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A Textbook on Ordinary Differential Equations

Second Edition

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A Textbook on Ordinary Differential Equations

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Professor Shair Ahmad wishes to thank his wife, Carol, for her continued and loving support, patience and understanding, which go far beyond what might normally be expected. He also wishes to acknowledge his grandson, Alton Shairson, as a source of infusion of energy and optimistic enthusiasm for life.

Preface

One of the authors' main motivations for writing this book has been to provide students and faculty with a more economical option when selecting an introductory textbook on ordinary differential equations (ODEs). This book is a primer on the theory and applications of ODEs. It is aimed at students of Mathematics, Physics, Engineering, Statistics, Information Science, etc. who already possess a sufficient knowledge of calculus and a minimal knowledge of linear algebra.

The first chapter starts with the simplest first-order linear differential equations and builds on this to lead to the more general equations. The concepts of initial values and existence and uniqueness of solutions are introduced early in this chapter. Ample examples, using simple integration, are provided to motivate and demonstrate these concepts. Almost all of the assertions are proved in elementary and simple terms.

The important concepts of the Cauchy problem and the existence and uniqueness of solutions are covered in detail and demonstrated by many examples. Proofs are given in an Appendix. There is also a rigorous treatment of some qualitative behavior of solutions. This chapter is important from a pedagogical point of view because it introduces students to rigor and fosters an understanding of important concepts at an early stage.

In a chapter on nonlinear first-order equations, students will learn how to explicitly solve certain types of equations, such as separable, homogeneous, exact, Bernoulli and Clairaut equations. Further chapters are devoted to linear higher order equations and systems, with several applications to mechanics and electrical circuit theory. Also included is an elementary but rigorous introduction to the theory of oscillation.

A chapter on phase plane analysis deals with finding periodic solutions, solutions of simple boundary value problems, and homoclinic and heteroclinic trajectories. There is also a section on the Lotka–Volterra system in the area of population dynamics.

Subsequently, the book deals with the Sturm–Liouville eigenvalues, Laplace transform and finding series solutions, including fairly detailed treatment of Bessel functions, which are important in Engineering.

Although this book is mainly addressed to undergraduate students, consideration is given to some more advanced topics, such as stability theory and existence of solutions to boundary value problems, which might be useful for more motivated undergraduates or even beginning graduate students.

A chapter on numerical methods is included as an Appendix, where the importance of computer technology is pointed out. Otherwise, we do not encourage the use of computer technology at this level. We believe that, at this stage, students should practice their prior knowledge of algebra and calculus instead of relying on technology, thus sharpening their mathematical skills in general.

Each chapter ends with a set of exercises that are intended to test the student's understanding of the concepts covered. Solutions to selected exercises are included at the end of the book.

We wish to acknowledge with gratitude the help of Dung Le, Rahbar Maghsoudi and Vittorio Coti Zelati, especially with regard to technical issues.

San Antonio and Trieste
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Shair Ahmad
Antonio Ambrosetti

Preface to the Second Edition

This edition contains corrections of errata and additional carefully selected exercises and provides more lucid explanations of some of the topics addressed. Although the book is written in a rigorous and thorough style, it offers instructors the flexibility to skip some of the rigor and theory and concentrate on methods and applications, should they wish to do so. This makes the book suitable not only for students studying Mathematics but also for those in other areas of Science and Engineering. We wish to thank Weiming Cao and Erik Whalén for several useful comments.

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Shair Ahmad
Antonio Ambrosetti

Contents

Notation

The following are some notations that are used in the book.

- $\mathbb{N}$ denotes the set of natural numbers $0, 1, 2 \ldots$
- $\mathbb{Z}$ denotes the set of integer numbers $0, \pm 1, \pm 2 \ldots$
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.
- If $a, b \in \mathbb{R}$, $[a, b]$ denotes the closed interval $\{a \leq t \leq b\}$; (a, b), or $]a, b[$, denotes the open interval $\{a < t < b\}$. Moreover $(a, b]$, or $]a, b]$, denotes the interval $\{a < t \leq b\}$, while $[a, b)$, or $[a, b[$, denotes the interval $\{a \leq t < b\}$.
- If $\overline{x}, \overline{y} \in \mathbb{R}^n$, $(\overline{x} \mid \overline{y}) = \sum x_i y_i$ denotes the euclidean scalar product of the vectors $\overline{x}, \overline{y}$, with components x_i, y_i, $i = 1, \ldots, n$. In some case we will also use $\overline{x} \cdot \overline{y}$ or $(\overline{x}, \overline{y})$ instead of $(\overline{x} \mid \overline{y})$. The corresponding euclidean norm is denoted by $|\overline{x}| = \sqrt{(\overline{x} \mid \overline{x})} = \sqrt{\sum x_i^2}$. If $n = 1$ then $|x|$ is the usual absolute value.
- $\frac{d^k f}{dt^k} = f^{(k)}$ denotes the k-th derivative of $f(t)$.
- $\frac{\partial f}{\partial x_i} = \partial_{x_i} f = f_{x_i}$ denotes the partial derivative of $f(x_1, \ldots, x_n)$ with respect to x_i.
- If $\Omega \subseteq \mathbb{R}^n$, $C(\Omega, \mathbb{R})$, or simply $C(\Omega)$, is the class of continuous real valued functions $f : \Omega \mapsto \mathbb{R}$ defined on Ω. $C(\Omega, \mathbb{R}^m)$ is the class of continuous functions $\overline{f}$ defined on Ω with values in $\mathbb{R}^m$.
- If $\Omega \subseteq \mathbb{R}^n$ is an open set, $C^k(\Omega, \mathbb{R})$, or simply $C^k(\Omega)$, is the class of real valued functions $f : \Omega \mapsto \mathbb{R}$ which are k times continuously differentiable. $C(\Omega, \mathbb{R}^m)$ is the class of functions $\overline{f} : \Omega \mapsto \mathbb{R}^m$, each component of which is k times continuously differentiable. Functions that are differentiable infinitely many times are often called *regular*.
- $W(f_1, \ldots, f_n)(t) = W(f_1(t), \ldots, f_n(t)) = W(t)$ represents the Wronskian of the functions $f_1, \ldots, f_n$.
- J_m = Bessel function of order m.

- $f * g$ = convolution of the functions f and g.
- $\delta(t)$ = the Dirac delta function.
- Det(A) = determinant of the matrix A.
- A_{kl} = Minor of the element a_{kl}, C_{kl} = cofactor of the element a_{kl}.
- $\mathcal{L}\{f(t)\}(s) = F(s)$ = the Laplace transform of the function f.
- $\nabla V(\overline{x}) = (V_{x_1}(\overline{x}), \ldots, V_{x_n}(\overline{x}))$, $\overline{x} \in \mathbb{R}^n$, denotes the gradient of the real valued function V.
- $(\nabla V(\overline{x}) \mid \overline{f}(\overline{x})) = \sum_1^n V_{x_i}(\overline{x}) f_{x_i}(\overline{x})$ = scalar product of $\nabla V(\overline{x})$ and $\overline{f}(\overline{x})$.

1

First order linear differential equations

1.1 Introduction

A *differential equation* is an equation involving an unknown function and its derivatives. By a *solution* of a differential equation we mean a function that is differentiable and satisfies the equation on some interval. For example, $x' - x = 0$ is a differential equation involving an unknown function x and its first derivative with respect to an independent variable that we may call t, s, etc. We notice that $(e^t)' - e^t = e^t - e^t = 0$ for all t in the interval $I = (-\infty, \infty)$. Therefore, $x(t) = e^t$ is a solution of the differential equation on the interval I.

A differential equation involving ordinary derivatives is called an *ordinary differential equation* and one involving partial derivatives is called a *partial differential equation*. For example, $x'' - t^2 x' + 2x = 0$ is an ordinary differential equation, while $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is a partial differential equation. In this book, we deal with ordinary differential equations.

By the *order* of a differential equation we mean the order of the highest derivative appearing in the equation. For example, $x''' + 2x'' - 3x' + 2x = 0$ is a third order differential equation while $x'' + x = 0$ is second order.

Differential equations play a central and important role not only in mathematics but also in almost all areas of science and engineering, economics, and social sciences. A differential equation may describe the flow of current in a conductor, or the motion of a missile, the behavior of a mixture, the spread of diseases, or the growth rate of the population of some species, etc. Often, we will have $x(t)$ describing a physical quantity, depending on time t, whose rate of change $x'(t)$ is given by the function $f(t, x(t))$ depending on time t and $x(t)$.

In the sequel we will discuss differential equations on a broader basis, including higher order equations and/or systems. In this first chapter, however, we start with the simplest, but very important, class of differential equations, namely first order *linear* equations.

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1.2 A simple case

Let us begin with the very specific and simple equation

$$x' = kx, \qquad k \in \mathbb{R}. \tag{1.1}$$

We will demonstrate a precise method for solving such equations below. But first we use our intuition and familiarity with the derivative of the exponential function to solve the simple equation (1.1).

Let us first take $k = 1$. We seek a function whose derivative is equal to itself: $x' = x$. One such function is $x(t) \equiv 0$. We also know that the exponential function $x = e^t$ has this feature. Actually, for every constant c, the function $x = ce^t$ is a solution of $x' = x$. This leads us to the slightly more general case $x' = kx$, which has $x = ce^{kt}$ as a solution, for any constant c. Furthermore, as we will see below, these are the only types of solutions that this differential equation can have.

We now illustrate a general procedure that will be used later to solve the most general first order linear differential equations. First suppose that $x(t)$ satisfies the equation

$$x'(t) = kx(t).$$

Multiplying both sides of the equivalent equation $x'(t) - kx(t) = 0$ by e^{-kt}, we have

$$x'(t)e^{-kt} - kx(t)e^{-kt} = 0.$$

We note that the left-hand side is the derivative of $(x(t)e^{-kt})$. Hence we have $(x(t)e^{-kt})' = 0$. Integrating, we obtain $x(t)e^{-kt} = c$, $\forall\, t \in \mathbb{R}$, where c is a constant. Hence $x(t) = ce^{kt}$.

On the other hand, by substituting any function of the form $x(t) = ce^{kt}$ into the equation (1.1), we see that $x(t)$ is a solution of (1.1). Therefore, $x(t)$ is a solution of (1.1) **if and only if** $x(t) = ce^{kt}$ for some constant c. We say that $x(t) = ce^{kt}$ is the *general solution* of (1.1), that is, it represents the family of all solutions of this equation.

Example 1.2.1. Consider the problem of finding $x(t)$ such that

$$x' = 2x, \quad x(0) = 1. \tag{1.2}$$

This is called an *initial value problem*. It is asking for a function $x(t)$ that satisfies the differential equation **and** $x(0) = 1$. We have shown above that $x(t) = ce^{2t}$ is the general solution. So, the desired solution, if it exists, must be of the form ce^{2t}. Substituting $t = 0$ in the equation $x(t) = ce^{2t}$, we obtain $1 = ce^0$ or $c = 1$. Therefore, $x(t) = e^{2t}$ is a solution to the initial value problem (1.2). Since every solution to the initial value problem (1.2) is of the form $x(t) = ce^{2t}$ and since by substituting the initial values in this general solution we obtain only one constant that satisfies the initial value problem, we conclude that the solution to the initial value problem (1.2) *exists* and it is *unique*. ∎

1.3 Some examples arising in applications

In spite of its simplicity, equation (1.1) arises in many fields of applied sciences. Below we discuss a couple of them.

1.3.1 Population dynamics

Let:

- t denote the time.
- $x(t)$ denote the number of individuals of a population at time t.
- b denote the birth rate of the population.
- d the death rate of the population.

The simplest model of population growth, due to Malthus[1] in the discrete version, assumes that b and d are constant and that the increment of the population $x(n+1)-x(n)$ between $t=n$ and $t=n+1$ is equal to the number of new-born individuals $b\cdot x(n)$ minus the number of deaths $d\cdot x(n)$, namely $x(n+1)-x(n)=bx(n)-dx(n)=(b-d)x(n)$. Introducing the number $k=b-d$ as the unit growth rate, that is the growth rate per unit time, we find the recursive equation

$$x(n+1)-x(n)=kx(n), \qquad n=0,1,2,... \tag{1.3}$$

which allows us to find $x(n)$ for any positive integer n. To pass to continuous time variables, we take an infinitesimal change of time Δt. Then the change of population $x(t+\Delta t)-x(t)$ between t and $t+\Delta t$ is given by the unit growth rate k, times the population size $x(t)$, times the interval of time Δt. Thus equation (1.3) becomes $x(t+\Delta t)-x(t)=kx(t)\Delta t$. Dividing by Δt we get

$$\frac{x(t+\Delta t)-x(t)}{\Delta t}=kx(t).$$

The left-hand side is the incremental ratio of $x(t)$. Letting $\Delta t\to 0$, we find

$$x'(t)=kx(t), \qquad k=b-d,$$

a first order linear differential equation like (1.1), whose solutions are $x(t)=ce^{kt}$. If $x(0)=x_0>0$, then $c=x_0>0$ and $x(t)=x_0e^{kt}$. If the birth rate b is equal to the death rate d, then $k=b-d=0$ and $x(t)=x_0e^0=x_0$ for all $t\geq 0$, as one would expect. If $b>d$ then $k=b-d>0$ and $x(t)=x_0e^{kt}$ grows exponentially and approaches $+\infty$ as $t\to+\infty$. On the other hand, if $k<0$ then $x(t)$ decays exponentially to 0 as $t\to+\infty$ and the population goes to extinction. See Figure 1.1.

This model is rather rough in the sense that it does not take into account the fact that b, d, and hence the growth rate k, might depend on the population size. In Section 3.1.1 of Chapter 3, we will discuss a more realistic model of population growth,

[1] Thomas R. Malthus (1766–1834).

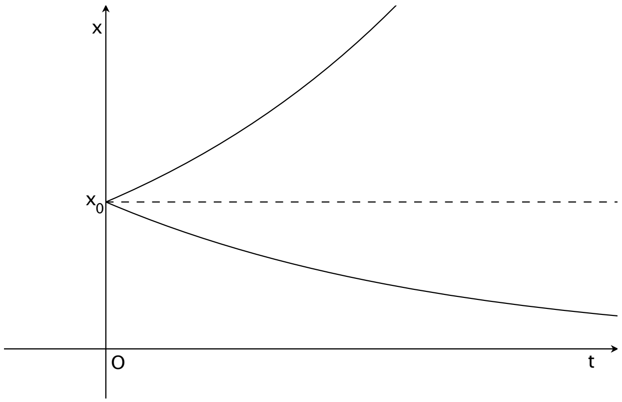


Fig. 1.1. Solutions of (1.3), with $k > 0$ (*upper curve*) and $k < 0$ (*lower curve*) and $k = 0$ (*dotted curve*)

which gives rise to the so called "logistic equation" having the form $x' = x(\alpha - \beta x)$, α, β positive constants.

1.3.2 An RC electric circuit

Let us consider an RC circuit with resistance R and capacity C with no external current or voltage source.

If we denote by $x(t)$ the capacitor voltage ($x(t) = V(t)$ in Figure 1.2) and by $I(t)$ the current circulating in the circuit, then, according to the Kirchhoff's law, we have

$$R \cdot I(t) + x(t) = 0.$$

Moreover, the constitutive law of capacitor yields

$$I(t) = C \cdot \frac{dx(t)}{dt}.$$

Substituting $I(t)$ in the first equation, we obtain the first order differential equation

$$RC \cdot x'(t) + x(t) = 0,$$

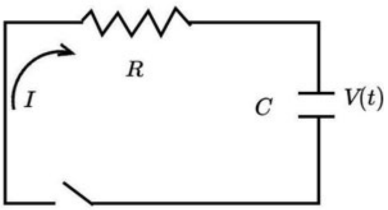


Fig. 1.2. An RC circuit

namely

$$x'(t) + \frac{x(t)}{RC} = 0,$$

which is of the form (1.1) with $k = -1/RC$. Also here we can look for a solution $x(t)$ satisfying the *initial condition* $x(0) = x_0$, which means that the initial voltage is x_0. The solution is given by

$$x(t) = x_0 e^{-t/RC}.$$

We can see that the capacitor voltage $x(t) = V(t)$ decays exponentially to 0 as $t \to +\infty$, in accordance with the experience. The number $\tau = RC$ is usually called the RC *time constant* and is the time after which the voltage $x(t) = V(t)$ decays to $V(\tau) = x_0 e^{-1}$. Moreover we can say that the bigger τ is, the slower the decay. As for the intensity of current $I(t)$, one finds

$$I(t) = Cx'(t) = -C \cdot \frac{x_0}{RC} e^{-t/RC} = -\frac{x_0}{R} e^{-t/RC}.$$

Other equations arising in the electric circuit theory will be discussed in Example 1.4.3 below, in Example 5.5.5 of Chapter 5 and in Section 11.6 of Chapter 11.

1.4 The general case

Now, we solve the general first order linear differential equation

$$x' + p(t)x = q(t) \tag{1.4}$$

where $p(t),\ q(t)$ are continuous functions on an interval $I \subseteq \mathbb{R}$.

If $q(t) \equiv 0$ the linear differential equation (1.4) is called *homogeneous*, otherwise it is called *nonhomogeneous* or *inhomogeneous*.

Motivated by the above discussion, we try to find a differentiable function $\mu(t)$, $\mu(t) > 0$ for $t \in I$, such that

$$\mu(t)x'(t) + \mu(t)p(t)x(t) = (\mu(t)x(t))'.$$

Such a function $\mu(t)$ is called an *integrating factor* of the equation (1.4). It has the property that if we multiply (1.4) by $\mu(t)$, it renders the left side of the equation to be equal to $(\mu(t)x(t))'$, which can be easily integrated.

Although based on the discussion in the preceding section, one might guess such an integrating factor, we prefer giving a precise method for finding it.

Let $x(t)$ be a solution of (1.4). Multiplying the equation by $\mu(t)$ we have

$$\mu x' + \mu p x = \mu q.$$

Now we wish to find μ such that

$$\mu x' + \mu p x = (\mu(t)x(t))'.$$

Expanding the right-hand side, we have

$$\mu(t)x'(t) + \mu(t)p(t)x(t) = \mu(t)x'(t) + \mu'(t)x(t).$$

Canceling $\mu(t)x'(t)$ from both sides, we obtain

$$\mu(t)p(t)x(t) = \mu'(t)x(t).$$

Assuming that $x(t) \neq 0$ and dividing both sides by $x(t)$, we find

$$\mu(t)p(t) = \frac{d\mu}{dt}.$$

Since $\mu(t) > 0$ we infer

$$\frac{d\mu}{\mu(t)} = p(t)dt.$$

Then, taking the indefinite integrals we obtain $\int \frac{d\mu}{\mu(t)} = \int p(t)dt$ and we find (recall that $\mu(t) > 0$) that $\ln(\mu(t)) = \int p(t)dt$.

Thus

$$\mu(t) = e^{P(t)}, \quad P(t) = \int p(t)dt.$$

In order to obtain the general solution of (1.4), we take the indefinite integral of both sides of

$$(\mu(t)x(t))' = \mu(t)q(t) \tag{1.5}$$

obtaining

$$\mu(t)x(t) = c + \int \mu(t)q(t)dt$$

where c is a constant. Substituting $\mu(t) = e^{P(t)}$, we have

$$x(t) = e^{-P(t)}\left[c + \int e^{P(t)}q(t)dt\right], \qquad P(t) = \int p(t)dt. \tag{1.6}$$

We have seen that if $x(t)$ solves (1.4), then there exists $c \in \mathbb{R}$ such that $x(t)$ has the form (1.6). Moreover, it is easy to check that for all $c \in \mathbb{R}$, $x(t)$ given above solves (1.4). This is why $x(t)$ in (1.6) is called the *general solution* of (1.4).

Now, suppose we are interested in solving the initial value problem

$$x' + p(t)x = q(t), \;\; x(t_0) = x_0.$$

Then we can substitute $t = t_0$, $x = x_0$ in the general solution and solve for the constant c. Another way is to take the definite integral of (1.5) from t_0 to t instead of the indefinite integral. Doing so, we have

$$\mu(t)x(t) - \mu(t_0)x(t_0) = \int_{t_0}^{t} \mu(s)q(s)ds.$$

We can also choose $P(t) = \int_{t_0}^{t} p(s)ds$ and then

$$\mu(t) = e^{\int_{t_0}^{t} p(s)ds}.$$

Hence $\mu(t_0) = 1$ and

$$x(t) = e^{-\int_{t_0}^{t} p(s)ds} \left[x_0 + \int_{t_0}^{t} e^{\int_{t_0}^{s} p(s)ds} q(s)ds \right]. \tag{1.7}$$

Remark 1.4.1. We prefer not to have to memorize (1.7) but rather go through this simple procedure each time, starting with integrating factors. ∎

As a special case of (1.6), when $q = 0$, the general solution of the homogeneous equation

$$x' + p(t)x = 0 \tag{1.8}$$

is

$$x(t) = c\, e^{-P(t)}, \quad P(t) = \int p(t)dt, \quad t \in I.$$

For $c = 0$ we obtain the *trivial solution* $x(t) \equiv 0$.

If we are searching for a solution satisfying the initial condition $x(t_0) = x_0$, then we can solve $x_0 = x(t_0) = c\, e^{-P(t_0)}$. If we take $P(t) = \int_{t_0}^{t} p(s)ds$, then $P(t_0) = 0$ and we find $c = x_0$. Thus

$$x(t) = x_0\, e^{-\int_{t_0}^{t} p(s)ds}$$

is the solution of $x' + p(t)x = 0$ such that $x(t_0) = x_0$, and it is **unique**. As a consequence, if t_0 is any number in I and $x(t_0) = x_0$, then

1. $x(t) = 0$, $\forall\, t \in I$, if and only if $x_0 = 0$.
2. $x(t) > 0$, $\forall\, t \in I$, if and only if $x_0 > 0$.
3. $x(t) < 0$, $\forall\, t \in I$, if and only if $x_0 < 0$.

In other words, if $x(t)$ is a solution of (1.8), then it is either identically zero, or it is always positive or it is always negative. In particular, if $x(t)$ vanishes somewhere in I, then it has to be the trivial solution $x(t) = 0$, for all $t \in I$.

The above arguments lead to the following existence and uniqueness result for (1.4), namely for $x' + p(t)x = q(t)$.

Theorem 1.4.2. *Let $p(t), q(t)$ be continuous in $I \subseteq \mathbb{R}$. Then*

1. *The general solution of* (1.4) *is given, for all $t \in I$, by*

$$x(t) = e^{-P(t)} \left[c + \int e^{P(t)} q(t)dt \right], \qquad P(t) = \int p(t)dt,$$

c a constant.

2. *There is exactly one solution $x(t)$ satisfying the initial value $x(t_0) = x_0$ for any numbers $t_0 \in I$ and $x_0 \in \mathbb{R}$. Precisely,*

$$x(t) = e^{-\int_{t_0}^{t} p(s)ds} \left[x_0 + \int_{t_0}^{t} e^{\int_{t_0}^{s} p(s)ds} q(s)ds \right], \qquad t \in I. \tag{1.9}$$

This theorem can also be deduced from general existence and uniqueness results stated in Chapter 2, Section 2.2.2.

We end this section by demonstrating, through examples, how to solve linear equations.

Example 1.4.3. Find the solution of

$$x'(t) + kx(t) = h, \qquad x(0) = x_0, \tag{1.10}$$

where h, k are constant. Equation (1.10) arises in the RC circuit when there is a generator of constant voltage $h = V_0$, see Figure 1.3.

Here $p(t) \equiv k$ and hence an integrating factor is e^{kt}. Multiplying the equation by e^{kt} yields

$$e^{kt}x' + ke^{kt}x(t) = he^{kt},$$

or

$$\frac{d}{dt}\left(x(t)e^{kt}\right) = he^{kt}.$$

Integrating, we find

$$x(t)e^{kt} = \frac{h}{k}e^{kt} + c$$

where c is an arbitrary constant. Thus the general solution is

$$x(t) = ce^{-kt} + \frac{h}{k}.$$

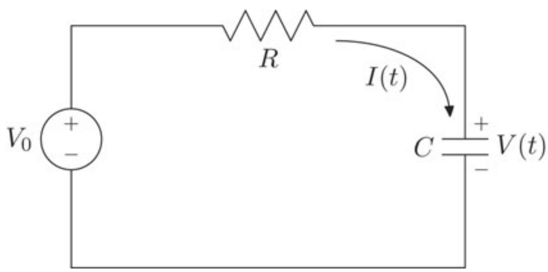


Fig. 1.3. RC circuit with a generator of voltage

To find a solution that satisfies the initial condition $x(0) = x_0$ we might simply substitute $t = 0$ in the preceding equation, finding

$$x_0 = c + \frac{h}{k} \quad \text{and hence} \quad c = x_0 - \frac{h}{k}.$$

Hence the unique solution of (1.10) is

$$x(t) = \left(x_0 - \frac{h}{k}\right) e^{-kt} + \frac{h}{k}. \tag{1.11}$$

Alternatively, we can use (1.9) yielding

$$x(t) = e^{-kt}\left[x_0 + \int_0^t e^{ks} h\, ds\right]$$
$$= e^{-kt}\left[x_0 + h \cdot \frac{1}{k} \cdot (e^{kt} - 1)\right] = \left(x_0 - \frac{h}{k}\right) e^{-kt} + \frac{h}{k}$$

as before.

Notice that, as $t \to +\infty$, $x(t) \to \frac{h}{k}$, from below if $x_0 < \frac{h}{k}$ (see Figure 1.4a) and from above if $x_0 > \frac{h}{k}$ (see Figure 1.4b).

The solution (1.11) implies that in this case the capacitor voltage $x(t) = V(t)$ does not decay to 0 but tends, as $t \to +\infty$, to the constant voltage $h/k = V_0/RC$. ∎

Example 1.4.4. Find the general solution of

$$x'(t) + 4tx(t) = 8t$$

and the solution such that $x(0) = 1$.

(a) Here $p(t) = 4t$ and hence we can take $P(t) = 2t^2$. We start by multiplying the equation by the integrating factor e^{2t^2}, which results in the equation

$$e^{2t^2} x' + 4te^{2t^2} x(t) = 8te^{2t^2},$$

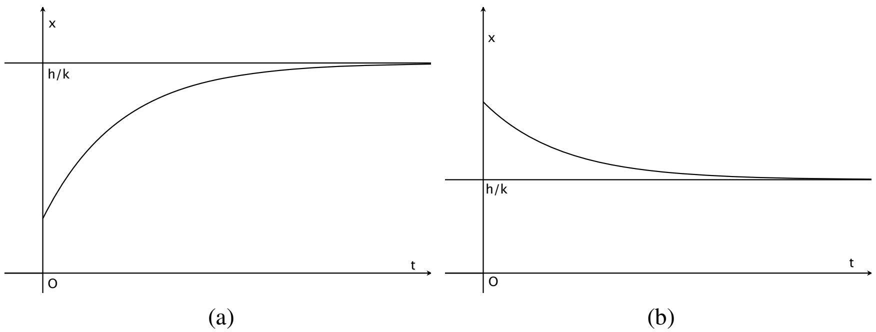


Fig. 1.4. Solutions of (1.10). (a) $x_0 < \frac{h}{k}$; (b) $x_0 > \frac{h}{k}$

which is the same as

$$\frac{d}{dt}\left(x(t)e^{2t^2}\right) = 8te^{2t^2}.$$

Integrating both sides, we obtain

$$x(t)e^{2t^2} = 2e^{2t^2} + c$$

where c is an arbitrary constant. Therefore, the general solution is

$$x(t) = 2 + c\,e^{-2t^2}.$$

(b) If we require that $x(0) = 1$, then the constant c is uniquely determined by the equation $1 = 2 + c\,e^{-2\cdot 0} = 2 + c$, that is $c = -1$ and hence

$$x(t) = 2 - e^{-2t^2}.$$

Alternatively, we can use the general formula (1.9) finding

$$\begin{aligned} x(t) &= e^{-4\int_0^t s ds}\left[1 + \int_0^t e^{4\int_0^s s ds}\, 8\cdot s\, ds\right] = e^{-2t^2}\left[1 + \int_0^t 8\,s\,e^{2s^2}\,ds\right] \\ &= e^{-2t^2}\left[1 + (2e^{2t^2} - 2)\right] = e^{-2t^2}\left[2e^{2t^2} - 1\right] = 2 - e^{-2t^2}. \end{aligned}$$

We make a couple of interesting observations concerning this equation.

1. We note that for $c = 0$, we obtain the constant solution $x = 2$. Furthermore, this solution divides all the other solutions into two groups: those that approach it from the top and those that approach it from the bottom, as $t \to \pm\infty$. See Figure 1.5.

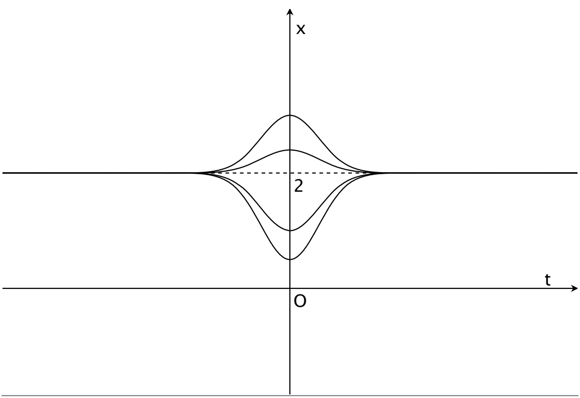


Fig. 1.5. Graphs of $x = 2 + ce^{-2t^2}$

2. We could have found the constant solution $x(t) = 2$ without even solving the equation, by simply noting that if $x(t)$ is a constant, then $x'(t) \equiv 0$ and therefore $x'(t) + 4tx(t) \equiv 8t$ implies $x(t) = 2$ for all t.

Example 1.4.5. Find the general solution of

$$t^2 x' + (1+t)x = \frac{1}{t}\, e^{1/t}, \qquad t > 0. \tag{1.12}$$

The first thing we notice is that the above equation is not in the form of equation (1.4) for which we found the integrating factor μ. To apply our method, we must put it in the proper form. So, we divide both sides of the equation by $t^2 \neq 0$, which yields

$$x' + \frac{(1+t)}{t^2} x = \frac{1}{t^3} \cdot e^{1/t} \qquad t > 0. \tag{1.13}$$

We know that an integrating factor μ can be determined as

$$\mu = e^{P(t)}, \quad \text{where} \quad P'(t) = \frac{(1+t)}{t^2}.$$

To find $P(t)$ we evaluate the indefinite integral

$$\int \frac{(1+t)}{t^2}\, dt = \int \left(\frac{1}{t^2} + \frac{1}{t}\right) dt = -\frac{1}{t} + \ln t + c, \qquad t > 0.$$

Taking $c = 0$ we find that an integrating factor is given by

$$\mu(t) = e^{-1/t + \ln t} = e^{-1/t}.e^{\ln t} = te^{-1/t}.$$

Multiplying (1.13) by this integrating factor, we obtain the equation

$$(t\, e^{-1/t} x)' = t\, e^{-1/t}.\frac{1}{t^3}\, e^{1/t} = \frac{1}{t^2}\,.$$

Integrating both sides, we get

$$te^{-1/t} x(t) = -\frac{1}{t} + c.$$

The general solution is

$$x(t) = \frac{-e^{1/t}}{t^2} + \frac{ce^{1/t}}{t} = \frac{e^{1/t}}{t}\left(c - \frac{1}{t}\right), \qquad t > 0. \tag{1.14}$$

It is clear that *all the solutions tend to* 0 *as* t *tends to* $+\infty$. On the other hand, it is easy to verify that for any constant c, $x(t)$ given by (1.14) satisfies equation (1.13). This means that $x(t)$ given by (1.14) is a solution of (1.13), and hence of (1.12), if and only if $x(t)$ is of the form (1.14), that is the general solution of (1.12). ■

If we want to solve the equation $x' + \frac{(1+t)}{t^2}x = \frac{1}{t^3} \cdot e^{1/t}$ for $t \neq 0$ we should distinguish the two cases $t > 0$ and $t < 0$ separately. As an exercise, the reader might repeat the calculations for $t < 0$.

Example 1.4.6. Solve the following initial value problem and show that the solution is defined for $t > 0$ and is unique:

$$t^2x' + (1+t)x = \frac{1}{t}e^{\frac{1}{t}}, \quad x(1) = 0. \tag{1.15}$$

We have shown that the general solution of (1.12) for $t > 0$ is $x(t)$ given by (1.14), where c is a constant.

Now in order to solve the initial value problem, since all solutions are included in (1.14), we simply substitute the initial values into the equation (1.14), obtaining $0 = -e + ce$, and hence $c = 1$. Therefore,

$$x(t) = \frac{-e^{\frac{1}{t}}}{t^2} + \frac{e^{\frac{1}{t}}}{t} = \frac{e^{\frac{1}{t}}}{t}\left(1 - \frac{1}{t}\right).$$

The graph of $x(t)$ is reported in Figure 1.6. The uniqueness follows from the fact that there is only one value of c for which $x(t)$ obtained from the general solution (1.14) satisfies the initial value $x(1) = 0$.

The reader can check, as an exercise, that the same result holds if we use the general formula (1.9). ■

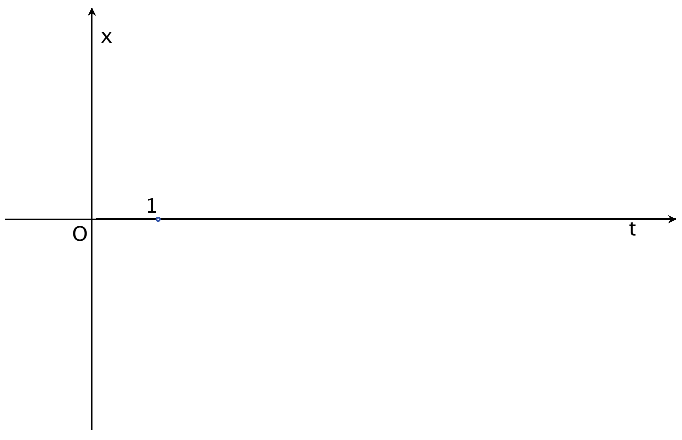


Fig. 1.6. $x(t) = \frac{e^{\frac{1}{t}}}{t}\left(1 - \frac{1}{t}\right), t > 0$

1.5 Exercises

1. Find the equation whose general solution is $x = c\, e^{-5t}$.
2. Solve $x' + (\ln 3)x = 0$.
3. Solve $x' + 4x = 4$.
4. Find all the solutions to the initial value problem $x' + \frac{2t^3+\sin t+5}{t^{12}+5}x = 0, x(0) = 0$.
5. Solve $x' = -2x + 3$ and find the solution satisfying $x(1) = 5$.
6. Find k such that there exists a solution of $x' = kx$ such that $x(0) = 1$ and $x(1) = 2$.
7. Explain why the solution to the problem
$$x' - 2(\cos t)x = \cos t, \quad x(0) = \frac{1}{2}$$
must oscillate, i.e. it must have arbitrarily large zeros.
8. In each of the following, find the maximum interval of existence of the solution, guaranteed by the existence theorem
   (a) $x' + \dfrac{1}{t^2 - 1}x = 0, \quad x(-2) = 1,$
   (b) $x' + (\sec t)\, x = \dfrac{1}{t-1}, \quad x\left(\frac{\pi}{4}\right) = 1.$
9. Solve $tx' + x = 2t^2$.
10. Show that there is an infinite family of solutions to the problem
$$t^2x' - 2tx = t^5, \quad x(0) = 0,$$
all of which exist everywhere on $(-\infty, \infty)$. Does this violate the uniqueness property of such equations?
11. Solve $x' = 2tx$ and find the solution satisfying $x(0) = 4$.
12. Solve $x' = -t^2x$.
13. Solve $x' + ax = bt$.
14. Solve: (a) $x' = x + 2t$, (b) $x' - 2x = 3t$, (c) $x' + 3x = -2t$.
15. Find the solution of $x' + ax = bt$ satisfying $x(t_0) = x_0$.
16. Solve the initial value problems (a) $x' - x = \frac{1}{2}t$, $x(0) = 1$, (b) $x' + x = 4t$, $x(1) = 0$, (c) $x' - 2x = 2t$, $x(0) = 3$.
17. Given $h, k \in \mathbb{R}$, $k > 0$, find the limits as $t \to +\infty$ of the solutions of $x' + kx = h$.
18. Consider $x' + kx = 1$, where k is a constant.
   (a) For what value of k will all solutions tend to 2 as $t \to +\infty$?
   (b) Is there any value of k for which there exists a non-constant solution $x(t)$ such that $x(t) \to -3$ as $t \to +\infty$? Explain.

19. Find the limits as $t \to \pm\infty$ of the solution of $x' = \frac{1}{1+t^2}\, x$, $x(0) = 1$.

20. Consider $x' + kx = h$, with $k \neq 0$. Find conditions on the constants h, k such that
    (a) all solutions tend to 0 as t tends to + infinity,
    (b) it will have only one solution bounded on $(0, +\infty)$,
    (c) all solutions are asymptotic to the line $x = 3$.

21. Show that for any differentiable function $f(t)$, $t \in \mathbb{R}$, all solutions of $x' + x = f(t) + f'(t)$ tend to $f(t)$ as t tends to $+\infty$.

22. Find a continuous function $q(t)$, $t \in \mathbb{R}$, such that all solutions of $x' + x = q(t)$
    (a) approach the line $x = 7t - 5$ as $t \to +\infty$,
    (b) approach the curve $x = t^2 - 2t + 5$ as $t \to +\infty$.

23. Show that if p is differentiable and such that $\lim_{t\to+\infty} p(t) = +\infty$, then all the solutions of $x' + p'(t)x = 0$ tend to zero as $t \to +\infty$.

24. If $k \neq 0$, show that the constant solution $x(t) = -\frac{1}{k^2}$ is the only solution of $x' - k^2 x = 1$ such that the $\lim_{t\to+\infty} x(t)$ is finite.

25. Let $k \neq 0$ and let $q(t)$ be continuous and such that $\lim_{t\to+\infty} q(t) = 0$, and $\int_0^{+\infty} e^{-k^2 s} q(s)ds = 0$. Show that the solution $x(t)$ of the ivp problem

$$x' - k^2 x = q(t), \quad x(0) = x_0,$$

    tends to 0 as $t \to +\infty$ if and only if $x_0 = 0$.

26. Show that the solution of $x' = k^2 x$, $x(t_0) = x_0$, is increasing if $x_0 > 0$ and decreasing if $x_0 < 0$.

27. Show that the solution of $x' = kx$, $x(t_0) = x_0$ is increasing if $kx_0 > 0$ and decreasing if $kx_0 < 0$.

28. Find the locus of minima of the solutions of $x' + 2x = 6t$.

29. Find the locus of maxima and minima of the solutions of $x' + x = at$, $a \neq 0$.

2

Theory of first order differential equations

Before discussing methods of solving more general classes of differential equations, it is convenient to present a theoretical overview of first order equations and their solutions, which will set a rigorous layout for the rest of the book.

2.1 Differential equations and their solutions

In Chapter 1 we introduced the notion of a differential equation and the meaning of the solution of such an equation. This will now be explained in more detail and in greater generality in the present section.

Consider the first order differential equation

$$x' = f(t, x) \tag{2.1}$$

where $f(t, x)$ is continuous, $(t, x) \in \Omega,\ \Omega \subset \mathbb{R}^2$.

A solution of (2.1) is a differentiable real valued *function* $x(t)$ defined on an interval $I \subseteq \mathbb{R}$ such that

$$x'(t) \equiv f(t, x(t)), \text{ for } (t, x(t)) \in \Omega. \tag{2.2}$$

An equation in this form is said to be in *normal form*, to distinguish it from more general differential equations that will be introduced later on.

One of the simplest examples of a first order differential equation is $x' = h(t)$, where h is continuous on an interval $I \subseteq \mathbb{R}$. If $H(t)$ is an antiderivative so that $H'(t) = h(t)$, then all the the solutions are given by $x(t) = H(t) + c$, c a real constant.

We have seen in Chapter 1 that all solutions of the linear equation $x' + p(t)x = q(t)$ form a family of functions depending on a constant. We will show in the sequel that this is a general fact: *solutions of* $x' = f(t, x)$ *form a one parameter family of functions*, although, as we will see, in the nonlinear case there could be some isolated cases of solutions that are not included in such a family.

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_2

1. The reader should recall that a solution of (2.2) is a *function*, in contrast to the *algebraic* equations, whose solutions are real (or complex) numbers. Moreover, it is important to note that (2.2) is an identity; it holds for all t in the domain of $x(t)$.
2. The domain of definition of a solution of (2.2) is *a priori* unknown and may depend on several facts. It could happen that, even if $f(t,x)$ makes sense for all real t, x, the solutions may be defined only for t in a proper subset of $\mathbb{R}$, see Example 2.2.2 below.

From a geometrical point of view, a solution of (2.1) is a curve $x = x(t)$, contained in the set Ω, such that the tangent at each point $(t^*, x(t^*))$ on the curve has slope equal to $f(t^*, x(t^*))$ and hence its equation is

$$x = f(t^*, x(t^*))(t - t^*) + x(t^*).$$

For example, (see Figure 2.1) consider the curve $x = e^t$ in the plane (t, x), which is a solution of $x' = x$. A generic point P^* on this curve has co-ordinates $P^* = (t^*, e^{t^*})$. The tangent to $x = e^t$ at P^* has equation

$$x = e^{t^*}(t - t^*) + e^{t^*}.$$

Remark 2.1.1. We have used t as the independent variable and x as the dependent one. But any other choice makes sense. For example, we could just as well name the dependent variable y and the independent variable x. With this notation, a first order differential equation would have the form

$$y' = \frac{dy}{dx} = f(x, y)$$

a solution of which would be a differentiable function $y(x)$ such that $y'(x) = f(x, y(x))$ for all x where $y(x)$ is defined. In any case, the equation will make it clear which one is the independent or dependent variable. ■

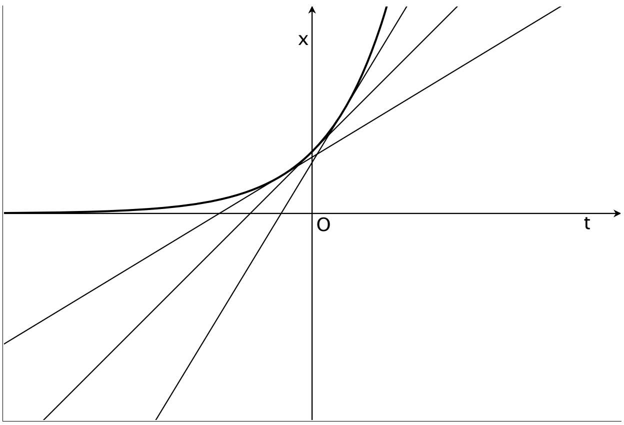


Fig. 2.1. Tangents to $x = e^t$

Dealing with a first order equation, one can distinguish between:

- *Linear* and *nonlinear* equations, according to whether $f(t,x)$ is linear with respect to x or not.
- *Autonomous* and *non-autonomous* equations according to whether f is independent of t or not.

For example, $x' = kx + c$ is linear and autonomous, $x' = x^2 + kx + c$ is nonlinear and autonomous; while $x' = e^t x + \sin t - 4$ is linear and non-autonomous, and $x' = tx^2 - tx + 3$ is nonlinear and non-autonomous.

Notice that, even if f is independent of t, the domain Ω has to be considered as an appropriate subset of $\mathbb{R}^2$. For example, in the equation $x' = \sqrt{x}$, $f(x) = \sqrt{x}$ is defined for $x \geq 0$ and hence $\Omega = \mathbb{R} \times \{x \geq 0\}$. Similarly, in the equation $x' = \sqrt{1-x^2}$, $f(x) = \sqrt{1-x^2}$ is defined for $-1 \leq x \leq 1$ and hence Ω is the horizontal strip $\mathbb{R} \times \{-1 \leq x \leq 1\}$.

More generally, let $F(t,x,p)$ be a real function of 3 real variables, defined on a set $R \subseteq \mathbb{R}^3$. Consider the first order differential equation

$$F(t,x,x') = 0,$$

whose solution is a differentiable real valued function $x(t)$ defined on an interval $I \subseteq \mathbb{R}$ such that

$$F(t,x(t),x'(t)) \equiv 0, \quad \forall\, t \in I. \tag{2.3}$$

If $F(t,x,p)$ is of the form $F(t,x,p) = p - f(t,x)$, we can write the equation $F(t,x,x') = 0$ in normal form $x' = f(t,x)$.

Even more generally, we may consider systems of n first order equations and n unknowns. We may also consider more general scalar equations $F(t,x,x',x'',\ldots,x^{(n)}) = 0$ of order n.

In this chapter we deal with first order equations. Higher order equations and systems will be discussed starting with Chapter 4.

2.2 The Cauchy problem: Existence and uniqueness

The problem of solving an equation in normal form $x' = f(t,x)$ coupled with the initial condition $x(t_0) = x_0$,

$$\begin{cases} x' & = f(t,x) \\ x(t_0) & = x_0 \end{cases}$$

is called a *Cauchy*[1] *problem* or an *initial value problem*, ivp in short.

Solve the previous ivp means finding a function $x(t)$ and an Interval I containing t_0, such that $x' = f(t,x)$ for all $t \in I$ and $x(t_0) = x_0$.

[1] Augustin-Louis Cauchy (1789–1857).

In this section we discuss some theoretical aspects of existence and uniqueness theorems for the Cauchy problems. The proofs are given in the Appendix 2.5 at the end of the chapter.

Existence and uniqueness of solutions is important not only from a theoretical point of view but also in applications. For example, in using a numerical method or some software such as Math Lab to find a solution, it is important to know whether or not such a solution exists in the first place. And if it does exist, is there more than one solution?

2.2.1 Local existence and uniqueness

Theorem 2.2.1 (Local existence and uniqueness). *Suppose that f is continuous in $\Omega \subseteq \mathbb{R}^2$ and has continuous partial derivative f_x with respect to x. Let (t_0, x_0) be a given point in the interior of Ω. Then there exists a closed interval I containing t_0 in its interior such that the Cauchy problem*

$$\begin{cases} x' & = f(t,x) \\ x(t_0) & = x_0 \end{cases} \tag{2.4}$$

has a unique solution, defined in I.

We will see that this is a particular case of a more general result, see Theorem 2.4.4 below.

We are going to outline the proof of the existence part by using a method introduced by Picard,[2] which is based on an approximation scheme.

We define a sequence of approximate solutions by setting

$$x_0(t) = x_0, \; x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))ds, \qquad k = 1, 2, \dots .$$

One shows that, under the given assumptions, there exists $\delta > 0$ such that $x_k(t)$ converges to some function $x(t)$, uniformly in $[t_0 - \delta, t_0 + \delta]$. Passing to the limit one finds that x satisfies

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Then $x(t)$ is differentiable and, using the Fundamental Theorem of Calculus, we get $x'(t) = f(t, x(t))$, for all $t \in [t_0 - \delta, t_0 + \delta]$. It is also clear that $x(t_0) = x_0$ and hence $x(t)$ is a solution of (2.4), defined in $[t_0 - \delta, t_0 + \delta]$. For details see the Appendix.

[2] Charles Emile Picard (1856–1941).

Let us show what happens in the particular case $f(t,x) = x$, $t_0 = 0$ and $x_0 = 1$. The sequence is constructed as follows:

$$\begin{aligned}
x_0(t) &= 1, \\
x_1(t) &= 1 + \int_0^t x_0(s)ds = 1 + \int_0^t ds = 1 + t, \\
x_2(t) &= 1 + \int_0^t x_1(s)ds = 1 + \int_0^t (1+s)ds = 1 + t + \frac{1}{2}t^2, \\
x_3(t) &= 1 + \int_0^t x_2(s)ds = 1 + \int_0^t (1 + s + \frac{1}{2}s^2)ds = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3, \\
\dots & \quad \dots \\
x_k(t) &= 1 + \int_0^t x_{k-1}(s)ds = 1 + t + \frac{1}{2}t^2 + \dots + \frac{1}{k!}t^k.
\end{aligned}$$

The sequence $x_k(t)$ converges uniformly to $x(t) = \sum \frac{1}{k!}t^k = e^t$, which is the solution to $x' = x$, $x(0) = 1$.

It is important to note that Theorem 2.2.1 is *local*, because it ensures that the solution exists (and is unique) in a *suitable* interval around t_0. The following example shows that the solution may not be defined on all of $\mathbb{R}$ even if the equation makes sense everywhere.

Example 2.2.2. Consider the ivp

$$\begin{cases} x' &= x^2 \\ x(0) &= a \neq 0. \end{cases} \tag{2.5}$$

Let us first solve the equation $x' = x^2$. This is a so-called "separable equation" and will be discussed in Section 3.1 of Chapter 3. Here, instead of using the general method, we will find the solutions by a direct argument which uses some intuition.

We have to find functions $x(t)$ whose derivatives are equal to $x^2(t)$. One choice could be $x(t) = -\frac{1}{t}$, because $x' = \frac{1}{t^2}$ which equals $x^2 = \frac{1}{t^2}$. More generally, consider the functions

$$\phi(t,c) = -\frac{1}{t-c}, \qquad c \in \mathbb{R}.$$

Since

$$\frac{d}{dt}\phi(t,c) = \frac{1}{(t-c)^2} = \phi^2(t,c),$$

it follows that, for all real constants c, the functions $\phi(t,c)$ solve $x' = x^2$.

To find the solution $x_a(t)$ of the Cauchy problem (2.5) we impose the requirement that $\phi(0,c) = a$, that is

$$a = -\frac{1}{0-c} = \frac{1}{c} \iff c = \frac{1}{a}.$$

Thus we find

$$x_a(t) = -\frac{1}{t - \frac{1}{a}} \,. \tag{2.6}$$

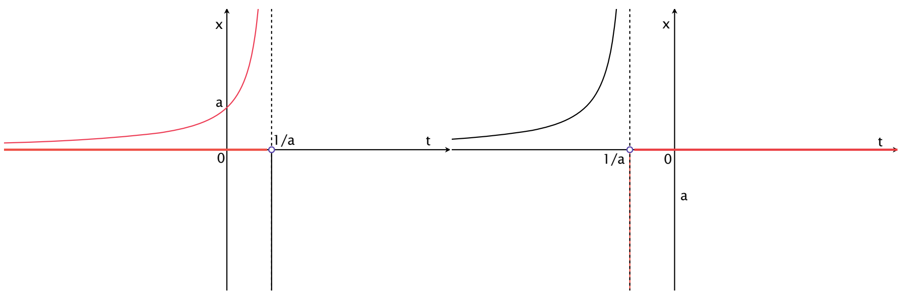
Let us notice explicitly that the solution is not defined for all t but only for $t \neq \frac{1}{a}$, despite the fact that the function x^2 in the equation is defined and continuous everywhere. This is one of the peculiarities of nonlinearity, compared with linear.

Saying that (2.6) solves (2.5) is in some sense an abuse of language because we did not specify the interval where (2.6) has to be considered. To be precise, the solution of the ivp (2.5) is (see Figure 2.2)

$$\text{if } a > 0: \; x_a(t) = -\frac{1}{t - \frac{1}{a}}, \; t < \frac{1}{a}\,, \qquad \text{if } a < 0: \; x_a(t) = -\frac{1}{t - \frac{1}{a}}, \; t > \frac{1}{a}\,.$$

The reason why we cannot take all of $\mathbb{R} \setminus \{\frac{1}{a}\}$ is that a solution of a differential equation is differentiable and hence - in particular - continuous, which is not the case if we consider (2.6) as defined on $\mathbb{R} \setminus \{\frac{1}{a}\}$. ∎

Now we want to extend the preceding discussion to the general case of the Cauchy problem (2.5). Recall that solving the Cauchy problem (2.5) means to find a function $x_a(t)$ and an interval I containing t_0, such that $x'(t) = f(t, x(t))$ for all $t \in I$ and $x(t_0) = x_0$. In other words, we have to find not only the function $x(t)$ such that $x' = f(t, x)$ and $x(t_0) = x_0$, but we also have to specify the interval I where the solution has to be considered. It is important to point out that it might happen that this interval I is strictly contained in the set S of all t where $x(t)$ makes sense. As explained before, the reason is that $x(t)$ could be discontinuous (or not differentiable) on S while, as a solution of a differential equation, $x(t)$ has to be continuous, differentiable indeed. The problem (2.5) discussed before is just an example in which $I \subsetneq S$. Actually, in that case $S = \{t \in \mathbb{R} : t \neq \frac{1}{a}\}$ while $I = (-\infty, \frac{1}{a})$ (if $a > 0$) or $I = (\frac{1}{a}, +\infty))$ (if $a < 0$).



(a) (b)

Fig. 2.2. Solutions of $x' = x^2$, $x(0) = a$. (a) plot of $x_a(t)$, $a > 0$; (b) plot of $x_a(t)$, $a < 0$

The following definition is in order.

Definition 2.2.3. We say that $J \subset \mathbb{R}$ is the *maximal interval* of definition of the solution $x(t)$ of the Cauchy problem (2.4) if $x(t)$ is defined in J, and $x(t)$ cannot be extended in an interval greater than J.

For example, the maximal interval of definition of the solution $x_a(t)$ of $x' = x^2,\ x(t_0) = a \neq 0$, discussed in the preceding Example 2.2.2, is the half line $(-\infty, \frac{1}{a})$ (if $a > 0$) or the half line $(\frac{1}{a}, +\infty)$ (if $a < 0$).

Lemma 2.2.4. *Let $x_0(t)$ be a solution of $x' = f(t,x)$. Suppose, for simplicity, that the set Ω where f is defined is all of $\mathbb{R}^2$. If J, the maximal interval of definition of the solution $x_0(t)$, is not all of $\mathbb{R}$, then it cannot be closed.*

Proof. By contradiction, let $J = [\alpha, \beta]$ or $J = (-\infty, \beta]$ or $J = [\alpha, +\infty)$. We deal with the first case, the other ones are similar.

Consider the new Cauchy problem

$$x' = f(t,x), \qquad x(\beta) = x_0(\beta)$$

in which the initial condition is prescribed at the point β and equals the value that $x_0(t)$ assumes at such a point. According to the local existence and uniqueness Theorem 2.2.1, the new ivp has a unique solution $x_1(t)$ defined in an interval $[\beta, \beta + \delta]$ for some $\delta > 0$. Consider the function obtained by gluing together x_0 and x_1 (see Figure 2.3), that is

$$x(t) = \begin{cases} x_0(t) & \text{if } \alpha \le t \le \beta \\ x_1(t) & \text{if } \beta \le t \le \beta + \delta. \end{cases}$$

Since $x_0(\beta) = x_1(\beta)$, the function $x(t)$ is continuous. Let us show that it is differentiable. This is clear for all $t \neq \beta$. At $t = \beta$ we consider the left and right limits of

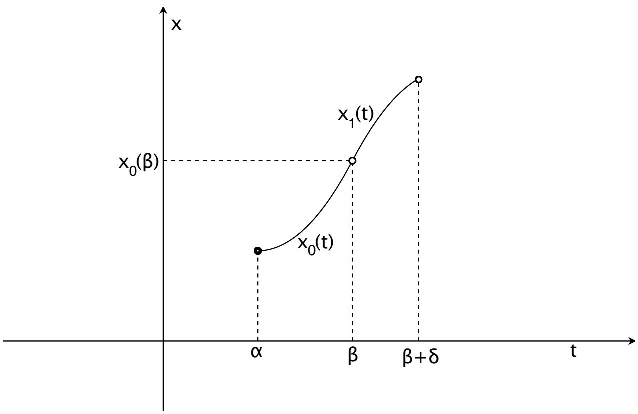


Fig. 2.3. Gluing $x_0(t)$ and $x_1(t)$

the difference quotients

$$\lim_{h\to 0-}\frac{x(\beta+h)-x(\beta)}{h},\qquad \lim_{h\to 0+}\frac{x(\beta+h)-x(\beta)}{h}.$$

For $h \le 0$, we have $x(\beta+h) = x_0(\beta+h)$ and hence

$$\lim_{h\to 0-}\frac{x(\beta+h)-x(\beta)}{h} = \lim_{h\to 0-}\frac{x_0(\beta+h)-x_0(\beta)}{h} = x_0'(\beta) = f(\beta, x_0(\beta)).$$

Similarly, for $h \ge 0$ we find

$$\lim_{h\to 0+}\frac{x(\beta+h)-x(\beta)}{h} = \lim_{h\to 0+}\frac{x_1(\beta+h)-x_1(\beta)}{h} = x_1'(\beta) = f(\beta, x_1(\beta)).$$

Since $x_0(\beta) = x_1(\beta)$, it follows that

$$\lim_{h\to 0-}\frac{x(\beta+h)-x(\beta)}{h} = \lim_{h\to 0+}\frac{x(\beta+h)-x(\beta)}{h},$$

and this means that $x(t)$ is differentiable at $t = \beta$.

We have found a solution of $x' = f(t,x)$ defined in $[\alpha, \beta+\delta]$ in contradiction with the fact that $J = [\alpha, \beta]$ is the maximal interval. The argument for the left end point α is the same. ∎

Proposition 2.2.5. *Let $f(t,x)$ satisfy the assumptions of the local existence and uniqueness theorem, with $\Omega = \mathbb{R}^2$. If $x(t)$ is a solution of $x' = f(t,x)$ which is monotone and bounded, then its maximal interval of definition J is all of $\mathbb{R}$.*

Proof. By contradiction, suppose that J is strictly contained in $\mathbb{R}$. For example, let us assume that J is bounded (the other cases are treated in a similar way). Let us show that J is closed. Let $\beta < +\infty$ be the right extreme of J. Since $x(t)$ is monotone and bounded, the limit $\lim_{t\to\beta-} x(t)$ exists and is finite. Thus $x(t)$ is defined also at $t = \beta$ and hence J contains β. Same argument for the left extreme $\alpha > -\infty$. We have shown that J is closed and this is in contradiction with the preceding Lemma. ∎

Concerning the fact that the solution of the ivp (2.4) is unique, we have required that f be differentiable with respect to x. The following example shows that if this condition is violated, the solution may not be unique.

Example 2.2.6. Consider the Cauchy problem

$$\begin{cases} x' &= 2\sqrt{x} \\ x(0) &= 0. \end{cases} \tag{2.7}$$

This is also a separable equation discussed in Section 3.1 of Chapter 3. One solution is given by $x(t) \equiv 0$. Another solution is given by

$$x(t) = t^2 \qquad t \ge 0,$$

because $\frac{d}{dt}(t^2) = 2t = 2\sqrt{t^2} = 2|t| = 2t$ for $t \geq 0$. Note that for $t < 0$ one has $|t| = -t$ and hence $x = t^2$ is not a solution for $t < 0$.

We have found two solutions that solve the ivp (2.7). Furthermore, one can verify that, for any $a > 0$, all the functions

$$x_a(t) = \begin{cases} 0, & \text{for } 0 \leq t \leq a \\ (t-a)^2, & \text{for } t \geq a \end{cases}$$

are solutions. So, (2.7) has infinitely many solutions. See Figure 2.4. Notice that $2\sqrt{|x|}$ is not differentiable at $x = 0$.

On the other hand, the ivp $x' = 2\sqrt{x}$, $x(0) = a$, has a unique solution provided $a > 0$. Actually, $f(x) = 2\sqrt{x}$ is differentiable in the *open* half plane $x > 0$ and Theorem 2.2.1 applies. One can check that the function $x^*(t) = \left(t + \sqrt{a}\right)^2$ solves the ivp, is defined for all t and is the unique solution. ■

Remark 2.2.7. An important consequence of the uniqueness result stated in Theorem 2.2.1 is that two solutions of $x' = f(t,x)$ cannot cross each other. In other words, if $v(t)$ and $z(t)$ are two solutions of $x' = f(t,x)$ defined on a certain interval $[a,b]$ and if there exists $t^* \in [a,b]$ such that $v(t^*) = z(t^*)$, then $v(t) = z(t)$ for all $t \in [a,b]$. The reason is that both v and z satisfy the same ivp

$$\begin{cases} x' &= f(t,x) \\ x(t^*) &= v(t^*) = z(t^*). \end{cases}$$

So, by uniqueness one must have $x(t) = z(t)$ on $[a,b]$. ■

We will see later on that we can also use the uniqueness result to deduce geometric properties of the solution of an ivp. See e.g. Section 2.3 below.

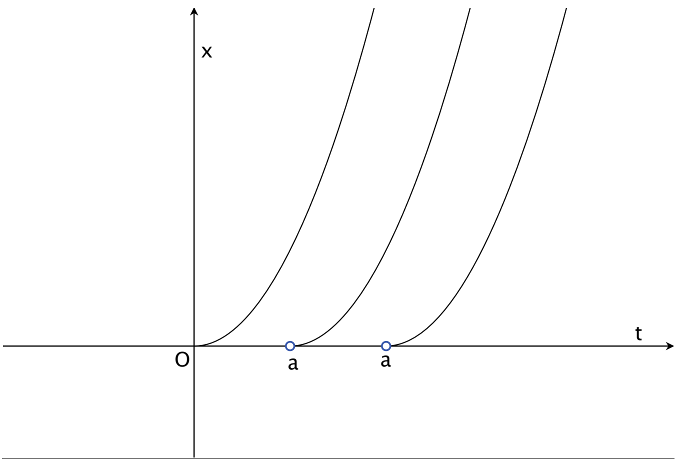


Fig. 2.4. Solutions of $x' = \sqrt{x}$

The following theorem, due to G. Peano[3] shows that the existence part (but not the uniqueness) of Theorem 2.2.1 requires *only continuity* of the function f. The proof of this theorem requires some more advanced topics, in particular the Ascoli[4] compactness theorem,[5] and is omitted.

Theorem 2.2.8 (Local existence). *Suppose that f is continuous in $\Omega \subseteq \mathbb{R}^2$ and let (t_0, x_0) be a point in the interior of Ω. Then the Cauchy problem*

$$\begin{cases} x' &= f(t,x) \\ x(t_0) &= x_0 \end{cases}$$

has at least one solution defined in a neighborhood of t_0.

For example, this result applies to the Cauchy problem $x' = 2\sqrt{x},\ x(0) = 0$, discussed earlier, and guarantees that it has at least one solution.

Remark 2.2.9. If the equation is $F(t, x, x') = 0$, we first have to put it in normal form, if possible, and then use the existence and uniqueness results sated above. For example, if $a(t) \neq 0$ for all t then $a(t)x' = f(t,x)$ is clearly equivalent to the equation

$$x' = \frac{f(t,x)}{a(t)}$$

which is in normal form. However, if $a(t)$ vanishes somewhere, solving $a(t)x' = f(t,x)$ is more complicated and requires a specific study.

For example, for all $c \in \mathbb{R}$ the straight lines $x = ct$ are solutions of $tx' = x$. In particular, there are infinitely many solutions passing through $(0,0)$. On the other hand, the solutions of $tx' = -x$ are given by $x = ct^{-1}$, $c \in \mathbb{R}$. If $c = 0$ we find $x = 0$ which is the only solution passing through $(0,0)$, while if $c \neq 0$ the solutions are not defined for $t = 0$, where the equation has a *singularity*.

The possible zeros of $a(t)$ are called *singular points* of the equation $a(t)x' = f(t,x)$. A discussion of some linear equations with singular points will be carried out in Chapter 10. ■

2.2.2 Global existence and uniqueness

As mentioned before, Theorem 2.2.1 is local. The next *global* result holds, provided the set Ω is a strip and f_x is bounded w.r.t. x.

Theorem 2.2.10 (Global Existence and Uniqueness). *Let Ω be the strip $\Omega = [a,b] \times \mathbb{R}$ and let (t_0, x_0) be a given point in the interior of Ω. Suppose that f is continuous in Ω and has continuous partial derivative with respect to x and that the*

[3] Giuseppe Peano (1858–1932).

[4] Guido Ascoli (1887–1957).

[5] A statement of the Ascoli Theorem is reported in Chapter 13.

partial derivative $f_x(t,x)$ is bounded in the strip. Then the Cauchy problem

$$\begin{cases} x' &= f(t,x) \\ x(t_0) &= x_0 \end{cases}$$

has a unique solution defined for all $t \in [a,b]$.

Corollary 2.2.11. *If $\Omega = \mathbb{R}^2$, $\Omega = [a,+\infty)\times\mathbb{R}$, or $\Omega = (-\infty,b]\times\mathbb{R}$, and $f_x(t,x)$ is bounded in Ω, then the solution is defined respectively on all of $\mathbb{R}$, on $[a,+\infty)$, or on $(-\infty,b]$.*

Theorem 2.2.10 and Corollary 2.2.11 are particular cases of the more general Theorem 2.4.5 in the next section.

The new feature of the preceding results is that now the solution is defined on the whole interval $[a,b]$.

Remark 2.2.12. Example 2.2.2 shows that the condition that f_x is bounded in the strip cannot be removed. ■

Example 2.2.13. Let $p,q \in C([a,b])$ and consider the linear equation $x' + p(t)x = q(t)$ discussed in Chapter 1. In this case, $f(t,x) = -p(t)x + q(t)$ and $f_x(t,x) = -p(t)$, which is bounded in $[a,b]$ and hence Theorem 2.2.10 applies. This provides an alternate proof of the existence and uniqueness result stated in Theorem 1.4.2 in Chapter 1. Note that the solutions of $x' + p(t)x = q(t)$ are defined on the whole interval $[a,b]$. Moreover, Corollary 2.2.11 implies that, if $p,q \in C(\mathbb{R})$ the solutions are defined on all of $\mathbb{R}$. ■

2.3 Qualitative properties of solutions

In this section we study some qualitative properties of solutions, using the Global Existence and Uniqueness result stated before.

In the sequel it is understood that the assumptions of this theorem are satisfied. Moreover, for simplicity, we will also assume that $\Omega = \mathbb{R}\times\mathbb{R}$.

We note that a certain qualitative behavior of a solution may be found without actually solving the corresponding equation explicitly. Apart from those equations that possess solutions in terms of elementary functions (which are very few: most of them are discussed in the next chapter), the procedure involving qualitative study is often the only way to understand certain features of the solutions, such as symmetry, monotonicity, asymptotic behavior, convexity, etc.

We start with simple symmetry results.

Lemma 2.3.1. *Let $f(x)$ be odd and let $x(t)$ be a solution of $x' = f(x)$. Then $-x(t)$ is also a solution.*

Proof. Setting $z(t) = -x(t)$ we find $z' = -x' = -f(x) = -f(-z)$ Since f is odd then $-f(-z) = f(z)$ and $z' = f(z)$. ■

Lemma 2.3.2. *Let $f(x)$ be even and let $x_0(t)$ be a solution of $x' = f(x)$ such that $x(0) = 0$. Then $x_0(t)$ is an odd function.*

Proof. Setting $z(t) = -x_0(-t)$ we find $z'(t) = x_0'(-t) = f(x_0(-t)) = f(-z)$ Since f is even then $f(-z) = f(z)$ and $z' = f(z)$. Moreover, $z(0) = x_0(0) = 0$. Thus, by uniqueness, $z(t) = x_0(t)$, namely $x_0(t) = -x_0(-t)$. ∎

Consider the autonomous equation

$$x' = f(x). \tag{2.8}$$

First of all, it is clear that for any number x_0, $f(x_0) = 0$ if and only if x_0 is a constant solution of (2.8). By uniqueness, the non constant solutions cannot cross the constant ones. Roughly, the possible constant solutions divide the plane (t, x) into horizontal strips (which could be bounded or not), filled up by non constant solutions of (2.8). More precisely, taken any $(t_0, x_0) \in \mathbb{R}^2$, let $\phi(t)$ denote the solution of the ivp

$$\begin{cases} x' &= f(x), \\ x(t_0) &= x_0. \end{cases} \tag{2.9}$$

If $f(x_0) = 0$ then $\phi(t) \equiv x_0$ is a constant solution. If $f(x_0) > 0$ then $\phi(t)$ is increasing, while if $f(x_0) < 0$ then $\phi(t)$ is decreasing. If $\phi(t)$ (is defined for all t and) is increasing or decreasing, it has limits $L_\pm$ (finite or infinite) as $t \to \pm\infty$. The finite limits are zeroes of f. To see this, let e.g. $L_+ < +\infty$. In such a case it is easy to check that $\phi'(t) \to 0$ as $t \to +\infty$. Passing to the limit in the identity $\phi'(t) = f(\phi(t))$ we find

$$0 = \lim_{t\to+\infty} f(\phi(t)) = \lim_{x\to L_+} f(x) = f(L_+).$$

Furthermore, if f is differentiable, then $\phi(t)$ is twice differentiable and one has

$$\phi''(t) = \frac{d\phi'}{dt} = \frac{d}{dt} f(\phi(t)) = f'(\phi(t))\phi'(t) = f(\phi(t))f'(\phi(t)). \tag{2.10}$$

This allows us to find the sign of $\phi''(t)$ and hence the convexity of ϕ.

Example 2.3.3. Let $\alpha, \beta, \alpha < \beta$ be two consecutive zeroes of f and suppose that $f(x) > 0$ for $\alpha < x < \beta$. Taking $x_0 \in (\alpha, \beta)$, from the discussion above it follows that $\phi(t)$ is increasing, $\alpha < \phi(t) < \beta$ for all t and

$$\lim_{t\to-\infty} \phi(t) = \alpha, \qquad \lim_{t\to+\infty} \phi(t) = \beta.$$

Moreover, let us assume that f is differentiable, and let $x^* \in (\alpha, \beta)$ be such that $f'(x^*) = 0$, $f'(x) > 0$ for $\alpha < x < x^*$ and $f'(x) < 0$ for $x^* < x < \beta$. Since $\phi(t)$ is increasing and $\alpha < \phi(t) < \beta$, then the equation $\phi(t) = x^*$ has a unique solution t^* and $\alpha < \phi(t) < x^*$ for $t < t^*$ while $x^* < \phi(t) < \beta$ for $t > t^*$. Hence (2.10)

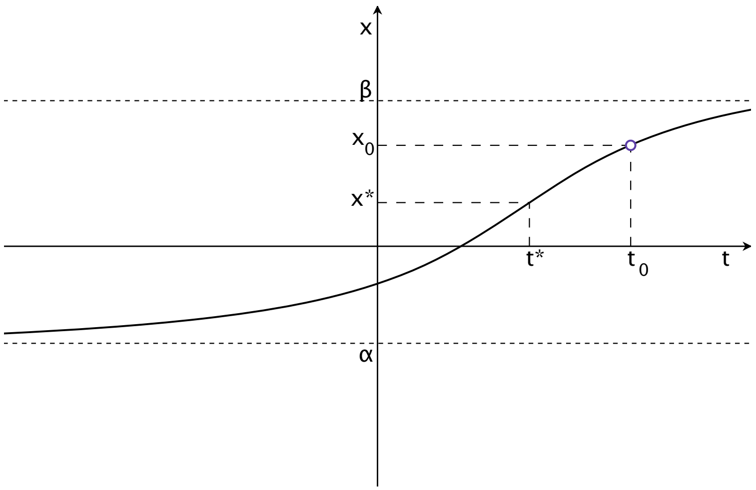


Fig. 2.5. Qualitative behavior of the solution $\phi(t)$ of (2.9)

implies that $\phi''(t^*) = 0$, $\phi''(t) > 0$ for $\alpha < t < t^*$ and $\phi''(t) < 0$ for $t < t^* < \beta$. See Figure 2.5 where we have taken $\alpha < 0 < x^* < x_0 < \beta$. ■

Dealing with a general non autonomous equation

$$x' = f(t, x)$$

the situation is quite different. We limit ourselves to discuss the possible maxima or minima of a solution $\phi(t)$. If t^* is such that $\phi'(t^*) = 0$ then, setting $x^* = \phi(t^*)$, and $M_0 = \{(t, x) \in \mathbb{R}^2 : f(t, x) = 0\}$ one has

$$f(t^*, x^*) = 0, \quad i.e. \ (t^*, x^*) \in M_0.$$

Moreover, if f is differentiable, from the identity $\phi'(t) = f(t, \phi(t))$ it follows that

$$\phi''(t) = f_t(t, \phi(t)) + f_x(t, \phi(t))\phi'(t) = f_t(t, \phi(t)) + f_x(t, \phi(t)) f(t, \phi(t)).$$

If t^* is such that $\phi'(t^*) = 0$, then

$$\phi''(t^*) = f_t(t^*, x^*). \tag{2.11}$$

Hence ϕ has a maximum or a minimum at $t = t^*$ depending on whether $f_t(t^*, x^*) < 0$ or $f_t(t^*, x^*) > 0$. In other words, letting

$$M_- = \{(t, x) \in \mathbb{R}^2 : f_t(t, x) < 0\}, \quad M_+ = \{(t, x) \in \mathbb{R}^2 : f_t(t, x) > 0\},$$

one has

(i) If $(t^*, x^*) = (t^*, \phi(t^*)) \in M_0 \cap M_-$ then $\phi(t)$ has a maximum at $t = t^*$.
(ii) If $(t^*, x^*) = (t^*, \phi(t^*)) \in M_0 \cap M_+$ then $\phi(t)$ has a minimum at $t = t^*$.

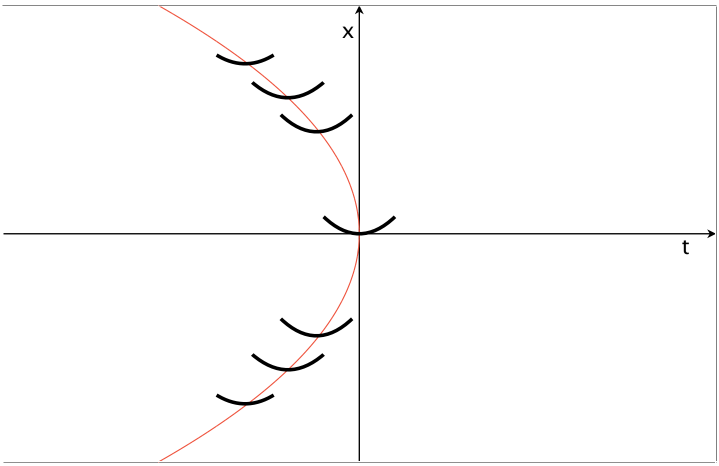


Fig. 2.6. The red curve is the parabola $x^2 = -4t$, where the solutions of $x' = x^2 + 4t$ have minima

Example 2.3.4. (i) Consider the equation $x' = x^2 + 4t$. Setting $f(t, x) = x^2 + 4t$ we find $M_0 = \{(t, x) \in \mathbb{R}^2 : x^2 = -4t\}$. Furthermore $f_t(t, x) = 4$ and thus $M_- = \emptyset$ and $M_+ = \mathbb{R}^2$. Therefore the solutions of $x' = x^2 + 4t$ have only minima located on the parabola $x^2 = -4t$. See Figure 2.6.

(ii) Show that the solution $\phi(t)$ of $x' = x^2 - t^2$, $x(a) = a \neq 0$ has a maximum at $t = a$, if $a > 0$, a minimum if $a < 0$.

One has $\phi'(a) = \phi^2(a) - a^2 = a^2 - a^2 = 0$. Moreover, using (2.11) we infer that $\phi''(a) = -2a$. Thus $\phi''(a) < 0$ or $\phi''(a) > 0$ depending on whether $a > 0$ or $a < 0$. Therefore if $a > 0$ then ϕ has a maximum at $t = a$; if $a < 0$ then ϕ has a minimum at $t = a$. See Figure 2.7. The case $a = 0$ is proposed as an exercise, see n.28 below. ■

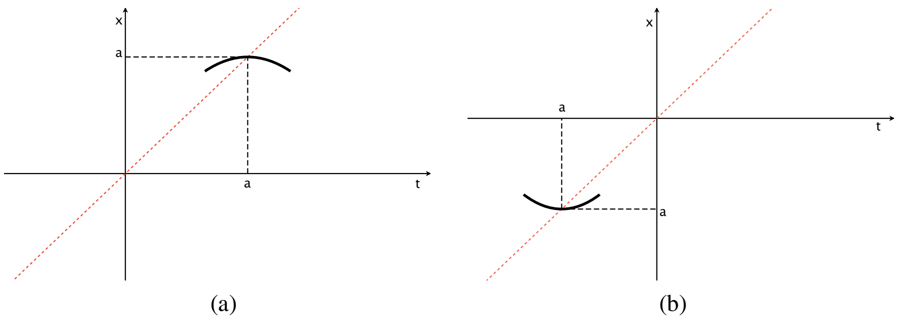


Fig. 2.7. Local behavior near $t = a$ of the solution of $x' = x^2 - t^2$, $x(a) = a \neq 0$. (a) $a > 0$; (b) $a < 0$

The following proposition is a symmetry result for a general non autonomous equation.

Proposition 2.3.5. *Suppose that $f(t, x)$ is odd with respect to t, that is $f(-t, x) = -f(t, x)$. Then the solutions of $x' = f(t, x)$ are even functions.*

Proof. Let $x(t)$ be any solution of $x' = f(t, x)$. Setting $z(t) = x(-t)$ one has

$$z'(t) = -x'(-t) = -f(-t, x(-t)) = -f(-t, z(t)).$$

Since, by assumption, $f(-t, z) = -f(t, z)$, we deduce

$$z' = f(t, z).$$

Thus $x(t)$ and $z(t)$ satisfy the same equation. Moreover x, z satisfy the same initial condition at $t = 0$ because one has $z(0) = x(0)$. Since f is continuously differentiable, the uniqueness statement in Theorem 2.4.4 applies and hence $x(t) = z(t)$, namely $x(t) = x(-t)$, proving that $x(t)$ is an even function, as required. ■

The next result allows us to compare solutions of two differential equations.

Theorem 2.3.6. *Let $x_a(t)$, $y_b(t)$ be solutions of the Cauchy problems*

$$\begin{cases} x' &= f(t, x) \\ x(t_0) &= a \end{cases} \qquad \begin{cases} y' &= g(t, y) \\ y(t_0) &= b \end{cases}$$

defined in a common interval $[t_0, \beta)$. If $a < b$ and $f(t, x) < g(t, x)$, then $x_a(t) < y_b(t)$ for all $t \in [t_0, \beta)$.

Proof. We argue by contradiction. Suppose that the set

$$S = \{t \in [t_0, \beta) : x_a(t) \geq y_b(t)\}$$

is not empty. Let σ be its infimum, namely its greatest lower bound. Since $x_a(t_0) = a < b = y_b(t_0)$, by the Sign Permanence Theorem of continuous functions, there exists $\epsilon > 0$ such that $x_a(t) < y_b(t)$ for all $t \in [t_0, t_0 + \epsilon)$ and thus $\sigma \geq t_0 + \epsilon > t_0$. Moreover, since σ is the infimum of the set S then there exists a sequence $t_j > \sigma$ with $t_j \to \sigma$ and such that $x_a(t_j) \geq y_b(t_j)$. Passing to the limit one finds $x_a(\sigma) \geq y_b(\sigma)$. But $x_a(\sigma)$ cannot be strictly greater than $y_b(\sigma)$ because, otherwise, using again the Sign Permanence Theorem, we would have $x_a(t) > y_b(t)$ in a left neighborhood of σ and this is in contradiction with the fact that $\sigma = \inf S$.

Recall that $x_a(\sigma + h) < y_b(\sigma + h)$ for $h < 0$ small, because $\sigma = \inf S$. Then, taking into account that $x_a(\sigma) = y_b(\sigma)$ and that $h < 0$ we deduce that the incremental ratios satisfy

$$\frac{x_a(\sigma + h) - x_a(\sigma)}{h} > \frac{y_b(\sigma + h) - y_b(\sigma)}{h}, \quad \forall h < 0, \text{ small}.$$

Passing to the limit as $h \to 0$, $h < 0$, we infer that $x_a'(\sigma) \geq y'(\sigma)$. But this is impossible, since, by assumption,

$$x_a'(\sigma) = f(\sigma, x_a(\sigma)) < g(\sigma, y_b(\sigma)) = y'(\sigma).$$

We have proved that S is empty and therefore that $x_a(t) < y_b(t)$ for all $t \in [t_0, \beta)$. ∎

The next examples show how we might apply the comparison theorem.

Example 2.3.7. (i) Let $x_a(t)$ be a positive solution of $x' = f(x)$ such that $x(t_0) = a$. Suppose that $f(x) < -kx$ for some $k > 0$, and that $x_a(t)$ is defined on an interval $[t_0 + \infty)$. Then $x_a(t)$ decays exponentially to 0 as $t \to +\infty$. To see this, let $y_b(t)$ be the solution of $y' = -ky$, $y(t_0) = b > \max\{a, 0\}$, namely $y_b(t) = be^{-k(t-t_0)}$. Then applying the previous Proposition with $g(y) = -ky$, it follows that $0 < x(t) \leq be^{-k(t-t_0)}$ and the result follows.

(ii) Let $y_b(t)$ be the solution of

$$\begin{cases} y' & = g(t, y) \\ y(t_0) & = b \end{cases}$$

and suppose it is defined on $[t_0 + \infty)$. If $g(t, y) > k > 0$ and $b > 0$, then $\lim_{t\to+\infty} y_b(t) = +\infty$. Applying the Proposition with $f(t, x) = k$ and $a = 0$, we infer $y_b(t) \geq k(t - t_0)$, from which the claim follows at once. ∎

2.4 Improving the existence and uniqueness results

The assumptions in the preceding theorems can be weakened, which means that the results can be extended to include a larger class of functions. We indicate below the main extension of this kind.

Definition 2.4.1. The function $f(t, x)$ defined in a set $\Omega \subseteq \mathbb{R}^2$, is *locally* lipschitzian[6] (or simply lipschitzian) at a point $(t_0, x_0) \in \Omega$ with respect to x, if there exists a neighborhood $U \subset \Omega$ of (t_0, x_0) and a number $L > 0$ such that

$$|f(t, x) - f(t, z)| \leq L|x - z|, \qquad \forall\ (t, x),\ (t, z) \in U.$$

We say that f is *globally* lipschitzian on Ω if there exists $L > 0$, such that

$$|f(t, x) - f(t, z)| \leq L|x - z|, \qquad \forall\ (t, x),\ (t, z) \in \Omega.$$

From the definition it immediately follows that any locally lipschitzian function is continuous at (t_0, x_0). Moreover, one has:

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[6] Rudolph Lipschitz (1832–1903).

Lemma 2.4.2. *Let $f(t,x)$ be continuously differentiable with respect to x in Ω. If there exists $\epsilon > 0$ such that $f_x(t,x)$ is bounded in $U = \{|t-t_0| < \epsilon\}\times\{|x-x_0| < \epsilon\}$, then f is lipschitzian on U.*

Proof. Applying the Mean Value Theorem to the function $f(t,x)$ we infer that

$$f(t,x) - f(t,z) = f_x(t,\xi)(x-z),$$

where $x < \xi < z$. Since $L = \sup\{|f_x(t,\xi)| : (t,\xi) \in U\}$ is finite by assumption, it follows that

$$|f(t,x) - f(t,z)| \le L|x-z|, \qquad \forall\ (t,x),\ (t,z) \in U,$$

proving the lemma. ∎

Example 2.4.3. (i) The function $f(x) = |x|$ is globally lipschitzian with constant $L = 1$, but is not differentiable at $x = 0$. Actually, $|f(x) - f(z)| = ||x| - |z|| \le |x - z|$ for all $x, z \in \mathbb{R}$. Moreover, since

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ \ \ x & \text{if } x > 0 \end{cases}$$

then the left derivative of f at $x = 0$ is -1, while the right derivative is $+1$. Thus f is not differentiable at $t = 0$.

(ii) The function $f(x) = x^2$ is locally lipschitzian at any point but not globally lipschitzian on $\mathbb{R}$. To prove this claim we first notice that $f(x)$ is differentiable with derivative $f'(x) = 2x$, which is bounded on every bounded subset of $\mathbb{R}$. Then, according to the previous lemma, f is locally lipschitzian at any point. If f were globally lipschitzian on $\mathbb{R}$, then there would exist $L > 0$ such that $|x^2 - z^2| \le L|x - z|$ for all $x, y \in \mathbb{R}$. Since $|x^2 - z^2| = |x + z| \cdot |x - z|$ it follows that $|x + z| \le L$ for all $x, z \in \mathbb{R}$, which is obviously false.

(iii) The function $f(x) = \sqrt{|x|}$ is not lipschitzian at $x = 0$. Otherwise, there would exist $\epsilon > 0$ and $L > 0$ such that $\left|\sqrt{|x|} - \sqrt{|z|}\right| \le L|x - z|$ for all $x, z \in (-\epsilon, \epsilon)$. In particular, taking $z = 0$, we get $\sqrt{|x|} \le L|x|$ for all $x \in (-\epsilon, \epsilon)$, which is obviously false. ∎

Using the previous definition, one can prove the following local and global existence result which holds for equations in normal form and extend the existence and uniqueness Theorems 2.2.1 and 2.2.10 as well as Corollary 2.2.11.

Theorem 2.4.4. *Let (t_0, x_0) be a given point in the interior of Ω. If f is locally lipschitzian with respect to x at (t_0, x_0), then the Cauchy problem*

$$\begin{cases} x' &= f(t,x) \\ x(t_0) &= x_0 \end{cases}$$

has a unique solution defined in a suitable neighborhood of t_0.

In Example 2.2.6 we have shown that the ivp $x' = 2\sqrt{|x|}$, $x(0) = 0$, has infinitely many solutions. Notice that the function $f(x) = 2\sqrt{|x|}$ is not lipschitzian at $x = 0$ (see Example 2.4.3(iii) above). This shows that the preceding result is sharp, in the sense that we cannot guarantee uniqueness of the Cauchy problem (2.4) if f is not lipschitzian at (t_0, x_0).

Theorem 2.4.5. *Suppose that* $\Omega = [a, b] \times \mathbb{R}$ *(resp.* $\Omega = \mathbb{R} \times \mathbb{R}$*), and* f *is globally lipschitzian in* Ω*. Let* $(t_0, x_0) \in \Omega$ *be given. Then the Cauchy problem*

$$\begin{cases} x' &= f(t, x) \\ x(t_0) &= x_0 \end{cases}$$

has a unique solution defined on all of $[a, b]$ *(resp. on all of* $\mathbb{R}$*).*

The proofs are given in the Appendix below.

Example 2.4.6. Since $|x|$ is globally lipschitzian, the Cauchy problem $x' = |x|$, $x(0) = x_0$ has a unique solution $x(t)$, defined for $t \in \mathbb{R}$. Precisely, if $x_0 = 0$ then $x(t) \equiv 0$. The other solutions $x(t)$ never vanish. If $x_0 > 0$, then $x(t) > 0$ and the equation becomes $x' = x$, and hence $x(t) = x_0 e^t$. If $x_0 < 0$ then $x(t) < 0$, the equation is $x' = -x$ and $x(t) = x_0 e^{-t}$. In any case $x(t)$ is increasing provided $x_0 \neq 0$. ■

2.5 Appendix: Proof of existence and uniqueness theorems

2.5.1 Proof of Theorem 2.4.5

Let us first prove Theorem 2.4.5 dealing with uniqueness and global existence of the Cauchy problem

$$\begin{cases} x' &= f(t, x) \\ x(t_0) &= x_0 \end{cases} \tag{2.12}$$

where $f(t, x)$ is defined in the strip $S = \{(t, x) \in \mathbb{R}^2 : a \leq t \leq b\}$, $(t_0, y_0) \in S$ and f is continuous and globally lipschitzian in S. Let us recall that this means that there exists $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L|x - y|, \qquad \forall\, (t, x), (t, y) \in S. \tag{2.13}$$

The strategy is to find a sequence of functions that converges to the solution of (2.12). For this, it is convenient to transform the ivp (2.12) into an equivalent integral equation.

Lemma 2.5.1. $x(t)$ *is a solution of* (2.12) *if and only if* $x(t)$ *satisfies*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \qquad \forall\, t \in [a, b]. \tag{2.14}$$

Proof. Let $x(t)$ be a solution of (2.12). This means that $x'(t) \equiv f(t, x(t))$ and hence integrating from t_0 to t we find

$$\int_{t_0}^{t} x'(t)dt = \int_{t_0}^{t} f(s, x(s))ds, \qquad \forall\ t \in [a, b].$$

Since $x(t_0) = x_0$ the first integral is equal to $x(t) - x(t_0) = x(t) - x_0$ and thus, $\forall\ t \in [a, b]$ one has

$$x(t) - x_0 = \int_{t_0}^{t} f(s, x(s))ds \quad \Longrightarrow \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,$$

namely $x(t)$ satisfies (2.14).

Conversely, let $x(t)$ satisfy (2.14). If for $t \in [a, b]$ we set

$$\phi(t) = \int_{t_0}^{t} f(s, x(s))ds$$

by the fundamental theorem of calculus ϕ is continuous, differentiable and

$$\phi'(t) = f(t, x(t)).$$

Thus $x(t) = x_0 + \phi(t)$ is differentiable in $[a, b]$ and

$$x'(t) = \phi'(t) = f(t, x(t)), \qquad \forall\ t \in [a, b].$$

Moreover,

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(s, x(s))ds = x_0$$

and hence $x(t)$ satisfies the ivp (2.12), completing the proof of the lemma. ■

Define by recurrence a sequence of functions such that for all $t \in [a, b]$ and all integers $k = 0, 1, 2, \ldots$

$$\begin{aligned}
x_0(t) &= x_0 \\
x_1(t) &= x_0 + \int_{t_0}^{t} f(s, x_0)ds \\
x_2(t) &= x_0 + \int_{t_0}^{t} f(s, x_1(s))ds \\
\ldots & \quad \ldots \\
x_{k+1}(t) &= x_0 + \int_{t_0}^{t} f(s, x_k(s))ds.
\end{aligned}$$

Lemma 2.5.2. *The sequence $x_k(t)$ is uniformly convergent in* $[a, b]$.

Proof. Let us start by showing by induction that for all $k = 1, 2, ..$

$$|x_k(t) - x_{k-1}(t)| \le \frac{M}{L} \cdot \frac{|t - t_0|^k L^k}{k!}, \quad \forall\ t \in [a, b] \tag{2.15}$$

where $M = \max\{|f(t, x_0)| : t \in [a, b]\}$.

To simplify notation, we carry out the proof taking $t \ge t_0$. The case $t \le t_0$ requires obvious changes[7]. For $k = 1$ we have, using the assumption that f is lipschitzian,

$$\begin{aligned} |x_2(t) - x_1(t)| &= \left| \int_{t_0}^{t} (f(s, x_1(s) - f(s, x_0))ds \right| \\ &\le \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_0)|ds \\ &\le L \int_{t_0}^{t} |x_1(s) - x_0|ds. \end{aligned}$$

On the other hand,

$$|x_1(s) - x_0| = \left| \int_{t_0}^{s} f(r, x_0)dr \right| \le \left| \int_{t_0}^{s} |f(r, x_0)|dr \right| \le M \cdot |s - t_0|$$

and thus

$$|x_2(t) - x_1(t)| \le ML \int_{t_0}^{t} |s - t_0|ds = \frac{ML}{2}|t - t_0|^2, \qquad \forall\ t \in [a, b],$$

which proves (2.15) for $k = 1$.

By induction, we assume that (2.15) holds for $k - 1$. Repeating the previous arguments, we find

$$\begin{aligned} |x_k(t) - x_{k-1}(t)| &\le \int_{t_0}^{t} |f(s, x_{k-1}(s)) - f(s, x_{k-2}(s))|ds \\ &\le L \int_{t_0}^{t} |x_{k-1}(s) - x_{k-2}(s)|ds, \qquad \forall\ t \in [a, b]. \end{aligned}$$

Using the induction hypothesis we find

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\le L \cdot \frac{M}{L} \cdot \frac{L^{k-1}}{(k-1)!} \int_{t_0}^{t} |s - t_0|^{k-1} ds \\ &\le \frac{M}{L} \frac{L^k}{(k-1)!} \frac{|t - t_0|^k}{k} = \frac{M}{L} \frac{|t - t_0|^k L^k}{k!}, \quad \forall\ t \in [a, b]. \end{aligned}$$

Therefore (2.15) holds for all natural numbers k.

Since (2.15) holds for all $t \in [a, b]$, then

$$\max_{t \in [a,b]} |x_{k+1}(t) - x_k(t)| \le \frac{M}{L} \frac{(b - a)^k L^k}{k!}. \tag{2.16}$$

[7] For example, in the next equation we should write $|x_2(t) - x_1(t)| \le \left| \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_0)|ds \right| \le \dots$

The sequence

$$\frac{(b-a)^k L^k}{k!} \to 0 \qquad (k \to +\infty)$$

because the series

$$\sum \frac{[L(b-a)]^k}{k!} \cdot t^k$$

is convergent to $e^{L(b-a)t}$. Thus (2.16) implies that the sequence $x_k(t)$ is uniformly convergent on $[a, b]$, as required. ∎

Proof of Theorem 2.4.5. By Lemma 2.5.2, the sequence $x_k(t) \to x(t)$, uniformly in $[a, b]$. Then $f(s, x_k(s)) \to f(s, x(s))$, uniformly in $[a, b]$ and we can pass to the limit under the integral in

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))ds$$

yielding

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

According to Lemma 2.5.1 it follows that $x(t)$ is a solution of the ivp (2.12).

