^{p,q}$. If $p+q=n$ for each summand $M^{p,q}$, then the Hodge structure is called pure of weight n.

A complex torus is a quotient $\mathbb{C}^g/\Lambda$, where $\Lambda \cong \mathbb{Z}^{2g}$ spans $\mathbb{C}^g$ as a vector space over $\mathbb{R}$. To give a complex torus is the same as to give an integral Hodge structure of type $\{(1,0),(0,1)\}$ on Λ. (The complex structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is defined by the action of $i \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$.) An abelian variety is a complex torus with a polarisation, which is an integral skew-symmetric form on Λ satisfying some conditions. (This can be also rephrased by saying that the integral Hodge structure is polarisable.)

An example of a complex torus is the Jacobian of a smooth projective curve. For a curve C of genus g the spaces $\mathrm{H}^{1,0} \cong \mathrm{H}^0(C,\Omega^1_C)$ and $\mathrm{H}^{0,1} \cong \mathrm{H}^1(C,\mathcal{O}_C)$ have dimension g, so the Hodge decomposition

$$\mathrm{H}^1(C,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^1(C,\mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}$$

gives rise to a complex torus. Explicitly, integrating g linearly independent holomorphic 1-forms over $2g$ elements of a $\mathbb{Z}$-basis of $\mathrm{H}_1(C,\mathbb{Z})$ produces a rank $2g$ free abelian group $\Lambda \subset \mathbb{C}^g$ which spans $\mathbb{C}^g$ as a real vector space. Then one shows that the complex torus $\mathbb{C}^g/\Lambda$ has a polarisation, so is an abelian variety. This is the Jacobian of C.

Hodge structures of K3 type

In a very rough analogy to the Jacobian of a curve, one would like to associate an abelian variety to a polarised K3 surface over $\mathbb{C}$ (a complex K3 surface X together with an ample line bundle whose class in $\mathrm{Pic}(X)$ is primitive, i.e. non-divisible). The Hodge decomposition on the second integral cohomology group of a complex K3 surface X is

$$\mathrm{H}^2(X,\mathbb{Z})_{\mathbb{C}} = \mathrm{H}^2(X,\mathbb{C}) = \mathrm{H}^{2,0} \oplus \mathrm{H}^{1,1} \oplus \mathrm{H}^{0,2},$$

where $\mathrm{H}^{2,0} \cong \mathrm{H}^0(X,\Omega^2_X)$ and $\mathrm{H}^{0,2} \cong \mathrm{H}^2(X,\mathcal{O}_X)$ are both 1-dimensional vector spaces over $\mathbb{C}$. This is a pure Hodge structure of weight 2.

Choose a non-zero $\omega \in \mathrm{H}^{2,0}$. Since $\mathrm{H}^{4,0}=0$ we have $\omega \cup \omega = 0$. The complex conjugate $\overline{\omega}$ is a non-zero element of $\mathrm{H}^{0,2}$. Since the pairing

$$\mathrm{H}^{2,0} \times \mathrm{H}^{0,2} \longrightarrow \mathrm{H}^{2,2} = \mathrm{H}^4(X,\mathbb{C}) \cong \mathbb{C}$$

is non-degenerate and the cup-product is symmetric, $\omega \cup \overline{\omega}$ is a non-zero real number. We have $\omega \cup \overline{\omega} > 0$, see [Voi02, Thm. 6.32]. Since $\mathrm{H}^{3,1} = \mathrm{H}^{1,3} = 0$, we have $\mathrm{H}^{2,0} \perp \mathrm{H}^{1,1}$ and $\mathrm{H}^{0,2} \perp \mathrm{H}^{1,1}$. It is convenient to twist this Hodge structure by 1 in order to obtain a Hodge structure of weight 0:

$$\mathrm{H}^2(X,\mathbb{Z}(1))_{\mathbb{C}} = \mathrm{H}^{1,-1} \oplus \mathrm{H}^{0,0} \oplus \mathrm{H}^{-1,1}.$$

The advantage of this is that now the image of $\mathbb{S}$ lies in $\mathrm{SO}(\mathrm{H}^2(X,\mathbb{Z}))_{\mathbb{R}}$. (Twisting by 1 also means rescaling the image of the integral cohomology inside the complex cohomology by $2\pi i$.)

The Picard group of a complex K3 surface X is a free abelian group, hence $\mathrm{Pic}(X) = \mathrm{NS}(X)$. Its rank ρ is called the *Picard number* of X. The cycle class map gives an embedding

$$\mathrm{NS}(X) \hookrightarrow \mathrm{H}^2(X, \mathbb{Z}(1)).$$

We have $\mathrm{NS}(X) = \mathrm{H}^2(X,\mathbb{Z}(1)) \cap \mathrm{H}^{(0,0)}$ by the Lefschetz $(1,1)$-theorem, see [Voi02, Thm. 7.2]. This implies $1 \le \rho \le 20$. The orthogonal complement $T(X) \subset \mathrm{H}^2(X,\mathbb{Z}(1))$ to $\mathrm{NS}(X)$ is the *transcendental lattice* of X, see Section 5.4.

Definition 16.4.1 *Let M be a lattice with symmetric bilinear form (x, y). An integral Hodge structure on M is called a* **Hodge structure of K3 type***, if the Hodge decomposition is*

$$M_{\mathbb{C}} = M^{1,-1} \oplus M^{0,0} \oplus M^{-1,1},$$

where $M^{1,-1} \perp M^{0,0}$, $\dim(M^{1,-1}) = 1$ and for a non-zero $\omega \in M^{1,-1}$ we have

$$(\omega^2) = 0, \quad (\omega, \overline{\omega}) > 0.$$

Recall that L denotes the K3 lattice (16.8). Take a primitive element $\lambda \in L$ such that $(\lambda^2) > 0$. Let $d = \frac{1}{2}(\lambda^2) \in \mathbb{Z}$. By [Huy16, Cor. 14.1.10], for any $d \in \mathbb{Z}$ primitive elements $x \in L$ with $(x^2) = 2d$ exist and form an orbit of $\mathrm{Aut}(L)$. Hence the isomorphism class of the orthogonal complement $\lambda^{\perp} \subset L$ depends only on d. Thus the lattice $\lambda^{\perp}$ is isomorphic to the orthogonal direct sum

$$L_d := \mathrm{E}_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus (-2d), \tag{16.9}$$

where $(-2d)$ denotes the abelian group $\mathbb{Z}$ equipped with the quadratic form $(n, n) = -2dn^2$. The signature of L_d is $(2, 19)$. Thus a K3 surface with a primitive polarisation of degree $2d$ gives rise to an integral Hodge structure of K3 type on the lattice L_d.

Associating to an integral Hodge structure of K3 type on L_d the 1-dimensional complex subspace $\mathrm{H}^{1,-1} \subset L_{d,\mathbb{C}} := L_d \otimes_{\mathbb{Z}} \mathbb{C}$ defines a point in the *period domain*

$$\Omega_d = \{x \in \mathbb{P}(L_{d,\mathbb{C}}) \mid (x^2) = 0,\ (x, \bar{x}) > 0\},$$

see [Voi02, Thm. 7.18]. One identifies Ω_d with the Grassmannian of positive definite oriented 2-dimensional real subspaces of $L_d \otimes \mathbb{R} \simeq \mathbb{R}^{21}$, by attaching to x the plane spanned by $\mathrm{Re}(x), \mathrm{Im}(x)$ in this order. Thus

$$\Omega_d \cong \mathrm{SO}(2, 19)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R}) \times \mathrm{SO}(19)(\mathbb{R}).$$

The period domain Ω_d has two isomorphic connected components that are interchanged by complex conjugation (or reversing the orientation).

16.5 Kuga–Satake variety

Hodge structures of curves and K3 surfaces are quite different, so we cannot construct an analogue of Jacobian for K3 surfaces without more work. Nevertheless, we have the following very important result. The classical Torelli theorem can be stated as follows: the isometry class of the integral Hodge structure on $\mathrm{H}^1(C,\mathbb{Z})$, where C is a smooth and connected complex curve, uniquely determines C. The Torelli theorem for K3 surfaces of Piatetskii-Shapiro and Shafarevich [PSS71] leads to the following result: the isometry class of the integral Hodge structure on $\mathrm{H}^2(X,\mathbb{Z})$, where X is a complex K3 surface, uniquely determines X, see [Huy16, Thm. 7.5.3].

