# Intersection Theory course notes

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## 1. Lectures 1-2: examples and tools

1.1. **Motivation.** Intersection theory had been developed in order to give a rigorous foundation for methods of enumerative geometry. Here is a typical question considered in enumerative geometry.

*How many lines in 3-space intersect 4 given lines in general position?*

Here is Schubert's solution. Choose 4 lines $l_1$, $l_2$, $l_3$, $l_4$, so that $l_1$ and $l_2$ lie in the same plane, and so do $l_3$ and $l_4$. It is easy to check that in this case there are exactly two lines intersecting all 4 lines, namely, the line passing through the intersection points $l_1 \cap l_2$ and $l_3 \cap l_4$ and the intersection line of planes that contain $l_1$, $l_2$ and $l_3$, $l_4$. Then by "conservation of number principle" the number of solutions in the general case is also two (see Section 9 for more rigorous applications of the conservation of number principle).

To solve problems in enumerative geometry, Schubert developed calculus of *conditions* (the original German word for condition is "Bedingung"). It is now called Schubert calculus. An example of *condition* is the condition that a line in 3-space intersects a given line. Two conditions can be added and multiplied. For instance, denote by $\sigma_i$ the condition that a line intersects a given line $l_i$. Then $\sigma_1 + \sigma_2$ is the condition that a line intersects either $l_1$ or $l_2$ , and $\sigma_1 \cdot \sigma_2$ is the condition that a line intersects both $l_1$ and $l_2$. Then the above question can be reformulated as follows: find the product $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$ of four conditions.

We will discuss interpretation of Schubert calculus via intersection theory on Grassmannians in Section 4.2. For instance, we will see that each condition $\sigma_i$ defines a hypersurface in the variety of lines in $\mathbb{P}^3$ (=Grassmannian $G(2,4)$ of planes passing through the origin in $\mathbb{C}^4$), and $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$ is the intersection index (or the number of common points) of four hypersurfaces.

In general, problems of enumerative geometry can be rigourously solved using intersection theory on suitable varieties. Here is another example.

*How many smooth conics intersect 5 given conics in general position?*

This problem was posed by Steiner. He also gave an incorrect answer: 7776 (=$6^5$). The first correct solution was found by Chasles. There are 3264 such conics. A rigorous solution involves intersection theory on the *space of complete conics* and will be discussed in Section 6.

Problems on the number of common zeroes of $n$ polynomials $f_1,\ldots,$ $f_n$ in $n$ variables can also be studied using intersection theory. For instance, the classical Bézout theorem answers the following question.

*How many solutions does a generic system* $f_1 = \ldots = f_n = 0$ *have?*

The answer is $\deg f_1 \cdot \ldots \cdot \deg f_n$ and can be obtained using intersection theory on $\mathbb{P}^n$.

There are more general versions of Bézout theorem proved by Koushnirenko (see Theorem 2.2), Bernstein and Khovanskii. They involve intersection theory on *toric varieties.*

1.2. **Goals.** Let $X$ be an algebraic variety over an algebraically closed field $\mathbb{K}$, and $M$ and $N$ two algebraic subvarieties in $X$ of complementary dimensions (i.e. $\dim M + \dim N = \dim X$). In all our examples, X will be an affine or projective variety over the field $\mathbb{C}$ of complex numbers.

*Reminder.* Recall that there is *Zariski topology* on the affine space $\mathbb{K}^n$: a subset $X \subset \mathbb{K}^n$ is *closed* if

$$X = \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f_1(x_1, \ldots, x_n) = \ldots = f_k(x_1, \ldots, x_n) = 0\},$$

where $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_n]$ are polynomials on $\mathbb{K}^n$.

**Exercise 1.1.** *Show that if $\mathbb{K} = \mathbb{C}$ then a Zariski closed subset is closed with respect to the usual topology on $\mathbb{C}^n$. The converse is not true.*

Recall that there is a bijective correspondence between affine varieties $X \subset \mathbb{K}^n$ and prime ideals $I_X \subset \mathbb{K}[x_1, \ldots, x_n]$. Namely, $X = \{x \in \mathbb{K}^n \mid f(x) = 0\ \forall f \in I_X\}$. Similarly, there is a bijective correspondence between projective varieties $X \subset \mathbb{P}^n$ and homogeneous prime ideals

$$\mathcal{I}_X = \bigoplus_{k=0}^{\infty} \left(\mathcal{I}_X \cap \mathbb{K}^{(k)}[x_0, x_1, \ldots, x_n]\right),$$

where $\mathbb{K}^{(k)}[x_0, x_1, \ldots, x_n]$ denotes the space of homogeneous polynomials of degree $k$.