It remains to prove the uniqueness. We will first consider an interval $|t - t_0| \le \delta$ where δ is such that $L\delta < 1$ (hereafter it is also understood that $t \in [a, b]$) and show that two solutions $x(t), y(t)$ of (2.12) coincide therein. One has

$$x(t) - y(t) = \int_{t_0}^{t} (f(s, x(s)) - f(s, y(s))ds$$

and hence

$$\begin{aligned} |x(t) - y(t)| &\le \left| \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|ds \right| \le L \left| \int_{t_0}^{t} |x(s) - y(s)|ds \right| \\ &\le L|t - t_0| \max_{|t-t_0| \le \delta} |x(t) - y(t)| \le L\delta \max_{|t-t_0| \le \delta} |x(t) - y(t)|. \end{aligned}$$

Taking the maximum of the left-hand side on $|t - t_0| \le \delta$ we find

$$\max_{|t-t_0| \le \delta} |x(t) - y(t)| \le \delta L \max_{|t-t_0| \le \delta} |x(t) - y(t)|.$$

Letting $A = \max_{|t-t_0! \le \delta} |x(t) - y(t)| > 0$, we divide by A finding $1 \le L\delta$, a contradiction because we have chosen δ such that $L\delta < 1$. Thus

$$\max_{|t-t_0| \le \delta} |x(t) - y(t)| = 0,$$

which implies that $x(t) = y(t)$ on the interval $|t - t_0| \le \delta$. In particular, $x(t_0 \pm \delta) = y(t_0 \pm \delta)$. We can now repeat the procedure in the interval $[t_0 + \delta, t_0 + 2\delta]$ and

$[t_0 - 2\delta, t_0 - \delta]$. Then we find that $x(t) = y(t)$ for all $t \in [t_0 - 2\delta, t_0 + 2\delta]$. After a finite number of times we find that $x(t) = y(t)$ for all $t \in [a, b]$. This completes the proof. ∎

2.5.2 *Proof of Theorem 2.4.4*

Here we will prove Theorem 2.4.4 on the local existence and uniqueness result for the ivp (2.12), where it is assumed that $f(t, x)$ is defined in $\Omega \subset \mathbb{R}^2$ and is locally lipschitzian near (t_0, x_0). Let us recall that this means that there exists a neighborhood $U \subset \Omega$ of (t_0, x_0) and a number $L > 0$ such that

$$|f(t, x) - f(t, y)| \le L|x - y|, \qquad \forall\, (t, x), (t, y) \in U. \tag{2.17}$$

Without loss of generality we can take $U = U_r$ as the (closed) square $\{(t, x) \in \Omega : |t - t_0| \le r,\ |x - x_0| \le r\}$, for some $r > 0$. We will deduce Theorem 2.4.4 from Theorem 2.4.5 proved in the preceding section. To this end, let us consider the strip

$$S_r := \{(t, x) \in \mathbb{R}^2 : |t - t_0| \le r\}$$

and extend f to the function $f^* : S_r \mapsto \mathbb{R}$ defined by setting

$$f^*(t, x) = \begin{cases} f(t, x_0 - r) & \text{if } (t, x) \in S_r \text{ and } x \le x_0 - r \\ f(t, x) & \text{if } (t, x) \in U_r \\ f(t, x_0 + r) & \text{if } (t, x) \in S_r \text{ and } x \ge x_0 + r. \end{cases}$$

It is easy to check that f^* is globally lipschitzian on S_r. For example, if x, y are such that $x_0 - r < x < x_0 + r \le y$, then $f^*(t, x) = f(t, x)$, $f^*(t, y) = f(t, x_0 + r)$ and one has

$$|f^*(t, x) - f^*(t, y)| = |f(t, x) - f(t, x_0 + r)| \le L|x - x_0 - r| \le L|x - y|.$$

Since f^* is globally lipschitzian on S_r, the global Theorem 2.4.5 yields a solution $x(t)$, defined on $[t_0 - r, t_0 + r]$, of the ivp

$$\begin{cases} x' = f^*(t, x) \\ x(t_0) = x_0. \end{cases}$$

The range of the function $x(t)$ could be outside $[x_0 - r, x_0 + r]$, where $f^* \ne f$. To overcome this problem we use the fact that $x(t)$ is continuous and $x(t_0) = x_0$. Then there exists $\delta > 0$ such that

$$t \in [t_0 - \delta, t_0 + \delta] \quad \Longrightarrow \quad |x(t) - x_0| \le r.$$

Therefore, for $t \in [t_0 - \delta, t_0 + \delta]$ one has that $f^*(t, x(t)) = f(t, x(t))$ and hence $x(t)$, restricted to such an interval, solves the ivp (2.12). ∎

2.6 Exercises

1. Check that the local existence and uniqueness theorem applies to the ivp $x' = t + x^2$, $x(0) = 0$.
2. Show that the function $f(x) = |x|^p$ is not lipschitzian at $x = 0$ if $0 < p < 1$.
3. Find a such that the existence and uniqueness theorem applies to the ivp $x' = \frac{3}{2}|x|^{1/3}$, $x(0) = a$.
4. Check that for all t_0, a the existence and uniqueness theorem applies to the ivp $\ln x' = x^2$, $x(t_0) = a$.
5. Explain why $x' + \frac{\sin t}{e^t+1}x = 0$ cannot have a solution $x(t)$ such that $x(1) = 1$ and $x(2) = -1$.
6. Transform the equation $e^{x'} = x$ into an equation in normal form and show that it has a unique solution such that $x(t_0) = a$, for all t_0 and all $a > 0$.
7. Find the equation whose solution is the catenary $x(t) = \cosh t = \frac{1}{2}(e^t + e^{-t})$.
8. Check that the functions $x(t) \equiv 1$ and

$$\phi(t) = \begin{cases} \sin t & \text{if } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ 1 & \text{if } t > \frac{\pi}{2} \end{cases}$$

   are solutions of the ivp $x' = \sqrt{1 - x^2}$, $x(\frac{\pi}{2}) = 1$.
9. Find $a \geq 0$ such that the Cauchy problem $x' = |x|^{1/4}$, $x(0) = a$ has a unique solution.
10. Show that if $p > 1$ the solution of $x' = x^p$, $x(0) = a > 0$, is not defined for all $t \geq 0$.
11. Show that if $0 < p \leq 1$, the solution of $x' = |x|^p$, $x(0) = a > 0$, is defined for all $t \geq 0$.
12. Show that the solutions of $x' = \sin x$ are defined on all $t \in \mathbb{R}$.
13. Show that the solutions of $x' = \arctan x$ are defined on all $t \in \mathbb{R}$.
14. Show that the solutions of $x' = \ln(1 + x^2)$ are defined on all $t \in \mathbb{R}$.
15. Show that the ivp $x' = \max\{1, x\}$, $x(0) = 1$, has a unique solution defined for all t and find it.
16. Show that the ivp $x' = \max\{1, -x\}$, $x(0) = -1$, has a unique solution defined for all t and find it.
17. Show that the solution of $x' = t^2x^4 + 1$, $x(0) = 0$ is odd.
18. Show that, if $f(x) > 0$, resp. $f(x) < 0$, the solutions of $x' = f(x)$ cannot be even.

19. Show that the solution $x(t)$ of the Cauchy problem $x' = 2 + \sin x$, $x(0) = 0$, cannot vanish for $t > 0$.

20. Let $f(x)$ be continuously differentiable and such that $f(0) = 0$. Show that the solutions of $x' = f(x)h(t)$ cannot change sign.

21. Show that the solutions of $x' = \sin(tx)$ are even.

22. Find the limits, as $t \to \pm\infty$, of the solution $\phi(t)$ of the ivp $x' = (x+2)(1-x^4)$, $x(0) = 0$.

23. Show that for every a, $-1 < a < 1$ the solution $\phi(t)$ of the ivp $x' = x^3 - x$, $x(0) = a$ is such that $\lim_{t\to+\infty} \phi(t) = 0$.

24. Show that the solutions of $x' = \arctan x + t$ cannot have maxima.

25. Show that the solutions of $x' = e^x - t$ cannot have minima.

26. Let $\phi(t)$ be the solution of the ivp $x' = tx - t^3$, $x(a) = a^2$ with $a \neq 0$. Show that ϕ has a maximum at $t = a$.

27. Let $\phi(t)$ be the solution of the ivp $x' = tx - t^3$, $x(0) = a^2$ with $a \neq 0$. Show that ϕ has a minimum at $t = 0$.

28. Show that the solution $\phi(t)$ of $x' = x^2 - t^2$, $x(0) = 0$, has an inflection point at $t = 0$.

29. Suppose that $g(x)$ is continuously differentiable and let $\phi(t)$ be the solution of $x' = tg(x)$, $x(0) = a$. If $g(a) > 0$, show that the function $\phi(t)$ has a minimum at $t = 0$, for all $a \in \mathbb{R}$.

30. Show that the solution $x_a(t)$ of $x' = 2t + g(x)$, $x_a(0) = a > 0$ satisfies $x_a(t) \geq t + t^2$ for $t > 0$, provided $g(x) \geq 1$.

31. Let $x_a(t)$ be the solution of of $x' = -t + g(x)$, $x_a(0) = a$,with $0 < a < 2$. If $x_a(t)$ is defined for all $t > 0$ and $g(x) \leq -x$, show using the comparison Theorem 2.3.6 that the equation $x_a(t) = 0$ has at least one positive solution in $(0, 2)$.

3

First order nonlinear differential equations

The main focus of this chapter is on learning how to solve certain classes of nonlinear differential equations of first order.

3.1 Separable equations

An equation of the form

$$x' = h(t)g(x) \tag{3.1}$$

is called a *separable equation*. Let us assume that $h(t)$ is continuous with $h(t) \not\equiv 0$ and $g(x)$ is continuously differentiable in the domain being considered, so that the local existence and uniqueness Theorem 2.2.1 of Chapter 2 applies.

If $x = k$ is any zero of g, $g(k) = 0$, then $x(t) \equiv k$ is a constant solution of (3.1). On the other hand, if $x(t) = k$ is a constant solution, then we would have $0 = h(t)g(k),\ \ t \in \mathbb{R}$, and hence $g(k) = 0$ since $h(t) \not\equiv 0$. Therefore, $x(t) = k$ is a constant solution (or *equilibrium solution*) if and only if $g(k) = 0$. There are no other constant solutions. All the non-constant solutions are separated by the straight lines $x = k$. Hence if $x(t)$ is a non-constant solution then $g(x(t)) \neq 0$ for any t, and we can divide

$$x' = h(t)g(x)$$

by $g(x)$ yielding

$$\frac{x'(t)}{g(x(t))} = h(t).$$

We integrate both sides with respect to t and obtain

$$\int \frac{x'(t)}{g(x(t))} dt = \int h(t) dt.$$

Since $x' = \frac{dx}{dt}$, we have

$$\int \frac{dx}{g(x)} = \int h(t) dt + c. \tag{3.2}$$

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We wish to point out that while it is very easy to express solutions of a separable equation implicitly in terms of integrals, it may be difficult or even impossible to perform the actual integration in terms of simple and familiar functions. In such cases, one can carry out a qualitative analysis to get some information about the behavior of solutions, see for example Section 2.3 in the previous chapter. Otherwise, if needed, one could use numerical methods or computer software to obtain approximate solutions to a reasonable or needed degree of accuracy.

If we want to solve the initial value problem

$$\begin{cases} x' &= h(t)g(x) \\ x(t_0) &= x_0 \end{cases} \tag{3.3}$$

we simply substitute the initial value $x(t_0) = x_0$ into (3.2) and solve for c. Note that this equation has a unique solution, according to Theorem 2.2.1 of Chapter 2.

Essentially, the idea behind solving separable equations is to separate the variables and then integrate.

Example 3.1.1. (i) Consider the equation $x' = h(t)x$. We notice that this is a linear homogeneous first order equation, and we learned in Chapter 1 how to solve such equations. But this is also a separable equation and we can solve it by the method described above. Separating the variables and then integrating, we obtain $\int \frac{dx}{x} = \int h(t)dt + c$, which yields $\ln|x| = \int h(t)dt + c$. Thus, letting $c_1 = \pm e^c$, we obtain the general solution

$$x(t) = c_1\, e^{\int h(t)dt}$$

in accordance with the result stated in Theorem 1.4.2 of Chapter 1.

(ii) Solve $x' = \frac{t^2}{1+3x^2}$.

There are no constant solutions. Separating the variables and integrating, we have $\int(1 + 3x^2)dx = \int t^2 dt + c$ and hence

$$x + x^3 = \frac{t^3}{3} + c$$

which defines the solutions implicitly. Moreover, since the function $\Phi(x) = x + x^3$ is increasing and its image is all of $\mathbb{R}$, it has an (increasing) inverse φ defined on all of $\mathbb{R}$. Thus $\Phi(x) = \frac{t^3}{3} + c$ yields $x(t) = \varphi(\frac{t^3}{3} + c)$. Note that the solutions are defined globally on $\mathbb{R}$. The reader might check that this also follows from the Global Existence Theorem 2.2.10 of Chapter 2.

(iii) Find the solution of the initial value problem $x' = 2tx^3$, $x(0) = 1$.
The only constant solution is $x \equiv 0$. Therefore if $x(t)$ is a solution such that $x(0) = 1$, then, by uniqueness, $x(t)$ cannot assume the value 0 anywhere. Since $x(0) = 1 > 0$, we infer that the solution is always positive. Using (3.2) we find

$$\int \frac{dx}{x^3} = \int 2t\, dt + c.$$

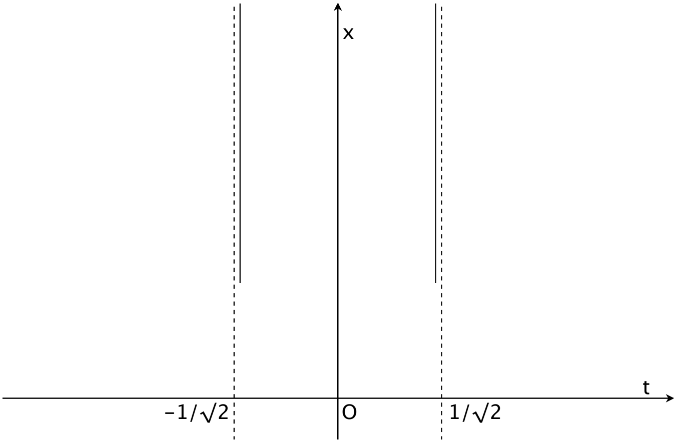

Fig. 3.1. Plot of $x = \frac{1}{\sqrt{1-2t^2}}$

Carrying out the integration it follows that

$$-\frac{1}{2x^2} = t^2 + c.$$

The initial condition $x(0) = 1$ yields $c = -\frac{1}{2}$ and hence

$$-\frac{1}{2x^2} = t^2 - \frac{1}{2}.$$

Solving for x, and recalling that $x > 0$, we find

$$x = \frac{1}{\sqrt{1-2t^2}}.$$

The graph of the solution is reported in Figure 3.1. Notice that in the present case the maximal interval of definition is given by $-\frac{1}{\sqrt{2}} < t < \frac{1}{\sqrt{2}}$. ■

3.1.1 The logistic equation

As a remarkable example of a separable equation, we consider a model, due to P.F. Verhulst[1], in which one assumes that the variation x' of the number of individuals in a population is proportional to x, but through a factor of $(\alpha - \beta x)$. This leads to the equation

$$x'(t) = x(t)(\alpha - \beta x(t)), \quad \alpha, \beta > 0. \tag{3.4}$$

Contrary to the Malthusian model discussed in the first chapter, here the constant factor k is substituted by the function $\alpha - \beta x$. The fact that this function is decreas-

[1] Pierre François Verhulst (1804 - 1849).

ing with respect to x may be explained by the observation that the bigger x is the more difficult it will be for an individual to find resources, such as food, space, etc, and hence the more difficult it will be to survive. This equation is called the *logistic equation*.

Since in this model, $x(t)$ represents the population of some species, we are interested in solutions $x(t)$ such that $x(t) \geq 0$. Solving $x(\alpha - \beta x) = 0$ we see that $x = 0$ and $x = \frac{\alpha}{\beta}$ are the equilibrium solutions. Such solutions play an important role in analyzing the trajectories of solutions in general. We now study solutions $x(t)$, where $x(t) > 0$ for all $t \geq 0$.

By uniqueness, the other solutions cannot cross the trajectories (which are lines in this case) defined by the two equilibrium solutions. Hence for any non-constant solution $x(t)$, $x(t) \neq 0$ and $x(t) \neq \alpha/\beta$ for any t. Thus, the two lines $x = 0$ and $x = \frac{\alpha}{\beta}$ divide the trajectories into two classes: those that lie above the line $x = \frac{\alpha}{\beta}$ and those that lie between the lines $x = 0$ and $x = \frac{\alpha}{\beta}$.

In Section 2.3 of the previous chapter we discussed how one might extract information about the qualitative behavior of solutions without solving the corresponding equations. We now discuss methods of finding solutions either explicitly or implicitly.

In order to solve the logistic equation, we separate the variables and obtain

$$\frac{dx}{x(\alpha - \beta x)} = dt$$

assuming that $x \neq 0$ and $x \neq \frac{\alpha}{\beta}$. The left side can be integrated by partial fractions method. We search for constants A and B such that

$$\frac{1}{x(\alpha - \beta x)} = \frac{A}{x} + \frac{B}{\alpha - \beta x}$$

and find $A = \frac{1}{\alpha}$ and $B = \beta A = \frac{\beta}{\alpha}$. Therefore, we have

$$\frac{1}{\alpha}\int \frac{dx}{x} + \frac{\beta}{\alpha}\int \frac{dx}{\alpha - \beta x} = \int dt + c$$

which yields

$$\frac{1}{\alpha}\ln|x| - \frac{1}{\alpha}\ln|\alpha - \beta x| = \frac{1}{\alpha}\ln|\frac{x}{\alpha - \beta x}| = t + c$$

from which we obtain

$$\left|\frac{x(t)}{\alpha - \beta x(t)}\right| = e^{\alpha(t+c)} = ke^{\alpha t}, \quad (k = e^{\alpha c}). \tag{3.5}$$

This is the general solution in implicit form. To solve for x we distinguish the cases $0 < x(t) < \frac{\alpha}{\beta}$ and $x(t) > \frac{\alpha}{\beta}$. In the first case one has that $\frac{x(t)}{\alpha - \beta x(t)} > 0$. Then (3.5) becomes

$$\frac{x(t)}{\alpha - \beta x(t)} = ke^{\alpha t}$$

and, solving for x,

$$x(t) = \frac{\alpha k e^{\alpha t}}{1 + \beta k e^{\alpha t}}. \tag{3.6}$$

If $x(t) > \frac{\alpha}{\beta}$ one has that $\frac{x(t)}{\alpha - \beta x(t)} < 0$. Then (3.5) becomes

$$-\frac{x(t)}{\alpha - \beta x(t)} = k e^{\alpha t}$$

and, solving for x,

$$x(t) = \frac{-k \alpha e^{\alpha t}}{1 - k \beta e^{\alpha t}}. \tag{3.7}$$

In any case, one finds that

$$\lim_{t \to +\infty} x(t) = \frac{\alpha}{\beta}.$$

This shows that all non-constant solutions approach the equilibrium solution $x(t) = \frac{\alpha}{\beta}$ as $t \to \infty$, some from above the line $x = \frac{\alpha}{\beta}$ and others from below (see Figure 3.2). The reader will notice that the behavior of $x(t)$ is totally different from the one found in the Malthusian model.

Assuming that the initial size of the poulation is a, we can determine the size of the population at any time t by solving the Cauchy problem

$$\begin{cases} x' &= x(\alpha - \beta x), \\ x(0) &= a. \end{cases}$$

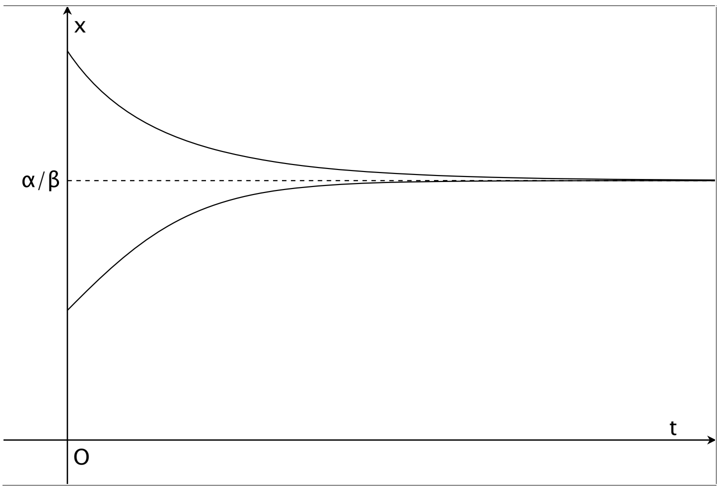

Fig. 3.2. Solutions of the logistic equation for $x(0) > \alpha/\beta$ and $x(0) < \alpha/\beta$

Assuming $a > 0$ and $a \neq \frac{\alpha}{\beta}$, we use (3.5) and the initial condition to find $k = \left|\frac{a}{\alpha - a\beta}\right|$. If $0 < a < \frac{\alpha}{\beta}$ then $k = \frac{a}{\alpha - a\beta}$, while if $a > \frac{\alpha}{\beta}$ then $k = -\frac{a}{\alpha - a\beta}$, namely $-k = \frac{a}{\alpha - a\beta}$. Therefore both (3.6) and (3.7) imply

$$x(t) = \frac{\dfrac{a\alpha}{\alpha - a\beta} e^{\alpha t}}{1 + \dfrac{a\beta}{\alpha - a\beta} e^{\alpha t}} = \frac{a\alpha e^{\alpha t}}{\alpha - a\beta + a\beta e^{\alpha t}}.$$

Example 3.1.2. Solve (i) $x' = x(2 - x)$, $x(0) = 1$ and (ii) $x' = x(2 - x)$, $x(0) = 3$. This is a logistic equation with $\alpha = 2$, $\beta = 1$, where the initial population size is given by $a = 1$ in case (i) and $a = 3$ in case (ii). Then we find

$$(i) \quad x(t) = \frac{2e^{2t}}{1 + e^{2t}}, \qquad (ii) \quad x(t) = \frac{6e^{2t}}{-1 + 3e^{2t}}.$$ ■

3.2 Exact equations

Consider the equations

$$N(x, y)\,\frac{dy}{dx} + M(x, y) = 0 \tag{3.8}$$

and

$$M(x, y)\,\frac{dx}{dy} + N(x, y) = 0. \tag{3.9}$$

Notice that in (3.8) we use y as dependent variable and x as the independent variable, while in (3.9) the roles of x and y are exchanged: x is now the dependent variable while y is the independent variable.

The preceding equations can be stated, in a differential form, as

$$M(x, y)dx + N(x, y)dy = 0. \tag{3.10}$$

Let us associate with (3.10) the differential form

$$\omega = M(x, y)dx + N(x, y)dy.$$

We say that (3.10) is an *exact equation* if ω is the exact differential of a function; that is, there exists an antiderivative $F(x, y)$ such that $dF = \omega$, which means that

$$\begin{cases} F_x(x, y) &= M(x, y) \\ F_y(x, y) &= N(x, y) \end{cases}$$

and hence

$$M(x, y)dx + N(x, y)dy = dF(x, y) = F_x(x, y)dx + F_y(x, y)dy.$$

Proposition 3.2.1. *Let M, N be continuous in $\Omega \subseteq \mathbb{R}^2$. Suppose that $M(x, y)dx + N(x, y)dy = 0$ is exact and let $F(x, y)$ denote an antiderivative of ω. If $y(x)$ is a solution of* (3.8) *then $F(x, y(x)) = c$, for some $c \in \mathbb{R}$. Conversely, if $y(x)$ is continuously differentiable and satisfies $F(x, y(x)) = c$, for some $c \in R$, then $y(x)$ satisfies* (3.8).

Proof. Let $y(x)$ be a solution of (3.8). The function $F(x, y(x))$ is differentiable and

$$\frac{d}{dx} F(x, y(x)) = F_x(x, y(x)) + F_y(x, y(x))\frac{dy}{dx}.$$

Since $dF = \omega$, then $F_x = M$, $F_y = N$ and we find

$$\frac{d}{dx} F(x, y(x)) = M(x, y(x)) + N(x, y(x)) \frac{dy}{dx}.$$

By assumption, $y(x)$ is a solution of (3.8) and hence

$$\frac{d}{dx} F(x, y(x)) = 0$$

which implies that $F(x, y(x))$ is constant, namely $F(x, (y(x)) = c, \;\; c \in \mathbb{R}$.

Conversely, let $y(x)$ be continuously differentiable and satisfy $F(x, y(x)) = c$ for some $c \in R$. Differentiating $F(x, y(x)) = c$, we find

$$\frac{d}{dx} F(x, y(x)) = F_x(x, y(x)) + F_y(x, y(x))\frac{dy}{dx} = 0.$$

Since $F_x = M$ and $F_y = N$ we deduce that

$$M(x, y(x)) + N(x, y(x))\frac{dy}{dx} = 0$$

and this means that $y(x)$ is a solution of (3.8). ■

Similarly, we have

Proposition 3.2.2. *Let M, N be continuous in $\Omega \subseteq \mathbb{R}^2$. Suppose that $M(x, y)dx + N(x, y)dy = 0$ is exact and let $F(x, y)$ denote an antiderivative of ω. If $x(y)$ is a solution of* (3.9) *then $F(x(y), y) = c$, for some $c \in \mathbb{R}$. Conversely, if $x(y)$ is continuously differentiable and satisfies $F(x(y), y) = c$, for some $c \in R$, then $x(y)$ satisfies* (3.9).

We have seen that the solutions of the exact equation (3.10) are those defined by $F(x, y) = c$, where F is an antiderivative of ω. We will say that $F(x, y) = c$ is the general solution of (3.10).

The constant c depends on the initial conditions. If $F(x, y) = c$ is the general solution of (3.10), the solution curve passing through $P_0 = (x_0, y_0)$ is given by $F(x, y) = F(x_0, y_0)$.

Notice that at any point $(x, y) \in \Omega$ such that $N(x, y) \neq 0$, equation (3.8) can be put in the normal form

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}. \tag{3.11}$$

Similarly, at any point $(x, y) \in \Omega$ such that $M(x, y) \neq 0$, equation (3.9) can be put in the normal form

$$\frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}. \tag{3.12}$$

To these equations we can apply the existence and uniqueness Theorems discussed in the preceding Chapter.

Remark 3.2.3. At points (x^*, y^*) such that $M(x^*, y^*) = N(x^*, y^*) = 0$ (3.10) is neither equivalent to (3.11) nor to (3.12). In Examples 3.2.4-3.2.5 below we illustrate some typical behavior of solutions near such points. ■

The simplest case of exact equations is when $M = M(x)$ and $N = N(y)$. In this case, one has $M_y = N_x = 0$. Notice that the corresponding equations

$$\frac{dy}{dx} = -\frac{M(x)}{N(y)} \ (N \neq 0), \qquad \frac{dx}{dy} = -\frac{N(y)}{M(x)} \ (M \neq 0)$$

are also separable equations. An antiderivative of $\omega = Mdx + Ndy$ is given by

$$F(x, y) = \int M(x)dx + \int N(x)dx,$$

since $F_x = M(x),\ F_y = N(y)$ by the Fundamental Theorem of Calculus.

We now discuss some examples of exact equations, starting with the simple case $M = M(x),\ N = N(y)$.

Example 3.2.4. (i) The equation $xdx + ydy = 0$ is exact and an antiderivative of $\omega = xdx + ydy$ is $F(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Then the general solution is given by

$$x^2 + y^2 = c.$$

If $c > 0$ this is a family of circles centered at $(0, 0)$. If $c = 0$ then $x^2 + y^2 = c$ reduces to the point $(0, 0)$. Notice that here $M = x$, $N = y$ and they both vanish at $(0, 0)$.

(ii) The equation $xdx - ydy = 0$ is also exact and the general solution is given by

$$x^2 - y^2 = c.$$

If $c \neq 0$ this is a family of hyperbola. If $c = 0$ then $x^2 - y^2 = 0$ is the pair of straight lines $y = \pm x$. Notice that here $M = x$, $N = -y$ and, as for (i), they both vanish at $(0, 0)$, the intersection point of the two straight lines. ■

Example 3.2.5. Find the solutions of $2xdx + 3(1 - y^2)dy = 0$ passing through the points $(0, 1)$ and $(0, -1)$ and discuss their behavior.

Here $M = 2x$ and $N = 3(1-y^2)$ and hence the equation is exact. An antiderivative of $\omega = Mdx + Ndy$ is

$$F(x, y) = 2\int xdx + 3\int (1 - y^2)dy = x^2 + 3y - y^3.$$

Thus the general solution is

$$x^2 + 3y - y^3 = c.$$

If $x = 0, y = 1$ we find $c = 2$. Solving $x^2 = y^3 - 3y + 2$ with respect to x, we find

$$x = \phi_{\pm}(y) = \pm\sqrt{y^3 - 3y + 2}.$$

Since $y^3 - 3y + 2 = (y + 2)(y - 1)^2$ then $\phi_{\pm}$ is defined for $y \geq -2$ and one has

$$\phi_{\pm}(y) = \pm\sqrt{(y + 2)(y - 1)^2} = \pm|y - 1| \cdot \sqrt{y + 2}.$$

This makes it clear that $\phi_{\pm}(y)$ have a cusp point at $y = 1$, while $\lim_{y\to -2^+} \phi'_{\pm}(y) = \pm\infty$.

If $x = 0, y = -1$ we find $c = -2$. Solving $x^2 = y^3 - 3y - 2$ with respect to x, we find

$$x = \psi_{\pm}(y) = \pm\sqrt{y^3 - 3y - 2}.$$

Since $y^3 - 3y - 2 = (y - 2)(y + 1)^2$ then $\psi_{\pm}$ is defined on the set $\{-1\} \cup \{y \geq 2\}$ and one has

$$\psi_{\pm}(y) = \pm\sqrt{(y - 2)(y + 1)^2} = \begin{cases} 0, & \text{if } y = -1 \\ \pm|y + 1| \cdot \sqrt{y - 2}, & \text{if } y \geq 2. \end{cases}$$

In this case the $(0, -1)$ is an isolated point of the graph of $x^2 = y^3 - 3y - 2$, while $\lim_{y\to 2^+} \psi'_{\pm}(y) = \pm\infty$. The graph of $x^2 - y^3 + 3y = c$ with $c = \pm 2$ is plotted in Figure 3.3. Notice that at $x = 0$ and $y = \pm 1$ both M and N vanish. ∎

We have become somewhat familiar with the concept of an exact equation, but now we need to know (1) how to recognize an exact equation, (2) knowing that it is exact, how do we solve it? The following theorem and its proof provide the answers.

Theorem 3.2.6. *Assume that $M(x, y)$ and $N(x, y)$ are continuous, with continuous partial derivatives with respect to x and y, on $\Omega = (\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$.*

(i) *If $\omega = M(x, y)dx + N(x, y)dy$ is exact, then $M_y(x, y) = N_x(x, y)$.*
(ii) *If $M_y(x, y) = N_x(x, y)$, then $\omega = M(x, y)dx + N(x, y)dy$ is exact.*

Remark 3.2.7. The reader should be aware that in the previous theorem we assume that M, N are defined in a rectangular region Ω, only for simplicity. In general, one

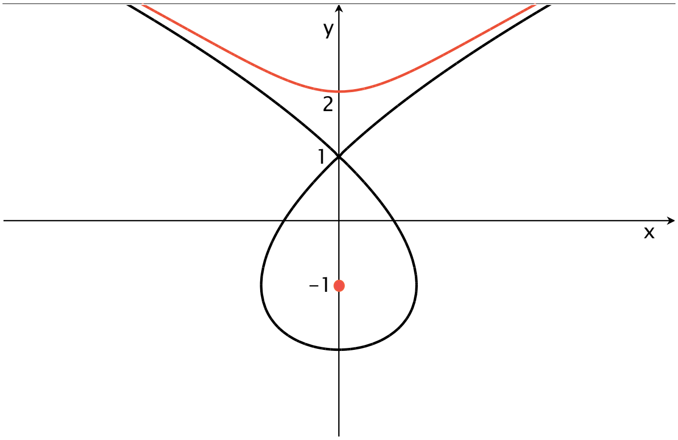

Fig. 3.3. Plot of $x^2 - y^3 + 3y = 2$ (*black*) and $x^2 - y^3 + 3y = -2$ (*red*)

could take any domain $\Omega \subset \mathbb{R}^2$ with the property that for any closed continuous curve γ contained in Ω, the set enclosed by γ is all contained in Ω. For example, any convex domain Ω satisfies this condition. On the contrary, $\mathbb{R}^2 \setminus \{0\}$ does not. ■

Proof of Theorem 3.2.6. (i) First let us assume that ω is exact. Then there exists a differentiable function $F(x, y)$ such that $dF = \omega$. This means that

$$\begin{cases} F_x(x,y) &= M(x,y), \\ F_y(x,y) &= N(x,y). \end{cases}$$

Therefore, we have

$$\begin{cases} F_{xy}(x,y) &= M_y(x,y), \\ F_{yx}(x,y) &= N_x(x,y). \end{cases}$$

Since the mixed second derivatives of F are equal, that is $F_{xy}(x, y) = F_{yx}(x, y)$, we deduce that $M_y(x, y) = N_x(x, y)$.

We provide two methods for proving part (ii), which can also be used for solving exact equations in general.

(ii-1) Now, we assume that $M_y(x, y) = N_x(x, y)$ and seek a function $F(x, y)$ such that $F_x(x, y) = M(x, y)$ and $F_y(x, y) = N(x, y)$. Let

$$F(x,y) = \int M(x,y)dx + h(y)$$

where $h(y)$ is a differentiable function of y, to be determined. We note that $F(x, y)$ already satisfies half of the requirement, since $F_x(x, y) = M(x, y)$ by the Fundamental Theorem of Calculus. We wish to show that there exists a function $h(y)$ such that $F_y(x, y) = N(x, y)$. But $F_y(x, y) = N(x, y)$ if and only if

$$\frac{\partial}{\partial y} F(x, y) = \frac{\partial}{\partial y} \int M(x, y)dx + h'(y) = N(x, y) \Longleftrightarrow$$
$$h'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx.$$

This means that if we choose $h(y)$ to be any antiderivative of $N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx$, then $F(x, y)$ will be the desired antiderivative of ω and we are done. But we can choose $h(y)$ in this manner only if we can show that $N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx$ is a function of y only. Otherwise, we would have $h'(y)$, a function of y, on the left side and a function of two variables x and y on the right side, which does not make sense. In order to show the right side is a function of y only, we will show that its derivative with respect to x is 0. To this end, since the function $\int M(x, y)dx$ has continuous mixed partial derivatives, we have

$$\frac{\partial}{\partial x}\left[N(x, y) - \frac{\partial}{\partial y} \int M(x, y)dx\right] = \frac{\partial}{\partial x} N(x, y) - \frac{\partial}{\partial x}\frac{\partial}{\partial y} \int M(x, y)dx$$
$$= \frac{\partial}{\partial x} N(x, y) - \frac{\partial}{\partial y}\frac{\partial}{\partial x} \int M(x, y)dx = N_x(x, y) - M_y(x, y) = 0.$$

In the above proof, we could have just as easily chosen

$$F(x, y) = \int N(x, y)dy + h(x)$$

and determined $h(x)$ as we obtained $h(y)$ above.

(ii-2) Let (x_0, y_0), (x, y) be two arbitrary points in the rectangle Ω. Consider the path $\Gamma = ([x_0, x] \times \{y_0\}) \cup (\{x\} \times [y_0, y])$, which is contained in Ω, see Figure 3.4, and define $F(x, y)$ by

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)ds + \int_{y_0}^{y} N(x, s)ds, \tag{3.13}$$

which corresponds to integrating the differential form ω along the path Γ. Let us show that F is an antiderivative of ω, that is, $F_x = M$, $F_y = N$. Using the fundamental theorem of Calculus, we find

$$F_x(x, y) = M(x, y_0) + \frac{\partial}{\partial x} \int_{y_0}^{y} N(x, s)ds.$$

We may recall from Calculus, or show independently, by using the definition of the derivative and the Mean Value Theorem, that

$$\frac{\partial}{\partial x} \int_{y_0}^{y} N(x, s)ds = \int_{y_0}^{y} \frac{\partial}{\partial x} N(x, s)ds.$$

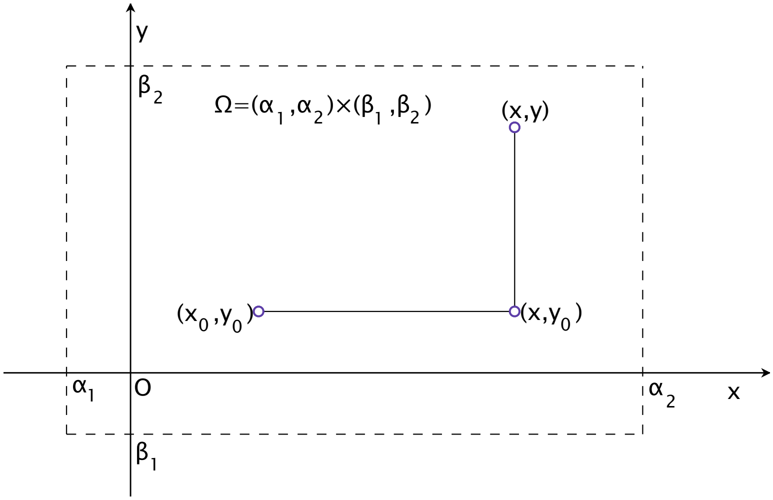

Fig. 3.4. The path Γ

Since, by assumption, $N_x = M_y$ we infer that

$$\begin{aligned} F_x(x,y) &= M(x,y_0) + \int_{y_0}^{y} M_y(x,s)ds \\ &= M(x,y_0) + M(x,y) - M(x,y_0) = M(x,y). \end{aligned}$$

To prove that $F_y = N$, we notice that the first integral is a function of x only. So,

$$F_y = \frac{\partial}{\partial y}\int_{y_0}^{y} N(x,s)ds = N(x,y).$$

In the above discussion, we could have also taken the path $\Gamma_1 = (\{x_0\} \times [y_0, y]) \cup ([x_0, x] \times \{y\})$ yielding

$$F(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,s)ds. \qquad \blacksquare$$

Example 3.2.8. Solve $(2x+y)dx + (x+2y)dy = 0$.

The equation is exact because

$$\frac{\partial(2x+y)}{\partial y} = 1 = \frac{\partial(x+2y)}{\partial x}.$$

Using (3.13), with $x_0 = y_0 = 0$, we have

$$F(x,y) = \int_0^x 2s ds + \int_0^y (x+2s)ds = x^2 + xy + y^2.$$

Therefore the general solution is given by

$$x^2 + xy + y^2 = c.$$

Notice that from $xy \geq -\frac{1}{2}(x^2 + y^2)$ it follows that $x^2 + xy + y^2 \geq \frac{1}{2}(x^2 + y^2) \geq 0$ for all x, y and hence $c \geq 0$. If $c > 0$, this is a family of ellipses centered at the origin. To see this it is convenient to make a change of coordinates by setting

$$\begin{cases} x &= u + v \\ y &= u - v. \end{cases}$$

In the (u, v) plane, we have $(u + v)^2 + (u^2 - v^2) + (u - v)^2 = c$ or $u^2 + v^2 + 2uv + u^2 - v^2 + u^2 - 2uv + v^2 = c$, whence $3u^2 + v^2 = c$. Hence, if $c > 0$, $x^2 + xy + y^2 = c$ is a family of ellipses centered at the origin as claimed, see Figure 3.5.

If $c = 0$ the ellipse degenerates to the point $(0, 0)$, the only point where $M = 2x + y$ and $N = x + 2y$ vanish. If $c < 0$ the equation $x^2 + xy + y^2 = c$ has no real solution. ∎

Example 3.2.9. Solve

$$2xy\,dx + (x^2 + y^2)dy = 0.$$

Here $M(x, y) = 2xy$ and $N(x, y) = x^2 + y^2$. Since $M_y = 2x = N_x$, the equation is exact. We give four solutions, using the four methods discussed above.

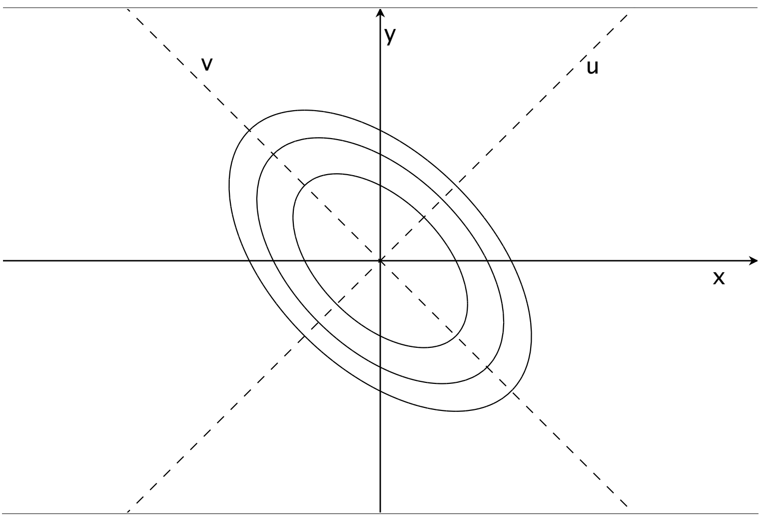

Fig. 3.5. $x^2 + xy + y^2 = c > 0$

Method 1. Let

$$F(x, y) = \int 2xy\, dx + h(y) = x^2 y + h(y).$$

Then clearly, $F_x = 2xy = M(x, y)$. We wish to determine $h(y)$ so that $F_y = x^2 + y^2 = N(x, y)$. Since $F(x, y) = x^2 y + h(y)$, this is equivalent to having $x^2 + h'(y) = x^2 + y^2$, which yields $h'(y) = y^2$ and hence $h(y) = \frac{1}{3}y^3 + k$. Therefore $F(x, y) = x^2 y + \frac{1}{3}y^3$ and the solution to the given differential equation is given by

$$x^2 y + \frac{1}{3}y^3 = c.$$

Notice that in the equation $h(y) = \frac{1}{3}y^3 + k$, we took $k = 0$. Otherwise, we would have $F(x, y) = x^2 y + \frac{1}{3}y^3 + k = c$. and $c - k$ would still be an arbitrary constant that we could call some thing like l, and then we would have $x^2 y + \frac{1}{3}y^3 = l$, which only changes the name c to l.

Method 2. Let

$$F(x, y) = \int (x^2 + y^2)dy + h(x) = x^2 y + \frac{1}{3}y^3 + h(x).$$

It is clear that $F_y = x^2 + y^2 = N(x, y)$. We wish to determine $h(x)$ so that $F_x = 2xy = M(x, y)$. Since $F(x, y) = x^2 y + \frac{1}{3}y^3 + h(x)$, this is the same as having $2xy + h'(x) = 2xy$, which gives us $h(x) = k$. As explained in Method 1, it is convenient to take k and hence $h(x)$ to be 0. Therefore, $F(x, y) = x^2 y + \frac{1}{3}y^3$ and the general solution is

$$x^2 y + \frac{1}{3}y^3 = c.$$

Method 3. We now use the method where $F(x, y)$ is given by

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)ds + \int_{y_0}^{y} N(x, s)ds.$$

We notice that if we take $y_0 = 0$, then $M(x, y_0) = 0$ for all x. Hence $F(x, y)$ would involve only one integral. Then since the first integral would be 0 anyway, we need not worry about x_0. So, let $y_0 = 0$. Then

$$F(x, y) = \int_0^y N(x, s)ds = \int_0^y (x^2 + s^2)ds = x^2 y + \frac{1}{3}y^3$$

and the solutions of the differential equation are again given implicitly by

$$x^2 y + \frac{1}{3}y^3 = c.$$

Method 4. Here we take $F(x, y) = \int_{x_0}^{x} M(s, y)ds + \int_{y_0}^{y} N(x_0, s)ds$. Letting $x_0 =$

0, $y_0 = 0$, we have

$$F(x, y) = \int_0^x 2sy ds + \int_0^y s^2 ds = x^2 y + \frac{1}{3} y^3.$$

Thus the general solution is, as before, by $x^2 y + \frac{1}{3} y^3 = c$.

Note that, for all x, the function $\phi(y) = x^2 y + \frac{1}{3} y^3$ is monotone and its range is $\mathbb{R}$. As a consequence, the equation $x^2 y + \frac{1}{3} y^3 = c$ has a positive solution $y_c(x)$ if $c > 0$ and a negative solution if $c < 0$. Such y_c solve

$$\frac{dy}{dx} = -2xy/(x^2 + y^2)$$

for all $c \neq 0$. ∎

Example 3.2.10. Find the solution of $(x^2 y + 1)dx + (\frac{1}{2} y + \frac{1}{3} x^3)dy = 0$ passing through $(\alpha, 0)$.

Since

$$\frac{\partial}{\partial y}\left(x^2 y + 1\right) = x^2 = \frac{\partial}{\partial x}\left(\frac{1}{2} y + \frac{1}{3} x^3\right)$$

the equation is exact.

Let us use Method 3, with $x_0 = y_0 = 0$. Then $M(x, 0) = 1$ and

$$F(x, y) = \int_0^x ds + \int_0^y \left(\frac{1}{2} s + \frac{1}{3} x^3\right) ds = x + \frac{1}{4} y^2 + \frac{1}{3} x^3 y$$

and the general solution is given implicitly by

$$x + \frac{1}{4} y^2 + \frac{1}{3} x^3 y = c.$$

Substituting $x = \alpha$ and $y = 0$, we obtain $c = \alpha$. Therefore the solution to the initial value problem is

$$x + \frac{1}{4} y^2 + \frac{1}{3} x^3 y = \alpha.$$

Alternate Solution. It may be convenient to take x_0 and y_0 as described by the initial values. So, letting $x_0 = \alpha$ and $y_0 = 0$, we have

$$F(x, y) = \int_\alpha^x ds + \int_0^y \left(\frac{1}{2} s + \frac{1}{3} x^3\right) ds = x - \alpha + \frac{1}{4} y^2 + \frac{1}{3} x^3 y = c.$$

Substituting $x = \alpha$ and $y = 0$, we get $c = 0$; so

$$x - \alpha + \frac{1}{4} y^2 + \frac{1}{3} x^3 y = 0$$

as before. ∎

Example 3.2.11. Solve

$$\frac{y^2+1}{x}dx + 2y \ln x dy = 0, \quad (x > 0).$$

We note that in order to use Method 3 or 4, here we cannot take the fixed point (0, 0). So, let us take the point (1, 1). Then since $\ln 1 = 0$, using Method 3, we easily obtain

$$F(x,y) = \int_1^x \frac{y^2+1}{s}ds = (y^2+1)\ln x, \quad (x > 0).$$

Thus the general solution is $(y^2 + 1)\ln x = c$. ∎

3.3 The integrating factor

In this section we learn how to deal with equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{3.14}$$

when it is not exact. It is possible that an equation of this type may not be exact but it becomes exact after it is multiplied by some suitable function. For example, the equation $ydx - xdy = 0, x > 0,\ y > 0$, is not exact. But after multiplying it by the function $\frac{1}{y^2}$, the resulting equation $\frac{1}{y}dx - \frac{x}{y^2}dy = 0$ becomes exact.

A nonzero function $\mu(x, y)$ is called an *integrating factor* of (3.14) if it has the property that when (3.14) is multiplied by this function, it becomes an exact equation.

Integrating factors exist, in general, but determining them may be quite difficult. Nevertheless, in some special cases finding an integrating factor can be fairly simple and it may be worth a try. We also point out that, as the following example shows, an integrating factor need not be unique.

Example 3.3.1. The reader should verify that for $x, y > 0$, all of the three functions

$$\frac{1}{xy}, \quad \frac{1}{x^2}, \quad \frac{1}{y^2},$$

are integrating factors of $y\,dx - x\,dy = 0$. ∎

One of the cases where finding an integrating factor can be quite simple is when the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

has an integrating factor that is either a function of x only or a function of y only. Let us assume that it has an integrating factor, which is a function of x only. Multiplying the equation by $\mu(x)$, we obtain

$$\mu(x)M(x,y)\,dx + \mu(x)N(x,y)\,dy = 0.$$

In order for this equation to be exact, we must have (notice that $\partial\mu(x)/\partial y = 0$)

$$\mu(x)M_y(x, y) = \mu'(x)N(x, y) + \mu(x)N_x(x, y).$$

If $N(x, y) \neq 0$, then we have

$$\mu'(x) = \frac{M_y(x, y) - N_x(x, y)}{N(x, y)} \cdot \mu(x). \tag{3.15}$$

Let

$$\Psi = \frac{M_y(x, y) - N_x(x, y)}{N(x, y)}.$$

If Ψ is a function of x only, then integrating $\mu'(x) = \Psi(x) \cdot \mu(x)$, we obtain

$$\mu = e^{\int \Psi(x)dx}.$$

If Ψ is not a function of x only, then we may try to find an integrating factor $\mu(y)$ which is a function of y only.

Multiplying the differential equation by $\mu(y)$ and following the same procedure, we obtain the equation

$$\mu'(y) = \frac{N_x(x, y) - M_y(x, y)}{M(x, y)} \cdot \mu(y).$$

Let

$$\Psi = \frac{N_x(x, y) - M_y(x, y)}{M(x, y)}.$$

If Ψ is a function of y only, then integrating $\mu'(y) = \Psi(y)\mu$, we obtain

$$\mu = e^{\int \Psi(y)dy}.$$

Example 3.3.2. Find an integrating factor for the equation

$$x \sin\, y\, dx + (x + 1) \cos\, y\, dy = 0.$$

Let us first check to see if we can find an integrating factor $\mu(x)$. We can use (3.15) to determine if such an integrating factor exists, but we recommend that students do not memorize this formula and instead go through the process each time. Thus, multiplying by $\mu(x)$, we have

$$\mu(x)x \sin\, y\, dx + \mu(x)(x + 1) \cos\, y\, dy = 0,$$
$$\mu(x)x \cos\, y = [\mu'(x)(x + 1) + \mu(x)] \cos\, y.$$

Dividing by $\cos y$ (assuming $\cos y \neq 0$), we have

$$x\mu(x) = (x + 1)\mu'(x) + \mu(x)$$

and hence

$$\mu'(x) = \frac{x-1}{x+1} \cdot \mu(x) = \left(1 - \frac{2}{x+1}\right) \mu(x), \qquad (x \neq -1).$$

Integrating,

$$\mu'(x) = \left(1 - \frac{2}{x+1}\right) \mu(x), \qquad (x \neq -1)$$

we obtain

$$\mu(x) = \frac{e^x}{(x+1)^2}, \qquad (x \neq -1).$$

Multiplying the given equation $x \sin\, y\, dx + (x+1) \cos\, y\, dy = 0$ by this $\mu(x)$ we have

$$\frac{e^x x}{(x+1)^2} \sin\, y\, dx + \frac{e^x}{(x+1)} \cos\, y\, dy = 0, \qquad (x \neq -1)$$

which is now exact. Thus, we may use (3.13) either on the half space $\{x > -1\}$ or on $\{x < -1\}$. Since $M(x, y) = \frac{e^x x}{(x+1)^2} \sin\, y$, if we take $x_0 = y_0 = 0$, we see that $M(x, y_0) = 0$ for all x. This implies that $\int_{x_0}^{x} M(s, y_0) ds = 0$.

Consequently,

$$F(x, y) = \int_0^y \frac{e^x}{(x+1)} \cos\, s\, ds = \frac{e^x}{(x+1)} \sin\, y.$$

Thus the general solution is

$$\frac{e^x}{1+x} \sin y = c, \qquad (x \neq -1).$$

Notice that for $c = 0$ the solutions are straight lines given by $y = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

We could have found these constant solutions without solving the equation, simply by observing that

$$\frac{dy}{dx} = -\frac{x \sin\, y}{(x+1) \cos\, y}.$$

For example, it is easy to see that $y(x) \equiv \pi$ is a solution since $y' = 0$ and also

$$-\frac{x \sin \pi}{(x+1) \cos \pi} = 0. \qquad \blacksquare$$

Example 3.3.3. The equation

$$(y + xy + y^2)dx + (x + 2y)dy = 0$$

is not exact because $M_y = 1 + x + 2y$ while $N_x = 1$. Let us try to find an integrating factor $\mu(x)$. We consider $\mu(x)(y + xy + y^2)dx + \mu(x)(x + 2y)dy = 0$ and equate the partial derivatives. Then we have

$$\mu(x)(1+x+2y)=\mu'(x)(x+2y)+\mu(x)$$

and hence $(x+2y)\mu(x)=(x+2y)\mu'(x)$. Therefore $\mu'(x)=\mu(x)$ and so we can take $\mu(x)=e^x$. Now $\omega=e^x(y+xy+y^2)dx+e^x(x+2y)dy$ is exact. Here $\Omega=\mathbb{R}^2$ and we can use Method 3 to find an antiderivative. Since $e^x(y+xy+y^2)=0$ for $y=0$, one has

$$F(x,y)=\int_0^y e^x(x+2s)ds=e^x(xy+y^2)$$

and hence the general solution is $e^x(xy+y^2)=c$. ∎

Example 3.3.4. Consider the equation $ydx+(2x+3y)dy=0$. Since $M_y=1\neq N_x=2$ the equation is not exact. Here it is convenient to look for an integrating factor of the type $\mu(y)$. The equation $\mu(y)ydx+\mu(y)(2x+3y)dy=0$ is exact provided

$$y\,\frac{d\mu(y)}{dy}+\mu(y)=2\mu(y)\quad\Longrightarrow\quad y\,\frac{d\mu(y)}{dy}=\mu(y)$$

which yields $\mu(y)=y$. An antiderivative of $y^2dx+y(2x+3y)dy=0$ is

$$F(x,y)=\int_0^y s(2x+3s)ds=xy^2+y^3$$

and hence $xy^2+y^3=c$ is the general solution of our equation. If $c=0$ we find $y=0$ and $y=-x$. If $c>0$, then $y^2(x+y)=c>0$ implies $y>-x$, while if $c<0$, then $y^2(x+y)=c<0$ implies $y<-x$. See Figure 3.6. ∎

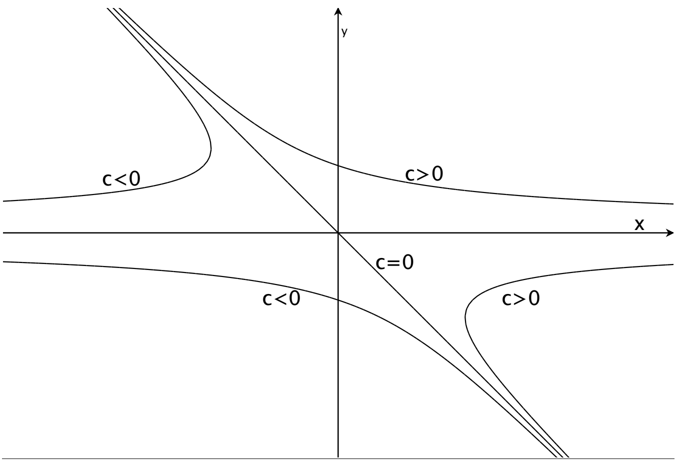

Fig. 3.6. Plot of $xy^2+y^3=c$

3.4 Homogeneous equations

The equation

$$x' = f(t, x)$$

is called homogeneous if $f(t, x)$ can be expressed as a function of the variable $\frac{x}{t}$, $t \neq 0$. For example,

$$x' = \frac{x^3 + t^3}{tx^2}, \; t \neq 0,$$

is homogeneous because if we divide the numerator and denominator by t^3, we obtain

$$x' = \frac{\left(\frac{x}{t}\right)^3 + 1}{\left(\frac{x}{t}\right)^2}$$

and the right side is a function of the variable $\frac{x}{t}$.

On the other hand,

$$x' = x^2 \sin t$$

is not homogeneous because it is impossible to express it as a function of $\frac{x}{t}$.

So, a homogeneous equation has the form

$$x' = g\left(\frac{x}{t}\right). \tag{3.16}$$

Equation (3.16) can be transformed into a separable equation by making the substitution $x(t) = tz(t)$. This follows since we would have $x' = z + tz' = g(z)$ and the equation $z + tz' = g(z)$ can be written as

$$z' = \frac{1}{t}(g(z) - z)$$

which is separable.

Example 3.4.1. Consider the equation

$$x' = \frac{t^2 + x^2}{tx}, \qquad tx \neq 0.$$

If we divide the numerator and denominator by t^2 and then let $x = tz$, we obtain

$$tz' + z = \frac{1 + z^2}{z},$$

$$tz' = \frac{1 + z^2}{z} - z = \frac{1 + z^2 - z^2}{z} = \frac{1}{z}$$

and hence

$$zz' = \frac{1}{t}$$

or equivalently

$$zdz = \frac{dt}{t}.$$

Integrating, we get

$$\tfrac{1}{2}z^2 = \ln|t| + c,$$

$$z^2 = 2(\ln|t| + c).$$

Now, what remains is to express the answer in terms of the original variables t and x. Since $x = zt$, $z = x/t$ and

$$\left(\frac{x}{t}\right)^2 = 2\ln|t| + k \quad k = 2c,$$

which gives rise, for all c, to a solution of our equation in implicit form. In this case, if we want to, we can find the solutions explicitly. ■

Let us now consider the more general equation

$$x' = \frac{M(t,x)}{N(t,x)} \tag{3.17}$$

where M, N are homogeneous functions of the same order k. Let us recall that $M = M(t,x)$ is a k*-homogeneous function* if

$$M(\lambda t, \lambda x) = \lambda^k M(t,x) \tag{3.18}$$

for all λ such that $M(\lambda t, \lambda x)$ makes sense.

If both M and N are k-homogeneous, then we have

$$\frac{M(t, t\cdot x/t)}{N(t, t\cdot x/t)} = \frac{t^k M(1, x/t)}{t^k N(1, x/t)} = \frac{M(1, x/t)}{N(1, x/t)}.$$

If we define

$$g\left(\frac{x}{t}\right) := \frac{M(1, x/t)}{N(1, x/t)}$$

we deduce that

$$x' = \frac{M(t,x)}{N(t,x)} = g\left(\frac{x}{t}\right),$$

which shows that (3.17) is homogeneous.

It will be helpful to remember that if an equation is a quotient of two polynomials, then it is homogeneous if and only if the sum of the exponents of the variables in each term, which we call *the total exponent*, both in the numerator and denominator, is the same. For example,

$$x' = \frac{t^3x^2 - x^5}{tx^4 + t^3x^2}$$

has two terms in the numerator and two in the denominator, all of total exponent 5. Therefore it is homogeneous. On the other hand,

$$x' = \frac{t^3x^2 - x^5}{tx^4 + t^2x^2}$$

is a quotient of two polynomials, but three of the terms have total exponent equal to 5 and one has total exponent equal to 4; therefore it is not homogeneous.

The proof of this rule easily follows from dividing the numerator and denominator by a certain power of t, as we did in Example 3.4.1.

Example 3.4.2. Solve

$$x' = \frac{x^2 + tx + t^2}{t^2}, \qquad t \neq 0.$$

All the terms in the numerator and the denominator have the same total exponent equal to 2. Therefore, it is homogenous.

Letting $x = tz$, one finds

$$z + tz' = \frac{t^2z^2 + t^2z + t^2}{t^2} = z^2 + z + 1,$$

$$tz' = z^2 + 1 \implies \frac{z'}{z^2+1} = \frac{1}{t}$$

and integrating, we have

$$\arctan z = \ln|t| + c \implies z = \tan(\ln|t| + c).$$

In conclusion the solution is

$$x = tz = t\ \tan(\ln|t| + c),$$

which is defined for $t \neq 0$ and $\ln|t| + c \neq \frac{\pi}{2} + k\pi$, k integer. ■

3.5 Bernoulli equations

A generalization of the linear equation is the *Bernoulli equation*

$$x' + p(t)x = q(t)x^{k+1} \tag{3.19}$$

where $p, q \in C([a, b])$ or $p, q \in C(\mathbb{R})$. The equation has the trivial solution $x = 0$ and we are interested in finding nontrivial solutions. Of course, we will consider only solutions such that x^{k+1} makes sense.

If $k = 0, -1$ or if $q(t) \equiv 0$ the equation becomes linear. With the exception of these three cases, the Bernoulli equation is nonlinear. Let us show that the change of variable $z = x^{-k}$, ($x \neq 0$ if $k > 0$), transforms the Bernoulli equation into a linear equation.

We have $z' = -kx^{-k-1}x'$. Since $x' = -p(t)x + q(t)x^{k+1}$, we get

$$z' = -kx^{-k-1}\left(-p(t)x + q(t)x^{k+1}\right) = kp(t)x^{-k} - kq(t).$$

Since $z = x^{-k}$, we have

$$z' - kp(t)z = -kq(t),$$

which is a linear equation. If z is a solution of this equation, then $x(t) = z^{-1/k}(t)$ is a solution of (3.19).

Example 3.5.1. The equation

$$x' - x = tx^2$$

is a Bernoulli equation with $k = 1$, $p = -1$, $q = t$. One solution is $x \equiv 0$. If $x \neq 0$, we set $z = x^{-1}$. Then we have

$$z' + z = -t$$

which is linear; and solving it, we have

$$z(t) = c\,e^{-t} + 1 - t, \qquad c \in \mathbb{R}$$

and finally, if $c\,e^{-t} + 1 - t \neq 0$,

$$x(t) = \frac{1}{z(t)} = \frac{1}{c\,e^{-t} + 1 - t}, \qquad c \in \mathbb{R}. \tag{3.20}$$

If we want to solve an initial value problem such as

$$x' - x = tx^2, \qquad x(0) = 1,$$

we substitute the initial value into (3.20) and solve for c. One finds

$$1 = \frac{1}{c\,e^0 + 1} = \frac{1}{c + 1}$$

whence $c = 0$. Thus the solution is $x(t) = \frac{1}{1-t}$, restricted to $t < 1$. The more general case that $x(0) = a > 0$ is discussed in Exercise no. 46. ■

Example 3.5.2. Solve $x' + 2x = e^t\sqrt{x}$, $x(0) = 1$. This is a Bernoulli equation with $p = 2$, $q = e^{-t}$ and $k + 1 = \frac{1}{2}$, namely $k = -\frac{1}{2}$. Setting $z = \sqrt{x}$, $x \geq 0$, we find

$$z' + z = \frac{1}{2}e^t.$$

Solving this linear equation, we get

$$z = c\,e^{-t} + \frac{1}{4}e^t.$$

Notice that $z \geq 0$ implies $c\,e^{-t} + \frac{1}{4}e^t \geq 0$, that is

$$c \geq -\frac{1}{4}e^{2t}. \tag{3.21}$$

Substituting $z = \sqrt{x}$, namely $x = z^2$, we find

$$x(t) = \left(c\,e^{-t} + \frac{1}{4}e^t\right)^2.$$

Inserting an initial condition such as $x(0) = 1$, we find $1 = \left(c + \frac{1}{4}\right)^2$. Solving, we have $c + \frac{1}{4} = \pm 1$ and hence $c = \frac{3}{4}$ or $c = -\frac{5}{4}$. Since (3.21), with $t = 0$, implies $c \geq -\frac{1}{4}$, we find that $c = \frac{3}{4}$. Thus

$$x(t) = \left(\frac{3}{4}e^{-t} + \frac{1}{4}e^t\right)^2$$

is the solution of our initial value problem. ■

3.6 Appendix. Singular solutions and Clairaut equations

A solution $x = \gamma(t)$ of a first order differential equation $F(t, x, x') = 0$ is called a *singular solution* if for each (t_0, x_0) with $\gamma(t_0) = x_0$, there exists a solution $\psi(t) \neq \gamma(t)$ of $F(t, x, x') = 0$, passing through (t_0, x_0), namely such that $\psi(t_0) = x_0$. In particular, $\gamma(t)$ and $\psi(t)$ have the same derivative at $t = t_0$ and thus they are tangent at (t_0, x_0). Since this holds for every point (t_0, x_0) this means that $x = \gamma(t)$ is the *envelope* of a family of solutions of $F(t, x, x') = 0$.

Recall that the envelope of a family of curves given implicitly by $g(t, x, c) = 0$ is a curve of implicit equation $\eta(t, x) = 0$ that can be found solving the system

$$\begin{cases} g(t,x,c) &= 0 \\ g_c(t,x,c) &= 0. \end{cases}$$

When $g(t, x, c) = x - h(t, c)$ the system becomes

$$\begin{cases} x &= h(t, c) \\ h_c(t, c) &= 0. \end{cases}$$

This is an easy example.

Example 3.6.1. Find the envelope of the family of parabolas $x = (t - c)^2$. The preceding system becomes

$$\begin{cases} x &= (t-c)^2 \\ -2(t-c) &= 0. \end{cases}$$

The second equation yields $c = t$ and, substituting into the first equation, we find $x = 0$, which is the envelope of the parabolas. ■

Let $x = \phi(t, c)$, $c \in \mathbb{R}$, be a one parameter family of solutions of $F(t, x, x') = 0$. Differentiating the identity $F(t, \phi(t, c), \phi'(t, c)) \equiv 0$ with respect to c, we find

$$F_x(t, \phi(t, c), \phi'(t, c))\partial_c\phi(t, c) + F_{x'}(t, \phi(t, c), \phi'(t, c))\partial_c\phi'(t, c) \equiv 0.$$

If $F_x(t, \phi(t, c), \phi'(t, c)) \not\equiv 0$ and $F_{x'}(t, \phi(t, c), \phi'(t, c)) \equiv 0$, we infer that $\partial_c\phi(t, c) \equiv 0$. Therefore, the singular solution solves the differential system

$$\begin{cases} F(t, x, x') &= 0 \\ F_{x'}(t, x, x') &= 0 \end{cases}$$

and is such that $F_x(t, x, x') \not\equiv 0$.

Example 3.6.2. Find the singular solutions of $x^2(1 + x'^2) = 1$. Here $F(t, x, x') = x^2(1 + x'^2) - 1$. Then $F_{x'} = 2x^2x'$ and the preceding system becomes

$$\begin{cases} x^2(1 + x'^2) &= 1 \\ 2x^2x' &= 0 \end{cases}$$

whose solutions are $x = \pm 1$. Since for $x = \pm 1$ one has $F_x = x = \pm 1$, thus the singular solutions are $x = \pm 1$.

Now, by substitution, we find that $\phi(t, c) = \pm\sqrt{1 - (t - c)^2}$ is a one parameter family of solutions of $x^2(1 + x'^2) = 1$. They are a family of circles $g(t, x, c) = (t - c)^2 + x^2 = 1$ centered at $(c, 0)$ with radius 1. Let us check that $x = \pm 1$ are the envelope of $\phi(t, c)$. We have to solve the system

$$\begin{cases} x &= \phi(t, c) \\ \phi_c(t, c) &= 0. \end{cases}$$

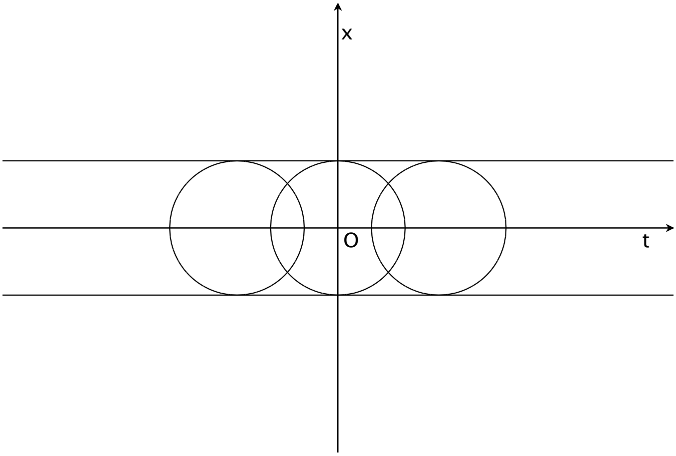

Fig. 3.7. Solutions of $x^2(1 + x'^2) = 1$

In this case we have $\phi(t, c) = \pm\sqrt{1 - (t - c)^2}$ and hence

$$\phi_c(t, c) = \pm\frac{t - c}{\sqrt{1 - (t - c)^2}}.$$

So $\phi_c(t, c) = 0$ for $c = t$. Substituting into $x = \phi(t, c) \pm \sqrt{1 - (t - c)^2}$ we find $x = \pm 1$, which are exactly the singular solutions found before. See Figure 3.7. ■

3.6.1 Clairaut equations

A typical equation where singular solutions can arise is the *Clairaut equation* which is a differential equation of the form

$$x = tx' + g(x') \tag{3.22}$$

where, say, $g \in C(\mathbb{R})$. Let us note that (3.22) is not in normal form.

If we let $x' = c$, for any $c \in \mathbb{R}$, we find the family of straight lines

$$x(t) = ct + g(c), \qquad c \in \mathbb{R}$$

which clearly solve (3.22).

Remark 3.6.3. If g is defined in a subset of $\mathbb{R}$, the solutions $x = ct + g(c)$ make sense only for c in the domain of g. See, e.g. Exercises nos. 40 and 41 below. ■

A specific feature of the Clairaut equation is that it might have singular solutions. According to the preceding discussion, we set $F(t, x, x') = tx' + g(x') - x$ and solve the system

$$\begin{cases} F(t,x,x') &= 0 \\ F_{x'}(t,x,x') &= 0 \end{cases} \implies \begin{cases} tx' + g(x') - x &= 0 \\ t + g'(x') &= 0. \end{cases}$$

Notice that $F_x = -1$ and hence the condition $F_x \neq 0$ is always satisfied. Let us suppose that $g \in C^1(\mathbb{R})$, and that g' is invertible. Recall that a function ϕ, defined on a set R with range in a set S, is invertible if there exists a function ψ defined on S with range in R such that $\phi(r) = s$ if and only if $r = \psi(s)$. The function ψ, denoted by ϕ^{-1}, is unique and satisfies $\phi^{-1}(\phi(r)) = r$ for all $r \in R$.

Setting $h = (g')^{-1}$, the second equation of the preceding system, that is $g'(x') = -t$, yields

$$x' = h(-t).$$

Substituting in the first equation we find

$$x(t) = th(-t) + g(h(-t)).$$

Therefore, this is the singular solution we were looking for.

Example 3.6.4. The equation

$$x = tx' + x'^2 \tag{3.23}$$

is a Clairaut equation with $g(x') = x'^2$. The function $g'(x') = 2x'$ is obviously invertible. Solving $2x' = -t$ we find $x' = -\frac{1}{2}t$. Hence the singular solution is

$$\gamma(t) = t \cdot \left(-\frac{t}{2}\right) + g\left(-\frac{t}{2}\right) = -\frac{t^2}{2} + \left(-\frac{t}{2}\right)^2 = -\frac{1}{4}t^2$$

and turns out to be the envelope of the family of straight lines

$$x(t) = ct + c^2, \qquad c \in \mathbb{R}.$$

Consider now the Cauchy problem

$$x = tx' + x'^2, \qquad x(a) = b. \tag{3.24}$$

A straight line $x = ct + c^2$ solves (3.24) provided

$$b = ca + c^2.$$

The second order algebraic equation in c, $c^2 + ac - b = 0$, can be solved yielding

$$c = \frac{-a \pm \sqrt{a^2 + 4b}}{2}$$

and hence there is a solution whenever

$$a^2 + 4b \geq 0, \implies b \geq -\tfrac{1}{4}a^2.$$

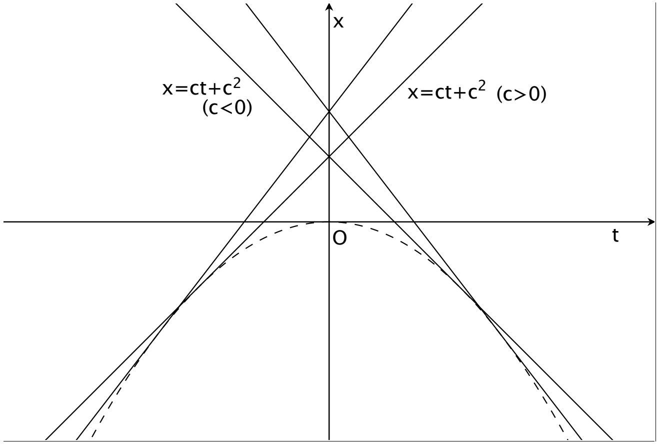

Fig. 3.8. Solutions of $x = tx' + x'^2$. The dotted curve is the singular solution $x = -\frac{1}{4}t^2$

This means that, unlike equations in normal form, (3.24) has a solution if and only if the initial values belong to the set

$$\{(t,x) \in \mathbb{R}^2 : x \geq -\tfrac{1}{4}t^2\},$$

above the singular solution $\gamma(t) = -\frac{1}{4}t^2$. See Figure 3.8. Precisely, one has:

(i) For all (a,b) such that $b > -\frac{1}{4}a^2$, the equation $b = ca + c^2$ has two solutions and hence there are two straight lines of the family $x = ct + c^2$ that satisfy (3.24).

(ii) If $b = -\frac{1}{4}a^2$, the equation $b = ca + c^2$ becomes $c^2 + ac = -\frac{1}{4}a^2$ and has only one solution given by $c = -\frac{1}{2}a$. Then there is only one solution among the family $\phi(t,c)$ that satisfies (3.24): that is $x = -\frac{1}{2}at + \frac{1}{4}a^2$. This straight line is tangent to $x = \gamma(t)$ at $(a,b) = (a, -\frac{1}{4}a^2)$, due to the fact that the singular solution is the envelope of the solution family $x = ct + c^2$.

(iii) For all (a,b) such that $b < -\frac{1}{4}a^2$, the equation $b = ca + c^2$ has no solution and hence there is no straight line of the family $x = ct + c^2$ that satisfies (3.24). ■

Remark 3.6.5. If g' is not invertible, there could be no singular solution. For example, the solutions of $x = tx' + x'$ are the family of straight lines $x = ct + c$ passing through $(-1,0)$ and the solution of the system $F = 0,\ F_{x'} = 0$, that is $x = tx' + x',\ 0 = t + 1$, reduces to the point $(-1,0)$. ■

Remark 3.6.6. A solution of the family $x = ct + g(c)$ solves the initial value problem

$$x = tx' + g(x'), \qquad x(a) = b,$$

whenever $b = ac + g(c)$. If we assume that g is twice differentiable and $g''(p) \neq 0$ (notice that this implies that g' is invertible so that the previous discussion applies),

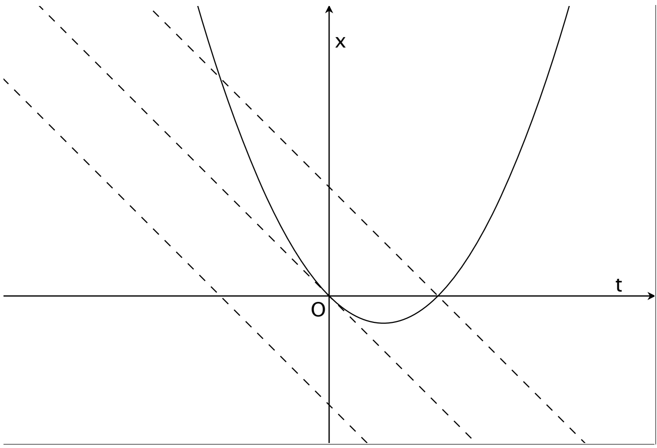

Fig. 3.9. The equation $g(c) = b - ac$ with $a > 0$

then g is either concave or convex and the equation $g(c) = b - ac$, in the unknown c, has two, one or no solution, see Figure 3.9, showing that what we saw in the Example 3.6.4 is a general fact. ∎

Remark 3.6.7. The Clairaut equation is a particular case of the D'Alambert–Lagrange equation $x = tf(x') + g(x')$, $f, g \in C(I)$. See two examples in Exercises nos. 42 and 43 below. ∎

3.7 Exercises

1. Solve $x' = x^2 + 1$.
2. Solve the ivp $x' = x^2 - 1$, $x(0) = 0$.
3. Solve the ivp $x' = x^2 + x$, $x(1) = 1$.
4. Solve the ivp $x' = \frac{x^2+x}{2x+1}$, $x(0) = 1$.
5. Let $\phi(t)$ be the solution of the ivp $x' = \frac{x^2-x}{2x-1}$, $x(0) = 2$. Find $\lim_{t\to-\infty} \phi(t)$ and $\lim_{t\to+\infty} \phi(t)$.
6. Solve $x' = 4t^3x^4$.
7. Solve $x' = -tx^2$.
8. Solve $x' = e^t(1 + x^2)$, $x(0) = 0$.
9. Solve $x' = \frac{t}{x}$ and find the solution such that $x(\sqrt{2}) = 1$.
10. Solve $x' = -\frac{t}{4x^3}$ and find the solution such that $x(1) = 1$.

11. Solve $x' = -t^2x^2$ such that $x(1) = 2$.
12. Solve $x' = 5t\sqrt{x}$, $x \geq 0$, $x(0) = 1$.
13. Solve $x' = 4t^3\sqrt{x}$, $x \geq 0$, $x(0) = 1$.
14. Solve and discuss uniqueness for the ivp $x' = 2t\sqrt{x}$, $x(a) = 0$, $x \geq 0$.
15. Find p such that the solutions of $x' = -(p+1)t^p x^2$ tend to 0 as $t \to +\infty$.
16. Find the limit, as $t \to +\infty$, of the solutions of $x' = -(p+1)t^p x^2$ when $p + 1 \leq 0$.
17. Solve $x' = \sqrt{1-x^2}$ and find the singular solutions. Explain why uniqueness does not hold.
18. Solve $(2x^2+1)dx = (y^5-1)dy$.
19. Solve $(x+3y)dx + (3x+y)dy = 0$ and sketch a graph of the solutions.
20. Solve $(x+y)dx + (x-y)dy = 0$.
21. Solve $(ax^p + by)dx + (bx + dy^q)dy = 0$, $p, q > 0$.
22. Solve $(3x^2 - y)dx + (4y^3 - x)dy = 0$ and find the solution passing through $(1, 1)$.
23. Solve $(y - x^{1/3})dx + (x + y)dy = 0$ and find the solution passing through $(0, 1)$.
24. Solve $(e^x - \frac{1}{2}y^2)dx + (e^y - xy)dy = 0$ and find the solution passing through $(0, 0)$.
25. Solve $(x + \sin y)dx + x\cos y\,dy = 0$ and find the solution passing through $(2, \pi)$.
26. Solve $(x^2 + 2xy - y^2)dx + (x-y)^2dy = 0$ and find the solution passing through $(1, 1)$.
27. Solve $(x^2 + 2xy + 2y^2)dx + (x^2 + 4xy + 5y^2)dy = 0$. Show that there exists a such that $y = ax$ is a solution passing through $(0, 0)$.
28. Solve $x\,dx - 2y^3dy = 0$ and describe the behavior of the solutions.
29. Find the number a such that $(x^2 + y^2)dx + (axy + y^4)dy = 0$ is exact and then solve it.
30. Find the coefficients a_i, b_i such that $(x^2 + a_1xy + a_2y^2)dx + (x^2 + b_1xy + b_2y^2)dy = 0$ is exact, and solve it.
31. Find a function $A(y)$ such that $(2x + A(y))dx + 2xy\,dy = 0$ is exact and solve it.
32. Find a function $B(x)$ such that $(x + y^2)dx + B(x)y\,dy = 0$ is exact and solve it.

33. Show that for any differentiable $f(y) \neq 0$ and any continuous $g(y)$, there exists an integrating factor $\mu = \mu(y)$ for the equation $f(y)dx + (xy + g(y))dy = 0$.
34. Show that $\mu = x$ is an integrating factor for $(x + y^2)dx + xydy = 0$ and solve it.
35. Find an integrating factor $\mu(x)$ by multiplying the equation

$$(2y + x)dx + (x^2 - 1)dy = 0$$

by $\mu(x)$ and determine it so that it satisfies the condition for exactness, and then solve it.
36. Solve $(x + 2y)dx + (x - 1)dy = 0$.
37. Solve $y^2dx + (xy + 3y^3)dy = 0$.
38. Solve $(1 + y^2)dx + xydy = 0$.
39. Solve $x' = (x + 2t)/t$, $t \neq 0$.
40. Solve $x' = tx/(t^2 + x^2)$.
41. Solve $x' = \frac{3x^2-2t^2}{tx}$.
42. Solve $x' = \frac{x^2+t^2}{2tx}$.
43. Solve $x' = \frac{x-t+1}{x-t+2}$.
44. Solve $x' = \frac{x-t}{x-t+1}$.
45. Solve $x' = -\frac{x+t+1}{x-t+1}$.
46. Solve the Cauchy problem $x' - x = tx^2$, $x(0) = a > 0$ and describe the solutions relative to a.
47. Find the nontrivial solutions of $x' + 2tx = -4tx^3$.
48. Find the nontrivial solutions of $x' - tx = x^2$.
49. Show that the circle $x^2 + t^2 = 1$ is the singular solution of $x'^2 = x^2 + t^2 - 1$.
50. Solve $x'^2 = 4(1 - x)$ and show that $x = 1$ is the singular solution.
51. Find a singular solution of $x'^2 - tx' + x = 0$.
52. Solve the Clairaut equation $x = tx' - x'^2$ and find the singular solution.
53. Solve the Clairaut equation $x = tx' + e^{x'}$ and find the singular solution.
54. Solve the Clairaut equation $x = tx' - \ln x'$ and find the singular solution.
55. Solve the Clairaut equation $x = tx' + \frac{1}{x'}$ and find the singular solution.
56. Find α, β such that $x(t) = \alpha t + \beta$ solves the D'Alambert-Lagrange equation $x = th(x') + g(x')$, $h, g \in C(I)$.
    (a) Show that the equation $x = t(1 + x') + x'$ has no solution of the form $x = \alpha t + \beta$.
    (b) Solve the equation by setting $x' = z$.

4

Existence and uniqueness for systems and higher order equations

In this chapter we extend (without proof) to systems and higher order equations, the existence and uniqueness theorems stated in Chapter 2.

4.1 Systems of differential equations

If $f_1, \ldots, f_n$ are functions of the $n+1$ variables $(t, x_1, \ldots, x_n)$, we consider the system of differential equations in normal form

$$\begin{cases} x_1' &= f_1(t, x_1, \ldots, x_n) \\ \ldots &= \ldots \\ x_n' &= f_n(t, x_1, \ldots, x_n). \end{cases} \tag{4.1}$$

To write this system in a compact form, we introduce

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \qquad \bar{f}(t, \overline{x}) = \begin{pmatrix} f_1(t, \overline{x}) \\ f_2(t, \overline{x}) \\ \vdots \\ f_n(t, \overline{x}) \end{pmatrix} \in \mathbb{R}^n.$$

If $(t, \overline{x}) \in \Omega \subseteq \mathbb{R}^{n+1}$, then $\bar{f}$ is a function from Ω to $\mathbb{R}^n$. With this notation, the preceding system becomes

$$\overline{x}' = \bar{f}(t, \overline{x})$$

which is formally like the first order equation $x' = f(t, x)$.

For example, if $n = 2$, a 2×2 system is

$$\begin{cases} x_1' &= f_1(t, x_1, x_2) \\ x_2' &= f_2(t, x_1, x_2). \end{cases}$$

If f_i, $i = 1, 2, \ldots, n$, do not depend on t, the system is *autonomous*. If f_i, $i = 1, 2, \ldots, n$, are linear with respect to $x_1, \ldots, x_n$, the system is *linear*.

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If $n = 2$ and f_1, f_2 are linear and depend only on x_1, x_2,

$$\begin{cases} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{cases}$$

is a 2×2 *linear, autonomous* system. To write this system in the vectorial form, it suffices to introduce the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

With this notation the linear 2×2 system becomes

$$\overline{x}' = A\overline{x}, \qquad \overline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2.$$

In general, a linear autonomous system has the form

$$\overline{x}' = A\overline{x}, \qquad \overline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

where A is an $n \times n$ matrix.

Example 4.1.1. If $\overline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

the system $\overline{x}' = A\overline{x}$ becomes

$$\begin{cases} x' &= x + 2y + 3z \\ y' &= 4x + 5y + 6z \\ z' &= 7x + 8y + 9z. \end{cases}$$

■

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Given $(t_0, \alpha) \in \Omega$, an initial value, or a Cauchy, problem for the system $\overline{x}' = \bar{f}(t, \overline{x})$ is

$$\begin{cases} \overline{x}' &= \bar{f}(t, \overline{x}) \\ \overline{x}(t_0) &= \alpha \end{cases} \tag{4.2}$$

or, in terms of the components

$$\begin{cases} x_i' &= f_i(t, x_1, \ldots, x_n) \\ x_i(t_0) &= \alpha_i \end{cases} \qquad i = 1, \ldots, n.$$

In general, one prescribes n initial values for an $n \times n$ first order system.

For example, an initial value problem for a 2×2 system is

$$\begin{cases} x_1' &= f_1(t, x_1, x_2) \\ x_2' &= f_2(t, x_1, x_2) \\ x_1(t_0) &= \alpha_1, \; x_2(t_0) = \alpha_2. \end{cases}$$

4.1.1 Existence and uniqueness results for systems

Below we state the existence and uniqueness theorems for systems in normal form.

Theorem 4.1.2. *(Local existence) Let $\Omega \subseteq \mathbb{R}^{n+1}$, let $\bar{f} : \Omega \mapsto \mathbb{R}^n$ be continuous and let $(t_0, \alpha) = (t_0, \alpha_1, \dots, \alpha_n)$ be a given point in the interior of Ω. Then the initial value problem*

$$\begin{cases} \overline{x}' &= \bar{f}(t, \overline{x}) \\ \overline{x}(t_0) &= \alpha \end{cases} \tag{4.2}$$

has at least one solution defined in a suitable interval $|t - t_0| < \delta$.

The function $\bar{f}$ is locally lipschitzian at $(t_0, \alpha) \in \Omega$ with respect to $\overline{x}$ if there exists a neighborhood $U \subset \Omega$ of (t_0, α) and a number $L > 0$ such that, denoting by $|\cdot|$ the euclidean norm in $\mathbb{R}^n$, one has

$$|\bar{f}(t, \overline{x}) - \bar{f}(t, \overline{z})| \leq L|\overline{x} - \overline{z}|,$$

for every $(t, \overline{x}), \; (t, \overline{z}) \in U$. If the preceding inequalities hold for all $(t, \overline{x}), (t, \overline{z}) \in \Omega$, then $\bar{f}$ is said to be globally lipschitzian in Ω, with respect to $\overline{x}$.

Theorem 4.1.3 (Uniqueness). *If $\bar{f}$ is continuous and locally lipschitzian with respect to $\overline{x}$, then* (4.2) *has a unique solution, defined in a suitable neighborhood of t_0.*

Theorem 4.1.4 (Global existence). *If $\Omega = [a, b] \times \mathbb{R}^n$ and $\bar{f}$ is continuous and globally lipschitzian in Ω with respect to $\overline{x}$, then the solution of* (4.2) *is defined on all $[a, b]$.*

Proof. (Sketch) As in the proof of the existence and uniqueness theorem for a single equation, see the Appendix 2.5 in Chapter 2, one checks that (4.2) is equivalent to the system of integral equations

$$x_i(t) = x_i(0) + \int_0^t f_i(t, x_1(t), \dots, x_n(t))dt, \quad i = 1, 2, \dots, n$$

which can be written in compact form as

$$\overline{x}(t) = \overline{x}_0 + \int_0^t \bar{f}(t, \overline{x}(t))dt$$

and is formally like the integral equation (2.14). Repeating the arguments carried in the aforementioned Appendix, one proves the local and global existence and uniqueness for (4.2). ∎

4.2 Higher order equations

If $n = 2$ and $f_1(t, x_1, x_2) = x_2$, the system $\overline{x}' = \bar{f}(t, \overline{x})$ becomes

$$\begin{cases} x_1' &= x_2 \\ x_2' &= f_2(t, x_1, x_2). \end{cases}$$

Then $x_1'' = x_2' = f_2(t, x_1, x_2) = f_2(t, x_1, x_1')$ or, setting $x_1 = x$ and $f_2 = f$,

$$x'' = f(t, x, x')$$

which is a second order differential equation in normal form.

In general, consider the $n \times n$ system

$$\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ \dots &= \dots \\ x_{n-1}' &= x_n \\ x_n' &= f_n(t, x_1, \dots, x_n). \end{cases}$$

We find $x_1'' = x_2'$, $x_1''' = (x_2')' = x_3'$, etc. $x_1^{(n)} = x_n'$. Hence, calling $x = x_1$ and $f_n = f$, we find the *n-th order equation in normal form*

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}), \quad \left(x^{(k)} = \frac{d^k x}{dt^k} \right). \tag{4.3}$$

To understand what the natural initial value problem is for an n-th order equation, we simply go back to its equivalent system. We know that, given $(t_0, \alpha_1, \dots, \alpha_n) \in \Omega$ an initial value problem for an $n \times n$ system consists of requiring that $x_i(t_0) = \alpha_i$, for $i = 1, 2 \dots, n$. Since $x_1 = x$, $x' = x_2, \dots, x^{(n-1)} = x_n$, an initial value problem for the n-th order equation in normal form $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$ is

$$\begin{cases} x^{(n)} &= f(t, x, x', \dots, x^{(n-1)}) \\ x(t_0) &= \alpha_1, \ x'(t_0) = \alpha_2, \dots, \ x^{n-1}(t_0) = \alpha_n \,. \end{cases} \tag{4.4}$$

So, we prescribe at a certain $t = t_0$ the value $x(t_0)$, together with its derivatives up to order $n-1$, that is $x'(t_0), \dots, x^{(n-1)}(t_0)$. For example, in an initial value problem for a second order equation, we prescribe at $t = t_0$ the unknown $x(t)$ and its derivative $x'(t)$, that is

$$\begin{cases} x'' &= f(t, x, x') \\ x(t_0) &= \alpha_1 \\ x'(t_0) &= \alpha_2. \end{cases}$$

Similar to the case for first order equations, one could consider n-th order equations in the form $F(t, x, x', \dots, x^{(n)}) = 0$, which is not the normal form; but for the sake of simplicity we choose to work with the normal form. Some second order equations such as $F(t, x, x', x'') = 0$ will be briefly discussed at the end of Chapter 5.

4.2.1 Existence and uniqueness for n-th order equations

From the local existence result for systems stated before, we can deduce immediately the following theorems for n-th order equations in normal form.

Theorem 4.2.1 (Local existence). *Let $f : \Omega \mapsto \mathbb{R}$ be continuous and let $(t_0, \alpha_1, \ldots, \alpha_n)$ be a given point in the interior of Ω. Then the initial value problem*

$$\begin{cases} x^{(n)} &= f(t, x, x', \ldots, x^{(n-1)}) \\ x(t_0) &= \alpha_1,\ x'(t_0) = \alpha_2, \ldots,\ x^{(n-1)}(t_0) = \alpha_n \end{cases} \tag{4.4}$$

has at least one solution defined in a suitable interval $|t - t_0| < \delta$.

Proof. It suffices to remark that if f is continuous, then

$$\bar{f}(t, \overline{x}) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ f(t, x_1, ..., x_n) \end{pmatrix}$$

is also continuous. ■

Theorem 4.2.2 (Uniqueness). *If $f : \Omega \mapsto \mathbb{R}$ is continuous and locally lipschitzian with respect to $(x_1, \ldots, x_n)$, then* (4.4) *has a unique solution, defined in a suitable neighborhood of t_0.*

Proof. It is evident that if f is locally lipschitzian with respect to $(x_1, \ldots, x_n)$, then $\bar{f}(t, \overline{x})$ is also locally lipschitzian with respect to $\overline{x}$. ■

For example, the previous Theorems imply that the ivp

$$\begin{cases} x'' &= f(t, x) \\ x(t_0) &= \alpha_1 \\ x'(t_0) &= \alpha_2 \end{cases}$$

has a unique solution, defined in a suitable neighborhood of t_0, provided that f is locally lipschitzian with respect to x.

For the same reason, the global existence result for systems implies

Theorem 4.2.3 (Global existence). *If $\Omega = [a, b] \times \mathbb{R}^n$ and $f : \Omega \mapsto \mathbb{R}$ is continuous and globally lipschitzian in Ω with respect to $(x_1, \ldots, x_n)$ then the solution of* (4.4) *is defined on all $[a, b]$.*

As for first order equations, the uniqueness result can be used to find some properties of the solutions. We illustrate this with two examples.

Example 4.2.4. Let $f(x)$ be locally lipschitzian. Show that the solution of $x'' = f(x)$, $x(0) = 0$, $x'(0) = 0$, is even. Setting $z(t) = x(-t)$ we have $z''(t) = x''(-t) = f(x(-t)) = f(z)$. Moreover, $z(0) = x(0) = 0$ and $z'(0) = -x'(0) = 0$. Then, by uniqueness, $x(t) = z(t)$, that is $x(t) = x(-t)$. ■

Example 4.2.5. Let $f(x)$ be locally lipschitzian and let $x(t)$ be a solution of $x'' = f(x)$, defined for all $t \in \mathbb{R}$, satisfying $x(0) = x(T)$, $x'(0) = x'(T)$. Then $x(t)$ is periodic with period T. Setting $z(t) = x(t+T)$ one has $z''(t) = x''(t+T) = f(x(t+T)) = f(z(t))$ for all $t \in \mathbb{R}$. Moreover, $z(0) = x(T) = x(0)$ and $z'(0) = x'(T) = x'(0)$. Then, by uniqueness, $x(t) = z(t)$, that is $x(t) = x(t+T)$, for all $t \in \mathbb{R}$, which means that $x(t)$ is T-periodic. ■

4.3 Exercises

1. Show that the Cauchy problem $x'' = x|x|$, $x(0) = a$, $x'(0) = b$, has a unique solution, for all $a, b \in \mathbb{R}$.
2. Show that the Cauchy problem $x'' = \max\{0, x|x|\}$, $x(0) = a$, $x'(0) = b$, has a unique solution, for all $a, b \in \mathbb{R}$.
3. Show that for $p \geq 2$ the Cauchy problem $x'' = |x|^p$, $x(0) = a$, $x'(0) = b$, has a unique solution, for all $a, b \in \mathbb{R}$.
4. Let x, y be the unique solution of
$$\begin{cases} x' &= y \\ y' &= -x \\ x(0) &= 0, \ \ y(0) = 1. \end{cases}$$
Show that x, y verify $x^2 + y^2 \equiv 1$.
5. Do the same as in Problem 4 when x, y solve
$$\begin{cases} x' &= \theta y \\ y' &= -\theta x \\ x(0) &= 0, \ \ y(0) = 1. \end{cases}$$
6. Prove that the solutions x, y of
$$\begin{cases} x' &= H_y(x, y) \\ y' &= -H_x(x, y), \end{cases}$$
where $H : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is smooth, satisfy $H(x(t), y(t)) = c$.
7. Find the second order equation a solution of which is $x(t) = e^t + e^{-t}$.
8. Same for $x = te^t$.
9. Let x be the solution of $x'' + x = 0$, $x(0) = 0$, $x'(0) = 1$. Prove that $x = \sin t$.