Another obstacle is that the cup-product pairing on $\mathrm{H}^2(X,\mathbb{Z})$ is symmetric, whereas for an abelian variety one would need a skew-symmetric pairing, such as that given by the cup-product on $\mathrm{H}^1(C,\mathbb{Z})$. To overcome this issue one employs the Clifford algebra of the quadratic form on $\mathrm{H}^2(X,\mathbb{Z})$.

Clifford algebra and spinor group

Let us recall the general construction of the Clifford algebra and the spinor group, see [BouIX, §9].

Let M be a finitely generated free abelian group with a non-degenerate quadratic form $q\colon M\to\mathbb{Z}$. Define the Clifford algebra $\mathrm{Cl}(M)$ as the quotient of the full tensor algebra $\oplus_{n\geq 0}M^{\otimes n}$ by the two-sided ideal I generated by the elements $x\otimes x-q(x)$ for $x\in M$. There is an isomorphism of abelian groups $\mathrm{Cl}(M)\simeq\oplus_{n=0}^{\mathrm{rk}(M)}\wedge^n M$, hence $\mathrm{rk}(\mathrm{Cl}(M))=2^{\mathrm{rk}(M)}$. Multiplication by -1 on M acts on $\oplus_{n\geq 0}M^{\otimes n}$. Since $x\otimes x-q(x)$ is invariant, we have $I=I^+\oplus I^-$, where I^+ is the subgroup of invariant elements and I^- is the subgroup of anti-invariant elements. Thus we can define

$$\mathrm{Cl}^+(M)=(\oplus_{n\geq 0}M^{\otimes 2n})/I^+,\quad \mathrm{Cl}^-(M)=(\oplus_{n\geq 0}M^{\otimes 2n+1})/I^-,$$

where the first equality is the quotient of a ring by an ideal, whereas the second one is the quotient of a (left or right) $(\oplus_{n\geq 0}M^{\otimes 2n})$-module $\oplus_{n\geq 0}M^{\otimes 2n+1}$ by the submodule I^-. The natural embedding of M into $\oplus_{n\geq 0}M^{\otimes n}$ gives rise to an injective map $M\to\mathrm{Cl}^-(M)$. Define the *Clifford group*

$$\mathrm{GSpin}(M)=\{g\in\mathrm{Cl}^+(M)^*|gMg^{-1}=M\}.$$

The group $\mathrm{GSpin}(M)$ acts by conjugation on M preserving the quadratic form. This gives an exact sequence of algebraic groups over $\mathbb{Q}$:

$$1 \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \longrightarrow \mathrm{GSpin}(M)_{\mathbb{Q}} \longrightarrow \mathrm{SO}(M)_{\mathbb{Q}} \longrightarrow 1.$$

The *adjoint* action of $\mathrm{GSpin}(M)$ on $\mathrm{Cl}^+(M)$, i.e. the action by conjugations, gives rise to a representation of $\mathrm{GSpin}(M)_{\mathbb{Q}}$, which is isomorphic to the direct sum of $\wedge^{2n} M_{\mathbb{Q}}$ for $n \geq 0$.

The *spinor group* $\mathrm{Spin}(M)_{\mathbb{Q}}$ is the algebraic group over $\mathbb{Q}$ defined by the exact sequence

$$1 \longrightarrow \mathrm{Spin}(M)_{\mathbb{Q}} \longrightarrow \mathrm{GSpin}(M)_{\mathbb{Q}} \longrightarrow \mathbb{G}_{m,\mathbb{Q}} \longrightarrow 1,$$

where the third arrow is the spinor norm.

It is instructive to consider the case of an orthogonal direct sum of n hyperbolic planes $U^{\oplus n}$, i.e. rank 2 lattices with $\mathbb{Z}$-basis e_i, f_i such that $q(e_i) = q(f_i) = 0$, $(e_i, f_i) = 1$, for $i = 1, \ldots, n$, where $(a, b) = q(a + b) - q(a) - q(b)$ is the associated bilinear form. Let Λ be the full exterior algebra of the abelian group $\mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$. Then $\mathrm{Cl}(U^{\oplus n})$ is isomorphic to $\mathrm{End}(\Lambda)$, see the proof of [BouIX, §9, no. 4, Thm. 2]. Next, $\mathrm{Cl}^+(U^{\oplus n})$ is isomorphic to $\mathrm{End}(\Lambda^+) \oplus \mathrm{End}(\Lambda^-)$, where Λ^+ and Λ^- are the even and odd parts of Λ, respectively.

From this it follows that if $\mathrm{rk}(M)$ is *even*, then $\mathrm{Cl}(M_{\mathbb{C}})$ is isomorphic to a matrix algebra. The unique simple module of this simple algebra is called the *spinor representation*. We have $\mathrm{Cl}(M_{\mathbb{C}}) = \mathrm{End}_{\mathbb{C}}(W)$. The restriction of W to $\mathrm{Cl}^+(M_{\mathbb{C}})$ splits into the direct sum of two non-isomorphic *semi-spinor representations*, so that $\mathrm{Cl}^+(M_{\mathbb{C}}) = \mathrm{End}_{\mathbb{C}}(W_1) \oplus \mathrm{End}_{\mathbb{C}}(W_2)$, where we have $W = W_1 \oplus W_2$ and $\dim(W_1) = \dim(W_2)$. The spaces W_1 and W_2 are non-isomorphic representations of $\mathrm{Spin}(M)_{\mathbb{C}}$.

If $\mathrm{rk}(M)$ is *odd*, then $\mathrm{Cl}^+(M_{\mathbb{C}})$ is isomorphic to a matrix algebra $\mathrm{End}_{\mathbb{C}}(W)$, where W is called the spinor representation. In this case the full Clifford algebra $\mathrm{Cl}(M_{\mathbb{C}})$ is the direct sum of two isomorphic matrix algebras, see [BouIX, §9, no. 4, Thm. 3]. More precisely, if $e_0, \ldots, e_{2n}$ is an orthonormal basis of q over $\mathbb{C}$, then one can choose a sign so that $\tau = \pm e_0 \ldots e_{2n}$ is in the centre of $\mathrm{Cl}(M_{\mathbb{C}})$ and $\tau^2 = 1$. Then $\mathrm{Cl}(M_{\mathbb{C}})$ is the direct sum of its two-sided ideals $\mathrm{Cl}^+(M_{\mathbb{C}})(1+\tau)$ and $\mathrm{Cl}^+(M_{\mathbb{C}})(1-\tau)$. Thus $\mathrm{Cl}(M_{\mathbb{C}})$ is isomorphic to $\mathrm{End}_{\mathbb{C}}(W)^{\oplus 2}$, that is, the two resulting representations of $\mathrm{Spin}(M)_{\mathbb{C}}$ are isomorphic to W.

Kuga–Satake construction, I

The preceding considerations can be applied to the second cohomology group of a polarised complex K3 surface X.

Fix a primitive ample class $\lambda \in \mathrm{H}^2(X, \mathbb{Z}(1))$ and define P as the orthogonal complement to λ in $\mathrm{H}^2(X, \mathbb{Z}(1))$, so that $\mathrm{rk}(P) = 21$. We have

$$P_{\mathbb{C}} = P^{1,-1} \oplus P^{0,0} \oplus P^{-1,1}.$$

Kuga and Satake [KS67] showed how to define a canonical complex structure on the real vector space $\mathrm{Cl}^+(P_{\mathbb{R}})$. We can normalise $\omega \in P^{1,-1}$ so that $(\omega, \overline{\omega}) = 2$. Write $\omega = \omega_1 + i\omega_2$, where $\omega_1, \omega_2 \in \mathrm{H}^2(X, \mathbb{R})$. Then $(\omega_1^2) = (\omega_2^2) = 1$ and $(\omega_1, \omega_2) = 0$. By the definition of the Clifford algebra, the following holds in $\mathrm{Cl}(P_{\mathbb{R}})$:

$$\omega_1^2 = \omega_2^2 = 1, \quad \omega_1\omega_2 = -\omega_2\omega_1.$$

Let $I = \omega_1\omega_2 \in \mathrm{Cl}^+(P_{\mathbb{R}})$. (It is immediate to check that I does not depend on ω.) Then $I^2 = -1$, so left multiplication by I defines a complex structure on the real vector space $\mathrm{Cl}^+(P_{\mathbb{R}})$, thus making $\mathrm{Cl}^+(P_{\mathbb{R}})/\mathrm{Cl}^+(P)$ a complex torus. It has a polarisation [Huy16, Ch. 4, 2.2], so is an abelian variety.