Note that the value of a homogeneous polynomial $f$ at a point $x \in \mathbb{P}^n$ is not well-defined, since $x = (x_0 : \ldots : x_n) = (\lambda x_0 : \ldots : \lambda x_n)$ for any nonzero $\lambda \in \mathbb{K}$. However, we can say whether $f$ vanishes at $x$ or not. All homogeneous polynomials in $\mathcal{I}_X$ are required to vanish on $X$.

For more details on affine and projective varieties see [17].

Our first goal is to define the *intersection index* $M \cdot N$ of $M$ and $N$. We will assign to each pair $(M, N)$ an integer number $M \cdot N$ satisfying the "conservation of number principle", that is, if we move subvarieties $M$ and $N$ inside $X$ then their intersection index does not change. We will formulate this principle explicitly for some interesting examples and see why it is useful.

Let us first consider a naive definition of the intersection index, namely, set $M \cdot N$ to be the number of points $|M \cap N|$ in the intersection of $M$ and $N$. The following example illustrates what is wrong with this definition and how it can be improved.

Take an affine plane $X = \mathbb{C}^2$ with coordinates $x$ and $y$, and let $M = \{f(x, y) = 0\}$ and $N = \{g(x, y) = 0\}$ be two curves in $X$. Consider four cases. In the first three cases, $M$ is a fixed parabola and $N$ is a line. Let us translate and rotate $N$ continuously and see how the number $|M \cap N|$ changes.

(1) $f = x^2 - y$, $g = y - 2$; then $|M \cap N| = 2$

(2) $f = x^2 - y$, $g = y$; then $|M \cap N| = 1$ so the conservation of number principle fails. However, the intersection point $(0, 0)$ is a point of tangency of curves $M$ and $N$, so this point should be counted with multiplicity two.
(3) $f = x^2 - y$, $g = x$; then $|M \cap N| = 1$, because the other intersection point went to infinity. So to preserve the intersection index we need to find a way to count intersection points at infinity or consider only compact $X$. For instance, if we take $\mathbb{CP}^2$ instead of $\mathbb{C}^2$ we recover the missing intersection point. Indeed, the point (0:1:0) (in homogeneous coordinates $(x : y : z)$) satisfies both $x = 0$ and $x^2 - yz = 0$ (homogenization of $f$)
(4) $f = x^2 - 1$, $g = x - 1$; then $|M \cap N| = \infty$ since $M \cap N = \{x - 1 = 0\}$ is a line. However, if we rotate $N$ a little bit we again get exactly two intersection points.

Cases 2 and 4 suggest to replace the subvariety $N$ with a family $\{N_t\}$ of subvarieties parameterized by a parameter $t$ so that $|M \cap N_t|$ is the same for generic $t$. More precisely, $t$ runs through points of an affine or projective variety $T$, and $t$ is *generic* if $t$ belongs to a Zariski open and dense subset of $T$. We will give a definition of the intersection index using this geometric approach whenever possible. In this particular example, $T$ is the variety of all lines in $\mathbb{P}^2$ (that is, also $\mathbb{P}^2$). A point $t \in T$ is generic if the corresponding line $N_t$ is not tangent to $M$.

**Exercise 1.2.** *Show that all lines tangent to $M$ form an algebraic curve in $T = \mathbb{P}^2$. Hence, the complement is Zariski open and dense.*

There is also an algebraic definition of multiplicity of an intersection point $p \in M \cap N$. Take the *local ring* $\mathcal{O}_p$ (the ring of all rational functions on $\mathbb{C}^2$ that do not have a pole at $p$) and quotient it by the ideal $(f, g)$. We get a finite-dimensional complex space. The *multiplicity* of $p$ is the dimension of $\mathcal{O}_p/(f, g)$. For instance, in case 3 we have

$$\mathcal{O}_{(0,0)}/(f, g) = \mathbb{C}[[x, y]]/(x, x^2 - y) = \langle 1 \rangle = \mathbb{C},$$

so the multiplicity is one. This agrees with the fact that the intersection is transverse at $p = (0, 0)$. In case 3, we have

$$\mathbb{C}[[x, y]]/(y, x^2 - y) = \langle 1, x \rangle = \mathbb{C}^2,$$

so the multiplicity is two. The multiplicity rises because curves are tangent at the origin.