10. Let x be the solution of $x'' + 4x = 0$, $x(0) = 1, x'(0) = 0$. Prove that $x = \cos 2t$.
11. Let $f(t, x)$ be T-periodic with respect to t and let $x(t)$ be a solution of $x'' = f(t, x)$, defined for all $t \in \mathbb{R}$, such that $x(0) = x(T)$ and $x'(0) = x'(T)$. Show that $x(t)$ is T-periodic.
12. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be smooth and let $x : \mathbb{R} \mapsto \mathbb{R}$ satisfy $x'''' = f(x)$, $x'(0) = 0$, $x'''(0) = 0$. Show that $x(t)$ is even.
13. Show that if f is continuous and $f(x) > 0$ for all $x \in \mathbb{R}$, then the solutions of $x''' = f(x)$, $x''(0) = 0$, have an inflection point at $t = 0$.

5

Second order equations

This chapter is devoted to second order equations and is organized as follows. First we deal with general linear homogeneous equations, including linear independence of solutions and the reduction of order. Then we discuss general linear nonhomogeneous equations. Sections 5.5 and 5.6 deal with the constant coefficients case. Section 5.7 is devoted to the study of oscillation theory and the oscillatory behavior of solutions. Finally, in the last section we deal with some nonlinear equations.

5.1 Linear homogeneous equations

The equation

$$a_0(t)x'' + a_1(t)x' + a_2(t)x = g(t)$$

represents the most general second order linear differential equation. When $g(t) \equiv 0$ it is called *homogeneous*; otherwise it is called *nonhomogeneous*.

For simplicity and convenience, we will assume that $a_0(t) \neq 0$, so that we can divide the equation by $a_0(t)$ and write the equation in the form

$$x'' + p(t)x' + q(t)x = f(t). \tag{5.1}$$

The values of t where $a_0(t)$ vanishes are called *singular* points. Notice that we have used the term "singular point" also in the case of exact equations $Mdx + Ndy = 0$, with a somewhat different meaning. Second order equations with singular points will be discussed in Chapter 10 in connection with Bessel equations.

Before starting the theoretical study of linear second order equations, we discuss an example which highlights the importance of these equations.

Example 5.1.1 (The harmonic oscillator). Consider a body P on the x axis at the free end of an elastic spring which is attached at the origin O.

Assuming that P has unitary mass and that there is neither friction nor external force acting on the body, Hooke's law states that the force F acting on the body is

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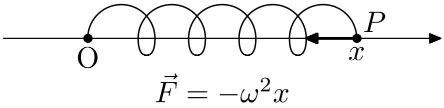

Fig. 5.1. The elastic spring

proportional, with a negative proportionality constant $-\omega^2 < 0$, to the distance x of the body to O, that is $F = -\omega^2 x$. Notice that the minus sign is due to the fact that the force is a restoring one, namely it brings back the body to the initial position, acting oppositely to the motion of P. See Figure 5.1.

Denoting by $x(t)$ such a distance, dependent on time t, and by $x''(t)$ its acceleration, Newton's second law, Force=Mass × Acceleration, yields $x'' = -\omega^2 x$, or

$$x'' + \omega^2 x = 0, \qquad (\omega \neq 0).$$

This equation is of the type (5.1) with $p = 0$, $q = \omega^2$, $f = 0$ and is usually referred to as the equation of the *free harmonic oscillator*. We will see that the solution is a superposition of sine and cosine functions and hence the body P at the free end of the spring oscillates and its motion is periodic, as expected.

We anticipate that a similar equation arises when we study the mathematical pendulum (see Example 5.5.4).

If there is an external force $f(t)$ acting on the body, the equation becomes

$$x'' + \omega^2 x = f(t)$$

which is a second order nonhomogeneous equation. In particular, we will study the case in which $f(t) = \sin \omega_1 t$ which yields

$$x'' + \omega^2 x = \sin \omega_1 t.$$

The solutions of this equation depend on the relationship between ω and ω_1 and give rise to interesting phenomena, like resonance or beats. See Section 5.6.1.

In the presence of friction proportional to the velocity x' of P, the equation becomes

$$x''(t) + kx'(t) + \omega^2 x(t) = f(t)$$

which is of the type (5.1) with $p = k$ and $q = \omega^2$. It is usually referred to as the equation of the *damped harmonic oscillator*, the damping term being kx'. Among other applications, equations of this type arise in the theory RLC electrical circuits (see Example 5.5.5). ■

Now we concentrate on the homogeneous case and state the existence and uniqueness result for such equations. The following theorem follows directly from Theorems 4.2.2 and 4.2.3 of Chapter 4.

Theorem 5.1.2 (Existence and Uniqueness). *If $p(t)$, $q(t)$ are continuous on an interval $I \subseteq \mathbb{R}$, then for any number t_0 in I and any numbers α and β, there exists a* *unique solution $x(t)$ of*

$$x'' + p(t)x' + q(t)x = 0 \tag{5.2}$$

satisfying the initial conditions $x(t_0) = \alpha$, $x'(t_0) = \beta$. Furthermore, this solution exists for all t in I.

Recall that a solution of (5.2) is a twice differentiable function $x(t)$ that satisfies equation (5.2).

We note that if α and β are both 0 in the above theorem, then the solution $x(t)$ guaranteed by the theorem must be the trivial solution, that is $x(t) \equiv 0$. This follows from the fact that the zero function is also a solution of (5.2) satisfying the same initial conditions as $x(t)$. Since there can be only one such solution, then we must have $x(t) \equiv 0$. We state this fact as

Corollary 5.1.3. *If $x(t)$ is any solution of* (5.2) *such that $x(t_0) = 0 = x'(t_0)$, then $x(t) \equiv 0$.*

Remark 5.1.4. Unlike the solutions of the first order linear homogeneous equations, here nontrivial solutions may vanish; in fact, they may vanish infinitely many times, as indicated by the examples below. So, it is no longer true that the solutions are either always positive or always negative. However, what *is* true is that in view of the above Corollary 5.1.3, the maximum and minimum points of the solutions cannot lie on the t-axis; so they are either above or below the line $t = 0$. ■

Example 5.1.5. The function $x = t^2e^t$ cannot be a solution of the differential equation (5.2). The reason is that x and its derivative $x' = t^2e^t + 2te^t$ both vanish at $t = 0$ and being a solution would contradict Corollary 5.1.3. ■

We would like to point out that occasionally a second order equation, linear or nonlinear, may be written as a first order equation, and then one can try and see if the methods developed for first order equations can be applied to solve it. We illustrate this in the following example.

Example 5.1.6. Consider the differential equation

$$x'' + x = 0. \tag{5.3}$$

This is a special case of the equation on the harmonic oscillator, discussed in Example 5.1.1.

In spite of its appearance, this equation is essentially a first order equation. To see this, let $z = x' = \frac{dx}{dt}$. Then by using the Chain Rule,

$$x'' = \frac{dx'}{dt} = \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = z\frac{dz}{dx}.$$

Now we can write equation (5.3) as

$$z\frac{dz}{dx} + x = 0$$

which is a first order separable equation and, by using the differential notation, it can be written as

$$z\,dz + x\,dx = 0.$$

Integrating, we obtain $\frac{z^2}{2} + \frac{x^2}{2} = c$, which can be written as $z^2 + x^2 = k_1$, where $k_1 = 2c$ is a non-negative constant. If $k_1 > 0$, solving for z, we get $z = \pm\sqrt{k_1 - x^2}$. Since $z = x'$, we have $x' = \frac{dx}{dt} = \pm\sqrt{k_1 - x^2}$, where the variables can be separated. In order to separate the variable, we assume that $-\sqrt{k_1} < x < \sqrt{k_1}$ so that $k_1 - x^2 > 0$. Then, using the differential notation, we have

$$\frac{\pm\,dx}{\sqrt{k_1 - x^2}} = dt. \tag{5.4}$$

We recall from Calculus that the plus sign in the above equation leads to $sin^{-1}\frac{x}{\sqrt{k_1}} = t + k_2$, or $\frac{x}{\sqrt{k_1}} = \sin(t + k_2)$. If we let $k_2 = 0$ and $\sqrt{k_1} = c_1$ we get the family of solutions

$$x = c_1 \sin t.$$

If in equation (5.4) we choose the negative sign, the same steps carried out above will lead to $cos^{-1}\frac{x}{\sqrt{k_2}} = t + k_3$, and if we let $k_3 = 0$ and $\sqrt{k_2} = c_2$, we obtain another family of solutions

$$x = c_2 \cos t.$$

We will see later in Example 5.2.10 that the all the solutions of (5.3) are given by $x = c_1\, sin\, t + c_2\, cos\, t$. ■

Remark 5.1.7. In the above discussion, in order to separate the variables, we assumed $k_1 - x^2 \neq 0$, which was necessary so that we would be able to divide both sides by $\sqrt{k_1 - x^2}$, thus obtaining the solution $x = \sqrt{k_1}\sin t$. But now we see that $x = \sqrt{k_1}\sin t$ is defined for all real numbers t, including those where $k_1 - x^2 = 0$, which can occur, for example, at $t = \frac{\pi}{2}$. Therefore, $x = \sqrt{k}\sin t$ satisfies the equation $x'' + x = 0$ for all t. This phenomenon is not all that uncommon in solving differential equations, where one makes an assumption in order to carry out a certain operation but later it turns out that the solution is valid even without the assumed restriction. So, at the end of solving an equation, it is worthwhile to check to see if the restrictions assumed to carry on the operations can be lifted. ■

Theorem 5.1.8. *If x_1 and x_2 are any solutions of* (5.2)*, then for any constants c_1 and c_2, the linear combination $c_1x_1 + c_2x_2$ is also a solution.*

Proof. Let $x = c_1x_1 + c_2x_2$. Substituting x' and x'' in equation (5.2) and regrouping terms, we have

$$\begin{aligned}(c_1x_1 + c_2x_2)'' + p(t)(c_1x_1 + c_2x_2)' + q(t)(c_1x_1 + c_2x_2) &= \\ [c_1x_1'' + p(t)c_1x_1' + c_1q(t)x_1] + [c_2x_2'' + p(t)c_2x_2' + c_2q(t)x_2] &= \\ c_1[x_1'' + p(t)x_1' + q(t)x_1] + c_2[x_2'' + p(t)x_2' + q(t)x_2] &= 0\end{aligned}$$

because x_1 and x_2 are solutions and hence

$$x_1'' + p(t)x_1' + q(t)x_1 = 0 = x_2'' + p(t)x_2' + q(t)x_2,$$

proving the theorem. ■

Remark 5.1.9. The property that linear combinations of solutions is a solution is particular to linear homogeneous equations. This is an important property of such equations, often referred to as the *Principle of Superposition*. In particular, it is not true for nonhomogeneous equations or nonlinear equations. For example, as can be easily verified, $x_1 = 1$ and $x_2 = e^t + 1$ are both solutions of $x'' - 3x' + 2x = 2$, but their sum $x_1 + x_2 = 1 + (e^t + 1)$ is not a solution. ■

5.2 Linear independence and the Wronskian

The goal of this section is to find the general solution of (5.2) which is, by definition, a family $x = \phi(t, c_1, c_2)$ depending on two real parameters c_1, c_2 such that:

1. For all c_1, c_2, the function $x = \phi(t, c_1, c_2)$ is a solution of (5.2).
2. If $\overline{x}(t)$ is a solution of (5.2), there exist $\overline{c}_1, \overline{c}_2$ such that $\overline{x}(t) = \phi(t, \overline{c}_1, \overline{c}_2)$.

Remark 5.2.1. Similar to the case for linear first order equations, here also the general solution includes *all the solutions* of (5.2). ■

To find the general solution of (5.2) we first introduce the notion of linear independence of functions.

Let $f(t)$ and $g(t)$ be two functions defined on an interval I. We say that f and g are *linearly independent* on I if the only way we can have $c_1f(t) + c_2g(t) = 0$ for all t is to have c_1 and c_2 both equal to 0. That is, if c_1 and c_2 are constants such that $c_1f(t) + c_2g(t) = 0$ for all t in I, then $c_1 = 0 = c_2$. Functions that are not linearly independent are said to be *linearly dependent*.

Remark 5.2.2. First we note that if there exist constants c_1 and c_2 such that $c_1f(t) + c_2g(t) = 0$ for all t in I, and f and g are not identically zero, then if one of the constants is zero, so is the other. Suppose $c_1 = 0$ and $c_2 \neq 0$. Then we have $c_2g(t) = 0$ and this implies that $g(t) = 0$ for all t. Similarly, the case $c_2 = 0$ leads to $c_1 = 0$.

Using the contrapositive of the statement defining linear independence, we see that f and g are linearly dependent if and only if there exist nonzero constants c_1

and c_2 such that $c_1 f(t) + c_2 g(t) = 0$ for all t in I. Thus, for any two functions, linear dependence means that one of them is a constant multiple of the other. But such a simplification is not possible for more than two functions. Therefore, we advise students to learn how to use the above definition in order to be prepared to deal with a higher number of functions later. ■

Example 5.2.3. Let us prove that $f(t) = \sin t$ and $g(t) = \cos t$ are linearly independent. We start by assuming that c_1 and c_2 are constants such that $c_1 \sin t + c_2 \cos t = 0$ for all t. Next, we show that $c_1 = c_2 = 0$. There are several ways to accomplish this. We explain three of them. Sometimes one of the methods is more convenient than the others, depending on the problem.

First method. We substitute some number t that will make one of the terms $c_1 \sin t$ and $c_1 \cos x$ become 0. Let, for example, $t = 0$. Then $c_1 \sin 0 + c_2 \cos 0 = 0$ implies that $c_2 = 0$. Next, we let $t = \frac{\pi}{2}$ and obtain $c_1 = 0$.

Second method. We notice that if $x = c_1 \sin t + c_2 \cos t = 0$ for all t, then so is $x' = c_1 \cos t - c_2 \sin t = 0$ for all t. So, we simply solve the system of equations

$$\begin{cases} c_1 \sin t + c_2 \cos t = 0 \\ c_1 \cos t - c_2 \sin t = 0 \end{cases}$$

for c_1 and c_2, using some method we have learned in Algebra. For example, multiplying the first equation by $\cos t$, the second one by $-\sin t$ and adding, we obtain $c_2(\sin^2 t + \cos^2 t) = 0$. Since $\sin^2 t + \cos^2 t = 1$, we must have $c_2 = 0$. Now, returning to the equation $c_1 \sin t + c_2 \cos t = 0$, we are left with $c_1 \sin t = 0$, which implies that $c_1 = 0$ since $\sin t$ cannot be 0 for all t. We note that this method is not applicable if the functions are not differentiable.

Third method. Once we obtained the system

$$\begin{cases} c_1 \sin t + c_2 \cos t = 0 \\ c_1 \cos t - c_2 \sin t = 0 \end{cases}$$

above, we need not solve for c_1 and c_2 but simply determine whether the coefficient determinant is zero or nonzero. Since the coefficient determinant

$$\begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t = -1$$

is nonzero, we recall from Algebra or Linear Algebra that this system has a unique solution in c_1 and c_2. Since the pair $c_1 = 0$ and $c_2 = 0$ is a solution, then this is the only solution and there cannot be any nonzero solutions c_1 and c_2. ■

Next, we will see that the coefficient determinant mentioned above plays an important role in the study of linear homogeneous equations.

5.2.1 Wronskian

The *Wronskian* of two differentiable functions $f(t)$ and $g(t)$ is defined as

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t).$$

Note: Sometimes, instead of $W(f,g)(t)$, we may interchangeably use the notation $W(f(t), g(t))$; and when there is no confusion about what the functions f and g are, we may simply use the notation $W(t)$; in other words, $W(f,g)(t) = W(f(t), g(t)) = W(t)$.

Theorem 5.2.4 (Abel's Theorem). *If x_1 and x_2 are any solutions of*

$$x'' + p(t)x' + q(t)x = 0 \tag{5.2}$$

on a given interval I where $p(t)$ and $q(t)$ are continuous, then the Wronskian of x_1 and x_2 is given by

$$W(t) = ce^{-\int p(t)\,dt}$$

where c is a constant.

Proof. Taking the derivative of $W(t) = x_1x_2' - x_1'x_2$, we obtain $W'(t) = x_1x_2'' - x_2x_1''$. Since $x_1'' = -p(t)x_1' - q(t)x_1$ and $x_2'' = -p(t)x_2' - q(t)x_2$ from equation (5.2), by substituting, we obtain $W'(t) = [-p(t)x_2' - q(t)x_2]x_1 - [-p(t)x_1' - q(t)x_1]x_2 = -p(t)[x_1x_2' - x_1'x_2] = -p(t)W(t)$. Solving the first order linear equation $W'(t) = -p(t)W(t)$ by the method of integrating factor, we obtain $W(t) = ce^{-\int p(t)\,dt}$. ∎

Note: In the above proof, instead of obtaining $W(t)$ in terms of the antiderivative, we could obtain it in terms of the definite integral $W(t) = ce^{-\int_{t_0}^{t} p(t)dt}$, where t_0 is any point in the interior of the interval I . This should be clear, since in solving $W'(t) = -p(t)W(t)$, one could multiply both sides of the equation $W'(t) + p(t)W(t) = 0$ by the integrating factor $\int_{t_0}^{t} p(t)dt$ instead of $\int p(t)dt$.

Corollary 5.2.5. *The Wronskian of two solutions is either always zero or never zero.*

Proof. Since for any solutions z_1 and z_2, by Abel's Theorem, $W(z_1, z_2)(t) = ce^{\int -p(t)\,dt}$ and $e^{\int -p(t)\,dt}$ is never zero, then the only way that the Wronskian can be zero at any point is to have $c = 0$, in which case the Wronskian is equal to zero for all t. ∎

Example 5.2.6. The functions $x_1 = e^t$ and $x_2 = \sin t$ cannot be solutions of the differential equation (5.2) on $I = (-\pi, \pi)$, given that $p(t)$ and $q(t)$ are continuous on I. To see this, we examine their Wronskian $W(\sin t, e^t) = e^t \sin t - e^t \cos t$. We see that $W(0) = -1$ and $W(\frac{\pi}{4}) = 0$, contradicting Corollary 5.2.5. ∎

Before establishing our next important theorem concerning Wronskian, let us recall some algebraic facts. Consider the system of equations

$$\begin{cases} ax + by = 0 \\ cx + dy = 0. \end{cases}$$

We can always obtain one solution, called the *trivial solution*, by letting $x = y = 0$. But, does it have any other types of *nontrivial* solutions? The answer depends on the determinant of the coefficients; namely the system has a unique solution if and only if the coefficient determinant is nonzero, that is if and only if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

This means that if the coefficient determinant is nonzero, then $x = y = 0$ is the only solution to the system; so it has no nontrivial solutions. Furthermore, since the condition on the coefficient determinant is both necessary and sufficient, it follows that the system has a nontrivial solution in x and y if and only if the coefficient determinant is zero.

The next theorem gives us a convenient criterion for determining if two solutions are linearly dependent.

Theorem 5.2.7. *Two solutions x_1 and x_2 of* (5.2) *are linearly dependent if and only if $W(x_1, x_2)(t) = 0$ for all t in I.*

Proof. Two solutions x_1 and x_2 are linearly dependent if and only if there exist nonzero constants c_1 and c_2 such that $c_1x_1(t) + c_2x_2(t) = 0$ for all t in I. We note that if such numbers c_1 and c_2 exist, then we also have $c_1x_1'(t) + c_2x_2'(t) = 0$ for all t in I, since the derivative of the zero function is the zero function. Let t_0 be some number in I and let us look at the system of two algebraic equations

$$\begin{cases} c_1x_1(t_0) + c_2x_2(t_0) = 0 \\ c_1x_1'(t_0) + c_2x_2'(t_0) = 0\,. \end{cases}$$

As pointed out earlier, such a system will have a nontrivial solution in c_1 and c_2 if and only if

$$\begin{vmatrix} x_1(t_0) & x_2(t_0) \\ x_1'(t_0) & x_2'(t_0) \end{vmatrix} = 0.$$

But this determinant happens to be the Wronskian of x_1 and x_2 evaluated at t_0. Therefore, we can say that x_1 and x_2 are linearly dependent if and only $W(x_1, x_2)(t_0) = 0$. Finally, since, by Abel's theorem, the Wronskian of two solutions is either identically zero or never zero, we can say that x_1 and x_2 are linearly dependent if and only if $W(x_1, x_2)(t) = 0$ for all t in I. ∎

Example 5.2.8. The above theorem is not valid for arbitrary functions that are not solutions of the linear equation (5.2). Let x_1 and x_2 be defined as follows:

$$x_1(t) = \begin{cases} -t^3 \; if \; -\infty < t \le 0 \\ \;\; t^3 \; if \;\; 0 \le t < \infty \end{cases}$$

and $x_2(t) = t^3$ for $-\infty < t < \infty$. The functions x_1 and x_2 are linearly independent on the interval $(-\infty, \infty)$ and yet $W(x_1, x_2)(t) = 0$ for all t. The student should verify this. ∎

If x_1 and x_2 are linearly independent solutions of (5.2) then x_1 and x_2 are said to form *fundamental solutions* of (5.2).

Theorem 5.2.9. *The general solution of* (5.2) *is given by* $c_1x_1 + c_2x_2$*, provided* x_1 *and* x_2 *are fundamental solutions.*

Proof. According to Theorem 5.1.8, $c_1x_1 + c_2x_2$ is a solution of (5.2). Next, if $y(t)$ is any solution of (5.2), we have to show that there exist constants $\overline{c}_1$ and $\overline{c}_2$ such that $y(t) = \overline{c}_1x_1(t) + \overline{c}_2x_2(t)$. Let t_0 be any fixed number in I and consider the system

$$\begin{cases} c_1x_1(t_0) + c_2x_2(t_0) = y(t_0) \\ c_1x_1'(t_0) + c_2x_2'(t_0) = y'(t_0) \end{cases}$$

where c_1 and c_2 are the unknowns. This system has a unique solution $\overline{c}_1, \overline{c}_2$ if and only if the determinant of the coefficients is different from 0. This determinant is precisely the Wronskian $W(x_1, x_2)(t)$ which is not zero because x_1, x_2 are fundamental solutions. ∎

Example 5.2.10. Show that $x = c_1 \sin t + c_2 \cos t$ is the general solution of $x'' + x = 0$ and find a different fundamental set of solutions that can be used to obtain the general solution.

As we saw in Example 5.1.6, $x_1 = \sin t$ and $x_2 = \cos t$ are solutions. Since $W(\sin t, \cos t) = (\sin t)(-\sin t) - (\cos t)(\cos t) = -(\sin^2 t + \cos^2 t) = -1 \neq 0$, it follows from Theorem 5.2.9 that they form a fundamental set of solutions and hence $x = c_1 \sin t + c_2 \cos t$ is the general solution.

To answer the second part, any pair of solutions whose Wronskian is different from zero would generate the same general solution. Thus, any pair of solutions that are not constant multiples of each other would work. For example, if $x_1 = 2\sin t$ and $x_2 = 3\cos t$, then $W(2\sin t, 3\cos t) = (2\sin t)(-3\sin t) - (3\cos t)(2\cos t) = -6(\sin^2 x + \cos^2 x) = -6$ and hence they form a fundamental set of solutions. Therefore, $y = c_1(2\sin t) + c_2(3\cos t)$ is the general solution. We note that if we replace the constant $2c_1$ by another constant k_1 and $3c_2$ by k_2, then the general solution becomes $x = k_1 \sin t + k_2 \cos t$. ∎

5.3 Reduction of the order

Consider again the general linear homogeneous equation

$$x'' + p(t)x' + q(t)x = 0 \tag{5.2}$$

where $p(t)$ and $q(t)$ are continuous in the interval $I \subseteq \mathbb{R}$. If x_1 is a solution, then we know that any constant multiple $x = cx_1$ is also a solution. Can c be replaced by a variable function $v(t)$ such that $x_2 = v(t)x_1(t)$ is also a solution? The answer is yes. As shown in the theorem below, substituting x_2 in the equation (5.2) reduces the order of the differential equation and is hence called the *Reduction of Order Method.*

Theorem 5.3.1. *If* $x_1(t)$ *is a solution of* (5.2) *in* I, $x_1(t) \neq 0$, *then*

$$x_2(t) = x_1(t) \int \frac{e^{-\int p(t)\,dt}}{x_1^2(t)}\, dt$$

is another solution. Furthermore, $x_1(t)$ *and* $x_2(t)$ *form a fundamental set of solutions.*

Proof. Substituting $v(t)x_1(x)$ in equation (5.2), we have

$$(v''x_1 + v'x_1' + vx_1'' + v'x_1') + p(v'x_1 + vx_1') + vqx_1 =$$
$$x_1v'' + (2x_1' + px_1)v' + v(x_1'' + px_1' + qx_1) = x_1v'' + (2x_1' + px_1)v' = 0$$

since $x_1'' + px_1' + qx_1 = 0$, as x_1 is a solution of (5.2). Now, if we let $w = v'$, we obtain the first order differential equation $x_1w' + (2x_1' + px_1)w = 0$ or $w' + (\frac{2x_1'}{x_1} + p)w = 0$. The integrating factor for this first order equation is $e^{2\ln x_1 + \int p\,dt} = {x_1}^2 e^{\int p\,dt}$. Therefore, assuming that $x_1 \neq 0$, we obtain

$$w = \frac{c}{{x_1}^2 e^{\int p\,dt}} = \frac{ce^{-\int p\,dt}}{x_1^2}.$$

Recall that $v' = w$. Since we only need one function $v(t)$ so that vx_1 is a solution, we can let $c = 1$ and hence

$$w = \frac{e^{-\int p\,dt}}{x_1^2}, \quad \text{and} \quad x_2 = x_1 v = x_1 \int \frac{e^{-\int p\,dt}}{x_1^2}\, dt.$$

To see that x_1 and x_2 form a fundamental set of solutions, we note that according to Theorem 5.2.9, x_1 and x_2 are fundamental solutions if $W(x_1, x_2) \neq 0$ or equivalently if they are linearly independent (see Theorem 5.2.7). Thus it suffices to show that they are linearly independent; or, equivalently, that one of them is not a constant multiple of the other. To verify the last statement, suppose that $x_2 = cx_1$, c a constant. Then

$$x_1 \int \frac{e^{-\int p\,dt}}{x_1^2} dt = cx_1$$

and hence

$$\int \frac{e^{-\int p\,dt}}{x_1^2} dt = c.$$

Now, taking the derivatives of both sides, we obtain

$$\frac{e^{-\int p\,dt}}{x_1^2} = 0$$

which is a contradiction since $e^{-\int p\,dt}$ is nonzero. ■

Example 5.3.2. Knowing that $x_1 = t$ is one solution, find the general solution of

$$x'' - \frac{1}{t}x' + \frac{1}{t^2}x = 0, \quad t > 0.$$

To find another linearly independent solution, using Theorem 5.3.1, we obtain

$$t \int \frac{e^{-\int -\frac{1}{t}\,dt}}{t^2} dt = t \ln t.$$

The general solution is $x = c_1 t + c_2 t \ln t$. ■

Equations like the one in the previous exercise are called Euler equations and will be discussed in Subsection 5.5.1.

5.4 Linear nonhomogeneous equations

In this section we study the nonhomogeneous equation

$$x'' + p(t)x' + q(t)x = f(t). \tag{5.5}$$

First we state the existence and uniqueness of solutions, which follow immediately from Theorems 4.2.2–4.2.3 of Chapter 4.

Theorem 5.4.1 (Existence and Uniqueness). *Suppose that p, q, and f are continuous functions on an interval $I \subseteq \mathbb{R}$. Equation* (5.5) *has a unique solution $x(t)$ such that $x(t_0) = \alpha$, $x'(t_0) = \beta$, where t_0 is any number in I and α and β are any real numbers. Furthermore, this solution is defined for all t, t in I.*

Similar to the case for homogeneous equations, the general solution of (5.5) is defined as the family of all solutions of such an equation.

The next theorem shows that in order to find the general solution of the nonhomogeneous equation, all we need is the general equation of the homogeneous equation and one solution of the nonhomogeneous equation. For the constant coefficient case, we have already learned how to find the general solution of the homogeneous equation. In the next section we will learn how to find a particular equation of the nonhomogeneous equation and thus get the general solution of the nonhomogeneous equation.

Lemma 5.4.2. *If x_1 and x_2 are two solutions of the nonhomogeneous equation* (5.5)*, then $x_1 - x_2$ is a solution of the corresponding homogeneous equation*

$$x'' + p(t)x' + q(t)x = 0. \tag{5.6}$$

Proof. The proof is straightforward . Since

$$x_1'' + p(t)x_1' + q(t)x = f(t)$$
$$x_2'' + p(t)x_2' + q(t)x_2 = f(t),$$

then by subtracting the second equation from the first, we obtain

$$(x_1'' - x_2'') + p(t)(x_1' - x_2') + q(t)(x_1 - x_2) = 0,$$

which proves the assertion. ■

Theorem 5.4.3. *If $x = c_1x_1 + c_2x_2$ is the general solution of the homogeneous equation* (5.6) *and x_p is any solution of the nonhomogeneous equation* (5.5)*, then $z = c_1x_1 + c_2x_2 + x_p$, $c_1, c_2 \in \mathbb{R}$, is the general solution of* (5.5).

Proof. Let z be any solution of the nonhomogeneous equation (5.5). We want to show that there exist constants k_1 and k_2 such that $z = k_1x_1 + k_2x_2 + x_p$. But since, by Lemma 5.4.2, $z - x_p$ is a solution of the homogeneous equation (5.2), there exist constants k_1 and k_2 such that $z - x_p = k_1x_1 + k_2x_2$ because $x = c_1x_1 + c_2x_2$ is given to be the general solution of the homogeneous equation. Solving for z, we get the desired result $z = k_1x_1 + k_2x_2 + x_p$. ■

Example 5.4.4. Consider the nonhomogeneous equation

$$x'' - \frac{1}{t}x' + \frac{1}{t^2}x = 4t, \quad t > 0.$$

In Example (5.2), we found the general solution of the corresponding homogeneous equation to be $x = c_1t + c_2t \ln t$. We also see that $x_p = t^3$ is a particular solution of the given nonhomogeneous equation. Therefore,

$$x = c_1t + c_2t \ln t + t^3, \quad t > 0$$

is its general solution. ■

So, it seems that solving the nonhomogeneous second order linear equation is quite simple *if* we can find the general solution of the corresponding homogeneous equation *and* a particular solution of the nonhomogeneous equation. But, except by inspection whenever possible, we have not yet developed a method for finding a particular solution of the nonhomogeneous equation. Next, we discuss a method for finding a particular solution of the nonhomogeneous equation.

5.4.1 Variation of parameters

As we saw in Section 5.3, given one solution x_1, we were able to find a function $v(t)$ so that vx_1 was also a solution. Here, given a pair of fundamental solutions x_1 and x_2 of the homogeneous equation (5.2), we try to find functions $v_1(t)$ and $v_2(t)$ such that $x = v_1x_1 + v_2x_2$ is a solution of (5.5). To this end, let $z = v_1x_1 + v_2x_2$. Calculating z' and z'', we have $z' = v_1x_1' + v_1'x_1 + v_2x_2' + v_2'x_2$. Now, since we have two unknowns v_1 and v_2, we would like to also have two equations involving these unknowns. Furthermore, we realize that substituting z in (5.5) will give us one equation. So, at this point we make the decision to let one of the equations be

$$v_1'x_1 + v_2'x_2 = 0 \tag{5.7}$$

which will also make it convenient and simpler to calculate z''. We have now reduced z' to $z' = v_1x_1' + v_2x_2'$ from which we obtain $z'' = v_1'x_1' + v_1x_1'' + v_2'x_2' + v_2x_2''$. In order to find a second equation involving v_1' and v_2', we substitute z in equation (5.5), obtaining

$$[v_1'x_1' + v_1x_1'' + v_2'x_2' + v_2x_2''] + p(t)[v_1x_1' + v_2x_2'] + q(t)[v_1x_1 + v_2x_2] = f(t)$$

which, after regrouping terms, can be written as

$$v_1[x_1'' + p(t)x_1' + q(t)x_1] + v_2[x_2'' + p(t)v_2' + v_2x_2] + [v_1'x_1' + v_2'x_2'] = f(t).$$

Since x_1 and x_2 are solutions of the homogeneous equation (5.6) and hence satisfy the equation, the preceding equation is reduced to

$$v_1'x_1' + v_2'x_2' = f(t). \tag{5.8}$$

Thus we have reduced the problem of finding v_1 and v_2 to solving the algebraic system of equations (5.7)–(5.8)

$$\begin{cases} v_1'x_1 + v_2'x_2 &= 0 \\ v_1'x_1' + v_2'x_2' &= f(t) \end{cases}$$

for v_1' and v_2' and then integrating to obtain v_1 and v_2. Solving for v_1' and v_2', we have

$$v_1' = \frac{\begin{vmatrix} 0 & x_2 \\ f(t) & x_2' \end{vmatrix}}{W(t)} = \frac{-x_2(t)f(t)}{W(t)}, \qquad v_2' = \frac{\begin{vmatrix} x_1 & 0 \\ x_1' & f(t) \end{vmatrix}}{W(t)} = \frac{x_1f(t)}{W(t)}$$

where $W(t) = W(x_1, x_2)(t)$. Therefore, the particular solution z of (5.5) is given by

$$z = x_1(t)\int \frac{-x_2(t)f(t)}{W(t)}dt \, + \, x_2(t)\int \frac{x_1(t)f(t)}{W(t)}dt. \tag{5.9}$$

We do not advise that one memorize these integrals for v_1 and v_2 but instead one should start with the system of equations (5.7) and (5.8) and go through the procedure outlined above.

Example 5.4.5. Consider the equation

$$x'' - x = f(t) \tag{5.10}$$

where f is a continuous function on an interval I.

To find the general solution of the associated homogeneous equation $x'' - x = 0$, namely $x'' = x$, we notice that $x_1 = e^t$ and $x_2 = e^{-t}$ solve the equation. Since the Wronskian of e^t, e^{-t} is

$$W = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2$$

they form a fundamental set of solutions. Thus the general solution of $x'' - x = 0$ is

$$x(t) = c_1 e^t + c_2 e^{-t}.$$

Finding a specific solution $z(t)$ of (5.10) in the form $z(t) = v_1(t)e^t + v_2(t)e^{-t}$ leads to the system

$$\begin{cases} v_1'(t)e^t + v_2'(t)e^{-t} &= 0 \\ v_1'(t)e^t - v_2'(t)e^{-t} &= f(t) \end{cases}$$

where the determinant of the coefficients is $W = -2$. Then

$$v_1'(t) = \frac{-e^{-t} f(t)}{-2} = \tfrac{1}{2} e^{-t} f(t), \quad v_2'(t) = \frac{e^t f(t)}{-2} = -\tfrac{1}{2} e^t f(t).$$

Integrating

$$v_1(t) = \tfrac{1}{2} \int e^{-t} f(t), \quad v_2(t) = -\tfrac{1}{2} \int e^t f(t)$$

and hence

$$z(t) = \tfrac{1}{2} e^t \int e^{-t} f(t) - \tfrac{1}{2} e^{-t} \int e^t f(t).$$

This formula gives the particular solution in an implicit way and it holds for *any* function $f(t)$. ■

Example 5.4.6. Find the general solution of

$$x'' + x = \frac{1}{\cos t}, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

We have already seen in Example 5.2.10 that the general solution of the associated homogeneous equation $x'' + x = 0$ is $x(t) = c_1 \sin t + c_2 \cos t$. Moreover, the Wronskian $W(x_1, x_2)$ of $x_1 = \sin t$, $x_2 = \cos t$ is equal to -1. To find a specific solution z of $x'' + x = \frac{1}{\cos t}$ we set $z = v_1 \sin t + v_2 \cos t$ and solve the system

$$\begin{cases} v_1'(t)\sin t + v_2'(t)\cos t &= 0 \\ v_1'(t)\cos t - v_2'(t)\sin t &= \frac{1}{\cos t} \end{cases} \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

where the coefficient determinant is just the Wronskian $W = -1$. Then Cramer's rule yields

$$v_1'(t) = \cos t\,\frac{1}{\cos t} = 1, \qquad v_2'(t) = -\sin t\,\frac{1}{\cos t} = -\frac{\sin t}{\cos t}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Integrating, we get

$$v_1(t) = \int dt = t + c_1$$

and

$$v_2(t) = -\int \frac{\sin t}{\cos t}\,dt = \int \frac{d\cos t}{\cos t} = \ln(\cos t) + c_2, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

For convenience we can take $c_1 = c_2 = 0$ since we need only one function v_1 and one function v_2. Thus a specific solution is $z(t) = t\sin t + \cos t \cdot \ln(\cos t)$. Finally, the general solution is

$$x(t) = c_1 \sin t + c_2 \cos t + t\sin t + \cos t \cdot \ln(\cos t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}. \qquad \blacksquare$$

Remark 5.4.7. The method of Variation of Parameters has a drawback and that is that the integration may be messy or even impossible to carry out in order to find the solution explicitly. But we can always find the solution implicitly as long as we can find the general solution of the homogeneous equation. ■

5.5 Linear homogeneous equations with constant coefficients

A general second order homogeneous equation with constant coefficients has the form

$$ax'' + bx' + cx = 0, \tag{5.11}$$

where $a \neq 0$. In searching for a solution, we recall that the exponential functions have the property that their derivatives involve the same exponential functions. So, we might try to find solutions of the form $x = e^{mt}$. We also see that if we substitute this exponential function in the differential equation, every term on the left side will have a constant times e^{mt} and hence we can eliminate it by dividing both sides by it. Now we end up with an algebraic quadratic equation that we can handle. To this end, substituting $y = e^{mt}$ into the equation, we obtain

$$a\,m^2 e^{mt} + b\,me^{mt} + c\,e^{mt} = 0.$$

Dividing by e^{mt}, we obtain the algebraic equation

$$a\,m^2 + b\,m + c = 0. \tag{5.12}$$

This shows that if $x = e^{mt}$ is a solution of (5.11), then m is a solution of (5.12). Conversely, if m is solution of (5.12), then, by reversing the steps, it follows that e^{mt} is a solution of (5.11).

Equation (5.12) is called the *characteristic* or *auxiliary* equation corresponding to equation (5.11). We have now reduced the problem of solving (5.11) to that of solving the characteristic equation and then analyzing the corresponding solutions. Solving (5.12), we have

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We have to consider three cases: (1) $b^2 - 4ac > 0$, (2) $b^2 - 4ac = 0$ and (3) $b^2 - 4ac < 0$.

(1) The case $b^2 - 4ac > 0$ (real distinct roots). In this case the characteristic equation has two distinct real roots

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The corresponding solutions of (5.11) are given by $x_1 = e^{m_1 t}$ and $x_2 = e^{m_2 t}$. We claim that x_1 and x_2 are a fundamental set of solutions. To see this, we simply evaluate their Wronskian. $W(x_1, x_2)(t) = e^{m_1 t} m_2 e^{m_2 t} - m_1 e^{m_1 t} e^{m_2 t} = e^{m_1 t} e^{m_2 t}(m_1 - m_2) \neq 0$ since m_1 and m_2 are distinct roots. This means that the general solution of (5.11) is given by

$$x = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

Example 5.5.1. Solve the initial value problem

$$2x'' + x' - x = 0, \;\; x(0) = 1, \;\; x'(0) = 2.$$

By substituting $x = e^{mt}$, we obtain the characteristic equation

$$2m^2 + m - 1 = 0.$$

Solving for m, we set $2m^2 + m - 1 = 0 = (2m - 1)(m + 1)$ and find the two roots $m_1 = -1$, $m_2 = \frac{1}{2}$. Since the roots of the characteristic equation are real and distinct, the general solution of the differential equation is $x = c_1 e^{-t} + c_2 e^{\frac{1}{2}t}$. In order to get the solution to the initial value problem, we set $x(0) = 1, \; x'(0) = 2$ and solve for c_1 and c_2. Solving,

$$\begin{cases} c_1 + c_2 & = 1 \\ -c_1 + \frac{1}{2}c_2 & = 2 \end{cases}$$

we obtain $c_1 = -1$ and $c_2 = 2$. Therefore,

$$x = -e^{-t} + 2e^{\frac{1}{2}t}$$

is the desired solution. ■

(2) The case $b^2 - 4ac = 0$ (repeated roots). In this case, $m = -\frac{b}{2a}$ is a repeated root of

$$ax'' + bx' + cx = 0$$

and we have only one solution $x_1 = e^{-\frac{b}{2a}t}$. In order to find another linearly independent solution, we either use the method of Reduction of Order directly or we use Theorem 5.3.1. Using the theorem, we let $p(t) = b/a$, $x_1 = e^{-\frac{b}{2a}t}$, and obtain another linearly independent solution

$$x_2 = e^{-\frac{b}{2a}t} \int \frac{e^{-\frac{b}{a}t}}{(e^{-\frac{b}{2a}t})^2} dt = e^{-\frac{b}{2a}t} \int \frac{e^{-\frac{b}{a}t}}{e^{-\frac{b}{a}t}} dt = e^{-\frac{b}{2a}t} \int dt = te^{-\frac{b}{2a}t}$$

taking the constant of integration to be zero. One can easily verify that $x_1 = e^{-\frac{b}{2a}t}$ and $x_2 = te^{-\frac{b}{2a}t}$ are linearly independent and hence form a fundamental set of solutions. Therefore $x = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}$ is the general solution.

Example 5.5.2. Find the general solution of $x'' - 6x' + 9x = 0$. The corresponding characteristic equation is $m^2 - 6m + 9 = 0$. We see that $m^2 - 6m + 9 = (m-3)^2 = 0$ has a repeated root $m = 3$. Therefore, $x = e^{3t}$ and te^{3t} are two independent solutions and hence the general solution is given by $x = c_1 e^{3t} + c_2 te^{3t}$. ■

(3) The case $b^2 - 4ac < 0$ (complex roots). We first recall a couple of simple facts about complex numbers.

1. If $\mu + i\lambda$ is a root of the characteristic equation $am^2 + bm + c = 0$, then so is its conjugate $\mu - i\lambda$.

2. A complex number can be equal to 0 only if its real and imaginary parts are both 0, that is $\mu + i\lambda = 0$ implies $\mu = \lambda = 0$.

When the discriminant $b^2 - 4ac$ is negative, the characteristic equation has two complex conjugate roots

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

But, for a complex number $\mu + i\lambda$, what is $e^{(\mu + i\lambda)t}$ and how do we extract real solutions out of this? We proceed as follows.

First, we write, formally, the Taylor expansion for $e^{i\lambda}$, obtaining

$$\begin{aligned} e^{i\lambda} = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} &= 1 + i\lambda + \frac{i^2\lambda^2}{2!} + \frac{i^3\lambda^3}{3!} + \frac{i^4\lambda^4}{4!} + \ldots \\ &= 1 + i\lambda - \frac{\lambda^2}{2!} - \frac{i\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots . \end{aligned}$$

Regrouping the real and imaginary terms in the above series, we obtain

$$e^{\lambda i} = \sum_{n=0}^{\infty} (-1)^k \frac{\lambda^{2k}}{(2k)!} + i \sum_{n=0}^{\infty} (-1)^k \frac{\lambda^{(2k+1)}}{(2k+1)!}.$$

We recognize the first sum to be $\cos\lambda$ and the second one to be $\sin\lambda$. Therefore, it seems reasonable to define $e^{\lambda i}$ as $e^{\lambda i} = \cos\lambda + i\sin\lambda$, and consistent with the exponential laws, $e^{\mu+\lambda i} = e^{\mu}e^{\lambda i} = e^{\mu}(\cos\lambda + i\sin\lambda)$.

Next, given a complex valued solution, how do we extract a real solution? To see this, let $\mu(t) + i\lambda(t)$ be a given complex valued solution of

$$x'' + p(t)x' + q(t)x = 0.$$

Then, substituting, we have

$$(\mu(t) + i\lambda(t))'' + p(t)(\mu(t) + i\lambda(t))' + q(t)(\mu(t) + i\lambda(t)) = 0.$$

Now grouping the terms involving μ and those involving λ, we obtain

$$(\mu'' + p(t)\mu' + q(t)\mu) + i(\lambda'' + p(t)\lambda' + q(t)\lambda) = 0.$$

This implies that the real and complex parts must be 0, that is $\mu'' + p(t)\mu' + q(t)\mu = 0 = \lambda'' + p(t)\lambda' + q(t)\lambda$. Therefore, $x_1 = \mu(t)$ and $x_2 = \lambda(t)$ are real valued solutions.

Returning to our differential equation $ax'' + bx' + cx = 0$, where $b^2 - 4ac < 0$, we see that if the characteristic equation has two complex valued roots, $m = \mu \pm i\lambda$. Then the corresponding real solutions of the differential equation $ax'' + bx' + cx = 0$ will be given by $x_1 = e^{\mu t}\cos\lambda t$ and $x_2 = e^{\mu t}\sin\lambda t$. The fact that they are linearly independent is obvious since they are not constant multiples of each other. Therefore they form a fundamental set of solutions and the general solution is

$$x = c_1 e^{\mu t}\cos\lambda t + c_2 e^{\mu t}\sin\lambda t = e^{\mu t}(c_1\cos\lambda t + c_2\sin\lambda t).$$

Summarizing, the general solution of $ax'' + bx' + cx = 0$ includes three types of solutions, depending on whether the discriminant of the characteristic equation $am^2 + bm + c = 0$ is positive, negative or zero.

Example 5.5.3. Solve the initial value problem

$$x'' + 2x' + 2x = 0, \; x(0) = 3, \; x'(0) = 7.$$

The roots of the characteristic equation $m^2 + 2m + 2 = 0$ are the complex numbers $m = -1 \pm \sqrt{-1} = -1 \pm i$. Therefore, $x_1 = e^{-t} \cos t$ and $x_2 = e^{-t} \sin t$ are the corresponding linearly independent solutions and the general solution is

$$x(t) = e^{-t}(c_1 \sin t + c_2 \cos t).$$

We point out that in this example, all solutions approach 0 as $t \to \infty$.

Calculating the derivative of the general solution, we have

$$x'(t) = e^{-t}(c_1 \cos t - c_2 \sin t) - e^{-t}(c_1 \sin t + c_2 \cos t).$$

Now, in order to find the solution satisfying the required initial values, we notice that if we let $t = 0$ in the general equation, we obtain $x(0) = c_2$. Therefore, we have to find $c_2 = 3$. To find c_1, we set $t = 0$ in the derivative x' of the general solution and obtain

$$x'(0) = c_1 - c_2 = 7,$$

which gives us $c_1 = 7 + c_2 = 10$. Therefore,

$$x = e^{-t}\,(10 \sin t + 3 \cos t)$$

is the desired solution. ■

Example 5.5.4. (The pendulum equation) Consider a point P of mass m suspended from a pivot by a chord of fixed length L so that P moves along a circle of radius L in a vertical plane passing through the pivot. On the point P acts the gravity force g and there is no friction.

Referring to Figure 5.2, the tangential component of the force acting on P is $-mg \sin\theta$ (the minus sign takes into account that the angle θ increases in the counterclockwise sense), while the tangential component of the acceleration is $L\theta''$. Thus Newton's law yields $mL\theta'' = -mg \sin\theta$, that is

$$L\,\theta'' + g\,\sin\theta = 0.$$

Recall that the Taylor expansion of $\sin\theta$ is

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\,.$$

Then, for small oscillations, we can approximate $\sin\theta$ by θ and the solutions of the pendulum equation are, up to a small error, those of

$$L\,\theta'' + g\,\theta = 0,$$

which is the equation of the harmonic oscillator with $\omega^2 = \frac{g}{L}$. The characteristic equation is $L\,m^2 + g = 0$, whose roots are $\pm i\,\sqrt{g/L}$. Then the solutions are

$$\theta(t) = c_1 \sin\sqrt{g/L}\;t + c_2 \cos\sqrt{g/L}\;t,$$

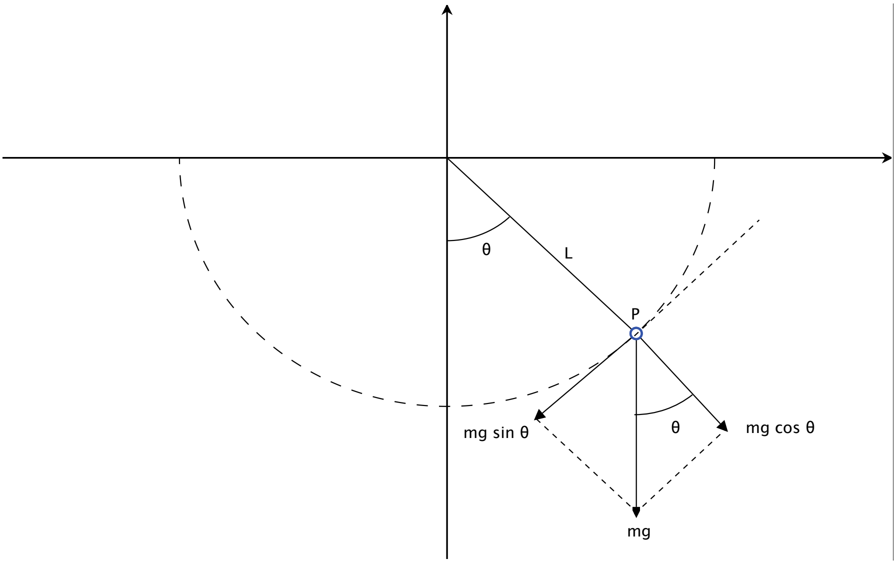

Fig. 5.2. The pendulum

which are periodic oscillations with period $T = 2\pi\sqrt{L/g}$. Notice that T depends only on L, not on the initial position of P. This property is the so-called *isochronism of the pendulum*. It is worth pointing out that isochronism is valid for the approximated equation, not for the true pendulum equation. ■

Example 5.5.5. (An RLC electrical circuit) In an RLC circuit with resistance R, inductance L, capacitance C and with a source with constant voltage V, see Figure 5.3, the intensity of the circulating current is governed by the second order equation

$$x''(t) + \frac{R}{L}\,x'(t) + \frac{1}{LC}\,x(t) = 0. \tag{5.13}$$

Here, to keep notation uniform with that used before, we have denoted by $x(t)$ the current intensity usually named $I(t)$ or $i(t)$.

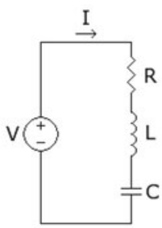

Fig. 5.3. RLC circuit

The reader will notice that (5.13) is also the equation of a damped harmonic oscillator, with $k = R/L > 0$, $\omega = 1/\sqrt{LC} > 0$ and $f = 0$, see Example 5.1.1, discussed in the first section. In such a case kx' represented a friction force.
Equation (5.13) is of the form of the general linear equation with constant coefficients (5.11), with $a = 1$, $b = k$ and $c = \omega^2$. Setting $k = 2\gamma$, the characteristic equation associated to (5.13) is

$$m^2 + 2\gamma m + \omega^2 = 0$$

whose roots are

$$m_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}.$$

Recalling that both ω and γ are positive, $m_{1,2}$ are real or complex conjugates depending on whether $\gamma \geq \omega$ or $\gamma < \omega$.

(1) Overdamped response. If $\gamma > \omega > 0$, m_1, m_2 are real, distinct and negative (because $\sqrt{\lambda^2 - \omega^2} < \gamma$), then the general solution of (5.13) is

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}.$$

Since $m_1, m_2 < 0$ these are decaying functions without oscillations, see Figure 5.4, blue curve. Here and below, the constants c_1, c_2 can be found if we impose initial conditions.

(2) Critically damped response. If $\gamma = \omega, m_1 = m_2 = -\gamma$ and the general solution is

$$x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t},$$

which implies fast decaying solutions without oscillations, see Figure 5.4, red curve.

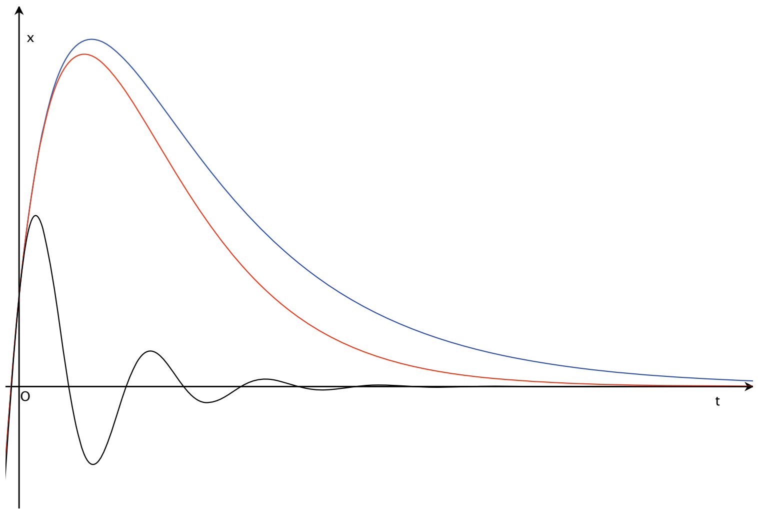

Fig. 5.4. Overdamped (*blue*), critically damped (*red*) and underdamped (*black*) response

(3) Underdamped response. If $0 < \gamma < \omega$, the roots of the characteristic equation are complex conjugates, namely $m_{1,2} = -\gamma \pm i\theta$, where $\theta = \sqrt{\omega^2 - \gamma^2}$. Then the general solution is

$$x(t) = e^{-\gamma t} \left(c_1 \sin \theta t + c_2 \cos \theta t\right),$$

which implies decaying oscillations, see Figure 5.4, black curve. ■

Remark 5.5.6. (i) Equation (5.13) is independent of the constant voltage V.

(ii) The decay is due to the presence of $k = R/L > 0$. In other words, it is the presence of the resistor R that, dissipating energy, induces a decay of the current intensity. If there is no resistance, that is if $R = 0$, then we have an LC circuit. In this case we have $\gamma = 0$ and $\theta = \omega$. The solution becomes $x(t) = c_1 \sin \omega t + c_2 \cos \omega t$, which means that the current intensity is sinusoidal and oscillates without any decay. ■

5.5.1 The Euler equation

An equation of the form

$$at^2x'' + btx' + cx = 0, \qquad t > 0,$$

is called a (homogeneous) Euler equation. Such an equation can be changed to one with constant coefficients by making the substitution $t = e^s$, or equivalently $s = \ln t$, as follows.

We note that $\frac{ds}{dt} = \frac{1}{t} = \frac{1}{e^s}$. For convenience, we let $\frac{dx}{ds} = \dot{x}$ to distinguish it from $x' = \frac{dx}{dt}$. Then, $x' = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \dot{x}\,\frac{1}{e^s}$. Therefore,

$$tx' = e^s\,\dot{x}\,\frac{1}{e^s} = \dot{x}.$$

Now,

$$x'' = \frac{dx'}{dt} = \frac{dx'}{ds}\frac{ds}{dt} = \frac{d(\dot{x}\frac{1}{e^s})}{ds}\frac{ds}{dt} = \frac{e^s\ddot{x} - e^s\dot{x}}{e^{2s}}\frac{1}{e^s} = \frac{\ddot{x} - \dot{x}}{e^{2s}}.$$

Therefore,

$$t^2x'' = \ddot{x} - \dot{x}.$$

We see that making the substitutions for x' and x'' will convert the given differential equation to the linear equation with constant coefficients

$$a(\ddot{x} - \dot{x}) + b\dot{x} + cx = 0,$$

or

$$a\ddot{x} + (b - a)\dot{x} + cx = 0.$$

Example 5.5.7. Solve

$$2t^2x'' + tx' - 3x = 0, \qquad t > 0.$$

Using the substitutions above, we have

$$2(\ddot{x} - \dot{x}) + \dot{x} - 3x = 2\ddot{x} - \dot{x} - 3x = 0.$$

The corresponding characteristic equation is $2m^2 - m - 3 = 0$ and the roots are $m = -1, \frac{3}{2}$. Therefore, the general solution in terms of s is $x(s) = c_1 e^{-s} + c_2 e^{\frac{3}{2}s}$. Finally, substituting $s = \ln t$, we have $x(t) = c_1 t^{-1} + c_2 t^{\frac{3}{2}}$. ∎

Example 5.5.8. Solve

$$t^2 x'' + tx' + x = 0, \qquad t > 0.$$

Making the substitution $s = \ln t$, we obtain

$$\ddot{x} + x = 0$$

whose general solution is $x(s) = c_1 \sin s + c_2 \cos s$. Since $s = \ln t$ we have $x(t) = c_1 \sin(\ln t) + c_2 \cos(\ln t)$. ∎

Nonhomogeneous Euler equations

$$at^2 x'' + btx' + cx = h(t) \qquad t > 0,$$

can be handled in a similar way and are briefly discussed in Remark 5.6.10 in the next section.

5.6 Linear nonhomogeneous equations – method of undetermined coefficients

Consider the equation

$$ax''(t) + bx'(t) + cx(t) = f(t), \tag{5.14}$$

where the coefficients a, b, c are constants and $a \neq 0$. Let us consider a specific case where a particular solution $z(t)$ of this nonhomogeneous equation can be found by inspection. This may happen, e.g., if $f(t)$ is a polynomial of degree n, or an exponential $e^{\lambda t}$, or a trigonometric function like $\sin \lambda t$, $\cos \lambda t$, or a linear combination of these. In such cases, one can try to find z, by careful guessing, as a function of the same type as $f(t)$. This is known as the *method of undetermined coefficients*.

Instead of carrying out a general discussion, we prefer to demonstrate this method by considering appropriate examples.

We first consider the case in which $f(t) = P(t)e^{\lambda t}$, where P is a polynomial. We can try to find a solution of

$$ax'' + bx' + cx = P(t)e^{\lambda t} \tag{5.15}$$

by setting $z = Q(t)e^{\lambda t}$, where Q is a polynomial to be determined. Since $z' = Q'e^{\lambda t} + \lambda Q e^{\lambda t}$ and $z'' = Q''e^{\lambda t} + 2\lambda Q' e^{\lambda t} + \lambda^2 Q e^{\lambda t}$, then z solves (5.15) provided

$$a(Q''e^{\lambda t} + 2\lambda Q'e^{\lambda t} + \lambda^2 Q e^{\lambda t}) + b(Q'e^{\lambda t} + \lambda Q e^{\lambda t}) + cQe^{\lambda t} = Pe^{\lambda t}.$$

Canceling $e^{\lambda t}$ we find $a(Q'' + 2\lambda Q' + \lambda^2 Q) + b(Q' + \lambda Q) + cQ = P$ or equivalently, rearranging,

$$a(Q'' + 2\lambda Q') + bQ' + (a\lambda^2 + b\lambda + c)Q = P. \tag{5.16}$$

This equation allows us to find Q by means of the polynomial identity principle.

To establish the degree $d(Q)$ of the unknown polynomial Q, it is convenient to distinguish whether λ is a root of the characteristic equation $am^2 + bm + c = 0$ or not. The former case is called *resonant*, the latter *non resonant*.

Equation (5.15)***: the non resonant case.*** In the non resonant case λ is not a root of the characteristic equation, namely $a\lambda^2 + b\lambda + c \neq 0$, and we can look for a polynomial Q such that $d(Q) = d(P)$, where $d(P)$ denotes the degree of P.

Example 5.6.1. (i) Find a particular solution of $2x'' - x' + 3x = 2t$. In this case $P(t) = 2t$ and $\lambda = 0$. Setting $z(t) = At + B$, we determine A, B such that z satisfies $2z'' - z' + 3z = 2t$. Since $2z'' - z' + 3z = -A + 3(At + B)$ then z solves the given equation whenever $3At + 3B - A = 2t$, namely $3A = 2$ and $3B - A = 0$. Thus we find $A = \frac{2}{3}$ and $B = \frac{2}{9}$ and hence $z = \frac{2}{3}t + \frac{2}{9}$.

(ii) Find a particular solution of $x'' + x = 3e^{2t}$. Here $P = 3$ and $\lambda = 2$. Taking $z = Ae^{2t}$ and substituting in the equation, we find $4Ae^{2t} + Ae^{2t} = 3e^{2t}$, and hence $5A = 3$, namely $A = \frac{3}{5}$. Thus $z = \frac{3}{5}e^{2t}$. ∎

Equation (5.15)***: the resonant case.*** In the resonant case λ is a root of the characteristic equation, namely

$$a\lambda^2 + b\lambda + c = 0.$$

If this holds (5.16) becomes

$$aQ'' + (2a\lambda + b)Q' = P. \tag{5.17}$$

Notice that the degree of the left hand side of (5.17) is $d(Q) - 1$ if $2a\lambda + b \neq 0$, otherwise it is equal to $d(Q) - 2$. Since both sides of (5.17) have the same degree, it follows that $d(P)$ is equal respectively to $d(Q) - 1$ or to $d(Q) - 2$, namely that $d(Q) = d(P) + 1$ or, respectively, $d(Q) = d(P) + 2$. Roughly, the feature of the resonant case is that seeking a solution of $ax'' + bx' + cx = Pe^{\lambda t}$ in the form $z = Qe^{\lambda t}$, the degree of Q has to be greater than the degree of P.

Moreover, the polynomial on the left hand side of (5.17) contains only the derivatives of Q and therefore the term of order zero plays no role and can be taken to be zero. In other words, we can look for a polynomial $Q(t)$ whose lower order term has degree one. For the same reason, if λ is a root of $a\lambda^2 + b\lambda + c = 0$ and, in addition,

$2a\lambda + b = 0$, then (5.17) becomes $aQ'' = P$ and hence we can search for Q such that its lower order term has degree two, see Example 5.6.3 below.

The following simple examples can be easily extended to more general cases.

Example 5.6.2. Find a particular solution of $2x'' + 3x' = 1 + 4t$.

Here $P(t) = 1+4t$ and $\lambda = 0$, which is a (simple) root of the characteristic equation $2m^2 + 3m = 0$. We are in the resonant case. Looking for a particular solution z of the form $z = At^2 + Bt$. Then we have $2z'' + 3z' = 4A + 6At + 3B = 1 + 4t$. Using the polynomial identity principle it follows that

$$\begin{cases} 4A + 3B & = 1, \\ 6A & = 4, \end{cases}$$

whence $A = \frac{2}{3}$ and $B = -\frac{5}{9}$ and $z = \frac{2}{3}t^2 - \frac{5}{9}t$.

Notice that $d(P) = 1$ while the degree of z is 2. ■

Example 5.6.3. (Inclined plane) Consider a body of mass m on a frictionless inclined plane with slope α, see Figure 5.5.

The component of the gravitational force mg parallel to the inclined plane is $F = mg\sin\alpha$. If x denotes the distance of the body from the top of the inclined plane (the point O in Figure 5.5), then, according to Newton's second law of motion, we deduce that the acceleration x'' of the body satisfies $mx'' = mg\sin\alpha$. Canceling m, we find

$$x'' = g\sin\alpha.$$

A particular solution is given by $z(t) = \frac{1}{2}g\,t^2\sin\alpha$. Notice that the constant forcing term $g\sin\alpha$ is resonant. Since the general solution of the corresponding homogeneous equation $x'' = 0$ is $x(t) = c_1 + c_2 t$, it follows that the general solution of $x'' = g\sin\alpha$ is given by

$$x(t) = c_1 + c_2 t + \frac{1}{2}g\,t^2\sin\alpha.$$

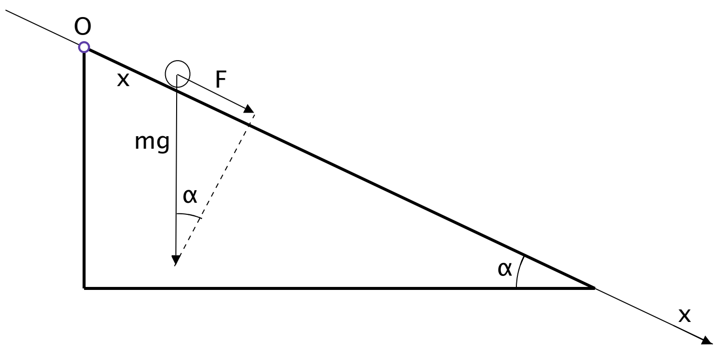

Fig. 5.5. The inclined plane with slope α

Noticing that $c_1 = x(0)$ and $c_2 = x'(0)$ we can write the solution as

$$x(t) = x(0) + x'(0)t + \frac{1}{2}g\,t^2 \sin\alpha.$$

For example, if the body is initially at rest on the top of the inclined plane, then the initial conditions are $x(0) = x'(0) = 0$ and we get $x(t) = \frac{1}{2}g \sin\alpha\, t^2$. Notice that, as it is well known, the motion of the body is independent of its mass m. ■

The following example deals with other resonant forcing terms.

Example 5.6.4. (i) Find a particular solution of $x'' - x = 2e^t$. Here the forcing term is $f(t) = P(t)e^{\lambda t}$ with $P(t) = 2$ and $\lambda = 1$. Since $\lambda = 1$ is a root of the characteristic equation $m^2 - 1 = 0$, we are in the resonant case. Let us try to find a particular solution of the form $z = Ate^t$. One finds $z'' = 2Ae^t + Ate^t$ and hence $z'' - z = 2Ae^t + Ate^t - Ate^t = 2Ae^t$. Then $z'' - z = 2e^t$ yields $A = 1$ and hence $z = te^t$.

(ii) Find a particular solution of $x'' - x' = (t + 2)e^t$. This is also a resonant case, because $\lambda = 1$ is a root of the characteristic equation $m^2 - m = 0$. Setting $z = t(At + B)e^t = (At^2 + Bt)e^t$, we find

$$z' = (2At + B)e^t + (At^2 + Bt)e^t = [At^2 + (2A + B)t]e^t$$

and

$$\begin{aligned} z'' &= (2At + 2A + B)e^t + [At^2 + (2A + B)t]e^t \\ &= [At^2 + (4A + B)t + 2A + B]e^t. \end{aligned}$$

Then $z'' - z' = (t + 2)e^t$ yields

$$[At^2 + (4A + B)t + 2A + B] - [At^2 + (2A + B)t] = t + 2,$$

whence

$$2At + 2A + B = t + 2.$$

Thus we find $A = \frac{1}{2}$ and $2A + B = 2$, whence $B = 1$. In conclusion, $z = (\frac{1}{2}t^2 + t)e^t$. ■

Remark 5.6.5. If $f(t) = f_1(t) + \dots + f_n(t)$, a particular solution can be sought as $z = z_1 + \dots + z_n$, where z_i solve $ax'' + bx' + cx = f_i$. For example, to find a particular solution of $x'' - x = e^t + e^{2t}$ we can solve separately $z_1'' - z_1 = e^t$ and $z_2'' - z_2 = e^{2t}$. The former is resonant and we find $z_1 = \frac{1}{2}te^t$. The second equation is not resonant and we find $z_2 = \frac{1}{3}e^{2t}$. Thus $z = \frac{1}{2}te^t + \frac{1}{3}e^{2t}$ is a particular solution of $x'' - x = e^t + e^{2t}$. ■

We now consider the equation

$$ax'' + bx' + cx = P_1(t)\sin\lambda t + P_2(t)\cos\lambda t \tag{5.18}$$

where P_1, P_2 are polynomials and $\lambda \neq 0$ (this will be always understood in the sequel, otherwise the forcing term becomes a polynomial, a case discussed previously). We can try to find a particular solution of (5.18) by setting $z = Q_1(t)\sin\lambda t + Q_2(t)\cos\lambda t$, where Q_1, Q_2 are polynomials to be determined.

First of all, let us see what what the resonant case is for (5.18).

If the forcing term is $Pe^{\lambda t}$ we have seen that the resonant case arises when λ is a root of the characteristic equation $am^2 + bm + c = 0$.

Dealing with (5.18) and recalling the Euler formula $e^{i\lambda t} = \cos\lambda t + i\sin\lambda t$, it is natural to say that $P_1(t)\sin\lambda t + P_2(t)\cos\lambda t$ is a resonant forcing term provided λi is a root of the characteristic equation $am^2 + bm + c = 0$, namely provided $-a\lambda^2 + b\lambda i + c = 0$. Since a complex number is zero whenever its real and imaginary parts are zero, it follows that we are in the resonant case if and only if $b = 0$ (recall that $\lambda \neq 0$) and $c = a\lambda^2$.

We will see that resonance for (5.18) has the same feature as resonance when the forcing term is $Pe^{\lambda t}$: in the latter case seeking a particular solution in the form $Qe^{\lambda t}$, the degree of Q is greater than the degree of P; similarly, dealing with a resonant equation like (5.18), looking for a particular solution in the form $Q_1\sin\lambda t + Q_2\cos\lambda t$, the degrees of Q_i are greater than the degrees of P_i. This confirms that the definition of resonance for (5.18) given above is the correct one.

Equation* (5.18)*: the non resonant case. We first discuss some examples dealing with the non resonant case, namely when either $b \neq 0$, or $c \neq a\lambda^2$, or both. As we will see, in this case we can seek a particular solution in the form $z = Q_1\sin\lambda t + Q_2\cos\lambda t$, where the degrees of Q_i are the same as those of P_i.

Example 5.6.6. Find a particular solution of the following equations with non resonant forcing terms.

(i) $x'' + x' + x = \sin t$. Setting $z = A\sin t + B\cos t$ we find $z' = A\cos t - B\sin t$ and $z'' = -A\cos t - B\cos t$, whence $z'' + z' + z = (-A\cos t - B\cos t) + (A\cos t - B\sin t) + (A\sin t + B\cos t) = -B\sin t + A\cos t$. Then from $z'' + z' + z = \sin t$ we infer that $A = 0$ and $B = -1$. Hence $z = -\cos t$.

(ii) $x'' + x' + x = \sin 2t$. Setting $z = A\sin 2t + B\cos 2t$ one has $z' = 2A\cos 2t - 2B\sin 2t$ and $z'' = -4A\sin 2t - 4B\cos 2t$, whence

$$\begin{aligned} z'' + z' + z &= (-4A\sin 2t - 4B\cos 2t) + (2A\cos 2t - 2B\sin 2t) \\ &\quad + A\sin 2t + B\cos 2t \\ &= (-3A - 2B)\sin 2t + (2A - 3B)\cos 2t. \end{aligned}$$

It follows that $z'' + z' + z = \sin 2t$ provided one has $(-3A - 2B)\sin 2t + (-3B + 2A)\cos 2t = \sin 2t$, for all $t \in \mathbb{R}$. Taking first $2t = \frac{\pi}{2}$ and after $t = 0$ we find that A, B satisfy the algebraic system

$$\begin{cases} -3A - 2B &= 1, \\ 2A - 3B &= 0. \end{cases}$$

This system has a unique solution given by $A = -\frac{3}{13}$ and $B = -\frac{2}{13}$. Thus $z = -\frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t$.

(iii) $x'' + x = \sin 3t$. Setting $A\sin 3t + B\cos 3t$ and repeating the previous calculations we find

$$z'' + z = -8A\sin 3t - 8B\cos 3t.$$

Then $z'' + z = \sin 3t$ implies $A = -\frac{1}{8}$, $B = 0$ and hence $z = -\frac{1}{8}\sin 3t$. ■

The next example deals with non constant polynomials P_1, P_2.

Example 5.6.7. Find a particular solution of $x'' - x = t\sin t$. Also in this case the forcing term is not resonant because i is not a root of the characteristic equation $m^2 - 1 = 0$. If we set $z = (A_1t + B_1)\sin t + (A_2t + B_2)\cos t$ we find

$$\begin{aligned} z' &= A_1t\sin t + (A_1t + B_1)\cos t + A_2\cos t - (A_2t + B_2)\sin t \\ z'' &= 2A_1\cos t - (A_1t + B_1)\sin t - 2A_2\sin t - (A_2t + B_2)\cos t. \end{aligned}$$

Then $z'' - z = t\sin t$ yields

$$\begin{aligned} &2A_1\cos t - (A_1t + B_1)\sin t - 2A_2\sin t - (A_2t + B_2)\cos t \\ &\quad - (A_1t + B_1)\sin t - (A_2t + B_2)\cos t = t\sin t. \end{aligned}$$

Rearranging we find

$$[2A_1 - 2(A_2t + B_2)]\cos t - [2(A_1t + B_1) + 2A_2]\sin t = t\sin t.$$

Then it follows that A_1, A_2, B_1, B_2 satisfy the algebraic system

$$\begin{cases} -2A_1t - 2B_1 - 2A_2 &= t, \\ -2A_2t + 2A_1 - 2B_2 &= 0, \end{cases}$$

which implies $A_1 = -\frac{1}{2}$, $A_2 = 0$, $B_1 = 0$ and $B_2 = -\frac{1}{2}$. In conclusion we get $z = -\frac{1}{2}t\sin t - \frac{1}{2}\cos t$. ■

Remark 5.6.8. As we have seen in the preceding examples, even if the forcing term is $P_1\sin\lambda t$ or $P_2\cos\lambda t$, we cannot - in general - find a particular solution containing only sine or cosine terms. ■

Equation* (5.18)*: the resonant case. Next, we consider the resonant case which arises if $b = 0$ and $c - a\lambda^2 = 0$ and the equation (5.18) becomes

$$ax'' + a\lambda^2 x = P_1(t)\sin\lambda t + P_2(t)\cos\lambda t. \tag{5.19}$$

As anticipated before, looking for a particular solution of (5.19) in the form $z = Q_1\sin\lambda t + Q_2\cos\lambda t$, we have to seek polynomials $Q_1 = A_1t + A_2t^2 + ...$ and $Q_2 = B_1t + B_2t^2 + ...$ whose degrees are greater than the degrees of P_1, P_2.

Example 5.6.9. Find a particular solution of the following equations with a resonant forcing term: (i) $x'' + x = \sin t$ and (ii) $x'' + x = t \sin t$.

(i) Taking $z = At \sin t + Bt \cos t$, we find

$$\begin{aligned}
z' &= A \sin t + B \cos t + At \cos t - Bt \sin t \\
z'' &= A \cos t - B \sin t + A \cos t - B \sin t - At \sin t - Bt \cos t \\
&= 2A \cos t - 2B \sin t - At \sin t - Bt \cos t \\
z'' + z &= 2A \cos t - 2B \sin t - At \sin t - Bt \cos t + At \sin t + Bt \cos t \\
&= 2A \cos t - 2B \sin t.
\end{aligned}$$

Then $z'' + z = \sin t$ implies $A = 0$ and $B = -\frac{1}{2}$ and thus $z = -\frac{1}{2}t \cos t$.

(ii) If take $z = (A_1 t + A_2 t^2) \sin t + (B_1 t + B_2 t^2) \cos t$, we can write $z = z_1 + z_2$, where $z_1 = A_1 t \sin t + B_1 t \cos t$ and $z_2 = A_2 t^2 \sin t + B_2 t^2 \cos t$. Using the calculations in (i) we find $z_1'' + z_1 = 2A_1 \cos t - 2B_1 \sin t$. As for z_2 we find

$$\begin{aligned}
z_2'' &= (2A_2 - 4B_2 t - A_2 t^2) \sin t + (2B_2 + 4A_2 t - B_2 t^2) \cos t \\
z_2'' + z_2 &= (2A_2 - 4B_2 t) \sin t + (2B_2 + 4A_2 t) \cos t.
\end{aligned}$$

Then

$$\begin{aligned}
z'' + z &= z_1'' + z_1 + z_2'' + z_2 \\
&= (2A_2 - 4B_2 t - 2B_1) \sin t + (2B_2 + 4A_2 t + 2A_1) \cos t
\end{aligned}$$

and the equation $z'' + z = t \sin t$ yields

$$\begin{cases}
2A_2 - 4B_2 t - 2B_1 &= t, \\
2B_2 + 4A_2 t + 2A_1 &= 0.
\end{cases}$$

Solving this algebraic system we find $A_2 = B_1 = 0$, $B_2 = -\frac{1}{4}$, and $A_1 = \frac{1}{4}$. In conclusion,

$$z = \frac{1}{4} t \sin t - \frac{1}{4} t^2 \cos t. \qquad \blacksquare$$

So far we have taken $f = Pe^{\lambda t}$ or $f = P_1 \sin \lambda t + P_2 \cos \lambda t$ but, more generally, we could deal with a forcing term $f = f_1 + \ldots + f_n$, where each f_i is in one of the forms discussed above. As in Remark 5.6.5, a particular solution of $ax'' + bx' + cx = f$ can be found as $z_1 + ... + z_n$ where $az_i'' + bz_i' + cz_i = f_i$. For example, to find a particular solution of $x'' - 2x = -e^t + \sin t$ we set $z = z_1 + z_2$, where $z_1'' - 2z_1 = -e^t$ and $z_2'' - 2z_2 = \sin t$. With calculations similar to the previous ones we find $z_1 = e^t$ and $z_2 = -\frac{1}{3} \sin t$. Thus a particular solution is given by $z = z_1 + z_2 = e^t - \frac{1}{3} \sin t$.

Remark 5.6.10. (Nonhomogeneous Euler equations) As for the homogeneous Euler equation, the substitution $t = e^s$ transforms the nonhomogeneous Euler equation

$$at^2 x'' + btx' + cx = h(t), \qquad t > 0,$$

into

$$a\,\frac{d^2x}{ds^2} + (b-a)\,\frac{dx}{ds} + cx = h(e^s),$$

which is a linear nonhomogeneous equation with constant coefficients and can be handled either by the method of Variation of Parameters or using the method of Undetermined Coefficients. ■

Example 5.6.11. Find a particular solution $z(t)$ of $t^2x'' + tx' - x = t^2 - t, t > 0$. Setting $t = e^s$ we get

$$\frac{d^2x}{ds^2} - x = e^{2s} - e^s.$$

Let us first find a solution of $\frac{d^2x}{ds^2} - x = e^{2s}$. To this end, we let $x_1 = Ae^{2s}$. Substituting, we obtain $4Ae^{2s} - Ae^{2s} = e^{2s}$ and hence $A - \frac{1}{3}$, which yields $x_1 = \frac{1}{3}e^{2s}$.

Next, we find a solution of $\frac{d^2x}{ds^2} - x = -e^s$. Now we are in the resonant case. So, we let $x_2 = Ase^s$. Substituting, we obtain $Ase^s + 2Ae^s - Ase^s = -e^s$, which implies that $A = -\frac{1}{2}$ and $x_2 = -\frac{1}{2}se^s$.

Using Remark 5.6.5, it follows that a particular solution of $\frac{d^2x}{ds^2} - x = e^{2s} - e^s$ is $x_1 + x_2 = \frac{1}{3}e^{2s} - \frac{1}{2}se^s$.

Substituting $t = e^s$, namely $s = \ln t$, we find that a particular solution of $t^2x'' + tx' - x = t^2 - t, t > 0$, is given by $z(t) = \frac{1}{3}t^2 - \frac{1}{2}t\ln t$. ■

In the next subsection we discuss a remarkable example arising in applications.

5.6.1 The elastic spring

Let us consider the second order nonhomogeneous equation

$$x'' + \omega^2 x = \sin\omega_1 t. \tag{5.20}$$

As we saw in Example 5.1.1, this equation models the motion of a body attached to a fixed point by an elastic spring, under the assumption that the body is subjected to a sinusoidal external force $f(t) = \sin\omega_1 t$.

We have already seen that the general solution of the associated homogeneous equation $x'' + \omega^2 x = 0$ is

$$x(t) = c_1 \sin\omega t + c_2 \cos\omega t.$$

To find a solution of the nonhomogeneous equation it is convenient to distinguish the cases whether $\omega \neq \omega_1$ or not.

(1) Case $\boldsymbol{\omega \neq \omega_1}$. Setting $z(t) = \alpha\sin\omega_1 t$, one finds $z'' = -\alpha\omega_1^2\sin\omega_1 t$. Then $z'' + \omega^2 z = \sin\omega_1 t$ yields

$$-\alpha\omega_1^2\sin\omega_1 t + \omega^2\alpha\sin\omega_1 t = \sin\omega_1 t.$$

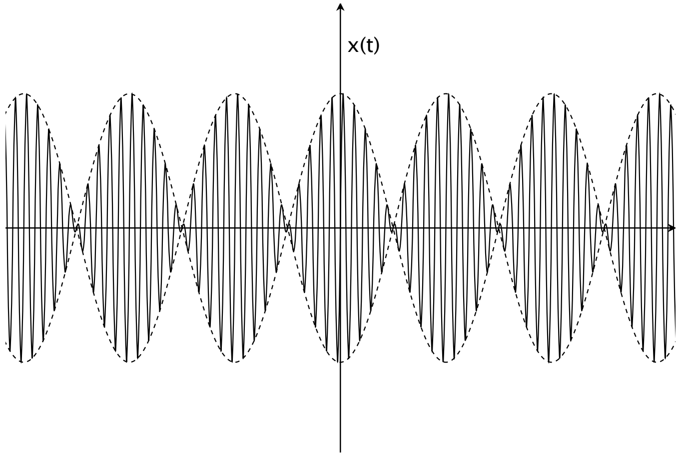

Fig. 5.6. Beats: solutions of (5.20) when $\omega_1 \sim \omega$

Dividing through by $\sin \omega_1 t$, we get $(\omega^2 - \omega_1^2)\alpha = 1$. Since $\omega_1^2 \neq \omega^2$ we find $\alpha = 1/(\omega^2 - \omega_1^2)$ and hence

$$z(t) = \frac{1}{\omega^2 - \omega_1^2} \sin \omega_1 t. \tag{5.21}$$

Thus the general solution of (5.20) is given by

$$x(t) = c_1 \sin \omega t + c_2 \cos \omega t + \frac{1}{\omega^2 - \omega_1^2} \sin \omega_1 t.$$

The resulting wave is a superposition of two oscillations with frequency ω_1 and ω. Particularly interesting is the case shown in Figure 5.6 in which the two frequencies are very close. This phenomenon is called *beat*.

(2) Case $\omega = \omega_1$. This is the *resonant case* when the equation becomes

$$x'' + \omega^2 x = \sin \omega t. \tag{5.22}$$

As in Example 5.6.4 - (i), we can find a particular solution in the form $z = At \sin \omega t + Bt \cos \omega t$. Repeating the calculations carried out therein, one finds

$$z'' + \omega^2 z = 2A\omega \cos \omega t - 2B\omega \sin \omega t.$$

Then $z'' + \omega^2 z = \sin \omega t$ yields $A = 0,\ B = -\frac{1}{2\omega}$ and hence

$$z(t) = -\frac{1}{2\omega}\, t \cos \omega t \tag{5.23}$$

is a particular solution of the nonhomogeneous equation.

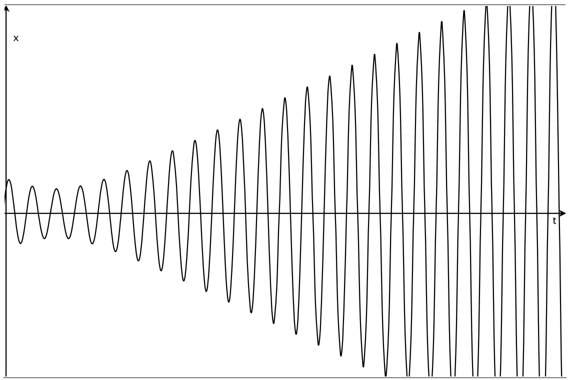

Fig. 5.7. Resonance: solution of (5.22) for $t > 0$

Therefore the general solution of (5.22) is given by

$$x(t) = c_1 \sin \omega t + c_2 \cos \omega t - \frac{t}{2\omega} \cos \omega t.$$

The graph of the solutions is shown in Figure 5.7 and shows that the presence of $\frac{t}{2\omega} \cos \omega t$ has the effect of producing oscillations of increasing amplitude.
Let us check this claim. To simplify the notation, we take $c_1 = \omega = 1$ and $c_2 = 0$ so that $x(t) = \sin t - \frac{t}{2} \cos t$. The general case is quite similar and is left as an exercise.

If we let $s_n = 2n\pi$, we have $\sin s_n = 0$ and $\cos s_n = 1$ so that $x(s_n) = -n\pi$ which tends to $-\infty$. If we let $t_n = (2n+1)\pi$, then $\sin t_n = 0$ and $\cos t_n = -1$ so that $x(t_n) = n\pi + \frac{\pi}{2}$ which tends to $+\infty$. This implies that $\liminf_{t\to+\infty} x(t) = -\infty$ and $\limsup_{t\to+\infty} x(t) = +\infty$. Moreover, by the Intermediate Value Theorem, between s_n and t_n there are zeros of $x(t)$.

5.7 Oscillatory behavior of solutions

Consider the second order linear homogeneous equation

$$x''(t) + p(t)x(t) = 0. \tag{5.24}$$

For simplicity, we assume that $p(t)$ is continuous everywhere. Obviously, we can restrict it only to the relevant interval, if we wish.

We say that a nontrivial solution $x(t)$ of (5.24) is *oscillatory* (or it *oscillates*) if for any number T, $x(t)$ has infinitely many zeros in the interval (T, ∞); or equivalently, for any number τ, there exists a number $\xi > \tau$ such that $x(\xi) = 0$. We also call the equation (5.24) *oscillatory* if it has an oscillatory solution.

We will see below that simple observations about the coefficient $p(t)$ can give us very interesting and important information about the oscillatory behavior of the solutions of (5.24).

First let us consider the special case

$$x'' + k^2 x = 0$$

which is the well-known equation for *harmonic oscillator*. If k is a nonzero constant, then the roots of the characteristic equation are given by $m = \pm k\,i$ and hence $x_1 = \sin k\,t$ and $x_2 = \cos k\,t$ are two linearly independent oscillatory solutions.

To start with, let us note that for $k = 1$, $x_1 = \sin t$ is a solution of $x'' + x = 0$ and this solution has exactly one zero in the interval $(0, 2\pi)$, namely at $t = \pi$.

For $k = 2$, $x_2 = \sin 2t$ is a solution of $x'' + 4x = 0$ and it has three zeros in the interval $(0, 2\pi)$, one at $t = \pi/2$, one at $t = \pi$ and one at $t = 3\pi/2$.

Based on the above observation, one would estimate that the larger the constant k is, the faster the solutions oscillate. Actually, this happens to be a general fact that was discovered by Jacques Charles Francois Sturm in 1836, and it has laid the foundation for the theory of oscillation. We now state and prove two beautiful and simple theorems due to Sturm.[1]

The first theorem below shows that the zeros of solutions are interlaced, that is, between any two zeros of a given solution, there is a zero of any other linearly independent solution. In the constant coefficient case, we see that this is true, since for any $k \neq 0$, $x_1 = \sin kt$ and $x_2 = \cos kt$ have this property; then it can be verified that all solutions have this property (see Example 5.7.5 below).

Theorem 5.7.1 (Sturm Separation Theorem). *Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of* (5.24) *and suppose a and b are two consecutive zeros of $x_1(t)$, with $a < b$; that is $x_1(a) = x_1(b) = 0$ and $x_1(t) \neq 0$ on (a, b). Then $x_2(t)$ has exactly one zero in the interval (a, b).*

Proof. Notice that $x_2(a) \neq 0 \neq x_2(b)$, otherwise x_1 and x_2 would have a common zero and hence their Wronskian would be 0 and they could not be linearly independent.

Suppose, by way of contradiction, that $x_2(t) \neq 0$ on the open interval (a, b). Then $x_2(t) \neq 0$ on the closed interval $[a, b]$. Let

$$h(t) = \frac{x_1(t)}{x_2(t)}.$$

Then h is differentiable on $[a, b]$ and $h(a) = h(b) = 0$. Therefore by Rolle's lemma, there exists a number c, $a < c < b$, such that $h'(c) = 0$. But $h'(c) = 0$ implies that

$$\frac{x_2(c)x_1'(c) - x_1(c)x_2'(c)}{x_2^2(c)} = 0.$$

This implies that $x_2(c)x_1'(c) - x_1(c)x_2'(c) = 0$, which in turn implies that the Wronskian of $x_1(t)$ and $x_2(t)$ vanishes at $t = c$, contradicting their linear independence.

[1] Sturm, C.: Mémoire sur les équations différentielles linéaires du second order. J. Math. Pures Appl. **1**, 106–186 (1836).

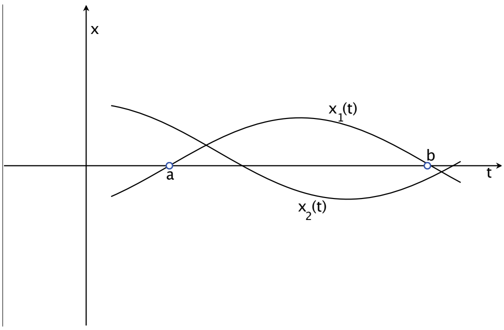

Fig. 5.8. The zeros of $x_1(t)$ and $x_2(t)$

This proves that $x_2(t)$ vanishes in the interval (a, b). What remains to be shown is that it cannot have more than one zero in this interval. See Figure 5.8.

Suppose that there exist two numbers t_1 and t_2 in the interval (a, b) such that $x_2(t_1) = x_2(t_2) = 0$. Then by what we have just proved, there would exist a number d between t_1 and t_2 such that $x_1(d) = 0$, contradicting the fact that a and b are consecutive zeros of $x_1(t)$. ∎

An immediate consequence of this theorem is

Corollary 5.7.2. *If* (5.24) *has one oscillatory solution, then all of its solutions are oscillatory.*

Theorem 5.7.3 (Sturm Comparison Theorem). *Consider the two equations*

$$x'' + p(t)x = 0, \tag{5.25}$$

$$y'' + q(t)y = 0. \tag{5.26}$$

Suppose that $x(t)$ is a nontrivial solution of (5.25) *with consecutive zeros at $x = a$ and $x = b$. Assume further that $p(t)$ and $q(t)$ are continuous on $[a, b]$ and $p(t) \leq q(t)$, with strict inequality holding at least at one point in the interval $[a, b]$. If $y(t)$ is any nontrivial solution of* (5.26) *such that $y(a) = 0$, then there exists a number c, $a < c < b$, such that $y(c) = 0$.*

Proof. Assume that the assertion of the theorem is false. First of all, we can assume, without any loss of generality, that $x(t) > 0$ on the interval (a, b), otherwise we can replace $x(t)$ by $-x(t)$ which is also a solution of the same equation and has the same zeros as $x(t)$. Similarly we can assume that $y(t) > 0$ on the interval (a, b). Multiplying the first equation by $y(t)$, the second equation by $x(t)$ and subtracting the resulting second equation from the first equation, we obtain

$$y(t)x''(t) - x(t)y''(t) + (p(t) - q(t))x(t)y(t) = 0.$$

Since $yx'' - xy'' = (yx' - xy')'$, if we integrate the above equation from a to b, we obtain

$$(yx' - xy')|_a^b = \int_a^b (q(t) - p(t))x(t)y(t)dt.$$

Since $x(a) = x(b) = y(a) = 0$, the above equation can be written as

$$y(b)x'(b) = \int_a^b (q(t) - p(t)x(t)y(t))dt. \tag{5.27}$$

Since $x(t) > 0$ to the left of b and $x(b) = 0$, we must have $x'(b) \leq 0$. Furthermore, since $y(t)$ is continuous and $y(t) > 0$ for t to the left of b, we must have $y(b) \geq 0$. Therefore, on the left-hand side of (5.27), we have $y(b)x'(b) \leq 0$.

Since, by assumption, $q(\bar{t}) - p(\bar{t}) > 0$ for some $\bar{t}$ in the interval $[a, b]$ and $q(t) - p(t)$ is continuous, then it will stay positive on some subinterval of $[a, b]$ containing $\bar{t}$. Since $(q(t) - p(t))x(t)y(t) \geq 0$ in $[a, b]$ and $(q(t) - p(t))x(t)y(t) > 0$ in some subinterval of $[a, b]$, it follows from the definition of the Riemann integral that

$$\int_a^b (q(t) - p(t))x(t)y(t))dt > 0.$$

We have shown that the right-hand side of (5.27) is positive and the left-hand side is less than or equal to 0. This contradiction proves the theorem. ∎

Corollary 5.7.4. *All solutions of* (5.26) *vanish between a and b.*

Proof. Let $z(t)$ be a given solution of (5.26). We have shown that $y(t)$ vanishes at a and at some number c, $a < c < b$. By the Sturm Separation Theorem, $z(t)$ has a zero between a and c and hence between a and b if z and y are linearly independent. If they are linearly dependent, then they are constant multiples of each other and have the same zeros. Since $y(t)$ has a zero in (a, b), then so does z. ∎

Example 5.7.5. Show that between any two zeros of cos t there is a zero of $2 \sin t - 3 \cos t$.

We recall that sin t and cos t are two linearly independent solutions of $x'' + x = 0$. In view of the Sturm Separation Theorem, it suffices to show that cos t and $(2 \sin t - 3 \cos t)$ are linearly independent solutions. Evaluating their Wronskian, we have

$$W(\cos t, 2 \sin t - 3 \cos t) = 2 \sin^2 t + 2 \cos^2 t = 2.$$

Therefore the two functions are linearly independent. ∎

Example 5.7.6. $x_1 = e^t$ and $x_2 = (t^2 - 1)e^{2t}$ cannot be solutions of (5.24) for any continuous function $p(t)$.