Kuga–Satake construction, II, d'après Deligne

In Deligne's version [Del72] one equips $\mathrm{Cl}^+(P)$ with an integral Hodge structure of type $\{(1,0),(0,1)\}$ as follows. Since $\mathbb{S}$ preserves the quadratic form on $P_{\mathbb{R}}$, we have a homomorphism $h\colon \mathbb{S} \to \mathrm{SO}(P)_{\mathbb{R}}$ whose kernel is $\{\pm 1\}$. For any $a, b \in \mathbb{R}$, not both equal to 0, we have $a + bI \in \mathrm{GSpin}(P)(\mathbb{R})$. Deligne points out that this is a canonical lifting of $h\colon \mathbb{S} \to \mathrm{SO}(P)_{\mathbb{R}}$ to $\widetilde{h}\colon \mathbb{S} \hookrightarrow \mathrm{GSpin}(P)_{\mathbb{R}}$. (Indeed, if we write $z = a + bi$, then $a + bI \in \mathrm{Cl}^+(P_{\mathbb{R}})$ and $x \mapsto (a+bI)x(a+bI)^{-1}$ acts on ω as multiplication by $z\bar{z}^{-1}$, on $\overline{\omega}$ as multiplication by $\bar{z}z^{-1}$, and on $P^{0,0} \cap P_{\mathbb{R}}$ as the identity.) This means that the adjoint action of $\mathrm{GSpin}(P_{\mathbb{Q}})$ on P induces our original Hodge structure of K3 type on P.

Lemma 16.5.1 *The left action of* $\mathrm{GSpin}(P)_{\mathbb{Q}}$ *on* $\mathrm{Cl}^+(P_{\mathbb{Q}})$ *induces an integral Hodge structure of type* $\{(1,0),(0,1)\}$ *on* $\mathrm{Cl}^+(P_{\mathbb{Q}})$*. The same is true for* $\mathrm{Cl}(P_{\mathbb{Q}})$*.*

Proof. The adjoint representation of $\mathrm{GSpin}(P)_{\mathbb{Q}}$ on $\mathrm{Cl}^+(P_{\mathbb{Q}})$ is isomorphic to the direct sum of $\wedge^{2n} P_{\mathbb{Q}}$ for $n \geq 0$. The Hodge structure on P is of K3 type, hence the induced Hodge structure on each $\wedge^{2n} P$ is of Hodge type $\{(1,-1),(0,0),(-1,1)\}$. Thus the Hodge structure on $\mathrm{Cl}^+(P)$, is also of Hodge type $\{(1,-1),(0,0),(-1,1)\}$.

The action of $\mathbb{S} \subset \mathrm{GSpin}(P)_{\mathbb{R}}$ by left multiplication induces an integral Hodge structure on $\mathrm{Cl}^+(P)$. We would like to determine its type. Note that the $\mathbb{C}$-algebra $\mathrm{Cl}^+(P_{\mathbb{C}})$ can be identified with a matrix algebra $\mathrm{End}_{\mathbb{C}}(W)$, where the complex vector space W is the unique simple module of $\mathrm{Cl}^+(P_{\mathbb{C}})$. Hence the action of $\mathrm{GSpin}(P)_{\mathbb{C}}$ on $\mathrm{Cl}^+(P_{\mathbb{C}})$ by left multiplication is isomorphic to $W^{\dim(W)}$. The adjoint representation $\mathrm{GSpin}(P)_{\mathbb{C}}$ on $\mathrm{Cl}^+(P_{\mathbb{C}})$

is isomorphic to $\mathrm{End}_{\mathbb{C}}(W) = W \otimes_{\mathbb{C}} W^*$, where $W^* = \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$. Thus the type of the Hodge structure on $\mathrm{Cl}^+(P)$ defined by left multiplication of $\mathbb{S} \subset \mathrm{GSpin}(P)_{\mathbb{R}}$ must be $\{(a, b), (b, a)\}$ with $a - b = \pm 1$, otherwise the Hodge structure on $W \otimes_{\mathbb{C}} W^*$ cannot be of type $\{(1, -1), (0, 0), (-1, 1)\}$. But $\mathbb{R}^* \subset \mathbb{C}^*$ acts on $\mathrm{Cl}^+(P_{\mathbb{C}})$ tautologically, so the weight of W is $a + b = 1$. Thus the type is $\{(1, 0), (0, 1)\}$.

The right multiplication by $x \in M$, $q(x) \neq 0$, is an isomorphism of $\mathbb{Q}$-vector spaces $\mathrm{Cl}^+(P_{\mathbb{Q}}) \to \mathrm{Cl}^-(P_{\mathbb{Q}})$ which preserves the left action of $\mathrm{GSpin}(P)_{\mathbb{Q}}$. This shows that the integral Hodge structure on $\mathrm{Cl}(P_{\mathbb{Q}})$ is of type $\{(1, 0), (0, 1)\}$. □

It can be shown that the integral Hodge structures on $\mathrm{Cl}^+(P)$ and $\mathrm{Cl}(P)$ are polarisable, so we actually obtain abelian varieties and not just complex tori. The complex abelian variety $\mathrm{Cl}^+(P_{\mathbb{R}})/\mathrm{Cl}^+(P)$ is sometimes called the *even Kuga–Satake variety* of (X, λ). The complex abelian variety $\mathrm{Cl}(P_{\mathbb{R}})/\mathrm{Cl}(P)$ is usually called the *Kuga–Satake variety* of (X, λ).

16.6 Moduli spaces of K3 surfaces and Shimura varieties

Moduli spaces of polarised K3 surfaces

Let $O(L_d)$ be the orthogonal group of the lattice L_d defined in (16.9) with associated period domain Ω_d. Write $L_d^* = \mathrm{Hom}(L_d, \mathbb{Z})$. We have a natural injective map $L_d \to L_d^*$. Its cokernel is the discriminant group of L_d. Define

$$\widetilde{O}(L_d) = \{g \in O(L_d) \mid g \text{ acts trivially on } L_d^*/L_d \simeq \mathbb{Z}/2d\}.$$

Equivalently, $\widetilde{O}(L_d)$ is the stabiliser of λ in $O(L)$. The key (difficult) facts are:

(1) $\widetilde{O}(L_d) \backslash \Omega_d$ is a quasi-projective irreducible variety over $\mathbb{C}$ (Baily–Borel);
(2) there is a coarse moduli space M_d of K3 surfaces with a primitive polarisation of degree $2d$;
(3) M_d is a Zariski open subscheme of $\widetilde{O}(L_d) \backslash \Omega_d$.

Note that M_d is not smooth, though it is smooth as an orbifold (or as a Deligne–Mumford stack). It is constructed as a categorical quotient of the open subscheme of the relevant Hilbert scheme parameterising K3 surfaces in a given projective space by the action of the projective linear group. Fact (3) uses local and global Torelli theorems, and surjectivity of the period map, see [Huy16, Cor. 6.4.3]. This description is a K3 analogue of the coarse moduli space of elliptic curves $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathcal{H}$ or the moduli space of dimension g principally polarised abelian varieties $\mathcal{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathcal{H}_g$, where $\mathcal{H}$ is the usual upper half-plane and $\mathcal{H}_g$ is the Siegel upper half-plane. Here Ω_d, $\mathcal{H}$, $\mathcal{H}_g$ are Hermitian symmetric domains, so property (1) follows from the Baily–Borel theorem about quotients of Hermitian symmetric domains by torsion-free arithmetic subgroups of their automorphism groups. (One needs to first apply the Baily–Borel theorem to a torsion-free finite index subgroup of $\widetilde{O}(L_d)$, and then take a quotient of a variety by a finite group action.)