**Exercise 1.3.** *Let $f = x^2 - y$ and $g = xy$. Compute both geometrically and algebraically the multiplicity of the intersection point* $(0, 0)$*.*

1.3. **Tools.** A basic notion of intersection theory is the *degree* of a projective variety. Let $X^d \subset \mathbb{P}^n$ be a subvariety of dimension $d$ in a projective space. We say that a subspace $\mathbb{P}^{n-d}$ of codimension $d$ is *generic* with respect to $X$ if it intersects $X$ transversally. Generic subspaces with respect to $X$ always exist and form a Zariski open dense set in the space of all subspaces. This follows from Bertini's theorem, which is an algebro-geometric analog of Sard's lemma (e.g. see [17, Section 2.6, Theorem 2]).

The *degree* of $X$ is the number of intersection points with a generic subspace:

$$\deg X = |X \cap \mathbb{P}^{n-d}|.$$

Note that an analogous definition makes sense for an affine subvariety in $\mathbb{A}^d \subset \mathbb{P}^d$, since for almost all generic subspaces all intersection points $X \cap \mathbb{P}^{d-n}$ lie in $\mathbb{A}^d$.

*Remark* 1.4. Let $H = X \cap \mathbb{P}^{n-1}$ be a *hyperplane section* of $X \subset \mathbb{P}^n$, that is, the intersection of $X$ with a hyperplane. Then $\deg X$ can be thought of as the self-intersection index of $H$. Indeed, all hyperplane sections form a family parameterized by points of $(\mathbb{P}^{n-1})^*$ (=hyperplanes in $\mathbb{P}^n$), and $d$ generic hyperplane sections from this family intersect transversally in $\deg X$ points.

Here are examples when degree is easy to compute.

*Example* 1.5.
(1) **Hypersurface.** Let $X = \{f = 0\}$ be a zero set of a homogeneous polynomial $f$ in $\mathbb{P}^k$. Then $\deg X = \deg f$.
(2) **Veronese embedding of $\mathbb{P}^1$.** Embed $\mathbb{P}^1$ into $\mathbb{P}^n$ by sending a point $(x_0 : x_1)$ to the collection $(x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d)$ of all monomials of degree $d$ in $x_0$ and $x_1$. Let $X \subset \mathbb{P}^d$ be the image of $\mathbb{P}^1$ under this embedding. Then $\deg X = d$ by the Fundamental Theorem of Algebra.
(3) **Veronese embedding of $\mathbb{P}^2$.** Embed $\mathbb{P}^2$ to $\mathbb{P}^5$ by mapping a point $(x_0 : x_1 : x_2)$ to the collection $(x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_1x_2 : x_0x_2)$ of all monomials of degree 2 in $x_0$, $x_1$ and $x_2$. Let $X \subset \mathbb{P}^5$ be the image of $\mathbb{P}^2$ under this embedding. Then the degree of $X$ is equal to the number of common zeroes of two generic quadratic polynomials in $\mathbb{P}^2$, that is, $\deg(X) = 4$ (two generic conics intersect at 4 points).

The following theorem allows one to compute the degree of a projective variety in many cases. Define the *Hilbert function* $h_X : \mathbb{N} \to \mathbb{N}$ of a projective variety $X \subset \mathbb{P}^n$ by setting $h_X(k)$ to be the number of linearly independent homogeneous polynomials of degree $k$ restricted to $X$. More formally,

$$h_X(k) := \dim \left(\mathbb{K}^{(k)}[x_0, \ldots, x_n]/(\mathcal{I}_X \cap \mathbb{K}^{(k)}[x_0, \ldots, x_n])\right)$$

is the dimension of the $k$-th component of the homogeneous coordinate ring $\mathbb{K}[x_0, \ldots, x_n]/\mathcal{I}_X$ of $X$ (here $\mathcal{I}_X$ is the homogeneous ideal defining $X$).

**Theorem 1.6** (Hilbert)**.** *The Hilbert function is asymptotically equal to a monomial in $k$:*

$$h_X(k) \sim ak^d \quad \textit{for } k \gg 0,$$

*where $d = \dim X$ and*

$$a = \frac{\deg X}{d!}.$$

Sometimes, this theorem is used as a definition of the degree and dimension of $X$.