This follows from the fact that $t = \pm 1$ are two zeros of x_2 but x_1 has no zero between 1 and -1, contradicting the Sturm Separation Theorem. ∎

Proposition 5.7.7. *If* $\lim_{t\to+\infty} p(t) > 1$, *then* $x'' + p(t)x = 0$ *is an oscillatory equation.*

Proof. Since $\lim_{t\to+\infty} p(t) > 1$, we can choose a number T such that for $t \geq T$, $p(t) > 1$. Comparing the solutions of $x'' + p(t)x = 0$ with those of $x'' + x = 0$, it follows that for $t \geq T$, every solution of $x'' + p(t)x = 0$ has a zero between any two zeros of $\sin t$. The assertion follows from the fact that the zeros of $\sin t$ are not bounded above. ∎

Although, for simplicity, in the above we chose $\lim_{t\to\infty} p(t) = 1$, the proof shows that for any $\alpha > 0$, $\lim_{t\to\infty} p(t) = \alpha$ would imply oscillation.

Example 5.7.8. Show that

$$x'' + \frac{2t^6 - 2t^4 + 3t - 1}{t^6 + 3t^2 + 1}x = 0 \tag{5.28}$$

is an oscillatory equation.

Dividing by t^6, we see that

$$\lim_{t\to\infty} \frac{2t^6 - 2t^4 + 3t - 1}{t^6 + 3t^2 + 1} = 2.$$

Using the preceding Proposition, we infer that (5.28) is oscillatory. ∎

Theorem 5.7.3 is completed by the following proposition.

Proposition 5.7.9. *If* $p(t) \leq 0$, $p(t) \not\equiv 0$, *then no solution of* (5.24) *can have more than one zero.*

Proof. Suppose that (5.24) has a solution $x_1(t)$ with two zeros t_1 and t_2. Then consider the equation $y'' + q(t)y = 0$, where $q(t) \equiv 0$, so that $y'' = 0$. Since $q(t) \geq p(t)$ and $q(t) \not\equiv p(t)$, by Corollary 5.7.4, every solution of $y'' = 0$ has a zero between t_1 and t_2. This is obviously false, since $y \equiv 1$ is a solution of $y'' = 0$. ∎

A careful examination of the above results shows that there is an obscure assumption that the zeros of solutions of (5.24) are isolated, that is, in any finite interval $[\alpha, \beta]$, there can be only a finite number of them. If this were not the case, then we would not be able to take two consecutive zeros, just as we cannot take two consecutive rational numbers. Recall that t_1 and t_2 are two consecutive zeros of $x(t)$ if $x(t_1) = x(t_2) = 0$ and $x(t) \neq 0$ in (t_1, t_2). How do we know that the interval (t_1, t_2) does not contain infinitely many zeros of $x_1(t)$ for any number $t_2 > t_1$?

We now give a proof of the fact that zeros of solutions of (5.24) are isolated. The proof can be easily followed by readers with adequate knowledge of introductory level Analysis. Those who do not have the proper background may skip the proof and simply note and use this property of the zeros of solutions, when needed.

Definition 5.7.10. A number α is a limit point (or accumulation point) of a set S of real numbers if every open interval containing α contains infinitely many points of the set S.

The following theorem is a special case of a theorem due to Bernard Bolzano and Karl Weierstrass, 1817. It can be found in almost all introductory level Analysis books. We skip the proof of this theorem and ask interested readers to consult an Analysis book.

Theorem 5.7.11 (Bolzano–Weierstrass). *Every infinite bounded set of real numbers has a limit point.*

Theorem 5.7.12. *Let $y(t)$ be a nontrivial solution of* (5.24) *and let $[a, b]$ be any closed interval. Then $y(t)$ has a finite number of zeros in $[a, b]$.*

Proof. Suppose that $y(t)$ has infinitely many zeros in the interval $[a, b]$. Let S be the set of zeros of $y(t)$ in $[a, b]$. Then by the Bolzano–Weierstrass Theorem, S has a limit point $\bar{t}$, which, by the definition of limit points, cannot be outside the interval $[a, b]$. By the definition of limit points, for every natural number k, the interval $(\bar{t} - 1/k, \bar{t} + 1/k)$ contains a point of S distinct from $\bar{t}$, denoted by t_k. It is then clear that the sequence (t_k) converges to $\bar{t}$ as $k \to \infty$.

By Rolle's Lemma, in each interval $(\bar{t} - 1/k, \bar{t} + 1/k)$, there is a number s_k such that $y'(s_k) = 0$. This follows from the fact that the interval $(\bar{t} - 1/k, \bar{t} + 1/k)$ contains infinitely many zeros of y; applying Rolle's lemma to any two zeros of y in this interval will give us a number s_k where y' vanishes. Again, it is clear that the sequence (s_k) converges to $\bar{t}$ as $k \to \infty$.

It follows from continuity of $y(t)$ and $y'(t)$ that $y(t_k) \to y(\bar{t})$ and $y'(s_k) \to y'(\bar{t})$. Now, since for each k, $y(t_k) = y'(s_k) = 0$, it follows that $y(\bar{t}) = y'(\bar{t}) = 0$.

Since $z(t) \equiv 0$ is also a solution of (5.24) satisfying the initial conditions $z(\bar{t}) = z'(\bar{t}) = 0$, it follows from the uniqueness of solutions that $y(t) \equiv 0$, contradicting the assumption that $y(t)$ is nontrivial. ∎

We wish to point out an important fact concerning the results in this section and that is the fact that studying equations of the form (5.24) instead of

$$x'' + p(t)x' + q(t)x = 0 \tag{5.29}$$

is not a great disadvantage. This is because any equation of the form (5.29) can be transformed into an equation of the form (5.24) by making the substitution

$$x(t) = y(t)e^{-\frac{1}{2}\int p(t)dt}$$

assuming that $p'(t)$ and $q(t)$ are continuous. Notice that $x(t)$ and $y(t)$ have the same set of zeros. The proof is left as an exercise.

5.8 Some nonlinear second order equations

In this section we briefly deal with some special classes of nonlinear second order equations that can be solved by a reduction of the order.

5.8.1 Equations of the type $F(t, x', x'') = 0$

Consider the equation

$$F(t, x', x'') = 0 \tag{5.30}$$

where the dependent variable x is missing.

We let $z = x'$ and get $z' = x''$, and we find $F(t, z, z') = 0$ which is a first order equation. If $z(t) = \phi(t, c)$ is a family of solutions of this equation, then integrating $x' = z = \phi(t, c)$ we find

$$x(t) = \int \phi(t, c)dt + c'$$

which is a solution of $F(t, x', x'') = 0$, for all $c' \in \mathbb{R}$.

Example 5.8.1. Solve the initial value problem $x'' = 2tx'$, $x(0) = 0, x'(0) = 1$. The equation $x'' = 2tx'$ is of the form (5.30). Setting $z = x'$ we reduce the problem to the first order separable equation $z' = 2tz$. Then $z(t) = c\, e^{t^2}$. For $t = 0$ from $x'(0) = z(0) = 1$ it follows that $1 = c$. Since $x' = z$ we find $x'(t) = e^{t^2}$. Integrating, we find $x(t) = \int_0^t e^{t^2} dt$ which takes into account the initial condition $x(0) = 0$. ∎

5.8.2 Equations of the type $F(x, x', x'') = 0$

Consider the equation

$$F(x, x', x'') = 0 \tag{5.31}$$

where the independent variable t is missing.

As in Example 5.1.6, we let $z = x'$. But now we use the Chain Rule, obtaining

$$x'' = \frac{dx'}{dt} = \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = z\frac{dz}{dx}. \tag{5.32}$$

Substituting in Equation (5.31), we obtain the first order equation

$$F(x, z, z\frac{dz}{dx}) = 0. \tag{5.33}$$

Let $z = \phi(x, c_1)$ be a family of solutions of (5.33), depending on a constant c_1. Then from $z = x'$ we infer

$$\frac{dx}{dt} = \phi(x, c_1)$$

which is a separable equation that can be integrated. Assuming that $\phi(x, c_1)$ never vanishes we obtain solutions of (5.31) in the form

$$\int \frac{dx}{\phi(x, c_1)} = t + c_2, \qquad c_2 \in \mathbb{R}.$$

An important class of equations that can be solved using the preceding method is $x'' = f(x)$. In this case we find the separable equation $z\,\frac{dz}{dx} = f(x)$ that can be integrated. Equations like $x'' = f(x)$ will be discussed more extensively in Chapter 8.

Example 5.8.2. Solve $x'' = 2xx',\ x(0) = 0,\ x'(0) = 1$. The equation $x'' = 2xx'$ is of the form (5.31). We let $z = x'$ and then using the Chain Rule, we have $x'' = \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = z\frac{dz}{dx}$. Now we have reduced the problem to solving the first order equation

$$z\frac{dz}{dx} = 2xz.$$

One solution is $z \equiv 0$, but it does not satisfy the initial condition $z(0) = 1$. Dividing by z, we find $\frac{dz}{dx} = 2x$, hence $z(x) = x^2 + c_1$. For $x = 0$ we have $z(0) = 1$ and hence $c_1 = 1$. The problem becomes $x' = x^2 + 1$ with $x(0) = 0$. Integrating we find

$$\int \frac{dx}{x^2 + 1} = t + c_2$$

namely $\arctan x = t + c_2$. The initial condition $x(0) = 0$ yields $c_2 = 0$. Thus $\arctan x = t$ and finally $x(t) = \tan t$, $|t| < \frac{\pi}{2}$. ■

5.8.3 *Equations of the form* $F(t, x, x', x'') = 0$ *with* F *homogenous*

Consider the equation

$$F(t, x, x', x'') = 0 \tag{5.34}$$

where F is a homogeneous function of degree k with respect to x, x', x'', namely $F(t, \lambda x, \lambda x', \lambda x'') = \lambda^k F(t, x, x', x'')$, for all $\lambda \in \mathbb{R}$ for which λ^k makes sense.

The homogeneity of F suggests to try to find solutions such that $x' = xz$. Setting $x'(t) = x(t)z(t)$ we find $x'' = zx' + xz' = xz^2 + xz'$ and hence $F(t, x, x', x'') = 0$ yields $F(t, x, xz, xz^2 + xz') = 0$. Using the homogeneity of F one finds

$$F(t, x, xz, xz^2 + xz') = x^k F(t, z, z^2 + z'),$$

yielding the first order equation

$$F(t, z, z^2 + z') = 0.$$

For example, if the given equation is $x'' = f(t, x, x')$ and f is homogeneous of degree 1 with respect to x, x', we find $z^2 + z' = f(t, z)$.

If $\phi(t, c_1)$, $c_1 \in \mathbb{R}$, is a family of solutions of $F(t, z, z^2 + z') = 0$, then $x'(t) = x(t)\phi(t, c_1)$ yields $x(t) = c_2\, e^{\phi(t,c_1)}$, $c_2 \in \mathbb{R}$. To this two parameter family of solutions we have to add the trivial solution $x(t) \equiv 0$.

If we want to solve the initial value problem

$$F(t, x, x', x'') = 0, \qquad x(t_0) = x_0 \neq 0, \quad x'(t_0) = x_1,$$

then from $x(t) = x'(t)z(t)$ we infer $x_1 = x_0 z(t_0)$ that is $z(t_0) = x_1/x_0$. So we have to solve the ivp $F(t, z, z^2 + z') = 0$, $z(t_0) = x_1/x_0$.

Example 5.8.3. Solve

$$xx'' - x'^2 - 2tx^2 = 0, \qquad x(0) = 1, \quad x'(0) = 0.$$

Here $F(t, x, x', x'') = xx'' - x'^2 - 2tx^2$ is homogeneous of degree 2. Setting $x' = xz$ we find $x'' = xz^2 + xz'$. Hence

$$x(xz^2 + xz') - x^2 z^2 - 2tx^2 = 0$$

and canceling $x^2 z^2$, we get $x^2 z' - 2tx^2 = 0$, namely

$$x^2(z' - 2t) = 0.$$

Notice that in the present case, the trivial solution $x(t) \equiv 0$ does not satisfy the initial condition $x(0) = 1$. The general integral of the first order equation $z' - 2t = 0$ is $z = \phi(t, c_1) = t^2 + c_1$. For $t = 0$, one has $z(0) = x'(0) = 0$ and hence $c_1 = 0$. Then $z(t) = t^2$ and $x' = xz$ yields the separable equation

$$x' = t^2 x \qquad \Longrightarrow \qquad x(t) = c_2\, e^{\frac{1}{3}t^3}.$$

Using the initial condition $x(0) = 1$, we obtain $c_2 = 1$. Thus

$$x(t) = e^{\frac{1}{3}t^3}$$

is the solution we were seeking. ∎

For second order equations one can consider problems like

$$x'' = f(t, x, x'), \quad x(a) = \alpha, \quad x(b) = \beta,$$

that are called *boundary value problems* because we require that the solution assumes some given values for t at the boundary of the interval $[a, b]$.

One can also take the interval to be all the real line and seek solutions that have a prescribed limit as $t \to \pm\infty$. Problems of this kind related to the equation $x'' = f(x)$ will be discussed in Chapters 8 and 13.

5.9 Exercises

A. Linear independence and Wronskian

A1. Show that $x_1 = t^3 - t^2$ and $x_2 = t^3 - 3t$ are linearly independent.

A2. Consider the functions $f(t) = \sin t$ and $g(t) = t^2$.
   a) Using the definition of linear independence, explain why they are linearly independent.
   b) Using a Wronskian argument, explain why they are linearly independent.
   c) Explain why they cannot be solutions of a differential equation $x'' + p(t)x' + q(t)x = 0$, where p and q are continuous functions.

A3. Show that if x_1 and x_2 are linearly independent, then so are their linear combinations $z_1 = 2x_1 + 3x_2$ and $z_2 = 2x_1 - 3x_2$.

A4. a) Prove that if the Wronskian of two differentiable functions $f(t)$ and $g(t)$, not necessarily solutions of differential equations, is nonzero at one point of an interval I, then they are linearly independent.
   b) Prove that if they are linearly dependent, then their Wronskian is identically equal to 0.

A5. Show that $x_1 = \tan t$ and $x_2 = \sin t$ are linearly independent on the interval $(0, \pi)$.

A6. Solve the initial value problem

$$W(t^2 + 1, f(t)) = 1, \quad f(0) = 1.$$

A7. Show that if $x_1(t)$ and $x_2(t)$ are two linearly independent functions, and $z(t)$ is a function such that $z(t) > 0$ on I, then zx_1 and zx_2 are also linearly independent on I.

A8. Give an example to show that the following statement is false: if two functions f_1 and f_2 are linearly independent in an interval I, then they are also independent in any subinterval J of I.

A9. Show that if x_1 and x_2 are linearly dependent on an interval I, then they are linearly dependent in any subinterval J of I.

A10. Show that if two solutions of a second order homogeneous differential equation with continuous coefficients on I have a common zero then all their zeros are in common.

A11. Let x_1 and x_2 be two solutions of $x'' + \frac{x'}{t} + q(t)x = 0,\ t > 0$, where $q(t)$ is a continuous function. Given that $W(6) = 7$, find $W(7)$.

B. Homogeneous equations with constant coefficients

Solve each of the following:

B1. $2x'' + x' - x = 0$.

B2. $x'' + 2x' + 2x = 0$.

B3. $x'' + 8x' + 16x = 0$.

B4. $x'' + 2x' - 15x = 0,\ x(0) = 1,\ x'(0) = 1.$

B5. $x'' - 3x' + 2x = 0,\ \ x(1) = 0,\ \ x'(1) = 1.$

B6. $4x' + 2x'' = -5x,\ \ x(0) = 0,\ \ x'(0) = 1.$

B7. $x'' - 6x' + 9x = 0,\ \ x(0) = 0,\ \ x'(0) = 1.$

B8. Show that for $\beta \geq 0$, $x'' + x' - \beta x = 0$ will always have some solutions that do not approach 0 as $t \to +\infty$.

B9. For which values of β will all solutions of

$$x'' + x' - \beta x = 0$$

go to 0 as t goes to ∞?

B10. Show that all the solutions of $x'' + 4x' + kx = 0$ go to 0 as $t \to +\infty$ if and only if $k > 0$.

B11. Show that the equation $x'' + bx' + cx = 0$ has infinitely many solutions and none of them, except the trivial solution, can have a maximum or a minimum point on the t-axis.

B12. Find a second order linear homogeneous equation whose corresponding characteristic equation has $m = 3 - 7i$ as one of its roots.

B13. Show that any solution of $x'' + 5x' + 6x = 0$ tends to zero as $t \to +\infty$.

B14. Show that if $p > 0$ then any solution of $x'' + px' = 0$ tends to a constant as $t \to +\infty$, while if $p < 0$ only constant solutions tend to constants.

B15. Find a such that the solution of $x'' + x' - 2x = 0$, $x(0) = a,\ x'(0) = 1$ tends to zero as $t \to +\infty$.

B16. Show that all solutions of $x'' - 2x' + 2x = 0$ are bounded on $(-\infty, 0]$, and unbounded on as $[0, \infty)$.

B17. Find conditions on a, b such that the solutions of $x'' - 2ax' + bx = 0$ are oscillating functions.

B18. Find $\lambda \neq 0$ such that the boundary value problem $x'' + \lambda^2 x = 0$, $x(0) = x(\pi) = 0$, has nontrivial solutions.

B19. Find $a \neq b$ such that the boundary value problem $x'' + x = 0$, $x(a) = x(b) = 0$, has nontrivial solutions.

B20. Show that the boundary value problem $x'' - x = 0$, $x(0) = x(1) = 0$, has only the trivial solution.

B21. Show that the problem $x'' + x' - 2x = 0$, $x(0) = 0$, $\lim_{t \to +\infty} x(t) = 0$ has only the trivial solution $x(t) = 0$.

B22. Solve $x'' - 2x' + 5x = 0$, $x(0) = 1$, $x(\pi/4) = 0$.

B23. Find θ such that $x'' - 2x' + 5x = 0$, $x(0) = 0$, $x'(\theta) = 0$, has only the trivial solution.

B24. Solve $x'' + 2x' = 0$, $x(0) = 0$, $\lim_{t \to +\infty} x(t) = a$.

C. Nonhomogeneous equations with constant coefficients

C1. Solve $x'' - 4x = t$.

C2. Solve $x'' - 4x = 4t^2$.

C3. Solve $x'' + x = t^2 - 2t$.

C4. Solve $x'' + x = 3t^2 + t$.

C5. Solve $x'' - x = e^{-3t}$.

C6. Solve $x'' - x = 3e^{2t}$.

C7. Solve $x'' - x = te^{2t}$.

C8. Solve $x'' - 3x' - x = t^2 + t$.

C9. Solve $x'' - 4x' + 13x = 20e^t$.

C10. Solve $x'' - x' - 2x = 2t + e^t$.

C11. Solve $x'' + 4x = \cos t$.

C12. Solve $x'' + x = \sin 2t - \cos 3t$.

C13. Solve $x'' + 2x' + 2x = \cos 2t$.

C14. Solve $x'' + x = t \sin 2t$.

C15. Solve $x'' - x' = t$.

C16. Solve $x'' - x = e^{kt}$, $k \in \mathbb{R}$.

C17. Solve $x'' - x' - 2x = 3e^{-t}$.

C18. Solve $x'' - 3x' + 2x = 3te^t$.

C19. Solve $x'' - 4x' + 3x = 2e^t - 5e^{2t}$.

C20. Solve $x'' + 2x = \cos \sqrt{2}t$.

C21. Solve $x'' + 4x = \sin 2t$.

C22. Solve $x'' + x = 2\sin t + 2\cos t$.

C23. Solve $x'' + 9x = \sin t + \sin 3t$.

C24. Solve the boundary value problem $x'' - x = t$, $x(0) = x(1) = 0$.

C25. Find k such that the solution of $x'' + 4x' + x = k$, $x(0) = 0$, $x'(0) = 0$ tends to $-\infty$ as $t \to +\infty$.

C26. Show that if $\lambda \neq 0$ and $h(t) > 0$ then any (possible) nontrivial solution of the boundary value problem $x'' - \lambda^2 x = h(t)$, $x(a) = x(b) = 0$ cannot be non negative in $(a.b)$.

C27. Show that for all $a \neq 0$ the boundary value problem $x'' - 2x = 2e^t$, $x(0) = x(a) = 0$, has a unique solution.

D. Miscellanea

D1. Show that

$$x'' + \frac{t^5+1}{t^4+5}x = 0$$

is an oscillatory equation.

D2. Which one of the following two equations has solutions that oscillate more rapidly?

$$x'' + \sqrt{t^6+3t^5+1}\,x = 0,$$
$$x'' + 2t^3x = 0.$$

D3. Explain why no nontrivial solution of (5.24) can vanish at each of the numbers $0, 1, 1/2, 1/3, \ldots 1/n \ldots$.

D4. Consider the boundary value problem

$$x'' - p(t)x = q(t), \quad x(a) = x(b) = 0.$$

Show that if $p(t)$ and $q(t)$ are continuous, with $p(t) > 0$, on the interval $[a, b]$, then there is a unique solution of this boundary value problem.

D5. Show that, assuming that $p'(t)$ and $q(t)$ are continuous, the substitution

$$x(t) = y(t)e^{-\frac{1}{2}\int p(t)dt}$$

transforms the equation

$$x'' + p(t)x' + q(t)x = 0$$

into the form (5.24).

D6. Determine the oscillation of

$$x'' + x' + x = 0$$

in two ways
(a) by transforming it to the form $x'' + p(t)x = 0$,
(b) by solving the equation explicitly.

D7. Determine the oscillation of

$$x'' - \frac{1}{4}tx' + x = 0.$$

D8. Let $u'' + p_1(t)u = 0$ and $v'' + p_2(t)v = 0$, with $v(t) \neq 0$ in $[a, b]$.
(a) Prove the *Picone Identity*

$$\left(\frac{u}{v}(u'v - uv')\right)' = (p_2 - p_1)u^2 + \left(u' - v'\frac{u}{v}\right)^2.$$

b) Use this to prove the Sturm comparison theorem.

D9. Let $u'' + p_1(t)u = 0$ and $v'' + p_2(t)v = 0$ with $p_2(t) > p_1(t)$ in (a, b). Suppose that $u(a) = v(a) = 0$, $u'(a) = v'(a) = \alpha > 0$. Show that there exists $\epsilon > 0$ such that $v(t) > u(t)$ in $(a, a + \epsilon)$.

D10. Solve the initial value problem $x'' = \dfrac{x'}{t}$, $x(1) = 0$, $x'(1) = 1$.

D11. Solve $x'' = 2x'(x - 1)$, $x(0) = 0$, $x'(0) = 1$.

D12. Solve $x'' = 2x'^3 x$.

D13. Solve $xx'' - 2x'^2 - x^2 = 0$.

D14. Solve (a) $xx'' - x'^2 + e^t x^2 = 0$, $x(0) = 1$, $x'(0) = -1$, and (b) $xx'' - x'^2 + e^t x^2 = 0$, $x(0) = -1$, $x'(0) = -1$.

D15. Solve the Euler equation $t^2 x'' - 2x = 0, \; t > 0$.

D16. Solve $t^2 x'' + atx' + x = 0$, $t > 0$.

D17. Solve $t^2 x'' - tx' - 3x = 0, \; x(1) = 0, \; x'(1) = 1$, $t > 0$.

D18. Solve the nonhomogeneous Euler equation $t^2 x'' + tx' + x = t, \; t > 0$.

D19. Solve $t^2 x'' + 3tx' - 3x = t^2, \qquad t > 0$.

D20. Show that a solution of $x'' - tx' + 3x = 0$ is a polynomial P of degree 3.

6

Higher order linear equations

6.1 Existence and uniqueness

Almost everything we learned in Chapter 5 about second order equations can be easily seen to be true for the corresponding higher order equations. Therefore, in order to avoid unnecessary repetition, here, for the most part, we simply state the more general results and give examples. In a few cases, when the generalizations are not so obvious, we will provide the explanations and proofs.

First we state the existence and uniqueness theorem, which follows from Theorems 4.2.2 and 4.2.3 in Chapter 4.

Theorem 6.1.1. *Consider the equation*

$$p_0(t)x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \ldots + p_n(t)x(t) = f(t), \tag{6.1}$$

where the coefficient functions $p_i(t)$, $0 \le i \le n$, *and* $f(t)$ *are continuous on a given interval* $I \subseteq \mathbb{R}$, *with* $p_0(t) \neq 0$. *Then for any number* t_0 *in* I, *there exists a unique solution* $x(t)$ *of* (6.1) *satisfying the initial conditions*

$$x(t_0) = \alpha_1, x'(t_0) = \alpha_2, \ldots, x^{(n-1)}(t_0) = \alpha_n,$$

where α_i, $1 \le i \le n$ *are any real numbers. Furthermore, this solution exists for all* t *in* I.

In equation (6.1), we normally take the leading coefficient $p_0(t)$ to be equal to one, which is the same as dividing the equation by $p_0(t)$. The above theorem treats the most general linear nonhomogeneous equation. If we take $f(t) \equiv 0$, we have the existence and uniqueness theorem for the most general linear homogeneous equation.

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UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_6

6.2 Linear independence and Wronskian

Similar to the case for second order equations, functions $f_1, f_2, \ldots, f_n$ are said to be *linearly independent* on an interval I if for any n constants $c_1, c_2, \ldots, c_n$, $c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \equiv 0$, t in I, implies that $c_1 = c_2 = \ldots = c_n = 0$. Functions that are not linearly independent are said to be *linearly dependent*, i.e. they are linearly dependent if there exist constants c_i, $1 \le i \le n$, not all 0, such that $c_1 f_1(t) + c_2 f_2(t) + \ldots + c_n f_n(t) \equiv 0$.

We recall that linear dependence had an easy and very useful characterization in the second order case, that is two functions are linearly dependent if and only if one of them is a constant multiple of the other. For higher order equations, an analogous statement would be that n functions are linearly dependent if and only if any one of them is a linear combination of the others, which is not as useful as in the second order case. However, it is useful to know that if any subset of two or more of a given set of n functions are linearly dependent, then all n of them are linearly dependent. The converse is, of course, false.

Example 6.2.1. The functions $f_1(t) = \sin t,\ f_2(t) = \cos t,\ f_3(t) = e^t,\ f_4(t) = \sqrt{2}\sin t$ are linearly dependent since f_1 and f_4 are linearly dependent. We note that since $(-\sqrt{2}) \cdot \sin t + (1) \cdot \sqrt{2}\sin t \equiv 0$, we can write $(-\sqrt{2}) \cdot \sin t + (0) \cdot \cos t + (0) \cdot e^t + (1) \cdot \sqrt{2}\sin t \equiv 0$, which satisfies the definition of linear dependence of f_1, f_2, f_3, f_4 with $c_1 = -\sqrt{2}, c_2 = 0, c_3 = 0, c_4 = 1$. ∎

We now extend the notion of the Wronskian to n functions and write

$$W(f_1, f_2, \ldots, f_n)(t) = \begin{vmatrix} f_1(t) & f_2(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & \cdots & f_n'(t) \\ \vdots & & & \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}.$$

Theorem 6.2.2 (Abel's Theorem). *If $x_1, x_2, \ldots, x_n$ are solutions of*

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \ldots + p_n(t)x(t) = 0 \tag{6.2}$$

on some interval I, where the coefficients p_i, $1 \le i \le n$, are continuous, then

$$W(x_1, x_2, \ldots, x_n)(t) = c\, e^{\int -p_1(t)dt}.$$

As a consequence, $W(x_1, x_2, \ldots, x_n)(t)$ is either identically zero or it never vanishes.

We give the proof for $n = 3$. The proof for higher order equations is identical but cumbersome.

Proof. Using the formula for the derivative of a determinant and the fact that $x_i''' = -p_1 x_i'' - p_2 x_i' - p_3 x_i$, $i = 1, 2, 3$, we have

$$W'(t) = \frac{d}{dt}\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} = \begin{vmatrix} x_1' & x_2' & x_3' \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1'' & x_2'' & x_3'' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1''' & x_2''' & x_3''' \end{vmatrix} =$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -p_1 x_1'' - p_2 x_1' - p_3 x_1 & -p_1 x_2'' - p_2 x_2' - p_3 x_2 & -p_1 x_3'' - p_2 x_3' - p_3 x_3 \end{vmatrix}.$$

Now, if we multiply the first row in the last determinant by p_3 and add it to the third row and then multiply the second row by p_2 and add it to the third row, we obtain

$$W'(t) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -p_1 x_1'' & -p_1 x_2'' & -p_1 x_3'' \end{vmatrix} = -p_1(t)W(t).$$

This shows that $W' + p_1(t)W = 0$. Solving this linear first order equation for W, the assertion of the theorem follows. ■

We now summarize some obvious generalizations of the second order equations.

1. Any linear combination $x(t) = c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t)$ of solutions $x_1, x_2, \ldots x_n$ of (6.2) is also a solution.
2. The Wronskian of solutions of (6.3) is either always zero or it is never zero in the interval where the solutions are defined.
3. If the Wronskian of arbitrary functions $f_1, f_2, \ldots, f_n$ is different from zero at one point of an interval where it is defined, then the functions are linearly independent on that interval. The contrapositive statement would be that if they are linearly dependent, then their Wronskian is identically equal to zero.
4. If $x_1, x_2, \ldots, x_n$ are solutions of (6.2), then they are linearly independent if and only if their Wronskian is different from zero.
5. If $x_1, x_2, \ldots, x_n$ are solutions of (6.2), whose Wronskian is different from zero, then they are a fundamental set of solutions, that is,

   $$x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

   is the general solution of (6.2).

6.3 Constant coefficients

Consider

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = 0, \tag{6.3}$$

where a_i, $1 \le i \le n$, are constant real numbers. As in the case of second order equations, in order to solve (6.3) we substitute $x = e^{mt}$ in the equation, which gives rise to the *characteristic* (or *auxiliary*) equation

$$C(m) = m^n + a_1 m^{n-1} + \ldots + a_n = 0.$$

The biggest difference between the second order equations with constant coefficients and the more general equations is that for the second order, we could always solve the characteristic equation by the Quadratic Formula, whereas for the more general case there is no method by which we can explicitly solve the above characteristic equation. Nevertheless, reducing the original differential equation to an algebraic equation is still simpler to deal with and has important theoretical implications.

The following is a summary of the extensions. We give proofs when the extensions do not follow from arguments similar to those given in the second order case.

Theorem 6.3.1. (i) *Let $m_1, m_2, \ldots, m_r$ be the distinct roots of the characteristic equation corresponding to* (6.3), *and let q_i represent the multiplicity of m_i. Then $t^k e^{m_i t}$ is a solution for $k = 1, \ldots, q_i - 1$.*

(ii) *The solutions $t^k e^{m_i t}$, $k = 0, 1, \ldots, q_i - 1$; $i = 1, 2, \ldots, r$ are linearly independent.*

We prove this theorem for $n = 3$.

Suppose that the characteristic equation $m^3 + a_1 m^2 + a_2 m + a_3 = 0$ has distinct roots m_1, m_2, m_3.

First of all, it is easy to check the fact that $e^{m_1 t}$, $e^{m_2 t}$, $e^{m_3 t}$ are solutions of the given differential equation. To show that they are linearly independent, we suppose, by contradiction, that there exist constants c_1, c_2, c_3, not all zero, such that $c_1 e^{m_1 t} + c_2 e^{m_2 t} + c_3 e^{m_3 t} = 0$. Suppose, without loss of generality, $c_3 \neq 0$. Then multiplying both sides of the equation by $e^{-m_1 t}$, we obtain $c_1 + c_2 e^{(m_2 - m_1)t} + c_3 e^{(m_3 - m_1)t} = 0$. Taking the derivative of both sides and multiplying by $e^{m_1 t}$, we get $c_2(m_2 - m_1)e^{m_2 t} + c_3(m_3 - m_1)e^{m_3 t} = 0$. We multiply both sides by $e^{-m_2 t}$, obtaining $c_2(m_2 - m_1) + c_3(m_3 - m_1)e^{(m_3 - m_2)t} = 0$. Taking the derivative again and multiplying both sides by $e^{m_2 t}$, we have $c_3(m_3 - m_1)(m_3 - m_2)e^{m_3 t} = 0$. Since m_1, m_2, m_3 are all distinct, i.e. not equal to each other, we must have $c_3 = 0$, which is a contradiction.

An alternate proof of linear independence consists of using Abel's theorem. Let $W(t)$ be the Wronskian of $e^{m_1 t}$, $e^{m_2 t}$, $e^{m_3 t}$. One has

$$W(t) = \begin{vmatrix} e^{m_1 t} & e^{m_2 t} & e^{m_3 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} & m_3 e^{m_3 t} \\ m_1^2 e^{m_1 t} & m_2^2 e^{m_2 t} & m_3^2 e^{m_3 t} \end{vmatrix}.$$

For $t = 0$ one has

$$W(0) = \begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ m_1^2 & m_2^2 & m_3^2 \end{vmatrix}.$$

Multiplying the first row by $-m_1$ and adding it to the second row, and multiplying the first row by $-m_1^2$ and adding it to the third row one obtains $W(0) = (m_3 - m_2)(m_3 - m_1)(m_2 - m_1)$. Since $m_i \neq m_j$ if $i \neq j$, then $W(0) \neq 0$ proving that $e^{m_1 t}$, $e^{m_2 t}$, $e^{m_3 t}$ are linearly independent.

Next, suppose that one of the roots is a simple root and one of them is a double root. Without loss of generality, assume that m_1 is a simple root and m_2 is a double root. To show that the corresponding solutions $e^{m_1 t}$, $e^{m_2 t}$, $te^{m_2 t}$ are linearly independent, we show that their Wronskian at $t = 0$ is nonzero.

$$W(t) = \begin{vmatrix} e^{m_1 t} & e^{m_2 t} & te^{m_2 t} \\ m_1 e^{m_1 t} & m_2 e^{m_2 t} & e^{m_2 t} + tm_2 e^{m_2 t} \\ m_1^2 e^{m_1 t} & m_2^2 e^{m_2 t} & 2m_2 e^{m_2 t} + tm_2^2 e^{m_2 t} \end{vmatrix}$$

and hence

$$W(0) = \begin{vmatrix} 1 & 1 & 0 \\ m_1 & m_2 & 1 \\ m_1^2 & m_2^2 & 2m_2 \end{vmatrix} = \begin{vmatrix} m_2 & 1 \\ m_2^2 & 2m_2 \end{vmatrix} - \begin{vmatrix} m_1 & 1 \\ m_1^2 & 2m_2 \end{vmatrix}$$

$$= 2m_2^2 - m_2^2 - 2m_1 m_2 + m_1^2 = m_2^2 - 2m_1 m_2 + m_1^2 = (m_2 - m_1)^2 \neq 0.$$

Finally, suppose that the characteristic equation has one triple root m_1, i.e. $(m - m_1)^3 = 0$ is the characteristic equation. Again, we show that the Wronskian of the corresponding solutions e^{mt}, te^{mt} $t^2 e^{mt}$ is nonzero at $t = 0$:

$$W(t) = \begin{vmatrix} e^{mt} & te^{mt} & t^2 e^{mt} \\ me^{mt} & e^{mt} + mte^{mt} & mt^2 e^{mt} + 2te^{mt} \\ m^2 e^{mt} & m^2 te^{mt} + 2me^{mt} & m^2 t^2 e^{mt} + 4mte^{mt} + 2e^{mt} \end{vmatrix},$$

$$W(0) = \begin{vmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ m^2 & 2m & 2 \end{vmatrix} = 2.$$

Example 6.3.2. Given that $1, 2, 5 \pm 6i, 5 \pm 6i$ are the roots of the characteristic equation, the general solution of the corresponding differential equation can be determined to be $x = c_1 e^t + c_2 e^{2t} + c_3 e^{5t} \sin 6t + c_4 e^{5t} \cos 6t + c_5 te^{5t} \sin 6t + c_6 te^{5t} \cos 6t$. ■

Example 6.3.3. Find the general solution of the differential equation

$$x'''' + 4x = 0$$

which has application to the vibrating rod problem.

In order to solve this equation, we need to find all the roots of the characteristic equation $m^4 + 4 = 0$, which is equivalent to finding the fourth roots of the number −4. For this we use de Moivre's formula.

$$(-4)^{1/4} = \sqrt{2}(-1)^{1/4} = \sqrt{2}\,[\cos(\pi + 2n\pi) + i\,\sin(\pi + 2n\pi)]^{1/4} =$$
$$\sqrt{2}\left[\cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right) + i\,\sin\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)\right].$$

Letting $n = 0, 1, 2, 3$, we obtain the roots $m = 1 + i,\ -1 + i,\ -1 - i,\ 1 - i$. Therefore, the general solution of the differential equation $x'''' + 4x = 0$ is $x(t) = e^t(c_1 \sin t + c_2 \cos t) + e^{-t}(c_3 \sin t + c_4 \cos t)$. ■

6.4 Nonhomogeneous equations

The following theorem shows that, as with the second order equations, in order to find the general solution of the nonhomogeneous equation (6.1), we need the general solution of the homogeneous equation (6.2) and one solution of the nonhomogeneous equation. The proof is similar to the case for the second order equations.

Theorem 6.4.1. *If $y = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is the general solution of the homogeneous equation* (6.2) *and x_p is any solution of the nonhomogeneous equation* (6.1)*, then $x = y + x_p$ is the general solution of* (6.1).

As in the second order case, we may use the method of Variation of Parameters or Undetermined Coefficients to find a particular solution of the nonhomogeneous equation. These methods are straightforward generalizations of the second order equations.

Method of Variation of Parameters. In using the method of Variation of Parameters, given that $x_1, x_2, \ldots, x_n$ are linearly independent solutions of (6.2), one finds functions $v_1, v_2, \ldots, v_n$ such that $x_p = v_1x_1 + v_2x_2 + \ldots + v_nx_n$ is a solution of (6.1). This is accomplished by solving the following system of n equations for $v_1', \ldots, v_n'$ and then integrating each:

$$\begin{array}{c} v_1'x_1 + v_2'x_2 + \ldots + v_n'x_n = 0 \\ v_1'x_1' + v_2'x_2' + \ldots + v_n'x_n' = 0 \\ v_1'x_1'' + v_2'x_2'' + \ldots + v_n'x_n'' = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v_1'x_1^{n-1} + v_2'x^{n-1} + \ldots + v_n'x_n^{n-1} = f(t) \end{array}$$

Now we give an illustrative example of this method.

Example 6.4.2. Use the method of Variation of Parameters to find the general solution of

$$x''' - x'' + x' - x = e^t.$$

First we find the general solution of the homogeneous equation

$$x''' - x'' + x' - x = 0.$$

In order to do so, we find the roots of the characteristic equation $m^3 - m^2 + m - 1 = 0$. This equation can be factored as $(m-1)(m^2+1) = 0$, yielding the roots $m = 1, i, -i$; which in turn gives us the general solution of the homogeneous equation to be $y(t) = c_1 e^t + c_2 \sin t + c_3 \cos t$.

In order to find a particular solution of the nonhomogeneous equation, we set $x_p = v_1 e^t + v_2 \sin t + v_3 \cos t$ and require that the functions v_1, v_2, v_3 satisfy the following equations:

$$\begin{aligned} e^t\, v_1' + (\sin t)\, v_2' + (\cos t)\, v_3' &= 0, \\ e^t\, v_1' + (\cos t)\, v_2' - (\sin t)\, v_3' &= 0, \\ e^t\, v_1' - (\sin t)\, v_2' - (\cos t)\, v_3' &= e^t. \end{aligned}$$

Solving for v_1', v_2', v_3', we obtain $v_1' = \frac{1}{2}$, $v_2' = -\frac{1}{2}(e^t \sin t + e^t \cos t)$, $v_3' = -\frac{1}{2}(e^t \cos t - e^t \sin t)$. Integrating, we have

$$v_1 = \frac{1}{2}t, \quad v_2 = -\frac{1}{2}\left[\frac{1}{2}e^t(\sin t - \cos t) + \frac{1}{2}e^t(\sin t + \cos t)\right] = -\frac{1}{2}e^t \sin t$$

$$v_3 = -\frac{1}{2}\left[\frac{1}{2}e^t(\sin t + \cos t) - \frac{1}{2}e^t(\sin t - \cos t)\right] = -\frac{1}{2}e^t \cos t.$$

Consequently,

$$x_p = \frac{1}{2}te^t - \frac{1}{2}e^t \sin^2 t - \frac{1}{2}e^t \cos^2 t = \frac{1}{2}te^t - \frac{1}{2}e^t = \frac{1}{2}e^t(t-1)$$

and the general solution of the given nonhomogeneous equation is $x(t) = c_1 e^t + c_2 \sin t + c_3 \cos t + \frac{1}{2}e^t(t-1)$. ■

Method of Undetermined Coefficients. Recall that this method depends on making a good guess and can be much simpler, when it works.

At first glance, it seems reasonable to try $x_p = ae^t$ and determine a so that x_p satisfies the equation $x''' - x'' + x' - x = e^t$. But when we substitute, we get 0 on the left side. This is because ae^t is a solution of the corresponding homogeneous equation. So, we try $x_p = ate^t$. Then setting $x_p''' - x_p'' + x_p' - x_p = e^t$, we obtain $2ae^t = e^t$, which gives $a = \frac{1}{2}$. Thus $x_p = \frac{1}{2}te^t$.

Remark 6.4.3. Notice that the answer we got by the second method is not the same as the one we got by using the first method. This should not be surprising since we were only asking for solutions without specifying any initial values. There are infinitely many solutions of the nonhomogeneous equation because each initial value problem has a unique solution.

Also, if we think about it, the solution we got by using the first method can be reduced to the one we got by using the second method. This is because $\frac{1}{2}e^t(t-1) = \frac{1}{2}te^t - \frac{1}{2}e^t$ and hence $x(t) = c_1e^t + c_2\sin t + c_3\cos t + \frac{1}{2}e^t(t-1)$ becomes $x(t) = (c_1 - \frac{1}{2})e^t + c_2\sin t + c_3\cos t + \frac{1}{2}e^t$. Calling $c_1' = c - \frac{1}{2}$, we find $x(t) = c_1'e^t + c_2\sin t + c_3\cos t + \frac{1}{2}e^t$. ∎

Example 6.4.4 (The Euler–Bernoulli beam equation). As an important problem where higher order equations arise, we discuss the Euler–Bernoulli theory of the (static) bending of an elastic uniform beam. According to this theory, the cross sections of the beam under deflection remain plane and normal to the deflected centroid axis of the beam. See Figure 6.1. Experience shows that this assumption is realistic at least for small deflections. The internal forces acting on each cross section keep attached the two parts in which the section divides the beam.

One finds that the deflection of the beam $z = z(x)$ satisfies the 4th order equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2z}{dx^2}\right) = f(x),$$

where E is the elastic Young's modulus of the beam, I is the second moment of area, $f(x)$ is the distributed load. Note that here x is the independent variable and z the dependent one. If both EI and $f(x) = \gamma$ are constant, we find

$$EI\,\frac{d^4z}{dx^4} = \gamma.$$

The characteristic equation is $m^4 = 0$ whose root is $m = 0$, with multiplicity 4. Thus the general solution is $z(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + z_p$ where $z_p(x)$ is a partic-

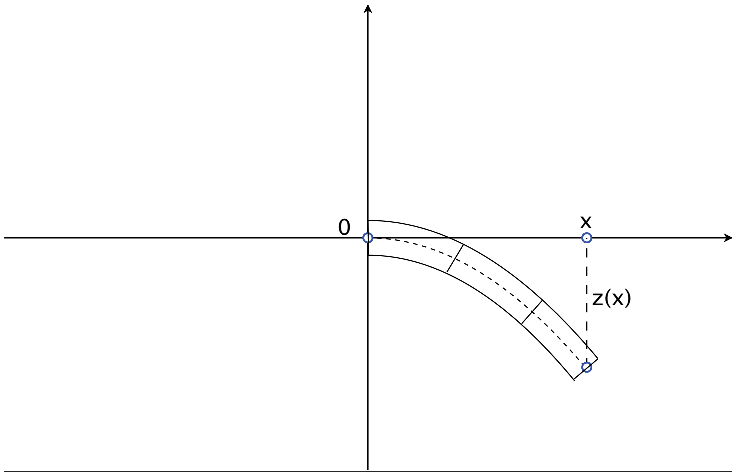

Fig. 6.1. The elastic beam

ular solution of the equation. It is easy to check that we can take $z_p(x) = \frac{\gamma}{24EI}\, x^4$. Then

$$z(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{\gamma}{24EI}\, x^4.$$

If we prescribe the deflection of the beam and its slope at the endpoint $x = 0$ to be zero, we have to impose the conditions

$$z(0) = z'(0) = 0$$

which yield $c_1 = c_2 = 0$ and hence

$$z(x) = c_3 x^2 + c_4 x^3 + \frac{\gamma}{24EI}\, x^4. \qquad \blacksquare$$

6.5 Exercises

1. Find the general solution of $2x''' = 0$.
2. Find the general solution of $x''' - x' = 0$.
3. Find the general solution of $x''' + 5x' - 6x = 0$.
4. Find the general solution of $x''' - 4x'' + x' - 4x = 0$.
5. Find the general solution of $x''' - 3x'' + 4x = 0$.
6. Find the solution to the initial value problem $x''' + 4x' = 0, x(0) = 1,\ x'(0) = -1,\ x''(0) = 2$.
7. Find the solution of $x''' - x' = 0$ satisfying $x(0) = 1$ and $\lim_{t\to+\infty} x(t) = 0$.
8. Find the solution of $x''' - x' = 0$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 1$.
9. Solve the initial value problem
$$x''' + x'' - 2x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1.$$
10. Show that there exists a solution of $x''' + ax'' + bx' + cx = 0$ such that $\lim_{t\to+\infty} x(t) = 0$, provided $c > 0$.
11. Show that for $0 < k < 2$ the equation $x''' - 3x' + kx = 0$ has a unique solution such that $x(0) = 1$ and $\lim_{t\to+\infty} x = 0$.
12. Find the general solution of $x'''' - 6x'' + 5x = 0$.
13. Find the solution of $x'''' - x = 0$, $x(0) = 1$, $x'(0) = x''(0) = x'''(0) = 0$.
14. Find the solutions of $x'''' - x'' = 0$, $x(0) = 1$, $x''(0) = 0$.
15. Find the solution $x(t)$ of $x'''' - 4x'' + x = 0$ such that $\lim_{t\to+\infty} x(t) = 0$ and $x(0) = 0,\ x'(0) = 1$.
16. Show that the only solution of $x'''' - 8x''' + 23x'' - 28x' + 12x = 0$, such that $\lim_{t\to+\infty} x(t) = 0$, is the trivial solution $x(t) \equiv 0$.

17. Show that $x'''' + 2x'' - 4x = 0$ has one periodic solution such that $x(0) = 1$, $x'(0) = 1$.
18. Find the general solution of $x^{(5)} - x' = 0$.
19. Find the general solution of $x^{(5)} + x'''' - x' - x = 0$.
20. Show that $x^{(5)} + x = 0$ has at least one solution such that $\lim_{t\to+\infty} x(t) = 0$.
21. Find the general solution of $x^{(6)} - x'' = 0$.
22. Find the general solution of $x^{(6)} - 64x = 0$.
23. Find the general solution of $x'''' + 3x''' + 2x'' = e^t$
    (a) by the method of Variation of Parameters,
    (b) by the method of Undetermined Coefficients.
24. Find the general solution of $x''' + 4x' = \sec 2t$.
25. Solve $x''' - x'' = 1$.
26. Solve $x''' - x' = t$, $x(0) = x'(0) = x''(0) = 0$.
27. Solve $x'''' + x''' = t$, $x(0) = x'(0) = x''(0) = 0$.
28. Explain whether the functions

$$5,\ t,\ t^2,\ t^3,\ \sin t,\ 3 - t^2,\ \cos t,\ e^t,\ e^{-t}$$

    are linearly dependent or independent.
29. Explain why $x(t) = \sin t^5$ cannot be a solution of a fourth order linear homogeneous equation with continuous coefficients.
30. Evaluate $W(t,\ t^2,\ t^3,\ \sin t,\ \cos t,\ t^4, e^t,\ e^{-t},\ t^4 - t^2)$.
31. Consider $x'''' - 3x''' + 2x' - 5x = 0$. If x_1, x_2, x_3, x_4 are solutions and $W(x_1, x_2, x_3, x_4)(0) = 5$, find $W(x_1, x_2, x_3, x_4)(6)$.
32. Explain why e^t, $\sin t$, t cannot be solutions of a third order homogeneous equation with continuous coefficients.
33. Solve $t^3x''' + 4t^2x'' + 3tx' + x = 0,\ \ t > 0$.
34. Show that if $x_1(t)$ satisfies the equation $x''' + p_1(t)x'' + p_2(t)x' + p_3(t)x = 0$, then the substitution $x = vx_1$ reduces the order of the equation from 3 to 2.

7

Systems of first order equations

7.1 Preliminaries: A brief review of linear algebra

In this preliminary section we recall some facts from Linear Algebra, mainly concerning matrices and vectors. We limit ourselves to discuss only the topics that will be used in this book. For more details and further developments we refer to any book on Linear Algebra, such as the one by H. Anton and C. Rorres (see References).

7.1.1 Basic properties of matrices

A *matrix* (a_{ij}), $1 \le i \le n,\ 1 \le j \le m$, is a rectangular array displayed as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

where the real number a_{ij} (we consider here only matrices with real entries) is the element belonging to the $i\ th$ row and $j\ th$ column. Such a matrix is said to be $n \times m$, where n refers to the number of rows and m refers to the number of columns. An $n \times 1$ matrix is called a *column vector* while a $1 \times n$ matrix is called a *row vector* . We shall be mostly interested in $n \times n$ matrices, called *square* matrices, and column vectors, simply referred to as vectors.

We use capital letters to represent matrices and small letter with bars on top, such as $\overline{v}$, to represent vectors. When it is clear from the context, we will simply use 0 to represent matrices all of whose elements are 0.

Addition and subtraction of square matrices are performed element-wise. For example,

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 4 & 12 \end{pmatrix}.$$

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_7

If $A = (a_{ij})$ and $B = (b_{ij})$ are two $n \times n$ matrices, then their product is defined as $AB = C$, where $C = (c_{ij})$ is the matrix such that

$$c_{ij} = \sum_h a_{ih} b_{hj} \quad i.e. \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

For example

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & -3 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 3 & -6 & -5 \\ -2 & -6 & -5 \end{pmatrix}.$$

We note that the product of an $n \times n$ matrix and an n-dimensional vector is an n-dimensional vector. For example

$$\begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 3 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 8 \end{pmatrix}.$$

It follows from the definition that multiplication of matrices is associative, that is

$$A(BC) = (AB)C.$$

However, unlike multiplication of numbers, multiplication of matrices is not commutative, that is, AB is not necessarily equal to BA, as shown by the following simple example.

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -1 & 3 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 5 \end{pmatrix}.$$

For any natural number n, the $n \times n$ matrix

$$I_n = I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

whose diagonal elements are 1 and the rest are 0, is called the *identity matrix*. It can easily be checked that for any $n \times n$ matrix A, $AI = IA = A$.

7.1.2 Determinants

We define the determinants of square $n \times n$ matrices by induction as follows:

If $n = 1$, that is $A = (a)$ consists of one element, the determinant is defined as $det\ A = a$.

Assuming to have defined the determinant of an $(n-1)\times(n-1)$ matrix, we define $det\ A$ of an $n \times n$ matrix A as follows:

1. Choose any entry a_{kl} of A and consider the matrix A_{kl} obtained by eliminating the row and the column to which a_{kl} belongs (that is, the k-th row and l-th column). The determinant of A_{kl}, called the *minor* of a_{kl}, is defined by induction hypothesis since it is $n-1 \times n-1$.

Setting $C_{kl} = (-1)^{k+l} det\ A_{kl}$ (C_{kl} is called the *cofactor* of a_{kl}), we define

$$det\ A = \sum_{1\le l\le n} a_{kl}C_{kl}.$$

For example, if we choose the elements a_{kl} along the second row, then $det\ A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + \ldots + a_{2n}C_{2n}$.

Let us indicate the calculation for $n = 2, 3$. The determinant of a 2×2 matrix is

$$det\ A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

because the cofactor of a_{11} is $\det(a_{22})$ and the cofactor of a_{12} is $-\det(a_{21}) = -a_{21}$.

The determinant of a 3×3 matrix $A = (a_{ij})$, is

$$det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Here we have chosen the first row. If we decide to use, say, the second column, then $det\ A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$, etc.

We state the following important general rule without proof.

The sum of the products of the elements by their corresponding cofactors along any row or column is the same.

Example 7.1.1. Let us evaluate the determinant

$$\begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix}$$

first along the second row and then along the third column.

Along the second row, we have:

$$\begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = -3\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 4.$$

Along the third column, we have:

$$\begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 4.$$

In concrete examples, it is convenient to choose a row or a column involving zero entries, whenever possible. For example, in order to evaluate

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

it is convenient to choose either the second row or the second column since that will involve only two nonzero terms to add, instead of three. Let us evaluate it along the second column. Then

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \\ -1 & 1 & 4 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -2 \cdot (12+2) - (2-3) = -27.$$

For example, if A is a triangular matrix, namely if $a_{ij} = 0$ for all $j > i$, then choosing the first column we find

$$det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{33} & \dots & a_{3n} \\ .. & .. & .. & .. \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}.$$

To evaluate the last determinant again we choose its first column, yielding

$$\begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{33} & \dots & a_{3n} \\ .. & .. & .. & .. \\ 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{22}\begin{vmatrix} a_{33} & a_{34} & \dots & a_{3n} \\ 0 & a_{44} & \dots & a_{4n} \\ .. & .. & .. & .. \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}.$$

Repeating the procedure we find

$$det\ A = a_{11}a_{22}\dots a_{nn}. \tag{7.1}$$

We now recall some additional properties of determinants.

1. Multiplying a row or a column of a determinant by a constant k is the same as multiplying the determinant by k.
2. Exchanging two rows (or columns) of a determinant changes the sign of the determinant.
3. Multiplying a row (or column) by a constant and adding it to another row (or column) does not change the value of the determinant.
4. $det\ (AB) = det\ (A) \cdot det\ (B)$.
5. If two rows (or columns) of a determinant are constant multiples of each other, then the value of the determinant is zero.

6. $det\ A \neq 0$ if and only if the rows (and columns) of A form a set of linearly independent vectors.

For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 1/3 & 2/3 & 1 \\ 5 & 6 & 7 \end{vmatrix} = 0$$

since the first row is 3 times the second row. Since the determinant is 0, item 6 implies that the rows of the matrix are linearly dependent.

The preceding properties allow us to simplify the calculation in evaluating $det\ A$ by making all the elements of A, except one, of a certain row (or column) to be 0. For example, let us evaluate

$$\begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \\ -5 & 1 & -4 \end{vmatrix}.$$

Suppose we decide to make the second two elements of the first column 0. We can do this by adding -3 times the first row to the second and 5 times the first row to the third, obtaining

$$\begin{vmatrix} 1 & -2 & 1 \\ 0 & 7 & -1 \\ 0 & -9 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -1 \\ -9 & 1 \end{vmatrix} = 7 + 9 = 16.$$

7.1.3 Inverse of a matrix

We call a matrix B *the inverse* of a matrix A if $AB = BA = I$. The inverse, if it exists, is unique. For $AB = BA = I$ and $AC = CA = I$, would imply $AB = AC$ and hence $B(AB) = B(AC)$ which can be regrouped as $(BA)B = (BA)C$. Since $BA = I$ by assumption, we have $B = C$.

Not all nonzero matrices have inverses. For example, let $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. Suppose $C = (c_{ij})$ is a matrix such that $AC = I$. Then we would have

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying the first row of A by the first column of C, we obtain $c_{11} + 2c_{21} = 1$. But multiplying the second row of A by the first column of C, we get $c_{11} + 2c_{21} = 0$, which is impossible. Therefore, A has no inverse.

When a matrix A has an inverse, it is called *nonsingular*, otherwise it is called *singular*. The reader familiar with Linear Algebra my recall that

"A is singular if and only is its determinant is 0."

The next question is: when a matrix does have an inverse, how do we find it? There are several ways and short cuts to find the inverse of a matrix. Here we explain the method of using cofactors. The inverse of a matrix (a_{ij}), when it exists, is the matrix (c_{ij}), where

$$c_{ij} = \frac{C_{ji}}{det A}.$$

For example, let us determine the inverse of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

In order to determine the determinant, we choose to use the cofactors along the last column. Since two of the elements of this column are 0, we can ignore them and see immediately that

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6.$$

Since the determinant is nonzero, we know that A has an inverse.

We now list the cofactors:

$$C_{11} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3.$$

Similarly, $C_{21} = -1$, $C_{22} = 2$, $C_{23} = 1$, $C_{31} = 0$, $C_{32} = 0$, $C_{33} = 6$. Now recalling that the ij-th element of A^{-1} is C_{ji} divided by the value of the determinant, which is 6, we have

$$A^{-1} = \begin{pmatrix} 1/2 & -1/6 & 0 \\ 0 & 1/3 & 0 \\ -1/2 & 1/6 & 1 \end{pmatrix}.$$

7.1.4 Eigenvalues and eigenvectors

If a real or complex number λ and a nonzero vector $\overline{v}$ satisfy the equation

$$A\overline{v} = \lambda\overline{v},$$

then λ is called an *eigenvalue* of the $n \times n$ matrix $A = (a_{ij})$, $a_{ij} \in \mathbb{R}$, and $\overline{v} \neq \bar{0}$ is called an *eigenvector* associated with λ (or the corresponding eigenvector). We note that the above equation can be written in the equivalent form

$$(A - \lambda I)\overline{v} = 0.$$

To find the eigenvalues of A we have to solve the equation $det(A - \lambda I) = 0$. If this equation has no solution, then Kramer's rule implies that the equation $(A - \lambda)\overline{v} = 0$ has only the trivial solution $\overline{v} = 0$. The determinant of $A - \lambda I$ is a polynomial of degree n in λ, called the *characteristic polynomial* and the equation $det(A - \lambda I) = 0$ is called the *characteristic equation* or the *auxiliary equation* of A.

For example, if A is a triangular matrix, then (7.1) yields

$$det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

Hence the eigenvalues of A are $\lambda_i = a_{ii}, i = 1, 2, \ldots, n$.

An eigenvector associated with the eigenvalue λ_j is found by solving the system $(A - \lambda_j I)\overline{v} = \bar{0}$. This system has nontrivial solutions if λ_j is a solution of $det(A - \lambda I) = 0$.

Of course, if $\overline{v}$ is an eigenvector of A, then so is $\alpha\overline{v}$ for all $\alpha \in \mathbb{R}, \alpha \neq 0$. The space $E_{\lambda_j} = \{\overline{x} \in \mathbb{R}^n : A\overline{x} = \lambda_j \overline{x}\}$ is called the *eigenspace corresponding to* λ_j. If $\overline{x}, \overline{y} \in E_{\lambda_j}$ then $\alpha\overline{x} + \beta\overline{y} \in E_{\lambda_j}$ for all $\alpha, \beta \in \mathbb{R}$, because

$$A(\alpha\overline{x} + \beta\overline{y}) = \alpha A\overline{x} + \beta A\overline{y} = \alpha\lambda_j\overline{x} + \beta\lambda_j\overline{y} = \lambda_j(\alpha\overline{x} + \beta\overline{y}).$$

Thus E_{λ_j} is a closed subspace of $\mathbb{R}^n$. If λ_j is a real number and $k \stackrel{def}{=} dim(E_{\lambda_j}) > 1$ it means that there are k linearly independent real eigenvectors corresponding to λ_j. The $dim(E_{\lambda_j})$ is called the geometric multiplicity of λ_j. The algebraic multiplicity of λ_j is defined as the multiplicity of the root $\lambda = \lambda_j$ of the characteristic polynomial. For example, the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is $(\lambda - 2)^2(\lambda - 3)$ and hence 2 is an eigenvalue of algebraic multiplicity 2, while 3 is an eigenvalue of algebraic multiplicity 1. To evaluate their geometric multiplicity, we have to solve the system $(A - \lambda)\overline{v} = 0$ namely

$$\begin{cases} (2-\lambda)v_1 + v_2 &= 0 \\ (2-\lambda)v_2 &= 0 \\ (3-\lambda)v_3 &= 0. \end{cases}$$

If $\lambda = 2$ we find $v_2 = v_3 = 0$, while if $\lambda = 3$ we find $v_1 = v_2 = 0$. Thus the corresponding eigenspaces are spanned by $(1, 0, 0)$ and $(0, 0, 1)$ respectively. As a consequence, the geometric multiplicity of both the eigenvalues is 1. On the other hand, if we consider the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

the characteristic polynomial is still $(\lambda - 2)^2(\lambda - 3)$ and hence, as before, the algebraic multiplicity of $2, 3$ is 2 and 1, respectively. To evaluate their geometric multiplicity we solve the system

$$\begin{cases} (2-\lambda)v_1 & = 0 \\ (2-\lambda)v_2 & = 0 \\ (3-\lambda)v_3 & = 0. \end{cases}$$

It follows that the eigenspace E_2 is 2-dimensional and spanned by $(1, 0, 0)$ and $(0, 1, 0)$, while the eigenspace E_3 is one-dimensional and spanned by $(0, 0, 1)$. Thus the geometric multiplicity of $\lambda = 2$ is 2 and that of $\lambda = 3$ is 1.

It can be shown that, in general, the geometric multiplicity is less than or equal to the algebraic multiplicity.

Let us point out that the eigenvalues of A might be complex numbers. However, since the coefficients of the characteristic polynomial of A are real, if $\alpha + i\beta$ is an eigenvalue of A so is $\alpha - i\beta$. If $\overline{u} + i\overline{v}$, with $\overline{u}, \overline{v} \in \mathbb{R}^n$, is an eigenvector corresponding to $\alpha + i\beta$, then it is easy to check that $\overline{u} - i\overline{v}$ is an eigenvector corresponding to $\lambda = \alpha - i\beta$.

The following result will be used later.

Theorem 7.1.2. *If $\overline{v}_1, \ldots, \overline{v}_j$ are eigenvectors of A corresponding to distinct real eigenvalues, $\lambda_1, \ldots, \lambda_j$, then they are linearly independent.*

Proof. The proof is based on the Induction Principle. For $j = 1$, it is trivially true. Suppose it is true for $j = k$, $k \geq 1$, that is, any k vectors with distinct eigenvalues are linearly independent. We will now show that the statement is also true for $j = k + 1$. Suppose not. Then there exist $k + 1$ linearly dependent eigenvectors $v_1, \ldots, v_{k+1}$, with corresponding distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$. Therefore, there exist constants $c_1, \ldots, c_{k+1}$, not all zero, such that

$$c_1\overline{v}_1 + c_2\overline{v}_2 + \ldots + c_{k+1}\overline{v}_{k+1} = 0. \tag{7.2}$$

Multiplying this equation by A we obtain

$$\begin{aligned} c_1A\overline{v}_1 + c_2A\overline{v}_2 + \ldots + c_{k+1}A\overline{v}_{k+1} = \\ c_1\lambda_1\overline{v}_1 + c_2\lambda_2\overline{v}_2 + \ldots + c_{k+1}\lambda_{k+1}\overline{v}_{k+1} = 0. \end{aligned} \tag{7.3}$$

If we multiply equation (7.2) by λ_1 and subtract it from the second equation in (7.3), we obtain

$$c_2(\lambda_2 - \lambda_1)\overline{v}_2 + \ldots + c_{k+1}(\lambda_{k+1} - \lambda_1)\overline{v}_{k+1} = 0.$$

But this is a linear combination of k eigenvectors $\overline{v}_2, \ldots, \overline{v}_{k+1}$ with distinct eigenvalues $\lambda_2, \ldots, \lambda_{k+1}$. Therefore, by induction hypothesis, we must have $c_2 = \ldots = c_{k+1} = 0$. This changes (7.2) to $c_1\overline{v}_1 = 0$, which implies that $c_1 = 0$ (recall that an eigenvector is nonzero by definition). This contradiction completes the proof. ∎

7.1.5 The Jordan normal form

If A is a nonsingular matrix, there exist two nonsingular matrices J and B such that $A = B^{-1}JB$, or equivalently $BA = JB$. J is called the *Jordan normal form* (or simply *Jordan matrix*) of A. The Jordan matrix J is triangular (but not necessarily diagonal).

If λ^* is a real eigenvalue of A the Jordan block relative to λ^* is the $p \times p$ matrix

$$J = \begin{pmatrix} \lambda^* & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & & 0 \\ \vdots & & & & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & 0 & \lambda^* \end{pmatrix}$$

where all the entries are zero, except the entries $a_{m,m+1}$ above the diagonal $a_{m,m}$ which are 1. Its characteristic polynomial is $(\lambda - \lambda^*)^p$ and hence J has a unique eigenvalue λ^* with algebraic multiplicity p. Moreover, solving the system $(J - \lambda^* I)\overline{v} = 0$ we find that the corresponding eigenspace is one-dimensional and spanned by $(1, 0, \dots, 0)$ so that the geometric multiplicity of λ^* (with respect to J) is 1.

If the eigenvalues of A are real, it is possible to show that the Jordan normal form associated with A is the $n \times n$ matrix

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_h \end{pmatrix}$$

where the sub-matrices $J_1, \dots, J_h$ are Jordan blocks relative to the eigenvalues.

As we will see later on, Jordan matrices are useful when we deal with linear systems $\overline{x}' = A(\overline{x})$.

Let us show what happens if $n = 2$, which is the case we will deal with in the sequel. Let A be a 2×2 matrix with eigenvalues λ_1, λ_2. Then the Jordan matrix is as follows.

1. If λ_1, λ_2 are real and distinct, then their algebraic and geometric multiplicity is 1 and hence
$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{J1}$$

2. If $\lambda_1 = \lambda_2$ is real, then its algebraic multiplicity is 2. Either its geometric multiplicity is also 2, a case where
$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \tag{J2.1}$$
or its geometric multiplicity is 1, a case where
$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \tag{J2.2}$$

Furthermore, if the eigenvalues are complex conjugate, $\lambda_{1,2} = \alpha \pm i\beta$, then one can show that

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{J3}$$

Moreover, let $\overline{v} \neq \bar{0}$ be an eigenvector of A corresponding to the eigenvalue λ, namely such that $A\overline{v} = \lambda\overline{v}$. Using the Jordan normal form, we find $B^{-1}JB\overline{v} = \lambda\overline{v}$ whence $JB\overline{v} = \lambda B\overline{v}$. In other words,

$\overline{v}$ is an eigenvector of A if and only if $B\overline{v}$ is an eigenvector of J.

7.2 First order systems

Consider the system of first order differential equations

$$x_i' = f_i(t, x_1, x_2, \ldots, x_n), \quad i = 1, 2, \ldots, n. \tag{7.4}$$

By a solution of such a system we mean n functions $y_1(t), y_2(t), \ldots, y_n(t)$ such that $y_i'(t) = f_i(t, y_1(t), y_2(t), \ldots, y_n(t)), \quad i = 1, 2, \ldots, n$. The corresponding initial value problem can be expressed as

$$x_i' = f_i(t, x_1, x_2, \ldots, x_n), \quad x_i(t_0) = x_{i0}, \quad i = 1, 2, \ldots, n$$

where t_0 is some point in the domain being considered.

Systems of differential equations arise in many areas such as Chemistry, Biology, Physics and Engineering. Examples arising in Population Dynamics are discussed in Chapter 8. In what follows we will study some important systems that are significant both theoretically and practically and we will develop methods of solving certain types of systems.

However, some systems can be solved by simply rewriting them and then using known methods to deal with them. We start with a couple of such systems.

Example 7.2.1. Solve the system

$$\begin{cases} x' &= 3x + y \\ y' &= -2x. \end{cases}$$

Taking the derivative of the first equation, we have $x'' - 3x' - y' = 0$ and then substituting $-2x$ for y', we obtain

$$x'' - 3x' + 2x = 0.$$

The characteristic equation $m^2 - 3m + 2 = 0$ has roots $m = 1, 2$ and hence the general solution for x is $x = c_1e^t + c_2e^{2t}$. Therefore, $y = x' - 3x = c_1e^t + 2c_2e^{2t} - 3(c_1e^t + c_2e^{2t}) = -2c_1e^t + -c_2e^{2t}$. Thus $x = c_1e^t + c_2e^{2t}$ and $y = -2c_1e^t + -c_2e^{2t}$ solve the given system. ∎

Example 7.2.2. Solve the initial value problem

$$\begin{cases} x' &= y, \quad x(0) = 1 \\ y' &= x^2, \quad y(0) = 2. \end{cases}$$

We note that $x'' = y' = x^2$. We recall that we can solve the equation $x'' = x^2$ by letting $v = x'$; and using the Chain Rule to get $x'' = v\frac{dv}{dx}$, which results is a first order equation $v\frac{dv}{dx} = x^2$. Solving this first order equation for v and then integrating v, and determining the constants from the initial values, we obtain

$$x(t) = \left(\frac{-1}{2}\sqrt{\frac{2}{3}}t + 1\right)^{-2}. \qquad \blacksquare$$

As we have seen in Chapter 4, any system of the form

$$x = x_1,\ x_1' = x_2,\ x_2' = x_3,\ \ldots, x_{n-1}' = x_n,\ x_n' = f(t, x_1, x_2, \ldots, x_n)$$

can be written as a single equation $x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$. Conversely, any equation of the form $x^{(n)} = f(t, x, x', \ldots, x^{(n-1)})$ can be written as a system of first order equations as follows: Let $x = x_1$. Then we can write the system as $x_1' = x_2,\ x_2' = x_3,\ \ldots, x_{n-1}' = x_n,\ x_n' = f(t, x, x', \ldots, x^{(n-1)}) = f(t, x_1, x_2, \ldots, x_n)$.

Example 7.2.3. Write the following initial value problem as a system.

$$x''' + 2x'' - (x')^3 + x = t^2 + 1,\ \ x(0) = 0,\ x'(0) = 1,\ x''(0) = 1.$$

Let $x = x_1,\ x' = x_2,\ x'' = x_3$. Then we have the system

$$\begin{cases} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -2x_3 + x_2^3 - x_1 + t^2 + 1 \end{cases}$$

subject to the initial conditions $x_1(0) = 0,\ x_2(0) = 1,\ x_3(0) = 1.$ ∎

We have already seen that single higher order equations can be written as first order systems, also higher order systems generally may be written as first order systems. But the resulting system is normally more complicated. We demonstrate this in the following example.

Example 7.2.4. Write the system

$$\begin{cases} x''' + x' = t \\ y'' - y^2 = 1 \end{cases}$$

as a first order system.

Let $x = x_1$, $x_1' = x_2$, $x_2' = x_3$, and $y = y_1$, $y_1' = y_2$. Then we can write the system

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ x_3' + x_2 = t \\ y_1' = y_2, \\ y_2' - y_1^2 = 1. \end{cases}$$

■

7.3 Linear first order systems

The following is the general form of a first order linear system:

$$\begin{cases} x_1' = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + f_1(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + f_n(t). \end{cases} \tag{7.5}$$

The functions $f_i(t)$, $1 \le i \le n$, are called the *forcing functions*. When there are no forcing functions in the system, i.e. $f_i(t) \equiv 0$, $1 \le i \le n$, the system is called *homogeneous*, otherwise it is called *nonhomogeneous*.

The next theorem and some of the concepts developed here are fairly similar to the high order linear homogeneous differential equations.

Theorem 7.3.1. *Suppose that a_{ij}, $1 \le i, j \le n$, and f_i, $1 \le i \le n$, are continuous in an interval I. If $t_0 \in I$, then for any numbers $x_{10}, x_{20}, \ldots, x_{n0}$, there is exactly one solution $x_1, x_2, \ldots, x_n$ of* (7.4) *satisfying the initial condition $x_1(t_0) = x_{10}$, $x_2(t_0) = x_{20}, \ldots, x_n(t_0) = x_{n0}$. Furthermore, this solution is defined everywhere in I.*

Proof. It follows immediately from Theorems 4.2.2 and 4.2.3 of Chapter 4. ■

For convenience and efficiency, we write the system (7.5) in terms of matrices and vectors. In particular we let

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \overline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \bar{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

Then the system (7.5) can be written in the equivalent form

$$\overline{x}'(t) = A(t)\overline{x}(t) + \bar{f}(t) \tag{7.6}$$

with the corresponding homogeneous system

$$\overline{x}'(t) = A(t)\overline{x}(t). \tag{7.7}$$

Example 7.3.2. The system

$$\begin{cases} x_1'(t) = 2tx_1(t) - e^t x_2(t) + 2tx_3(t) + \sin(t) \\ x_2'(t) = x_1(t) - 3x_2(t) + 5t^2 x_2(t) + \cos(t) \\ x_3'(t) = 4x_1(t) - 5tx_2(t) + 5t^3 x_3(t) + e^t \end{cases}$$

can be written as

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix} = \begin{pmatrix} 2t & -e^t & 2t \\ 1 & -3 & 5t^2 \\ 4 & -5t & 5t^3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} \sin t \\ \cos t \\ e^t \end{pmatrix}.$$ ■

Theorem 7.3.1 can be stated in matrix form as:

Theorem 7.3.3. *If $A(t)$ and $\bar{f}(t)$ are continuous in an interval I, then there exists exactly one solution $\overline{x}(t)$ of* (7.6) *satisfying the initial condition $\overline{x}(t_0) = \overline{x}_0$, where*

$$\overline{x}_0 = \begin{pmatrix} x_{10} \\ \vdots \\ x_{n0} \end{pmatrix}$$

is any vector with n components, consisting of arbitrary real numbers. Furthermore, $\overline{x}(t)$ is defined everywhere in I.

Theorem 7.3.4. *If $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ are solutions of* (7.7)*, then any linear combination $c_1\overline{x}_1 + c_2\overline{x}_2 + \ldots + c_n\overline{x}_n$ of these solutions is also a solution of* (7.7).

Proof. It suffices to prove it for $n = 2$; the general case will then easily follow from the Principle of Mathematical Induction. Let $\overline{x}_1$, $\overline{x}_2$ be two solutions of (7.7). Then $(c_1\overline{x}_1 + c_2\overline{x}_2)' = c_1\overline{x}_1' + c_2\overline{x}_2' = c_1(A(t)\overline{x}_1) + c_2(A(t)\overline{x}_2) = A(t)(c_1\overline{x}_1 + c_2\overline{x}_2)$. This shows that $c_1\overline{x}_1 + c_2\overline{x}_2$ is a solution of (7.7). ■

Definition 7.3.5. Vectors $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ are said to be linearly independent if for any constants $c_1, c_2, \ldots, c_n$, $c_1\overline{x}_1 + c_2\overline{x}_2 + \ldots + c_n\overline{x}_n = 0$ implies $c_1 = c_2 = \ldots = c_n = 0$.

If they are not linearly independent, then they are called linearly dependent.

Example 7.3.6. Check the vectors

$$\overline{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \overline{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \overline{x}_3 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

for linear independence.

The idea is similar to what we did in the case of a single higher order equation. We indicate two ways to solve this problem.

1. $c_1\overline{x}_1 + c_2\overline{x}_2 + c_3\overline{x}_2 = \bar{0}$ is equivalent to the system

$$\begin{cases} c_1 + 2c_2 + 4c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0. \end{cases}$$

We solve the system for c_1, c_2, c_3 by subtracting the last equation from the first and obtain $c_2 + c_3 = 0$, which is the same as the second equation and has infinitely many solutions. For each such pair of solutions, we can solve for c_1. For example, let $c_2 = 1$. Then, $c_3 = -1$. Substituting these values of c_2 and c_3 in the first equation, we have $c_1 + 2 - 4 = 0$, or $c_1 = 2$. Therefore, $2\overline{x}_1 + \overline{x}_2 - \overline{x}_3 = 0$. This shows that $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ are linearly dependent.

2. Instead of solving for c_1, c_2, c_3 in the above system, we simply evaluate the determinant of their coefficients

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix}.$$

Multiplying the first row by -1 and adding it to the last row we get

$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{vmatrix} = -1 + 1 = 0.$$

Therefore the above system has nontrivial solutions in c_1, c_2, c_3, which implies that the vectors $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ are linearly dependent. ∎

7.3.1 Wronskian and linear independence

Consider the linear homogeneous scalar differential equation

$$x'''(t) + a_1(t)x''(t) + a_3x(t) = 0.$$

We defined the Wronskian of solutions x, y, z of this equation as

$$W(x, y, z) = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

We also explained above how we can write the scalar differential equation $x'''(t) + a_1(t)x''(t) + a_3x = 0$ as the system $x_1 = x, x_1' = x_2, x_2' = x_3$. $y'''(t) + a_1(t)y''(t) + a_3y = 0$ can be written as the system $y_1 = y, y_1' = y_2, y_2' = y_3$. Similarly, if y and z are solutions, we let $y_1 = y$, $y_1' = y_2$, $y_2' = y_3$ and $z_1 = z$, $z_1' = z_2$, $z_2' = z_3$. This suggests that the definition of Wronskian may be extended to three vector functions as follows:

$$W(\overline{x}, \overline{y}, \overline{z}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

where

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \overline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \ \overline{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

So, we extend the definition of Wronskian to vector functions and define the Wronskian of n vector functions

$$\overline{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ \overline{x}_2 = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \ \cdots, \overline{x}_n = \begin{pmatrix} x_{1n}(t) \\ x_{2n}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix} \tag{7.8}$$

as

$$W(\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n)(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \vdots & \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{vmatrix} \tag{7.9}$$

in which the i-th column is the vector $\overline{x}_i(t)$.

Theorem 7.3.7. *Vector functions $\overline{x}_1(t), \overline{x}_2(t), \cdots, \overline{x}_n(t)$ are linearly independent if their Wronskian is nonzero at some point t_0.*

Proof. Suppose $c_1\overline{x}_1(t) + c_2\overline{x}_2(t) + \ldots + c_n\overline{x}_n(t) = 0$, where $\overline{x}_i, \ 1 \le i \le n$, are denoted as in (7.8). Then, at t_0, this is equivalent to the system

$$\begin{cases} c_1x_{11}(t_0) + c_2x_{12}(t_0) + \ldots + c_nx_{1n}(t_0) = 0 \\ c_1x_{21}(t_0) + c_2x_{22}(t_0) + \ldots + c_nx_{2n}(t_0) = 0 \\ \vdots \\ c_1x_{n1}(t_0) + c_2x_{n2}(t_0) + \ldots + c_nx_{nn}(t_0) = 0. \end{cases}$$

This algebraic system of equations has no nontrivial solution in $c_1, \ c_2, \ \ldots, c_n$ if the coefficient determinant is nonzero. But the coefficient determinant is precisely the Wronskian of $\overline{x}_1, \ \overline{x}_2, \ldots, \ \overline{x}_n$ at $t = t_0$. Therefore, $c_1 = c_2 = \ldots = c_n = 0$ and the proof is complete. ■

Theorem 7.3.8. *Suppose that $\overline{x}_1, \ \overline{x}_2, \ \ldots, \overline{x}_n$ are solutions of*

$$\overline{x}' = A(t)\overline{x}$$

where $A(t)$ is an $n \times n$ matrix, continuous on an interval I. Then their Wronskian $W(t)$ is given by

$$W(t) = W(t_0)e^{\int_{t_0}^{t}(a_{11}(s)+a_{22}(s)+\ldots+a_{nn}(s))ds}$$

where $a_{11}, a_{22}, \ldots a_{nn}$ are the diagonal elements of $A(t)$.

Proof. We give the proof for $n = 2$. The proof for the general case follows exactly the same way, but the notations become cumbersome. Suppose that

$$A(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; \overline{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}, \; \overline{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

The Wronskian of $\overline{x}_1, \overline{x}_2$ is given by

$$W(t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}.$$

Then

$$W'(t) = \begin{vmatrix} x'_{11}(t) & x'_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix} + \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x'_{21}(t) & x'_{22}(t) \end{vmatrix}.$$

Since $\overline{x}'_1 = A(t)\overline{x}$, it follows that $x'_{11} = a_{11}x_{11} + x_{21}a_{12}$, $x'_{21} = a_{21}x_{11} + a_{22}x_{21}$, $x'_{12} = a_{11}x_{12} + a_{12}x_{22}$, $x'_{22} = a_{21}x_{12} + a_{22}x_{22}$. Making these substitutions in the above equation for $W'(t)$, we obtain

$$W'(t) = \begin{vmatrix} a_{11}x_{11} + a_{12}x_{21} & a_{11}x_{12} + a_{12}x_{22} \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} \\ a_{21}x_{11} + a_{22}x_{21} & a_{21}x_{12} + a_{22}x_{22} \end{vmatrix}.$$

Now, we multiply the second row of the first determinant by $-a_{12}$ and add it to the first row. We also multiply the first row of the second determinant by $-a_{21}$ and add it to the second row. Then we obtain

$$W'(t) = \begin{vmatrix} a_{11}x_{11} & a_{11}x_{12} \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{12} \\ a_{22}x_{21} & a_{22}x_{22} \end{vmatrix} = a_{11}W + a_{22}W = (a_{11} + a_{22})W.$$

Integrating $W' = (a_{11} + a_{22})W$ from t_0 to t proves the theorem. ■

Corollary 7.3.9. *The Wronskian of n solutions of* (7.7) *is either always zero or never zero.*

The sum of the diagonal elements of a matrix is called the *trace* of the matrix and denoted by $tr(A)$. With this notation we can write

$$W(t) = W(t_0)e^{\int_{t_0}^{t} tr(A(s))ds}.$$

Example 7.3.10. Consider the scalar differential equation

$$x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = 0.$$

Recall that if $x_1, x_2, \ldots, x_n$ are solutions of this differential equation, then their Wronskian is given by $W(t) = ce^{\int -a_1(s)ds}$, or in terms of definite integral, $W(t) = W(t_0)e^{\int_{t_0}^{t} -a_1(s)ds}$. As indicated above, we can convert this equation to a system by letting $x_1 = x,\ x_1' = x_2,\ \ldots, x_{n-1}' = x_n,\ x_n' = -a_1x_n - a_2x_{n-1} - \ldots - a_nx_1$. This system can be written in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & a_{n-3} & \cdots & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

As we can see, the trace of the matrix is $-a_1(t)$. So, applying Theorem 7.3.8, we obtain

$$W(t) = W(t_0)e^{\int_{t_0}^{t} -a_1(s)ds}$$

which agrees with what we found by the method of scalar equations. ■

Theorem 7.3.11. *Let $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ be linearly independent solutions of* (7.7)*, on an interval I where $A(t)$ is continuous. Then the general solution of* (7.7) *is given by $\overline{x}(t) = c_1\overline{x}_1 + c_2\overline{x}_2 + \ldots + c_n\overline{x}_n$, $c_i \in \mathbb{R}$.*

Proof. By Theorem 7.3.4 we know that $\overline{x} = \sum c_i\overline{x}_i$ is a solution of (7.7). We have to show that given any solution $\overline{y}$ of (7.7), there then exist constants $c_1, c_2, \ldots, c_n$ such that $\overline{y} = c_1\overline{x}_1 + \ldots + c_n\overline{x}_n$. For this, let

$$\overline{x}_i(t) = \begin{pmatrix} x_{1i}(t) \\ x_{2i}(t) \\ \vdots \\ x_{ni}(t) \end{pmatrix}, \quad i = 1, 2, \ldots, n, \quad \text{and } \overline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

and let t_0 be any number in I. Then $c_1\overline{x}_1(t_0) + c_2\overline{x}_2(t_0) + \ldots + c_n\overline{x}_n(t_0) = \overline{y}(t_0)$ is equivalent to the system

$$\begin{cases} c_1x_{11}(t_0) + c_2x_{12}(t_0) + \ldots + c_nx_{1n}(t_0) = y_1(t_0) \\ c_1x_{21}(t_0) + c_2x_{22}(t_0) + \ldots + c_nx_{2n}(t_0) = y_2(t_0) \\ \vdots \\ c_1x_{n1}(t_0) + c_2x_{n2}(t_0) + \ldots + c_nx_{nn}(t_0) = y_n(t_0). \end{cases}$$

This system will have a unique solution in $c_1, c_2, \ldots, c_n$ if the coefficient determinant

$$\begin{vmatrix} x_{11}(t_0) & x_{12}(t_0) & \cdots & x_{1n}(t_0) \\ x_{21}(t_0) & x_{22}(t_0) & \cdots & x_{2n}(t_0) \\ \vdots & \vdots & \vdots & \\ x_{n1}(t_0) & x_{n2}(t_0) & \cdots & x_{nn}(t_0) \end{vmatrix}$$

is nonzero. But this determinant is precisely the Wronskian of $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$, which is nonzero by assumption. Now it follows from the uniqueness theorem that $c_1\overline{x}_1(t) + c_2\overline{x}(t) + \ldots + c_n\overline{x}_n(t) = \overline{y}(t)$ for all t in I. ∎

7.4 Constant systems – eigenvalues and eigenvectors

In this section we consider the homogeneous system

$$\overline{x}' = A\overline{x} \tag{7.10}$$

where $A = (a_{ij})$ is a constant matrix, that is the entries a_{ij} , $1 \leq i, j \leq n$, are constants, and it is nonsingular. We recall that in the case of homogeneous scalar equations with constant coefficients, we were able to find the general solution by substituting e^{mt} for the dependent variable. This suggests that we try substituting $\overline{x} = e^{\lambda t}\overline{v}$ in (7.10). Doing so, we obtain $\overline{x}' = \lambda e^{\lambda t}\overline{v} = Ae^{\lambda t}\overline{v}$, which gives rise to the equation $A\overline{v} = \lambda\overline{v}$. The last equation may be written as

$$(A - \lambda I)\overline{v} = \bar{0} \tag{7.11}$$

where I is the $n \times n$ identity matrix and $\bar{0} \in \mathbb{R}^n$ is the zero vector. It is now clear that $\overline{x} = e^{\lambda t}\overline{v}$ will be a solution of (7.10) if λ and $\overline{v}$ satisfy equation (7.11). Using the notation introduced in Section 7.1, this means that λ is an eigenvalue of the matrix A and $\overline{v} \neq \bar{0}$ is an eigenvector associated with λ. We have also seen that we must have that λ is a solution of the characteristic polynomial

$$det(A - \lambda I) = 0. \tag{7.12}$$

Example 7.4.1. Let

$$A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}.$$

(a) Find the characteristic equation of A.
(b) Find the eigenvalues and the corresponding eigenvectors of A.
(c) Find the solutions of $\overline{x}' = A\overline{x}$ corresponding to each eigenvalue.
(d) Show that the solutions in (c) are linearly independent.
(e) Find solution $\overline{y}(t)$ satisfying the initial condition

$$\overline{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution. (a)

$$det\,(A-\lambda I)=\begin{vmatrix}1-\lambda & -3 & 0\\ 0 & 2-\lambda & 0\\ 1 & 1 & 3-\lambda\end{vmatrix}=(1-\lambda)(2-\lambda)(3-\lambda).$$

Therefore, the characteristic equation is $(1-\lambda)(2-\lambda)(3-\lambda)=0$.

(b) The eigenvalues of A are $\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$. To find an eigenvector $\overline{v}_1$ corresponding to $\lambda_1=1$, we substitute $\lambda_1=1$ in (7.11) and solve for $\overline{v}=\overline{v}_1$. If the components of $\overline{v}_1$ are x, y, z, then

$$(A-\lambda I)\overline{v}_1=\begin{pmatrix}0 & -3 & 0\\ 0 & 1 & 0\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

We obtain $y=0$ and $z=-\frac{1}{2}x$. Thus,

$$\overline{v}_1=\begin{pmatrix}x\\ 0\\ -\frac{1}{2}x\end{pmatrix}.$$

We can take x to be any nonzero number. So, let $x=1$. As mentioned before, any $\alpha\overline{v}$, $\alpha\in\mathbb{R}$, $\alpha\neq 0$, is also an eigenvector. Hence, taking $\alpha=-2$ we have

$$\overline{v}_1=\begin{pmatrix}-2\\ 0\\ 1\end{pmatrix}.$$

Similarly, substituting $\lambda=2$ and $\lambda=3$ in (7.11) and solving for $\overline{v}_2$ and $\overline{v}_3$, respectively, we obtain

$$\overline{v}_2=\begin{pmatrix}3\\ -1\\ -2\end{pmatrix},\quad \overline{v}_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Once more, we point out that there are many ways, in fact infinitely many ways, to choose an eigenvector. We should try to choose options that seem convenient.

(c) The corresponding solutions are

$$\overline{x}_1=\begin{pmatrix}-2\\ 0\\ 1\end{pmatrix}e^{t},\ \ \overline{x}_2=\begin{pmatrix}3\\ -1\\ -2\end{pmatrix}e^{2t},\ \ \overline{x}_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}e^{3t}.$$

(d) In order to show that these solutions are linearly independent, we show that their Wronskian

$$W(t)=\begin{vmatrix}-2e^{t} & 3e^{2t} & 0\\ 0 & -e^{2t} & 0\\ e^{t} & -2e^{2t} & e^{3t}\end{vmatrix}$$

is nonzero. Expanding the determinant above along the third column, we see that $W(t) = e^{3t} \cdot (2e^{3t}) = 2e^{6t}$, which is never zero for any t.

We point out that in general we only need to show that the Wronskian is nonzero at some convenient point. It will then follow from Corollary 7.3.9 that it is always nonzero.

(e) By part (d), $y(t) = c_1\overline{x}_1(t)+c_2\overline{x}_2(t)+c_3\overline{x}_3(t)$ is the general solution. Therefore, there exist constants c_1, c_2, c_3 such that $c_1\overline{x}_1(0) + c_2\overline{x}_2(0) + c_3\overline{x}_3(0)$ satisfies the given initial condition. We find these constants by solving the system

$$c_1\begin{pmatrix}-2\\0\\1\end{pmatrix}+c_2\begin{pmatrix}3\\-1\\-2\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$$

which is equivalent to the system

$$\begin{cases} -2c_1+3c_2 &= 1\\ -c_2 &= 0\\ c_1-2c_2+c_3 &= 1.\end{cases}$$

We see that $c_1 = -\frac{1}{2},\ c_2 = 0,\ c_3 = \frac{3}{2}$. Therefore, the desired solution is

$$-\frac{1}{2}e^t\begin{pmatrix}-2\\0\\1\end{pmatrix}+\frac{3}{2}e^{3t}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}e^t\\0\\-\frac{1}{2}e^t\end{pmatrix}+\begin{pmatrix}0\\0\\\frac{3}{2}e^{3t}\end{pmatrix}=\begin{pmatrix}e^t\\0\\-\frac{1}{2}e^t+\frac{3}{2}e^{3t}\end{pmatrix}. \qquad \blacksquare$$

We first deal with the case when the eigenvalues of A are real and distinct.

Theorem 7.4.2. *If $\overline{v}_1, \dots, \overline{v}_n \in \mathbb{R}^n$ are n eigenvectors of the $n \times n$ matrix A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are real and distinct, then $\overline{x}(t) = c_1\overline{v}_1e^{\lambda_1 t} + \dots + c_n\overline{v}_ne^{\lambda_n t}$ is the general solution of* (7.10).

Proof. To any $\lambda_i \in \mathbb{R}$ and $\overline{v}_i \in \mathbb{R}^n$, $i = 1, \dots, n$, we can associate a function $\overline{x}_i(t) = \overline{v}_ie^{\lambda_i t}$ which is a solution of $\overline{x}' = A\overline{x}$. According to Theorem 7.3.11, $\overline{x}(t)$ is the general solution provided $\overline{x}_i$ are linearly independent. Since $\overline{x}_i(0) = \overline{v}_i$, then their Wronskian at $t = 0$ is the determinant of the matrix whose columns are the vectors $\overline{v}_1, \dots, \overline{v}_n$. Thus $\overline{x}_i$ are linearly independent if and only if $\overline{v}_i$ are so. On the other hand, by Theorem 7.1.2 proved in Section 7.1, $\overline{x}_i$ are linearly independent provided $\lambda_1, \dots, \lambda_n$ are distinct. This completes the proof. ∎

Notice that part (d) of Example 7.4.1 is an immediate consequence of this Theorem.

In Theorem 7.4.2 it was shown that if the eigenvalues of A are distinct (hence simple), then the n corresponding eigenvectors are linearly independent. It follows from the proof of this theorem that any k distinct eigenvalues, $1 \le k \le n$, give rise to k linearly independent eigenvectors. The situation for repeated eigenvalues is not so simple. For example, a double eigenvalue may or may not yield two linearly independent eigenvectors. In other words, if A has some repeated eigenvalues, it may or may not have n linearly independent eigenvectors. One class of matrices that have the property that to each eigenvalue that is repeated k times, there correspond k linearly independent eigenvectors, is the class of symmetric matrices. We give below some examples that illustrate both possibilities. The student can easily understand how to handle other similar cases.

Example 7.4.3. Find the general solution of $\overline{x}' = A\overline{x}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = -1$ (double). Moreover,

$$\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

are 3 linearly independent eigenvectors.

Then $\overline{x}_1 = \overline{v}_1 e^t$, $\overline{x}_2 = \overline{v}_2 e^{-t}$ and $\overline{x}_3 = \overline{v}_3 e^{-t}$ solve $\overline{x}' = A\overline{x}$ and are linearly independent because $\overline{x}_i(0) = \overline{v}_i$, $i = 1, 2, 3$, are so. Thus the general solution is $\overline{x} = c_1\overline{v}_1 e^t + c_2\overline{v}_2 e^{-t} + c_3\overline{v}_3 e^{-t}$.

As an exercise, the student can find the same result noticing that the components x_1, x_2, x_3 of $\overline{x}$ satisfy the uncoupled system

$$\begin{cases} x_1' &= x_1, \\ x_2' &= -x_2, \\ x_3' &= -x_3. \end{cases}$$ ■

Example 7.4.4. Find the general solution of $\overline{x}' = A\overline{x}$, where

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

Since

$$det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda),$$

the eigenvalues of A are $\lambda = 3$ (double) and $\lambda = 2$.