Replacing $O(L_d)$ by the index 2 subgroup $\mathrm{SO}(L_d)$ gives rise to an unramified cover $\widetilde{M}_d \to M_d$. Here $\widetilde{M}_d$ is a Zariski open subset of $\widetilde{\mathrm{SO}(L_d)}\backslash \Omega_d$, where $\widetilde{\mathrm{SO}(L_d)} = \mathrm{SO}(L_d) \cap \widetilde{O}(L_d)$. This replaces the non-connected orthogonal group by the connected special orthogonal group. The degree of $\widetilde{M}_d \to M_d$ is 2 unless $d = 1$. In the exceptional case $d = 1$ the group $O(L_1) = \widetilde{O}(L_1)$ contains -1 which acts trivially on Ω_1, hence $O(L_1) = \{\pm 1\} \times \mathrm{SO}(L_1)$ and thus $\widetilde{M}_1 \to M_1$ is an isomorphism.

We have seen that a point in Ω_d is a homomorphism $\mathbb{S} \to \mathrm{SO}(L_d)_{\mathbb{R}}$. The action of $\mathrm{SO}(L_d)(\mathbb{R})$ on Ω_d is transitive, so Ω_d can be identified with the conjugacy class of h in $\mathrm{Hom}(\mathbb{S}, \mathrm{SO}(L_d)(\mathbb{R}))$. This is similar to the classical identification of $\mathcal{H} = \mathrm{SL}(2)(\mathbb{R})/\mathrm{SO}(2)(\mathbb{R})$ with the conjugacy class of $\mathbb{S} \subset \mathrm{GL}(2)^+_{\mathbb{R}}$. (Here $\mathrm{GL}(2)^+_{\mathbb{R}}$ is given by the condition $\det(x) > 0$; note that $\mathrm{GL}(2)(\mathbb{R})/\mathbb{S} = \mathcal{H}^{\pm}$.)

In modern language $\mathcal{A}_g$ and $\widetilde{\mathrm{SO}(L_d)}\backslash \Omega_d$ are the sets of complex points of Shimura varieties. To exploit the connection between moduli spaces of primitively polarised K3 surfaces and Shimura varieties, we now give a very brief introduction to Shimura varieties, referring the reader to Deligne's foundational paper [Del79] and Milne's lecture notes [Mil05] for a systematic treatment.

Orthogonal Shimura varieties

A *Shimura datum* is a pair (G, X), where G is a connected reductive algebraic group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}, G(\mathbb{R}))$ satisfying certain axioms ensuring that each connected component of X is a Hermitian symmetric domain. Morphisms of Shimura data are defined in the obvious way. In the K3 case, let $\mathrm{SO}(L_d)$ be the group scheme over $\mathbb{Z}$ whose functor of points associates to a ring R the group $\mathrm{SO}(L_d \otimes_{\mathbb{Z}} R)$. Then $(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$ is a Shimura datum. In the case of principally polarised abelian varieties the Shimura datum is $(\mathrm{GSp}_{2g,\mathbb{Q}}, \mathcal{H}^{\pm}_g)$.

A *congruence subgroup* is a subgroup of $G(\mathbb{Q})$ cut out by a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$, where $\mathbf{A}_{\mathbb{Q},\mathrm{f}} = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the ring of finite adèles. Deligne's definition of the Shimura variety defined by the Shimura datum (G, X) and a compact open subgroup $K \subset G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ is

$$\mathrm{Sh}_K(G, X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})/K,$$

where $G(\mathbb{Q})$ acts diagonally on both factors on the left, whereas K acts on $G(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ on the right. The crucial fact is that any Shimura variety $\mathrm{Sh}_K(G, X)$

descends to a certain natural variety over a number field; it is the so-called *canonical model*. The set $\mathrm{Sh}_K(G,X)(\mathbb{C})$ is a disjoint union of the quotients $\Gamma\backslash X_+$, where X_+ is a connected component of X and Γ is a congruence subgroup of the stabiliser of X_+ in $G(\mathbb{Q})$.

We now go back to the K3 surfaces Shimura datum $(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d)$. Let $\mathbb{K} \subset \mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ be a compact open subgroup. The canonical model of the associated Shimura variety $\mathrm{Sh}_K(L_d) := \mathrm{Sh}_{\mathbb{K}}(\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_L)$ is a quasi-projective variety over $\mathbb{Q}$. By construction, the $\mathbb{C}$-points of $\mathrm{Sh}_K(L_d)$ parameterise $\mathbb{Z}$-Hodge structures on L_d of K3 type, see Definition 16.4.1.

Suppose that $\mathbb{K}$ is neat. (See R. Pink's thesis [Pin, pp. 4–5] for the definition of neatness and the fact that every compact open subgroup of $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ contains a neat subgroup of finite index.) Then for each prime ℓ there is a lisse $\mathbb{Z}_\ell$-sheaf $L_{d,\ell}$ on $\mathrm{Sh}_K(L_d)$ defined by the inverse system of finite étale covers $\mathrm{Sh}_{\mathbb{K}(\ell^m)}(L_d)\to\mathrm{Sh}_K(L_d)$, where $\mathbb{K}(\ell^m)$ is the largest subgroup of $\mathbb{K}$ that acts trivially on L/ℓ^m. Thus, to a k-point x of $\mathrm{Sh}_K(L_d)$ there corresponds a representation $\mathrm{Gal}(\bar{k}/k)\to\mathrm{SO}(L_d\otimes_{\mathbb{Z}}\mathbb{Z}_\ell)$. Putting together these representations for all ℓ gives a representation

$$\phi_x\colon \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}). \tag{16.10}$$

We refer to this as the *monodromy representation*.

Spin Shimura varieties

From a lattice with signature $(2,n)$, $n \geq 1$, one can also construct a spin Shimura variety. Recall that $\mathrm{Cl}(L_d)$ is the Clifford algebra of L_d, and $\mathrm{Cl}^+(L_d) \subset \mathrm{Cl}(L_d)$ is the even Clifford algebra. Let $\mathrm{GSpin}(L_d)$ be the $\mathbb{Z}$-group scheme whose functor of points associates to a commutative ring R the group of invertible elements g of $\mathrm{Cl}^+(L_d\otimes_{\mathbb{Z}}R)$ such that $g(L_d\otimes_{\mathbb{Z}}R)g^{-1} = L_d\otimes_{\mathbb{Z}}R$.

Recall that $h\colon \mathbb{S}\to\mathrm{SO}(L_d)_{\mathbb{R}}$ canonically lifts to $\widetilde{h}\colon \mathbb{S}\to\mathrm{GSpin}(L_d)_{\mathbb{R}}$. It follows that the $\mathrm{GSpin}(L_d)(\mathbb{R})$-conjugacy class of $\widetilde{h}\colon \mathbb{S}\to\mathrm{GSpin}(L_d)_{\mathbb{R}}$ maps bijectively to Ω_d, which is the $\mathrm{SO}(L_d)(\mathbb{R})$-conjugacy class of h. This shows that the homomorphism $\mathrm{GSpin}(L_d)\to\mathrm{SO}(L_d)$ naturally extends to a morphism of Shimura data

$$(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d) \longrightarrow (\mathrm{SO}(L_d)_{\mathbb{Q}}, \Omega_d).$$

If $\widetilde{\mathbb{K}} \subset \mathrm{GSpin}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$ is a compact open subgroup, we write $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ for the Shimura variety $\mathrm{Sh}_{\widetilde{\mathbb{K}}}(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d)$. We can take $\mathbb{K}$ to be the image of $\widetilde{\mathbb{K}}$ in $\mathrm{SO}(L)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$; indeed, by [And96, 4.4] this image is compact and open in $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$. The natural group homomorphism $\mathrm{GSpin}(L_d)_{\mathbb{Q}}\to\mathrm{SO}(L_d)_{\mathbb{Q}}$ induces a morphism $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)\to\mathrm{Sh}_{\mathbb{K}}(L_d)$. This morphism is finite and surjective, and is defined over $\mathbb{Q}$, see [And96, App. 1].

