Intersection multiplicity

The purpose of intersection multiplicity is to generalise the notion of the multiplicity of a zero of a polynomial.

Definition. Let V be an irreducible affine variety and let $P \in V$. The *local ring of V at P* is $\mathcal{O}_{V,P} = \{\phi \in K(V) \mid \phi \text{ is defined at } P\}$. $\mathcal{O}_{V,P}$ consists of the rational functions which can be written as $\phi = f/g$ with $g(P) \neq 0$.

The function $\varepsilon_P : \mathcal{O}_{V,P} \to K$, $\phi \mapsto \phi(P)$ is a ring homomorphism. It is surjective, therefore its kernel $m_{V,P} = \{\phi \in \mathcal{O}_{V,P} \mid \phi(P) = 0\}$ is a maximal ideal. Moreover, any element of $\mathcal{O}_{V,P} \setminus m_{V,P}$ is invertible, so any proper ideal of $\mathcal{O}_{V,P}$ must be contained in $m_{V,P}$, therefore $m_{V,P}$ is the unique maximal ideal of $\mathcal{O}_{V,P}$.

Definition. Let f, $g \in K[x, y]$ and let $P \in \mathbb{A}^2$. The *intersection multiplicity of f, g at P* is $I_P(f, g) = \dim \mathcal{O}_{\mathbb{A}^2,P}/\langle f, g\rangle$.
If C and D are curves in $\mathbb{A}^2$, then let $I(C) = \langle f\rangle$ and $I(D) = \langle g\rangle$, and the *intersection multiplicity of C and D at P* is $I_P(C, D) = I_P(f, g)$.
For homogeneous polynomials in $K[X, Y, Z]$ or for curves C and D in $\mathbb{P}^2$, their intersection multiplicity at $P \in \mathbb{P}^2$ is defined to be the intersection multiplicity of their dehomogenisations or of the corresponding affine curves in an affine piece containing P.

Properties. 1. $I_P(f, g)$ only depends on the ideal $\langle f, g\rangle \triangleleft \mathcal{O}_{\mathbb{A}^2,P}$.
2. $I_P(f, g)$ is finite if and only if f and g have no common factor vanishing at P.
3. $I_P(f, g) = 0$ if and only if $f(P) \neq 0$ or $g(P) \neq 0$.
4. $I_P(f, g) = 1$ if and only if neither f, nor g has a repeated factor vanishing at P, P is a non-singular point of the curves $f = 0$ and $g = 0$, and the tangent lines to these curves at P are distinct.
5. If neither f, nor g has a repeated factor vanishing at P, which is the case when calculating intersection multiplicities of curves, $1 < I_P(f, g) < \infty$ if and only if $f(P) = g(P) = 0$, and either at least one of the curves $f = 0$, $g = 0$ is singular at P or they are both non-singular and have the same tangent line at P.
6. $I_P(f_1 f_2, g) = I_P(f_1, g) + I_P(f_2, g)$, this property can be used in combination with the previous ones to calculate $I_P(f, g)$.