It is easy to check that A has 3 eigenvectors given by

$$\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \overline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \overline{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which are linearly independent because

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1.$$

Then $\overline{x}_1 = \overline{v}_1 e^{3t}$, $\overline{x}_2 = \overline{v}_2 e^{3t}$ and $\overline{x}_3 = \overline{v}_3 e^{2t}$ solve $\overline{x}' = A\overline{x}$ and are linearly independent because $\overline{x}_i(0) = \overline{v}_i$, $i = 1, 2, 3$, are so.

Thus the general solution is $\overline{x} = c_1\overline{v}_1 e^{3t} + c_2\overline{v}_2 e^{3t} + c_3\overline{v}_3 e^{2t}$. ■

Example 7.4.5. Find the general solution of $\overline{x}' = A\overline{x}$ where

$$A = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix},$$

with $\lambda, a \neq 0$. Now, λ is a double eigenvalue but the eigenspace corresponding to λ is one-dimensional and spanned by $\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, which yields $\overline{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}$, but it is not obvious how to find a second linear independent solution.

Let us take $\overline{x}_2 = \overline{v}_1 t e^{\lambda t} + \overline{u} e^{\lambda t}$ and determine $\overline{u}$ such that $\overline{x}_1, \overline{x}_2$ are linearly independent and $\overline{x}_2' = A\overline{x}_2$. As before, from $\overline{x}_1(0) = \overline{v}_1, \overline{x}_2(0) = \overline{u}$, it follows that for $\overline{x}_1, \overline{x}_2$ to be linearly independent it suffices that $\overline{v}_1, \overline{u}$ are so.

On the other hand, the equation $\overline{x}_2' = A\overline{x}_2$ is equivalent to

$$\lambda t e^{\lambda t}\overline{v}_1 + e^{\lambda t}\overline{v}_1 + \lambda e^{\lambda t}\overline{u} = A(te^{\lambda t}\overline{v}_1 + e^{\lambda t}\overline{u}) = A\overline{v}_1\, te^{\lambda t} + A\overline{u}\, e^{\lambda t}.$$

Since $A\overline{v}_1 = \lambda\overline{v}_1$, we have

$$\lambda t e^{\lambda t}\overline{v}_1 + e^{\lambda t}\overline{v}_1 + \lambda e^{\lambda t}\overline{u} = \lambda\overline{v}_1\, te^{\lambda t} + A\overline{u}\, e^{\lambda t}.$$

Canceling $\lambda t e^{\lambda t}\overline{v}_1$, we obtain $e^{\lambda t}\overline{v}_1 + \lambda e^{\lambda t}\overline{u} = A\overline{u}\, e^{\lambda t}$ and hence

$$\overline{v}_1 + \lambda\overline{u} = A\overline{u}.$$

This can be written as

$$(A - \lambda I)\overline{u} = \overline{v}_1$$

namely

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad i.e. \quad \begin{cases} au_2 & = 1, \\ 0 & = 0. \end{cases}$$

Thus $\overline{u} = \begin{pmatrix} 0 \\ 1/a \end{pmatrix}$, which is obviously linearly independent from $\overline{v}_1$. In conclusion the general solution of $\overline{x}' = A\overline{x}$ is

$$\begin{aligned} \overline{x} &= c_1\overline{x}_1 + c_2\overline{x}_2 = c_1\overline{v}_1 e^{\lambda t} + c_2\left[\overline{v}_1 te^{\lambda t} + \overline{u}e^{\lambda t}\right] \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\left[c_1 e^{\lambda t} + c_2 te^{\lambda t}\right] + c_2\begin{pmatrix} 0 \\ 1/a \end{pmatrix} e^{\lambda t}, \end{aligned}$$

that is,

$$\begin{cases} x_1 &= c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \\ x_2 &= \frac{c_2}{a} e^{\lambda t}. \end{cases}$$

The reader will recognize that the preceding procedure is similar to that carried out for the scalar second order equation $x'' - 2\lambda x' + \lambda^2 x = 0$ whose characteristic equation has the double root $m = \lambda$. ■

Our last example deals with the case in which A has complex eigenvalues.

Example 7.4.6. Find the general solution of $\overline{x}' = A\overline{x}$, where

$$A = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}.$$

The characteristic equation is given by

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 + 4 = \lambda^2 - 2\lambda + 5 = 0.$$

Solving the quadratic equation $\lambda^2 - 2\lambda + 5 = 0$, we find the eigenvalues to be $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Let

$$\overline{v} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

In order to find the eigenvector $\overline{v}$ corresponding to $1 + 2i$, we set $A\overline{v} = (1 + 2i)\overline{v}$, which is equivalent to the system

$$\begin{cases} x + 4y = (1 + 2i)x, \\ -x + y = (1 + 2i)y. \end{cases}$$

Simplifying the equations in this system, we obtain $2y = ix$ and $-x = 2iy$. We note that if we multiply the first equation by i, we get the second equation. Therefore, we actually have only one equation and two unknowns. This means that we can assign an arbitrary value to one of the unknowns and then solve for the other. To this end, let $x = 2$. Then $y = i$. This means that the solution corresponding to the eigenvalue $1 + 2i$ is

$$\overline{x} = \begin{pmatrix} 2 \\ i \end{pmatrix} e^{(1+2i)t}.$$

Now, using Euler's formula $e^{(1+2i)t} = e^t \cos 2t + i e^t \sin 2t$, we extract real solutions:

$$\overline{x} = e^{(1+2i)t} \begin{pmatrix} 2 \\ i \end{pmatrix} = \begin{pmatrix} 2(e^t \cos 2t + i e^t \sin 2t) \\ i(e^t \cos 2t + i e^t \sin 2t) \end{pmatrix} = \begin{pmatrix} 2e^t \cos 2t + i(2e^t \sin 2t) \\ -e^t \sin 2t + i(e^t \cos 2t) \end{pmatrix}$$
$$= \begin{pmatrix} 2e^t \cos 2t \\ -e^t \sin 2t \end{pmatrix} + i \begin{pmatrix} 2e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}.$$

Now we can take the two real solutions to be

$$\overline{x}_1 = \begin{pmatrix} 2e^t \cos 2t \\ -e^t \sin 2t \end{pmatrix}, \quad \overline{x}_2 = \begin{pmatrix} 2e^t \sin 2t \\ e^t \cos 2t \end{pmatrix}.$$

Evaluating their Wronskian, we see that

$$W(t) = \begin{vmatrix} 2e^t \cos 2t & 2e^t \sin 2t \\ -e^t \sin 2t & e^t \cos 2t \end{vmatrix} = 2e^{2t} \neq 0.$$

Therefore, $\overline{x}_1$ and $\overline{x}_2$ are linearly independent and $\overline{x} = c_1\overline{x}_1 + c_2\overline{x}_2$ is the general solution. ∎

7.5 Nonhomogeneous systems

Consider the nonhomogeneous system

$$\overline{x}' = A(t)\overline{x} + \bar{f}(t) \tag{7.13}$$

where the coefficient matrix $A(t)$ and the forcing function $\bar{f}(t)$ are continuous in an interval I. Let $\overline{x}_1, \ldots, \overline{x}_n$ be a fundamental set of solutions of the homogeneous equation

$$\overline{x}' = A(t)\overline{x} \tag{7.14}$$

and let $\overline{x}_p$ be a particular solution of (7.13). If $\overline{y}$ is any solution of (7.13), then, as in the scalar case, it is easy to see that $\overline{y} - \overline{x}_p$ is a solution of the homogeneous equation (7.14). Therefore, there exist constants $c_1, \ldots, c_n$ such that $\overline{y} - \overline{x}_p = c_1\overline{x}_1 + \ldots + c_n\overline{x}_n$; and hence $\overline{y} = \overline{x}_p + c_1\overline{x}_1 + \ldots + c_n\overline{x}_n$. We state this as

Theorem 7.5.1. *If $\overline{x}_1, \ldots, \overline{x}_n$ is a fundamental set of solutions of* (7.14) *and $\overline{x}_p$ is any particular solution of* (7.13)*, then $\overline{y} = \overline{x}_p + c_1\overline{x}_1 + \ldots + c_n\overline{x}_n$ is the general solution of* (7.13).

Once again it is important to find a particular solution of the nonhomogeneous equation. This may be accomplished by the Method of Undetermined Coefficients, which is pretty much a selective guessing scheme.

Method of undetermined coefficients. This method involves making a calculated guess for each situation. It may be used when the functions involved are familiar functions whose derivatives share some similarity with them. For example, the derivative of a polynomial is a polynomial with one degree less. Other such functions are $\sin t$, $\cos t$, and exponential functions.

Example 7.5.2. Find the general solution of the system

$$\overline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t - 2 \\ -4t \end{pmatrix}.$$

Here it seems reasonable to try

$$\overline{x}_p = \begin{pmatrix} at + b \\ ct + d \end{pmatrix}$$

and determine the constants a, b, c, d. Substituting $\overline{x}_p$ in the above system, we obtain the algebraic system

$$\begin{cases} a = (at + b) + (ct + d) + 2t - 2 \\ c = -3(at + b) + 5(ct + d) - 4t \end{cases}$$

which reduces to the system

$$\begin{cases} (a + c + 2)t + (b + d - 2 - a) = 0 \\ (3a - 5c + 4)t + (c + 3b - 5d) = 0. \end{cases}$$

In each of the above equations, we must have the coefficients of t and the constants equal to zero. Setting these equal to zero, we obtain the algebraic system

$$\begin{cases} a + c + 2 = 0 \\ 3a - 5c + 4 = 0 \\ b + d - 2 - a = 0 \\ c + 3b - 5d = 0. \end{cases}$$

Solving the first two equations, we find $a = -\frac{7}{4}, c = -\frac{1}{4}$. Substituting these values of a and c in the third and fourth equations, we find $b = \frac{3}{16}$ and $d = \frac{1}{16}$. Therefore,

$$\overline{x}_p = \begin{pmatrix} \frac{-7}{4}t + \frac{3}{16} \\ \frac{-1}{4}t + \frac{1}{16} \end{pmatrix}$$

is a particular solution of the nonhomogeneous equation.

Now, we need the general solution of the corresponding homogeneous equation $\overline{x}' = A\overline{x}$. We see that $\lambda = 2$ and $\lambda = 4$ are the roots of the characteristic polynomial

$$\begin{vmatrix} 1 - \lambda & 1 \\ -3 & 5 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 = 0.$$

The corresponding eigenvectors may be taken as

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The general solution of the nonhomogeneous system is then

$$c_1 \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{4t} \\ 3e^{4t} \end{pmatrix} + \begin{pmatrix} -\frac{7}{4}t + \frac{3}{16} \\ -\frac{1}{4}t + \frac{1}{16} \end{pmatrix} = \begin{pmatrix} c_1e^{2t} + c_2e^{4t} - \frac{7}{4}t + \frac{3}{16} \\ c_1e^{2t} + 3c_2e^{4t} - \frac{1}{4}t + \frac{1}{16} \end{pmatrix}. \quad \blacksquare$$

Variation of parameters. First of all, let us note that if $\overline{x}_1, \ldots, \overline{x}_n$ are vectors that satisfy the system $\overline{x}' = A(t)\overline{x}$, these equations can be compactly expressed as

$$X' = A(t)X,$$

where X, resp. X', is the matrix whose columns consist of the vectors $\overline{x}_i$, resp. $\overline{x}'$, i=1,...,n.

Let $A(t)$ be a 2×2 matrix and let

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \overline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

be solutions of the system $\overline{x}' = A(t)\overline{x}$. Then

$$\overline{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \overline{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

which is equivalent to

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \\ y_1' = a_{11}y_1 + a_{12}y_2 \\ y_2' = a_{21}y_1 + a_{22}y_2. \end{cases}$$

It is easy to see that this system can be expressed in terms of matrices as $X' = AX$, that is

$$\begin{pmatrix} x_1' & y_1' \\ x_2' & y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

Let $\overline{x}_1, \ldots, \overline{x}_n$ be linearly independent solutions of

$$\overline{x}' = A\overline{x}.$$

We now describe a method for finding a particular solution of the nonhomogeneous system

$$\overline{x}' = A\overline{x} + \bar{f}.$$

Since for any constants $c_1, \ldots, c_n$, $\overline{x} = c_1\overline{x}_1 + \ldots + c_n\overline{x}_n$ is a solution of this system, we are motivated, as in the scalar case, to try to find variable functions $u_1(t), \ldots, u_n(t)$ such that $\overline{y} = u_1\overline{x}_1 + \ldots + u_n\overline{x}_n$ is a solution of the nonhomogeneous system. To this end, we first write

$$\overline{y} = X\overline{u}$$

where $\overline{u}$ is the vector whose j-th component is u_j, $1 \le j \le n$, and X is the matrix described above, that is, its columns consist of the vectors $\overline{x}_i$, $1 \le i \le n$. Substituting $X\overline{u}$ in the nonhomogeneous system, we obtain

$$X\overline{u}' + X'\overline{u} = AX\overline{u} + \bar{f}.$$

Since, as explained above, $X' = AX$, the above equation is reduced to

$$X\overline{u}' = \bar{f}. \tag{7.15}$$

Since the columns of X are linearly independent, X^{-1} exists. Multiplying both sides by X^{-1}, we have

$$\overline{u}' = X^{-1}\bar{f}. \tag{7.16}$$

Integrating both sides and taking the constant of integration to be zero, we obtain

$$\overline{u} = \int X^{-1}(t)\bar{f}(t)dt \tag{7.17}$$

and hence

$$\overline{y} = X\overline{u} = X\int X^{-1}(t)\bar{f}(t)dt. \tag{7.18}$$

Example 7.4.1 revisited. Let us try to solve

$$\overline{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2t-2 \\ -4t \end{pmatrix}$$

by the method of Variation of Parameters. Recall that the general solution of the corresponding homogeneous system is

$$c_1\begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} + c_2\begin{pmatrix} e^{4t} \\ 3e^{4t} \end{pmatrix}.$$

We let

$$X = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix}.$$

Then

$$X^{-1} = \frac{1}{2}e^{-6t}\begin{pmatrix} 3e^{4t} & -e^{4t} \\ -e^{2t} & e^{2t} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}e^{-2t} & -\frac{1}{2}e^{-2t} \\ -\frac{1}{2}e^{-4t} & \frac{1}{2}e^{-4t} \end{pmatrix}.$$

Therefore,

$$X^{-1}\bar{f} = \begin{pmatrix} \frac{3}{2}e^{-2t} & -\frac{1}{2}e^{-2t} \\ -\frac{1}{2}e^{-4t} & \frac{1}{2}e^{-4t} \end{pmatrix}\begin{pmatrix} 2t-2 \\ -4t \end{pmatrix} = \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix}.$$

Using (7.17),

$$\overline{u} = \int \begin{pmatrix} 5te^{-2t} - 3e^{-2t} \\ -3te^{-4t} + e^{-4t} \end{pmatrix} dt = \begin{pmatrix} -\frac{5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix}.$$

Therefore,

$$\overline{y} = X\overline{u} = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix} \begin{pmatrix} -\frac{5}{2}te^{-2t} + \frac{1}{4}e^{-2t} \\ \frac{3}{4}te^{-4t} - \frac{1}{16}e^{-4t} \end{pmatrix} = \begin{pmatrix} -\frac{7}{4}t + \frac{3}{16} \\ -\frac{1}{4}t + \frac{1}{16} \end{pmatrix}.$$

Remark 7.5.3. First of all, in order to find $\overline{u}$ it is not necessary to calculate the inverse of the matrix X. One can simply solve the system (7.10) for $\overline{u}'$ and then integrate. In the above example, we would have the system

$$X\overline{u}' = \begin{pmatrix} e^{2t} & e^{4t} \\ e^{2t} & 3e^{4t} \end{pmatrix} \overline{u}' = \bar{f}$$

which is equivalent to the system

$$\begin{aligned} e^{2t}u_1' + e^{4t}u_2' &= 2t - 2 \\ e^{2t}u_1' + 3e^{4t}u_2' &= -4t. \end{aligned}$$

Secondly, finding the inverse of a 2×2 matrix, when it exists, is trivial. Here is the formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bd} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$ ■

7.6 Exercises

These exercises are divided in 4 parts. The first 3 deal with linear systems with constant coefficients: A) when the matrix A is a 2×2 Jordan matrix J; B) when A is a general 2×2 constant matrix; C) when A is a 3×3 constant matrix. The last set of exercises D) deals with general linear and/or nonlinear first order systems.

A1. Solve $\overline{x}' = J\overline{x}$ where $J = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$.

A2. Solve $\overline{x}' = J\overline{x}$ where $J = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, with $a \neq 0$.

A3. Solve $\overline{x}' = J\overline{x}$ where $J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, with $a \neq 0$.

A4. Solve $\overline{x}' = J\overline{x}$, $\quad J = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$.

A5. Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

A6. Solve $\overline{x}' = J\overline{x}$, where $J = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$.

A7. Solve the Cauchy problem

$$\overline{x}' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overline{x}, \qquad \overline{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A8. Solve the Cauchy problem

$$\overline{x}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\overline{x}, \qquad \overline{x}(0) = \begin{pmatrix} a \\ 0 \end{pmatrix}.$$

A9. Solve

$$\begin{cases} x' = \ \ y, & x(0) = 0 \\ y' = -x, & y(0) = 1. \end{cases}$$

A10. Solve

$$\begin{cases} x' = 3x + t \\ y' = -y + 2t. \end{cases}$$

A11. Solve

$$\begin{cases} x' = x + t^2 \\ y' = y + 1. \end{cases}$$

A12. Solve

$$\begin{cases} x' = x + y + 1 \\ y' = -x + y - 5. \end{cases}$$

B1. Show that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has 0 as an eigenvalue if and only if A is singular.

B2. Show that the symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

has two distinct real eigenvalues if $b \neq 0$.

B3. Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}$$

and write the general solution of the system $\overline{x}' = A\overline{x}$.

B4. Find the general solution of the system

$$x' = 2x + 6y$$
$$y' = x + 3y.$$

B5. Find the general solution of

$$x' = 2x + 6y + e^t$$
$$y' = x + 3y - e^t.$$

B6. Solve the initial value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

B7. Solve

$$\begin{cases} x' = x + 2y + 2t \\ y' = 3y + t^2. \end{cases}$$

B8. Solve

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}.$$

B9. Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where $A = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}$.

B10. Solve

$$\overline{x}' = A\overline{x}, \qquad A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

B11. Solve

$$\begin{cases} x' = x + 3y \\ y' = x - y. \end{cases}$$

B12. Solve the Cauchy problem

$$\begin{cases} x' = x + y, & x(0) = 1 \\ y' = x - y, & y(0) = 0. \end{cases}$$

B13. Solve the Cauchy problem

$$\begin{cases} x' = x + y, & x(0) = -1 \\ y' = -y, & y(0) = 2. \end{cases}$$

B14. Solve

$$\begin{cases} x' = x + 3y + 2t \\ y' = x - y + t^2. \end{cases}$$

B15. Solve

$$\begin{cases} x' = x + 2y + e^t \\ y' = x - 2y - e^t. \end{cases}$$

C1. Let A, B be similar $n \times n$ matrices. Show that $det\ A = det\ B$ and that they have the same eigenvalues.

C2. Solve

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 2 \end{pmatrix}.$$

C3. Solve $\overline{x}' = A\overline{x}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

C4. Find $\overline{x}$ solving the Cauchy problem

$$\overline{x}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \overline{x}, \quad \overline{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

C5. Solve $\overline{x}' = A\overline{x}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

C6. Recall that the characteristic equation of the differential equation

$$x''' - 2x'' + 3x' + x = 0$$

is $m^3 - 2m^2 + 3m + 1 = 0$. Change the differential equation to a system and then show that its characteristic equation as a system remains the same.

C7. Solve the Cauchy problem

$$\begin{cases} x' = x + z & x(0) = 0 \\ y' = -y + z & y(0) = 1 \\ z' = y - z & z(0) = 0. \end{cases}$$

C8. Find $a \in \mathbb{R}$ such that the system

$$\overline{x}' = A\overline{x}, \qquad A = \begin{pmatrix} a & 0 & 0 \\ b_1 & b_2 & 0 \\ b_4 & b_5 & b_6 \end{pmatrix}$$

has a nontrivial solution $\overline{x}(t)$ satisfying $|\overline{x}(t)| \to 0$ as $t \to +\infty$ for all $b_i \in \mathbb{R}$.

C9. If

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & -1 \end{pmatrix},$$

find a nontrivial solution of $\overline{x}' = A\overline{x}$ such that $\lim_{t \to +\infty} |\overline{x}(t)| = 0$.

C10. Find a such that all the solutions of

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} a-2 & 1 & 0 \\ -1 & a-2 & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

satisfy $\lim_{t\to+\infty} |x_i(t)| = 0$, $i = 1, 2, 3$.

D1. Solve

$$\begin{cases} x' + ty &= -1, \\ y' + x' &= 2. \end{cases}$$

D2. Solve

$$\begin{cases} x' + y &= 3t, \\ y' - tx' &= 0. \end{cases}$$

D3. Solve

$$\begin{cases} x' - ty &= 1, \\ y' - tx' &= 3. \end{cases}$$

D4. Solve

$$\begin{cases} t^2x' - y &= 1, \\ y' - 2x &= 0. \end{cases}$$

D5. Solve

$$\begin{cases} x' - y &= 3, \\ y' - 3x' &= -2x. \end{cases}$$

D6. Solve

$$\begin{cases} tx' + y' &= 1, \\ y' + x + e^{x'} &= 1. \end{cases}$$

D7. Solve

$$\begin{cases} xx' + y &= 2t, \\ y' + 2x^2 &= 1. \end{cases}$$

8

Qualitative analysis of 2 × 2 systems and nonlinear second order equations

In this chapter we study

1. Planar hamiltonian systems.
2. Lotka–Vilterra prey-predator systems.
3. Second order equations of the form $x'' = f(x)$.

We investigate the existence of periodic solutions, called *closed* trajectories, and non-periodic solutions, called *open* trajectories such as homoclinic and heteroclinic solutions, and so on. Our approach is based on phase plane analysis and geometric considerations and leads to information about the qualitative behavior of solutions without explicitly solving the equations. The common feature of the problems we address is the fact that there exists a quantity that is conserved along the solutions.

First of all let us state an important property of autonomous systems.

Lemma 8.0.1. *If $\overline{x}(t)$ is a solution of the autonomous system*

$$\overline{x}' = \overline{f}(\overline{x}),$$

then $\overline{x}(t+h)$ is also a solution, $\forall\, h \in \mathbb{R}$.

Proof. Setting $\overline{x}_h(t) := \overline{x}(t+h)$, one has $\overline{x}_h'(t) = \overline{x}'(t+h) = \overline{f}(\overline{x}(t+h)) = \overline{f}(\overline{x}_h(t))$, which means that $\overline{x}_h$ solves $\overline{x}' = \overline{f}(\overline{x})$. ■

In general, the preceding property does not hold for non-autonomous systems. For example, in the case of a single equation such as $x' = 2tx$, we have that $x(t) = e^{t^2}$ is a solution, but $x_h(t) = x(t+h) = e^{(t+h)^2}$ is not a solution for any $h \neq 0$. Actually $x_h'(t) = 2(t+h)e^{(t+h)^2} = 2te^{(t+h)^2} + he^{(t+h)^2} = 2tx_h(t) + hx_h(t)$.

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8.1 Planar hamiltonian systems

In this section we deal with an important class of autonomous systems

$$\begin{cases} x' & = H_y(x, y) \\ y' & = -H_x(x, y) \end{cases} \tag{HS}$$

where $H(x, y)$ is a twice differentiable function defined at $(x, y) \in \mathbb{R}^2$. The function H is called hamiltonian and the system is called a *hamiltonian system*. Hamiltonian systems are *conservative* because there is a quantity that is conserved along its solutions.

In the sequel we will always assume that solutions $x(t), y(t)$ of (HS) are defined for all $t \in \mathbb{R}$.

Lemma 8.1.1. *If $x(t), y(t)$ is a solution of (HS), then there exists $c \in \mathbb{R}$ such that $H(x(t), y(t)) = c$.*

Proof. Taking the derivative one finds

$$\frac{d}{dt} H(x(t), y(t)) = H_x(x(t), y(t))x'(t) + H_y(x(t), y(t))y'(t).$$

Since $x'(t) = H_x(x(t), y(t))$ and $y'(t) = -H_y(x(t), y(t))$, we have

$$H_x(x(t), y(t))H_x(x(t), y(t)) - H_y(x(t), y(t))H_x(x(t), y(t)) = 0.$$

Therefore $H(x(t), y(t))$ is constant. ■

Consider the set in the plane defined by

$$\Lambda_c = \{(x, y) \in \mathbb{R}^2 : H(x, y) = c\}.$$

From the preceding Lemma it follows that any solution $x(t), y(t)$ of (HS) satisfies $H(x(t), y(t)) = c$ for some constant c and thus $(x(t), y(t)) \in \Lambda_c$ for all t. Let $x(t), y(t)$ be the (unique) solution of (HS) satisfying the initial condition $x(t_0) = x_0, y(t_0) = y_0$. If (x_0, y_0) belongs to Λ_c for some c, then $c = c_0 = H(x_0, y_0)$ and $(x(t), y(t))$ belongs to Λ_{c_0} for all t. Recall that, since the system (HS) is autonomous, if $x(t), y(t)$ is a solution, then so is $x(t + h), y(t + h)$, for all $h \in \mathbb{R}$. Therefore, given $(x_0, y_0) \in \Lambda_{c_0}$ we can shift the time t and assume without loss of generality that $t_0 = 0$, namely that $x(0) = x_0$, $y(0) = y_0$. In other words, any (nonempty) curve Λ_c singles out a unique solution of (HS), the one such that $x(0) = x_0, y(0) = y_0$, with $(x_0, y_0) \in \Lambda_c$.

Remark 8.1.2. If $H_x(x, y)$ and $H_y(x, y)$ do not vanish simultaneously for $(x, y) \in \Lambda_c$ then $H(x, y) = c$ is a regular curve. A proof of this claim is carried out in a particular case in Lemma 8.3.3 in Section 8.3. Notice that the points $(x^*, y^*) \in \mathbb{R}^2$ such that $H_x(x^*, y^*) = H_y(x^*, y^*) = 0$ are precisely the equilibria of the hamiltonian system (HS). ■

Example 8.1.3. If $H(x, y) = Ax^2 + Bxy + Cy^2$, the only equilibrium is $(0, 0)$. If $c \neq 0$, the curve Λ_c is a conic. Precisely:

1. If $B^2 - 4AC < 0$ and $c > 0$, then Λ_c is an ellipse.
2. If $B = 0$, $A = C$ and $c > 0$, then Λ_c is a circle.
3. If $B^2 - 4AC > 0$ (and $c \neq 0$), then Λ_c is a hyperbola.

In any case Λ_c is a regular curve. If $c = 0$ or $B^2 = 4AC$ the conic is degenerate and can be a pair of straight lines or it reduces to a point.

Let $x_c(t), y_c(t)$ be the solution of (HS) such that $H(x(t), y(t)) = c$. Set $P_c(t) \equiv (x_c(t), y_c(t))$.

In the sequel we are interested in the existence of periodic solutions of (HS).

Lemma 8.1.4. *If there exists $T > 0$ such that $P_c(T) = P_c(0)$, then $x_c(t), y_c(t)$ is a T-periodic solution,*

Proof. By assumption, there exists $T > 0$ such that $P_c(T) = P_c(0)$, namely $x_c(T) = x_c(0)$ and $y_c(T) = y_c(0)$. We now argue as in Example 4.2.5 in Chapter 4. Setting $\tilde{x}_c(t) = x_c(t + T)$, $\tilde{y}_c(t) = y_c(t + T)$ we see that

$$\begin{cases} \tilde{x}_c'(t) &= x_c'(t + T) = H_y(x_c(t + T), y_c(t + T)) = H_y(\tilde{x}_c(t), \tilde{y}_c(t)), \\ \tilde{y}_c'(t) &= y_c'(t + T) = -H_x(x_c(t + T), y_c(t + T)) = -H_x(\tilde{x}_c(t), \tilde{y}_c(t)). \end{cases}$$

Moreover, $\tilde{x}_c(0) = x_c(T) = x(_c0)$ and $\tilde{y}_c(0) = y_c(T) = y_c(0)$. By uniqueness, it follows that $\tilde{x}_c(t) = x_c(t)$ and $\tilde{y}_c(t) = y_c(t)$ for all t, that is $x_c(t + T) = x_c(t)$, $y_c(t + T) = y_c(t)$. This means that $(x_c(t), y_c(t))$ is a T-periodic solution. ∎

We conclude this section stating, without proof, the following result.

Theorem 8.1.5. *Suppose that $\Lambda_c \neq \emptyset$ is a compact curve that does not contain equilibria of (HS). Then $(x_c(t), y_c(t))$ is a periodic solution of (HS).*

Let us point out that, according to Remark 8.1.2, Λ_c does not contain equilibria of (HS) if and only if H_x and H_y do not vanish simultaneously on Λ_c. Moreover, using the Implicit Function Theorem, it is possible to show that if the curve $\Lambda_c = \{H(x, y) = 0\} \neq \emptyset$ is compact and H_x and H_y do not vanish simultaneously on Λ_c, then Λ_c is a loop (i.e. a continuous curve $\gamma : [a, b] \to \mathbb{R}^2$ such that $\gamma(a) = \gamma(b)$) diffeomorphic to a circle.

Example 8.1.6. Show that the solution of the ivp

$$\begin{cases} x' &= 2x + 3y, \quad x(0) = 0 \\ y' &= -3x - 2y, \quad y(0) = 1 \end{cases}$$

is periodic.

Here $H_y = 2x + 3y$ and $H_x = 3x + 2y$. We note that $H_y = 2x + 3y$ implies that $H = 2xy + \frac{3}{2}y^2 + h(x)$, where we take $h(x)$ as the constant of integration with respect to y. Therefore, $H_x = 2y + h'(x) = 3x + 2y$ yielding $h = \frac{3}{2}x^2$ and hence $H(x, y) = 2xy + \frac{3}{2}y^2 + \frac{3}{2}x^2$.

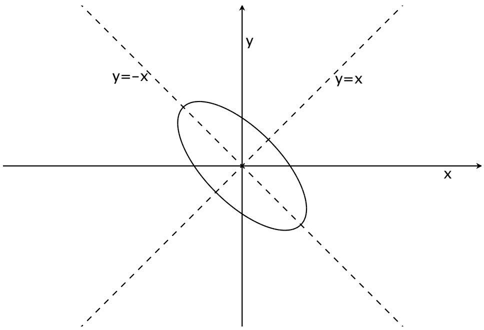


Fig. 8.1. $3y^2 + 4xy + 3x^2 = 3$

The curve Λ_c has equation $\frac{3}{2}x^2 + 2xy + \frac{3}{2}y^2 = c$. Using the initial values, we find $c = \frac{3}{2}$. The curve defined by $\frac{3}{2}x^2 + 2xy + \frac{3}{2}y^2 = \frac{3}{2}$, or $3y^2 + 4xy + 3x^2 = 3$, is an ellipse that does not contain the equilibrium $(0,0)$, see Figure 8.1. Hence the solution of the ivp is periodic. ∎

Remark 8.1.7. The next examples show that the assumptions that Λ_c is compact and does not contains equilibria cannot be eliminated.

(i) If Λ_c is unbounded the solution $(x_c(t), y_c(t))$ cannot be periodic because $x_c(t)$ and/or $y_c(t)$ are unbounded.

(ii) Consider $H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$, which corresponds to the system

$$\begin{cases} x' &= y, \\ y' &= x - x^3. \end{cases}$$

For $c = 0$, the curve $\Lambda_0 = \{H(x,y) = 0\}$ is compact but contains the singular point $(0,0)$. In Subsection 8.4.1 we will see that the corresponding solution $x_0(t), y_0(t)$ satisfies $\lim_{t\to\pm\infty} x_0(t) = 0$ and hence is not periodic. ∎

8.2 A prey-predator system

In this section we will study a system arising in population dynamics in the presence of prey (e.g. sheep) and predators (e.g. wolves).

Let $x(t) > 0$ denote the number of prey at time t and $y(t) > 0$ the number of predators at time t. It is assumed that the prey has an unlimited food supply (e.g. grass) while wolves may feed on sheep, for example. The change in the number of prey and predators is modeled by the so called Lotka–Volterra[1] system

$$\begin{cases} x' &= ax - bxy \\ y' &= -cy + dxy \end{cases} \qquad (LV)$$

[1] Alfred J. Lotka (1880–1949); Vito Volterra (1860–1940).

where a, b, c, d are *strictly positive* constants which depend on the skills of the prey and predators, the environment and the challenges for predators to kill their prey, and so on. The meaning of this model is, roughly, the following. In the absence of predators, the prey is assumed to follow the Malthusian model $x' = ax$ with $a > 0$ and hence it grows exponentially (recall that we are supposing that the prey has an unlimited food supply). The presence of predators reduces the increasing rate of the prey by a factor of $-bxy$ (say, the number of encounters between the sheep and the wolves). In other words, the larger the population of the predators, the smaller the growth rate of the prey.

The second equation models the growth rate of the predators. In the absence of prey, the predators also follow a Malthusian model $y' = -cy$, but with a negative coefficient and they will eventually go to extinction due to lack of sufficient food supply. The presence of prey modifies the growth rate of the predators by a factor of dxy: the larger the number of prey, the greater the food supply for predators and hence the bigger their growth rate.

Heuristically we can guess that the number of predators and prey oscillate. Neither of the two can increase beyond a certain threshold. For example, wolves cannot increase after a threshold because when they become too many, they have to compete harder for their food supply. But the sheep population cannot decrease too much, because the smaller their population, the smaller the survival rate of the wolf population; and consequently the prey can prosper.

We want to prove this claim rigorously, by studying the behavior over time of the number of prey and predators. Roughly, we try to find a constant of motion and use it to deduce the properties of the solutions of (LV).

First of all let us find the equilibria of the system. Putting $x' = y' = 0$ it follows that

$$\begin{cases} ax - bxy & = 0 \\ -cy + dxy & = 0. \end{cases}$$

Solving this system, we see that the solutions are either the trivial solution $x = y = 0$ or $x = c/d, y = a/b$.

These equilibria correspond to two constant solutions: $x(t) \equiv y(t) \equiv 0$ and $x(t) \equiv \frac{c}{d}$, $y(t) \equiv \frac{a}{b}$ (recall that a, b, c, d are strictly positive). In other words, if the initial number of prey is $x(0) = \frac{c}{d}$ and the initial number of predators is $y(0) = \frac{a}{b}$, then their numbers remain the same for all $t > 0$.

Next, let us show that (LV) possesses a one parameter family of positive periodic solutions. Following what we did earlier, it would be useful to find the counterpart of the energy constant, looking for a function $H(x, y)$ such that $H(x(t), y(t)) = k$, for some $k \in \mathbb{R}$.

Let us check that such a function is given by

$$H(x, y) = dx + by - c\ln x - a\ln y, \quad x > 0,\ y > 0.$$

To this end, take the derivative of $H(x(t), y(t))$ (for brevity we understand the dependence on t without indicating it each time)

$$\frac{d}{dt} H(x, y) = H_x x' + H_y y' = \left(d - \frac{c}{x}\right) x' + \left(b - \frac{a}{y}\right) y'.$$

Substituting $x' = ax - bxy$, $y' = -cy + dxy$, we deduce

$$\frac{d}{dt} H(x, y) = \left(d - \frac{c}{x}\right)(ax - bxy) + \left(b - \frac{a}{y}\right)(-cy + dxy)$$
$$= (ad\, x - ac - bd\, xy + bc\, y) + (-bc\, y + ac + bd\, xy - ad\, x) = 0,$$

proving that $H(x, y) = k$, for some $k \in \mathbb{R}$, along the solutions of (LV).

As before, the solutions of (LV) are defined implicitly by

$$H(x, y) = dx + by - c \ln x - a \ln y = k, \quad x > 0,\ y > 0,$$

provided, of course, that the set $\{H(x, y) = k\}$ is not empty. Set

$$\kappa = H\left(\frac{c}{d}, \frac{a}{b}\right) = c + a - c \ln\left(\frac{c}{d}\right) - a \ln\left(\frac{a}{b}\right).$$

It is possible to show that the set $\{H(x, y) = k\}$ is not empty and defines a compact curve, surrounding the equilibrium $(c/d, a/b)$ if and only if $k > \kappa$.

To give a sketch of the proof of this claim, we need the theory of functions of two variables. The reader who is not interested in the proof, or does not have sufficient background to understand it, may skip the details given in small letters below.

Let us study the surface $z = H(x, y)$, $x > 0$, $y > 0$. The stationary points are the solutions of $H_x = 0$, $H_y = 0$. Since

$$H_x(x, y) = d - \frac{c}{x}, \qquad H_y(x, y) = b - \frac{a}{y}$$

we find

$$d - \frac{c}{x} = 0, \qquad b - \frac{a}{y} = 0.$$

Then the only stationary point is the equilibrium $\left(\frac{c}{d}, \frac{a}{b}\right)$. The Hessian matrix H'' of H is given by

$$H'' = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix} = \begin{pmatrix} cx^{-2} & 0 \\ 0 & ay^{-2} \end{pmatrix}.$$

Then, for all $x > 0$, $y > 0$, the eigenvalues of $H''(x, y)$ are both positive and this implies that $z = H(x, y)$ is a strictly convex surface. In particular, $\left(\frac{c}{d}, \frac{a}{b}\right)$ is the unique global minimum of H. Letting

$$\kappa = H\left(\frac{c}{d}, \frac{a}{b}\right) = c + a - c \ln\left(\frac{c}{d}\right) - a \ln\left(\frac{a}{b}\right)$$

it follows that: (i) for $k < \kappa$ the set $\{H(x, y) = k\}$ is empty; (ii) for $k = \kappa$ the set $\{H(x, y) = k\}$ reduces to the equilibrium point; (iii) for all $k > \kappa$ the equation $H(x, y) = k$ is a level curve of the surface $z = H(x, y)$ and hence it defines a compact curve. This latter statement is also a consequence of the fact that $H(x, y) \to +\infty$ as $x \to 0^+$ or $y \to 0^+$ as well as $x \to +\infty$ as $y \to +\infty$.

Since, for all $k > \kappa$, the curve $\{H(x, y) = k\}$ is compact and does not contain the equilibria of (LV), see Figure 8.2, one shows as in Theorem 8.1.5 that it carries a periodic solution of (LV).

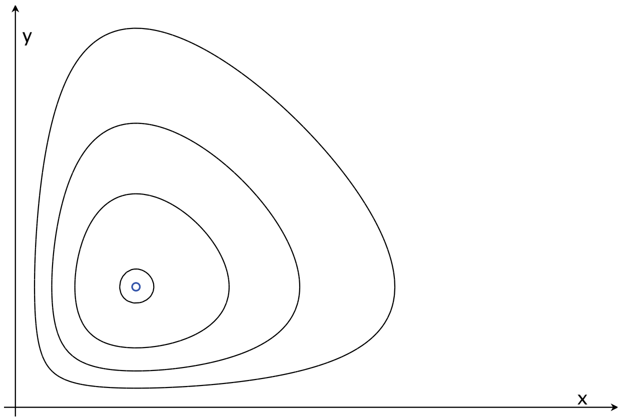


Fig. 8.2. The curves $H(x, y) = k, k > \kappa$

If $x(t), y(t)$ is a T-periodic solution of (LV), an important quantity is their mean value

$$\overline{x} := \frac{1}{T}\int_0^T x(t)dt, \qquad \overline{y} := \frac{1}{T}\int_0^T y(t)dt.$$

The following result shows that, in the mean, the number of prey and predators equal the equilibria.

Theorem 8.2.1. *One has*

$$\overline{x} = \frac{c}{d}, \qquad \overline{y} = \frac{a}{b}. \tag{8.1}$$

Proof. From the first equation we infer (recall that $x(t) > 0$)

$$\int_0^T \frac{x'(t)dt}{x(t)} = \int_0^T (a - by(t))dt.$$

Since $x(T) = x(0)$, then

$$\int_0^T \frac{x'(t)dt}{x(t)} = \ln x(T) - \ln x(0) = 0.$$

It follows

$$aT - b\int_0^T y(t)dt = 0 \quad \Longrightarrow \quad \int_0^T y(t)dt = \frac{aT}{b}$$

whence $\overline{y} = \frac{a}{b}$. In a similar way, using the second equation one proves that $\overline{x} = \frac{c}{d}$. ■

Let us consider the specific case in which $a = 2, b = 1, c = 3, d = 1$. The equilibrium is the point $P = (3, 2)$ and the system becomes

$$\begin{cases} x' &= 2x - xy \\ y' &= -3y + xy. \end{cases}$$

Moreover

$$H(x, y) = x + y - 3 \ln x - 2 \ln y,$$

and $\kappa = 5 - 3\ln 3 - 2\ln 2 = 5 - \ln(3^3 \cdot 2^2) = 5 - \ln(108) > 0$.

From the preceding equations it follows that we can distinguish 4 regions, see Figure 8.3:

$S^{+-} = \{x < 3, y < 2\}$ where $x' > 0, y' < 0$;
$S^{++} = \{x > 3, y < 2\}$ where $x' > 0, y' > 0$;
$S^{-+} = \{x > 3, y > 2\}$ where $x' < 0, y' > 0$;
$S^{--} = \{x < 3, y > 2\}$ where $x' < 0, y' < 0$.

Let us take the initial values to be $Q = (2, 1)$. Letting $k^* = H(2, 1) = 3 - 3\ln 2$, the equation $H(x, y) = k^*$ defines a closed curve γ^* which carries a solution $x^*(t), y^*(t)$ of the system, such that $x^*(0) = 2, y^*(0) = 1$.

Referring to Figure 8.3, we fix the points A, B, C, D on γ^*. Let x_m, x_M be such that $B = (x_M, 2), D = (x_m, 2) \in \gamma^*$ and let y_m, y_M be such that $A = (3, y_m), C = (3, y_M) \in \gamma^*$. To find x_m, x_M it suffices to solve the equation $H(x, 2) = 3 - 3\ln 2$, that is

$$x + 2 - 3\ln x - 2\ln 2 = 3 - 3\ln 2 \quad \Longrightarrow \quad x - 3\ln x = 1 - \ln 2.$$

Similarly, y_m, y_M are the solutions of $H(3, y) = 3 - 3\ln 2$, namely

$$3 + y - 3\ln 3 - 2\ln y = 3 - 3\ln 2 \quad \Longrightarrow \quad y - 2\ln y = 3\ln 3 - 3\ln 2.$$

The curve γ^* is contained in the rectangle $[x_m, x_M] \times [y_m, y_M]$ (the dotted box in Figure 8.3) and $x^*(t), y^*(t)$ are oscillating functions with minimal amplitudes x_m, y_m, respectively, and maximal amplitudes x_M, y_M, respectively. Notice that $x_m > 0$ as well as $y_m > 0$.

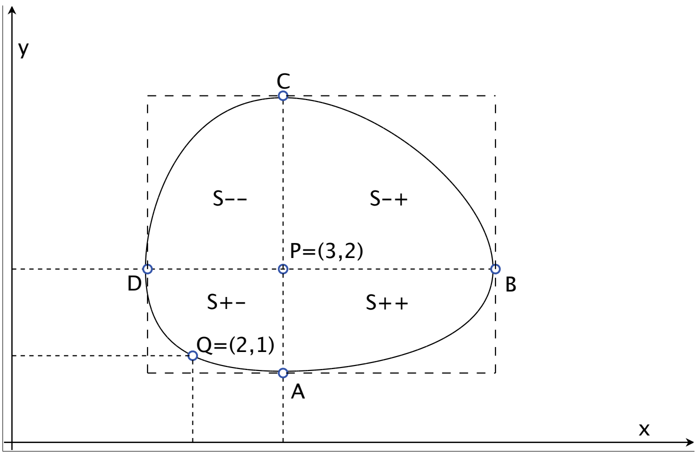


Fig. 8.3. The curve γ^*

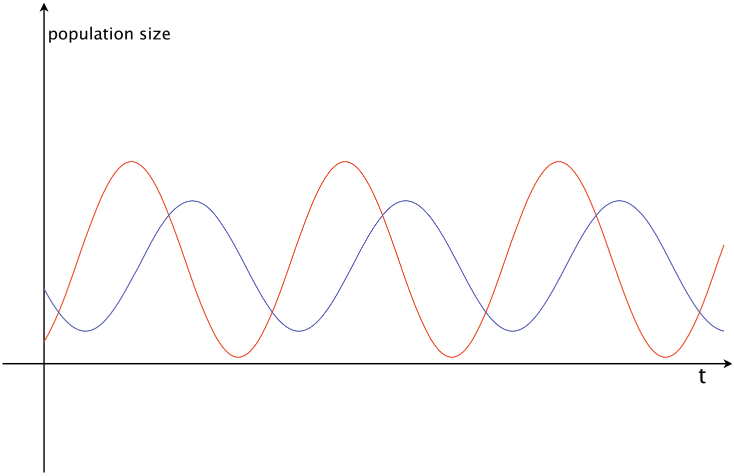


Fig. 8.4. Plot of possible oscillations of prey $x(t)$ (*red*) and predator $y(t)$ (*blue*) population size

In our model, the initial value $Q = (2, 1)$ belongs to S^{+-} where $x^*(t)$ increases while $y^*(t)$ decreases. The point $(x^*(t), y^*(t))$ "moves" on γ^* in counterclock direction and at a certain time t_1 it reaches A, where one has $x^*(t_1) = 3$ and $y^*(t_1) = y_m$. At this time, the number of prey is enough to let the predators increase. Actually, for $t > t_1$ one enters into the region S^{++} where both $x^*(t)$ and $y^*(t)$ increase, even if with different slopes. At some $t = t_2$ the point on γ^* reaches B: the number of prey achieves its maximum x_M while $y^*(t_2) = 2$. Now the number of wolves is sufficiently large to cause the sheep population to decrease: for $t > t_2$, $x^*(t)$ decreases while $y^*(t)$ increases until the point on γ^* reaches C at a time t_3 such that $x^*(t_3) = 3$ and $y^*(t_3) = y_M$. But the number of predators cannot increase without any limit because their big numbers would reduce the population of the prey and thus cause a shortage of food supply. As a consequence, the number of predators decays. For a while, prey still decreases, but at a lower rate. At $t = t_4$ where $x^*(t_4) = x_m$, $y^*(t_4) = 2$, the point on γ^* is $D = (x_m, 2)$ and the wolves are so few that the sheep population starts increasing until $(x^*(t), y^*(t))$ once again reaches the starting initial value $Q = (2, 1)$.

In the Figure 8.4 the cyclic fluctuation of preys and predators over time t, is illustrated.

8.2.1 *The case of fishing*

The original research of Volterra was carried out in order to understand why, after the end of the first world war, in the Adriatic sea the number of small fish, like sardines (the prey) increased while the number of big fish (the predators) decreased. The explanation was that the phenomenon was due to the fact that after the war there was increased fishing activity. Roughly, fishing kills some prey and some predators and this modifies the model as follows

$$\begin{cases} x' &= ax - bxy - \epsilon x = (a-\epsilon)x - bxy \\ y' &= -cy + dxy - \epsilon y = -(c+\epsilon)y + dxy. \end{cases}$$

This can be explained by noticing that the ϵ increase of fishing causes a bigger death rate of predators, namely $c+\epsilon$, and a smaller birth rate of preys, namely $a-\epsilon$, while the parameters, b, d, describing the interaction of the two species, remain unchanged. The new equilibrium is

$$x_\epsilon = \frac{c+\epsilon}{d} > \frac{c}{d}, \qquad y_\epsilon = \frac{a-\epsilon}{b} < \frac{a}{b}$$

and, according to (8.1), the number of sardines and predators are, in the mean,

$$x_\epsilon > \overline{x} = \frac{c}{d}, \qquad y_\epsilon < \overline{y} = \frac{a}{b}.$$

So, according to the Lotka–Volterra model, a small increment of fishing causes, in the mean, a growth of the sardines and a smaller number of predators.

In Figure 8.5 we indicate the effect of an ϵ-increment of fishing. The equilibrium $P = \left(\frac{c}{d}, \frac{a}{b}\right)$ is transformed into the new equilibrium $P_\epsilon = \left(\frac{c}{d} + \frac{\epsilon}{d}, \frac{a}{b} - \frac{\epsilon}{b}\right)$. A level set $H(x,y) = k$ is modified into the level set of

$$H_\epsilon = dx + by - (c+\epsilon)\ln x - (a-\epsilon)\ln y$$

(the red curve in Figure 8.5) which is shifted to the right with respect to x and down with respect to y. This shows that the number of prey increases while the number of predator decreases.

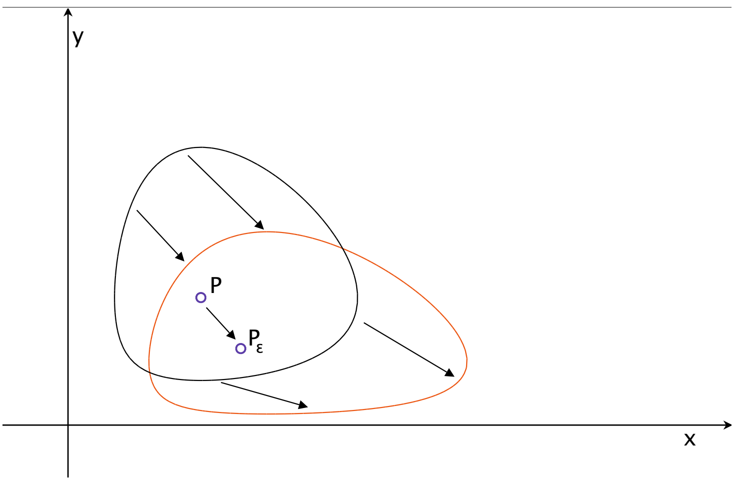


Fig. 8.5. Effect of an ϵ-increment of fishing. The black curve is the level set $H = k$, the red curve is the level set $H_\epsilon = k$

8.2.2 Improving (LV)

A more realistic version of the (LV) system is given by

$$\begin{cases} x' &= (a - hx)x - bxy \\ y' &= -cy + dxy. \end{cases} \tag{LV$'$}$$

In the first equation of this system the term ax is replaced by $(a - hx)x$, with $h > 0$ (if $h = 0$ (LV$'$) becomes the previous (LV)). This means that, in absence of predators, namely if $y = 0$, the prey species growth is more realistic, being governed by the logistic equation $x' = (a - hx)x$ instead of $x' = ax$.

The equilibria of system (LV$'$) are the solution of the algebraic system

$$\begin{cases} (a - hx)x - bxy &= 0 \\ -cy + dxy &= 0. \end{cases}$$

From the second equation we get $y = 0$ or $x = \frac{c}{d}$. Substituting $y = 0$ in the first equation we find $(a-hx)x = 0$ yielding two equilibria $(0, 0)$ and $\left(\frac{a}{h}, 0\right)$. Substituting $x = \frac{c}{d}$ in the first equation we get

$$y = \frac{a - hx}{b} = \frac{a - h \cdot \frac{c}{d}}{b} = \frac{ad - hc}{bd},$$

which is positive provided $ad > hc$. In conclusion, if $0 < h < \frac{ad}{c}$ then (LV$'$) has 3 equilibria in the first quadrant $x \geq 0, y \geq 0$ given by

$$(0, 0), \qquad \left(\frac{a}{h}, 0\right), \qquad \left(\frac{c}{d}, \frac{ad - hc}{bd}\right).$$

As for (LV), also here the first quadrant is divided by the two dotted straight-lines $dx = c$ and $by = a - hx$ (see Figure 8.6) into 4 sectors: (I) where $x' < 0, y' > 0$, (II) where $x' < 0, y' < 0$, (III) where $x' > 0, y' < 0$ and (IV) where $x' > 0, y' > 0$. But the dynamic of this modified model could be different from the one of (LV).

First of all, let us consider the following initial value problem for the modified Lotka-Volterra system

$$\begin{cases} x' &= (a - hx)x - bxy, \quad x(0) = \alpha > 0, \\ y' &= -cy + dxy, \qquad\quad y(0) = 0. \end{cases} \tag{8.2}$$

It is immediate to check that a solution of (8.2) is $(x_\alpha(t), 0)$, where $x_\alpha(t)$ is the solution of $x' = (a - hx)x$, such that $x(0) = \alpha$. Moreover, notice that by uniqueness this is the only solution of (8.2). Recall that, according to the discussion carried out in Subsection 3.1.1 of Chapter 3, one has that $\lim_{t\to+\infty} x_\alpha(t) = \dfrac{a}{h}$. In other words, for any initial condition $(\alpha, 0)$, $\alpha > 0$, the solution $(x_\alpha(t), 0)$ (8.2) tends to $\left(\dfrac{a}{h}, 0\right)$ as $t \to +\infty$.

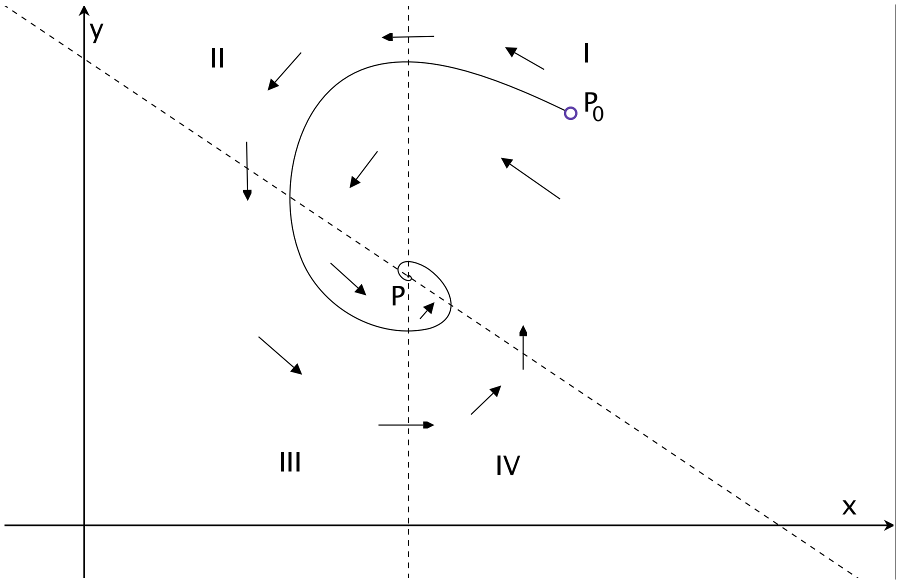


Fig. 8.6. Possible behavior of a trajectory of (LV′). The dotted straight-lines have equation, respectively, $x = \frac{c}{d}$ and $by = a - hx$; $P_0 = (x(0), y(0))$, $P = \left(\frac{c}{d}, \frac{ad-hc}{bd}\right)$

Another important difference is that, in general, we do not have periodic solutions any more. A possible behavior of the solutions of (LV′) is shown in Figure 8.6, where the trajectory $(x(t), y(t))$, with initial conditions $P_0 = (x(0), y(0))$, $x(0) > 0$, $y(0) > 0$, tends asymptotically to the equilibrium $P = \left(\frac{c}{d}, \frac{ad-hc}{bd}\right)$ as $t \to +\infty$, in the sense that

$$\lim_{t\to+\infty} x(t) = \frac{c}{d}, \qquad \lim_{t\to+\infty} y(t) = \frac{ad-hc}{bd}.$$

8.3 Phase plane analysis

In this section we study the nonlinear system

$$\begin{cases} x' &= y \\ y' &= f(x) \end{cases} \tag{8.3}$$

where $f \in C^\infty(\mathbb{R})$. In the sequel it will be always understood that the solutions of (8.3) are defined for all $t \in \mathbb{R}$.

The plane (x, y) is called *phase plane* and the study of the system (8.3) is called *phase plane analysis*.

System (8.3) is a hamiltonian system with hamiltonian (called here E) given by

$$E(x, y) = \frac{1}{2}y^2 - F(x),$$

where F is such that $F'(x) = f(x)$. Actually, $E_y = y$ and $E_x = -f(x)$. Note that, in this case, the equilibria of the hamiltonian system are the points $(x_0, 0) \in \mathbb{R}^2$ such that $f(x_0) = 0$, that correspond to the constant solutions $x(t) = x_0$, $y(t) = 0$ of (8.3).

The hamiltonian E is the sum of the kinetic energy $\frac{1}{2}y^2 = \frac{1}{2}x'^2$ and the potential energy $-F(x)$ and is therefore the total energy of the system.

From Lemma 8.1.1 proved in Section 8.1 it follows:

Lemma 8.3.1. *If $(x(t), y(y))$ is a solution of (8.3), then $E(x(t), y(t))$ is constant.*

For $c \in \mathbb{R}$, let

$$\Lambda_c := \{(x, y) \in \mathbb{R}^2 : E(x, y) = c\} = \{(x, y) \in \mathbb{R}^2 : \tfrac{1}{2}y^2 - F(x) = c\}.$$

Remark 8.3.2. The following properties hold:

(i) Λ_c is symmetric with respect to y: $(x, y) \in \Lambda_c$ if and only if $(x, -y) \in \Lambda_c$.
(ii) A point $(x, 0)$ belongs to Λ_c if and only if $F(x) = -c$.
(iii) A point $(0, y)$ belongs to Λ_c if and only if $c \geq 0$. In this case one has $y = \sqrt{2c}$.
(iv) If a point $(x_0, y_0) \in \Lambda_c$, then $c = \frac{1}{2}y_0^2 - F(x_0)$.

The proof is left to the reader as an easy exercise. ■

Lemma 8.3.3. *If Λ_c does not contain any equilibria of (8.3), then it is a regular curve in the phase plane, in the sense that in a neighborhood of each point $(x_0, y_0) \in \Lambda_c$, Λ_c is either a differentiable curve of equation $y = \phi(x)$ or $x = \psi(y)$.*

Proof. (Sketch) One has $c = c_0 = \frac{1}{2}y_0^2 - F(x_0)$ and hence $E(x, y) = c_0$ yields $y^2 = 2F(x) + 2c_0$. Since (x_0, y_0) is not singular, then either $y_0 \neq 0$ or $y_0 = 0$ and $f(x_0) \neq 0$. In the former case there exists a neighborhood U of x_0 such that $2F(x) + 2c_0 > 0$ for all $x \in U$ and thus $y = \pm\sqrt{2F(x) + 2c_0}$, $x \in U$, where the sign $\pm$ is the same as the sign of y_0. This shows that in U the set Λ_c is a curve of equation $y = \phi(x)$.

If $y_0 = 0$, then $y^2 = 2(F(x) - F(x_0))$, namely $F(x) = \frac{1}{2}y^2 + F(x_0)$ Since $(x_0, 0)$ is not singular, then $F'(x_0) = f(x_0) \neq 0$. By continuity we infer that $F'(x) \neq 0$ in a neighborhood V of x_0. Then F is invertible in V with inverse Φ, and this yields $x = \Phi(\frac{1}{2}y^2 + F(x_0))$. Hence in V Λ_c is a curve of equation $x = \psi(y)$. ■

If $E(x, y) = x^2 - y^2$ and $c = 0$, $\Lambda_0 = \{x^2 - y^2 = 0\}$ is the pair of straight lines $x + y = 0$ and $x - y = 0$ and cannot be represented by any cartesian curve in any neighborhood of $(0, 0)$, which is the equilibrium of the corresponding system $x' = -2y,\ y' = -2x$. This shows that the preceding Lemma can be false if Λ_c contains an equilibrium.

8.4 On the equation $x'' = f(x)$

In this section we deal with the second order equations of the form $x'' = f(x)$. The importance of this class of differential equations is linked e.g. to the Newton Law. Actually, $x''(t)$ is the acceleration of a body with unit mass of position $x(t)$ at time t and $f(x)$ is the force acting on the body, depending on its position.

Here we focus on periodic, homoclinic and heteroclinic solutions, see definitions later on. Boundary value problems such as $x'' = f(x), x(a) = x(b) = 0$, will be discussed in Section 13.1 of Chapter 13.

Let us start by proving

Lemma 8.4.1. *The second order equation*

$$x'' = f(x) \tag{8.4}$$

is equivalent to the system

$$\begin{cases} x' &= y \\ y' &= f(x). \end{cases} \tag{8.5}$$

Moreover, the initial conditions $x(0) = x_0,\ y(0) = y_0$ *for* (8.5) *correspond to the initial conditions* $x(0) = x_0,\ x'(0) = y_0$ *for* (8.4).

Proof. Suppose that x, y is a solution of (8.5), with $x(0) = x_0, y(0) = y_0$. Then $x' = y$ implies $x'' = y' = f(x)$ and $x(0) = x_0, y(0) = y_0$ imply $x(0) = x_0$, $x'(0) = y(0) = y_0$. This shows that (8.5) implies (8.4).

Now, suppose that $x'' = f(x)$. Then if we let $x' = y$, we obtain $y' = x'' = f(x)$ and hence the system (8.5). Furthermore, the initial conditions $x(0) = x_0, x'(0) = y_0$ imply $x(0) = x_0, y(0) = x'(0) = y_0$. ∎

As a consequence of the preceding Lemma we can apply to $x'' = f(x)$ all the results of the preceding section. In particular:

1. The total energy

   $$E(x, y) = \frac{1}{2}y^2 - F(x), \quad y = x'$$

   is constant along the solutions of $x'' = f(x)$. We let $x_c(t)$ denote the solution of $x'' = f(x)$ with energy c, carried by $E = c$.
2. If $E = c$ is a compact curve which does not contain any zero of f, then it carries a periodic solution of $x'' = f(x)$. Notice that the zeros of f are the equilibria of the system (8.5).

In the sequel we will discuss two specific examples that show the typical features of the arguments.

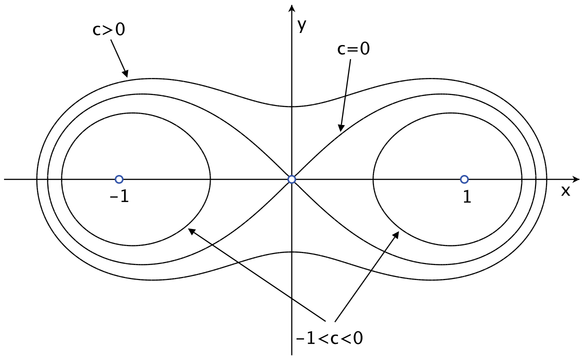


Fig. 8.7. $2y^2 - 2x^2 + x^4 = c$

8.4.1 A first example: The equation $x'' = x - x^3$

Consider the equation

$$x'' = x - x^3. \tag{8.6}$$

Here $f(x) = x - x^3$ and hence there are 3 equilibria: $0, \pm 1$. The conservation of the energy becomes

$$E(x, y) = 2y^2 - 2x^2 + x^4 = c, \quad y = x'. \tag{8.7}$$

Notice that $E(x, y) = c$ is symmetric with respect to x and y. Writing (8.7) as

$$y = \pm\sqrt{\frac{c + 2x^2 - x^4}{2}}$$

it follows that (see Figure 8.7):

(i) $E(0, y) = 2y^2 = c$ yields $c \geq 0$ and $y = \pm\sqrt{c/2}$. In particular, if $c = 0$, then $y = 0$.

(ii) If $y = 0$, then $E(x, 0) = c$ becomes $x^4 - 2x^2 = c$. Plotting the graph of the function $2x^2 - x^4$ we see that $2x^2 - x^4 \leq 1$ and $2x^2 - x^4 = 1$ for $x = \pm 1$. Then it follows that $c = x^4 - 2x^2 \geq -1$. Moreover, for $c = -1$, $E(x, 0) = 1$ provided $x = \pm 1$.

(iii) For all $c > 0$, $E(x, y) = c$ is a compact curve that does not contain equilibria and that crosses both $x = 0$ and $y = 0$.

(iv) For all $-1 < c < 0$, $E(x, y) = c$ is the union of two compact curves that do not contain equilibria and that do not cross $x = 0$.

(1) Periodic solutions. If $c > -1$, $c \neq 0$, then according to (iii–iv) the curve $E(x, y) = c$ is compact and does not contain equilibria and then $x_c(t)$ is periodic. We have proved

Theorem 8.4.2. *If $c > -1$, $c \neq 0$, the equation $x'' = x - x^3$ has a periodic solution $x_c(t)$ such that $E(x_c(t), x_c'(t)) = c$.*

(2) Homoclinic solutions.

Definition 8.4.3. We say $x(t)$ is a *homoclinic* to x_0 (relative to the equation $x'' = f(x)$), if $x(t)$ is a solution such that $\lim_{t\to\pm\infty} x(t) = x_0$.

We are going to show that $x'' = x - x^3$ has homoclinics to 0.

Let $c = 0$ and let $x_0^+(t) > 0$ be the solution of (8.6) carried by the branch of

$$E(x, y) = 2y^2 - 2x^2 + x^4 = 0, \quad y = x'$$

contained in the half plane $x \geq 0$. This curve crosses the x axis at $x = \sqrt{2}$ and, without loss of generality, we can assume that $x_0^+(0) = \sqrt{2}$ (and $(x_0^+)'(0) = 0$.).

Recall that $x'' = x - x^3$ is equivalent to the system

$$\begin{cases} x' &= y \\ y' &= x - x^3 \end{cases}$$

whose solution is denoted by $x_0^+(t), y_0^+(t)$ and satisfies $E(x_0^+(t), y_0^+(t)) = 0$. For all $t < 0$, the point $(x_0^+(t), y_0^+(t))$ remains in the first quadrant. Then $y_0^+(t) > 0$ for $t < 0$ and hence $\frac{d}{dt}x_0^+(t) = y_0^+(t) > 0$. Similarly, $\frac{d}{dt}x_0^+(t) = y_0^+(t) < 0$ for $t > 0$.

As a consequence, $x_0^+(t)$ is decreasing for $t > 0$ and hence converges to a limit L as $t \to +\infty$ and $L < x_0^+(0) = \sqrt{2}$. Moreover, $y_0^+(t) = \frac{d}{dt}x_0^+(t) \to 0$. From the conservation of energy we deduce

$$E(x_0^+(t), y_0^+(t)) = 2(y_0^+(t))^2 - 2(x_0^+(t))^2 + (x_0^+(t))^4 = 0.$$

Passing to the limit as $t \to +\infty$ we infer that $0 - 2L^2 + L^4 = 0$, that is $L^4 = 2L^2$. Since $L < \sqrt{2}$, it follows that $L = 0$, namely

$$\lim_{t\to+\infty} x_0^+(t) = 0.$$

Similarly, as $t \to -\infty$, one has

$$\lim_{t\to-\infty} x_0^+(t) = 0.$$

Moreover, $E(x, y) = 0$ is symmetric with respect to x and y and hence $x_0^-(t) = -x_0^+(t)$ is the solution carried by the branch contained in the half plane $x \leq 0$. The graph of $x_0^\pm(t)$ is reported in Figure 8.8. We have proved:

Theorem 8.4.4. *Equation $x'' = x - x^3$ possesses one positive and one negative symmetric homoclinic to* 0.

Notice that $x_0^\pm(t + h)$ are also homoclinics to 0, for all $h \in \mathbb{R}$. Actually, according to Lemma 8.0.1, $x_0^\pm(t + h)$ is a solution of $x'' = x - x^3$ and $\lim_{t\to\pm\infty} x_0^\pm(t + h) = 0$.

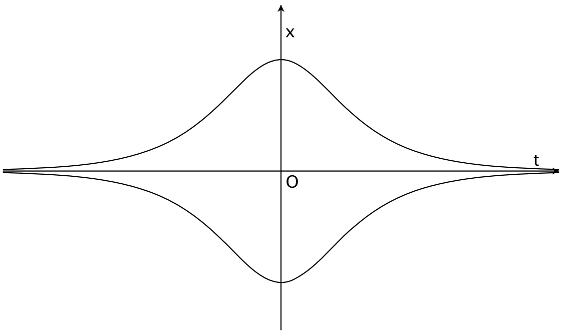


Fig. 8.8. Homoclinic solutions of $x'' = x - x^3$

Remark 8.4.5. In general, if $x'' = f(x)$ has a homoclinic to x_0, then x_0 is an equilibrium. ■

8.4.2 A second example: The equation $x'' = -x + x^3$

Consider the equation

$$x'' = -x + x^3. \tag{8.8}$$

As before, there are 3 equilibria $0, \pm 1$. The equation $E(x, y) = c$ becomes

$$2y^2 + 2x^2 - x^4 = c, \quad y = x'.$$

The corresponding curves, which are symmetric with respect to x and y, are plotted in Figure 8.9. If $0 < c < 1$, then $E(x, y) = c$ is a closed curve surrounding the origin and hence the corresponding solution is periodic.

The curve $E(x, y) = c$ passes through $(\pm 1, 0)$ provided $c = 1$. This gives rise to a new type of solutions, as we are going to see. Let $\widetilde{\Lambda} \subset \{E(x, y) = 1\}$ be the

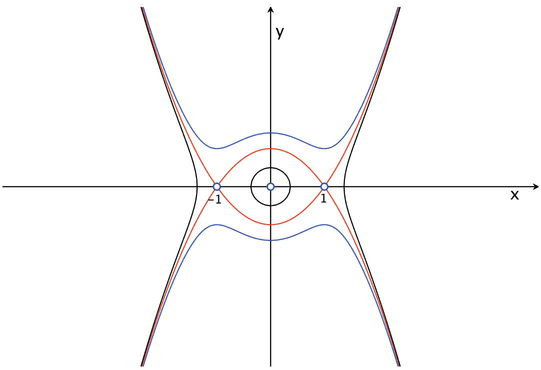


Fig. 8.9. $2y^2 + 2x^2 - x^4 = c$: $c = 1$ (*red*); $0 < c < 1$ (*black*); $c > 1$ (*blue*)

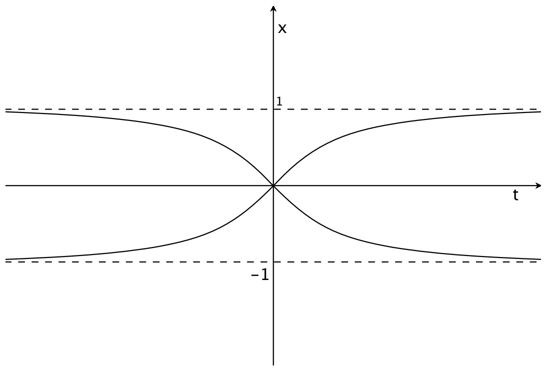


Fig. 8.10. Symmetric heteroclinics of $x'' = -x + x^3$

arc contained in the upper half plane $y > 0$, joining the points $(-1, 0)$ and $(1, 0)$. The corresponding solution $\widetilde{x}(t)$ is strictly increasing, because $y > 0$. Repeating the arguments carried out in the homoclinic case, one shows that

$$\lim_{t\to-\infty} \widetilde{x}(t) = -1, \qquad \lim_{t\to+\infty} \widetilde{x}(t) = 1.$$

Of course, for all $h \in \mathbb{R}$, any $\widetilde{x}(t+h)$ as well as $-\widetilde{x}(t+h)$ is also a solution of $x'' = -x + x^3$ with the property that they tend to different limits as $t \to -\infty$ and $t \to +\infty$. See Figure 8.10. These solutions that join two different equilibria are called *heteroclinics*.

We can state

Theorem 8.4.6. *The equation $x'' = -x + x^3$ possesses infinitely many heteroclinics.*

8.5 Exercises

1. Find the equilibrium of

$$\begin{cases} x' &= x + 1 \\ y' &= x + 3y - 1. \end{cases}$$

2. Find a, b such that the equilibrium of

$$\begin{cases} x' &= x + 3y + a \\ y' &= x - y + b \end{cases}$$

is $(1, 2)$.

3. Find α, β such that

$$\begin{cases} x' &= \alpha x + y \\ y' &= -2x + \beta y \end{cases}$$

is hamiltonian.

4. Discuss the family of conics $x^2 + Bxy + y^2 = c$ depending on B, c.
5. Discuss the family of conics $Ax^2 - xy + y^2 = c$ depending on A, c.
6. Find C such that the system

$$\begin{cases} x' &= x + y \\ y' &= -2Cx - y \end{cases}$$

has no periodic solution but the equilibrium $x(t) = y(t) \equiv 0$.

7. Show that if $AC < 0$ then all the nontrivial solutions of the system

$$\begin{cases} x' &= Bx + Cy \\ y' &= -Ax - By \end{cases}$$

are not periodic.

8. Find B such that the system

$$\begin{cases} x' &= Bx + 3y \\ y' &= -3x - By \end{cases}$$

has periodic solutions.

9. Show that the solution of the system

$$\begin{cases} x' &= x + y \\ y' &= -2x - y \\ x(0) &= 1 \\ y(0) &= 0 \end{cases}$$

is periodic.

10. Show that the solution of the system

$$\begin{cases} x' &= x - 6y \\ y' &= -2x - y \\ x(0) &= 1 \\ y(0) &= 0 \end{cases}$$

is unbounded.

11. Draw the phase plane portrait of the pendulum equation

$$Lx'' + g \sin x = 0$$

and discuss the behavior of the solutions.

12. Find the equilibria of the Lotka–Volterra system

$$\begin{cases} x' &= x - xy \\ y' &= -y + xy. \end{cases}$$

13. Find the nontrivial equilibrium (x_ϵ, y_ϵ) of

$$\begin{cases} x' &= 2x - 7xy - \epsilon x \\ y' &= -y + 4xy - \epsilon y. \end{cases}$$

14. Prove that there exists a periodic solution of the system
$$\begin{cases} x' &= 2x - 2xy \\ y' &= -y + xy \end{cases}$$
such that $x + 2y - 4 = \ln(xy^2)$.

15. Prove that there exists a periodic solution of the system
$$\begin{cases} x' &= x - 4xy \\ y' &= -2y + xy \end{cases}$$
such that $x + 4y - 4 = \ln(x^2 y)$.

16. Let $x(t), y(t)$ be a T-periodic solution of the Lotka–Volterra system
$$\begin{cases} x' &= x(3 - y) \\ y' &= y(x - 5). \end{cases}$$
Show that $\frac{1}{T}\int_0^T x(t)dt = 5$ and $\frac{1}{T}\int_0^T y(t)dt = 3$.

17. Show that $U(t) = \sqrt{2}/\cosh t$ is a homoclinic of the equation $x'' = x - x^3$.

18. Let $x_0(t)$ be a homoclinic of $x'' = x - x^3$. Show that $x_0'''(t) \to 0$ as $t \to \pm\infty$. Extend the result to any derivative of $x_0(t)$.

19. Prove the preceding result for the heteroclinics of $x'' = -x + x^3$.

20. Show that the solution of $x'' = -x + x^3$, $x(0) = 0$, $x'(0) = \frac{1}{2}$ is periodic.

21. Discuss the behavior of the solution of $x'' = -x + x^3$ such that $x(0) = 2$, $x'(0) = 0$.

22. Discuss the behavior of the solutions $x'' = -x + x^3$ such that $x(0) = 0$, $x'(0) = 1$.

23. Show that the solution of $x'' = x - x^3$ such that $x(0) = 0$, $x'(0) = 1$ is periodic.

24. Show that the solution of $x'' = x - x^3$ such that $x(0) = 2$, $x'(0) = 0$ is periodic.

25. Show that the solution of $x'' = x - x^3$ such that $x(0) = 1/\sqrt{2}$, $x'(0) = 0$ is periodic.

26. Show that for all $a \neq 0$ the solution of $x'' + x + 8x^7 = 0$, $x(0) = 0$, $x'(0) = a$ is periodic.

27. Discuss the behavior of the solution of $x'' + x + \frac{1}{3}x^2 = 0$, $x(0) = 1$, $x'(0) = 0$.

28. Show that the solution of $x'' - x + 3x^2 = 0$, $x(0) = \frac{1}{2}$, $x'(0) = 0$ is homoclinic to $x = 0$.

29. Show that the solution of $x'' - x + 3x^2 = 0$, $x(0) = \frac{1}{4}$, $x'(0) = 0$ is periodic.

30. Discuss the behavior of the solution of $x'' - x + 3x^2 = 0$, $x(0) = 0$, $x'(0) = 1$.

31. Discuss the behavior of the solution of $x'' - x + 3x^2 = 0$ such that $x(0) = -\frac{1}{4}$, $x'(0) = 0$.

9

Sturm Liouville eigenvalue theory

In this chapter we deal with Dirichlet boundary value problems as

$$\begin{cases} x''(t) + A(t)x'(t) + B(t)x(t) + \lambda C(t)x(t) &= 0 \\ x(a) &= 0 \\ x(b) &= 0 \end{cases}$$

where $a < b$, λ is a real parameter and A, B, C are continuous functions in $[a, b]$.

Multiplying the equation by the integrating factor $p(t) = e^{\int_0^t A(s)ds}$ one finds

$$p(t)x''(t) + p(t)A(t)x'(t) + p(t)B(t)x(t) + \lambda p(t)C(t)x(t) = 0.$$

Since $p' = Ap$, then

$$[px']' = Apx' + px'' = Apx' + p(-Ax' - Bx - \lambda Cx) = -pBx - \lambda pCx.$$

Hence, setting

$$r(t) = p(t)B(t), \qquad q(t) = p(t)C(t),$$

the equation becomes

$$\frac{d}{dt}\left[p(t)\frac{dx}{dt}\right] + r(t)x(t) + \lambda q(t)x(t) = 0.$$

From now on we will consider this equation where $p(t) > 0$ and it is continuously differentiable. We will also assume that $q(t) \not\equiv 0$. Moreover, in the above equation, there are two terms involving x. We simplify the equation by letting $r(t) \equiv 0$. This is equivalent to letting $B(t) \equiv 0$.

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_9

9.1 Eigenvalues and eigenfunctions

One of the solutions of

$$\begin{cases} (px')' + \lambda qx = 0 & \text{in } [a,b], \\ x(a) = x(b) = 0 \end{cases} \tag{9.1}$$

is obviously the trivial solution $x(t) \equiv 0$.

Definition 9.1.1. We say that λ is an *eigenvalue* of the system (9.1) if it has a non-trivial solution, called an *eigenfunction*, corresponding to λ.

Remark 9.1.2. If $\varphi(t)$ is an eigenfunction corresponding to an eigenvalue λ, so is $c\varphi(t)$ for all $c \neq 0$. ∎

Theorem 9.1.3. *If* $q(t) > 0$, *then the eigenvalues of* (9.1) *are strictly positive.*

Proof. Let λ be an eigenvalue of (9.1). Multiplying the equation by $x(t)$ and integrating on $[a,b]$ we find

$$\int_a^b (p(t)x'(t))'x(t)dt + \lambda \int_a^b q(t)x^2(t)dt = 0. \tag{9.2}$$

Integrating by parts, the first integral becomes

$$\int_a^b (p(t)x'(t))'x(t)dt = (p(b)x'(b))x(b) - (p(a)x'(a))x(a) - \int_a^b p(t)x'(t)x'(t)dt.$$

Since $x(a) = x(b) = 0$ we infer

$$\int_a^b (p(t)x'(t))'x(t)dt = -\int_a^b p(t)[x'(t)]^2 dt.$$

Since $p(t) > 0$ and $x(t) \not\equiv 0$, this integral is strictly negative. From (9.2) it follows that $\lambda \int_a^b q(t)x^2(t)dt > 0$. Taking again into account that $q(t) > 0$ and $x(t) \not\equiv 0$ it follows that $\lambda > 0$. ∎

Theorem 9.1.4. *Let* $\lambda_1 \neq \lambda_2$ *be two different eigenvalues of* (9.1) *and denote by* $\varphi_1(t)$, $\varphi_2(t)$ *their corresponding eigenfunctions. Then*

$$\int_a^b q(t)\varphi_1(t)\varphi_2(t)dt = 0.$$

Proof. Multiplying $(p\varphi_1')' + \lambda_1 q\varphi_1 = 0$ by φ_2 and integrating by parts from a to b, we obtain

$$\int_a^b p\varphi_1'\varphi_2' dt = \int_a^b \lambda_1\varphi_1(t)\varphi_2(t)q(t)dt.$$

Similarly, multiplying $(p\varphi_2')' + \lambda_2 q\varphi_2 = 0$ by φ_1 and integrating by parts from a to b, we obtain

$$\int_a^b p\varphi_1'\varphi_2' dt = \int_a^b \lambda_2\varphi_1(t)\varphi_2(t)q(t)dt.$$

Therefore,

$$\int_a^b \lambda_1\varphi_1(t)\varphi_2(t)q(t)dt = \int_a^b \lambda_2\varphi_1(t)\varphi_2(t)q(t)dt$$

which implies

$$\int_a^b \varphi_1(t)\varphi_2(t)q(t)dt = 0$$

if we assume that $\lambda_1 \neq \lambda_2$. ∎

Corollary 9.1.5. *Eigenfunctions corresponding to different eigenvalues are linearly independent.*

Proof. If $\varphi_2 = \alpha\varphi_1$ for some real number $\alpha \neq 0$ we would have

$$\int_a^b q(t)\varphi_1(t)\varphi_2(t)dt = \alpha \int_a^b q(t)\varphi_1^2(t)dt = 0,$$

a contradiction. ∎

9.2 Existence and properties of eigenvalues

Consider the case in which $p = q = 1$. The equation becomes $x'' + \lambda x = 0$, whose general solution is $x(t) = c_1 \sin\sqrt{\lambda}\,t + c_2\cos\sqrt{\lambda}\,t$. Imposing the boundary condition $x(a) = x(b) = 0$ we find the algebraic system in the unknowns c_1, c_2

$$\begin{cases} c_1 \sin\sqrt{\lambda}\,a + c_2\cos\sqrt{\lambda}\,a &= 0\\ c_1 \sin\sqrt{\lambda}\,b + c_2\cos\sqrt{\lambda}\,b &= 0.\end{cases}$$

The system has the trivial solution $c_1 = c_2 = 0$. According to Kramer's rule, the system has a nontrivial solution if and only if the determinant of the system is zero, that is

$$\begin{aligned}\begin{vmatrix} \sin\sqrt{\lambda}\,a & \cos\sqrt{\lambda}\,a\\ \sin\sqrt{\lambda}\,b & \cos\sqrt{\lambda}\,b\end{vmatrix} &= \sin\sqrt{\lambda}\,a\cos\sqrt{\lambda}\,b - \cos\sqrt{\lambda}\,a\sin\sqrt{\lambda}\,b\\ &= \sin\sqrt{\lambda}\,(a-b) = 0\end{aligned}$$

whence $\sqrt{\lambda}\,(a-b) = k\pi, k = 1, 2, \ldots$ Then for any $\lambda_k = \left(\frac{k\pi}{b-a}\right)^2, k = 1, 2, \ldots,$ the problem has nontrivial solutions and hence λ_k are the eigenvalues we were looking for.

Example 9.2.1. The eigenvalues of

$$\begin{cases} x'' + \lambda x = 0 \\ x(0) = x(\pi) = 0 \end{cases}$$

are $\lambda_k = k^2, k = 1, 2, \cdots$. The general solution is $x_k(t) = c_1 \sin kt + c_2 \cos kt$. The condition $x_k(0) = 0$ yields $c_2 = 0$ and hence the eigenfunctions are $\varphi_k(t) = C \sin kt$, $C \neq 0$ a constant. ∎

It is possible to extend the previous result to the general equation (9.1) yielding

Theorem 9.2.2. *Suppose that* $q(t) > 0$*. Then there exist infinitely many positive eigenvalues* λ_k *of* (9.1) *such that* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$*. Moreover,* $\lambda_k \to +\infty$*.*

Proof. (Sketch) We outline the proof in the general case. Let $x_{p,\lambda}(t)$ be the solution of the initial value problem

$$\begin{cases} (p(t)x'(t))' + \lambda q(t)x(t) &= 0 \\ x(a) &= 0 \\ x'(a) &= p. \end{cases}$$

If $p \neq 0$, then $x_{p,\lambda}(t) \not\equiv 0$. Thus if $x_{p,\lambda}(t)$ has a zero at $t = b$, then $x_{p,\lambda}(t)$ is an eigenfunction.

Notice that the solution is oscillatory. Denoting by $\alpha_k(p, \lambda)$ the k-th zero of $x_{p,\lambda}$, let us solve the equation $\alpha_k(p, \lambda) = b$. It is possible to show that, for each fixed p, $\alpha_k(p, \lambda)$ is continuous and increasing as a function of λ (see the graph plotted in Figure 9.1). Thus for each $k = 1, 2, \ldots$, the equation $\alpha_k(p, \lambda) = b$ has a solution giving rise to an eigenvalue λ_k. Moreover, one proves that $\alpha_1(p, \lambda) > \alpha_2(p, \lambda) > \ldots > \alpha_k(p, \lambda) > \ldots$ and this implies that $\lambda_1 < \lambda_2 < \cdots$. ∎

We will always assume that $q(t) > 0$ and denote by $\lambda_k[q]$ the eigenvalues of (9.1) and by $\varphi_k(t)$ a corresponding eigenfunction.

The smallest eigenvalue $\lambda_1[q]$ (also called the first or the *principal* eigenvalue) has a "variational characterization" that we are going to outline.

Multiplying $(p\varphi_1')' + \lambda_1[q]q\varphi_1 = 0$ by φ_1 and integrating by parts, one finds

$$-\int_a^b p(t)\varphi_1'^2(t)dt + \lambda_1[q] \int_a^b q(t)\varphi_1^2(t)dt = 0.$$

It follows (recall that we are assuming $q(t) > 0$) that

$$\lambda_1[q] = \frac{\int_a^b p(t)\varphi_1'^2(t)dt}{\int_a^b q(t)\varphi_1^2(t)dt}.$$

Let $\mathcal{C}$ denote the class of functions $\phi \in C^1(a, b)$ such that $\phi(a) = \phi(b) = 0$.

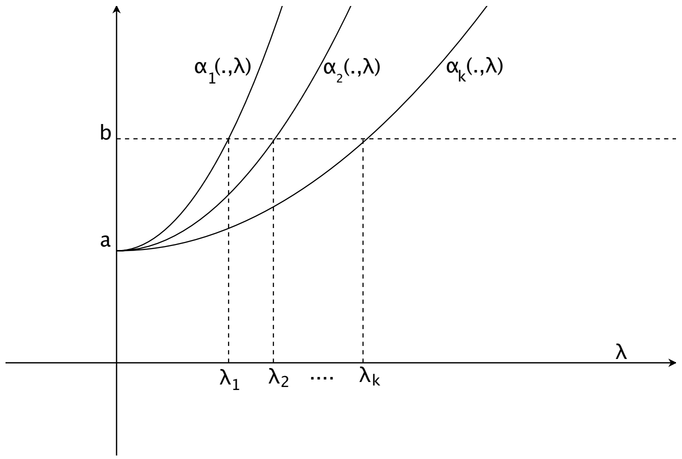


Fig. 9.1. Plot of $\alpha_k(p, \lambda)$ with $p > 0$

Theorem 9.2.3. *One has*

$$\lambda_1[q] \leq \frac{\int_a^b p(t)\phi'^2(t)dt}{\int_a^b q(t)\phi^2(t)dt}, \qquad \forall \phi \in \mathcal{C}. \tag{9.3}$$

Moreover,

$$\lambda_1[q] = \min \left[\frac{\int_a^b p(t)\phi'^2(t)dt}{\int_a^b q(t)\phi^2(t)dt} : \phi \in \mathcal{C} \right].$$

The proof requires advanced topics and is omitted.
The inequality in the preceding Theorem is known as the *Poincaré inequality*. The quotient

$$R(\phi) = \frac{\int_a^b p(t)\phi'^2(t)dt}{\int_a^b q(t)\phi^2(t)dt}$$

on the right-hand side is usually called the *Rayleigh Quotient*.

Example 9.2.4. If $p = q = 1, a = 0, b = \pi$, the problem becomes $x'' + \lambda x = 0$, $x(0) = x(\pi) = 0$ whose eigenvalues are $\lambda_k = k^2, k = 1, 2, \ldots$, see Example 9.2.1. Thus one has

$$\int_0^\pi \phi^2(t)dt \leq \int_0^\pi \phi'^2(t)dt, \qquad \forall \phi \in \mathcal{C}. \qquad \blacksquare$$

Theorem 9.2.5. *Let* $\lambda_k[q_i]$, $i = 1, 2$, *be the eigenvalues of* $(p(t)x')' + \lambda q_i(t)x = 0$, $x(a) = x(b) = 0$. *If* $q_1(t) \leq q_2(t)$ *for all* $t \in [a.b]$, *then* $\lambda_k[q_1] \geq \lambda_k[q_2]$ *for all* $k = 1, 2, \ldots$.

Proof. We prove the result for $k = 1$, using its variational characterization stated in the preceding theorem. Since $0 < q_1(t) \le q_2(t)$ on $[a, b]$, then for all $\phi \in \mathcal{C}$ one has

$$R_1(\phi) = \frac{\int_a^b p(t)\phi'^2(t)dt}{\int_a^b q_1(t)\phi^2(t)dt} \ge \frac{\int_a^b p(t)\phi'^2(t)dt}{\int_a^b q_2(t)\phi^2(t)dt} = R_2(\phi).$$

Since this inequality holds for all $\phi \in \mathcal{C}$, the same holds for the minima of both sides, minima that are achieved, according to Theorem 9.2.3. Then we get

$$\lambda_1[q_1] = \min_{\phi\in\mathcal{C}} R_1(\phi) \ge \min_{\phi\in\mathcal{C}} R_2(\phi) = \lambda_1[q_2],$$

completing the proof. ∎

Corollary 9.2.6. *If* $0 < m \le q(t) \le M$ *on* $[a, b]$*, then*

$$\frac{\pi^2}{M(b-a)^2} \le \lambda_1[q] \le \frac{\pi^2}{m(b-a)^2}.$$

Proof. One has $\lambda_1[M] \le \lambda_1[q] \le \lambda_1[m]$. Since $\lambda_1[m] = \frac{\pi^2}{m(b-a)^2}$ and $\lambda_1[M] = \frac{\pi^2}{M(b-a)^2}$, the result follows. ∎

Example 9.2.7. Let us show that the boundary value problem

$$\begin{cases} x'' + \lambda(x - x^3) = 0 \\ x(0) = x(\pi) = 0 \end{cases}$$

has only the trivial solution if $0 \le \lambda \le 1$. Multiplying the equation by $x(t)$, we get $xx'' = -\lambda(x^2 - x^4)$ and hence

$$\int_0^\pi xx''dt = -\lambda \int_0^\pi (x^2 - x^4)dt.$$

Integrating by parts the left integral and taking into account the boundary conditions $x(0) = x(\pi) = 0$ we infer

$$\int_0^\pi x(t)x''(t)dt = -\int_0^\pi x'^2(t)dt$$

and thus, if $\lambda \ge 0$,

$$\int_0^\pi x'^2(t)dt = \lambda \int_0^\pi (x^2(t) - x^4(t))dt \le \lambda \int_0^\pi x^2(t)dt.$$

If, by contradiction, there is λ, with $0 < \lambda < 1$, such that the boundary value problem has a solution $x(t) \not\equiv 0$, then

$$\int_0^\pi x'^2(t)dt < \int_0^\pi x^2(t)dt.$$

But the Poincaré inequality, in particular Example 9.2.4, yields $\int_0^\pi x^2(t)dt \leq \int_0^\pi x'^2(t)dt$, a contradiction. ∎

Finally, we state, without proof, a property concerning the zeros of eigenfunctions, that can easily be proved as an exercise in the specific case when $p = q = 1$.

Theorem 9.2.8. *Any eigenfunction $\varphi_k(t)$ of (9.1) has exactly $k - 1$ zeros in the open interval (a, b). In particular, φ_1 does not change sign in (a, b).*

Remark 9.2.9. In this chapter we have considered only the Dirichlet boundary conditions $x(a) = x(b) = 0$. It is worth mentioning that one could also consider the *Neumann boundary conditions* $x'(a) = x'(b) = 0$, or else general *mixed boundary conditions*

$$\begin{cases} \alpha_1 x(a) + \beta_1 x'(a) & = 0 \\ \alpha_2 x(b) + \beta_2 x'(b) & = 0 \end{cases}$$

where the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

is nonsingular, namely its determinant is different from zero. These cases require some changes. Some of them are proposed as exercises. ∎

9.3 An application to the heat equation

The heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{9.4}$$

is a partial differential equation that describes the variation of the temperature $u(t, x)$ at time $t \geq 0$ and at a given point x of a rod of length $\ell = \pi$, that is for $x \in [0, \pi]$. Notice that here x is an independent variable, in contrast with the notation used before.

Given

$$f(x) = \sum_{k=1}^{N} f_k \sin kx = f_1 \sin x + f_2 \sin 2x + \ldots + f_N \sin Nx,$$

we look for u satisfying the initial condition

$$u(0, x) = f(x), \qquad x \in [0, \pi], \tag{9.5}$$

that prescribes the temperature at $t = 0$. Moreover, we require that the temperature is zero at the extrema of the rod, that is

$$u(t, 0) = u(t, \pi) = 0, \qquad \forall\, t \geq 0. \tag{9.6}$$

Let us point out that u is not identically zero provided f is not, which we assume throughout in the sequel.

Let us look for a solution of (9.4) by separation of the variables, namely seeking $u(t,x)$ as a product of a function of t and a function of x, that is in the form $u(t,x) = \phi(t)\psi(x)$. Since

$$\frac{\partial u}{\partial t} = \phi'(t)\psi(x), \qquad \frac{\partial^2 u}{\partial x^2} = \phi(t)\psi''(x)$$

one finds

$$\phi'(t)\psi(x) = \phi(t)\psi''(x), \qquad \forall\, t \geq 0,\ \forall\, x \in [0,\pi]. \tag{9.7}$$

Since $u \neq 0$ then $\phi, \psi \neq 0$ and we infer

$$\frac{\phi'(t)}{\phi(t)} = \frac{\psi''(x)}{\psi(x)}, \qquad \forall\, t \geq 0,\ \forall\, x \in [0,\pi].$$

Since the left hand side is a function of t only, while the right hand side is a function of x only, it follows that they are constant. Calling $-\lambda$ this constant, we find

$$\frac{\phi'(t)}{\phi(t)} = \frac{\psi''(x)}{\psi(x)} = -\lambda, \qquad \forall\, t \geq 0,\ \forall\, x \in [0,\pi].$$

Therefore (9.7) implies

$$\phi'(t) + \lambda\phi(t) = 0, \qquad \forall\, t \geq 0 \tag{9.8}$$

and

$$\psi''(x) + \lambda\psi(x) = 0, \qquad \forall\, x \in [0,\pi]. \tag{9.9}$$

Roughly, we have transformed the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ into a couple of ordinary differential equations for the components $\phi(t), \psi(x)$ of u.