For a positive integer N let $\widetilde{\mathbb{K}}_N$ be the subgroup of $\mathrm{GSpin}(L_d)(\widehat{\mathbb{Z}})$ consisting of the elements of $\mathrm{GSpin}(L_d)(\widehat{\mathbb{Z}})$ that are congruent to 1 modulo N

in $\mathrm{Cl}^+(L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$. If $\widetilde{\mathbb{K}} \subset \widetilde{\mathbb{K}}_N$ for $N \geq 3$, then $\widetilde{\mathbb{K}}$ and $\mathbb{K}$ are neat and the morphism $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d) \to \mathrm{Sh}_{\mathbb{K}}(L_d)$ is étale. This morphism restricts to an isomorphism on each geometric connected component [Riz10, §5.5, (32)]. Thus $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d) \to \mathrm{Sh}_{\mathbb{K}}(L_d)$ has a section defined over a number field E which only depends on $\widetilde{\mathbb{K}}$.

Kuga–Satake construction, III: the Kuga–Satake abelian scheme

By Lemma 16.5.1, left action of $\mathrm{GSpin}(L_d)_{\mathbb{Q}}$ on $\mathrm{Cl}(L_{d,\mathbb{Q}})$ gives rise to an integral Hodge structure of type $\{(1,0),(0,1)\}$ on $\mathrm{Cl}(L_d)$. The choice of a polarisation of this Hodge structure defines a morphism of Shimura data

$$(\mathrm{GSpin}(L_d)_{\mathbb{Q}}, \Omega_d) \longrightarrow (\mathrm{GSp}_{2g,\mathbb{Q}}, \mathcal{H}_g^{\pm}),$$

where $g = 2^{20}$. Moreover, there is a finite morphism of Shimura varieties from $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ to a moduli space of abelian varieties, defined over $\mathbb{Q}$. In order to construct this, we find a skew-symmetric form on $\mathrm{Cl}(L_d)$ following [Huy16, Ch. 4, 2.2]. For this we choose orthogonal elements $f_1, f_2 \in L_d$ satisfying $(f_1^2), (f_2^2) > 0$ and define a skew-symmetric form on $\mathrm{Cl}(L_d)$ by $\pm\mathrm{Tr}(f_1 f_2 v^* w)$, where $\mathrm{Tr}(x)$ is the trace of the left multiplication by $x \in \mathrm{Cl}(L_d)$ on $\mathrm{Cl}(L_d)$. The action of $\mathrm{GSpin}(L_d)$ on this form is multiplication by the spinor norm (see [Huy16, Ch. 4, Prop. 2.5] for proofs of these facts, as well as the correct choice of sign). The group $\mathrm{GSpin}(L_d)$ injects into the group of symplectic similitudes $\mathrm{GSp}(\mathrm{Cl}(L_d))$ of this form.

If $\widetilde{\mathbb{K}} \subset \widetilde{\mathbb{K}}_N$, then we have a morphism from $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$ to the Shimura variety $\mathrm{Sh}_{\Gamma_N}(\mathrm{GSp}(\mathrm{Cl}(L_d))_{\mathbb{Q}}, \mathcal{H}^{\pm})$, where Γ_N is the subgroup of $\mathrm{GSp}(\mathrm{Cl}(L_d))(\widehat{\mathbb{Z}})$ consisting of the elements that are congruent to 1 modulo N. The latter Shimura variety is identified with the moduli variety $\mathcal{A}_{g,\delta,N}$ parameterising abelian varieties of dimension $g = 2^{n+1}$, polarisation type δ (explicitly computable in terms of L and f_1, f_2) and level structure of level N. If $N \geq 3$, then $\mathcal{A}_{g,\delta,N}$ is a fine moduli space carrying a universal family of abelian varieties.

Recall that E is a number field over which there exists a section of the morphism of Shimura varieties $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)_E \to \mathrm{Sh}_{\mathbb{K}}(L_d)_E$. The definition of the Kuga–Satake abelian scheme depends on the choice of E.

Definition 16.6.1 *The* **Kuga–Satake abelian scheme** $f\colon A \to \mathrm{Sh}_{\mathbb{K}}(L_d)_E$ *is defined as the pullback of the universal family of abelian varieties on* $\mathcal{A}_{g,\delta,N}$ *to* $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}}(L_d)$*, and then, after extending the ground field from* $\mathbb{Q}$ *to* E*, to* $\mathrm{Sh}_{\mathbb{K}}(L_d)_E$*.*

The left multiplication by the elements of $L_d \subset \mathrm{Cl}(L_d)$ on $\mathrm{Cl}(L_d)$ gives a homomorphism $L_d \hookrightarrow \mathrm{End}_{\mathbb{Z}}(\mathrm{Cl}(L_d))$ whose cokernel is torsion-free. Since $\mathrm{Cl}(L_d) = R^1 f_{\mathrm{an},*}\mathbb{Z}$ as sheaves on $\mathrm{Sh}_{\mathbb{K}}(L_d)_{\mathbb{C}}$, this gives rise to a morphism of variations of $\mathbb{Z}$-Hodge structures

$$L_d \hookrightarrow \mathrm{Cl}(L_d) \hookrightarrow \mathrm{End}_{\mathbb{Z}}(R^1 f_{\mathrm{an},*}\mathbb{Z}). \tag{16.11}$$

Via the comparison theorems we get a morphism of $\mathbb{Z}_\ell$-sheaves

$$L_{d,\ell} \hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(R^1 f_*\mathbb{Z}_\ell). \tag{16.12}$$

Back to moduli spaces of K3 surfaces

Recall that M_d introduced in the beginning of this section is the coarse moduli space over $\mathbb{Q}$ of primitively polarised K3 surfaces of degree $2d$; this is a quasi-projective variety defined over $\mathbb{Q}$. Let $\widetilde{M}_d$ be the coarse moduli space over $\mathbb{Q}$ of triples (X, λ, u) such that X is a K3 surface over a field of characteristic zero, λ is a primitive polarisation of X of degree $2d$, and u is an isometry

$$\det(P^2(\overline{X}, \mathbb{Z}_2(1))) \longrightarrow \det(L_d \otimes_{\mathbb{Z}} \mathbb{Z}_2),$$

where $P^2(\overline{X}, \mathbb{Z}_2(1))$ is the orthogonal complement of the image of λ in the 2-adic étale cohomology $\mathrm{H}^2(\overline{X}, \mathbb{Z}_2(1))$. We have an unramified cover $\widetilde{M}_d \to M_d$ (of degree 2 unless $d = 1$, when this is an isomorphism). By the work of Rizov and Madapusi Pera based on the Torelli theorem [PSS71], there is an open immersion $\widetilde{M}_d \hookrightarrow \mathrm{Sh}_{\mathbb{K}_d}(L_d)$ defined over $\mathbb{Q}$, where

$$\mathbb{K}_d = \{g \in \mathrm{SO}(L_d \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) : g \text{ acts trivially on } L_d^*/L_d\}. \tag{16.13}$$

For a proof that this immersion is defined over $\mathbb{Q}$, see [MP15, Cor. 5.4] (see also [Riz10, Thm. 3.9.1]).

To a polarised K3 surface (X, λ) defined over a field k of characteristic zero one can attach two Galois representations: the representation in étale cohomology and the monodromy representation. The first of them comes from the natural action of the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$ on $\mathrm{H}^2_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))$. For a prime ℓ define $P^2(\overline{X}, \mathbb{Z}_\ell(1))$ as the orthogonal complement to λ in $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$. Choose an isometry $u\colon \det(P^2(\overline{X}, \mathbb{Z}_2(1))) \xrightarrow{\sim} \det(L_d \otimes_{\mathbb{Z}} \mathbb{Z}_2)$. After replacing k by a quadratic extension we can assume that Γ acts trivially on $\det(P^2(\overline{X}, \mathbb{Z}_2(1)))$. By [Sai12, Cor. 3.3] the quadratic character through which Γ acts on the 1-dimensional vector space $\det(\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1)))$ does not depend on ℓ. Thus Γ acts trivially on $\det(P^2(\overline{X}, \mathbb{Z}_\ell(1)))$ for all primes ℓ, hence the representation $\rho_X : \Gamma \to \mathrm{O}(P^2(\overline{X}, \widehat{\mathbb{Z}}(1)))$ attached to X takes values in $\mathrm{SO}(P^2(\overline{X}, \widehat{\mathbb{Z}}(1)))$.

The triple (X,λ,u) defines a k-point x in $\widetilde{M}_{2d}\subset \mathrm{Sh}_{\mathbb{K}_d}(L_d)$. Choose a neat compact open subgroup $\mathbb{K}'_d\subset\mathbb{K}_d$ and let x' be a lift of x to $\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$, so that x' is defined over a finite extension k' of k. Let $\Gamma'=\mathrm{Gal}(\bar{k}/k')$ and let $\phi_{x'}\colon \Gamma'\to\mathrm{SO}(L_d\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})$ denote the monodromy representation associated with the point x', as defined at (16.10).

Lemma 16.6.2 *Let (X,λ) be a primitively polarised K3 surface over a field k of characteristic zero. There exists a finite extension k'/k of explicitly bounded degree such that the adelic Galois representations*

$$\rho_{X|\Gamma'}\colon \Gamma'\to\mathrm{SO}(P^2(\overline{X},\widehat{\mathbb{Z}}(1)))\ \textit{and}\ \phi_{x'}\colon \Gamma'\to\mathrm{SO}(L_d\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})$$

are isometric, where $\Gamma'=\mathrm{Gal}(\bar{k}/k')$.

Proof. This is an immediate consequence of [MP16, Prop. 5.6 (1)]. □

The conclusion of the work done in this section is the following proposition.

Proposition 16.6.3 *Let k be a field of characteristic zero, and let (X,λ) be a primitively polarised K3 surface over k. Let $P^2(\overline{X},\mathbb{Z}_\ell(1))$ be the orthogonal complement to λ in $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$, where ℓ is a prime. There exists a finite extension k'/k and an abelian variety A over k' with the following properties.*

(i) *Let $\Gamma'=\mathrm{Gal}(\bar{k}/k')$. For any prime ℓ there is an embedding of Γ'-modules:*

$$P^2(\overline{X},\mathbb{Z}_\ell(1))\hookrightarrow \mathrm{End}_{\mathbb{Z}_\ell}(\mathrm{H}^1_{\text{ét}}(\overline{A},\mathbb{Z}_\ell)). \tag{16.14}$$

(ii) *Let $k\subset\mathbb{C}$, and let $P^2(X_{\mathbb{C}},\mathbb{Z}(1))$ be the orthogonal complement to λ in $\mathrm{H}^2(X_{\mathbb{C}},\mathbb{Z}(1))$. There is an embedding of integral Hodge structures of weight 0:*

$$P^2(X_{\mathbb{C}},\mathbb{Z}(1))\hookrightarrow \mathrm{End}_{\mathbb{Z}}(\mathrm{H}^1(A_{\mathbb{C}},\mathbb{Z})). \tag{16.15}$$

The two embeddings are compatible via comparison isomorphisms between classical and ℓ-adic étale cohomology.

Proof. Let x be the k-point in M_d defined by (X,λ). After replacing k by a quadratic extension k'/k we can assume that x lifts to a k'-point on $\widetilde{M}_d\hookrightarrow \mathrm{Sh}_{\mathbb{K}_d}(L_d)$, where $\mathbb{K}_d$ is defined in (16.13). Choose a neat compact open subgroup $\widetilde{\mathbb{K}}_d$ in $\mathrm{GSpin}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$, for example, the subgroup of $\mathrm{GSpin}(L_d)(\hat{\mathbb{Z}})$ consisting of the elements of $\mathrm{GSpin}(L_d)(\hat{\mathbb{Z}})$ congruent to 1 modulo N in $\mathrm{Cl}^+(L_d\otimes\widehat{\mathbb{Z}})$, where $N\geq 3$. Let $\mathbb{K}'_d$ be the intersection of $\mathbb{K}_d$ with the image of $\widetilde{\mathbb{K}}_d$ in $\mathrm{SO}(L_d)(\mathbf{A}_{\mathbb{Q},\mathrm{f}})$. Then $\mathbb{K}'_d$ is a neat compact open subgroup of $\mathbb{K}_d$. We enlarge k' so that x comes from a k'-point s on the cover $\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$ of $\mathrm{Sh}_{\mathbb{K}_d}(L_d)$. We extend k' further to include the number field E over which there is a section of the morphism of Shimura varieties $\mathrm{Sh}^{\mathrm{spin}}_{\widetilde{\mathbb{K}}_d}(L_d)_E\to\mathrm{Sh}_{\mathbb{K}'_d}(L_d)_E$. Now we have the Kuga–Satake abelian scheme $f\colon A\to\mathrm{Sh}_{\mathbb{K}'_d}(L_d)$, so $A=f^{-1}(s)$ is an abelian variety over k'. Now (16.14) is just the specialisation of (16.12) at the k'-point s. Similarly, (16.15) is the specialisation of (16.11). □

Given a polarised K3 surface X over k we can call an abelian variety A from Proposition 16.6.3 a *Kuga–Satake variety* of X. Indeed, for $k \subset \mathbb{C}$, by construction $A_{\mathbb{C}}$ is isomorphic to the complex Kuga–Satake variety of the complex K3 surface $X_{\mathbb{C}}$ as defined at the end of the previous section. What we gain now is that A is defined over a finite extension of k.

It is worth noting that Lemma 16.6.2 replaces Proposition 6.4 and Lemma 6.5.1 in Deligne's pioneering work [Del72] (written before the machinery of Shimura varieties was fully developed) in establishing that (16.14) is an isomorphism of Galois modules, cf. [Del72, Prop. 6.5].

16.7 Tate conjecture and Brauer group of K3 surfaces

We continue the discussion of the previous section using the same assumptions and notation. By Proposition 16.6.3 a primitively polarised K3 surface X has a Kuga–Satake abelian variety A defined over a finite extension k' of k such that there is an embedding of $\mathrm{Gal}(\bar{k}/k')$-modules

$$P^2(\overline{X}, \mathbb{Q}_\ell(1)) \hookrightarrow \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1_{\text{ét}}(A \times_{k'} \bar{k}, \mathbb{Q}_\ell)). \tag{16.16}$$

Deligne used this to prove the Weil conjectures for K3 surfaces over finite fields (before he proved them for arbitrary varieties), but this theory has many other applications. For example, if k is finitely generated over $\mathbb{Q}$, then the semisimplicity of the Galois module $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ for a K3 surface X follows from (16.16) and semisimplicity for abelian varieties established by Faltings.

The Tate conjecture for divisors for K3 surfaces over finitely generated fields of characteristic zero was proved by S.G. Tankeev [Tan88] and Y. André [And96].

Theorem 16.7.1 (Tankeev, André) *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then the Tate conjecture holds for X, that is, we have*

$$\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))^{\Gamma} = \mathrm{NS}(\overline{X})^{\Gamma} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Proof. Let A be a Kuga–Satake abelian variety of X defined over a finite field extension k' of k, as constructed in Proposition 16.6.3.

The profinite, hence compact group Γ acts continuously on the discrete group $\mathrm{NS}(\overline{X})$, so by extending k' we can assume that this action factors through a finite quotient $\mathrm{Gal}(k'/k)$ of Γ, where k' is a finite Galois extension of k. Thus it is enough to prove the theorem under the additional assumption that Γ acts trivially on $\mathrm{NS}(\overline{X})$ and on $\mathrm{End}(\overline{A})$. We need to show that the Γ-invariant subspace of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ is $\mathrm{NS}(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$.