Conversely, if $\phi(t), \psi(x)$ satisfy (9.8) and (9.9) for the same constant λ, then $\phi(t)\psi(x)$ verifies (9.7). Moreover, $u(t,x) \neq 0$ whenever both $\phi(t) \neq 0$ and $\psi(x) \neq 0$.

Next, the boundary condition (9.6) yields $\psi(0) = \psi(\pi) = 0$. To have a nontrivial solution of (9.9) with these boundary conditions, λ has to be an eigenvalue of the problem $\psi'' + \lambda\psi = 0$, $\psi(0) = \psi(\pi) = 0$, namely $\lambda_k = k^2$, with $k = 1, 2, \ldots$. Thus nontrivial solutions ψ have the form $\psi_k(x) = A_k \sin kx$, A_k constant and k any positive integer.

For $\lambda = k^2 > 0$ the equation $\phi' + k^2\phi = 0$ yields $\phi(t) = B_k e^{-k^2 t}$, B_k constant, and thus, setting $C_k = A_k B_k$, we find that any

$$u_k(t,x) = C_k e^{-k^2 t} \sin kx, \qquad k = 1, 2, \ldots$$

is a solution of the heat equation (9.4) satisfying the boundary conditions (9.5). Of course, any finite sum of these u_k is a solution and so we can seek solutions in the form

$$u(t,x) = \sum_{k=1}^{\infty} C_k e^{-k^2 t} \sin kx.$$

We first proceed formally, assuming this series is uniformly convergent together with the series $\sum_{k=1}^{\infty} -k^2 C_k e^{-k^2 t} \sin kx$, obtained differentiating the series term by term once w.r.t. t or twice w.r.t x.

If this is the case, we can find the constants C_k, using the initial condition $u(0,x) = f(x)$, that is

$$\sum_{k=1}^{\infty} C_k \sin kx = f(x) = \sum_{k=1}^{N} f_k \sin kx.$$

Multiplying by $\sin hx$ and integrating (recall that under our assumption we can exchange $\int$ and $\sum$) we find

$$\sum_{k=1}^{\infty} C_k \int_0^{\pi} \sin hx \sin kx dx = \sum_{k=1}^{N} f_k \int_0^{\pi} \sin hx \sin kx dx.$$

It is known that

$$\int_0^{\pi} \sin hx \sin kx dx = 0, \quad \text{if } h \neq k, \qquad \int_0^{\pi} \sin^2 kx dx = \frac{\pi}{2}.$$

This implies that $C_k = f_k$ for $k = 1, 2, ..., N$ and $C_k = 0$ for all integer $k > N$ and hence the series $\sum_{k=1}^{\infty} C_k e^{-k^2 t} \sin kx$ reduces to the finite sum $\sum_{k=1}^{N} f_k e^{-k^2 t} \sin kx$.

This allows us to say that the previous formal procedure is consistent and we can conclude that a solution of (9.4) satifying (9.5) and (9.6) with $f = \sum_{k=1}^{N} f_k \sin kx$, is given by

$$u(t,x) = \sum_{k=1}^{N} f_k e^{-k^2 t} \sin kx.$$

For example, if $f(x) = 2\sin x - 5\sin 3x$ we find $u(t,x) = 2e^{-t} \sin x - 5e^{-9t} \sin 3x$.

We complete this section by proving the following uniqueness result.

Theorem 9.3.1. *The solution of* (9.4) *is uniquely determined by the boundary conditions* (9.5) *and the initial condition* (9.6).

Proof. If u, v are two solutions of the preceding problem, then $z = u - v$ satisfies the heat equation

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}$$

and the conditions

$$\begin{cases} z(0,x) & = 0, & \forall\, x \in [0,\pi] \\ z(t,0) & = z(t,\pi) = 0, & \forall\, t \geq 0. \end{cases}$$

The theorem follows if we show that $z(t,x) \equiv 0$. Let us set

$$I(t) = \int_0^\pi z^2(t,x)dx.$$

We have

$$I'(t) = 2\int_0^\pi z(t,x)z_t(t,x)dx = 2\int_0^\pi z(t,x)z_{xx}(t,x)dx.$$

Integrating by parts and taking into account the boundary conditions $z(t,0) = z(t,\pi) = 0$, we find

$$I'(t) = -2\int_0^\pi z_x^2(t,x)dx.$$

If, by contradiction, $z \not\equiv 0$ we have $I(t) > 0$ and $I'(t) < 0$ which implies $0 < I(t) < I(0)$. Since $z(0,x) = 0$ for all $x \in [0,\pi]$, then

$$I(0) = \int_0^\pi z^2(0,x)dx = 0$$

and we get a contradiction. ■

9.4 Exercises

1. If $\alpha > 0$, find the eigenvalues of $x'' + \lambda\alpha x = 0$, $x(0) = x(b) = 0$.
2. If $\beta > 0$, find the eigenvalues of $\beta x'' + \lambda x = 0$, $x(0) = x(b) = 0$.
3. Estimate the eigenvalues of $x'' + \lambda(1+t)x = 0$, $x(0) = x(1) = 0$.
4. Estimate the first eigenvalue of $x'' + \lambda e^t x = 0$, $x(0) = x(2) = 0$.
5. Show that the first eigenvalue λ_1 of $(t^2x')' + \lambda x = 0$, $x(0) = x(\pi) = 0$ is smaller or equal to π^2.
6. If $0 < \alpha \le p(t) \le \beta$ in $[a,b]$, estimate the first eigenvalue λ_1 of $(p(t)x')' + \lambda x = 0$, $x(a) = x(b) = 0$.
7. Estimate the first eigenvalue λ_1 of $(p(t)x')' + \lambda q(t)x = 0$, $x(a) = x(b) = 0$, under the assumption that $0 < \alpha \le p(t) \le \beta$ and $0 < m \le q(t) \le M$ in $[a,b]$.
8. Let $\lambda_1[q]$, resp. $\widetilde{\lambda}_1[q]$, be the first eigenvalue of $(p(t)x')' + \lambda q(t)x = 0$, resp. $(\widetilde{p}(t)x')' + \lambda q(t)x = 0$, with the boundary conditions $x(a) = x(b) = 0$. If $p(t) \le \widetilde{p}(t)$ for all $t \in [a,b]$, show that $\lambda_1[q] \le \widetilde{\lambda}_1[q]$.
9. Show that the eigenvalues of $x'' + \lambda x = 0$, $x'(a) = x'(b) = 0$ cannot be strictly negative.
10. Find the eigenvalues of $x'' + \lambda x = 0$, $x'(0) = x'(\pi) = 0$.
11. Find the eigenvalues of $x'' + \lambda x = 0$, $x(0) = x'(\pi) = 0$.

12. Let $x(t)$ be a solution of the nonhomogeneous problem $x'' + \lambda_k q(t)x = h(t)$, $x(a) = x(b) = 0$, where $\lambda_k = \lambda_k[q]$ is the n-th eigenvalue with corresponding eigenfunction φ_k. Prove that $\int_a^b h(t)\varphi_k(t)dt = 0$.

13. Setting $L(u) = (p(t)u')' + r(t)u$, show that $L(u)v - L(v)u = (p(uv' - vu'))'$. Deduce that if $u(a) = v(a) = u(b) = v(b) = 0$, then $\int_a^b L(u)v dt = \int_a^b L(v)u dt$.

14. Solve $u_t = u_{xx}$, $u(0,x) = \alpha \sin x$, $u(t,0) = u(t,\pi) = 0$.

15. Solve $u_t = c^2 u_{xx}$, $u(0,x) = \alpha \sin x$, with the boundary condition $u(t,0) = u(t,\pi) = 0$.

16. Solve $u_t = u_{xx}$, $u(0,x) = \alpha \sin\left(\frac{\pi x}{L}\right)$, with the boundary condition $u(t,0) = u(t,L) = 0$.

10

Solutions by infinite series and Bessel functions

10.1 Solving second order equations by series

It should be clear by now that the methods for solving differential equations thus far have been limited and applicable only to certain types of equations.

In this chapter we discuss methods of finding solutions of linear differential equations by using power series. The basic idea is to substitute, formally, an infinite power series $x(t) = \sum a_k t^k$ into the equation and use the fact that $\sum b_k t^k = \sum c_k t^k$ if and only if $b_k = c_k$ for all $k \in \mathbb{N}$. In this way, one tries to find a recursive formula that allows us to determine the coefficients of the desired series. One assumes that the series is absolutely convergent, that is analytic, in some interval I so that it can be differentiated term by term. After determining the coefficients of the power series, one tries to find its radius of convergence by some method such as the ratio test.

10.2 Brief review of power series

Recall the following properties of power series.

1. Shifting indices: one can easily verify that

$$\sum_{3}^{n} a_k t^k = \sum_{4}^{n+1} a_{k-1} t^{k-1} = \sum_{1}^{n-2} a_{k+2} t^{k+2}.$$

   Such shifting of indices is important in calculating series solutions. It is easy to remember that in order to increase (or decrease) the indices in the summation by m, we must decrease (or increase) the limits of summation by m.

2. With each power series

$$\sum_{k=0}^{\infty} a_k (t - t_0)^k \tag{10.1}$$

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_10

is associated a *radius of convergence* R, $R \geq 0$, with the following properties:

(a) $R > 0$ and the power series (10.1) is absolutely convergent if $|t - t_0| < R$ and it is divergent if $|t - t_0| > R$. For $|t - t_0| = R$, it can go either way depending on the particular power series. In this case we say that the *interval of convergence* is $|t - t_0| < R$.

(b) $R = 0$ and the power series converges only for $t = t_0$.

(c) $R = \infty$ and the power series converges for all t, with the interval of convergence $(-\infty, \infty)$. The easiest way to evaluate the radius of convergence is the *ratio test* which states that

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

provided the limit exists.

3. If a power series is absolutely convergent, then it is convergent. The converse is false.

4. When a function $f(t)$ has a power series representation at $t = t_0$, with a positive radius of convergence, then $f(t)$ is said to be analytic at $t = t_0$. In such a case, the series can be differentiated term by term infinitely many times, with the derivatives having the same radius of convergence.

5. An analytic function has a unique power series representation, within its radius of convergence, which is given by the Taylor series

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t_0)}{k!}(t - t_0)^k.$$

For example, in order to show that

$$\frac{1}{1-t} = \sum_{0}^{\infty} t^k = 1 + t + t^2 + \ldots + t^k + \ldots$$

is valid for $-1 < t < 1$, instead of using the Taylor expansion, we simply use long division and, dividing 1 by $1 - t$, obtain

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k.$$

In order to find its interval of convergence, we use the ratio test. Thus we have

$$\lim_{k\to\infty} \left| \frac{t^{k+1}}{t^k} \right| = |t|.$$

This shows that the radius of convergence is $R = 1$ and hence

$$\frac{1}{1-t} = \sum_{0}^{\infty} t^k$$

for $-1 < t < 1$. Furthermore, this representation, in terms of powers of t, is unique. So, if we use the Taylor expansion for $f(t) = 1/(1-t)$ around $t = 0$, we will get the same series. Lastly, we note that for $t > 1$, the above series representation does not hold. But we can find its Taylor expansion around $t = 3$, for example.

10.3 Series solutions around ordinary points

Consider the differential equation

$$a_0(t)x^{(n)} + a_1(t)x^{(n-1)} + ... + a_n(t)x(t) = 0$$

where the function $a_i(t)$, $1 \le i \le n$, is analytic at $t = t_0$, with convergent power series in an interval $R_i - t_0 < t < R_i + t_0$.

If $a_0(t_0) \neq 0$, then t_0 is called an *ordinary point*. If $a_0(t_0) = 0$, then it is called a *singular point*. At an ordinary point $t = t_0$, the above differential equation has a unique power series solution at $t = t_0$, for any initial value problem $x(t_0) = \alpha_0,\ ..., x^{(n-1)}(t_0) = \alpha_{n-1}$. The radius of convergence of the solution is at least as large as the smallest of the radii of convergence of the power series of the coefficients $a_i, 0 \le i \le n$.

Singular points are more of a problem. At such points there may not exist analytic solutions. Special cases of such points will be discussed later.

The examples below demonstrate the general procedure for determining series solutions at ordinary points.

Example 10.3.1. We know that the general solution of $x' = x$ is $x(t) = ce^t$. Let us find this by using infinite power series. Setting

$$x(t) = \sum_{k\ge 0} a_k t^k = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k + \ldots$$

we find

$$x'(t) = \sum_{k\ge 1} k a_k t^{k-1} = a_1 + 2a_2 t + \ldots + k a_k t^{k-1} + \ldots .$$

The equation $x' - x = 0$ yields

$$\sum_{k\ge 1} k a_k t^{k-1} - \sum_{k\ge 0} a_k t^k = 0.$$

Our goal now is to make the powers of t the same in both summations so that we can factor it out and set the coefficients equal to 0. We can accomplish this in more than one way. But let us increase the power of t in the first sum by 1, which means that we have to shift down the starting point by 1. Then we obtain

$$\sum_{k\ge 0}(k+1)a_{k+1}t^k - \sum_{k\ge 0} a_k t^k = \sum_{k\ge 0}[(k+1)a_{k+1} - a_k]t^k = 0.$$

Now setting the coefficients equal to 0, we have $(k+1)a_{k+1} - a_k = 0$, which gives us the recursive formula

$$a_{k+1} = \frac{1}{k+1} a_k, \quad k = 0, 1, \dots .$$

Thus

$$a_1 = a_0, \qquad a_2 = \frac{1}{2} a_1 = \frac{1}{2} a_0,$$

$$a_3 = \frac{1}{3} a_2 = \frac{1}{3}\frac{1}{2} a_0 = \frac{1}{3!} a_0, \qquad a_4 = \frac{1}{4} a_3 = \frac{1}{4!} a_0, \dots .$$

It is now clear that in general we have

$$a_k = \frac{a_{k-1}}{k} = \frac{a_{k-2}}{k(k-1)} = \cdots = \frac{a_0}{k(k-1)\cdots 2} = \frac{a_0}{k!}.$$

Therefore the general solution to the given differential equation is

$$x(t) = a_0 \sum_{k=0}^{\infty} \frac{1}{k!} t^k.$$

We note that the above sum is the Taylor expansion for e^t; therefore the general solution is $x(t) = a_0 e^t$, where a_0 is an arbitrary constant. ■

Example 10.3.2. Let us use power series to solve the initial value problem

$$x'' = x, \qquad x(0) = 0, \; x'(0) = c.$$

We set $x = \sum_{k \ge 0} a_k t^k = a_0 + a_1 t + a_2 t^2 + \ldots + a_k t^k + \ldots$. The condition $x(0) = 0$ implies that $a_0 = 0$.

We may use a slightly different procedure to find the recursive formula, as follows. We find

$$\begin{aligned} x' &= \sum_{k \ge 1} k a_k t^{k-1} = a_1 + 2a_2 t + 3a_3 t^2 + \ldots + k a_k t^{k-1} + \ldots \\ x'' &= \sum_{k \ge 2} k(k-1) a_k t^{k-2} \\ &= 2a_2 + 2 \cdot 3\, a_3 t + 3 \cdot 4\, a_4 t^2 + \ldots + k(k-1) a_k t^{k-2} + \ldots . \end{aligned}$$

Then $x'' = x$, $x(0) = 0$, yields $a_0 = 0$ and

$$\sum_{k \ge 2} k(k-1) a_k t^{k-2} = \sum_{k \ge 1} a_k t^k,$$

that is

$$\begin{aligned} &2a_2 + 2 \cdot 3\, a_3 t + 3 \cdot 4\, a_4 t^2 + \ldots + k(k-1) a_k t^{k-2} + \ldots \\ &\quad = a_1 t + a_2 t^2 + a_3 t^2 + \ldots + a_k t^k + \ldots . \end{aligned}$$

This equality implies

$$2a_2 = 0,\ 2 \cdot 3\, a_3 = a_1,\ 3 \cdot 4\, a_4 = a_2,\ \dots,\ k(k-1)a_k = a_{k-2}$$

that is

$$a_2 = 0,\ a_3 = \frac{a_1}{2 \cdot 3} = \frac{a_1}{3!},\ a_4 = \frac{a_2}{3 \cdot 4} = 0,\ a_5 = \frac{a_1}{5!},\ a_6 = 0, \dots .$$

In general, $a_k = 0$ if k is even, while if k is odd

$$a_k = \frac{a_{k-2}}{k(k-1)} = \frac{a_{k-4}}{k(k-1)(k-2)(k-3)} = \cdots = \frac{a_1}{k!}.$$

Thus, setting $c = a_1$, we find

$$x(t) = c\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots\right).$$

The series is absolutely convergent on all of $\mathbb{R}$. This can be verified by using the ratio test since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(2n+3)!}}{\frac{1}{(2n+1)!}} = \lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!} = 0.$$

Let us recall that $t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots = \sinh t$. In other words, $x(t) = c \sinh t$, which can also be found by using the methods discussed in Chapter 5. ∎

Example 10.3.3. Find $x(t) = \sum_{k=0}^{\infty} a_k t^k$ that solves $x'' = t^2 x$. Since $x''(t) = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2}$, the equation $x'' - t^2 x = 0$ yields

$$\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} - \sum_{k=0}^{\infty} a_k t^{k+2} = 0.$$

Once gain, our goal is to combine the two series into one series, factor out the powers of t, and then set the coefficients equal to zero, which will give us the desired recursive formula. To this end, let us increase the power of t in the first sum by 2, which requires that we decrease the lower limit by 2, and also shift down the power of t in the second series by 2, obtaining

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k - \sum_{k=2}^{\infty} a_{k-2}t^k = 0.$$

Now, everything is fine except that one of the sums starts at $k = 0$ and the other at $k = 2$, which means we still cannot combine the two sums. But this problem is easy to resolve by simply writing the first two terms of the first series separately. Thus we have

$$2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3\, t + \sum_{k=2}^{\infty} (k+2)(k+1)a_{k+2}t^k - \sum_{k=2}^{\infty} a_{k-2}t^k =$$
$$2a_2 + 6a_3\, t + \sum_{k=2}^{\infty} \left[(k+2)(k+1)a_{k+2} - a_{k-2}\right] t^k.$$

Now we set the coefficients equal to 0, obtaining $a_2 = a_3 = 0$ and $(k+2)(k+1)a_{k+2} - a_{k-2} = 0$, which gives us the recursive formula

$$a_{k+2} = \frac{a_{k-2}}{(k+2)(k+1)}, \quad k = 2, 3, \ldots$$

which can also be written as

$$a_{k+4} = \frac{a_k}{(k+4)(k+3)}, \qquad k \geq 0.$$

Now we can compute as many terms of the series solution as we wish, which with the aid of computer technology, gives us the approximate solution to any initial value problem to a desired degree of accuracy. So, our big task in solving differential equations by series is finding the recursive formula. However, it is instructive to continue analyzing and simplifying this problem further.

Recall from above that our recursive formula is complemented by $a_2 = a_3 = 0$, while a_0, a_1 remain undetermined. Notice that if $k = 0, 1, 2, 3$, we find

$$\begin{aligned}
a_4 &= \tfrac{a_0}{4(4-1)} = \tfrac{a_0}{4\cdot 3}\\
a_5 &= \tfrac{a_1}{5(5-1)} = \tfrac{a_1}{5\cdot 4}\\
a_6 &= \tfrac{a_2}{6(6-1)} = 0\\
a_7 &= \tfrac{a_3}{7(7-1)} = 0.
\end{aligned}$$

This suggests to distinguish four cases:

1. If $k = 4n$, then

$$\begin{aligned}
a_{4n+4} = a_{4(n+1)} &= \frac{a_{4n}}{(4n+4)\cdot(4n+4-1)}\\
&= \frac{a_{4(n-1)}}{4(n+1)\cdot[4(n+1)-1]\cdot 4n\cdot(4n-1)}\\
&= \cdots\\
&= \frac{a_0}{4(n+1)\cdot[4(n+1)-1]\cdots 4\cdot 3}.
\end{aligned}$$

2. If $k = 4n + 1$, then

$$
\begin{aligned}
a_{4(n+1)+1} &= \frac{a_{4n+1}}{[4(n+1)+1] \cdot 4(n+1)} \\
&= \frac{a_{4(n-1)+1}}{[4(n+1)+1] \cdot 4(n+1) \cdot 4(n-1) \cdot [4(n-1)-1]} \\
&= \cdots \\
&= \frac{a_1}{[4(n+1)+1] \cdot 4(n+1) \cdots 5 \cdot 4}.
\end{aligned}
$$

3. If $k = 4n + 2$, then

$$
\begin{aligned}
a_{4(n+1)+2} &= \frac{a_{4n+2}}{[4(n+1)+2] \cdot [4(n+1)+1]} \\
&= \cdots \\
&= \frac{a_2}{[4(n+1)+2] \cdot [4(n+1)+1] \cdots} = 0.
\end{aligned}
$$

4. If $k = 4n + 3$, then

$$
\begin{aligned}
a_{4(n+1)+3} &= \frac{a_{4n+3}}{[4(n+1)+3] \cdot [4(n+1)+2]} \\
&= \cdots \\
&= \frac{a_3}{[4(n+1)+3] \cdot [4(n+1)+2] \cdots} = 0.
\end{aligned}
$$

In conclusion, the solution depends on the two constants a_0, a_1 and has the form

$$
\begin{aligned}
x(t) &= a_0 \sum_{n \geq 0} \frac{t^{4n}}{4(n+1) \cdot [4(n+1)-1] \cdots 4 \cdot 3} \\
&\quad + a_1 \sum_{n \geq 0} \frac{t^{4n+1}}{[4(n+1)+1] \cdot 4(n+1) \cdots 5 \cdot 4} \\
&= a_0 \left(1 + \frac{t^4}{4 \cdot 3} + \frac{t^8}{8 \cdot 7 \cdot 4 \cdot 3} + ..\right) + a_1 \left(t + \frac{t^5}{5 \cdot 4} + \frac{t^9}{9 \cdot 8 \cdot 5 \cdot 4} + ..\right)
\end{aligned}
$$

which is the general solution of $x'' = t^2 x$. If we were interested, for example, in a solution $x(t)$ satisfying the initial conditions $x(0) = 1,\ x'(0) = 0$, then in the last equation we would simply let $a_1 = 0$ and $a_0 = 1$. ■

In general, one has to be careful not to generalize too quickly based only on a few terms. In the preceding example if one evaluates only the first 3 terms, one might be tempted to think that all the coefficients a_2, a_3, etc. are zero, which, of course, would be false.

10.4 The Frobenius method

In this and the next section we deal with some equations containing singular points.

Let us start by considering the first order linear equation

$$a_0(t)x' + a_1(t)x = 0$$

in which, say, $t = 0$ is a singular point.

Although it is obvious that this problem can be solved by the method of separation of variables, here, by way of motivation, we demonstrate a method that will yield the solutions in terms of infinite series.

For the sake of simplicity we consider

$$tx' + (3 + 4t)x = 0. \tag{10.2}$$

It can easily be verified that if we try to find series solutions for this problem by the earlier method of letting

$$x = \sum_{k=0}^{\infty} a_k t^k$$

we will only obtain the trivial solution. Let us try

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{r+k}$$

where r is to be determined. Notice that

$$tx'(t) = t \sum_{k=0}^{\infty} (r + k) a_k t^{r+k-1} = \sum_{k=0}^{\infty} (r + k) a_k t^{r+k}.$$

Substituting in (10.2) we get

$$\sum_{k=0}^{\infty} (r + k) a_k t^{r+k} + (3 + 4t) \sum_{k=0}^{\infty} a_k t^{r+k} = 0$$

and hence

$$\sum_{k=0}^{\infty} (r + k + 3) a_k t^{r+k} + \sum_{k=0}^{\infty} 4 a_k t^{r+k+1} = 0.$$

Shifting indices in the second series we deduce

$$\sum_{k=0}^{\infty} (r + k + 3) a_k t^{r+k} + \sum_{k=1}^{\infty} 4 a_{k-1} t^{r+k} = 0,$$

or

$$(r+3)a_0t^r + \sum_{k=1}^{\infty}[(r+k+3)a_k + 4a_{k-1}]t^{r+k} = 0.$$

Since all the coefficients of the series have to be zero, we set

$$(r+3)a_0 = 0, \quad (r+k+3)a_k + 4a_{k-1} = 0, \quad \text{for all } k \geq 1.$$

If $a_0 = 0$ then the recursive formula $(r+k+3)a_k = -4a_{k-1}$ yields $a_k = 0$ for all $k \in \mathbb{N}$ and hence we get the trivial solution.

If $a_0 \neq 0$, then we must have $r = -3$, which reduces the above equations to the simple equation

$$ka_k + 4a_{k-1} = 0, \quad i.e. \quad a_k = -\frac{4}{k}\,a_{k-1}, \quad k \geq 1.$$

Therefore,

$$\begin{aligned} a_k &= -\frac{4}{k}\,a_{k-1} = -\frac{4}{k}\cdot\left(-\frac{4}{k-1}\,a_{k-2}\right) = \frac{4^2}{k(k-1)}\,a_{k-2} \\ &= \frac{4^2}{k(k-1)}\cdot\left(-\frac{4}{k-2}\,a_{k-3}\right) = -\frac{4^3}{k(k-1)(k-2)}\,a_{k-3} \\ &= \ldots = (-1)^k\frac{4^k}{k!}\,a_0. \end{aligned}$$

Then we find that the solution to (10.2) is given by

$$x = \sum_{k=0}^{\infty}(-1)^k\frac{4^k}{k!}\,a_0\,t^{k-3} = a_0\,t^{-3}\sum_{k=0}^{\infty}(-1)^k\frac{4^k}{k!}\,t^k.$$

Since

$$\sum_{k=0}^{\infty}(-1)^k\frac{4^k}{k!}\,t^k = \sum_{k=0}^{\infty}\frac{(-4)^k}{k!}t^k = e^{-4t},$$

we get

$$x = a_0\,t^{-3}\,e^{-4t},$$

which includes, for $a_0 = 0$, the trivial solution found above.

Let us point out that, as anticipated, we can obtain the same result by integrating the separable equation

$$x' + \frac{3+4t}{t}x = 0, \quad i.e. \quad \frac{x'}{x} = -\frac{3}{t} - 4,$$

yielding $\ln|x| = -3\ln|t| - 4t$, whereby $x = c\,t^{-3}e^{-4t}$.

As an exercise, the student can repeat the preceding calculations to show that the solutions of $tx' + (q_0 + q_1 t)x = 0$ are given by $x = c\,t^{-q_0}e^{-q_1 t}$. Notice that if

$q_0 < 0$ there are infinitely many solutions passing through $(0, 0)$, while if $q_0 > 0$ no solution but the trivial one is defined for $t = 0$. In any case, there is no solution satisfying the initial condition $x(0) = x_0 \neq 0$. This can also be deduced by the fact that, setting $t = 0$ in the equation $tx' + (q_0 + q_1 t)x = 0$, we get $q_0 x(0) = 0$ whereby $x(0) = 0$.

Next, we consider the second order linear equation

$$p_0(t)x'' + p_1(t)x' + p_2(t)x = 0$$

where the coefficient functions $p_i(t)$ are continuous, and $t = 0$ is a singular point, i.e. $p_0(0) = 0$, and furthermore

$$t\,\frac{p_1(t)}{p_0(t)}, \quad t^2\,\frac{p_2(t)}{p_0(t)}$$

are analytic at $t = 0$. In such a case, $t = 0$ is called a *regular singular point*.

We briefly discuss a general method, due to F.G.Frobenius [1], to solve such problems by using infinite series. Precisely, extending the procedure discussed above for the first order linear equation $tx' + (3 + 4t)x = 0$, we are going to show that one can find a solution by substituting

$$t^r \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+r}$$

for some number r.

To be specific, let us consider the equation

$$t^2 x'' + tP(t)x' + Q(t)x = 0 \tag{10.3}$$

under the assumption that P, Q are polynomials. Clearly $t = 0$ is a regular singular point. If we look for solutions of the form $x(t) = t^r \sum_{k\geq 0} a_k t^k = \sum_{k\geq 0} a_k t^{k+r}$, with $r > 0$, we have:

$$\begin{aligned} tx' &= t \sum_{k\geq 0}(k + r)a_k t^{k+r-1} = \sum_{k\geq 0}(k + r)a_k t^{k+r} \\ t^2 x'' &= t^2 \sum_{k\geq 0}(k + r)(k + r - 1)a_k t^{k+r-2} = \sum_{k\geq 0}(k + r)(k + r - 1)a_k t^{k+r}. \end{aligned}$$

Thus, the equation becomes

$$\sum_{k\geq 0}(k + r)(k + r - 1)a_k t^{k+r} + P(t) \sum_{k\geq 0}(k + r)a_k t^{k+r} + Q(t) \sum_{k\geq 0} a_k t^{k+r} = 0,$$

[1] F.G. Frobenius (1849-1917).

whence

$$\sum_{k\geq 0}\left[(k+r)(k+r-1)a_k t^{k+r} + P(t)(k+r)a_k t^{k+r} + Q(t)a_k t^{k+r}\right] = 0.$$

or

$$\sum_{k\geq 0}\left[(k+r)(k+r-1) + P(t)(k+r) + Q(t)\right] a_k t^{k+r} = 0,$$

namely

$$\begin{aligned}&[r(r-1) + P(t)r + Q(t)]\, a_0 t^r \\ &\quad + \sum_{k\geq 1}\left[(k+r)(k+r-1) + P(t)(k+r) + Q(t)\right] a_k t^{k+r} = 0.\end{aligned}$$

To simplify calculations, let us consider the more specific case in which $P(t) = p_0$ and $Q(t) = q_0 + q_1 t$. It follows

$$\begin{aligned}&[r(r-1) + p_0 r + (q_0 + q_1 t)]\, a_0 t^r + \\ &\quad + \sum_{k\geq 1}\left[(k+r)(k+r-1) + p_0(k+r) + (q_0 + q_1 t)\right] a_k t^{k+r} \\ &= [r(r-1) + p_0 r + q_0]\, a_0 t^r + q_1 a_0 t^{r+1} + \\ &\quad + \sum_{k\geq 1}\left[(k+r)(k+r-1) + p_0(k+r) + q_0\right] a_k t^{k+r} + q_1 a_k t^{k+r+1} = 0.\end{aligned}$$

If we introduce the second order polynomial

$$F(r) = r(r-1) + rp_0 + q_0,$$

the preceding equation can be written as

$$\begin{aligned}F(r)a_0 t^r + q_1 a_0 t^{r+1} + F(r+1)a_1 t^{r+1} + q_1 a_1 t^{r+2} \\ + F(r+2)a_2 t^{r+2} + q_1 a_2 t^{r+3} + \ldots = 0.\end{aligned}$$

All the coefficients of the power t^{r+k} have to be zero and hence we deduce

$$\begin{aligned}F(r)a_0 &= 0 \\ F(r+1)a_1 + q_1 a_0 &= 0 \\ F(r+2)a_2 + q_1 a_1 &= 0 \\ \ldots\quad &\quad \ldots\end{aligned}$$

and, in general,

$$F(r)a_0 = 0, \quad F(r+k)a_k + q_1 a_{k-1} = 0, \quad k = 1, 2, \ldots .$$

If $F(r) \neq 0$ we find that $a_0 = 0$. Moreover, if $F(r+1) \neq 0$ then $F(r+1)a_1 = 0$ yields $a_1 = 0$ and so on: if $F(r+k) \neq 0$ for all $k \geq 0$ then all the a_k are 0 and the procedure gives the trivial solution.

The equation $F(r) = 0$ is called the *indicial equation* and is a second order algebraic equation. Its roots are called the *characteristic exponents* of (10.3).

Let r_1 be a root of $F(r) = 0$. Now, if we put $r = r_1$ in the preceding recursive formulae, a_0 remains undetermined, while

$$\begin{aligned} a_1\ F(r_1+1) &= -q_1 a_0 \\ a_2\ F(r_1+2) &= -q_1 a_1 \\ \dots &\quad \dots\dots \\ a_k\ F(r_1+k) &= -q_1 a_{k-1}. \end{aligned}$$

Then for all $k \geq 1$ such that $F(r_1 + k) \neq 0$, these equations allow us to find a_k while if $F(r_1 + k) = 0$ for some k, the corresponding a_k remains undetermined. It can be shown that the corresponding series converges uniformly on $\mathbb{R}$. So, the formal procedure is consistent and

$$x(t) = t^{r_1} \sum_{k \geq 0} a_k t^k$$

is a solution of (10.3). For example, if $F(r_1 + k) \neq 0$ for all $k \geq 1$, one finds

$$\begin{aligned} a_1 &= -q_1 \frac{a_0}{F(r_1+1)} \\ a_2 &= -q_1 \frac{a_1}{F(r_1+2)} = q_1^2 \frac{a_0}{F(r_1+1)F(r_1+2)} \\ a_3 &= -q_1 \frac{a_2}{F(r_1+3)} = -q_1^3 \frac{a_0}{F(r_1+1)F(r_1+2)F(r_1+3)} \\ \dots\dots &\quad \dots\dots \\ a_k &= (-1)^k q_1^k \frac{a_0}{F(r_1+1)F(r_1+2)\cdots F(r_1+k)} \end{aligned}$$

and the solution $x(t)$ will depend on the constant a_0.

Example 10.4.1. Find a solution of $t^2 x'' + tx' + (t - 1)x = 0$. Here $p_0 = 1$, $q_0 = -1$ and $q_1 = 1$. Thus the indicial equation is

$$F(r) = r(r-1) + r - 1 = r^2 - 1$$

whose roots are ± 1. Taking $r = 1$ we find $F(k + 1) = (k + 1)^2 - 1 = k(k + 2)$ which vanishes if and only if $k = 0$. Thus for $k = 1, 2 \dots$ we find

$$a_k = (-1)^k \frac{a_0}{3 \cdot 8 \cdots k(k+2)}.$$

Hence a solution is

$$x(t) = a_0 t \cdot \sum_{k \geq 1} (-1)^k \frac{t^k}{3 \cdot 8 \cdots k(k+2)},$$

which depends upon a constant a_0. ■

Finding a second solution $y(t)$, linearly independent of the preceding one, requires some caution. Referring e.g. to the book by E.L. Ince (see Bibliography) for a complete discussion, we limit ourselves to simply stating that if r_1, r_2 are the roots of the indicial equation, one should distinguish among 3 cases:

1. If $r_1 \neq r_2$ and they do not differ by an integer, then $y(t) = t^{r_2} \sum_{k\geq 0} b_k t^k$, where b_k satisfies the recursive formula $b_k F(r_2 + k) = -q_1 b_{k-1}$.
2. If $r_1 \neq r_2$ and they differ by an integer, then $y(t) = cx(t) \ln t + t^{r_2} \sum_{k\geq 0} b_k t^k$, where the constant c can be zero.
3. If $r_1 = r_2$, then $y(t) = x(t) \ln t + t^{r_1} \sum_{k\geq 0} b_k t^k$ where b_k has to be determined.

Example 10.4.2. Consider the equation $t^2 x'' + 2tx' - \ell(\ell+1)x = 0$ with $\ell > 0$, that arises in solving the 3D Laplace equation in spherical coordinates. This is an Euler equation. Here we use the Frobenius method to find solutions. The given equation is of the form (10.3) with $p_0 = 2$, $q_0 = -\ell(\ell+1)$ and $q_1 = 0$. Then we have

$$a_k \; F(r_1 + k) = 0, \qquad k = 1, 2, \ldots .$$

The indicial equation is $F(r) = r(r-1) + 2r - \ell(\ell+1) = 0$, that is $r^2 + r - \ell(\ell+1) = 0$, whose roots are $r_1 = \ell$, $r_2 = -(\ell+1)$. Taking $r = \ell$ one finds

$$\begin{aligned} F(\ell + k) &= (\ell + k)(\ell + k - 1) + 2(\ell + k) - \ell(\ell + 1) \\ &= k(4\ell + 3k + 1) > 0, \quad \forall\, k = 1, 2, \ldots . \end{aligned}$$

Thus $a_k = 0$ for all $k \geq 1$. Taking for example $a_0 = 1$, a solution is given by $x(t) = t^\ell$. Even if the two roots may differ by an integer, in this specific case a second linearly independent solution can be found in the form $y(t) = t^{-(\ell+1)}$. It is easy to check that the two functions t^ℓ and $t^{-(\ell+1)}$ are linearly independent and so the general solution is $x(t) = c_1 t^\ell + c_2 t^{-(\ell+1)}$. ■

10.5 The Bessel equations

In this Section we consider the Bessel[2] equations, namely

$$t^2 x'' + tx' + (t^2 - m^2)x = 0. \tag{10.4}$$

The number m is called the *order of the Bessel equation* and could be an integer or a rational number.

Bessel equations arise in applications when solving the Laplace equation or the wave equation, in 3-dimensional space, using the method of separation of variables. For completeness, we briefly outline how it works in the case of the Laplace equation.

---

[2] Friedrich Wilhelm Bessel (1784–1846).

The 3-D Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$ in cylindrical coordinates $x = r\cos\phi$, $y = r\sin\phi$, z can be written as

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\phi\phi} + u_{zz} = 0.$$

Looking for solutions in the form $u(r, \phi, z) = R(r)\Phi(\phi)Z(z)$ one checks that the equation becomes

$$\frac{1}{rR}(rR_r)_r + \frac{1}{r^2\Phi}\Phi_{\phi\phi} + \frac{1}{Z}Z_{zz} = 0.$$

The only term containing z is $\frac{1}{Z}Z_{zz}$, which therefore has to be constant, say k^2. Then $\frac{1}{Z}Z_{zz} = k^2$, namely $Z_{zz} = k^2 Z$, which yields

$$\frac{1}{rR}(rR_r)_r + \frac{1}{r^2\Phi}\Phi_{\phi\phi} + k^2 = 0$$

and, multiplying by r^2, we obtain

$$\frac{r}{R}(rR_r)_r + \frac{1}{\Phi}\Phi_{\phi\phi} + k^2 r^2 = 0.$$

The only term containing ϕ is $\frac{1}{\Phi}\Phi_{\phi\phi}$ and hence it has to be constant, say $\frac{1}{\Phi}\Phi_{\phi\phi} = A$, namely $\Phi_{\phi\phi} = A\Phi$. Since we expect that Φ is periodic with respect to ϕ, we take $A = -m^2$ so that we get $\Phi_{\phi\phi} + m^2\Phi = 0$, the equation of a harmonic oscillator. Then we can write the equation for $R(r)$ as

$$r(rR_r)_r + (k^2r^2 - m^2)R = 0$$

that is

$$r^2 R_{rr} + rR_r + (k^2r^2 - m^2)R = 0.$$

Finally, setting $t = kr$ and $x(t) = R(r/k)$, we find $x'(t) = k^{-1}R_r$, $x''(t) = k^{-2}R_{rr}$. Thus $rR_r = krx' = tx'$, $r^2R_{rr} = r^2k^2x'' = t^2x''$ whence

$$t^2x''(t) + tx'(t) + (t^2 - m^2)x(t) = 0,$$

which is the Bessel equation of order m.

Now, let us solve the Bessel equations. Even if $t = 0$ is a regular singular point, we do not follow the general Frobenius method, but we prefer to handle (10.4) in a simpler way by using the series $\sum_{k=0}^{\infty} a_k t^k$. Though in general this method gives rise to the trivial solution (i.e. $a_k = 0$ for all k), it works in the case of Bessel equations.

Setting $x = \sum a_k t^k$, we find

$$\begin{aligned} tx' &= \sum_{k\geq 1} ka_k t^k = a_1 t + 2a_2t^2 + 3a_3t^3 + \ldots + ka_kt^k + \ldots \\ t^2x'' &= \sum_{k\geq 2} k(k-1)a_k t^k \\ &= 2a_2t^2 + 3\cdot 2a_3t^3 + 4\cdot 3t^4 + \ldots + k(k-1)a_kt^k + \ldots . \end{aligned}$$

Then x solves the Bessel equation (10.4) provided

$$\sum_{k\geq 2} k(k-1)a_k t^k + \sum_{k\geq 1} k a_k t^k + (t^2 - m^2)\sum_{k\geq 0} a_k t^k = 0.$$

We can write this equality in the form

$$\sum_{k\geq 2} k(k-1)a_k t^k + \sum_{k\geq 1} k a_k t^k - m^2 \sum_{k\geq 0} a_k t^k + \sum_{k\geq 0} a_k t^{k+2} = 0. \tag{10.5}$$

In the sequel we will carry out a detailed discussion in the cases $m = 0, 1$. The other cases will be sketched only.

10.5.1 The Bessel equation of order 0

When $m = 0$ we have

$$t^2 x'' + t x' + t^2 x = 0 \tag{10.6}$$

which is the Bessel equation of order 0. If $m = 0$ the preceding equality (10.5) becomes

$$\sum_{k\geq 2} k(k-1)a_k t^k + \sum_{k\geq 1} k a_k t^k + \sum_{k\geq 0} a_k t^{k+2} = 0.$$

Since

$$\sum_{k\geq 1} k a_k t^k = a_1 t + \sum_{k\geq 2} k a_k t^k$$

and

$$\sum_{k\geq 0} a_k t^{k+2} = \sum_{k\geq 2} a_{k-2} t^k,$$

we infer

$$\sum_{k\geq 2} k(k-1)a_k t^k + \sum_{k\geq 2} k a_k t^k + \sum_{k\geq 2} a_{k-2} t^k + a_1 t = 0.$$

Simplifying, we have

$$\sum_{k\geq 2} k^2 a_k t^k + \sum_{k\geq 2} a_{k-2} t^k + a_1 t = 0.$$

Then $a_1 = 0$ and, for $k \geq 2$,

$$k^2 a_k + a_{k-2} = 0 \implies a_k = -\frac{a_{k-2}}{k^2}.$$

In other words,

$$a_k = 0 \quad \text{if } k \text{ is odd},$$

while, for k even, we find

$$a_2 = -\frac{a_0}{2^2}, \quad a_4 = \frac{a_2}{4^2} = \frac{a_0}{2^2 \cdot 4^2}, \ldots .$$

In general we have, for k even,

$$a_k = (-1)^{k/2} \frac{a_0}{2^2 \cdot 4^2 \cdots k^2}.$$

Then a first family of solutions of the Bessel equation with $m = 0$ is given by

$$x(t) = a_0 \left(1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right),$$

where the series turns out to be uniformly convergent on all $\mathbb{R}$.

If we set

$$\begin{aligned} J_0(t) &= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \\ &= 1 - \frac{1}{2}\left(\frac{t}{2}\right)^2 + \frac{1}{2^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(2 \cdot 3)^2}\left(\frac{t}{2}\right)^6 + \cdots \\ &= \sum_{k \ge 0} \frac{(-1)^k}{(k!)^2}\left(\frac{t}{2}\right)^{2k} \end{aligned}$$

we have that all the functions $x(t) = c J_0(t)$ solve the Bessel equation of order 0.

The function J_0 is analytic, even and $J_0(0) = 1$. Moreover, it is possible to show that J_0 has the following properties (as for the first two, see also Theorem 10.5.4):

1. It has infinitely many zeros.
2. It decays to zero at infinity.
3. It is integrable on all $\mathbb{R}^+$ and $\int_0^{+\infty} J_0(t)dt = 1$.

The graph of $J_0(t)$ is shown in Figure 10.1. The function J_0 is called the Bessel function of order 0, *of first type*, to distinguish it from another solution that we are going to find.

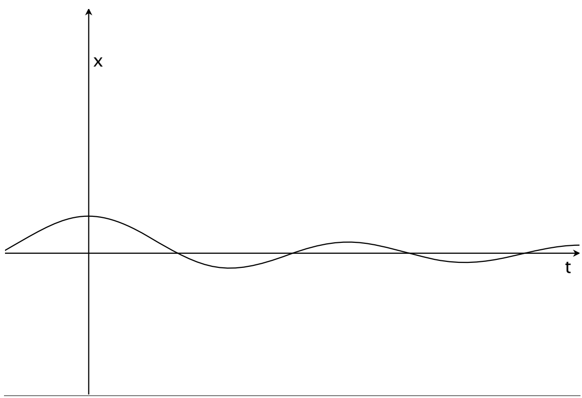


Fig. 10.1. Graph of $J_0(t)$

To find a second solution $Y_0(t)$ of (10.4) linearly independent of J_0 we can use the method of reduction of order discussed in Chapter 5, Section 5.3. We let $Y_0(t) = v(t)J_0(t)$. Then $Y_0' = v'J_0 + vJ_0'$ and $Y_0'' = v''J_0 + 2v'J_0' + vJ_0''$. Substituting into (10.4) we find

$$t^2(v''J_0 + 2v'J_0 + vJ_0'') + t(v'J_0 + vJ_0') + t^2 vJ_0 = 0.$$

Rearranging,

$$v(t^2J_0'' + tJ_0' + t^2J_0) + tJ_0(tv'' + v') = 0.$$

Since J_0 solves (10.4) it follows that

$$tJ_0(tv'' + v') = 0.$$

Solving $tv'' + v' = 0$ we find either $v = const.$ or, setting $z = v'$, $tz' = -z$ which is separable. Integrating we find $z = \frac{1}{t}$ and hence $v = \ln t$, $t > 0$. Then we have found

$$Y_0(t) = \ln t \cdot J_0(t), \qquad t > 0.$$

Similar to J_0, Y_0 also has infinitely many zeros (the same as J_0), but unlike J_0, Y_0 has a singularity at $t = 0$. It is named a Bessel function of order 0 of the *second kind*. The graph of Y_0 is shown below in Figure 10.2.

Since J_0 and Y_0 are linearly independent, the general solution of the Bessel equation of order 0 is given by

$$x(t) = c_1J_0(t) + c_2Y_0(t) = c_1J_0(t) + c_2 \ln t \cdot J_0(t).$$

Example 10.5.1. Find a solution of $t^2x'' + tx' + t^2x = 0$ such that $x(0) = 2$.

In the preceding formula of the general solution of the Bessel equation of order 0, the function $Y_0(t) \to -\infty$ as $t \to 0+$. Then the condition $x(0) = 2$ implies that $c_2 = 0$. Moreover, since $J_0(0) = 1$, then $c_1 = 2$. Thus $x(t) = 2J_0(t)$. ■

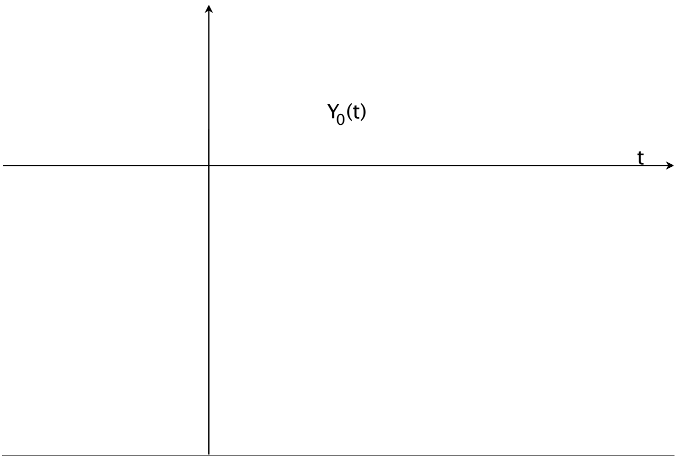


Fig. 10.2. Graph of $Y_0(t)$

10.5.2 The Bessel equation of order 1

When $m = 1$ we have the Bessel equation of order 1, namely

$$t^2 x'' + tx' + (t^2 - 1)x = 0. \tag{10.7}$$

As before, we first look for solutions of the form $x(t) = \sum a_k t^k$. Now the general equality (10.5) becomes

$$\sum_{k \geq 2} k(k-1)a_k t^k + \sum_{k \geq 1} k a_k t^k - \sum_{k \geq 0} a_k t^k + \sum_{k \geq 0} a_k t^{k+2} = 0.$$

If in all the infinite sums we take $k \geq 2$ we find

$$\sum_{k \geq 2} \left[k(k-1)a_k t^k + k a_k t^k - a_k t^k + a_{k-2} t^k \right] + a_1 t - a_0 - a_1 t = 0.$$

Simplifying, we have

$$\sum_{k \geq 2} \left[k^2 a_k - a_k + a_{k-2} \right] t^k - a_0 = 0.$$

Then $a_0 = 0$ and a_k satisfy the recurrence formula

$$k^2 a_k - a_k + a_{k-2} = 0 \implies a_k = -\frac{a_{k-2}}{k^2 - 1}, \quad k \geq 2.$$

Thus if k is even we find

$$a_2 = -\frac{a_0}{3} = 0, \quad a_4 = -\frac{a_2}{3 \cdot 5} = 0, \ \ldots \ a_k = 0.$$

If $k \geq 3$ is odd, the coefficients can be found in terms of a_1. For what follows it is convenient to set $a_1 = \frac{c}{2}$. With this notation one has

$$\begin{aligned}
a_3 &= -\frac{a_1}{8} = -\frac{a_1}{2^3} = -\frac{c}{2! \cdot 2^3}, \\
a_5 &= -\frac{a_3}{24} = \frac{c}{24 \cdot 2! \cdot 2^3} = \frac{c}{2! \cdot 3! \cdot 2^5}, \\
a_7 &= -\frac{a_5}{48} = -\frac{c}{48 \cdot 2! \cdot 3! \cdot 2^5} = -\frac{c}{3! \cdot 4! \cdot 2^7}, \\
... &= ...
\end{aligned}$$

in general we find

$$a_{2k+1} = (-1)^k \frac{c}{k!(k+1)! \cdot 2^{2k+1}}, \qquad k = 0, 1, 2, \ldots.$$

In conclusion,

$$x(t) = c \left(\frac{t}{2} - \frac{t^3}{2! \cdot 2^3} + \frac{t^5}{2! \cdot 3! \cdot 2^5} - \frac{t^7}{3! \cdot 4! \cdot 2^7} + \ldots \right),$$

the series being uniformly convergent on all of $\mathbb{R}$.

If we set

$$J_1(t) = \left(\frac{t}{2}\right) - \frac{1}{2!}\left(\frac{t}{2}\right)^3 + \frac{1}{2!\cdot 3!}\left(\frac{t}{2}\right)^5 + \ldots$$
$$= \sum_{k\geq 0} \frac{(-1)^k}{k!(k+1)!}\left(\frac{t}{2}\right)^{2k+1}$$

we can say that $x(t) = cJ_1(t)$ solve (10.6). Notice that J_1 is an odd function with $J_1(0) = 0$, $J_1'(0) = 1$. It has infinitely many zeros and decays to zero at infinity, like J_0. It is named *Bessel function of order 1, of the first kind*. The graph of J_1 is reported in Figure 10.3.

An interesting fact is that between two consecutive zeros of J_0 there is a zero of J_1, see Figure 10.4.

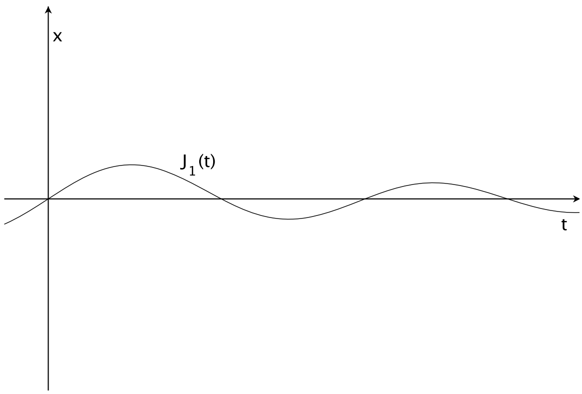


Fig. 10.3. Plot of J_1

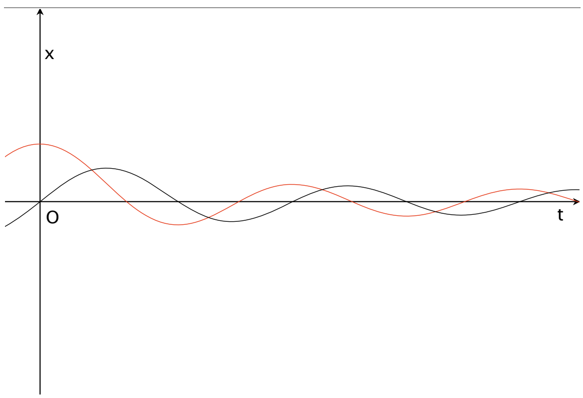


Fig. 10.4. J_0 (*red*) vs. J_1 (*black*)

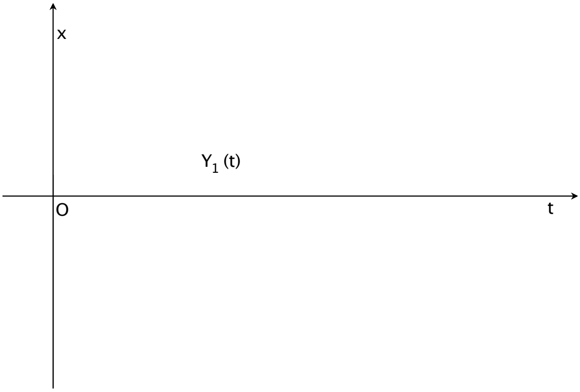


Fig. 10.5. Graph of $Y_1(t)$

This property and the oscillatory character of J_0, J_1 hold in general for all J_m and will be discussed in the next section, see Theorems 10.5.3 and 10.5.4.

As before, a solution of the Bessel equation of order 1, linearly independent of J_1, is given by

$$Y_1(t) = \ln t \cdot J_1(t), \qquad t > 0,$$

which is called Bessel function of order 1, of the *second kind*, see Figure 10.5.

As Y_0, also Y_1 has infinitely many zeros and possesses a singularity at $t = 0$.

It follows that the general solution is

$$c_1 J_1(t) + c_2 Y_1(t).$$

Example 10.5.2. Find a solution of $t^2 x'' + tx' + (t^2 - 1)x = 0$ such that $x(0) = 0$, $x'(0) = 2$. Since $x(0) = 0$, then $c_2 = 0$ because $Y_1(t) \to -\infty$ as $t \to 0+$. Thus $x(t) = c_1 J_1(t)$. From $x'(0) = c_1 J'(0)$, and $J_1'(0) = 1$, it follows that $c_1 = 2$ and the solution of the initial value problem is $x(t) = 2J_1(t)$. ■

10.5.3 Bessel equations of order m

If m is an integer, the Bessel functions of order m, of first kind can be defined as

$$J_m(t) = \sum \frac{(-1)^k}{k!(k+m)!} \left(\frac{t}{2}\right)^{2k+m}.$$

The functions J_m are solutions of the Bessel equation of order m. If m is an even integer, then $J_m(t)$ is an even function, while if m is odd, then $J_m(t)$ is an odd function.

Although for negative m the Bessel equation remains unaffected, it is customary to set

$$J_{-m}(t) = (-1)^m J_m(t).$$

It would be possible to define Bessel functions for any real number m. The expression of $J_m(t)$ is formally equal to the preceding one, giving an appropriate definition of $(m+k)!$, which can be done by means of the *Gamma function* Γ. But this is beyond the scope of this book.

10.5.4 Some properties of the Bessel functions

One can check that the following recurrence formula holds

$$J_{m+1}(t) = \frac{2m}{t} J_m(t) - J_{m-1}(t).$$

For example,

$$J_2(t) = \frac{2}{t} J_1(t) - J_0(t).$$

The function $J_m(t)$ is analytic and has infinitely many zeros. Furthermore, $J_m(0) = 0$ for all $m \neq 0$ and $J'_m(0) = 0$ for all $m \neq 1$.

Moreover, the following identity holds

$$\frac{d}{dt}(t^m J_m(t)) = t^m J_{m-1}(t). \tag{10.8}$$

As an exercise, let us prove (10.8) for $m = 1$. We know that

$$J_1(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!(k+1)!} \cdot \left(\frac{t}{2}\right)^{2k+1}$$

so that

$$t J_1(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!(k+1)!} 2 \cdot \left(\frac{t}{2}\right)^{2k+2}.$$

Recall that the above series is uniformly convergent and hence the derivative of $t J_1(t)$ equals the series obtained by differentiating each term. Thus, taking the derivative one finds

$$\begin{aligned}
(t J_1(t))' &= \sum_{k \geq 0} \frac{(-1)^k}{k!(k+1)!} 2 \cdot \frac{d}{dt} \left(\frac{t}{2}\right)^{2k+2} \\
&= \sum_{k \geq 0} \frac{(-1)^k}{k!(k+1)!} (2k+2) \cdot \left(\frac{t}{2}\right)^{2k+1} \\
&= \sum_{k \geq 0} \frac{(-1)^k}{k!(k+1)!} 2(k+1) \cdot \left(\frac{t}{2}\right)^{2k+1} \\
&= \sum_{k \geq 0} \frac{(-1)^k}{k!\,k!} 2 \cdot \left(\frac{t}{2}\right)^{2k+1}.
\end{aligned}$$

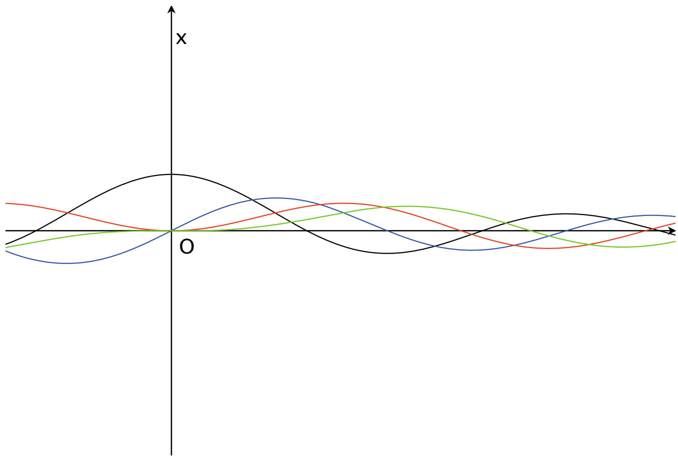


Fig. 10.6. Plots of J_0 (*black*), J_1 (*blue*), J_2 (*red*) and J_3 (*green*)

Since

$$tJ_0(t) = \sum_{k\geq 0} \frac{(-1)^k}{(k!)^2} 2 \cdot \left(\frac{t}{2}\right)^{2k+1}$$

the conclusion follows.

Another useful relationship is

$$\frac{d}{dt}\left(\frac{J_m(t)}{t^m}\right) = -\frac{J_{m+1}(t)}{t^m}. \tag{10.9}$$

One can use (10.8) and (10.9) to prove

Theorem 10.5.3. *Between two consecutive, positive (or negative), zeros of $J_m(t)$ there is one and only one zero of $J_{m+1}(t)$. See Figure 10.6.*

Proof. Let $\alpha_1 < \alpha_2$ be two consecutive, positive, zeros of J_m. Clearly, they are also consecutive zeros of $\frac{J_m(t)}{t^m}$. The Rolle theorem applied to $\frac{J_m(t)}{t^m}$ on the interval $[\alpha_1, \alpha_2]$ implies that there exists $\beta \in]\alpha_1, \alpha_2[$ such that the function $\left(\frac{J_m(t)}{t^m}\right)'$ vanishes at β. By (10.9), β is a zero of J_{m+1}.

Similarly, let $\beta_1 < \beta_2$ be two consecutive, positive, zeros of J_{m+1}. Applying the Rolle theorem to $t^{m+1}J_{m+1}(t)$ on the interval $[\beta_1, \beta_2]$, we find $\alpha \in (\beta_1, \beta_2)$ such that $(t^{m+1}J_{m+1}(t))'$ vanishes at α. Using (10.8) we deduce that $\alpha^{m+1}J_m(\alpha) = 0$, namely that α is a zero of J_m. ∎

Similar results can be given for the Bessel functions of second kind. For example, one has

$$Y_{m+1}(t) = \frac{2m}{t} Y_m(t) - Y_{m-1}(t), \qquad Y_{-m}(t) = (-1)^m Y_m(t),$$

$$\frac{d}{dt}(t^m Y_m(t)) = t^m Y_{m-1}(t), \qquad \frac{d}{dt}\left(\frac{Y_m(t)}{t^m}\right) = -\frac{Y_{m+1}(t)}{t^m}.$$

Each $Y_m(t)$ is singular at $t = 0$ and has infinitely many zeros that alternate between each other. See Figure 10.7.

As a further application, let us look for $\lambda > 0$ such that the problem

$$s\ \ddot{y}(s) + \lambda y(s) = 0, \qquad y(0) = 0, \quad \dot{y}(1) = 0 \tag{10.10}$$

has a nontrivial solution. Here $\dot{y} = \frac{d}{ds}$ and $\ddot{y} = \frac{d^2y}{ds^2}$.

First of all, let us show that the change of variable $t = 2\sqrt{\lambda s}$ and $y(s) = tx(t)$ transforms the equation into a Bessel equation of order 1. Actually, one has

$$\begin{aligned} \dot{y} = \frac{dy}{ds} &= \frac{d(tx(t))}{dt}\frac{dt}{ds} = \big(x(t) + tx'(t)\big)\sqrt{\frac{\lambda}{s}} \\ &= \frac{2\lambda}{t}\big(x(t) + tx'(t)\big) = 2\lambda\left(\frac{x(t)}{t} + x'(t)\right) \end{aligned}$$

and

$$\ddot{y}(s) = \frac{4\lambda^2}{t}\left(x''(t) + \frac{x'(t)}{t} - \frac{x(t)}{t^2}\right).$$

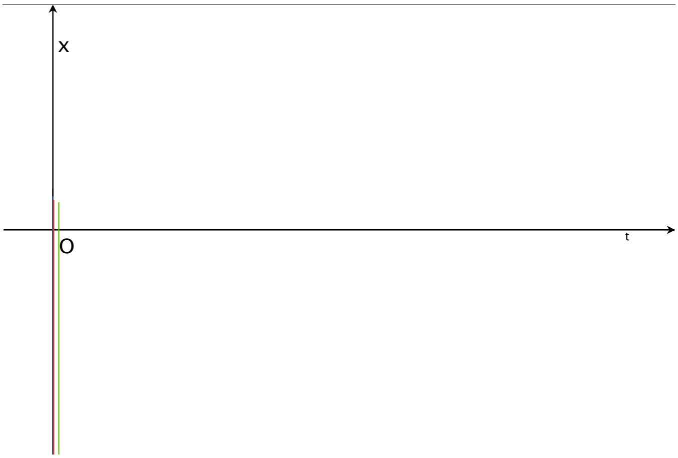


Fig. 10.7. Plots of Y_0 (*black*), Y_1 (*blue*), Y_2 (*red*) and Y_3 (*green*)

Recalling that $4\lambda s = t^2$, we find

$$s\,\ddot{y}(s) = \frac{t^2}{4\lambda}\,\frac{4\lambda^2}{t}\,\left(x''(t) + \frac{x'(t)}{t} - \frac{x(t)}{t^2}\right) = \lambda t\,\left(x''(t) + \frac{x'(t)}{t} - \frac{x(t)}{t^2}\right).$$

Thus $s\ddot{y}(s) + \lambda y(s) = 0$ becomes

$$\lambda t\,\left(x''(t) + \frac{x'(t)}{t} - \frac{x(t)}{t^2}\right) + \lambda t x(t) = 0.$$

Dividing by $\lambda > 0$ we get

$$t x''(t) + x'(t) - \frac{x(t)}{t} + t x(t) = 0$$

or

$$t^2 x''(t) + t x'(t) + (t^2 - 1)x(t) = 0$$

which is the Bessel equation of order 1. A family of solutions is $x(t) = c J_1(t)$, c a constant, whence

$$y(s) = 2c\sqrt{\lambda s}\;J_1(2\sqrt{\lambda s}).$$

For $s = 0$ we have $y(0) = 0$. Moreover

$$\dot{y}(s) = c\left(\frac{\sqrt{\lambda}}{2\sqrt{s}}\;J_1(2\sqrt{\lambda s}) + \lambda\;J_1'(2\sqrt{\lambda s})\right).$$

Recall that by (10.8), one has $J_1'(t) = J_0(t) - \frac{1}{t}J_1(t)$ and hence for $t = 2\sqrt{\lambda s}$

$$\frac{\sqrt{\lambda}}{2\sqrt{s}}\;J_1(2\sqrt{\lambda s}) + \lambda\;J_1'(2\sqrt{\lambda s}) = J_0(2\sqrt{\lambda s}).$$

Then $\dot{y}(s) = c\lambda\,J_0(2\sqrt{\lambda s})$ and the condition $\dot{y}(1) = 0$ yields $J_0(2\sqrt{\lambda}) = 0$.

In conclusion, if $0 < \alpha_1 < \alpha_2 < \cdots \alpha_n < \cdots$ denote the zeros of J_0, then for each $\lambda_n = \left(\frac{\alpha_n}{2}\right)^2$, the problem (10.10) has nontrivial solutions given by $y_n(s) = 2c\sqrt{\lambda_n s}J_1(2\sqrt{\lambda_n s})$.

We conclude this section by stating, without proof, the following result:

Theorem 10.5.4. *J_m decays to zero at infinity, changing sign infinitely many times.*

10.6 Exercises

1. Find $x(t) = \sum a_k t^k$ such that $tx'' = x$.
2. Find $x(t) = \sum_{k\geq 0} a_k t^k$ such that $tx'' = x'$.
3. Find $x(t) = \sum_{k\geq 0} a_k t^k$ such that $x'' = tx + 1$ and $x(0) = 0, x'(0) = 1$.
4. Solve $x'' + tx' + x = 0, x(0) = 1, x'(0) = 0$.
5. Using the Frobenius method, solve $4t^2x'' + 4tx' - x = 0, t > 0$.
6. Using the Frobenius method, solve $t^2x'' + 3tx' = 0, t > 0$.
7. Using the Frobenius method, solve $t^2x'' - 3tx' + (4 - t)x = 0$.
8. Using the Frobenius method, solve $t^2x'' + tx' + (t - 1)x = 0$.
9. Find the solution x_a of the Bessel equation $t^2x'' + tx' + t^2x = 0$ such that $x_a(0) = a$.
10. Find the solution x_a of the Bessel equation $t^2x'' + tx' + (t^2 - 1)x = 0$ such that $x_a'(0) = a$.
11. Find the positive integers m such that $t^2x'' + tx' + (t^2 - m^2)x = 0$ has a nontrivial solution such that $x(0) = 0$.
12. Prove that $(t^2 J_2(t))' = t^2 J_1(t)$.
13. Show that $J_0(t)$ has a maximum at $t = 0$.
14. Let α be a positive zero of $J_0(t)$. Show that if $J_1(\alpha) > 0$ then $J_0'(\alpha) < 0$.
15. Setting $Z(t) = J_0(t) - tJ_1(t)$, show that if α is a zero of $J_0(t)$ then $Z'(\alpha) = J_0'(\alpha)$.
16. Using the power expansions of J_0, J_1, J_2, prove that $J_2(t) = \frac{2}{t}J_1(t) - J_0(t)$.
17. Prove that $J_m'(t) = J_{m-1}(t) - \frac{m}{t}J_m(t)$, m an integer.
18. Prove if α_1 is the first positive zero of $J_0(t)$, then $J_1(\alpha_1) > 0$ and $J_2(\alpha_1) > 0$.
19. Let α_1 denote the first positive zero of J_0. Show that the only solution of $t^2x'' + tx' + (t^2 - 1)x = 0$ such that $x(0) = x(\alpha_1) = 0$ is $x(t) \equiv 0$.
20. Find $\lambda > 0$ such that the problem
$$s\,\frac{d^2y}{ds^2} + \lambda y(s) = 0, \qquad y(0) = 0, \quad y(1) = 0,$$
has a nontrivial solution. [Hint: use the change of variable $t = 2\sqrt{\lambda s}$ to transform the equation into the Bessel equation of order 1.]
21. Find the positive integer λ such that $t^2x'' + tx' + t^2x = \lambda x$ has a nontrivial solution satisfying $x(0) = 0, x'(0) = 1$.

11

Laplace transform

In this chapter we study some basic properties of the *Laplace Transform*, (or *L-Transform* for short), which plays an important role not only in mathematics but also in the sciences and engineering. This is a deep subject and a rigorous and thorough treatment of it would be beyond the scope of this book. So we will not venture too far beyond some of its basic properties. In particular, we will discuss its application in solving initial value problems for linear differential equations with constant coefficients. This powerful method essentially consists of converting an initial value problem to an algebraic equation involving the Laplace Transform X of the solution x to the differential equation, and then recovering the solution x by taking the inverse of X.

11.1 Definition and preliminary examples

Given a real valued function $f(t)$ defined on $[0, +\infty)$, the L-transform of f, denoted by $\mathcal{L}\{f(t)\}(s)$, or $F(s)$, is the function of s defined by

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t)dt,$$

provided the integral makes sense, that is

$$\lim_{r \to +\infty} \int_0^{r} e^{-st} f(t)dt$$

exists and is finite. The set of $s \in \mathbb{R}$ where this is true is called the region (or domain) of convergence of $\mathcal{L}\{f(t)\}(s)$. Often we will write $\mathcal{L}\{f(t)\}$ or simply $\mathcal{L}\{f\}$ instead of $\mathcal{L}\{f(t)\}(s)$.

We notice that one could define the L-transform of a complex valued function of the complex variable $s = \sigma + i\omega$. But for our purposes it is sufficient to limit our study to the real case.

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_11

The L-transform is well defined for a broad class of functions. Recall that f has a jump discontinuity at $t = a$ if both one-sided limits

$$\lim_{t\to a^+} f(t), \qquad \lim_{t\to a^-} f(t)$$

exist but are not equal to each other. For example, the function

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ t+1, & \text{if } t \geq 0 \end{cases}$$

has a jump discontinuity at $t = 0$ because $\lim_{t\to 0^-} f(t) = 0$ while $\lim_{t\to 0^+} f(t) = \lim_{t\to 0^+}(t+1) = 1$. On the other hand, the function

$$g(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{1}{t}, & \text{if } t > 0 \end{cases}$$

has a discontinuity at $t = 0$ which is not a jump discontinuity.

We say that f is *piecewise continuous* on its domain D if there exists a numerable set $\mathcal{S}$ such that f is continuous on $D \setminus \mathcal{S}$ and has a jump discontinuity at each point of $\mathcal{S}$. If $\mathcal{S}$ is empty, then f is just continuous. For example, the preceding function f is piecewise continuous, while g is not.

A function $f(t)$ is said to be of exponential order if there exist constants M and α such that

$$|f(t)| \leq Me^{\alpha t}. \tag{11.1}$$

For example, any *bounded* piecewise continuous function, any polynomial $P(t) = a_0 + a_1 t + \ldots + a_n t^n$, in general any function such that $f(t) = O(t^n)$ as $|t| \to +\infty$ (i.e. for some number A, $|f(t)| \leq At^n$ for t large enough), all satisfy (11.1) for $s > 0$ (hence $\mathcal{L}\{f\}(s)$ exists for $s > 0$). However, $f(t) = e^{t^2}$ does not satisfy (11.1): since $e^{t^2}e^{-\alpha t} \leq M$, is obviously false.

Theorem 11.1.1. *Suppose that f is piecewise continuous on $\mathbb{R}^+$ and satisfies* (11.1). *Then $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$.*

Proof. One has

$$\left|e^{-st} f(t)\right| dt \leq e^{-st} Me^{\alpha t} = Me^{(\alpha - s)t}.$$

We recall from Calculus that if $\int_0^\infty |f(t)|dt$ exists, then $\int_0^\infty f(t)dt$ also exists. Therefore $e^{-st} f(t)$ is integrable on $\mathbb{R}^+$. Moreover

$$\int_0^{+\infty} e^{-st} f(t)dt$$

exists and is finite, provided $s > \alpha$, which means that $\mathcal{L}\{f\}(s)$ exists for all $s > \alpha$. ∎

In the sequel, even if not explicitly stated, it will be assumed that the functions we treat satisfy (11.1).

Next we consider some examples of L-transform.

Example 11.1.2. Consider $f(t) = e^{\alpha t}$ which obviously satisfies (11.1) . Moreover one has

$$\int_0^r e^{-st} e^{\alpha t} dt = \int_0^r e^{(\alpha - s)t} dt = \frac{e^{(\alpha-s)r}}{\alpha - s} - \frac{1}{\alpha - s}.$$

For $s > \alpha$ one has that $e^{(\alpha-s)r} \to 0$ as $r \to +\infty$ and hence

$$\mathcal{L}\{e^{\alpha t}\} = \lim_{r \to +\infty} \int_0^r e^{(\alpha-s)t} dt = \frac{1}{s - \alpha}, \quad s > \alpha. \tag{11.2}$$

In particular, if $\alpha = 0$ then $f(t) = e^0 = 1$ and

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0. \tag{11.3}$$

■

Example 11.1.3. Consider the *Heaviside function* (or *step function*)

$$H(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1, & \text{if } t \geq 0 \end{cases}$$

and let, for $a \geq 0$,

$$H_a(t) := H(t - a) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } t \geq a. \end{cases}$$

Notice that H_a is bounded and piecewise continuous, with $\mathcal{S} = \{a\}$ and hence Theorem 11.1.1 applies with $\alpha = 0$. Taking into account the definition of H_a, we get for $s > 0$

$$\int_0^r e^{-st} H_a(t) dt = \int_a^r e^{-st} dt = -\frac{e^{-sr}}{s} + \frac{e^{-as}}{s}.$$

Thus

$$\int_a^{+\infty} e^{-st} dt = \lim_{r \to +\infty} \left(-\frac{e^{-sr}}{s} + \frac{e^{-as}}{s} \right) = \frac{e^{-as}}{s}.$$

Hence

$$\mathcal{L}\{H_a\} = \frac{e^{-as}}{s}, \qquad s > 0. \tag{11.4}$$

Of course, if $a = 0$ then $H(t) = 1$ for all $t \geq 0$ and we find that $\mathcal{L}\{H\}(s) = \mathcal{L}\{1\} = 1/s$, in agreement with (11.3). ■

Example 11.1.4. Consider the characteristic function

$$\chi_{[a,b]}(t) = \begin{cases} 0, & \text{if } t < a \\ 1, & \text{if } a \leq t \leq b \\ 0, & \text{if } t > b \end{cases}$$

over $[a, b], 0 \leq a < b$. We note that $\chi_{[a,b]}$ is bounded and piecewise continuous, with $\mathcal{S} = \{a, b\}$ and hence it possesses the L-transform defined for all $s > 0$. Performing calculations as in the previous example, we find that, for $r > b$,

$$\int_0^r e^{-st}\chi_{[a,b]}(t)dt = \int_a^b e^{-st}dt = -\frac{e^{-bs}}{s} + \frac{e^{-as}}{s}$$

and hence

$$\mathcal{L}\{\chi_{[a,b]}\} = \frac{e^{-as} - e^{-bs}}{s}, \qquad s > 0. \tag{11.5}$$

In particular, if $a = 0$ we find

$$\mathcal{L}\{\chi_{[0,b]}\} = \frac{1 - e^{-bs}}{s}, \qquad s > 0. \qquad \blacksquare$$

It is worth pointing out that from the definition it immediately follows that if f, g are piecewise continuous, satisfy (11.1) and differ on a numerable set, then

$$\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s) \tag{11.6}$$

for all s on their common region of convergence.

11.2 Properties of the Laplace transform

The following Proposition shows that the Laplace transform is a linear operator.

Proposition 11.2.1. *$\mathcal{L}$ is linear, that is*

$$\mathcal{L}\{af(t) + bg(t)\}(s) = a\mathcal{L}\{f\}(s) + b\mathcal{L}\{g\}(s),$$

for each s such that the right-hand side makes sense.

Proof. It follows immediately from the linearity of the integrals that:

$$\begin{aligned}
\mathcal{L}\{af(t) + bg(t)\} &= \int_0^{+\infty} e^{-st}[af(t) + bg(t)]dt \\
&= a\int_0^{+\infty} e^{-st}f(t)dt + b\int_0^{+\infty} e^{-st}g(t)dt \\
&= a\mathcal{L}\{f\} + b\mathcal{L}\{g\}.
\end{aligned} \qquad \blacksquare$$

For example,

$$\mathcal{L}\{k\} = \mathcal{L}\{k \cdot 1\} = k\mathcal{L}\{1\} = k \cdot \frac{1}{s} = \frac{k}{s}, \qquad s > 0.$$

As another example, let us note that $\chi_{[a.b]}(t) = H_a(t) - H_b(t)$. Thus, using (11.4),

$$\mathcal{L}\{\chi_{[a.b]}\} = \mathcal{L}\{H_a\} - \mathcal{L}\{H_b\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}$$

according to (11.5).

The L-transform has a smoothing effect.

Theorem 11.2.2. *Suppose that f satisfies* (11.1). *In its region of convergence, $F(s) = \mathcal{L}\{f\}$ is differentiable infinitely many times at each point. Precisely, one has*

$$F^{(n)}(s) = (-1)^n \mathcal{L}\{t^n f(t)\}(s)$$

for any $n = 1, 2, \ldots$.

The following result is important for applications in differential equations.

Theorem 11.2.3. *Suppose that f is differentiable for $t \geq 0$ and satisfies* (11.1). *If $\mathcal{L}\{f'\}$ exists then*

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Proof. By definition

$$\mathcal{L}\{f'\} = \int_0^{+\infty} e^{-st} f'(t)dt.$$

Integrating by parts we find

$$\int_0^r e^{-st} f'(t)dt = e^{-sr} f(r) - f(0) + s\int_0^r e^{-st} f(t)dt.$$

If we pass to the limit as $r \to +\infty$, (11.1) implies that $e^{-sr} f(r) \to 0$. Then it follows that $\mathcal{L}\{f'\}$ is equal to $s\mathcal{L}\{f\} - f(0)$. ∎

Now, it follows that

$$\begin{aligned} \mathcal{L}\{f''\} &= s\mathcal{L}\{f'\} - f'(0) \\ &= s[s\mathcal{L}\{f\} - f(0)] - f'(0) \\ &= s^2\mathcal{L}\{f\} - sf(0) - f'(0), \end{aligned}$$

provided $\mathcal{L}\{f'\}$ and $\mathcal{L}\{f''\}$ exist.

By using Mathematical Induction one can find the L-transform of $f^{(n)}$, see (P4) below.

Below we collect some properties of the L-transform. The domains of convergence can be determined in each case:

$(P1)$ $\mathcal{L}\{e^{\alpha t} f(t)\} = \mathcal{L}\{f\}(s-\alpha)$;

$(P2)$ $\mathcal{L}\{e^{-\beta t} - e^{-\alpha t}\} = \dfrac{\alpha - \beta}{(s+\alpha)(s+\beta)}$;

$(P3)$ $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \dfrac{\mathcal{L}\{f\}}{s}$;

$(P4)$ $\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)$;

$(P5)$ $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} \mathcal{L}\{f\}$ (see Theorem 11.2.2).

Properties $(P1 - P5)$ can be used to find other L-transforms. For example if $f(t) = 1$, $(P5)$ yields

$$\mathcal{L}\{t^n\} = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{1\} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}. \tag{11.7}$$

Let us use $(P4)$ to find $F(s) = \mathcal{L}\{\sin \omega t\}$. Recall that since $f(t) := \sin \omega t$ is smooth and bounded, then $F(s)$ exists for all $s > 0$. One has that $f'(t) = \omega \cos \omega t$ and $f''(t) = -\omega^2 \sin \omega t$, that is $f'' = -\omega^2 f$. Moreover, $f(0) = \sin 0 = 0$ and $f'(0) = \omega \cos 0 = \omega$. Now we take the L-transform yielding

$$\mathcal{L}\{f''(t)\} = -\omega^2 \mathcal{L}\{f\} = -\omega^2 F(s).$$

Property $(P4)$ implies

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0) = s^2 F(s) - \omega.$$

Then it follows that

$$s^2 F(s) - \omega = -\omega^2 F(s) \quad \Longrightarrow \quad (s^2 + \omega^2) F(s) = \omega.$$

Hence, $F(s) = \omega/(s^2 + \omega^2)$, that is

$$\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2}. \tag{11.8}$$

For the reader's convenience, let us find the L-transform of $\sin \omega t$ directly, using the definition. First, integrating by parts, we evaluate

$$\int_0^r e^{-st} \sin \omega t \, dt = -\frac{\cos \omega r}{\omega} e^{-sr} + \frac{1}{\omega} - \frac{s}{\omega} \int_0^r e^{-st} \cos \omega t \, dt.$$

Another integration by parts yields

$$\int_0^r e^{-st} \cos \omega t \, dt = \frac{\sin \omega r}{\omega} e^{-sr} + \frac{s}{\omega} \int_0^r e^{-st} \sin \omega t \, dt.$$

Passing to the limit as $r \to +\infty$ in the previous two equations we find

$$\int_0^{+\infty} e^{-st} \sin \omega t \; dt = \frac{1}{\omega} - \frac{s}{\omega} \int_0^{+\infty} e^{-st} \cos \omega t \; dt \tag{11.9}$$

and

$$\int_0^{+\infty} e^{-st} \cos \omega t \; dt = \frac{s}{\omega} \int_0^{+\infty} e^{-st} \sin \omega t \; dt.$$

Substituting the latter integral in (11.9) we get

$$\int_0^{+\infty} e^{-st} \sin \omega t \; dt = \frac{1}{\omega} - \frac{s^2}{\omega^2} \int_0^{+\infty} e^{-st} \sin \omega t \; dt.$$

Thus

$$\left(1 + \frac{s^2}{\omega^2}\right) \int_0^{+\infty} e^{-st} \sin \omega t \; dt = \frac{1}{\omega}$$

that is

$$\frac{\omega^2 + s^2}{\omega^2} \int_0^{+\infty} e^{-st} \sin \omega t \; dt = \frac{1}{\omega}$$

and finally

$$\mathcal{L}\{\sin \omega t\} = \int_0^{+\infty} e^{-st} \sin \omega t \; dt = \frac{1}{\omega} \cdot \frac{\omega^2}{\omega^2 + s^2} = \frac{\omega}{\omega^2 + s^2}$$

according to (11.8).

Similarly, with minor changes one finds

$$\mathcal{L}\{\cos \omega t\} = \frac{s}{\omega^2 + s^2}. \tag{11.10}$$

As a further application, let us consider the Bessel function of order 0, $J_0(t)$, which satisfies

$$t J_0''(t) + J_0'(t) + t J_0(t) = 0, \qquad J_0(0) = 1, \; J'(0) = 0. \tag{11.11}$$

For the properties of $J_0(t)$ we refer to Section 10.5.1 of the previous chapter. In particular, $J_0(t)$ is smooth and bounded and hence its L-transform $X(s) := \mathcal{L}\{J_0(t)\}(s)$ exists for all $s > 0$. Moreover, since $J_0(t)$ is integrable on $[0, +\infty)$ and

$$\int_0^{+\infty} J_0(t) dt = J_0(0) = 1,$$

then

$$X(s) = \int_0^{+\infty} e^{-st} J_0(t) dt,$$

which is a priori defined for $s > 0$, can be extended to $s = 0$ and one has

$$X(0) = \int_0^{+\infty} J_0(t)dt = 1.$$

We want to show that

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}. \tag{11.12}$$

We start taking the L-transform of (11.11):

$$\mathcal{L}\{tJ_0''(t) + J_0'(t) + tJ_0(t)\} = 0.$$

Using the linearity of $\mathcal{L}$ we infer

$$\mathcal{L}\{tJ_0''(t)\} + \mathcal{L}\{J_0'(t)\} + \mathcal{L}\{tJ_0\} = 0. \tag{11.13}$$

Then $(P4) - (P5)$ yield

$$\begin{aligned}
\mathcal{L}\{tJ_0''(t)\} &= -(s^2X(s) - J_0(0)s - J_0'(0))' \\
&= -(s^2X(s) - s)' = -2sX(s) - s^2X'(s) + 1, \\
\mathcal{L}\{J_0'(t)\} &= sX(s) - J_0(0) = sX(s) - 1, \\
\mathcal{L}\{tJ_0\} &= -X'(s).
\end{aligned}$$

Substituting into (11.13) we find

$$-2sX(s) - s^2X'(s) + 1 + sX(s) - 1 - X'(s) = 0,$$

or

$$(1+s^2)X'(s) + sX(s) = 0.$$

This is a separable equation. One finds

$$\frac{X'(s)}{X(s)} = -\frac{s}{1+s^2}.$$

Integrating we have

$$\ln\frac{|X(s)|}{|X(0)|} = -\frac{1}{2}\ln(1+s^2).$$

Taking into account that $X(0) = 1$ we get

$$\ln|X(s)| = \ln(1+s^2)^{-\frac{1}{2}}$$

whence

$$X(s) = (1+s^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+s^2}}$$

proving (11.12). Notice that in this case it would be more complicated to evaluate $\mathcal{L}\{J_0\}$ directly.

Table 11.1. L-transforms

$f(t)$	$F(s) = \mathcal{L}\{f\}$	$f(t)$	$F(s) = \mathcal{L}\{f\}$
1	$\frac{1}{s}$	t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$	e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$H_a(t)$	$\frac{e^{-as}}{s}$	$\chi_{[a,b]}(t)$	$\frac{e^{-as}-e^{-bs}}{s}$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$	$J_0(t)$	$\frac{1}{\sqrt{1+s^2}}$

Table 11.1 summarizes the L-transforms for some of the functions that we have discussed. The domain of convergence can be determined in each case.

11.3 Inverse Laplace transform

We start with an example dealing with the RC circuits discussed in Section 1.3.2.

Example 11.3.1. We have seen that an RC electric circuit gives rise to a first order differential equation as $RCx' + x = 0$, $x(0) = x_0$. We will use this example to motivate the introduction of the inverse L-transform.

To keep the notation consistent with that used in this chapter, here we write the ivp as

$$x'(t) + x(t) = 0, \qquad x(0) = x_0,$$

where we take $RC = 1$ in order to simplify notation. Let us assume for the moment that $x(t)$ satisfies (11.1). We will see later that this is indeed the case. Taking the L-transform of both sides, and recalling the linearity of $\mathcal{L}$, we find

$$\mathcal{L}\{x' + x\} = \mathcal{L}\{x'\} + \mathcal{L}\{x\} = 0.$$

Using property $(P5)$, and recalling that $x(0) = x_0$, we infer

$$sX(s) - x_0 + X(s) = 0, \qquad X(s) := \mathcal{L}\{x\}.$$

Then

$$X(s) = \frac{x_0}{1+s}$$

and $x(t)$ is the function such that its L-transform is $X(s)$. We will say that $x(t)$ is the inverse L-transform of $X(s)$. In this specific case, it is easy to find $x(t)$. Actually, in Example 11.1.2 we have shown that $\mathcal{L}\{e^{\alpha t}\} = 1/(s-\alpha)$. If we take $\alpha = -1$ we find

$$\mathcal{L}\{x_0 e^{-t}\} = x_0 \mathcal{L}\{e^{-t}\} = \frac{x_0}{s+1}$$

and hence $x(t) = x_0 e^{-t}$, in accordance with what we have found in Section 1.3.2. ∎

In the rest of this section we will discuss the inverse L-transform. Let us begin by stating a preliminary result on the injectivity of the L-transform, which is nothing but the converse of (11.6).

Proposition 11.3.2. *Let f, g be piecewise continuous on* $[0, +\infty)$ *and satisfy* (11.1). *If $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ on their common region of convergence, then $f(t) = g(t)$, for all $t \geq 0$, up to a numerable set of points. If f, g are continuous, then $f(t) = g(t)$ for all $t \geq 0$.*

Definition 11.3.3. Let $F(s)$ be given. If there exists a function $f(t)$ such that $\mathcal{L}\{f\}(s)$ exists and $\mathcal{L}\{f\}(s) = F(s)$, we say that F has an inverse L-transform given by $f(t)$. In this case we write $f(t) = \mathcal{L}^{-1}\{F(s)\}(t)$.

In other words, the idea of the inverse of L-transform is nothing new: $\mathcal{L}\{f(t)\}(s) = F(s)$ if and only if $f(t) = \mathcal{L}^{-1}\{F\}(s)$ (assuming that the inverse exists). For example, for $f(t) = t$, $\mathcal{L}\{t\} = \frac{1}{s^2}$ implies the equivalent relation $t = \mathcal{L}^{-1}\{\frac{1}{s^2}\}$. Often we will write $\mathcal{L}^{-1}\{F(s)\}$ or simply $\mathcal{L}^{-1}\{F\}$ instead of $\mathcal{L}^{-1}\{F(s)\}(t)$.

Proposition 11.3.2 shows that if $f(t)$ is piecewise continuous on $t \geq 0$ and satisfies (11.1), then it is uniquely determined by $F(s)$, up to the numerable set $\mathcal{S}$, and hence the preceding definition makes sense.

For example, (11.4) yields $\mathcal{L}\{H_a\} = e^{-as}/s$ and hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = H_a(t). \tag{11.14}$$

The following proposition says that $\mathcal{L}^{-1}$ is linear like $\mathcal{L}$.

Proposition 11.3.4. *Suppose that $F(s), G(s)$ have inverse L-transforms. Then for all $a, b \in \mathbb{R}$, $aF(s) + bG(s)$ has inverse L-transform and*

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}.$$

Proof. Let $f(t) = \mathcal{L}^{-1}\{F(s)\}$ and $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Then $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Moreover, using the linearity of $\mathcal{L}$, it follows that

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} = aF(s) + bG(s).$$

Taking the inverse L-transform of both sides we infer that

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t) = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\},$$

as required. ∎

Example 11.3.5. Using the linearity property and our familiarity with the Laplace transforms of $\sin t$ and $\cos t$, we have

$$\mathcal{L}^{-1}\left\{\frac{2s-7}{s^2+3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\sqrt{3})^2}\right\} - \frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right\}$$
$$= 2\cos\sqrt{3}\,t - \frac{7}{\sqrt{3}}\sin\sqrt{3}\,t. \quad ∎$$

Example 11.3.6. Using partial fractions, we see that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(2s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{2}{2s+3}\right\} =$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3/2}\right\} = e^{-t} - e^{(-3/2)t}. \quad ∎$$

In general, if

$$R(s) = \sum_1^m \frac{k_i}{s-\lambda_i}$$

then

$$\mathcal{L}^{-1}\{R(s)\} = \sum_1^m k_i e^{\lambda_i t}. \tag{11.15}$$

For example, if

$$R(s) = \frac{1}{(s-\alpha)(s-\beta)} = \frac{1}{\alpha-\beta}\left[\frac{1}{s-\alpha} - \frac{1}{s-\beta}\right]$$

we have $k_1 = \frac{1}{\alpha-\beta}$, $k_2 = -\frac{1}{\alpha-\beta}$, $\lambda_1 = \alpha$ and $\lambda_2 = \beta$. Thus we find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-\alpha)(s-\beta)}\right\} = \frac{e^{\alpha t}}{\alpha-\beta} - \frac{e^{\beta t}}{\alpha-\beta} = \frac{e^{\beta t}-e^{\alpha t}}{\beta-\alpha}. \tag{11.16}$$

Similarly, if

$$R(s) = \frac{s}{(s-\alpha)(s-\beta)} = \frac{1}{\alpha-\beta}\left[\frac{\alpha}{s-\alpha} - \frac{\beta}{s-\beta}\right]$$

we have $k_1 = \frac{\alpha}{\alpha-\beta}$, $k_2 = -\frac{\beta}{\alpha-\beta}$, $\lambda_1 = \alpha$ and $\lambda_2 = \beta$ and we find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-\alpha)(s-\beta)}\right\} = \frac{\alpha}{\alpha-\beta}e^{\alpha t} - \frac{\beta}{\alpha-\beta}e^{\beta t} = \frac{\beta e^{\beta t}-\alpha e^{\alpha t}}{\beta-\alpha}. \tag{11.17}$$

Now, let us state without proof a result on the inverse L-transform of rational functions.

Theorem 11.3.7. *Let $P(s)$, $Q(s)$ be two polynomials with degree $n < m$, respectively. If Q has m simple roots $\lambda_1, \dots, \lambda_m$, then*

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_1^m \frac{P(\lambda_i)}{Q'(\lambda_i)} e^{\lambda_i t}. \tag{11.18}$$

As a simple exercise, the student can establish (11.16) and (11.17) using the preceding theorem.

The counterpart of properties $(P1)$ and $(P3)$ can be found immediately. Below we set $F(s) = \mathcal{L}\{f\}$.

From $(P1)$ and $(P3)$ it follows that

$$(P1') \quad \mathcal{L}^{-1}\{F(s-\alpha)\} = e^{\alpha t} f(t), \qquad s > \alpha;$$

$$(P3') \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau.$$

One can also show that

$$(P1'') \quad \mathcal{L}^{-1}\{e^{-\alpha s} F(s)\} = H_\alpha(t) f(t-\alpha).$$

The following list of inverse L-transforms can be deduced from the table of L-transforms. See Table 11.2.

Table 11.2. inverse L-transforms

$F(s)$	$\mathcal{L}^{-1}\{F\}$	$F(s)$	$\mathcal{L}^{-1}\{F\}$
$\frac{1}{s}$	1	$\frac{1}{s^2}$	t
$\frac{n!}{s^{n+1}}$	t^n	$\frac{1}{s-a}$	e^{at}
$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
$\frac{\omega}{s^2-\omega^2}$	$\sinh \omega t$	$\frac{s}{s^2-\omega^2}$	$\cosh \omega t$
$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at} \sin \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$
$\frac{e^{-as}}{s}$	$H_a(t)$	$\frac{e^{-as}-e^{-bs}}{s}$	$\chi_{[a,b]}(t)$
$\frac{F(s)}{s}$	$\int_0^t f(\tau)d\tau$	$\frac{1}{\sqrt{1+s^2}}$	$J_0(t)$

11.3.1 Convolution

Let $f(t), g(t)$ be two piecewise continuous functions on $t \geq 0$.