By a theorem of Faltings, the Tate conjecture holds for A:

$$\big(\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1_{\text{ét}}(\overline{A},\mathbb{Q}_\ell))\big)^{\Gamma}=\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell.$$

Hence the image of $P^2(\overline{X},\mathbb{Q}_\ell(1))^{\Gamma}$ in $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{H}^1_{\text{ét}}(\overline{A},\mathbb{Q}_\ell))$ belongs to the $\mathbb{Q}_\ell$-span of the intersection of the image of $P^2(X_{\mathbb{C}},\mathbb{Q}(1))$ in $\mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}},\mathbb{Q}))$ with $\mathrm{End}(\overline{A})\otimes_{\mathbb{Z}}\mathbb{Q}\subset\mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}},\mathbb{Q}))$. Such elements of $\mathrm{End}_{\mathbb{Q}}(\mathrm{H}^1(A_{\mathbb{C}},\mathbb{Q}))$ have Hodge type $(0,0)$. Hence every element of $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Q}_\ell(1))^{\Gamma}$ is a $\mathbb{Q}_\ell$-linear combination of classes of type $(0,0)$ in $\mathrm{H}^2(X_{\mathbb{C}},\mathbb{Q}(1))$. By the Lefschetz theorem, each such class is algebraic. □

The following result was obtained in [SZ08], using Deligne's version of the Kuga–Satake construction [Del72].

Theorem 16.7.2 (Skorobogatov–Zarhin) *Let X be a K3 surface over a field k finitely generated over $\mathbb{Q}$. Then $\mathrm{Br}(\overline{X})^{\Gamma}$ is finite.*

Proof. The ℓ-primary torsion subgroup $\mathrm{Br}(\overline{X})^{\Gamma}\{\ell\}$ is finite for all primes ℓ as follows from Theorems 16.1.1 and 16.7.1. It remains to prove that $\mathrm{Br}(\overline{X})^{\Gamma}[\ell]=0$ for almost all ℓ. By Theorem 5.3.1 (iv) it is enough to show that for almost all ℓ we have $\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_\ell)^{\Gamma}=(\mathrm{NS}(\overline{X})/\ell)^{\Gamma}$.

Fixing an embedding of k into $\mathbb{C}$ we define the transcendental lattice $T(X_{\mathbb{C}})$ as the orthogonal complement to $\mathrm{NS}(X_{\mathbb{C}})$ in $\mathrm{H}^2(X_{\mathbb{C}},\mathbb{Z}(1))$ with respect to the cup-product pairing. As was discussed in Section 5.4, by the comparison theorem between classical and étale cohomology, we have isomorphisms

$$\mathrm{H}^2(X_{\mathbb{C}},\mathbb{Z})\otimes\mathbb{Z}_\ell(1)\cong\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$$

compatible with the cycle class map and the cup-product. Thus $T(X_{\mathbb{C}})\otimes\mathbb{Z}_\ell$ is the orthogonal complement to $\mathrm{NS}(\overline{X})\otimes\mathbb{Z}_\ell$ in $\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))$, so $T(X_{\mathbb{C}})\otimes\mathbb{Z}_\ell$ has a natural structure of a Γ-module. As in Theorem 5.3.1, for the primes ℓ not dividing the discriminant of the intersection pairing on $\mathrm{NS}(\overline{X})$ we have a direct sum decomposition of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X},\mathbb{Z}_\ell(1))=\big(\mathrm{NS}(\overline{X})\otimes\mathbb{Z}_\ell\big)\oplus\big(T(X_{\mathbb{C}})\otimes\mathbb{Z}_\ell\big).$$

These are free $\mathbb{Z}_\ell$-modules of finite rank, hence for these ℓ we have a direct sum decomposition of Γ-modules

$$\mathrm{H}^2_{\text{ét}}(\overline{X},\mu_\ell)=\big(\mathrm{NS}(\overline{X})/\ell\big)\oplus\big(T(X_{\mathbb{C}})/\ell\big).$$

Thus we need to prove that $(T(X_{\mathbb{C}})/\ell)^{\Gamma}=0$ for almost all ℓ. Let A be a Kuga–Satake abelian variety of X defined over a finite field extension k' of k, as in Proposition 16.6.3. As in the previous proof we assume that Γ acts trivially on $\mathrm{End}(\overline{A})$, so that $\mathrm{End}(\overline{A})=\mathrm{End}(A)=\mathrm{End}(A_{\mathbb{C}})$. By the Lefschetz theorem and the non-degeneracy of the intersection pairing on $\mathrm{NS}(\overline{X})$, the transcendental lattice $T(X_{\mathbb{C}})$ does not contain non-zero elements of Hodge type $(0,0)$. Hence the image of $T(X_{\mathbb{C}})$ in $\mathrm{End}_{\mathbb{Z}}(\mathrm{H}^1(A_{\mathbb{C}},\mathbb{Z}))$ has trivial intersection with $\mathrm{End}(A_{\mathbb{C}})$. It follows that for almost all ℓ the image of $T(X_{\mathbb{C}})/\ell$ in $\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])$ intersects trivially with $\mathrm{End}(\overline{A})/\ell = \mathrm{End}(A)/\ell$. By Faltings and Zarhin, for almost all ℓ we have

$$\mathrm{End}_{\mathbb{F}_\ell}(A[\ell])^{\Gamma} = \mathrm{End}(A)/\ell.$$

Thus $(T(X_{\mathbb{C}})/\ell)^{\Gamma} = 0$ for almost all ℓ. □

Theorem 16.7.2 and Corollary 5.5.3 then give

Corollary 16.7.3 *Let X be a K3 surface over a field k that is finitely generated over $\mathbb{Q}$. The group $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite.*

Remark 16.7.4 Let k be a field of characteristic $p > 0$ which is finitely generated over $\mathbb{F}_p$. Then the subgroups of $\mathrm{Br}(\overline{X})^{\Gamma}$ and $\mathrm{Br}(X)/\mathrm{Br}_0(X)$, which consist of the elements of order prime to p, are both finite. For $p > 2$ this is proved in [SZ15] using work of Rizov and Madapusi Pera [MP15], and Zarhin [Zar76, Zar77, Zar85]. For $p = 2$ this is proved by K. Ito in [Ito18] using instead of [MP15] a more recent work of W. Kim and Madapusi Pera proving the Tate conjecture and essentially establishing the Kuga–Satake construction in characteristic 2.

The following stronger variant of Theorem 16.7.2 was obtained by M. Orr and Skorobogatov, see [OS18, Thm. C].

Theorem 16.7.5 *Let X be a K3 surface over a field k that is finitely generated over $\mathbb{Q}$. Let n be a positive integer. There exists a constant $C_{n,X}$ depending only on n and X such that for every K3 surface Y defined over a field $K \subset \bar{k}$ of degree $[K : k] \leq n$ such that $Y \times_K \bar{k} \simeq \overline{X}$ we have $|\mathrm{Br}(\overline{Y})^{\mathrm{Gal}(\bar{k}/K)}| < C_{n,X}$.*

The main ingredients of the proof are the results of Cadoret and Moonen on the Mumford–Tate conjecture, which build on the previous work of many authors, including Serre, Wintenberger, Larsen, Pink, Cadoret and Kret, and the proof of the Mumford–Tate conjecture for K3 surfaces by Tankeev and André. Although the finiteness of $\mathrm{Br}(\overline{Y})^{\mathrm{Gal}(\bar{k}/K)}$ in Theorem 16.7.5 follows from Theorem 16.7.2, it is not used in the proof, so Theorem 16.7.5 also gives a different approach to the finiteness result of Theorem 16.7.2.

16.8 Diagonal surfaces

The aim of this section is to illustrate general methods for calculating the Brauer group of a variety over a number field in the particular case of surfaces in $\mathbb{P}^3_k$ given by diagonal equations, with focus on degree 4.