Definition 11.3.8. The *convolution* of $f(t)$ and $g(t)$, denoted by $f * g$, is the function defined by setting

$$(f * g)(t) = \int_0^t f(t-\theta)g(\theta)d\theta.$$

The reader should exercise caution when dealing with convolution. For example, in general, $1 * g \neq g$. This is the case for $g(t) = \sin t$, because $1 * \sin t = \int_0^t \sin\theta d\theta = 1 - \cos t$.

The following proposition is important in the sequel because it allows us to evaluate the inverse L-transform of a product.

Proposition 11.3.9. *Let f, g be piecewise continuous of exponential order, which means that they satisfy* (11.1). *Setting $F(s) = \mathcal{L}\{f\}$ and $G(s) = \mathcal{L}\{g\}$, one has*

$$\mathcal{L}\{f * g\} = F(s) \cdot G(s) \tag{11.19}$$

and hence

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \tag{11.20}$$

As an application let us evaluate the inverse L-transform of

$$\phi(s) = \frac{e^{-as} - e^{-bs}}{s(s+1)}.$$

Using (11.20) with $F(s) = 1/(s+1)$ and $G(s) = (e^{-as} - e^{-bs})/s$, one finds

$$\mathcal{L}^{-1}\{\phi(s)\} = (f * g)(t)$$

where (see (11.2))

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

and (see (11.5))

$$g(t) = \mathcal{L}^{-1}\left\{\frac{e^{-as} - e^{-bs}}{s}\right\} = \chi_{[a,b]}(t)$$

and hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-as} - e^{-bs}}{s(s+1)}\right\} = e^{-t} * \chi_{[a,b]}(t). \tag{11.21}$$

Proposition 11.3.9 can also be used to solve integral equations as

$$x(t) = k(t) + \int_0^t f(t-\theta)x(\theta)d\theta = k(t) + (f * x)(t). \tag{11.22}$$

It suffices to take the L-transform yielding (with obvious meaning of notation) $X(s) = \mathcal{L}\{k\}(s) + \mathcal{L}\{f * x\}(s) = K(s) + F(s) \cdot X(s)$. If we know K and F, solving with respect to X we find $x(t) = \mathcal{L}^{-1}\{X\}$.

Example 11.3.10. Solve

$$x(t) = e^t + \int_0^t e^{t-\theta} x(\theta) d\theta = e^t + e^t * x(t).$$

One has $X = \frac{1}{s-1} + \frac{1}{s-1} \cdot X$ and hence $(1 - \frac{1}{s-1})X = \frac{1}{s-1}$, namely $X = \frac{1}{s-2}$. Then $x = \mathcal{L}^{-1}\{\frac{1}{s-2}\} = e^{2t}$. ■

11.4 Laplace transform and differential equations

The Laplace transform is useful in solving linear differential equations. Let us start with a general linear second order equation with constant coefficients

$$x''(t) + ax'(t) + bx(t) = g(t), \qquad x(0) = x_0, \quad x'(0) = x_1.$$

Let us assume that the L-transform of $G(s) = \mathcal{L}\{g(t)\}(s)$ exists for $s > 0$.

If we take the L-transform of both sides and use $(P4)$ we find

$$s^2 X(s) - sx_0 - x_1 + a(sX(s) - x_0) + bX(s) = G(s)$$

where $X(s) = \mathcal{L}\{x(t)\}(s)$. As for $X(s)$, we proceed formally, assuming that it makes sense for $s > 0$. At the end we shall verify that this is indeed the case.

Solving for $X(s)$, we get

$$X(s) = \frac{sx_0 + x_1 + ax_0 + G(s)}{s^2 + as + b}.$$

Notice that $x(t) = \mathcal{L}^{-1}\{X\}$. Thus, taking the inverse L-transform, we find

$$x(t) = \mathcal{L}^{-1}\left\{\frac{sx_0 + x_1 + ax_0 + G(s)}{s^2 + as + b}\right\}$$

which can be found explicitly using the properties of the inverse L-transform.

Let us illustrate the procedure with an example.

Example 11.4.1. Consider the problem

$$x''(t) + x(t) = g(t), \qquad x(0) = 0, \quad x'(0) = k$$

and assume that the inverse L-transform $G(s)$ of $g(t)$ exists for $s > 0$. Then

$$X(s) = \frac{k + G(s)}{s^2 + 1}$$

so that $X(s)$ has an inverse L-transform and

$$x(t) = \mathcal{L}^{-1}\left\{\frac{k}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+1}\right\}.$$

Using (11.8)

$$x(t) = k\sin t + \mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+1}\right\}.$$

From (11.20), with $F(s) = 1/(s^2+1)$, we deduce

$$x(t) = k\sin t + \sin t * g(t) = k\sin t + \int_0^t \sin(t-\theta)g(\theta)d\theta,$$

which is the solution of our ivp for any L-transformable $g(t)$. ∎

The preceding discussion highlights the procedure one follows in the general case of

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_{n-1}x' + a_n x = g(t)$$

together with initial conditions

$$x(0) = x_0,\ x'(0) = x_1, \ldots, x^{(n-1)}(0) = x_{n-1}.$$

One takes the L-transform of the equation and uses $(P4)$ yielding

$$s^n X(s) - s^{n-1}x_0 - s^{n-2}x_1 - \ldots - x_{n-1} + \ldots + a_n X(s) = G(s).$$

This allows us to find

$$P(s)X(s) - Q(s) = G(s)$$

where

$$P(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-2}s^2 + a_{n-1}s + a_n$$

and

$$Q(s) = s^{n-1}x_0 + s^{n-2}x_1 + \ldots + a_{n-2}(sx_0 + x_1) + a_{n-1}x_0.$$

Then $X(s) = (G(s) + Q(s))/P(s)$ has an inverse L-transform $x(t)$ which solves the ivp.

Other equations that can be solved by means of the L-transform are linear equations with coefficients depending on time. Once again, we discuss some specific examples.

Consider the ivp

$$x'' + tx = 0, \qquad x(0) = 0,\ x'(0) = b.$$

Taking the L-transform and using $(P4)$ we get

$$s^2 X(s) - b + \mathcal{L}\{tx(t)\} = 0.$$

From $(P5)$ we infer that $\mathcal{L}\{tx(t)\} = -X'(s)$ and hence we deduce $s^2X(s) - b - X'(s) = 0$ that is

$$X'(s) - s^2X(s) = -b.$$

This is a linear first order equation that can be solved by the integrating factor method. Once $X(s)$ is found, the solution we are looking for is given by $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

As a further example let us consider the system

$$\begin{cases} x' &= x + y \\ y' &= x - y \end{cases}$$

with the initial conditions $x(0) = 1$ and $y(0) = 0$. Taking the L-transform and setting $X(s) = \mathcal{L}\{x\}$, $Y(s) = \mathcal{L}\{y\}$ we deduce

$$\begin{cases} \mathcal{L}\{x'\} = sX(s) - 1 &= X(s) + Y(s) \\ \mathcal{L}\{y'\} = sY(s) &= X(s) - Y(s) \end{cases}$$

whence

$$\begin{cases} (s-1)X(s) - Y(s) &= 1 \\ -X(s) + (s+1)Y(s) &= 0. \end{cases}$$

Finding $X(s) = (s+1)Y(s)$ from the second equation and substituting into the first one we find

$$(s-1)(s+1)Y(s) - Y(s) = 1$$

and hence

$$Y(s) = \frac{1}{s^2 - 2}\,.$$

Taking the inverse L-transform, see Table 11.2, we find

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-2}\right\} = \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\,.$$

Moreover,

$$X(s) = (s+1)Y(s) = \frac{s+1}{s^2-2} = \frac{s}{s^2-2} + \frac{1}{s^2-2}$$

and thus we get

$$x(t) = \cosh\sqrt{2}\,t + \frac{1}{\sqrt{2}}\sinh\sqrt{2}\,t\,.$$

11.5 Generalized solutions

The L-transform allows us to handle linear differential equations with a forcing term which may be discontinuous.

As an example, let us consider the first order equation

$$x'(t) + x(t) = \chi_{[0,b]}(t), \qquad x(0) = 0, \qquad (b > 0), \tag{11.23}$$

where $\chi_{[0,b]}$ is the step function introduced in Example 11.1.4. This problem models the voltage of an RC circuit when an initial step impulse is given at the capacitor.

Taking the L-transform of both sides, we find

$$\mathcal{L}\{x'\} + \mathcal{L}\{x\} = \mathcal{L}\{\chi_{[0,b]}\} = \frac{1 - e^{-bs}}{s}.$$

Using property $(P4)$ with $n = 1$ and setting $X(s) = \mathcal{L}\{x\}$ we infer

$$sX(s) + X(s) = \frac{1 - e^{-bs}}{s}.$$

Solving for $X(s)$, we get

$$X(s) = \frac{1 - e^{-bs}}{s(s+1)}.$$

Let us evaluate the inverse L-transform on the right-hand side. We use (11.21) with $a = 0$, obtaining

$$\mathcal{L}^{-1}\left\{\frac{1 - e^{-bs}}{s(s+1)}\right\} = e^{-t} * \chi_{[0,b]}(t) = \int_0^t e^{-t+\theta}\chi_{[0,b]}(\theta)d\theta.$$

To evaluate the integral, we distinguish between $0 \le t \le b$ and $t > b$. In the former case, $0 \le \theta \le t \le b$ and we find

$$\int_0^t e^{-t+\theta}\chi_{[0,b]}(\theta)d\theta = \int_0^t e^{-t+\theta}d\theta = 1 - e^{-t} \quad (0 \le t \le b).$$

For $t > b$ we have

$$\begin{aligned}\int_0^t e^{-t+\theta}\chi_{[0,b]}(\theta)d\theta &= \int_0^b e^{-t+\theta}\chi_{[0,b]}(\theta)d\theta + \int_b^t e^{-t+\theta}\chi_{[0,b]}(\theta)d\theta \\ &= \int_0^b e^{-t+\theta}d\theta = e^{-t+b} - e^{-t}.\end{aligned}$$

We have shown that

$$\mathcal{L}^{-1}\left\{\frac{1 - e^{-bs}}{s(s+1)}\right\} = \begin{cases} 1 - e^{-t}, & \text{if } 0 \le t < b \\ e^{-t+b} - e^{-t}, & \text{if } t \ge b. \end{cases}$$

Therefore, $x(t) = \mathcal{L}^{-1}\{X(s)\}$ exists and is given by

$$x(t) = \begin{cases} 1 - e^{-t}, & \text{if } 0 \le t \le b \\ e^{-t+b} - e^{-t}, & \text{if } t \ge b. \end{cases}$$

Let us check this result working directly on the equation. For $0 \le t \le b$ one has $\chi_{[0,b]}(t) = 1$ and the ivp becomes $x' + x = 1,\ x(0) = 0$. Solving this, we find

$x(t) = 1 - e^{-t}$, $0 \le t \le b$ and $x(b) = 1 - e^{-b}$. For $t \ge b$ one has $\chi_{[0,b]}(t) = 0$ and hence we are lead to solve the ivp

$$x' + x = 0, \qquad x(b) = 1 - e^{-b}, \quad (t \ge b)$$

which can be solved easily: the general solution of $x' + x = 0$ is $x(t) = ce^{-t}$. To find c we impose the initial condition $x(b) = 1 - e^{-b}$ yielding $ce^{-b} = 1 - e^{-b}$ and thus $c = (1 - e^{-b})/e^{-b} = e^b - 1$. In conclusion,

$$x(t) = \begin{cases} 1 - e^{-t}, & \text{if } 0 \le t < b \\ e^{-t}(e^b - 1), & \text{if } t \ge b \end{cases}$$

as before.

The solution is increasing for $0 < t \le b$ and then decays to 0 for $t > b$. See Figure 11.1. It is worth pointing out that $x(t)$ here is continuous (it suffices to check this at $t = b$) but is not differentiable at $t = b$ because the left derivative at $t = b$ is e^{-b} while the right derivative at the same point is $e^{-b} - 1$. In this case $x(t)$ is a "generalized" solution of our equation, in the sense that it solves the differential equation for all $t \ge 0$, except at $t = b$. This could have been expected because the right-hand side of the equation has a discontinuity at $t = b$.

More generally, consider the differential equation

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_{n-1}x' + a_n x = g(t)$$

where g is continuous with possibly the exception of a finite number of points $\mathcal{S} = \{a_1, \ldots, a_n\}$. We say that a continuous $x(t)$ is a *generalized solution* of the equation if it is continuously differentiable on $\mathbb{R} \setminus \mathcal{S}$ and satisfies the equation for all $t \in \mathbb{R} \setminus \mathcal{S}$. We could consider more general classes of generalized solutions, but this is out of the scope of this book.

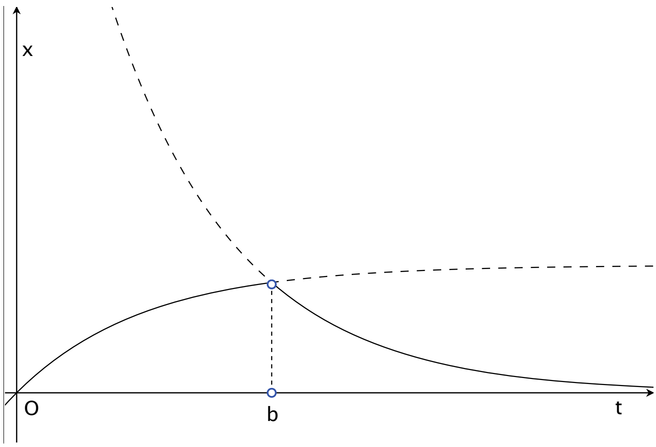


Fig. 11.1. Graph of $x(t) = e^{-t} * \chi_{[0,b]}(t)$

11.6 Appendix: The Dirac delta

A rigorous treatment of the Dirac[1] delta would require more advanced topics such as the Theory of Distribution and cannot be carried out here. However, its importance in applications makes it worthwhile to give at least a heuristic sketch of this topic. The reader should be aware that the exposition below is only an outline, not a very rigorous and complete, treatment of the Dirac delta.

Let us define a sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$ by setting

$$f_n(t) = \begin{cases} n, & \text{if } |t| \le \frac{1}{2n} \\ 0, & \text{if } |t| > \frac{1}{2n}. \end{cases}$$

For every fixed $t \neq 0$ we have that $f_n(t) \to 0$ as $n \to +\infty$. Of course, this is not true for $t = 0$. Indeed, since $f_n(0) = n$ we have that $f_n(0) \to +\infty$ as $n \to +\infty$. So, if we denote by $\delta(t)$ the pointwise limit of $f_n(t)$, this $\delta(t)$, called the *Dirac delta* or *δ-function*, is not a function as we are used to dealing with, but rather a "generalized" function or *distribution*.

Notice that

$$\int_{-\infty}^{+\infty} f_n(t)dt = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n\,dt = 1, \qquad \forall\, n \in \mathbb{N}.$$

If we "pass to the limit" under the integral, in an appropriate "generalized" sense, we find

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1. \tag{11.24}$$

Another important characteristic property of δ is that

$$\int_{-\infty}^{+\infty} \delta(t)\phi(t)dt = \phi(0) \tag{11.25}$$

for any smooth function $\phi : \mathbb{R} \mapsto \mathbb{R}$.

In order to show that δ has the L-transform and to find it, we evaluate

$$\int_0^{+\infty} e^{-st}\delta(t)dt.$$

Since $\delta(t) = 0$ for all $t < 0$, and using (11.25) with $\phi(t) = e^{-st}$, we infer

$$\int_0^{+\infty} e^{-st}\delta(t)dt = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = e^0 = 1.$$

Hence we can say that δ has an L-transform given by

$$\mathcal{L}\{\delta(t)\} = 1. \tag{11.26}$$

[1] Paul Dirac (1902–1984).

More generally, we can consider a shifted delta function by considering $\delta(t-a)$ which has the following properties

$$\delta(t-a)=0 \qquad \forall\, t \neq a \tag{11.27}$$

$$\int_{-\infty}^{+\infty} \delta(t-a)dt = 1 \tag{11.28}$$

$$\int_{-\infty}^{+\infty} \delta(t-a)\phi(t)dt = \phi(a) \tag{11.29}$$

for all smooth functions ϕ.

It is interesting to evaluate the convolution of $\delta(t-a)$ with a piecewise continuous function $g(t)$. One has

$$\delta(t-a) * g(t) = \int_0^t \delta(\theta-a)g(t-\theta)d\theta.$$

Let $a \geq 0$. Since $\delta(\theta-a)=0$ for $\theta \leq t < a$, we get

$$\int_0^t \delta(\theta-a)g(t-\theta)d\theta = 0.$$

On the other hand, for $t \geq a$ we can use (11.29) to infer

$$\int_0^t \delta(\theta-a)g(t-\theta)d\theta = \int_{-\infty}^{+\infty} \delta(\theta-a)g(t-\theta)d\theta = g(t-a).$$

In conclusion, we can say that

$$\int_0^t \delta(\theta-a)g(t-\theta)d\theta = \begin{cases} 0 & \text{if } 0 \leq t < a \\ g(t-a) & \text{if } t \geq a \end{cases}$$

that is

$$\delta(t-a) * g(t) = H_a(t)g(t-a). \tag{11.30}$$

Next let us find the L-transform of $\delta(t-a)$, $a \geq 0$. We argue as before and use (11.29) to yield

$$\int_0^{+\infty} e^{-st}\delta(t-a)dt = \int_{-\infty}^{+\infty} \delta(t-a)e^{-st}dt = e^{-as}$$

hence we can say that

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}. \tag{11.31}$$

We now want to show that the shifted Heaviside function $H_a(t)$ has the "derivative in a generalized sense" given by $\delta(t-a)$. To have a heuristic idea of this fact, one first observes that

$$\int_0^t \delta(t-a)dt = 0, \qquad \forall\, t < a,$$

while for $t \geq a$, (11.28) yields

$$\int_0^t \delta(t-a)dt = \int_{-\infty}^{+\infty} \delta(t-a)dt = 1, \qquad \forall\, t \geq a.$$

In other words,

$$\int_0^t \delta(t-a)dt = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases} = H_a(t).$$

So $H_a(t)$ is a kind of antiderivative of $\delta(t-a)$ and this gives rise, in a "generalized" sense, to $H_a'(t) = \delta(t-a)$.

It is worth pointing out that this agrees with the result we find by taking the L-transform. Actually, we know that

$$\mathcal{L}\{H_a(t)\} = \frac{e^{-as}}{s}.$$

Then

$$\mathcal{L}\{H_a'(t)\} = s\mathcal{L}\{H_a(t)\} = e^{-as}$$

which is exactly the L-transform of $\delta(t-a)$.

As for the inverse L-transforms, (11.31) implies

$$\mathcal{L}^{-1}\{e^{-as}\} = \delta(t-a).$$

In Physics, $k\delta(t-a)$ corresponds to a sudden force impulse of intensity k acting at the unique instant $t = a$. To illustrate its applications to differential equations, we will solve a couple of problems.

Example 11.6.1. For $a \geq 0$ solve the problem (arising in an RC circuit)

$$x' + x = k\delta(t-a), \qquad x(0) = 0, \qquad t \geq 0.$$

Taking the L-transform we find

$$sX(s) + X(s) = ke^{-as}.$$

Hence

$$X(s) = \frac{ke^{-as}}{s+1}.$$

Then, using the convolution property (11.20) of the inverse L-transform, we infer

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = k\mathcal{L}^{-1}\{e^{-s}\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = k\delta(t-a) * e^{-t}.$$

Then, by (11.30),

$$x(t) = kH_a(t)e^{-(t-a)} = \begin{cases} 0 & \text{if } 0 \leq t < a \\ \\ ke^{-(t-a)} & \text{if } t \geq a. \end{cases}$$

This function solves our problem in a "generalized" sense (slightly different from the one introduced before). It has a jump discontinuity at $t = a$. Moreover, for $0 \le t < a$, $x(t) \equiv 0$ and hence it solves $x' + x = 0$, $x(0) = 0$, while for $t \ge a$, $x(t)$ solves $x' + x = 0$ with initial condition $x(a) = k$, see Figure 11.2a. If $a = 0$ we find $x(t) = ke^{-t}$ which solves $x' + x = 0$ with initial condition $x(0) = k$, see Figure 11.2b. ■

In applications to an RC circuit, there is no circulating current in the circuit for $t \le a$ because the capacitor is discharged, corresponding to the initial condition $x(0) = 0$. For $t > a$, the sudden instantaneous impulse of intensity k generates a current as if the initial capacitor voltage is k. For $t > a$ the RC circuit works as usual and the voltage decays exponentially to zero as $t \to +\infty$.

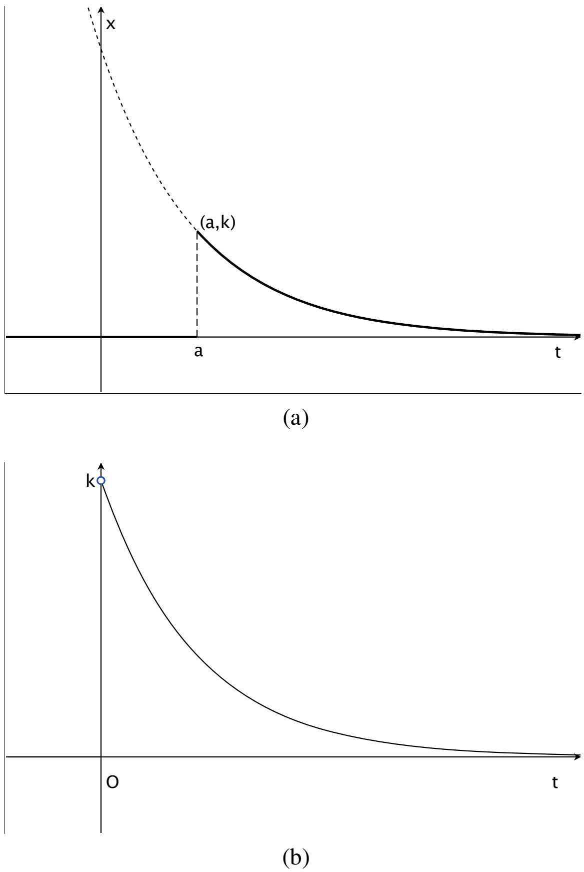


Fig. 11.2. (a) Solution of $x' + x = k\delta(t-a)$, $x(0) = 0, k > 0, a > 0$; (b) Solution of $x' + x = k\delta(t)$, $x(0) = 0, k > 0$

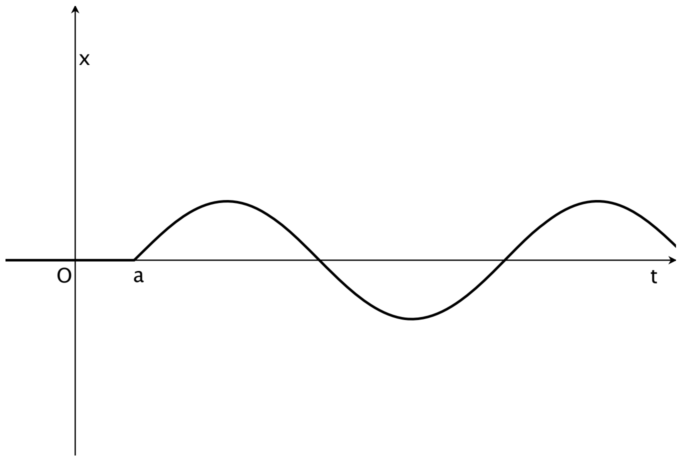


Fig. 11.3. Graph of $x(t) = kH_a(t)\sin(t-a)$ with $a > 0$

Example 11.6.2. For $a \geq 0$ let us consider the problem

$$x''(t) + x(t) = k\delta(t-a), \qquad x(0) = 0,\ x'(0) = 0, \quad t \geq 0$$

which models a harmonic oscillator with the forcing term $k\delta(t-a)$.

As before, we take the L-transform and find

$$s^2X(s) + X(s) = ke^{-as}.$$

Then

$$X(s) = \frac{ke^{-as}}{s^2+1}$$

and hence (11.20) yields

$$\begin{aligned} x &= \mathcal{L}^{-1}\{X\} = k\mathcal{L}^{-1}\{e^{-s}\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = k\delta(t-a) * \sin t \\ &= kH_a(t)\sin(t-a) = \begin{cases} 0 & \text{if } 0 \leq t < a \\ k\sin(t-a) & \text{if } t \geq a. \end{cases} \end{aligned}$$

In other words, if $a > 0$ the solution is 0 until $t = a$. After this time the impulse $k\delta(t-a)$ yields the (nontrivial) solution of the equation $x'' + x = 0$ satisfying the initial conditions $x(a) = 0$, $x'(a) = k$. See Figure 11.3. Moreover, notice that $x(t) = kH_a(t)\sin(t-a)$ is not differentiable at $t = a$ and hence the name "solution" has once again to be understood in a "generalized" sense. If $a = 0$ we find merely $x(t) = k\sin t$, $t \geq 0$, which solves $x'' + x = 0$ with the new initial conditions $x(0) = 0, x'(0) = k$. ■

11.7 Exercises

1. Find the L-transform of $\sinh \omega t = \frac{1}{2}(e^{\omega t} - e^{-\omega t})$ and $\cosh \omega t = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$.
2. Find the L-transform of $t \sin \omega t$ and $t \cos \omega t$.
3. Find the L-transform of $t \sinh \omega t$ and $t \cosh \omega t$.
4. Find the L-transform of $e^{\alpha t} \sin \omega t$ and $e^{\alpha t} \cos \omega t$.
5. Find $\mathcal{L}\{f\}$ where $f(t) = 1$, if $0 \le t \le 1$, $f(t) = 2$, if $3 \le t \le 4$ and $f(t) = 0$ otherwise.
6. Find the L-transform of $t * e^t$.
7. Find the L-transform of $t^2 * e^{at}$.
8. Let f be a piecewise continuous T-periodic function. Show that $\mathcal{L}\{f\}$ exists and
$$F(s) = \mathcal{L}\{f\} = \frac{1}{1 - e^{-sT}} \cdot \int_0^T e^{-st} f(t) dt.$$
9. Find the L-transform of the 2-periodic *square wave* function
$$f(t) = 1, \quad \text{if } 0 \le t < 1, \qquad f(t) = 0, \quad \text{if } 1 \le t < 2,$$
and $f(t+2) = f(t)$ for all $t \ge 2$.
10. Find the L-transform of the *saw-tooth* T-periodic function
$$f(t) = t, \quad \text{if } 0 \le t < T, \qquad f(t+T) = f(t), \ \forall\, t \ge T.$$
11. Let $F(s) = \mathcal{L}\{f\}(s)$ be defined for $s > 0$. Suppose that $|f(t)| \le C$ for all $t \ge 0$. Show that $\lim_{s \to +\infty} F(s) = 0$.
12. Let $F(s) = \mathcal{L}\{f\}(s)$ be defined for $s > 0$. Suppose that $f(t) \ge C > 0$ for all $t \ge 0$. Show that $\lim_{s \to 0+} F(s) = +\infty$.
13. Find the inverse L-transform of $\frac{2}{s^2-4}$ and $\frac{s}{s^2-4}$.
14. Find the inverse L-transform of $\frac{1}{s^2-2s+2}$ and $\frac{s-1}{s^2-2s+2}$.
15. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2-3s+2}\right\}$.
16. Find $\mathcal{L}^{-1}\left\{\frac{s-2}{s^3-s}\right\}$.
17. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^4-1}\right\}$.
18. Find $\mathcal{L}^{-1}\left\{\frac{s^2+3s+1}{s^2+s}\right\}$.
19. Let $F(s) = \mathcal{L}\{f\}$. Show that if $f(t) > 0$ then F is decreasing and concave upward.

20. Prove property $(P3)$ of the L-transform: if $g(t) = \int_0^t f(\tau)d\tau$ then $\mathcal{L}\{g\}(s) = \frac{\mathcal{L}\{f\}(t)}{s}$.

21. Use $(P3)$ to show that $\mathcal{L}\{t\} = s^{-2}$.

22. Use $(P4)$ to find the L-transform of $e^{\alpha t}$.

23. Let $J_0(t)$ be the Bessel function of order 0 satisfying $tx'' + x' + tx = 0$, such that $J_0(0) = 1$. Find $\mathcal{X}(s) = \mathcal{L}\{J_0\}$ such that $\mathcal{X}(0) = 1$.

24. Solve $x' + x = e^t,\ x(0) = 0$ using the L-transform.

25. Solve $x' + x = t,\ x(0) = -1$ using the L-transform.

26. Solve $x'' - 2x' + x = 0,\ x(0) = x'(0) = 1$ using the L-transform.

27. Solve $x'' - 4x' + 3x = 1,\ x(0) = x'(0) = 0$ using the L-transform.

28. Solve $x'''' + x'' = 0,\ x(0) = 0, x'(0) = 1, x''(0) = x'''(0) = 0$ using the L-transform.

29. Find the "generalized solution" of $x' - x = H_a(t),\ x(0) = 0$.

30. Find the "generalized solution" of $x' + x = H_a(t),\ x(0) = 0$.

31. Find the "generalized solution" of $x' - x = k\delta,\ x(0) = a, \qquad t \geq 0$.

32. Find the "generalized solution" of $x'' + x = g(t),\ x(0) = x'(0) = 0$, where g is any piecewise continuous function with L-transform $G(s)$ defined for $s > 0$. In particular, solve in the case that $g(t) = \chi_{[0,1]}(t)$.

33. Find the "generalized solution" of $x'' = \delta(t - a)$, $x(0) = 1$, $x'(0) = 0$, where $a > 0$.

34. Solve $x(t) = 1 + e^{2t} * x(t)$.

35. Solve $x(t) = t^3 + \sin t\ *\ x$.

36. Solve $x' - k * x = 1, x(0) = 0, k > 0$ a constant.

37. Solve $x' + (k^2) * x = 1, x(0) = 1, k \neq 0$.

38. Solve the system
$$\begin{cases} x' = 2x + y, & x(0) = 0 \\ y' = -x - 4y, & y(0) = 1. \end{cases}$$

39. Solve the system
$$\begin{cases} x' = -x + y, & x(0) = 1 \\ y' = x + y, & y(0) = 0. \end{cases}$$

40. Solve the system
$$\begin{cases} x' = x + y, & x(0) = 0 \\ y' = -y + \delta, & y(0) = 0. \end{cases}$$

12

Stability theory

In this chapter we present an introduction to the theory of stability. Since this is a very broad area which includes not only many topics but also various notions of stability, we mainly focus on Liapunov stability of equilibrium points and leave out topics such as the Poincaré–Bendixon theory, stability of periodic solutions, limit cycles, etc. Some of the proofs are omitted or carried out in special simple cases. For a more complete treatment the reader may consult sources such as the books by J. La Salle & S. Lefschetz, or by H. Amann, see References.

12.1 Definition of stability

Given $\overline{x} = (x_1.x_2, \dots, x_n) \in \mathbb{R}^n, \overline{f} = (f_1, \dots, f_n) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ (in this chapter we display the components of vectors horizontally instead of vertically, as was done earlier), let $\overline{x}(t, p)$ be the (unique) solution of the system

$$\overline{x}' = \overline{f}(\overline{x}), \qquad \overline{x}(0) = p \tag{12.1}$$

or equivalently

$$\begin{cases} x_i' &= f_i(x_1, \cdots, x_n) \\ x_i(0) &= p_i. \end{cases} \qquad (i = 1, 2, \dots, n).$$

We will suppose that the solution $\overline{x}(t, p)$ is defined for all $t \geq 0$ and for all $p \in \mathbb{R}^n$.

It is possible to show that the solution $\overline{x}(t, p)$ depends continuously on the initial condition p:

Theorem 12.1.1. *Suppose that $\overline{f}$ is continuously differentiable on $\mathbb{R}^n$. Given $p \in \mathbb{R}^n$, for each $\epsilon > 0$ and $T > 0$ there exists $r > 0$ such that $|\overline{x}(t, p) - \overline{x}(t, q)| < \epsilon$, for all $t \in [0, T]$ and all $|p - q| < r$.*

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed., UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_12

In other words, for q close to p, $\overline{x}(t, q)$ remains close to $\overline{x}(t, p)$ in any *finite interval*, that is, solutions that are close to each other initially remain close to each other for some finite time. However, stability deals with the behavior of $\overline{x}(t, p)$ for all $t \geq 0$, that is, if they are initially close to each other then they remain close to each other for *all time* $t \geq 0$.

Notation. $T_r(\overline{y}) = \{\overline{x} \in \mathbb{R}^n : |\overline{x} - \overline{y}| < r\}$ denotes the ball of radius $r > 0$ centered at $\overline{y} \in \mathbb{R}^n$. Recall that $|\overline{x}|$ is the euclidean norm in $\mathbb{R}^n$, namely $|\overline{x}|^2 = (\overline{x} \mid \overline{x}) = \sum x_i^2$.

Definition 12.1.2. Let $\overline{x}^* \in \mathbb{R}^n$ be such that $\overline{f}(\overline{x}^*) = 0$ so that $\overline{x}^*$ is an equilibrium point of (12.1):

1. $\overline{x}^*$ is stable if for every $r > 0$ there exists a neighborhood U of $\overline{x}^*$ such that $p \in U \implies \overline{x}(t, p) \in T_r(\overline{x}^*)$ for all $t \geq 0$.
2. $\overline{x}^*$ is asymptotically stable if there is a neighborhood U' of $\overline{x}^*$ such that $\lim_{t \to +\infty} x(t, p) = \overline{x}^*$ for all $p \in U'$.
3. $\overline{x}^*$ is unstable if it is not stable.

Of course, asymptotic stability implies stability. But there could be stable equilibria which are not asymptotically stable such as a Poincaré "center" – an equilibrium point surrounded by circular trajectories.

12.2 Liapunov direct method

At the beginning of the 1900's, the Russian mathematician Aleksandr Liapunov developed what is called the Liapunov Direct Method for determining the stability of an equilibrium point. We will describe this method and illustrate its applications.

Definition 12.2.1. Let $\overline{x}^* \in \mathbb{R}^n$ be an equilibrium point of (12.1). Let $\Omega \subseteq \mathbb{R}^n$ be an open set containing $\overline{x}^*$. A real valued function $V \in C^1(\Omega, \mathbb{R})$ is called a Liapunov function for (12.1) if

$(V1)$ $V(\overline{x}) > V(\overline{x}^*)$ for all $\overline{x} \in \Omega, \overline{x} \neq \overline{x}^*$.
$(V2)$ $\dot{V}(\overline{x}) \stackrel{def}{=} (\nabla V(\overline{x}) \mid \overline{f}(\overline{x})) \leq 0$, for all $\overline{x} \in \Omega$.

Recall that $(\overline{x} \mid \overline{y})$ denotes the euclidean scalar product of the vectors $\overline{x}, \overline{y}$, see Notations. Moreover, $\nabla V = (V_{x_1}, \cdots, V_{x_n})$ denotes the gradient of V and the subscripts denote partial derivatives.

Note that, since $\overline{x}'(t) = \overline{f}(\overline{x}(t))$, we have that

$$\begin{aligned}
\dot{V}(\overline{x}(t)) &= V_{x_1}(\overline{x}(t)) f_1(\overline{x}(t)) + V_{x_2}(\overline{x}(t)) f_2(\overline{x}(t)) \ldots + V_{x_n}(\overline{x}(t)) f_n(\overline{x}(t)) \\
&= V_{x_1}(\overline{x}(t))\overline{x}_1'(t) + V_{x_2}(\overline{x}(t))\overline{x}_2'(t) + \ldots + V_{x_n}(\overline{x}(t))\overline{x}_n'(t) \\
&= (\nabla V(\overline{x}(t)) \mid \overline{x}'(t)) = \frac{d}{dt}[V(\overline{x}(t))].
\end{aligned}$$

In other words, $\dot{V}(\overline{x}(t)) = \frac{dV(\overline{x}(t))}{dt}$ is nothing but the derivative of V along the trajectories $\overline{x}(t)$. Therefore (V2) implies that $V(\overline{x}(t))$ is non-increasing along the trajectories $\overline{x}(t)$.

Theorem 12.2.2 (Liapunov stabilty theorem).

(i) *If* (12.1) *has a Liapunov function, then* $\overline{x}^*$ *is stable.*
(ii) *If in* $(V2)$ *one has that* $\dot{V}(\overline{x}) < 0$, *for all* $\overline{x} \neq \overline{x}^*$, *then* $\overline{x}^*$ *is asymptotically stable.*

Proof. We will prove only the statement (i). By the change of variable $\overline{y} = \overline{x} - \overline{x}^*$, the autonomous system $\overline{x}' = \overline{f}(\overline{x})$ becomes $\overline{y}' = \overline{f}(\overline{y} + \overline{x}^*)$ which has $\overline{y} = 0$ as equilibrium. Thus, without loss of generality, we can assume that $\overline{x}^* = 0$. Moreover, still up to a translation, we can assume without loss of generality that $V(\overline{x}^*) = 0$. Finally, for simplicity, we will assume that $\Omega = \mathbb{R}^n$. The general case requires only minor changes. Set

$$\varphi_p(t) = V(\overline{x}(t, p)).$$

The function $\varphi_p(t)$ is defined for all $t \geq 0$ and all $p \in \mathbb{R}^n$. Moreover $\varphi_p(t)$ is differentiable and one has

$$\varphi_p'(t) = V_{x_1}x_1' + \cdots + V_{x_n}x_n' = (\nabla V(\overline{x}(t, p) \mid \overline{x}'(t, p)) = \dot{V}(\overline{x}(t, p)).$$

By $(V2)$ it follows that $\varphi_p'(t) \leq 0$ for all $t \geq 0$. Hence $\varphi_p(t)$ is non-increasing and thus

$$0 \leq V(\overline{x}(t, p)) \leq V(\overline{x}(0, p)) = V(p), \qquad \forall\, t \geq 0. \tag{12.2}$$

Given any ball T_r centered at $\overline{x} = 0$ with radius $r > 0$, let S_r denote its boundary. From $(V1)$ it follows that

$$m = m(r) = \min\{V(y) : y \in S_r\} > 0.$$

Let $U = \{p \in T_r,\ \ V(p) < m\}$. From $(V1)$ one has that U is a neighborhood of $\overline{x} = 0$. Moreover, by (12.2) it follows that $V(\overline{x}(t, p)) < m$ for all $t \geq 0$ and all $p \in U$. Since m is the minimum of V in S_r, the solution $\overline{x}(t, p)$ has to remain in T_r, provided $p \in U$, namely $p \in U \implies \overline{x}(t, p) \in T_r$ and this proves that $\overline{x} = 0$ is stable. ∎

Roughly, the Liapunov function V is a kind of potential well with the property that the solution with initial value p in the well remain confined therein for all $t \geq 0$.

Remark 12.2.3. If $\dot{V} = 0$ for all $t \geq 0$, then $V(\overline{x}(t, p))$ is constant, namely $V(\overline{x}(t, p)) = V(\overline{x}(0, p)) = V(p)$ for all $t \geq 0$. Then $\overline{x}(t, p)$ cannot tend to $\overline{x}^*$ as $t \to +\infty$. As a consequence, $\overline{x}^*$ is stable but not asymptotically stable. ∎

Example 12.2.4. As a first application we want to study the stability of the nontrivial equilibrium $x^* = \frac{c}{d}$, $y^* = \frac{a}{b}$ of the Lotka–Volterra system

$$\begin{cases} x' &= ax - bxy \\ y' &= -cy + dxy. \end{cases}$$

Recall (see Section 8.2) that letting $H(x, y) = dx + by - c \ln x - a \ln y$, $x > 0$, $y > 0$, one has that H is constant along the solutions of the system. Let us take $V = H$. Then $\dot{V} = \dot{H} = 0$ and hence $(V2)$ holds with equality. Moreover we know that H has a strict local minimum at (x^*, y^*) and thus $(V1)$ is satisfied. It follows that H is a Liapunov function and one deduces that (x^*, y^*) is stable (but not asymptotically stable, see Remark 12.2.3). We will see later on that the trivial equilibrium $(0, 0)$ is unstable. ■

Theorem 12.2.2 is countered with the following instability result

Theorem 12.2.5. *Suppose that there exists a scalar function $W \in C^1(\Omega, \mathbb{R})$ such that $W(\overline{x}^*) = 0$ and that $\dot{W}(\overline{x}) := (\nabla W(\overline{x}) \mid \overline{f}(\overline{x}))$ is either positive or negative for all $\overline{x} \neq \overline{x}^*$. Moreover, we assume that there exists a sequence $\overline{x}_k \in \Omega$, with $\overline{x}_k \to \overline{x}^*$ such that $W(\overline{x}_k)\dot{W}(\overline{x}_k) > 0$. Then $\overline{x}^*$ is unstable.*

12.3 Stability of linear systems and n-th order linear equations

In this section we will apply the previous theorems to study the stability of the linear system

$$\overline{x}' = A\overline{x}, \qquad \overline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

We start with linear 2×2 autonomous systems. Recall that these systems have been discussed in Chapter 7, but here they are studied from the point of view of stability.

12.3.1 Stability of 2×2 systems

Changing notation, we call $(x, y) \in \mathbb{R}^2$ the variable and write the system in the form

$$\begin{cases} x' = & a_{11}x + a_{12}y \\ y' = & a_{21}x + a_{22}y \end{cases} \tag{12.3}$$

where the coefficients a_{ij} are real numbers. Letting $u = (x, y)$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the system can be written as $u' = Au$. If A is nonsingular, which we always assume, the only equilibrium is $(x, y) = (0, 0)$. We are going to study the qualitative properties of the solutions of (12.3), in particular their asymptotic behavior, as $t \to +\infty$.

Referring to Chapter 7 for some more details, let us recall that the Jordan normal form of a nonsingular matrix A is a nonsingular matrix J with the property that there exists an invertible matrix B such that $BA = JB$. The Jordan matrix J exists and has the same eigenvalues λ_1, λ_2 as A. Moreover, if λ_1, λ_2 are real numbers, then

$$\lambda_1 \neq \lambda_2 \quad \Longrightarrow \quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{J1}$$

If $\lambda_1 = \lambda_2$ then either

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \tag{J2.1}$$

or

$$J = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}. \tag{J2.2}$$

If the eigenvalues are complex, $\lambda = \alpha \pm i\beta$, then

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}. \tag{J3}$$

Lemma 12.3.1. *The change of variable $z = B^{-1}u$ transforms the solutions $u(t)$ of $u' = Au$ into the solutions $z(t)$ of $z' = Jz$. Therefore, $(0,0)$ is stable or unstable relative to $u' = Au$ if and only if it is the same relative to $y' = Jz$.*

Proof. The first part of the lemma has been proved in Chapter 7 (with slightly different notation): from $z' = B^{-1}u' = B^{-1}Au = JBu$ it follows that $z' = Jz$. If we set $z = (z_1, z_2)$ this shows that the change of variable B^{-1} transforms a solution curve $u(t) = (x(t), y(t))$ of $u' = Au$ in the plane (x, y) into a solution curve $z(t) = (z_1(t), z_2(t))$ of $z' = Jz$ in the plane (z_1, z_2). Thus the qualitative properties of $(x(t), y(t))$ are the same as those of $(z_1(t), z_2(t))$. In particular, $(x(t), y(t)) \to (0,0)$ as $t \to \pm\infty$ if and only if $(z_1(t), z_2(t))$ does the same, and hence the point $(0,0)$ is stable or unstable relative to $u' = Au$ if and only if it is the same relative to $z' = Jz$. ∎

By the lemma, it suffices to study the system

$$u' = Ju.$$

Consider first the case when the eigenvalues are real and distinct. According to (J1) the system $u' = Ju$ becomes

$$\begin{cases} x' = \lambda_1 x \\ y' = \lambda_2 y \end{cases}$$

which is decoupled. Its solutions are given by $x(t) = c_1 e^{\lambda_1 t}$ and $y(t) = c_2 e^{\lambda_2 t}$, where $c_1 = x(0)$, $c_2 = y(0) \in \mathbb{R}$. If $\lambda_1 < 0$, resp. $\lambda_2 < 0$, then $x(t) \to 0$, resp. $y(t) \to 0$, as $t \to +\infty$. Therefore,

if both the eigenvalues are real and negative, the equilibrium $(0,0)$ is asymptotically stable, while if one of the eigenvalues is positive, $(0,0)$ is unstable.

We can write the solutions in the form $y = y(x)$. Precisely, if $c_1 = 0$ then $x(t) \equiv 0$. If $c_1 \neq 0$ we solve $x(t) = c_1 e^{\lambda_1 t}$ for t, obtaining

$$t = \frac{1}{\lambda_1} \ln \left| \frac{x}{c_1} \right| = \ln \left| \frac{x}{c_1} \right|^{\frac{1}{\lambda_1}}.$$

Substituting into $y(t)$ we get

$$y(t) = c_2 e^{\lambda_2 t} = c_2 \exp\left[\lambda_2 \ln\left|\frac{x}{c_1}\right|^{\frac{1}{\lambda_1}}\right] = c_2 \exp\left[\ln\left|\frac{x}{c_1}\right|^{\frac{\lambda_2}{\lambda_1}}\right] = \frac{c_2}{c_1'}\, x^{\frac{\lambda_2}{\lambda_1}}$$

where $c_1' = c_1^{\frac{\lambda_2}{\lambda_1}}$. The behavior of these functions depends on the sign of the eigenvalues and on their ratio. If $\lambda_1 < \lambda_2 < 0$, then the exponent of x is positive and greater that 1, see Figure 12.1a. If $\lambda_2 < \lambda_1 < 0$, then the exponent of x is positive and smaller than 1. In any case the origin is asymptotically stable and is called a *stable node*, see Figure 12.1b.

If λ_1 and λ_2 are positive, we have an *unstable node*. The graphs are plotted in Figures 12.2a–12.2b.

If $\lambda_1 \cdot \lambda_2 < 0$, the functions $y(x)$ are hyperbolas. The origin is unstable and is called a *saddle*, see Figure 12.3.

We now consider the case when $\lambda_1 = \lambda_2 := \lambda$ and is real. If $(J2.1)$ holds, then the system becomes

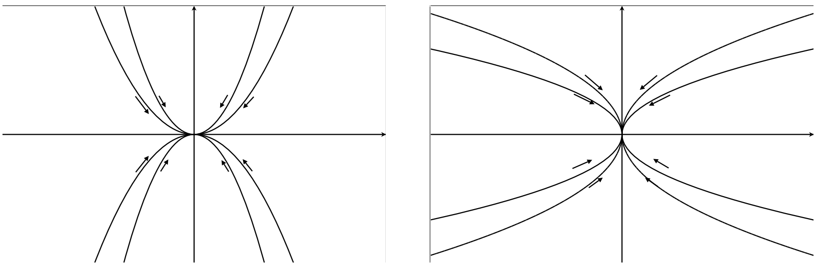
$$\begin{cases} x' = & \lambda x \\ y' = & \lambda y. \end{cases}$$

Thus $x(t) = c_1 e^{\lambda t}$, $y(t) = c_2 e^{\lambda t}$ and $(0, 0)$ is asymptotically stable provided $\lambda < 0$, otherwise $(0, 0)$ is unstable. It is still called a *stable or unstable node*. Here $y(x) = cx$, with $c = c_2/c_1$, see Figures 12.4a–12.4b.

If $(J2.2)$ holds, then the system becomes

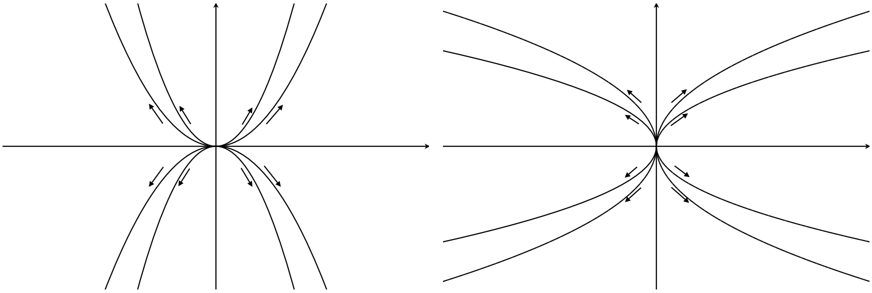
$$\begin{cases} x' = & \lambda x + y \\ y' = & \lambda y. \end{cases}$$

The solution of the second equation is $y(t) = c_2 e^{\lambda t}$. Substituting into the first one, we find $x' = \lambda x + c_2 e^{\lambda t}$ which is a linear first order non-autonomous equation. The solution is $x(t) = (c_1 + c_2 t)e^{\lambda t}$. Once again, if $\lambda < 0$ we have asymptotic stability. Otherwise, if $\lambda > 0$ we have instability. The origin is still a *node*.



(a) (b)

Fig. 12.1. Stable node. (a) $\lambda_1 < \lambda_2 < 0$; (b) $\lambda_2 < \lambda_1 < 0$



(a) (b)

Fig. 12.2. Unstable node. (a) $0 < \lambda_1 < \lambda_2$; (b) $0 < \lambda_2 < \lambda_1$

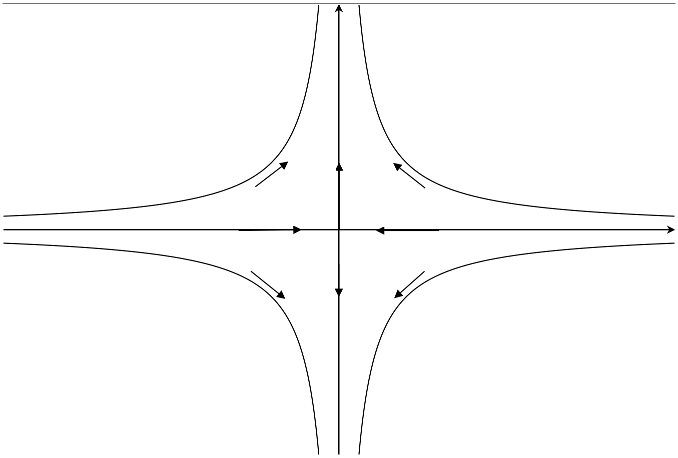
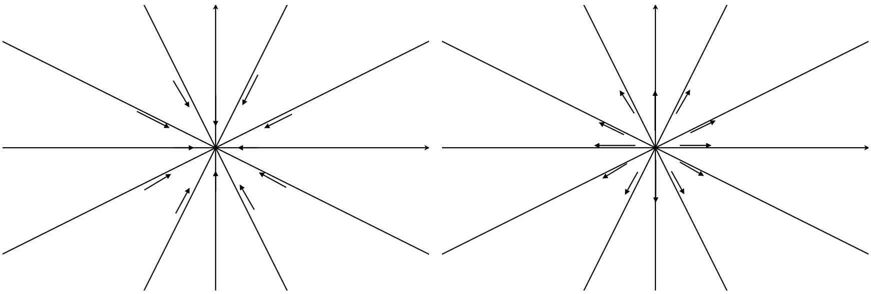


Fig. 12.3. Saddle, with $\lambda_1 < 0 < \lambda_2$



(a) (b)

Fig. 12.4. Case $(J2.1)$. (a) Stable node, with $\lambda_1 = \lambda_2 < 0$; (b) unstable node, with $\lambda_1 = \lambda_2 > 0$

If $c_2 = 0$ we find $y(t) \equiv 0$. If $c_2 \neq 0$, we have

$$e^{\lambda t} = \frac{y}{c_2} \implies t = \frac{1}{\lambda} \ln\left(\frac{y}{c_2}\right).$$

Thus from $x = (c_1 + c_2 t)e^{\lambda t}$ we infer

$$x = \left[c_1 + c_2 \frac{1}{\lambda} \ln\left(\frac{y}{c_2}\right)\right] \frac{y}{c_2}, \quad (c_2 \neq 0).$$

The graphs are shown in Figures 12.5a–12.5b.

We finally consider the case in which the eigenvalues are complex. From $(J3)$ it follows that the system becomes

$$\begin{cases} x' = & \alpha x - \beta y \\ y' = & \beta x + \alpha y. \end{cases}$$

Using polar coordinates $x(t) = r(t)\cos\theta(t)$, $y(t) = r(t)\sin\theta(t)$, we find

$$x' = r'\cos\theta - r\theta'\sin\theta, \quad y = r'\sin\theta + r\theta'\cos\theta$$

whereby

$$\begin{cases} r'\cos\theta - r\theta'\sin\theta & = \alpha(r\cos\theta) - \beta(r\sin\theta) \\ r'\sin\theta + r\theta'\cos\theta & = \beta(r\cos\theta) + \alpha(r\sin\theta) \end{cases}$$

Adding the first equation multiplied by $\cos\theta$ to the second equation multiplied by $\sin\theta$ we get $r' = \alpha r$. Similarly, subtracting the first equation multiplied by $\sin\theta$ from the second equation multiplied by $\cos\theta$ we get $\theta' = \beta$. In other words, the system in the unknowns r, θ is simply

$$\begin{cases} r' = & \alpha r \\ \theta' = & \beta \end{cases}$$

whose solutions are $r(t) = c_1 e^{\alpha t}$, $\theta(t) = \beta t + c_2$. Thus

$$x(t) = c_1 e^{\alpha t}\cos(\beta t + c_2), \qquad y(t) = c_1 e^{\alpha t}\sin(\beta t + c_2).$$

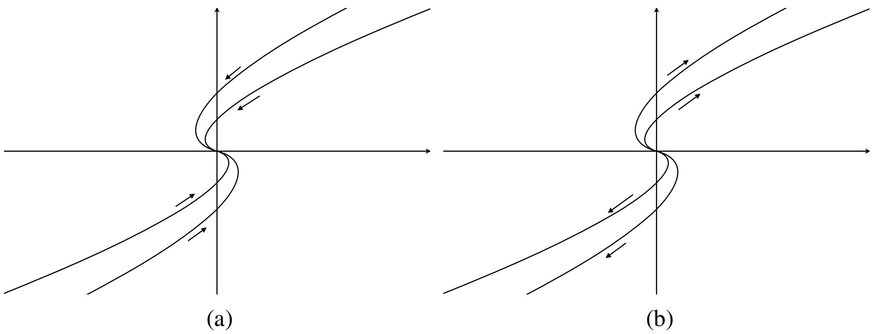


Fig. 12.5. Case $(J2.2)$. (a) Stable node, with $\lambda_1 = \lambda_2 < 0$; (b) unstable node, with $\lambda_1 = \lambda_2 > 0$

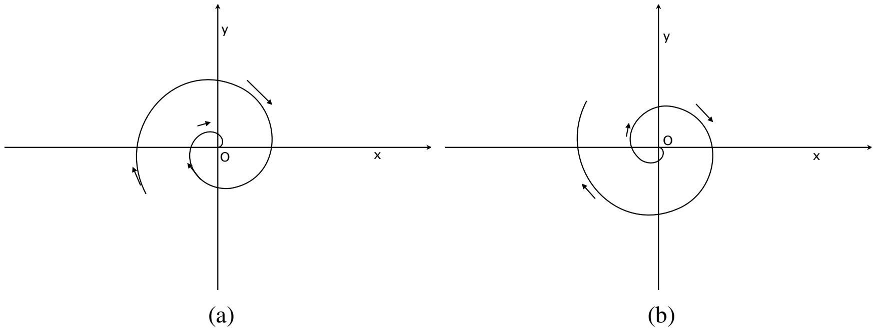


Fig. 12.6. Case ($J3$), with $\alpha \neq 0$. (a) $\alpha < 0$: stable focus; (b) $\alpha > 0$: unstable focus

Thus stability depends only on α. Precisely, one has:

if $\lambda_{1,2} = \alpha \pm i\beta$ *and* $\alpha < 0$*, then the origin is asymptotically stable, while if* $\alpha > 0$*, the origin is unstable.*

If $\alpha \neq 0$, the equilibrium is called a *focus*, see Figures 12.6a–12.6b. The curves are spirals.

If $\lambda_{1,2} = \alpha \pm i\beta$ and $\alpha = 0$ we find that $r(t)$ is a constant. Thus the solution curves are circles $r = c > 0$, namely $x^2 + y^2 = c$, centered at the origin, see Figure 12.7. Hence

if $\lambda_{1,2} = \alpha \pm i\beta$ *and* $\alpha = 0$*, the origin is stable, but not asymptotically stable.*

The equilibrium is called a *center*.

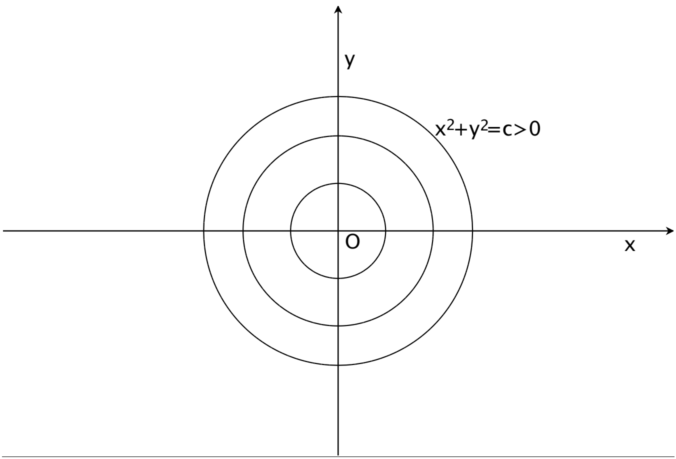


Fig. 12.7. Center: case (J3) with $\alpha = 0$

Table 12.1. Stability of equilibria of $u' = Au$; $\lambda_{1,2}$ are the eigenvalues of the 2×2 nonsingular matrix A

Eigenvalues	*Equilibrium*
$\lambda_{1,2} \in \mathbb{R},\ \lambda_1, \lambda_2 < 0$	asymptotically stable node
$\lambda_{1,2} \in \mathbb{R},\ \lambda_1, \lambda_2 > 0$	unstable node
$\lambda_{1,2} \in \mathbb{R},\ \lambda_1 \cdot \lambda_2 < 0$	unstable saddle
$\lambda_{1,2} = \alpha \pm i\beta,\ \alpha < 0$	asymptotically stable focus
$\lambda_{1,2} = \alpha \pm i\beta,\ \alpha > 0$	unstable focus
$\lambda_{1,2} = \pm i\beta,$	stable center

Remark 12.3.2. To complete the discussion, consider some cases in which one eigenvalue of A is zero.

If $A = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ with $b \neq 0$, the system is

$$\begin{cases} x' = & bx \\ y' = & 0 \end{cases}$$

whose solutions are $x(t) = c_1 e^{bt}$, $y(t) = c_2$ which, in the (x, y) - plane are half lines parallel to the x axis. If $A = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$, we find $x(t) = c_1$, $y(t) = c_2 e^{bt}$, namely a family of half lines parallel to the y axis. We have stability if $b < 0$ and instability if $b > 0$.

If $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ the system is

$$\begin{cases} x' = & by \\ y' = & 0 \end{cases}$$

whereby $y(t) = c_2$ and $x(t) = c_1 + c_2 bt$ which is a family of straight lines parallel to the x axis. Similarly, if $A = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$, then $x = c_1, y = c_2 + c_1 bt$. In both of these two last cases we have instability for any $b \neq 0$. ■

Table 12.1 summarizes the nature of the equilibrium (0, 0) when A is nonsingular.

12.3.2 Stability of n × n linear systems

Extending the previous results, we state the following theorem dealing with the linear autonomous $n \times n$ system $\overline{x}' = A\overline{x}$, where $\overline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Theorem 12.3.3. *Suppose that A is a constant $n \times n$ nonsingular matrix.*

(i) *If all the eigenvalues of A have negative real part, then $\overline{x}^* = 0$ is asymptotically stable. More precisely, for all $p \in \mathbb{R}^n$ one has that $\overline{x}(t, p) \to 0$ as $t \to +\infty$.*

(ii) *If one eigenvalue of A has positive real part, then $\overline{x}^* = 0$ is unstable.*

In the above statement, if an eigenvalue is real, its real part is the eigenvalue itself.

In the case that

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

where the eigenvalues λ_i are real (notice that we do not require that $\lambda_i \neq \lambda_j$ for $i \neq j$), the systems $\overline{x}' = A\overline{x}$ splits into n independent equations $x_i' = \lambda_i x_i$. These equations yield $x_i(t) = c_i e^{\lambda_i t}$. This immediately implies the asymptotic stability of $\bar{0}$ provided all $\lambda_i < 0$. Moreover, if one of the λ_i is positive, then $x_i(t) = c_i e^{\lambda_i t}$ does not tend to 0 as $t \to +\infty$ and we have instability.

If A is not diagonal, the proof of (i) is based on finding a Liapunov function for $\overline{x}' = A\overline{x}$ and on applying the Liapunov Stability Theorem 12.2.2. To avoid cumbersome calculations, we carry out the details in two specific examples in 3D. The general case follows from similar arguments.

Let $\overline{x} = (x, y, z) \in \mathbb{R}^3$, and consider the following two cases:

1. $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \gamma \end{pmatrix}$.
2. $A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}$.

We claim that $V(\overline{x}) = \frac{1}{2}(x^2 + y^2 + z^2)$ is a Liapunov function.

Clearly, $V(\overline{x}) > 0$ for all $\overline{x} \neq (0, 0, 0)$ and hence $(V1)$ holds. As for $(V2)$, we consider separately the two cases.

1. Since $A\overline{x} = (\lambda x + y, \lambda y, \gamma z)$ one infers

$$\dot{V} = (\nabla V \mid A\overline{x}) = \lambda[x^2 + xy + y^2] + \gamma z^2.$$

Notice that

$$x^2 + xy + y^2 = \frac{x^2}{4} + xy + y^2 + \frac{3x^2}{4} = \left(\frac{x}{2} + y\right)^2 + \frac{3x^2}{4} > 0, \ \forall (x, y) \neq (0, 0).$$

Thus, if $\lambda, \gamma < 0$ it follows that $\dot{V} < 0$ for all $(x, y, z) \neq (0, 0, 0)$ and hence $(V2)$ holds (with strict inequality).

2. Here the eigenvalues of A are $\alpha \pm i\beta$ and $\gamma \in \mathbb{R}$. We have that $A\overline{x} = (\alpha x + \beta y, -\beta x + \alpha y, \gamma z)$ and hence

$$\dot{V} = (\nabla V \mid A\overline{x}) = \alpha[x^2 + y^2] + \gamma z^2.$$

Thus $(V2)$ holds (with strict inequality) provided α and γ are both negative.

In each of the above cases, we can apply (*ii*) of the Liapunov Stability Theorem 12.2.2 to infer that $\overline{x}^* = (0, 0, 0)$ is asymptotically stable provided all the eigenvalues of A have negative real parts.

12.3.3 Stability of n-th order linear equations

Recall that the linear n-th order equation in the real variable x

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1\frac{dx}{dt} + a_0 x = 0 \qquad (12.4)$$

is equivalent to the system

$$\begin{cases} x_1' = & x_2 \\ x_2' = & x_3 \\ \dots & \dots \\ x_n' = & -a_{n-1}x_n - \cdots - a_1 x_2 + a_0 x_1. \end{cases} \qquad (12.5)$$

The stability of the trivial solution of (12.4) is the same as the stability of the point with coordinates $x_1 = x_2 = \cdots = x_n = 0$ for the equivalent system (12.5). In particular, the asymptotic stability of $x = 0$ means that for all $p \in \mathbb{R}$ near $x = 0$ one has

$$\lim_{t\to+\infty} x(t,p) = \lim_{t\to+\infty}\frac{dx(t,p)}{dt} = \cdots = \lim_{t\to+\infty}\frac{d^{n-1}x(t,p)}{dt^{n-1}} = 0.$$

One can check that the roots of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

of (12.4) coincide with the eigenvalues of the system (12.5). Let us show this claim in the case of the second order equation $x'' + ax' + bx = 0$, equivalent to the system

$$\begin{cases} x_1' = & x_2 \\ x_2' = & -ax_2 - bx_1. \end{cases}$$

The eigenvalues are the solutions of

$$\begin{vmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{vmatrix} = \lambda^2 + a\lambda + b = 0,$$

which is the same as the characteristic equation of $x'' + ax' + bx = 0$.

Using the preceding remark about the roots of the characteristic equation, we can use Theorem 12.3.3 to infer

Theorem 12.3.4. *The trivial solution $x = 0$ is asymptotically stable if all the roots of the characteristic equation have negative real parts, while it is unstable if at least one root of the characteristic equation a has positive real part.*

On the other hand, we might also work directly on the equation. Actually, the general solution of (12.4) is the superposition of terms $t^m e^{\lambda t}$ or $t^m e^{\alpha t}(\sin \beta t + \cos \beta t)$, where λ or $\alpha \pm i\beta$ are roots of the characteristic equation. These terms, together with their derivatives, tend to zero as $t \to +\infty$ if and only if $\lambda < 0$, or $\alpha < 0$.

12.4 Hamiltonian systems

Let $H : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ be continuously differentiable and consider the *hamiltonian system*

$$\begin{cases} x_i' &= -H_{y_i}(x_1, \dots, x_n, y_1, \dots, y_n) \\ y_i' &= \quad H_{x_i}(x_1, \dots, x_n, y_1, \dots, y_n) \end{cases} \qquad i = 1, 2, \cdots n,$$

or, in a compact form

$$\begin{cases} \overline{x}' &= -\nabla_y H(\overline{x}, \overline{y}) \\ \overline{y}' &= \quad \nabla_x H(\overline{x}, \overline{y}) \end{cases} \tag{HS}$$

where $\nabla_x H = (H_{x_1}, \cdots, H_{x_n})$ and $\nabla_y H = (H_{y_1}, \cdots, H_{y_n})$.

Planar hamiltonian systems have been discussed in Section 1 of Chapter 8. The following Lemma is the counterpart of Lemma 8.1.1 therein.

Lemma 12.4.1. *If $(\overline{x}(t), \overline{y}(t))$ is a solution of (HS), then $H(\overline{x}(t), \overline{y}(t))$ is constant.*

Proof. One has

$$\frac{d}{dt} H(\overline{x}(t), \overline{y}(t)) = (\nabla_x H(\bullet) \mid \overline{x}') + (\nabla_y H(\bullet) \mid \overline{y}')$$

where $(\bullet) = (\overline{x}(t), \overline{y}(t))$. Since $(\overline{x}(t), \overline{y}(t))$ satisfies (HS) it follows

$$\frac{d}{dt} H(\bullet) = -(\nabla_x H(\bullet) \mid \nabla_y H(\bullet)) + (\nabla_y H(\bullet) \mid \nabla_x H(\bullet)) = 0$$

and thus $H(\overline{x}(t), \overline{y}(t))$ is constant. ■

Theorem 12.4.2. *Let $H(0,0) = 0$ and suppose that $(0,0)$ is a local strict minimum of H, namely that there exists a neighborhood $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ of $(0,0)$ such that $H(\overline{x}, \overline{y}) > 0$ for all $(\overline{x}, \overline{y}) \in \Omega$, $(\overline{x}, \overline{y}) \neq (0,0)$. Then $(0,0)$ is stable (but not asymptotically stable).*

Proof. We claim that the restriction of H to Ω is a Liapunov function for (HS). First of all, by assumption, $(V1)$ holds. Moreover, Lemma 12.4.1 implies that $\dot{H} = 0$

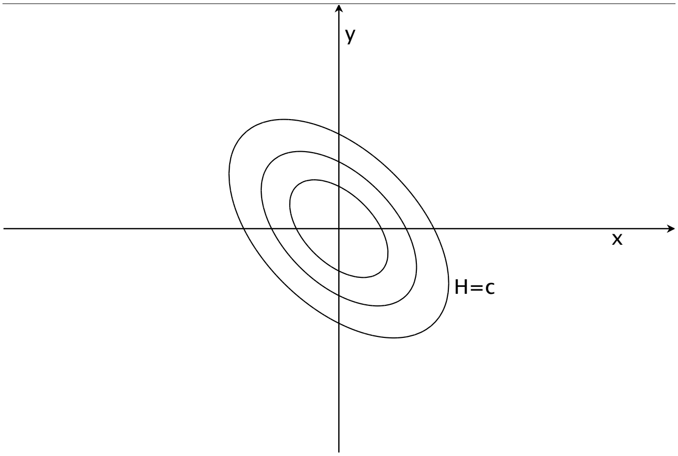


Fig. 12.8. Typical phase plane portrait of a hamiltonian system in 2D

and hence $(V2)$ holds. From Theorem 12.2.2 it follows that $(0, 0)$ is stable. More precisely, since $\dot{H} = 0$ then, according to Remark 12.2.3, $(0, 0)$ is stable but not asymptotically stable. If $n = 1$ the equilibrium is like a *stable center* for linear systems. See Figure 12.8. ∎

Remark 12.4.3. To show the stability of the nontrivial equilibrium of the Lotka–Volterra system (see Example 12.2.4), we could also use the preceding theorem. ∎

If

$$H(\overline{x}, \overline{y}) = \frac{1}{2}|\overline{y}|^2 + F(\overline{x}), \qquad F \in C^1(\mathbb{R}^n, \mathbb{R})$$

the hamiltonian system (HS) becomes

$$\begin{cases} x_i' = & -y_i \\ y_i' = & F_{x_i}(\overline{x}) \end{cases} \tag{12.6}$$

which is equivalent to the second order *gradient system*

$$\overline{x}'' + \nabla F(\overline{x}) = 0, \tag{12.7}$$

namely

$$x_i'' + F_{x_i}(x_1, \dots, x_n) = 0, \qquad i = 1, \dots, n.$$

Let us point out that an equilibrium $\overline{x}^*$ of a system such as (12.7) corresponds to the equilibrium $(\overline{x}^*, 0)$ of the equivalent first order system (12.6). Stability of $\overline{x}^*$ for (12.7) has to be understood as the stability of $(\overline{x}^*, 0)$ for (12.6). For example, the asymptotic stability of $\overline{x}^*$ means that $\overline{x}(t, p) \to \overline{x}^*$ and $\overline{x}'(t, p) \to 0$ as $t \to +\infty$, for all p close to $\overline{x}^*$.

The following theorem is known as the Dirichlet–Lagrange stability criterion.

Theorem 12.4.4. *Let* $F(\overline{x}^*) = 0$ *and suppose that* $\overline{x}^*$ *is a local strict minimum of* F*. Then the equilibrium* $\overline{x}^*$ *is stable with respect to* (12.7).

Proof. It suffices to remark that $H(\overline{x}, \overline{y}) = \frac{1}{2}|\overline{y}|^2 + F(\overline{x})$ has a strict local minimum at $(\overline{x}^*, 0)$ and apply Theorem 12.4.2. ■

12.5 Stability of equilibria via linearization

Given a system $\overline{x}' = \overline{f}(\overline{x})$ with equilibrium $\overline{x}^* = 0$, its *linearization* at $\overline{x}^* = 0$ is the linear system $\overline{x}' = A\overline{x}$, where $A = \nabla\overline{f}(0)$.

Developing f in Taylor's expansion we find $f(\overline{x}) = A\overline{x} + o(|\overline{x}|)$. Then the linearized system is $\overline{x}' = A\overline{x}$. We have seen that a sufficient condition for the asymptotic stability of $\overline{x} = 0$ for $\overline{x}' = A\overline{x}$ is that all the real parts of the eigenvalues of A be negative, whilst if at least one eigenvalue is positive, or has positive real part, then $\overline{x}^* = 0$ is unstable. This result is extended to the nonlinear case in the next theorem, whose proof is omitted.

Theorem 12.5.1. *Suppose that all the eigenvalues of* $\nabla\overline{f}(0)$ *have negative real parts. Then the equilibrium* $\overline{x}^* = 0$ *is asymptotically stable with respect to the system* $\overline{x}' = \nabla\overline{f}(0)\overline{x} + o(|\overline{x}|)$.

If at least one eigenvalue of $\nabla\overline{f}(0)$ *has positive real part, then the equilibrium* $\overline{x}^* = 0$ *is unstable.*

Example 12.5.2. Consider the Van der Pol system

$$\begin{cases} x' = & -y \\ y' = & x - 2\mu(x^2 - 1)y \end{cases}$$

with $|\mu| < 1$. Here the eigenvalues of

$$A = \nabla\overline{f}(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 2\mu \end{pmatrix}$$

are $\lambda_1 = \mu + \sqrt{\mu^2 - 1}$, $\lambda_2 = \mu - \sqrt{\mu^2 - 1}$. If $0 < \mu < 1$, both the eigenvalues have positive real part and the equilibrium $(0, 0)$ is unstable. On the other hand, if $-1 < \mu < 0$, both the eigenvalues have negative real part and the equilibrium $(0, 0)$ is asymptotically stable. ■

Example 12.5.3. We have seen in Example 12.2.4 that the nontrivial equilibrium of a Lotka–Volterra system

$$\begin{cases} x' & = ax - bxy \\ y' & = -cy + dxy \end{cases}$$

is stable. On the contrary, let us show that $(0,0)$ is unstable. Here

$$\overline{f}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} ax - bxy \\ -cy + dxy \end{pmatrix}.$$

Thus

$$\nabla \overline{f}(x,y) = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

and hence

$$A = \nabla \overline{f}(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

whose eigenvalues are $a > 0, -c < 0$. It follows that $(0,0)$ is unstable. ■

Consider the one-dimensional case of a single equation $x' = f(x)$, where f is continuously differentiable and $f(0) = 0$. The linearized equation is $x' = f'(0)x$ for which the stability of $x = 0$ is determined by the sign of $f'(0)$. Then the previous theorem yields that the equilibrium $x = 0$ is asymptotically stable if $f'(0) < 0$ and unstable if $f'(0) > 0$.

Example 12.5.4. If $f(x) = \lambda x - x^3$, $x = 0$ is an equilibrium of $x' = f(x)$. Since $f'(0) = \lambda$, it is asymptotically stable if $\lambda < 0$ and unstable if $\lambda > 0$. When λ becomes positive there is a change of stability and a pair of nontrivial equilibria branch off from $\lambda = 0$. These new equilibria are $\pm x_\lambda = \pm\sqrt{\lambda}$. Since $f'(x_\lambda) = \lambda - 3x_\lambda^2 = -2\lambda < 0$ for $\lambda > 0$, it follows that x_λ are asymptotically stable. This phenomenon is called a *pitchfork bifurcation*. See Figure 12.9.

The same bifurcation arises in the case of $x' = \lambda x + x^3$. Here the nontrivial equilibria $\pm x_\lambda = \pm\sqrt{(-\lambda)}$, $\lambda < 0$, are unstable and the branch is downward directed. ■

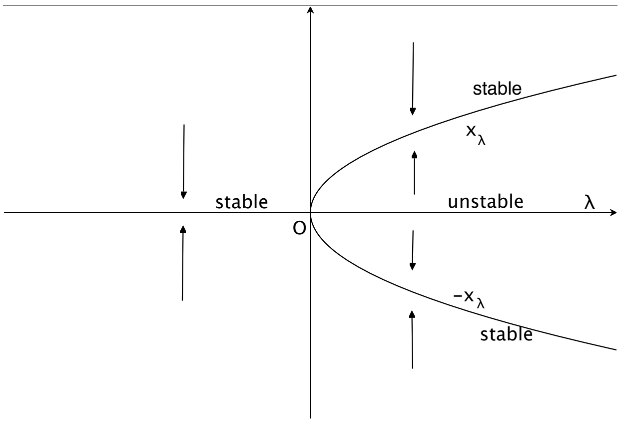


Fig. 12.9. Pitchfork bifurcation for $x' = \lambda x - x^3$

The following example shows that if the matrix $\nabla\overline{f}(0)$ has a pair of conjugate eigenvalues with zero real parts, the stability of $\overline{x}^* = 0$ cannot be deduced by the previous theorems, but depends on the higher order term in the Taylor's expansion of $\overline{f}(x)$.

Example 12.5.5. Consider the system

$$\begin{cases} x' = & y + \epsilon x(x^2 + y^2) \\ y' = & -x + \epsilon y(x^2 + y^2) \end{cases} \tag{12.8}$$

whose linear part has eigenvalues $\pm i$. Letting $V(x, y) = x^2 + y^2$, let us evaluate

$$\dot{V} = 2(xx' + yy').$$

Multiplying the first equation by x and the second by y and summing up, we get

$$xx' + yy' = \epsilon(x^2 + y^2)^2.$$

Therefore

$$\dot{V} = 2\epsilon(x^2 + y^2)^2,$$

whose sign depends on ϵ. The equilibrium $(0, 0)$ is asymptotically stable if $\epsilon < 0$, whilst it is unstable if $\epsilon > 0$. If $\epsilon = 0$, then $\dot{V} = 0$ and hence we have stability. As an exercise, the reader can transform (12.8) using polar coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$, and show that for $\epsilon \neq 0$ the trajectories are spirals, with a behavior like the one plotted in Figures 12.6a–12.6b. ■

12.5.1 Stable and unstable manifolds

The results below describe the behavior of the solutions near an unstable equilibrium in more detail.

Consider the linear system $\overline{x}' = A\overline{x}$. The equilibrium $\overline{x}^* = 0$ is called *hyperbolic* if the matrix A has no eigenvalues with zero real part.

Theorem 12.5.6. *Suppose that $\overline{x}^* = 0$ is a hyperbolic equilibrium: A has k eigenvalues $\lambda_1, \dots, \lambda_k$ with negative real parts and $n - k$ eigenvalues $\lambda_{k+1}, \dots, \lambda_n$ with positive real parts. Let e_i, $i = 1, \cdots, n$, denote an orthogonal system of eigenvectors corresponding to λ_i and let*

$$L^s = L^s_k = span\{e_1, \cdots, e_k\}, \quad L^u = L^u_{n-k} = span\{e_{k+1}, \cdots, e_n\}.$$

Then:

(i) *L^s and L^u are invariant, that is if $p \in L^s$, resp. $p \in L^u$, then $\overline{x}(t, p) \in L^s$, resp. $\overline{x}(t, p) \in L^u$, for all t.*
(ii) *$p \in L^s$ if and only if* $\lim_{t\to+\infty} \overline{x}(t, p) = 0$.
(iii) *$p \in L^u$ if and only if* $\lim_{t\to-\infty} \overline{x}(t, p) = 0$.

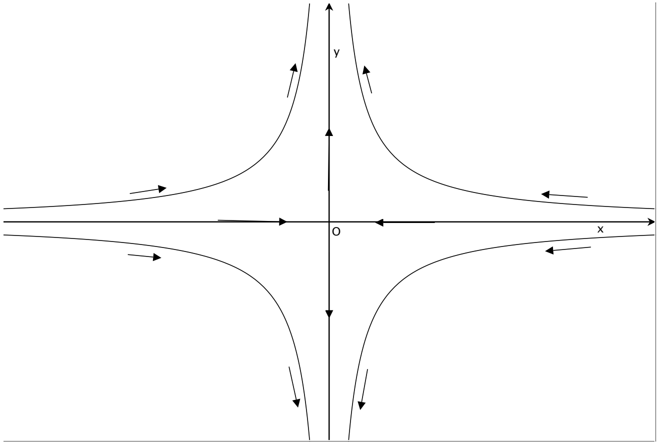


Fig. 12.10. Phase plane portrait of $\overline{x}' = A\overline{x}$, $\overline{x} = (x, y)$, with $A = \mathrm{diag}\{\lambda_1, \lambda_2\}$, $\lambda_1 < 0 < \lambda_2$

Proof. We prove the theorem in the simple case in which $A = \mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$, where $\lambda_1 \le \cdots \le \lambda_k < 0 < \lambda_{k+1} \le \cdots \le \lambda_n$. The system $\overline{x}' = A\overline{x}$ is decoupled into n independent equations $x_i' = \lambda_i x_i$. If $p = (p_1, \cdots, p_n)$, one finds that $\overline{x}(t, p) = (p_1 e^{\lambda_1 t}, \cdots, p_n e^{\lambda_n t})$. Then $\lim_{t\to+\infty} \overline{x}(t, p) = 0$ if and only if $p_{k+1} = \cdots = p_n = 0$. This implies that $L^s = span\{e_1, \cdots, e_k\}$. Similarly, $\lim_{t\to-\infty} \overline{x}(t, p) = 0$ if and only if $p_1 = \cdots = p_k = 0$ and hence $L^u = span\{e_{k+1}, \cdots, e_n\}$. ∎

If $n = 2$ and $A = \mathrm{diag}\{\lambda_1, \lambda_2\}$ we find the saddle plotted in Figure 12.10.

Example 12.5.7. Consider the system

$$\begin{cases} x' &= -x + y \\ y' &= y \end{cases}$$

The origin $(0, 0)$ is unstable because the eigenvalues of the coefficient matrix $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ are $\lambda_1 = -1, \lambda_2 = 1$. If $e_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ are two eigenvectors corresponding to λ_i, $i = 1, 2$, the stable, resp. unstable, manifold is the linear space spanned by e_1, resp. e_2, namely the straight lines

$$x = a_i t, \quad y = b_i t, \quad \text{or else} \quad y = \frac{b_i}{a_i} x.$$

Solving $Ae_i = \lambda_i e_i$, namely

$$\begin{cases} -a_i + b_i &= \lambda_i a_i \\ \qquad\; b_i &= \lambda_i b_i \end{cases}$$

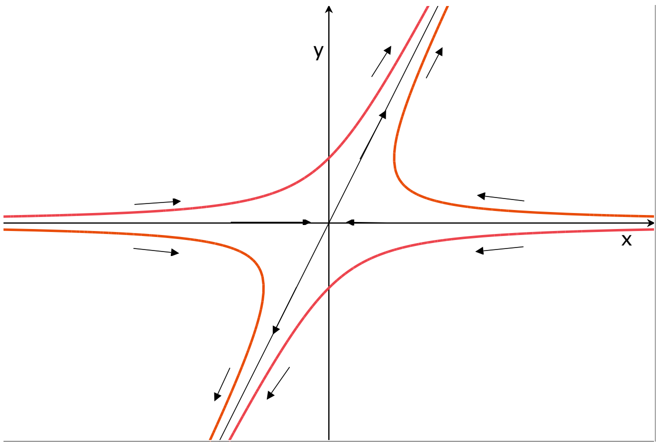


Fig. 12.11. The hyperbolas $2xy = 2c_1c_2 + y^2$ with asymptotes $y = 0$ (*stable manifold*) and $y = 2x$ (*unstable manifold*)

we can take

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

It follows that the stable manifold is $y = 0$, while the unstable manifold is $y = 2x$.

The reader should notice that the solutions of the given system are $x = c_1e^{-t} + \frac{1}{2}c_2e^t$, $y = c_2e^t$. From the last equation we get $e^t = y/c_2$. Substituting into the first equation it follows

$$x = \frac{c_1c_2}{y} + \frac{y}{2}$$

which yields the family of hyperbolas

$$2xy = 2c_1c_2 + y^2.$$

The stable and unstable manifolds are exactly the asymptotes of these hyperbolas. See Figure 12.11. ■

The previous result can be extended to the nonlinear system $\overline{x}' = \nabla\overline{f}(0)\overline{x} + o(|\overline{x}|)$.

Theorem 12.5.8. *Let $\overline{f}$ be smooth and suppose that the matrix $A = \nabla\overline{f}(0)$ has k eigenvalues with negative and $n-k$ eigenvalues with positive real parts. Then there are smooth surfaces $M^s = M^s_k$ and $M^u = M^u_{n-k}$, of dimension k and $n-k$ respectively, with $M^s \cup M^u = \{0\}$, defined in a neighborhood of $\overline{x}^* = 0$, which are tangent to L^s, resp. L^u, such that:*

(i) *M^s and M^u are invariant, that is, if $p \in M^s$, resp. $p \in M^u$, then $\overline{x}(t,p) \in M^s$, resp. $\overline{x}(t,p) \in M^u$, for all t.*
(ii) *$p \in M^s$ if and only if* $\lim_{t\to+\infty} \overline{x}(t,p) = 0$.
(iii) *$p \in M^u$ if and only if* $\lim_{t\to-\infty} \overline{x}(t,p) = 0$.

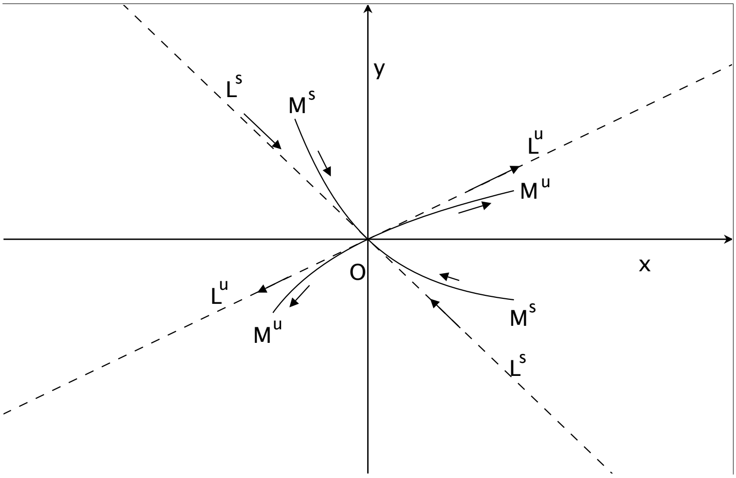


Fig. 12.12. Stable and unstable manifolds

The surface M^s is called the *stable manifold* of the system, whilst M^u is called the *unstable manifold*, see Figure 12.12. If $\overline{f}(\overline{x}) = A\overline{x}$ the stable and unstable manifolds are the linear spaces L^s, L^u. If all the eigenvalues have negative, resp. positive, real parts then $M^u = \emptyset$, resp. $M^s = \emptyset$.

Remark 12.5.9. If M^s, M^u are defined globally on $\mathbb{R}^n$, it is possible to prove that if $p \in \mathbb{R}^n \setminus M^s$, then $\overline{x}(t, p)$ approaches M^u asymptotically as $t \to +\infty$. ∎

12.6 Exercises

1. Show that $(0, 0)$ is asymptotically stable for the linear system
$$\begin{cases} x' = & -2x + y \\ y' = & 7x - 4y. \end{cases}$$
2. Show that $(0, 0)$ is asymptotically stable for the linear system
$$\begin{cases} x' = & -x - y \\ y' = & 4x - y. \end{cases}$$
3. Show that $(0, 0)$ is stable for the linear system
$$\begin{cases} x' = & x - y \\ y' = & 3x - y. \end{cases}$$
4. Study the stability of $(0, 0)$ for the system
$$\begin{cases} x' = & -2ax - y \\ y' = & (9 + a^2)x \end{cases}$$
depending on the parameter a.

5. Show that $(0, 0)$ is unstable for the system

$$\begin{cases} x' = & -x + 4y \\ y' = & 3x - 5y \end{cases}$$

and find the stable and unstable manifold.

6. Study the stability of the trivial solution of the equation $x'' + 2x' - x = 0$.

7. Study the stability of the trivial solution of the equation $x'' + 2x' + x = 0$.

8. Study the stability of the trivial solution of the equation $x'' + 2hx' + k^2x = 0$, $h, k \neq 0$.

9. Show that the equilibrium of the system

$$\begin{cases} x_1' = & -2x_1 + x_2 + x_3 \\ x_2' = & -2x_2 + x_3 \\ x_3' = & x_2 - 2x_3 \end{cases}$$

is asymptotically stable.

10. Study the stability of the equilibrium of the system

$$\begin{cases} x_1' = & ax_1 + 5x_3 \\ x_2' = & -x_2 - 2x_3 \\ x_3' = & -3x_3 \end{cases}$$

depending on $a \neq 0$.

11. Show that the equilibrium of the system

$$\begin{cases} x_1' = & x_1 + x_2 + x_3 \\ x_2' = & x_1 - 2x_2 - x_3 \\ x_3' = & x_2 - x_3 \end{cases}$$

is unstable.

12. Find a such that the equilibrium of the system

$$\begin{cases} x_1' = & ax_1 \\ x_2' = & ax_2 + x_3 \\ x_3' = & x_2 + ax_3 \end{cases}$$

is asymptotically stable.

13. Study the stability of the equilibrium of the system

$$\begin{cases} x_1' = & x_2 + x_4 \\ x_2' = & -x_2 + x_3 \\ x_3' = & x_2 + x_3 \\ x_4' = & x_1 - x_4. \end{cases}$$

14. Consider the third order equation $x''' + ax'' + bx' + cx = 0$ and prove that the roots of the characteristic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$ coincide with the

eigenvalues of the equivalent first order system

$$\begin{cases} x_1' = & x_2 \\ x_2' = & x_3 \\ x_3' = & -ax_3 - bx_2 - cx_1. \end{cases}$$

15. Study the stability of the trivial solution $x = 0$ for the equations $x''' + x = 0$ and $x''' - x = 0$.
16. Study the stability of the trivial solution of $x''' + 5x'' + 9x' + 5x = 0$.
17. Prove that the trivial solution of $x'''' + x''' - x' - x = 0$ is unstable.
18. Prove that $x = 0$ is stable for $x'''' + 8x''' + 23x'' + 28x' + 12 = 0$.
19. Prove that the equilibrium of the system

$$\begin{cases} x'' = & x' - 2y \\ y'' = & -3x + 2y' \end{cases}$$

is unstable.

20. Show that the equilibrium of the system

$$\begin{cases} x'' - 2y' = & ax + 3y \\ y'' + 2x' = & 3x + ay \end{cases}$$

is unstable provided $|a| < 3$.

21. Show that $x' = \lambda x - x^5$ has a pitchfork bifurcation.
22. Show that $x' = \lambda x - x^3 - x^5$ has a pitchfork bifurcation.
23. Show that $x' = \lambda x - x^3 - x^{2k+1}$ has a pitchfork bifurcation provided that $k > 1$.
24. Show that $(0, 0, 0)$ is unstable for the linear system

$$\begin{cases} x_1' = & -x_1 \\ x_2' = & -2x_2 \\ x_3' = & x_3 \end{cases}$$

and find the stable and unstable manifold.

25. Determine the stability of $(0, 0)$ of

$$\begin{cases} x' & = -x + y + y^2 \\ y' & = -2y - x^2. \end{cases}$$

26. Show that $V(x, y) = \frac{1}{4}(x^4 + y^4)$ is a Liapunov function for the system,

$$\begin{cases} x' & = -x^3 \\ y' & = -y^3 \end{cases}$$

and deduce the stability of $(0, 0)$.

27. Show that $V(x, y) = \frac{1}{2}(x^2 + y^2)$ is a Liapunov function for the system,

$$\begin{cases} x' &= y - x^3 \\ y' &= -x - y^3 \end{cases}$$

and deduce the stability of $(0, 0)$.

28. Show that $(0, 0)$ is unstable for

$$\begin{cases} x' &= y + x^3 \\ y' &= -x + y^3. \end{cases}$$

29. Consider the system

$$\begin{cases} x' &= y \\ y' &= -x(x + a) - y \end{cases}$$

where $a > 0$. Show that $(0, 0)$ is asymptotically stable.

30. For the same system, show that $(-a, 0)$ is unstable.

31. Study the stability of the equilibrium of gradient system

$$\begin{cases} x'' + 4x(x^2 + y^2) &= 0 \\ y'' + 4y(x^2 + y^2) &= 0. \end{cases}$$

32. Study the stability of the equilibrium of the equation $x'' + f(x') + g(x) = 0$ under the assumption that $f(0) = g(0) = 0$ and $yf(y) \geq 0$ and $xg(x) > 0$ for all $x \neq 0$.

33. Study the stability of the equilibrium of gradient system

$$\begin{cases} x'' + 2(x - 1) + 2xy^2 &= 0 \\ y'' + 2x^2y &= 0. \end{cases}$$

13

Boundary value problems

In this chapter we discuss boundary value problems for second order nonlinear equations. The linear case has been discussed in Chapter 9.

We first deal with autonomous and then with the non-autonomous equations.

13.1 Boundary value problems for autonomous equations

In this first section we consider the autonomous nonlinear boundary value problem

$$\begin{cases} x'' = f(x) \\ x(0) = x(b) = 0 \end{cases} \tag{13.1}$$

by using the phase plane analysis. We assume the student is familiar with this topic discussed in Section 8.3 of Chapter 8.

Consider the phase plane (x, y), $y = x'$, and the curve Λ_c of the equation $\frac{1}{2}y^2 - F(x) = c$, where $F(x) = \int_0^x f(s)ds$. We will assume that $\Lambda_c \neq \emptyset$ and does not contain any singular points. Thus Λ_c is a regular curve that carries a solution of $x'' = f(x)$ with energy c. Furthermore, suppose that Λ_c is a closed curve and let $\overline{\Lambda_c}$ be the arc of Λ_c, contained in the half plane $x \geq 0$, with endpoints $(0, Y_c)$, $(0, -Y_c)$, for some $Y_c > 0$ (recall that Λ_c is symmetric with respect to y), see Figure13.1a. Without loss of generality, we can assume that the solution $x_c(t)$, with energy c, is such that $x_c(0) = 0$ and $y_c(0) = Y_c$.

Let $x = X_c > 0$ be such that $(X_c, 0)$ is the point where $\overline{\Lambda_c}$ crosses the x axis and let $t_c > 0$ be such that $x_c(t_c) = X_c$ and $y_c(t_c) = 0$, see Figure 13.1a. To evaluate t_c we use the energy relationship $\frac{1}{2}y^2 - F(x) = c$ which yields $F(x) \geq -c$ and $y = y(x) = \pm\sqrt{2F(x) + 2c}$. Since $y(x) > 0$ for $0 < x < X_c$, we get $y = +\sqrt{2F(x) + 2c}$.

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UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_13

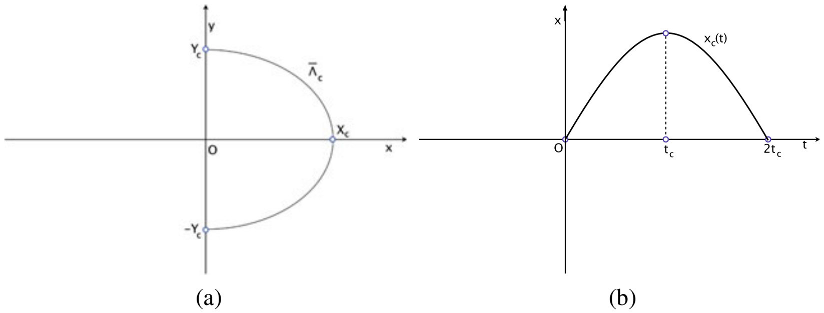


Fig. 13.1. (a) The arc $\overline{\Lambda}_c$; (b) The solution $x_c(t)$, $0 \le t \le 2t_c$

From $x' = y$ it follows that $dt = \frac{dx}{x'} = \frac{dx}{y}$. Moreover, as t ranges from 0 to t_c, x ranges from 0 to X_c. Therefore,

$$t_c = \int_0^{t_c} dt = \int_0^{X_c} \frac{dx}{y} =$$
$$= \int_0^{X_c} \frac{dx}{\sqrt{2F(x) + 2c}} = \frac{1}{\sqrt{2}} \int_0^{X_c} \frac{dx}{\sqrt{F(x) + c}}. \tag{13.2}$$

Lemma 13.1.1. *If X_c is not a singular point, then $t_c < +\infty$.*

Proof. Since $F(X_c) + c = 0$, its Taylor expansion is

$$F(x) + c = F'(X_c)(x - X_c) + o(|x - X_c|) = f(X_c)(x - X_c) + o(|x - X_c|).$$

Thus

$$\sqrt{F(x) + c} = \sqrt{f(X_c)} \cdot \sqrt{x - X_c} + o(|x - X_c|^{1/2}).$$

By assumption, X_c is not a singular point and hence $f(X_c) \neq 0$. Therefore, $(F(x) + c)^{-1/2}$ is integrable in the interval $[0, X_c]$, namely $\int_0^{X_c} (F(x) + c)^{-1/2} dx$ is finite. ∎

The reader should notice the difference with the homoclinic and heteroclinic case, discussed in Chapter 8, where we have shown that if X_c is a singular point, then $t_c = +\infty$.

Let τ_c be the time needed by the point $(x_c(t), y_c(t)) \in \overline{\Lambda}_c$ to go from $(X_c, 0)$ to $(0, -Y_c)$. By symmetry, one has that $\tau_c = t_c$. Let us check this fact. As before, $\tau_c = \int_0^{\tau_c} dt$. But now $y = -\sqrt{2F(x) + 2c}$. Moreover, as t ranges from 0 to τ_c, x ranges downwards from X_c to 0. Therefore,

$$\tau_c = \int_0^{\tau_c} dt = \int_{X_c}^0 \frac{dx}{y} = \frac{1}{\sqrt{2}} \int_{X_c}^0 \frac{dx}{-\sqrt{F(x) + c}}$$
$$= \frac{1}{\sqrt{2}} \int_0^{X_c} \frac{dx}{\sqrt{F(x) + c}} = t_c.$$

The function $x_c(t)$ has the following properties:

1. $x_c(0) = 0,\ x_c'(0) = Y_c > 0.$
2. $x_c(t_c) = X_c,\ x_c'(t_c) = 0.$
3. $x_c(2t_c) = 0,\ x_c'(2t_c) = -Y_c < 0.$

It follows that if c is such that $2t_c = b$, the corresponding $x_c(t)$ solves the boundary value problem (13.1), is positive and its maximum is X_c, achieved at $t = t_c = \frac{b}{2}$. See Figure 13.1b.

Theorem 13.1.2. *Let c be such that $\Lambda_c \neq \emptyset$ is a closed curve that does not contain any singular points. If c satisfies*

$$b = \sqrt{2}\int_0^{X_c} \frac{dx}{\sqrt{F(x)+c}}, \tag{13.3}$$

then the function $x_c(t)$ is a solution of the boundary value problem (13.1) *such that $x_c(t) > 0$ for all $a < t < b$.*

Proof. We have seen that a solution $x_c(t)$ corresponds to a c such that $b = 2t_c$. Since t_c is given by (13.2), we obtain (13.3), proving the theorem. ■

Remark 13.1.3. (i) More generally, if the equation $b = 2kt_c$ has a solution $c = c_k$, $k = 1, 2, \ldots$, (see Figure 13.2), we find a solution $x_{c_k}(t)$ that changes sign $k - 1$ times. Note that, in any case, $x_{c_k}'(0)(= Y_{c_k}) > 0$. For example, if $k = 2$, the solution $x_{c_2}(t)$, corresponding to the closed curve Λ_{c_2}, is positive for $0 < t < 2t_{c_2}$, negative for $2t_{c_2} < t < 4t_{c_2}$ and such that $x_{c_2}'(0) > 0$.

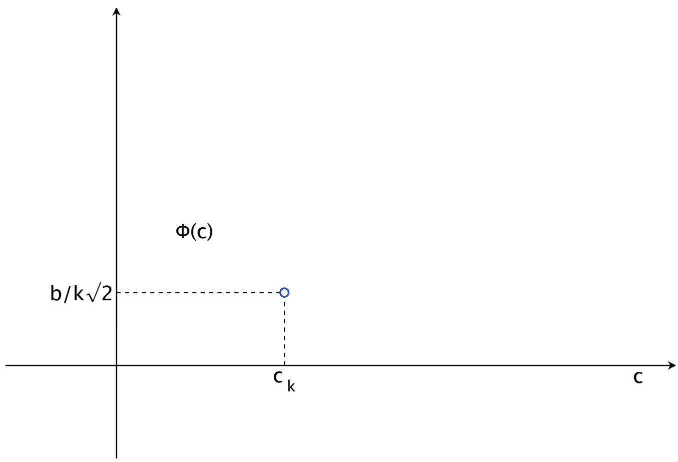


Fig. 13.2. Solutions of $\Phi(c) = \frac{b}{k\sqrt{2}}$

(ii) By a similar argument, one can show that the boundary value problem

$$\begin{cases} x'' = f(x) \\ x(a) = x(b) = 0 \end{cases}$$

has a solution that changes sign $k-1$ times in (a, b) provided the equation

$$b - a = \sqrt{2}\,k \int_0^{X_c} \frac{dx}{\sqrt{F(x) + c}}$$

has a solution $c = c_k$. ■

13.1.1 Examples

Below we demonstrate a couple of specific examples that show how to solve equation (13.3) in order to find solutions of the boundary value problems.

Proposition 13.1.4. *The problem*

$$\begin{cases} x'' + 4x^3 = 0 \\ x(0) = x(b) = 0 \end{cases} \tag{13.4}$$

has infinitely many solutions.

Proof. In this case Λ_c has equation $\frac{1}{2}y^2 + x^4 = c$. For all $c > 0$, Λ_c is nonempty and is a closed curve that does not contain the (unique) singular point $(0, 0)$. By Theorem 13.1.2 and Remark 13.1.3, if the equation

$$b = \sqrt{2}\,k \int_0^{X_c} \frac{dx}{\sqrt{c - x^4}}, \qquad k = 1, 2, \ldots,$$

has a solution $c = c_k$, then (13.4) has a solution that changes sign $k-1$ times in (a, b). Setting

$$\Phi(c) = \int_0^{X_c} \frac{dx}{\sqrt{c - x^4}}$$

the preceding equation becomes $\Phi(c) = \frac{b}{\sqrt{2}k}$. Notice that in this case X_c is the positive solution of $x^4 = c$, that is $X_c = c^{1/4}$. Thus

$$\Phi(c) = \int_0^{c^{1/4}} \frac{dx}{\sqrt{c - x^4}}.$$

Notice that $c - x^4 \geq 0$ for $0 \leq x \leq X_c = c^{1/4}$. Moreover, according to (13.2) and Lemma 13.1.1, the integral is finite. Let us study the function $\Phi(c)$.

The change of variable $x = c^{1/4}z$ yields $dx = c^{1/4}dz$ and hence, for $c > 0$ one finds

$$\Phi(c) = \int_0^1 \frac{c^{1/4}dz}{\sqrt{c - cz^4}} = c^{-1/4} \int_0^1 \frac{dx}{\sqrt{1 - z^4}}.$$

Thus $\Phi(c)$ is positive, decreasing and satisfies $\lim_{c\to 0^+} \Phi(c) = +\infty$ and $\lim_{c\to+\infty} \Phi(c) = 0$. It follows that for every $k = 1, 2, \ldots$, the equation $\Phi(c) = \frac{b}{k\sqrt{2}}$ has a solution $c_k > 0$ that gives rise to a solution of (13.4). ∎

Proposition 13.1.5. *If k is an integer such that $1 < k^2 < \lambda < (k+1)^2$, then the problem*

$$\begin{cases} x'' + \lambda(x - x^3) = 0 \\ x(0) = x(\pi) = 0 \end{cases} \tag{13.5}$$

has k pairs of nontrivial solutions $\pm x_j(t)$, $1 \le j \le k$, with $j - 1$ zeros in the open interval $(0, \pi)$. If $\lambda \le 1$, there is only the trivial solution.

Proof. Here the curve Λ_c is defined by the equation $\frac{1}{2}y^2 + \lambda(\frac{1}{2}x^2 - \frac{1}{4}x^4) = c$, that is $y^2 + \lambda(x^2 - \frac{1}{2}x^4) = 2c$. For $0 < 2c < \lambda$ the curve Λ_c is not empty, closed, symmetric with respect to x and y, and does not contain the singular points $(0, 0)$ and $(\pm 1, 0)$. According to Theorem 13.1.2, setting $X_c = \xi > 0$, we have to solve the equation

$$\pi = 2k \int_0^\xi \frac{dx}{\sqrt{2c - \lambda(x^2 - \frac{1}{2}x^4)}}.$$

The change of variable $x = \xi z$ yields

$$\pi = 2k \int_0^1 \frac{\xi dz}{\sqrt{2c - \lambda(\xi^2 z^2 - \frac{1}{2}\xi^4 z^4)}}.$$

Since ξ satisfies $2c = \lambda(\xi^2 - \frac{1}{2}\xi^4)$, we have

$$\pi = 2k \int_0^1 \frac{\xi dz}{\sqrt{\lambda(\xi^2 - \frac{1}{2}\xi^4) - \lambda(\xi^2 z^2 - \frac{1}{2}\xi^4 z^4)}}$$

and, factoring $\xi > 0$ in the denominator and then canceling it, we obtain

$$\pi = \frac{2k}{\sqrt{\lambda}} \int_0^1 \frac{dz}{\sqrt{1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4}}.$$

Let us study the behavior of the function $\Psi(\xi)$, defined for $\xi \ge 0$, by setting

$$\Psi(\xi) \stackrel{def}{=} \frac{2}{\sqrt{\lambda}} \int_0^1 \frac{dz}{\sqrt{1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4}}. \tag{13.6}$$

If $\xi = 0$, one has

$$\Psi(0) = \frac{2}{\sqrt{\lambda}} \int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{\pi}{\sqrt{\lambda}}.$$

Since

$$\frac{\partial}{\partial \xi} \frac{1}{\sqrt{1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4}} = -\frac{1}{2} \frac{-\xi + \xi z^4}{(1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4)^{3/2}} =$$

$$= \frac{1}{2} \frac{\xi(1-z^4)}{(1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4)^{3/2}},$$

then differentiating the quantity under the integral (13.6), we obtain

$$\Psi'(\xi) = \frac{1}{\sqrt{\lambda}} \int_0^1 \frac{\xi(1-z^4)dz}{(1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4)^{3/2}} > 0.$$

Moreover,

$$lim_{\xi \to 1-} \left(\frac{1}{\sqrt{1 - \frac{1}{2}\xi^2 - z^2 + \frac{1}{2}\xi^2 z^4}} \right) = \frac{1}{\sqrt{\frac{1}{2} - z^2 + \frac{1}{2}z^4}} =$$

$$= \sqrt{2} \frac{1}{\sqrt{1 - 2z^2 + z^4}} = \frac{\sqrt{2}}{\sqrt{(1-z^2)^2}} = \frac{\sqrt{2}}{1-z^2}.$$

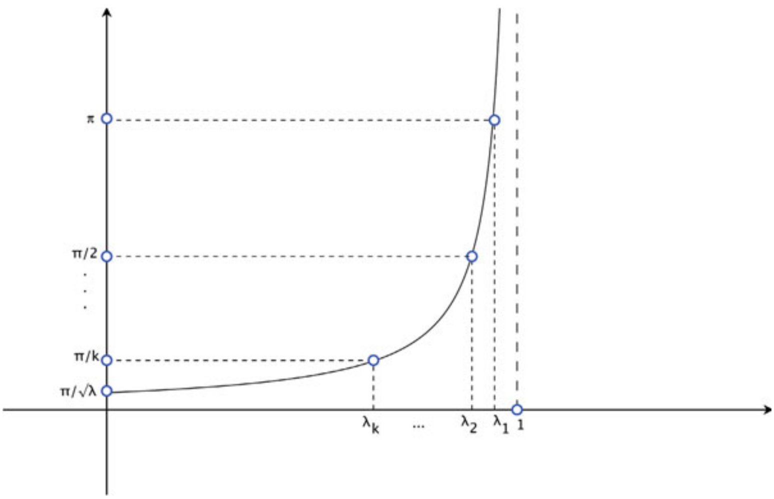


Fig. 13.3. Solutions of $\Psi(\xi) = \frac{\pi}{k}$, with $1 < k^2 < \lambda < (k+1)^2$

Thus

$$lim_{\xi\to 1-}\Psi(\xi) = \frac{2\sqrt{2}}{\sqrt{\lambda}}\int_0^1 \frac{dz}{1-z^2} = +\infty.$$

The graph of $\Psi(\xi)$ is shown in Figure 13.3.

Recall that we have to solve $\pi = k\Psi(\xi)$, namely $\Psi(\xi) = \frac{\pi}{k}$. From the graph of $\Psi(\xi)$ it follows that the equation $\Psi(\xi) = \frac{\pi}{k}$ has a solution if and only if $\frac{\pi}{k} > \frac{\pi}{\sqrt{\lambda}}$, namely whenever $\lambda > k^2$. Precisely, if $1 \le k^2 < \lambda < (k+1)^2$ the equation $\Psi(\xi) = \frac{\pi}{k}$ has k solutions $\xi_1, \cdots \xi_k$ and hence (13.5) has k nontrivial solutions $x_j(t)$, $1 \le j \le k$. Notice that Theorem 13.1.2, resp. Remark 13.1.3, imply that $x_1(t) > 0$ in (a, b), resp. x_j has $j-1$ zeros in $(0, \pi)$ and $x_j'(0) > 0$. Notice also that the solutions of (13.5) arise in pairs because if $x(t)$ is a solution, so is $-x(t)$. Finally, if $\lambda < 1$ then $\frac{\pi}{k} < \frac{\pi}{\lambda}$ for all $k = 1, 2 \ldots$ and hence the equation $\Psi(\xi) = \frac{\pi}{k}$ has no solution. Thus, in this case, problem (13.5) has only the trivial solution. ∎

Remark 13.1.6. The fact that (13.5) has only the trivial solution for $\lambda < 1$ could also be proved using the Poincaré inequality as in Example 9.2.7. The reader could carry out the details as an exercise. ∎

13.2 The Green function

To solve boundary value problems with a time dependent nonlinearity, it is useful to introduce the Green function. This is what we are going to do in this section.

Let p, r be functions satisfying:

1. $p(t) > 0$ and is continuously differentiable on the interval $[a, b]$.
2. $r(t) \ge 0$ and is continuous on the interval $[a, b]$.

These assumptions will be assumed throughout the rest of this chapter.

Define the differential operator L by setting

$$L[x] \stackrel{def}{=} \frac{d}{dt}\left(p(t)\frac{dx}{dt}\right) - r(t)x.$$

The operator L is linear, that is

$$L[c_1 x + c_2 y] = c_1 L[x] + c_2 L[y].$$

The reader will notice that L is the differential operator used in Chapter 9 with $q = 1$ and $-rx$ instead of rx (the choice of $-r$ is made for convenience, only: recall that no assumption on the sign of r was made there). In particular, from Theorem 9.2.2 of Chapter 9 (with $q(t) = 1$) it follows that the eigenvalue problem

$$\begin{cases} L[x] + \lambda x = 0, & t \in [a, b] \\ x(a) = x(b) = 0 \end{cases}$$

has a sequence λ_i, with $0 < \lambda_1 < \lambda_2 < \ldots$, of eigenvalues.

Let φ, ψ be the solutions of the Cauchy problems

$$\begin{cases} L[\varphi] &= 0,\ t \in [a,b] \\ \varphi(a) &= 0,\ \varphi'(a) = \alpha \neq 0 \end{cases} \qquad \begin{cases} L[\psi] &= 0,\ t \in [a,b] \\ \psi(b) &= 0,\ \psi'(b) = \beta \neq 0. \end{cases}$$

Notice that $\psi(a) \neq 0$, otherwise ψ would satisfy $L[\psi] = 0$ and $\psi(a) = \psi(b) = 0$ and this means that ψ is an eigenfunction of L with eigenvalue $\lambda = 0$, which is not possible. Of course, for the same reason we also have $\varphi(b) \neq 0$. Consider their Wronskian $W(t) = \varphi(t)\psi'(t) - \varphi'(t)\psi(t)$ and recall that, by the Abel theorem, $W(t) \equiv const$, say $W(t) = -C$.

From the definition of φ, ψ it follows that

$$-C = W(a) = -\varphi'(a)\psi(a) = W(b) = \varphi(b)\psi'(b) \neq 0.$$

In other words, φ and ψ are linearly independent.

The Green function of L (with boundary conditions $x(a) = x(b) = 0$) is the function $G(t,s)$ defined on the square $Q = [a,b] \times [a,b]$ by setting

$$G(t,s) = \begin{cases} \frac{1}{p(t)C}\ \varphi(t)\psi(s), & \text{if } t \in [a,s] \\[2ex] \frac{1}{p(t)C}\ \varphi(s)\psi(t), & \text{if } t \in [s,b]. \end{cases}$$

The function G is continuous on Q and

$$G(a,s) = \frac{\varphi(a)\psi(s)}{p(a)C} = 0, \qquad G(b,s) = \frac{\varphi(s)\psi(b)}{p(b)C} = 0. \tag{13.7}$$

Moreover, G is differentiable at $(t,s) \in Q$, $s \neq t$. In addition, for each s, setting $G_t(s^-,s) = \lim_{t\to s^-} \frac{d}{dt}G(t,s)$ and $G_t(s^+,s) = \lim_{t\to s^+} \frac{d}{dt}G(t,s)$, it is easy to check that $G_t(s^-,s) - G_t(s^+,s) = Cp(s)$.

Example 13.2.1. Let us calculate the Green function of $L[x] = x'' - x$ in the interval $[0,1]$. Here $p = r = 1$ and $[a,b] = [0,1]$. The general solution of $x'' - x = 0$ is $x = c_1e^t + c_2e^{-t}$. If $x(0) = c_1 + c_2 = 0$ and $x'(0) = c_1 - c_2 = 1$, we find $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ and we can take $\varphi = \frac{1}{2}(e^t - e^{-t}) = \sinh t$. If $x(1) = c_1e + \frac{c_2}{e} = 0$ and $x'(1) = c_1e - \frac{c_2}{e} = -1$, we find $c_1 = -\frac{1}{2e}, c_2 = \frac{e}{2}$ and we can take $\psi = -\left[\frac{1}{2e}e^t - \frac{e}{2}e^{-t}\right] = -\frac{1}{2}\left[\frac{e^t}{e} - ee^{-t}\right] = -\frac{1}{2}\left[e^{t-1} - e^{-(t-1)}\right] = -\sinh(t-1)$. Clearly φ, ψ are linearly independent and

$$C = \psi(0) = \frac{1}{2e} - \frac{e}{2} = \frac{1-e^2}{2e} < 0$$

and hence

$$G(t,s) = \begin{cases} \frac{2e}{e^2-1} \cdot \sinh t \cdot \sinh(s-1), & \text{if } t \in [0,s] \\[2ex] \frac{2e}{e^2-1} \cdot \sinh s \cdot \sinh(t-1), & \text{if } t \in [s,1] \end{cases}$$

is the Green function we are looking for. ■

The Green function of L can be used to transform a boundary value problem into an integral equation.

Theorem 13.2.2. *For any continuous function $h(t)$, the nonhomogeneous problem*

$$\begin{cases} L[x] + h(t) = 0, & t \in [a,b] \\ x(a) = x(b) = 0 \end{cases} \tag{13.8}$$

has a unique solution given by the function

$$x(t) = \int_a^b G(t,s)h(s)ds.$$

Proof. To simplify the notation, we take $p \equiv 1$. Using (13.7), we find $x(a) = \int_a^b G(a,s)h(s)ds = 0$ and $x(b) = \int_a^b G(b,s)h(s)ds = 0$, so that x satisfies the desired boundary conditions. Furthermore, splitting the integral $\int_a^b ds$ into $\int_a^t ds + \int_t^b ds$, one has

$$x(t) = \int_a^t G(t,s)h(s)ds + \int_t^b G(t,s)h(s)ds.$$

Since for $a \le s \le t$ one has that $G(t,s) = \frac{1}{C}(\varphi(s)\psi(t))$, while for $t \le s \le b$ one has that $G(t,s) = \frac{1}{C}(\varphi(t)\psi(s))$, it follows that

$$x(t) = \psi(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \varphi(t)\int_t^b \frac{\psi(s)h(s)}{C}ds.$$

Then $x(t)$ is differentiable and, using the fundamental theorem of Calculus, we get

$$\begin{aligned} x'(t) &= \psi'(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \frac{1}{C}\varphi(t)\psi(t)h(t) \\ &\quad + \varphi'(t)\int_t^b \frac{\psi(s)h(s)}{C}ds - \frac{1}{C}\psi(t)\varphi(t)h(t) \\ &= \psi'(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \varphi'(t)\int_t^b \frac{\psi(s)h(s)}{C}ds. \end{aligned}$$

Therefore x' is also differentiable and one has

$$\begin{aligned} x''(t) &= \psi''(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \frac{1}{C}\psi'(t)\varphi(t)h(t) \\ &\quad + \varphi''(t)\int_t^b \frac{\psi(s)h(s)}{C}ds - \frac{1}{C}\varphi'(t)\psi(t)h(t) \\ &= \psi''(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \varphi''(t)\int_t^b \frac{\psi(s)h(s)}{C}ds \\ &\quad + \frac{1}{C}\left(\psi'(t)\varphi(t) - \varphi'(t)\psi(t)\right)h(t). \end{aligned}$$

Notice that $\varphi'(t)\psi(t) - \psi'(t)\varphi(t) = W(t) = -C$. Thus

$$L[x] = x''(t) - rx = \psi''(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \varphi''(t)\int_t^b \frac{\psi(s)h(s)}{C}ds - h(t) - rx.$$

Since $L[\varphi] = \varphi'' - r\varphi = 0$ and $L[\psi] = \psi'' - r\psi = 0$, we find

$$\begin{aligned}
L[x] &= r\psi(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + r\varphi(t)\int_t^b \frac{\psi(s)h(s)}{C}ds - h - rx \\
&= r\left[\psi(t)\int_a^t \frac{\varphi(s)h(s)}{C}ds + \varphi(t)\int_t^b \frac{\psi(s)h(s)}{C}ds\right] - h - rx \\
&= r\left[\int_a^t \frac{\varphi(s)\psi(t)h(s)}{C}ds + \int_t^b \frac{\varphi(t)\psi(s)h(s)}{C}ds\right] - h - rx \\
&= r\int_a^b G(t,s)h(s)ds - h - rx = rx - h - rx = -h.
\end{aligned}$$

This proves the existence of a solution of (13.8). To prove the uniqueness, let x_1, x_2 be two solutions of (13.8). Then, setting $z = x_1 - x_2$, one has $L[z] = L[x_1] - L[x_2] = 0$ and $z(a) = z(b) = 0$. Since $\lambda = 0$ is not an eigenvalue of L with zero boundary conditions, it follows that $z(t) \equiv 0$, that is $x_1(t) = x_2(t)$ for all $t \in [a,b]$. ∎

Corollary 13.2.3. *If $f(t,x)$ is continuous, then*

$$x(t) = \int_a^b G(t,s)f(s,x(s))ds$$

is a solution of $L[x] + f(t,x) = 0$, $x(a) = x(b) = 0$.

13.3 Sub- and supersolutions

In this section we study the nonlinear boundary value problem

$$\begin{cases} -x'' = f(t,x), & t \in [a,b] \\ x(a) = x(b) = 0 \end{cases} \tag{13.9}$$

where $f(t,x)$ is a continuous real valued function defined on $[a,b] \times \mathbb{R}$. Notice that the equation can also be written as $x'' + f(t,x) = 0$, in which the differential operator x'' has the form $L[x]$ introduced in the previous section, with $p = 1$ and $r = 0$. In particular, according to Corollary 13.2.3, to find a solution of the preceding problem it suffices to find $x(t)$ solving the integral equation

$$x(t) = \int_a^b G(t,s)f(s,x(s))ds$$

where G is the Green function of x'' with boundary conditions $x(a) = x(b) = 0$.

Definition 13.3.1. A function $v \in C^2([a,b])$ is a subsolution of (13.9) if

$$\begin{cases} -v'' & \leq f(t,v), \qquad t \in [a,b] \\ v(a) & \leq 0 \\ v(b) & \leq 0. \end{cases}$$

A function $w \in C^2([a,b])$ is a supersolution of (13.9) if

$$\begin{cases} -w'' & \geq f(t,w), \qquad t \in [a,b] \\ w(a) & \geq 0 \\ w(b) & \geq 0. \end{cases}$$

Example 13.3.2. A negative constant $-c$ is a subsolution provided $f(t,-c) \geq 0$. Similarly, a positive constant c is a supersolution provided $f(t,c) \leq 0$. ■

The following Lemma is a sort of a "maximum principle". Since its interest goes beyond the topics discussed in this chapter, we prefer to consider a general differential operator $L[x] = (p(t)x')' - r(t)x$, where $p(t) > 0$ is continuously differentiable and $r(t) \geq 0$ is continuous, even if we use the simpler operator $x'' - mx$, $m \geq 0$, a constant.

Lemma 13.3.3. *If w is such that $-L[w] \geq 0$, $w(a) \geq 0$, $w(b) \geq 0$, then $w(t) \geq 0$ for all $t \in [a,b]$.*

Proof. Let λ_1 be the first eigenvalue of $L[x] + \lambda x = 0$, $x(a) = x(b) = 0$ and let ϕ_1 be an associated eigenfunction, that can be taken strictly positive in (a,b). Set $w_\epsilon = w + \epsilon\phi_1$. Since $\epsilon > 0$ and $\phi_1 > 0$ in $[a,b]$, then

$$-L[w_e] = -L[w] - \epsilon L[\phi_1] = -L[w] + \epsilon\lambda_1\phi_1 > 0, \quad \forall\, t \in (a,b)\,. \tag{13.10}$$

Moreover

$$w_\epsilon(a) = w(a) \geq 0, \qquad w_\epsilon(b) = w(b) \geq 0.$$

Let t_ϵ be the point where $w_\epsilon(t)$ achieves its minimum in $[a,b]$. If, by contradiction, $w(t_\epsilon) < 0$, then $a < t_\epsilon < b$ and thus $w'(t_\epsilon) = 0$. Furthermore, since $w'_\epsilon(t_\epsilon) = 0$ we find

$$\begin{aligned} -L[w_\epsilon(t_\epsilon)] &= -\big(p'(t_\epsilon)w'_\epsilon(t_\epsilon) + p(t_\epsilon)w''_\epsilon(t_\epsilon)\big) + r(t_\epsilon)w_\epsilon(t_\epsilon) \\ &= -p(t_\epsilon)w''_\epsilon(t_\epsilon) + r(t_\epsilon)w_\epsilon(t_\epsilon). \end{aligned}$$

From (13.10), it follows that

$$-p(t_\epsilon)w''_\epsilon(t_\epsilon) + r(t_\epsilon)w_\epsilon(t_\epsilon) > 0 \implies p(t_\epsilon)w''_\epsilon(t_\epsilon) < r(t_\epsilon)w_\epsilon(t_\epsilon).$$

Since $p(t_\epsilon) > 0, r(t_\epsilon) \geq 0, w_\epsilon(t_\epsilon) < 0$, it follows that $w''_\epsilon(t_\epsilon) < 0$. This is a contradiction to the fact that t_ϵ is a minimum point of w_ϵ, proving the theorem. ■

The next Theorem is a rather general existence result in the presence of ordered sub- and supersolutions.

Theorem 13.3.4. *Suppose that f is continuous on $[a,b]\times\mathbb{R}$ and*

$$\exists\, m>0, \text{ such that the function } mx+f(t,x) \text{ is increasing for all } t\in[a,b]. \quad (*)$$

If (13.9) *has a subsolution v and a supersolution w such that $v(t)\le w(t)$ for all $t\in[a,b]$, then* (13.9) *has a solution x with $v(t)\le x(t)\le w(t)$ for all $t\in[a,b]$.*

Proof. A solution of $x''+f(t,x)=0$, $x(a)=x(b)=0$ will be found by an iteration procedure that we are going to describe. First of all, setting $f_m(t,x)\stackrel{def}{=} mx+f(t,x)$, the equation $x''+f(t,x)=0$ is equivalent to $x''-mx+f_m(t,x)=0$. Let

$$L_m[x]\stackrel{def}{=} x''-mx.$$

Then the equation can be written in the form

$$L_m[x]+f_m(t,x)=0.$$

Notice that Theorem 13.2.2 holds for L_m. In particular, a solution of (13.9) can be found solving the integral equation

$$x=S[x],\quad S[x](t)\stackrel{def}{=}\int_a^b G_m(t,s)f_m(s,x(s))ds \tag{13.11}$$

where G_m denotes the Green function of L_m.

Let $v_1=v$ and, for $k=2,3\ldots$, we let v_k be the solution of

$$-L_m[v_k]=f_m(t,v_{k-1})=mv_{k-1}+f(t,v_{k-1}),\quad v_k(a)=v_k(b)=0,$$

which exists and is unique in view of Theorem 13.2.2, with $L=L_m$ and $h=f_m(t,v_{k-1})$. Using the notation introduced above we can say that

$$v_k=S[v_{k-1}].$$

By induction, one shows that for all $k=1,2\ldots$ one has

$$v(t)\le v_k(t)\le w(t),\qquad \forall\, t\in[a,b]. \tag{13.12}$$

Since $v_1=v\le w$, (13.12) holds for $k=1$. Suppose that (13.12) holds for k and set $z=v_{k+1}-v$. Then $-L_m[z]=-L_m[v_{k+1}]+L_m[v]=mv_k+f(t,v_k)+L_m[v]$. Since $-L[v]\le f(t,v)$ we get $-L_m[v]\le mv+f(t,v)$, and hence

$$-L_m[z]\ge mv_k+f(t,v_k)-mv-f(t,v)=f_m(t,v_k)-f_m(t,v).$$

By the inductive assumption, $v_k\ge v$. This and the fact that $f_m(\cdot,x)$ is increasing yield $-L_m[z]\ge 0$. Moreover, $z(a)=v_{k+1}(a)-v(a)\ge 0$ because $v_{k+1}(a)=0$ and $v(a)\le 0$. Similarly $z(b)\ge 0$. Applying the Maximum Principle, Lemma 13.3.3 (with $p\equiv 1$ and $r\equiv m$), it follows that $z(t)\ge 0$, namely $v_{k+1}(t)\ge v(t)$ in $[a,b]$. In the same way, using the fact that w is a supersolution, one finds $-L[w-v_{k+1}]\ge 0$

and $w(a) - v_{k+1}(a) \geq 0$, $w(b) - v_{k+1}(b) \geq 0$ which implies that $w(t) \geq v_{k+1}(t)$ in $[a, b]$. This proves (13.12).

To prove that v_k converges, up to a subsequence, uniformly in $[a, b]$ to a continuous function, we use the Ascoli Compactness Theorem which says:

If a sequence of continuous functions f_k defined in an interval $[a, b]$ is bounded uniformly with respect to k, and is continuous uniformly with respect to k, then there exists a subsequence converging uniformly in $[a, b]$ to a continuous function.

We have:

(i) v_k are bounded, uniformly with respect to k. From (13.12) it follows that

$$\min_{t\in[a,b]} v(t) \leq v_k(t) \leq \max_{t\in[a,b]} w(t), \qquad \forall\, k.$$

(ii) v_k are continuous uniformly with respect to k. Let us use $v_k = S v_{k-1}$ to infer that

$$\begin{aligned} |v_k(t) - v_k(t')| &\leq \int_a^b |G_m(t,s) - G_m(t',s)| \cdot |f_m(s, v_{k-1}(s))| ds \\ &\leq c \int_a^b |G_m(t,s) - G_m(t',s)| ds. \end{aligned}$$

Since G_m is uniformly continuous in the square $Q = [a, b] \times [a, b]$ it follows that the sequence v_k is continuous uniformly with respect to k. In view of these two properties we can use the Ascoli compactness theorem to infer that, up to a subsequence, $v_k(t)$ converges to a continuous function $x(t)$, uniformly in $[a, b]$. This allows us to pass to the limit in $v_k = S[v_{k-1}]$, yielding $x = S[x]$, namely

$$x(t) = \int_a^b G(t,s) f(s, y(s)) ds.$$

Thus, by Corollary 13.2.3, $x(t)$ solves (13.9). ■

Remark 13.3.5. Examples show that, in general, the condition $v \leq w$ cannot be eliminated. ■

The next two theorems are applications of the preceding general result.

Theorem 13.3.6. *Let $f(t, x)$ be continuously differentiable on $[a, b] \times \mathbb{R}$. Moreover, suppose*

$$\exists\, \alpha, \beta \geq 0 \,:\, f(t, -\alpha) \geq 0, \quad f(t, \beta) \leq 0, \quad \forall\, t \in [a, b]. \tag{13.13}$$

Then

$$\begin{cases} -x'' = f(t, x) \\ x(a) = x(b) = 0 \end{cases} \tag{13.14}$$

has a solution $x(t)$ such that $-\alpha \leq x(t) \leq \beta$.

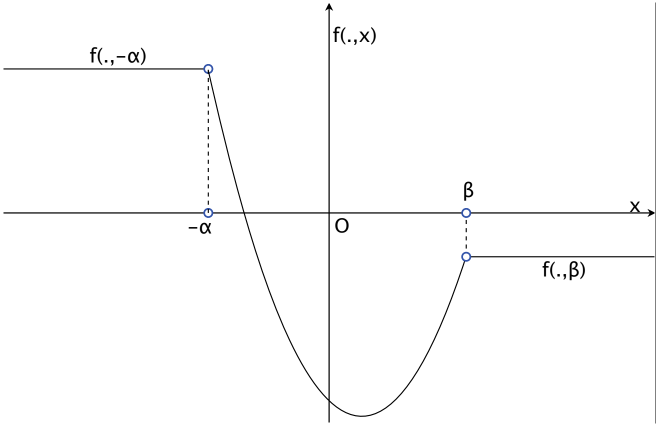


Fig. 13.4. Plot of $\tilde{f}(t,x)$

Proof. As mentioned before, $v(t) \equiv -\alpha < 0$ is a subsolution and $w(t) \equiv \beta > 0$ is a supersolution. To apply Theorem 13.3.4 we should have that f satisfies (*). This difficulty is overcome by using a truncation, which we are going to discuss. Define a truncated function $\tilde{f}(t,x)$ by setting (see Figure 13.4)

$$\tilde{f}(t,x) = \begin{cases} f(t,-\alpha) & \text{if } x \le -\alpha \\ f(t,x) & \text{if } a \le x \le \beta \\ f(t,\beta) & \text{if } x \ge \beta. \end{cases}$$

Lemma 13.3.7. *The function* $x(t)$ *solves*

$$\begin{cases} -x'' = \tilde{f}(t,x) \\ x(a) = x(b) = 0 \end{cases} \tag{13.15}$$

if and only if it solves (13.14).

Proof. Let x be a solution of (13.15) and let τ, τ' be such that

$$x(\tau) = \min_{t\in[a,b]} x(t), \qquad x(\tau') = \max_{t\in[a,b]} x(t).$$

We claim that $x(\tau) \ge -\alpha$. Otherwise, if $x(\tau) < -\alpha \le 0$, then $a < \tau < b$ and $\tilde{f}(\tau, x(\tau)) = f(\tau,-\alpha) > 0$, by definition. Thus $-x''(\tau) = \tilde{f}(\tau, x(\tau)) > 0$, which is a contradiction because τ is the minimum of x. In the same way one proves that $x(\tau') \le \beta$. As a consequence, we have that $-\alpha \le x(t) \le \beta$ and hence $\tilde{f}(t,x(t)) = f(t,x(t))$ so that x solves (13.14). The converse is trivial. ∎

Proof of Theorem 13.3.6 completed. Since $f'_x(t,x)$ is bounded in the rectangle $[a,b]\times[-\alpha,\beta]$, then the function $\tilde{f}$ satisfies (*). Furthermore, since $f = \tilde{f}$ for $-\alpha \le x \le \beta$, then $v = -\alpha$, resp. $w = \beta$, is a subsolution, resp. a supersolution, not only of

$-x'' = f(t,x)$ but also of $-x'' = \tilde{f}(t,x)$. In addition one has that $v \le w$. We can now apply Theorem 13.3.4 finding a solution $x(t)$ such that $-\alpha \le x(t) \le \beta$. ∎

From the preceding Theorem we can deduce:

Theorem 13.3.8. *If* $\lim_{x\to-\infty} f(t,x) > 0$ *and* $\lim_{x\to+\infty} f(t,x) < 0$*, uniformly with respect to* $t \in [a,b]$*, then the problem* (13.14) *has a solution.*

Proof. From the assumptions on the limits, it follows that (13.13) holds. ∎

Corollary 13.3.9. *Let* $f(t,x) = -x + g(t,x)$*, with* g *bounded. Then the problem* (13.14) *has a solution.*

Proof. One has $\lim_{x\to-\infty} f(t,x) = +\infty$ and $\lim_{x\to+\infty} f(t,x) = -\infty$. ∎

13.4 A nonlinear eigenvalue problem

Consider the *nonlinear eigenvalue problem*

$$\begin{cases} -x'' = \lambda x - g(t,x) \\ x(a) = x(b) = 0 \end{cases} \tag{13.16}$$

where λ is a real parameter and $g(t,0) \equiv 0$. Problem (13.16) has the trivial solution $x \equiv 0$ for all λ. The existence of a positive solution is established in the following theorem. By a positive solution, resp. sub/supersolution, of (13.16) we mean a solution, resp. sub/supersolution, $x(t)$ such that $x(t) > 0$ for all $a < t < b$.

Theorem 13.4.1. *Let* $g(t,x)$ *be continuous on* $[a,b]\times\mathbb{R}$ *and such that* $g(t,0) = 0$ *for all* $t \in [a,b]$*. Furthermore, suppose that*

$$\lim_{x\to 0} \frac{g(t,x)}{x} = 0, \quad \textit{uniformly w.r.t. } t \in [a,b] \tag{g1}$$

$$\lim_{x\to+\infty} \frac{g(t,x)}{x} = +\infty, \quad \textit{uniformly w.r.t. } t \in [a,b]. \tag{g2}$$

Then (13.16) *has a solution* $x(t) > 0$ *in* (a,b)*, provided* $\lambda > \lambda_1 = \frac{\pi}{b-a}$*, where* λ_1 *is the first eigenvalue of the linear problem* $x'' + \lambda x = 0$*,* $x(a) = x(b) = 0$*.*

Proof. Fix $\lambda > \frac{\pi}{b-a}$. The function $f(t,x) \stackrel{def}{=} \lambda x - g(t,x)$ is such that $f(t,0) = 0$ and, by (g2), $\lim_{x\to+\infty} f(t,x) = -\infty$. It follows that there exists $M_\lambda > 0$ such that $f(t,M_\lambda) < 0$. Clearly $w_\lambda = M_\lambda$ is a supersolution of (13.16). Actually $-w''_\lambda = 0 > f(t,M_\lambda) = f(t,w_\lambda)$.

Finding a positive subsolution is slightly more involved. Let $\phi_1 > 0$ be such that $\phi_1'' + \lambda_1\phi_1 = 0$, $\phi_1(a) = \phi_1(b) = 0$. Taking $\epsilon > 0$, let us show that $v_\epsilon(t) = \epsilon\phi_1(t)$

is a positive subsolution for $\epsilon > 0$ sufficiently small. To prove this, we evaluate

$$-v_\epsilon'' = -\epsilon\phi_1'' = \epsilon\lambda_1\phi_1 = \lambda_1 v_\epsilon \,.$$

From $(g1)$ it follows that

$$\lim_{\epsilon\to 0} \frac{g(t,\epsilon\phi_1(t))}{\epsilon\phi_1(t)} = 0, \quad \forall\, t\in[a,b].$$

Then, if $\lambda > \lambda_1$ one infers that there exists $\epsilon_0 > 0$ such that

$$\frac{g(t,\epsilon\phi_1(t))}{\epsilon\phi_1(t)} \le \lambda - \lambda_1, \quad \forall\, 0<\epsilon<\epsilon_0,\ \forall\, t\in[a,b].$$

Recalling that $v_\epsilon = \epsilon\phi_1$, it follows that

$$g(t,v_\epsilon) \le (\lambda-\lambda_1)v_\epsilon, \quad \forall\, 0<\epsilon<\epsilon_0,\ \forall\, t\in[a,b],$$

namely $\lambda_1 v_\epsilon \le \lambda v_\epsilon - g(t,v_\epsilon)$. Furthermore,

$$\epsilon\phi_1(t) \le \epsilon \max_{t\in[a,b]} \phi_1(t)$$

and hence, taking $\epsilon > 0$, possibly smaller, one has that $v_\epsilon(t) \le M_\lambda$ in $[a,b]$.

As in the proof of Theorem 13.3.6, we can replace f by a truncated function like

$$\tilde{f_\lambda}(t,x) = \tilde{f}(t,x) = \begin{cases} h(x) & \text{if } x \le 0 \\ f(t,x) = \lambda x - g(t,x) & \text{if } 0 \le x \le M_\lambda \\ f(t,M_\lambda) & \text{if } x \ge M_\lambda \end{cases}$$

where $h(x)$ is any smooth function such that $h(0)=0$, h' is bounded and $h(x)>0$ for $x<0$. Of course, $\tilde{f}$ satisfies (*). Moreover, from $0 \le v_\epsilon \le M_\lambda$ it follows that $\tilde{f}(t,v_\epsilon) = f(t,v_\epsilon)$. Thus $-x'' = \tilde{f}(t,x)$, $x(a)=x(b)=0$, possesses a positive super solution M_λ and a positive subsolution v_ϵ, with $v_\epsilon \le M_\lambda$. According to Theorem 13.3.4, the truncated problem has a solution such that $v_\epsilon(t) \le x(t) \le M_\lambda$ in $[a,b]$. Then $f(t,x(t)) = \tilde{f}(t,x(t))$ and hence $x(t)$ solves (13.16). Finally, from $x(t) \ge v_\epsilon(t)$ it follows that $x(t) > 0$ in (a,b). ■

Remark 13.4.2. As in Proposition 13.1.5 or Example 9.2.7, one can show that if $\lambda < \lambda_1$ the problem $-x'' = \lambda x - x^3$, $x(a)=x(b)=0$ has only the trivial solution. This shows that, in general, the condition $\lambda < \lambda_1$ cannot be removed. ■

13.5 Exercises

1. Show that the boundary value problem
$$\begin{cases} x'' - x^3 = 0 \\ x(0) = x(b) = 0 \end{cases}$$
has only the trivial solution $x(t) \equiv 0$.

2. Let $a < b$. Prove that the boundary value problem
$$\begin{cases} x'' + 4x^3 = 0 \\ x(a) = x(b) = 0 \end{cases}$$
has infinitely many solutions.

3. Show that the boundary value problem
$$\begin{cases} x'' + 6x^5 = 0 \\ x(0) = x(b) = 0 \end{cases}$$
has infinitely many solutions.

4. Show that for all $k \geq 0$ the boundary value problem
$$\begin{cases} x'' + (2p+2)x^{2p+1} = 0 \\ x(0) = x(b) = 0 \end{cases}$$
has infinitely many solutions.

5. Show that for $\lambda < \pi$ the boundary value problem
$$\begin{cases} x'' + \lambda x - x^3 = 0 \\ x(0) = x(1) = 0 \end{cases}$$
has only the trivial solution.

6. Show that the following boundary value problems
$$(a) \quad \begin{cases} x'' + 4x^3 = 0 \\ x(0) = 0, \ x(b) = 1 \end{cases} \qquad (b) \quad \begin{cases} x'' + 4x^3 = 0 \\ x'(0) = 0, \ x(b) = 0 \end{cases}$$
have a positive solution.

7. Prove that the preceding problems (a) and (b) have infinitely many solutions.

8. Find $b > 0$ such that the boundary value problem
$$\begin{cases} x'' + 4x^3 = 0 \\ x(0) = 0, \ x(b) = 1, \ x'(b) = 0 \end{cases}$$
has positive solutions.

9. Find the Green function of $L[x] = x''$ on $[0, 1]$ and solve the boundary value problem $x'' = 1, \ x(0) = x(1) = 0$.

10. Find the Green function of $L[x] = x''$ on $[-1, 1]$.

11. Find the Green function of $L[x] = x'' - k^2 x$ on $[0, 1]$ where $k \neq 0$.

12. Show that $-x'' = 1 - x - x^2$, $x(a) = x(b) = 0$ has a solution $x(t)$ such that $0 \leq x(t) \leq 1$.

13. Show that $-x'' + x = e^{-x}$, $x(a) = x(b) = 0$ has a solution such that $0 \leq x(t) \leq 1$.

14. Let $g(x)$ be continuous and such that $g(0) > 0$, $g(1) < 1$. Show that $-x'' + x = g(x)$, $x(a) = x(b) = 0$ has a solution.

15. Show that $-x'' + x = e^{-x^2}$, $x(a) = x(b) = 0$ has a positive solution.

16. Let $g(x)$ be continuous and such that $0 < g(x) \leq M$ for all x. Show that $-x'' = g(x) - x$, $x(a) = x(b) = 0$ has a positive solution.

17. Show that $-x'' = (1 + x^2)^{-1/2} - x$, $x(a) = x(b) = 0$, has a positive solution.

18. Show that $-x'' + x = \min\{e^x, 1\}$, $x(a) = x(b) = 0$, has a positive solution.

19. Prove that $-x'' = 2x - x^2$, $x(0) = x(\pi) = 0$, has a positive solution.

20. Show that if $b > \pi$, the problem $-x'' = \arctan x$, $x(0) = x(b) = 0$, has a positive solution.

Appendix A

Numerical methods

Many differential equations cannot be solved analytically; however, sometimes a numerical approximation to the solution is sufficient to serve one's need. Here we discuss some elementary algorithms that may be used to compute such approximations.

Let us consider the problem of approximating a solution to the initial problem

$$x' = f(t, x), \quad x(t_0) = x_0. \tag{A.1}$$

The unknown $x = x(t)$ could be a vector valued function so that (A.1) would be a system of first order ODEs. However, we will restrict ourselves to the scalar case in this text. With respect to (A.1), we assume that a unique solution exists, but that analytical attempts to construct it have failed.

In the Figure A.1, the blue curve is the graph of $x(t)$ and we want to find some approximation points connecting by red segments.

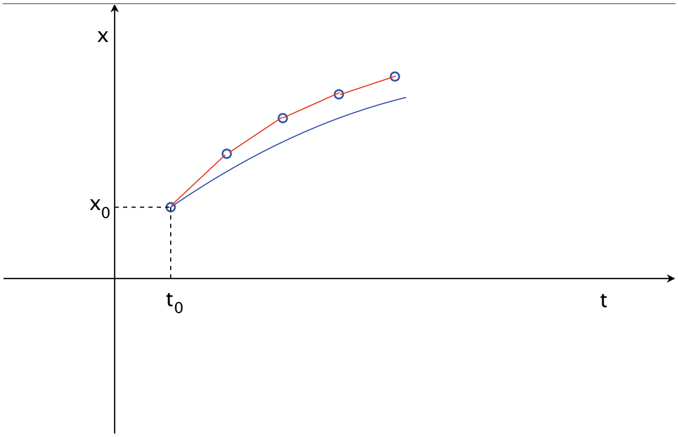


Fig. A.1. Exact solution curve (*blue*) and its approximation (*red*)

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3_A

In this chapter, we will discuss a very elementary method; namely, Euler's method and its improved version as well as a bit more advanced method – Runge–Kutta's method.

A.1 First order approximation: Euler's method

The basic idea is as follows: As we know,

$$x'(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}.$$

For sufficiently *small h* the above suggests that

$$x'(t) \sim \frac{x(t+h) - x(t)}{h},$$

and we can approximate $x(t+h)$ by $x(t+h) \sim x(t) + hx'(t)$. But as $x(t)$ satisfies the equation (A.1), $x'(t) = f(t, x(t))$, we then have

$$x(t+h) \sim x(t) + hf(t, x(t)).$$

Now, assume that we are already 'happy' with some approximate value X for $x(t)$, then the above would be a natural (and naive) approximation for $x(t+h)$:

$$\bar{X} = X + hf(t, X). \tag{A.2}$$

Repeating the process, we then come up with the following procedure (Euler's method).

a. Set $X_0 = x_0$ and pick a positive *step size* $h > 0$.
b. For each integer $i = 0, 1, 2, \ldots$, define

$$X_{i+1} = X_i + hf(t_i, X_i), \quad t_{i+1} = t_i + h. \tag{A.3}$$

Since a computer cannot calculate indefinitely, we can only approximate the solution $x(t)$ of (A.1) in a finite interval $[t_0, t_0 + L]$ of length $L > 0$, which is determined by the physics of the phenomenon under consideration. Suppose that we want to have n approximation points $x_1, \ldots, x_n$, then the *step size* h is given by L/n. See Figure A.2.

A generic algorithm for the Euler method is given by:

Step 1. Set the number n of points we wish to compute.
Step 2. Set the time step size $h = L/n$.
Step 3. Set $X = x_0$ and $t = t_0$.
Step 4. Set a counter $k = 1$.
Step 5. Compute $B = X$ and $C = hf(t, X)$.
Step 6. Compute $x_k = B + C$.

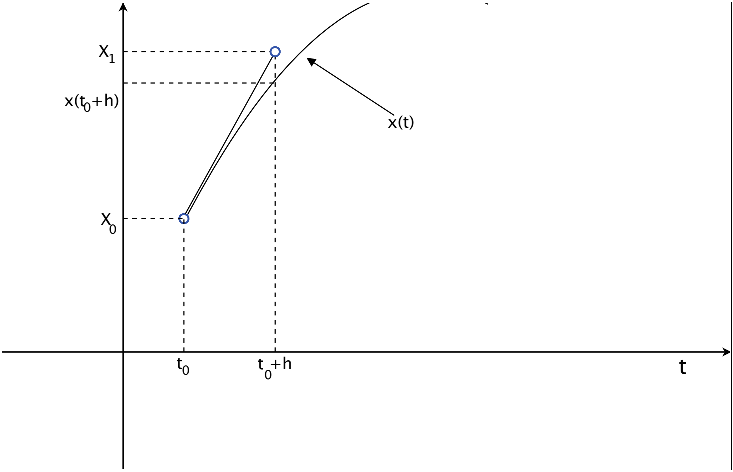


Fig. A.2. First approximation point: given a sufficiently small $h > 0$, we can start with the initial point $X_0 = x_0$ and t_0, dictated by the initial condition in (A.1), to construct the first approximation value $X_1 = X_0 + hf(t_0, X_0)$ for the true value $x(t_0 + h)$

Step 7. Set $X = x_k$ and $t = t + h$.
Step 8. Increase the counter k by 1.
Step 9. If $k < n$ then repeat steps 5–8. Otherwise, stop.

The following simple Maple code can be used to realize the above algorithm:

$x := x0; t := t0; X[0] :=$ Initial value
for i **from** 1 **by** 1 **to** N **do** Loop to compute x_i
$x := x + h * f(t, x); t := t + h;$ Compute the new value and time
$X[i] := x;$ Record the value
end do;

Let us apply the above algorithm to approximate the solution to

$$x' = x^2 + t^2, \quad x(0) = 1.$$

If we want to approximate the solution in the interval $[0, 0.8]$ by 8 points, then the step size $h = 0.8/8 = 0.1$ and the Euler algorithm gives rise to the following Table A.1.

In Figure A.3 these values are plotted against the graph of the solution. Note that in all the following figures the scale for the t and x axes is based on a ratio of 1 to 10.

A.1.1 Improved Euler's method

In the standard Euler method, we advance along the tangent of the solution curve to obtain the next point, a *predicted one*. We can possibly improve this by *correcting*

Table A.1. Euler method for $x' = x^2 + t^2, \quad x(0) = 1$. Step size $h = 0.1$

k	t_k	X_k	$X_{k+1} = X_k + h(X_k^2 + t_k^2)$
0	0	1	1.1
1	0.1	1.1	1.222
2	0.2	1.222	1.3753284
3	0.3	1.3753284	1.573481221
4	0.4	1.573481221	1.837065536
5	0.5	1.837065536	2.199546514
6	0.6	2.199546514	2.719347001
7	0.7	2.719347001	3.507831812
8	0.8	3.507831812	STOP

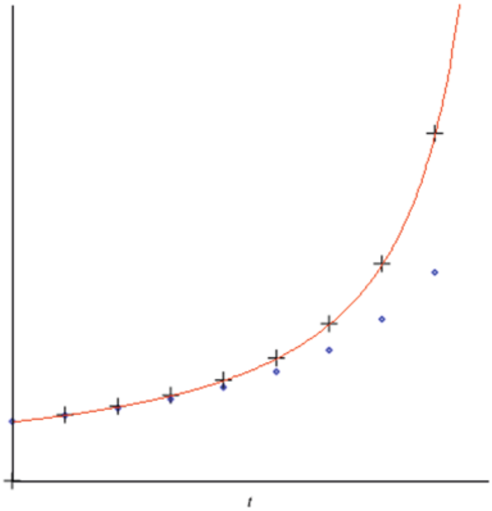


Fig. A.3. Approximation values against solution

the *predicted*. To this end, we can first compute the *predicted value* as before (see (A.2))

$$x_{k+1}^* = x_k + hf(t_k, x_k), \tag{A.4}$$

then correct it by

$$x_{k+1} = x_k + h\frac{f(t_k, x_k) + f(t_{k+1}, x_{k+1}^*)}{2}. \tag{A.5}$$

This simply means that we advance along the line between the tangents at the previous point (t_k, x_k) and the predicted point by Euler's method in order to obtain the

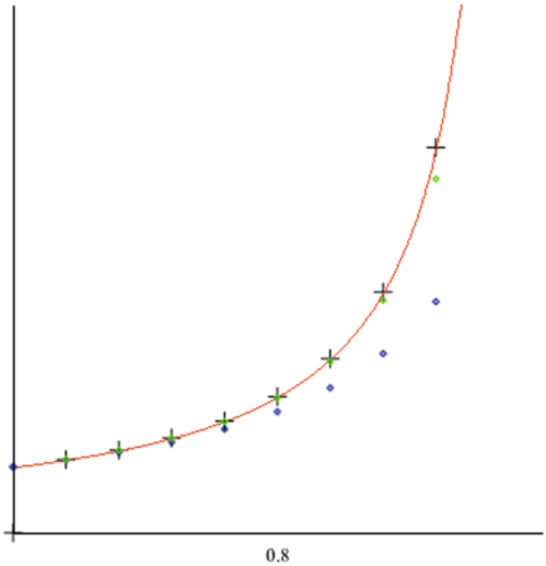


Fig. A.4. Plotting these values against the graph of the solution and the previous result obtained from the Euler method, we can see a great improvement

next point. See Figure A.4. In some cases, this seems to be a better approximation as we will see by applying this improved version to the previous example.

We apply the formulas (A.4) and (A.5) to obtain Table A.2.

An important note should be made here before we move on to the next section discussing more advanced numerical methods. There is no doubt that powerful computers can assist us to do tedious computation and, in many cases, provide almost what we practically need in applications. However, computers don't think! Yet efficiently

Table A.2. Improved Euler method for $x' = x^2 + t^2, \quad x(0) = 1$. Step size $h = 0.1$. Here, $f(t, x) = x^2 + t^2$

k	t_k	X_k	$X^*_{k+1} = X_k + hf(X_k, t_k)$	$X_{k+1} = X_k + h\frac{f(X_k,t_k)+f(X^*_{k+1},t_{k+1})}{2}$
0	0	1	1.1	1.111000000
1	0.1	1.111000000	1.235432100	1.251530674
2	0.2	1.251530674	1.412163577	1.436057424
3	0.3	1.436057424	1.651283516	1.688007333
4	0.4	1.688007333	1.988944209	2.048770724
5	0.5	2.048770724	2.493516872	2.600025118
6	0.6	2.600025118	3.312038179	3.529011494
7	0.7	3.529011494	4.823403706	5.371468766
8	0.8	5.371468766	STOP	STOP

but they simply do whatever we ask them to do. Therefore we cannot completely (and blindly) trust their output. A qualitative analysis needs to be done first before we can rely on any numerical method to do the messy and cumbersome job. The following simple example will be a good warning.

Let us consider the initial value problem

$$x'(t) = x^2 + 1, \quad x(0) = 1,$$

which can be solved easily by separating the variables, and we get

$$x(t) = \tan\left(t + \frac{\pi}{4}\right).$$

Obviously, the solution is only defined on the interval $[0, \frac{\pi}{4})$ as we need $t + \frac{\pi}{4} < \frac{\pi}{2}$. However, computers do not know this if we ask them to perform the discussed Euler methods on this problem. They would go on and compute 'values' of $x(t)$ for t beyond $\frac{\pi}{4}$!

Furthermore, the Euler method is often not accurate enough. In more precise terms, it only has order one. This caused us to look for higher-order methods. One possibility is to use not only the previously computed value x_k to determine x_{k+1}, but to make the solution depend more on past values. This yields the so-called *multistage methods*. We will discuss one such method, the Runge–Kutta, in the next section.

A.2 The Runge–Kutta method

We now study a more advanced and accurate Runge–Kutta method to approximate a solution to the initial problem (A.1), namely

$$x' = f(t, x), \quad x(t_0) = x_0.$$

In the Euler method, the next value x_{k+1} is computed by the previous x_k advancing along the *approximated* tangent. The Runge–Kutta method computes the next value x_{k+1} via multiple stages in order to obtain better approximations. To this end, x_{k+1} will be x_k plus a *weighted average* of a number s of *increments* (the number s is fixed and called the number of stages). Each increment is just a product of the step size h and an *estimated* slope of the solution curve specified by the right-hand side $f(t, x)$ in the equation (A.1).

For example, let us consider the *2-stage* method given by the formula

$$\begin{aligned} x_{k+1} &= x_k + h\left(\frac{1}{2}f(t_k, x_k) + \frac{1}{2}f(t_k + h, x_k + hf(t_k, x_k))\right) \\ &= x_k + \frac{1}{2}I_1 + \frac{1}{2}I_2. \end{aligned} \tag{A.6}$$

One can see that x_{k+1} is obtained by advancing x_k by the average of 2 increments I_1, I_2:

1. $I_1 = hf(t_k, x_k)$ is the increment based on the slope at the beginning of the interval, using the Euler method.
2. $I_2 = hf(t_k + h, x_k + hf(t_k, x_k))$ is the increment based on the slope at the end of the interval, using $x_k + hf(t_k, x_k)$.

A keen reader will notice that this method is just the improved Euler method discussed earlier!

Generalizing (A.6), we can take any number $\alpha \in (0, 1]$ and define

$$x_{k+1} = x_k + \left(1 - \frac{1}{2\alpha}\right) I_1 + \frac{1}{2\alpha} I_2, \tag{A.7}$$

where:

1. $I_1 = hf(t_k, x_k)$ is the increment based on the slope at the beginning of the interval. This increment is given the weight $\left(1 - \frac{1}{2\alpha}\right)$.
2. $I_2 = hf(t_k + \alpha h, x_k + \alpha I_1)$ is the increment based on the slope at the point $t_k + \alpha h$ of the interval, using $x_k + \alpha I_1 = x_k + \alpha hf(t_k, x_k)$.

The reader can easily check that (A.6) is a special case of this generalization when $\alpha = 1$. If one takes $\alpha = 1/2$ then (A.7) results in the so-called *midpoint method*

$$x_{k+1} = x_k + hf(t_k + \frac{1}{2}h, x_k + \frac{1}{2}hf(t_k, x_k)),$$

which looks similar to the formula in Euler's method but using the slope at midpoint of the interval.

Let us move on to another member of the family of Runge–Kutta methods which is so commonly used that it is often referred to as "RK4", "classical Runge–Kutta method" or simply as "the Runge–Kutta method".

The formula is as follows

$$x_{k+1} = x_k + \frac{1}{6}I_1 + \frac{1}{3}I_2 + \frac{1}{3}I_3 + \frac{1}{6}I_4. \tag{A.8}$$

Here, there are 4 increments (4 stages) and their weights are given by:

1. $I_1 = hf(t_k, x_k)$ is the increment based on the slope at the beginning of the interval. This increment is given the weight $\frac{1}{6}$.
2. $I_2 = hf(t_k + \frac{1}{2}h, x_k + \frac{1}{2}I_1)$ is the increment based on the slope at the midpoint $t_k + \frac{1}{2}h$ of the interval, using $x_k + \frac{1}{2}I_1$. Its weight is $\frac{1}{3}$.
3. $I_3 = hf(t_k + \frac{1}{2}h, x_k + \frac{1}{2}I_2)$ is the increment based on the slope at the midpoint $t_k + \frac{1}{2}h$ of the interval, but now using $x_k + \frac{1}{2}I_2$. Its weight is still $\frac{1}{3}$.
4. $I_4 = hf(t_k + h, x_k + I_3)$ is the increment based on the slope at the end of the interval, using $x_k + I_3$. Its weight is $\frac{1}{6}$.

We now describe the general s-stages method. We fix an integer $s \geq 1$ and define

$$x_{k+1} = x_k + \sum_{i=1}^{s} w_i I_i,$$

where $w_i \in [0, 1]$ are the weights whose sum must be 1. The increments $I_1, \dots, I_s$ are given by

$$\begin{aligned}
I_1 &= hf t_k, (x_k) \\
I_2 &= hf(t_k + c_2 h.x_k + a_{21} I_1) \\
I_3 &= hf(t_k + c_3 h, x_k + a_{31} I_1 + a_{32} I_2) \\
&\vdots \\
I_s &= hf(t_k + c_s h, x_k + a_{s1} I_1 + a_{s2} I_2 + \cdots + a_{s,s-1} I_{s-1}).
\end{aligned}$$

We can see that, for $m = 1, \dots, s$, I_m is the increment based on the slope at $t_k + c_m h$ and using x_k advancing by a weighted sum of previous increments $I_1, \dots, I_{m-1}$:

$$I_m = hf(t_k + c_m h, x_k + a_{m1} I_1 + a_{m2} I_2 + \cdots + a_{m,m-1} I_{m-1}),$$

which is the approximated slope at the time $t_k + c_m h$. It is then natural to require that the increments in x satisfy

$$a_{m1} + a_{m2} + \cdots + a_{m,m-1} = c_m.$$

In such a case, we say that the method is *consistent*.

It is clear that the Runge–Kutta is much more complicated than the primitive Euler method and it is not practical to perform the calculation on a handheld calculator without programming ability. If the reader has some knowledge in programming then the following Maple programming code can be used to generate the approximation values in the general Runge–Kutta method.

```
RKgenVal := proc(A, c, W, f, t0, x0, h, N, S)
   local x, t, X, i, j, k, INC, Inc;
     x := x0; t := t0; X[0] := x; Initial value
      for i from 1 by 1 to N do Compute the increments
         for j from 1 by 1 to S do
            INC := 0;
            for k from 1 by 1 to j - 1 do
               INC := INC + A[j, k] * Inc[k]
            end do ;
            Inc[j] := h * f(t + c[1, j] * h, x + INC)
         end do ;
        INC := 0;
```

for k **from** 1 **by** 1 **to** S **do** Weighted total increment
$INC := INC + W[1,k] * Inc[k];$
end do;
$x := x + INC; t := t + h; X[i] := x;$ Record the new value
end do;
$X;$
end proc

The above procedure requires the following inputs:

1. A matrix A holding the weights a_{ij}

$$A = \begin{bmatrix} 0 & 0 & \cdots & & 0 \\ a_{21} & 0 & \cdots & & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{a1} & a_{s2} & \cdots & a_{s,s-1} & 0 \end{bmatrix}.$$

2. A vector c holding the *time weights* c_i.
3. A vector W holding the increment weights w_i.
4. The right-hand side $f := f(t,x)$, the initial time $t0$, the initial condition $x0 = x(t_0)$, the step size h, number of points to compute N and the number of stages S.

For example, the matrices for a 2-stage method can be

$$A := \begin{bmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & \frac{2}{3} \end{bmatrix}, \quad W := \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

While for the classic RK4, we use

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \quad W := \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

Let us apply the above two methods with such parameters and revisit the example

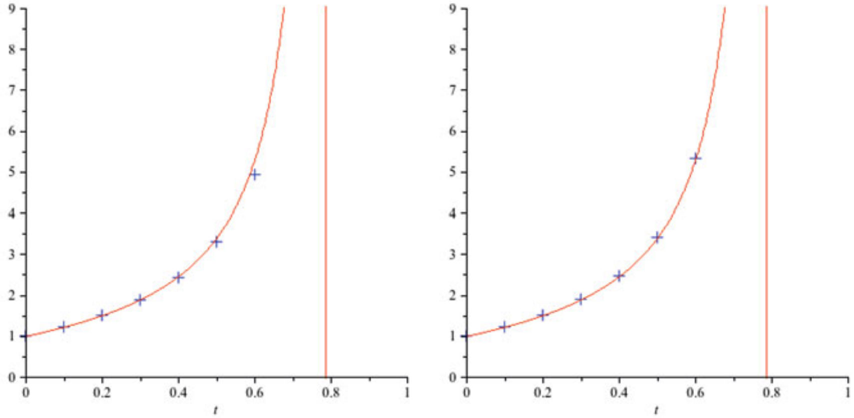
$$x' = x^2 + 1, \quad x(0) = 1.$$

Table A.3. approximation points

k	*2-stages*	*Errors*	*4-stages*	*Errors*
1	1.221333333	0.001715548	1.223048914	0.0000000330
2	1.502999707	0.005497940	1.508496167	0.000001480
3	1.881423779	0.014341347	1.895754160	0.000010966
4	2.427681154	0.037281607	2.464899687	0.000063074
5	3.300240967	0.107982483	3.407820425	0.000403025
6	4.928987792	0.402867449	5.327896817	0.003958424

We will use the step size $h = 0.1$ to compute $m = 6$ approximation points. The result is recorded in Table A.3.

We can see that the 4-stage method provides much smaller errors. Plotting the approximation points obtained by the two methods against the true solution $x(t) = \tan(t + \frac{\pi}{4})$, we can see that the 4-stage points in Figure A.5b are much closer to the graph of the true solution.



(a) (b)

Fig. A.5. The plots of the solution $x(t) = \tan(t + \frac{\pi}{4})$ and approximation points. (a) 2-stages; (b) 4-stages

Answers to selected exercises

Chapter 1

1. $x' = -5x$.
3. $x = c\,e^{-4t} + 1$.
4. Use the uniqueness property.
6. $k = \ln 2$.
7. Find the solution and then use the Intermediate Value Theorem.
8. (a) $(-\infty, -1)$, (b) $(-\pi/2, 1)$.
9. $x = \frac{2}{3}t^2 + \frac{c}{t},\ t \neq 0$.
11. $x = 4e^{t^2}$.
12. $x = c\,e^{-t^3/3}$.
13. $x = \frac{b}{a}\left(t - \frac{1}{a}\right) + Ce^{-at}$.
17. $\dfrac{h}{k}$.
18. (a) k=1/2, (b) No!
19. $e^{\pm\pi/2}$.
22. (a) $q(t) = 7t + 2$, (b) $q(t) = t^2 + 3$.
26. Recall that solutions of such equations do not change sign.

S. Ahmad, A. Ambrosetti, *A Textbook on Ordinary Differential Equations*. 2nd Ed.,
UNITEXT – La Matematica per il 3+2 88, DOI 10.1007/978-3-319-16408-3

28. $x = 3t$.

29. $x = at$: if $a > 0$ minima, if $a < 0$ maxima.

Chapter 2

2. Suppose $||x|^p - |y|^p| \leq L|x - y|$ and let $y = 0$.

4. All $a \neq 0$.

6. Check the conditions for existence and uniqueness for $x' = \ln x$ $(x > 0)$.

9. Note that $|x|^{1/4}$ is lipschitzian off $x > 0$.

12. Verify that $f(x) = \sin x$ satisfies the conditions of the Global Existence Theorem.

14. Verify that $f(x) = \ln(1 + x^2)$ satisfies the conditions of the Global Existence Theorem.

15. The function $f(x) = \max\{1, x\}$ is globally lipschitzian.

17. Use uniqueness to show that if $z(t) = -x(-t)$, then $x(t) = z(t)$.

18. Solutions are either increasing or decreasing.

19. Solutions are increasing.

20. Show that if it changed sign, it would violate uniqueness.

24. Show that if $x'(t^*) = 0$ then $x''(t^*) > 0$.

26. Show that $\phi'(a) = 0$ and $\phi''(a) < 0$.

28. Show that $\phi'(0) = \phi''(0) = 0$ and $\phi'''(0) < 0$.

30. Solve $y' = 1 + 2t, \;\; y(0) = 0$ and use the Comparison Theorem.

Chapter 3

2. $x = \frac{1-e^{2t}}{1+e^{2t}}$.

4. $x = \frac{-1+\sqrt{1+8e^t}}{2}$.

6. $x = -\left(3(t^4 + c)\right)^{-1/3}$.

7. $x = \frac{2}{t^2-2c}$, $t \neq \pm\sqrt{2c}$.

9. $x = \sqrt{t^2-1}, t \geq 1$.

11. $x = \frac{6}{2t^3+1}$, $t > -\left(\frac{1}{2}\right)^{1/3}$.

13. $x = \left(\frac{1}{2}t^4 + 1\right)^2$.

14. $x = \left(\frac{1}{4}t^2 - \frac{1}{4}a^2\right)^2$ and $x = 0$.

15. $p + 1 > 0$.

16. The limit is a constant, which depends on the initial conditions. (a), $c = 0$: (b) $-1/2$.

18. $\frac{2}{3}x^3 + x - \left(\frac{y^6}{6} - y\right) = c$.

21. $\frac{ax^{p+1}}{p+1} + bxy + \frac{dy^{q+1}}{q+1} = c$.

24. $e^x + e^y - \frac{1}{2}xy^2 = c$.

26. $x^3 + 3x^2y - 3xy^2 + y^3 = c$.

27. $x^3 + 3x^2y + 6xy^2 + 5y^3 = c$. a is the unique negative solution of $1 + 3a + 6a^2 + 5a^3 = 0$.

30. $a_1 = 2, b_1 = 2a_2$; $x^3 + 3x^2y + 3a_2xy^2 + b_2y^3 = c$.

31. $A(y) = y^2 + \kappa, \kappa$ constant; $x^2 + \kappa x + xy^2 = c$.

33. μ satisfies $\mu'(y) = \frac{y - f'(y)}{f(y)}$.

36. $\frac{1}{3}x^3 - \frac{1}{2}x^2 + y(x-1)^2 = c$.

38. Show that an integrating factor is $\mu(y) = \frac{1}{\sqrt{1+y^2}}$.

40. $x = 0$ or $-\frac{t^2}{2x^2} + \ln\left|\frac{x}{t}\right| = \ln\left|\frac{1}{t}\right| + c, \quad t \neq 0$.

42. $t - \frac{x^2}{t} = c$.

43. $\frac{1}{2}(x-t)^2 + 2x - t = c$.

44. $2(x+t) + \ln|2(x-t)+1| = c$.

47. $x = \pm\frac{1}{\sqrt{ce^{2t^2}-2}}$.

48. $x = z^{-1}$, with $z = e^{-t^2/2}\left(-\int e^{t^2/2}dt + c\right)$

49. Solve $\begin{cases} x'^2 &= x^2 + t^2 - 1 \\ 2x' &= 0. \end{cases}$

50. $x = \frac{1}{4}t^2$.

52. $x = ct - c^2$. The singular solution is $x = \frac{t^2}{4}$.

53. $x = ct + e^c$. The singular solution is $x = t\ln(-t) - t,\ t < 0$.

54. $x = ct - \ln c, c > 0$. The singular solution is $x = 1 + \ln t,\ t > 0$.

56. $\alpha = h(\alpha)$ and $\beta = g(\alpha)$.

Chapter 4

1. The function $x|x|$ has a continuous first derivative.

2. The function $\max\{0, x|x|\}$ has a continuous first derivative.

4. Show that $\frac{d}{dt}(x^2 + y^2) = 0$.

6. Show that $\frac{dH(x,y)}{dt} = 0$.

7. $x'' = x$.

11. Set $z(t) = x(t + T)$ and use the uniqueness of the ivp.

12. Set $z(t) = x(-t)$ and use uniqueness.

13. $x''(t)$ is increasing.

Chapter 5

A2. b) $W(\frac{\pi}{2}) = \pi \neq 0$, c) Use Abel's Theorem.

A5. $W(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} \neq 0$.

A6. $f(t) = (t^2 + 1)(\arctan t + 1)$.

A7 $W(zx_1, zx_2) = z^2 W(x_1, x_2)$.

A10. x_1 and x_2 are linearly dependent.

A11. $W(7) = 6$.

B2. $x = c_1 e^{-t} \sin t + c_2 e^{-t} \cos t.$

B3. $x = c_1 e^{-4t} + c_2 t e^{-4t}.$

B5. $x = -e^{t-1} + e^{2t-2}.$

B6. $x = \frac{2}{\sqrt{6}} e^{-t} \sin \frac{\sqrt{6}}{2} t.$

B8. One root of the characteristic equation is positive (if $\beta > 0$) or null (if $\beta = 0$).

B9. $\beta < 0.$

B11. Use the uniqueness of the ivp.

B15. $a = -\frac{1}{2}.$

B17. $a^2 < b.$

B18. $\lambda = k, k = 1, 2, \ldots .$

B19. $a - b = k\pi, k = 1, 2, \ldots .$

B24. $x = a - ae^{-2t}.$

C2. $x = c_1 e^{2t} + c_2 e^{-2t} - t^2 - \frac{1}{2}.$

C4. $x = c_1 \sin t + c_2 \cos t + 3t^2 + t - 6.$

C6. $x = c_1 e^t + c_2 e^{-t} + e^{2t}.$

C8. $x = c_1 e^{(\frac{3+\sqrt{13}}{2})t} + c_2 e^{(\frac{3-\sqrt{13}}{2})t} - t^2 + 5t - 17.$

C10. $x = c_1 e^{-t} + c_2 e^{2t} - t + \frac{1}{2} - \frac{1}{2} e^t.$

C12. $x = c_1 \sin t + c_2 \cos t + -\frac{1}{3} \sin 2t + \frac{1}{8} \cos 3t.$

C14. $x = c_1 \sin 2t + c_2 \cos 2t - \frac{1}{3} t \sin 2t - \frac{4}{9} \cos 2t.$

C16.
$$x = c_1 e^t + c_2 e^{-t} + \frac{1}{k^2 - 1} e^{kt}, \qquad (k \neq \pm 1)$$
$$x = c_1 e^t + c_2 e^{-t} + \frac{1}{2} t e^t, \qquad (k = 1)$$
$$x = c_1 e^t + c_2 e^{-t} - \frac{1}{2} t e^t, \qquad (k = -1)$$

C18. $x = c_1 e^t + c_2 e^{2t} - \left(3t + \frac{3}{2} t^2\right) e^t.$

C20. $x = c_1 \sin\sqrt{2}\,t + c_2 \cos\sqrt{2}\,t + \frac{t}{2\sqrt{2}} \sin\sqrt{2}\,t$.

C22. $x = c_1 \sin t + c_2 \cos t - t \cos t + t \sin t$.

C24. $x = \frac{e}{e^2-1}(e^t - e^{-t}) - t$.

C26. Multiply the equation by x and integrate in $[a, b]$.

D1. Compare with $x'' + x = 0$.

D2. The first equation.

D3. Show that $x(0) = x'(0) = 0$.

D4. Use the general solution of the nonhomogeneous equation to show that one can choose the constants to find the desired solution.

D8. (a) evaluate the derivative and use the equations, (b) by contradiction, using (a).

D9. Set $\phi = \frac{v(t)}{u(t)}$, $t \in (a, a+\epsilon)$ and show that $\phi' > 0$.

D11. $x = \frac{t}{t+1}$.

D13. $x = 0$ and $|x(t)| = \frac{e^{c_2}}{|\cos(t+c_1)|}$.

D14. (a) $x = e^{1-e^t}$, (b) $\ln|x(t)| = 1 + 2t - e^t$.

D16. Distinguish between $a < 1$ or $a > 5$, $1 < a < 5$ and $a = 1$ or $a = 5$.

D19. $x = c_1 t + \frac{c_2}{t^3} + \frac{1}{5} t^2$.

D20. $P(t) = -3a_3\,t + a_3\,t^3$.

Chapter 6

2. $x = c_1 + c_2 e^{-t} + c_3 e^t$.

5. $x = c_1 e^{-t} + c_2 e^{2t} + c_3 t e^{2t}$.

6. $x = \frac{3}{2} - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$.

7. $x = e^{-t}$.

9. $x = \frac{1}{5}e^t - e^{-t}\left(\frac{2}{5}\sin t + \frac{1}{5}\cos t\right)$.

10. The characteristic equation has at least one negative zero.

11. Check the max and min of the characteristic equation.

13. $x = \frac{1}{4}e^t + \frac{1}{4}e^{-t} + \frac{1}{2}\cos t$.

15. $x = \frac{e^{-bt}}{a-b} - \frac{e^{-at}}{a-b}$, where a, b $(a \neq b)$ are the two positive roots of $m^4 - 4m^2 + 1 = 0$.

16. The characteristic equation has only positive solutions.

18. $x = c_1 + c_2e^t + c_3e^{-t} + c_4 \sin t + c_5 \cos t$.

20. The characteristic equation has the negative root $m = -1$.

21. $x = c_1 + c_2t + c_3e^t + c_4e^{-t} + c_5 \sin t + c_6 \cos t$.

23. $x = c_1 + c_2t + c_3e^{-t} + c_4e^{-2t} + \frac{1}{6}e^t$.

24. Using the method of Variation of Parameters one finds $x = c_1 + c_2 \sin 2t + c_3 \cos 2t + \frac{1}{8} \ln|\sec 2t + \tan 2t| - \frac{1}{8} \ln|\cos 2t| \sin 2t - \frac{1}{4}t \cos 2t$.

26. $x = -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t} - \frac{1}{2}t^2$.

28. Find a proper subset of linearly dependent functions.

30. Show that $W(t^2, -t^2) = 0$ and explain why this implies

$$W(t,\, t^2,\, t^3,\, \sin t,\, \cos t,\, t^4, e^t,\, e^{-t},\, t^4 - t^2) = 0.$$

31. $W(6) = 5e^{18}$.

32. Use Abel's Theorem.

33. $x = \frac{c_1}{t} + c_2 \sin(\ln t) + c_3 \cos(\ln t),\ t > 0$.

Chapter 7

A1. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1e^t \\ c_2e^{-3t} \end{pmatrix}$.

A3. $\overline{x} = \begin{pmatrix} c_1e^{at} + c_2te^{at} \\ c_2e^{at} \end{pmatrix}$.

A5. $x = e^{2t}(c_1 \cos t + c_2 \sin t), \quad y = e^{2t}(-c_1 \sin t + c_2 \cos t).$

A7. $\overline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}.$

A8. $\overline{x} = \begin{pmatrix} a \cos 2t \\ a \sin 2t \end{pmatrix}.$

A10. $x = c_1 e^{3t} - \frac{1}{3}t - \frac{1}{3}, \quad y = c_2 e^{-t} + 2t - 2.$

A11. $x = c_1 e^{t} - t^2 - 2t - 2, \quad y = c_2 e^{t} - 1.$

B3. $\lambda = 1, \ -1, \overline{x}(t) = \begin{pmatrix} 2c_2 e^{-t} \\ c_1 e^{t} - 3c_2 e^{-t} \end{pmatrix}.$

B5. $x = 3c_1 + 2c_2 e^{5t} + 2e^{t}, \quad y = -c_1 + c_2 e^{5t} - \frac{1}{2}e^{t}.$

B6. $\overline{x} = \begin{pmatrix} -e^{t} + 3e^{3t} \\ 3e^{3t} \end{pmatrix}.$

B7. $x = c_1 e^{t} + c_2 e^{3t} - \frac{2}{27} - \frac{2}{9}t + \frac{2}{3}t^2, \quad y = c_2 e^{3t} - \frac{2}{27} - \frac{2}{9}t - \frac{1}{3}t^2.$

B8. $x = c_1 e^{2t} + c_2 e^{3t}, \qquad y = c_1 e^{2t}.$

B11. $x = 3c_1 e^{2t} + c_2 e^{-2t}, \qquad y = c_1 e^{2t} - c_2 e^{-2t}.$

C2. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$

C4. $\overline{x} = \begin{pmatrix} \frac{1}{3}e^{4t} + \frac{2}{3}e^{t} \\ 0 \\ e^{4t} \end{pmatrix}.$

C7. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + \frac{1}{2} \begin{pmatrix} \frac{1}{3} \\ 1 \\ -1 \end{pmatrix} e^{-2t}.$

C8. $a < 0.$

C10. $0 < a < 2.$

D2. $x = \frac{3}{2}t^2 + \int y(t)dt, \quad y = e^{-\frac{1}{2}t^2} \left(c_1 + 3 \int t^2 e^{\frac{1}{2}t^2} dt \right).$

D4. $x = c_1 t + c_2 t^{-2}, y = c_1 t^2 - 2c_2 t^{-1} - 1.$

D6. $x = c_1 t - e^{c_1}, \quad y = t - \frac{1}{2}c_1 t^2 + c_2$ as well as $x = t \ln t - t + c_1'$,
$y = t - \frac{1}{2}t^2 \ln t + \frac{1}{4}t^2 + c_2'.$

Chapter 8

2. $a = -7,\ b = 1.$

3. $\alpha = -\beta.$

6. $C < \frac{1}{2}.$

8. $B^2 - 9 < 0.$

9. Solutions satisfy $x^2 + xy + \frac{1}{2}y^2 = 1$ which is an ellipse.

10. Solutions satisfy $x^2 + xy - 3y^2 = 1$ which is a hyperbola.

13. $x_\epsilon = \frac{1+\epsilon}{4}, \qquad y_\epsilon = \frac{2-\epsilon}{7}.$

14. The solutions satisfy $H(x, y) = x + 2y - \ln x - 2\ln y = k$, $x, y > 0$; then take $k = 4$.

16. The nontrivial equilibrium of the system is $x = 5$, $y = 3$.

18. x_0 satisfies $x_0''' = x_0' - 3x_0^2 x_0'$.

20 Use the phase plane analysis.

21. $x(t) > 0$, it is increasing for $t > 0$, decreasing for $t < 0$ and $\lim_{t\to\pm\infty} x(t) = +\infty$.

23. The solution verifies $y^2 - x^2 + \frac{1}{2}x^4 = 1$, which is a compact curve that does not contain equilibria.

26. The solution is $y = \pm\sqrt{a^2 - x^2 - 2x^8}$.

29. The solution satisfies $y^2 - x^2 + 2x^3 = 0$.

31. The solution satisfies $y^2 - x^2 + 2x^3 = 1$.

Chapter 9

2. $\lambda_k = \frac{\beta k^2\pi^2}{b^2}, k = 1, 2, \ldots.$

3. $\frac{k^2\pi^2}{2} \leq \lambda_k[1 + t] \leq k^2\pi^2.$

4. $\frac{\pi^2}{4e} \leq \lambda_k[e^t] \leq \frac{\pi^2}{4}.$

5. Use the variational characterization of the first eigenvalue.

7. $\frac{\alpha\pi^2}{M(b-a)^2} \le \lambda_1 \le \frac{\beta\pi^2}{m(b-a)^2}$.

10. $\lambda_k = k^2$, with $k = 0, 1, 2\dots$. Notice that the eigenfunctions corresponding to $\lambda = 0$ are constants.

12. Multiply the equation by φ_k and integrate.

14. $u(t, x) = \alpha e^{-t} \sin x$.

15. $u(t, x) = \alpha e^{-c^2 t} \sin x$.

16. $u(t, x) = \alpha e^{-\frac{\pi^2}{L^2}t} \sin\left(\frac{\pi x}{L}\right)$.

Chapter 10

1. $x = a_1 \sum_{k\ge 1} \frac{t^k}{k!(k-1)!}$.

2. $a_1 = 0$, $a_k = 0$ for all $k \ge 3$. Hence $x(t) = a_0 + a_2 t^2$.

3. $$x = t + \sum_{n\ge 1} \frac{t^{3n+1}}{(3n+1)3n(3n-2)(3n-3)\cdots 4\cdot 3} + \frac{t^2}{2} + \sum_{n\ge 1} \frac{t^{3n+2}}{(3n+2)(3n+1)(3n-1)(3n-2)\cdots 5\cdot 4\cdot 2}.$$

4. $x = \sum_{n\ge 0} (-1)^n \frac{t^{2n}}{(2n)!!}$.

5. The roots of the indicial equation are $r = \pm 1/2$. If $r = 1/2$, $a_k = 0$, for all $k \ge 1$ and $x_1(t) = ct^{1/2}$. If $r = -1/2$, $a_k = 0$ for all $k \ge 2$ and $x(t) = t^{-1/2}(a_0 + a_1 t)$.

7. The indicial equation has a double root $r = 2$ yielding
$$x = a_0 t^2 \sum \frac{t^k}{2^2 \cdots k^2}.$$

8. $x = (4 - \lambda_1)A_1 e^{\lambda_1 t} + (4 - \lambda_2)A_2 e^{\lambda_2 t}$, $y = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$, where $\lambda_{1,2} = -1 \pm \sqrt{8}$ and $A_i = \dfrac{\lambda_i - 2}{2\lambda_i + 2}$.

9. $x_a(t) = aJ_0(t)$.

11. $x(t) = c_1 J_m(t)$.

14. Differentiating the series term by term, show that $\alpha J_1'(\alpha) = -J_1(\alpha)$.

18. Use the fact that between two consecutive zeros of J_1 there is a zero of J_0 to infer that $J_1(\alpha_1) > 0$. Moreover, $J_2(\alpha_1) = \frac{2}{\alpha_1} J_1(\alpha_1) > 0$.

20. $\lambda_n = \left(\frac{\beta_n}{2}\right)^2$ where $J_1(\beta_n) = 0$, $\beta_n > 0$; $y_n(s) = 2c\sqrt{\lambda_n s}J_1(2\sqrt{\lambda_n s})$.

21. $\lambda = 1$.

Chapter 11

3. $$\frac{2\omega s}{(s^2-\omega^2)^2}, \quad \frac{\omega^2+s^2}{(s^2-\omega^2)^2}.$$

4. $$\frac{\omega}{(s-\alpha)^2+\omega^2}, \quad \frac{s-\alpha}{(s-\alpha)^2+\omega^2}.$$

5. $\frac{1-e^{-s}}{s} + 2\,\frac{e^{-3s}-e^{-4s}}{s}$.

6. $\frac{1}{s^2} \cdot \frac{1}{s-1}$.

8. Show that $\mathcal{L}\{f\} = \sum_0^{+\infty} e^{-nTs} \cdot \int_0^T e^{-s\tau} f(\tau)d\tau$.

9. $\frac{s-e^{-s}}{s(1-e^{-2s})}$.

10. $\frac{1}{s^2} - \frac{Te^{-sT}}{s(1-e^{-sT})}$.

14. $e^{-t}\sin t, \quad e^{-t}\cos t$.

16. Use Theorem 11.3.7 with $P(s) = s-2$ and $Q(s) = s^3 - s$.

18. $\delta + 1 + e^{-t}$.

19. Show that $F'(s) < 0$ and $F''(s) > 0$.

20. Apply $(P4)$ with $g'(t) = f(t)$.

23. Use $(P4)$ to find $(1+s^2)\mathcal{X}'(s) + s\mathcal{X}(s) = 0$.

24. $x = \sinh t$.

25. $x = t - 1.$

27. $\mathcal{L}\{x\} = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}.$

29. $x = \int_0^t e^{t-\theta} H_a(\theta) d\theta = \begin{cases} 0 & \text{if } t < a \\ -1 + e^{t-a} & \text{if } t \geq a \end{cases}.$

31. $x = (k + a)e^t$. Remark that this solves $x'' - x = 0$ with the initial condition $x(0) = k + a$.

32. $x(t) = \sin t \, * \, g(t)$. If $g(t) = \chi_{[0,1]}(t)$,

$$x(t) = \begin{cases} 1 - \cos t & \text{if } t \in [0, 1] \\ \cos(t - 1) - \cos t, & \text{if } t > 1. \end{cases}$$

33. $x = 1 + H_a(t)(t - a) = \begin{cases} 1 & \text{if } 0 \leq t < a \\ 1 + t - a & \text{if } t \geq a. \end{cases}$

35. $x = 6\left(\frac{t^5}{5!} + \frac{t^3}{3!}\right) = \frac{t^5}{20} + t^3.$

36. $x(t) = \frac{1}{\sqrt{k}} \cdot \sinh(\sqrt{k}\, t).$

37. $x(t) = \cos k\, t + \frac{\sin k\, t}{k}.$

38. $x = \frac{15}{8}\left(e^{7t} - e^{-t}\right), \; y = \frac{1}{8}\left(3e^{-t} + 5e^{7t}\right).$

40. $x = \frac{1}{2}(e^t - e^{-t}), \quad y = e^{-t}.$

Chapter 12

1. The eigenvalues of the coefficient matrix are $\lambda = -3 \pm \sqrt{8}$.

3. The eigenvalues of the coefficient matrix are $\lambda = \pm i\sqrt{2}$.

4. If $a < 0$, unstable; if $a > 0$, asymptotically stable; if $a = 0$, stable, but not asymptotically stable.

6. Unstable.

10. If $a < 0$, the equilibrium is asymptotically stable. If $a > 0$ the equilibrium is unstable.

11. Show that at least one eigenvalue of the coefficient matrix is greater than 1.

12. $a < -1$.

13. Unstable.

15. Unstable.

18. The solutions of $\lambda^4 + 8\lambda^3 + 23\lambda^2 + 28\lambda + 12 = 0$ are $\lambda = -1, -2, -2, -3$.

19. Write the equivalent first order system and show that one eigenvalue of the coefficients matrix is positive.

21. $x = 0$ is asymptotically stable for $\lambda < 0$ and unstable for $\lambda > 0$.

24. The stable manifold is $x_3 = 0$: the unstable manifold is the x_3 axis.

25. The eigenvalues of the linearized system are $-1, -2$.

26. $V(x, y) > 0$ and $\dot{V} < 0$ for all $(x, y) \neq (0, 0)$.

28. Apply the Instability Theorem with $W(x, y) = \frac{1}{2}(x^2 + y^2)$.

29. The eigenvalues of the coefficient matrix of the linearized system are

$$\lambda = \frac{-1 \pm \sqrt{1 - 4a}}{2}.$$

30. Change variable $\widetilde{x} = x + a$ and show that $\widetilde{x} = 0, y = 0$ is unstable for the corresponding system.

31. The potential $F(x, y) = (x^2 + y^2)^2$ has a strict minimum at $(0, 0)$.

32. Show that $V(x, y) = \frac{1}{2}y^2 + \int_0^x g(s)ds$ is a Liapunov function.

Chapter 13

1. Multiply $x'' = x^3$ by x and integrate.

3. Letting $\Phi(c) = c^{-1/3} \int_0^1 \frac{dy}{\sqrt{1-y^6}}$, show that the equation $\Phi(c) = \frac{b}{\sqrt{2}k}$ has infinitely many solutions c_k.

5. $\lambda = \pi$ is the first eigenvalue of the linearized problem $x'' + \lambda x = 0,\ x(0) = x(1) = 0$.

6. (a) In the phase plane take the arc $\overline{\Lambda}_c$ of equation $\frac{1}{2}y^2 + x^4 = c$ in the first quadrant between $x = 0$ and $x = 1$.
(b) Consider the arc $\widehat{\Lambda}_c$ in the fourth quadrant.

8. $b = \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

9. $G(t,s) = \begin{cases} t(1-s), & \text{if } t \in [0,s] \\ s(1-t), & \text{if } t \in [s,1] \end{cases}; \quad x(t) = \frac{1}{2}t^2 - \frac{1}{2}t.$

11. $$G(t,s) = \begin{cases} \frac{-1}{k\sinh(k)} \cdot \sinh(kt) \cdot \sinh[k(s-1)], & \text{if } t \in [0,s] \\ \frac{-1}{k\sinh(k)} \cdot \sinh(ks) \cdot \sinh[k(t-1)], & \text{if } t \in [s,1] \end{cases}$$

12. $v \equiv 0$ is a subsolution and $w \equiv 1$ is a supersolution.

14. $v \equiv 0$ is a subsolution and $w \equiv 1$ is a supersolution.

16. $v = 0$ is a subsolution and $w = M$ is a supersolution. Positiveness follows by contradiction.

18. $0 < \min\{e^x, 1\} \leq 1$.

20. Write $\arctan x = x - g(x)$ with $g(x) = x - \arctan x$ and apply Theorem 13.4.1 of Chapter 13 with $\lambda = 1$.

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