Let k be a field, $\mathrm{char}(k) \neq 2$. Consider smooth surfaces $X \subset \mathbb{P}^3_k$ given by

$$x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4 = 0,$$

where $a_1, a_2, a_3 \in k^*$. The group $\mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ was computed by Bright in his thesis [Bri02, Bri06] using the fact that the 48 lines contained in X^{s} generate the abelian group $\mathrm{Pic}(X^{\mathrm{s}})$. (In fact, one can choose 20 lines that freely generate $\mathrm{Pic}(X^{\mathrm{s}})$, see [GvS].) When k is a number field, we have $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ by Remark 5.4.3 (3). An explicit form of the elements of $\mathrm{Br}_1(X)$ is not known in general, cf. [Bri11]. When $k \subset \mathbb{C}$, Ieronymou [Ier10, Thm. 3.1] constructed generators of $\mathrm{Br}(X_{\mathbb{C}})[2]$ and showed [Ier10, Prop. 4.9] that they come from 4-torsion elements of the Brauer group of the quartic Fermat surface over $\mathbb{Q}(i, \sqrt[4]{2})$ (in fact, they come from 2-torsion elements, see [GvS, Cor. 3.5]). He then obtained sufficient conditions on the diagonal quartic surface X over $\mathbb{Q}$ under which the 2-torsion subgroup of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is trivial [Ier10, Thm. 5.2]. See [GvS] for further references.

Over a number field k, computation of $\mathrm{Br}(X)$ can proceed in the following steps.

(a) Determine the action of $\Gamma = \mathrm{Gal}(k^{\mathrm{s}}/k)$ on the geometric Brauer group $\mathrm{Br}(\overline{X})$, hence compute $\mathrm{Br}(\overline{X})^{\Gamma}$. For a K3 surface with complex multiplication this involves identifying the Grössencharakter which describes the action of Γ on the Tate module of $\mathrm{Br}(\overline{X})$. For diagonal quartic surfaces, this has essentially been done by Pinch and Swinnerton-Dyer in [PSD91] using classical work of Weil [Weil52].

(b) Determine the image of $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})^{\Gamma}$, that is, the transcendental Brauer group. This uses a geometric description of the differentials given in Section 5.4.2, see Proposition 5.4.10.

(c) The group $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{H}^1(k, \mathrm{Pic}(\overline{X}))$ is finite; it can be determined if we know a finite generating set of $\mathrm{Pic}(\overline{X})$ and the action of Γ on it. The group $\mathrm{Br}(X)/\mathrm{Br}_0(X)$, an extension of the finite abelian group $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ by the finite abelian group $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$, is computed as the first Galois hypercohomology group with coefficients in the complex (5.27). See Remark 5.4.11.

The additional tool that we have in the case of diagonal surfaces is the description of the primitive complex cohomology of the Fermat hypersurface of degree d obtained by F. Pham and further developed by Looijenga. The idea of Pham is to consider a natural "vanishing cycle" in the complement to a hyperplane section of the Fermat hypersurface and show that its orbit under the action of diagonal automorphisms generates the primitive integral homology group. This allows one to compute the cup-product bilinear form and the Hodge decomposition. The Galois representation in the étale cohomology of the Fermat hypersurface was studied and computed by Weil, Katz, Shioda, and Ulmer, among others, see the references in [GvS].

To state the main result of this section, we call two diagonal quartic forms $\sum_{i=0}^{3} a_ix_i^4$, where $a_i \in k^*$, *equivalent* if one is obtained from another by permuting the variables x_0, x_1, x_2, x_3, multiplying the coefficients a_i by fourth powers in k^*, and multiplying all four coefficients by a common multiple in k^*. Diagonal surfaces given by equivalent forms will be called equivalent. It is clear that equivalent surfaces are isomorphic.

The following theorem gives a classification of the transcendental Brauer groups of diagonal quartic surfaces with coefficients in $\mathbb{Q}$ over the ground fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$. It was proved in [IS15, Thm. 1.1] in the case of torsion of odd order. A general method was given in [GvS].

Theorem 16.8.1 *Let $a_1, a_2, a_3 \in \mathbb{Q}^*$ and let $X \subset \mathbb{P}^3_{\mathbb{Q}}$ be the surface*

$$x_0^4 + a_1x_1^4 + a_2x_2^4 + a_3x_3^4 = 0.$$

(i) *Let $K = \mathbb{Q}(\sqrt{-1})$. If $X_K = X \times_{\mathbb{Q}} K$ is equivalent to the diagonal quartic surface with $(a_1, a_2, a_3) = (1, 2, -2)$, then the 2-primary torsion subgroup of $\mathrm{Br}(X_K)/\mathrm{Br}_1(X_K)$ is $\mathbb{Z}/2$; otherwise this subgroup is 0.*

The odd order torsion subgroup of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is 0, unless $-3a_1a_2a_3$ is in $\mathbb{Q}^{*4} \cup -4\mathbb{Q}^{*4}$ when this subgroup is $(\mathbb{Z}/3)^2$, or $125a_1a_2a_3$ is in $\mathbb{Q}^{*4} \cup -4\mathbb{Q}^{*4}$ when this subgroup is $(\mathbb{Z}/5)^2$.

(ii) *If X is equivalent to the diagonal quartic surface with coefficients $(1, 2, -2)$ or $(1, 8, -8)$, then the 2-primary torsion subgroup of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is $\mathbb{Z}/2$; otherwise this subgroup is 0.*

The odd order torsion subgroup of $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is 0, unless $-3a_1a_2a_3$ is in $\mathbb{Q}^{*4} \cup -4\mathbb{Q}^{*4}$ when this subgroup is $\mathbb{Z}/3$, or $125a_1a_2a_3$ is in $\mathbb{Q}^{*4} \cup -4\mathbb{Q}^{*4}$ when this subgroup is $\mathbb{Z}/5$.

Corollary 16.8.2 *Let X be a diagonal quartic surface over $\mathbb{Q}$ with coefficients a_1, a_2, a_3 such that $a_1a_2a_3$ is a square in $\mathbb{Q}$. Then $\mathrm{Br}(X) = \mathrm{Br}_1(X)$.*

Carrying out calculations of step (c) one can describe the structure of the extension of finite abelian groups

$$0 \to \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \longrightarrow \mathrm{Br}(X)/\mathrm{Br}_0(X) \longrightarrow \mathrm{Br}(X)/\mathrm{Br}_1(X) \to 0. \quad (16.17)$$

Since Γ acts on $\mathrm{Pic}(\overline{X})$ via a finite 2-group, the order of $\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong \mathrm{H}^1(k, \mathrm{Pic}(X^{\mathrm{s}}))$ is always a power of 2, so it is enough to consider the case of 2-primary torsion.

Supplement to Theorem 16.8.1 *Let X be the diagonal quartic surface with coefficients $(1, 2, -2)$ or $(1, 8, -8)$ over $\mathbb{Q}$.*

(i) *Let $K = \mathbb{Q}(\sqrt{-1})$. Then the exact sequence (16.17) for X_K over K is the extension*

$$0 \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/4 \longrightarrow (\mathbb{Z}/4)^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

(ii) *The exact sequence (16.17) for X over $\mathbb{Q}$ is the extension*

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Combined with Bright's classification of algebraic Brauer groups of diagonal quartic surfaces over number fields, this gives a classification of Brauer groups of diagonal quartic surfaces with coefficients in $\mathbb{Q}$ over the ground fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$.

The supplement illustrates a difficulty in lifting Galois-invariant elements of $\mathrm{Br}(\overline{X})[2]$ to $\mathrm{Br}(X)$: an element in the image of $\mathrm{Br}(X)\to\mathrm{Br}(\overline{X})$ may not lift to an element of $\mathrm{Br}(X)$ of the same order.

The above method can be applied to diagonal surfaces of other degrees, as well as to diagonal surfaces in weighted projective spaces, see [GLN]. We refer to [SZ12] for general methods and examples of calculation of the Brauer group of Kummer surfaces.

We refer the reader to [IS15] for explicit examples of Brauer–Manin obstruction to weak approximation on diagonal quartic surfaces X over $\mathbb{Q}$ given by (transcendental) elements of $\mathrm{Br}(X)$ of odd order (that is, of order 3 and 5). Ieronymou proved that the Fermat quartic surface over $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$ does not satisfy weak approximation [Ier10, Thm. 6.1]: there is a transcendental Brauer element of order 2 that takes both the zero and non-zero values when evaluated at local points at the unique prime of $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$ above 2. Similar examples for the diagonal quartic surfaces over $\mathbb{Q}$ with coefficients $(1, 2, -2)$ or $(1, 8, -8)$ are currently unknown.

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