**Exercise 1.7.** *Verify Hilbert's theorem for a hypersurface and for the Veronese embeddings of Example 1.5.*

*Remark* 1.8. We gave an abridged version of Hilbert's theorem. More is true: $h_X(k)$ not only grows as a polynomial but also coincides with a polynomial (called the *Hilbert polynomial* of $X$) for large enough values of $k$.

## 2. Lectures 3-4: Kushnirenko's theorem, divisors, Picard group

2.1. **Application of Hilbert's theorem to Kushnirenko's theorem.** Hilbert's theorem can be used to prove the following generalization of the Bézout theorem. Instead of the degree of a polynomial we need a finer combinatorial-geometric invariant called *Newton polytope.*

First, recall few definitions. A *Laurent polynomial* in $n$ variables $x_1,\ldots,\ x_n$ is a finite linear combination of *Laurent monomials* $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ where $k_1,\ldots,\ k_n$ are (possibly negative) integers. Assign to each Laurent monomial $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ its exponent $(k_1,\ldots,k_n)$ that can be regarded as a point in the integral lattice $\mathbb{Z}^n\subset\mathbb{R}^n$. For a Laurent polynomial $f$, define its *Newton polytope* $P_f\subset\mathbb{R}^n$ as the convex hull of all the exponents of the Laurent monomials occuring in $f$.

*Example* 2.1. Take $n=2$. Consider the Laurent polynomial $f=x_1+x_2+x_1^{-1}x_2^{-1}$. Its Newton polytope is the triangle with the vertices $(1,0)$, $(0,1)$ and $(-1,-1)$.

Note that the values of Laurent polynomials are defined if $x_1,\ \ldots,\ x_n$ do not vanish. So each Laurent polynomial is a regular function on the *complex torus* $(\mathbb{C}^*)^n=\{(x_1,\ldots,x_n)\mid x_i\neq 0\}\subset\mathbb{C}^n$.

**Theorem 2.2** (Kushnirenko's theorem)**.** *Fix a polytope $P\subset\mathbb{R}^n$ with integral vertices. Consider a generic collection of $n$ Laurent polynomials whose Newton polytope is $P$. Then the number of their common zeroes that lie inside $(\mathbb{C}^*)^n$ is equal to $n!$ times the volume of $P$.*

Consider an example for $n=2$.

*Example* 2.3. Let $P$ be the triangle with the vertices $(0,0)$, $(n,0)$ and $(0,n)$. Its area is equal to $n^2/2$. A generic Laurent polynomial with the Newton polygon $P$ is just a generic usual polynomial in two variables (since there are no negative exponents) of degree $n$. By homogenizing we get a generic polynomial on $\mathbb{CP}^2$. Note that the common zeroes in $\mathbb{CP}^2$ of two such polynomials will all lie in $(\mathbb{C}^*)^2\subset\mathbb{CP}^2$. Thus we get a partial case of the Bezout theorem for the projective plane $\mathbb{CP}^2$ and polynomials of equal degrees.

Kushnirenko's theorem can be proved using intersection theory on toric varieties. A *toric variety* of dimension $n$ is an algebraic variety with an action of a complex torus $(\mathbb{C}^*)^n$ that has an open dense orbit (isomorphic to $(\mathbb{C}^*)^n$). In particular, a projective toric variety can be viewed as a compactification of a complex torus.

For every Newton polytope $P$, we now construct a projective toric variety $X_P$ whose degree is given by Kushnirenko's theorem. Let $a_1, \dots, a_N$ be all integer points inside and at the boundary of the polytope $P$. Define the embedding

$$\varphi_P : (\mathbb{C}^*)^n \to \mathbb{P}^{N-1}; \quad \varphi_P : x \mapsto (x^{a_1} : \dots : x^{a_N}).$$

We use multiindex notation, that is, $x^a$ means $x_1^{k_1} \dots x^{k_n}$ for every $a = (k_1, \dots, k_n) \in \mathbb{Z}^n$. Define the variety $X_P$ as the Zariski closure of $\varphi((\mathbb{C}^*)^n)$.

**Exercise 2.4.** *Apply this construction to the triangle with the vertices* $(0,0)$, $(n,0)$, $(0,n)$ *and to the square with the vertices* $(0,0)$, $(n,0)$, $(0,n)$, $(n,n)$. *Show that the resulting projective varieties are* $\mathbb{P}^2$ *and* $\mathbb{P}^1 \times \mathbb{P}^1$, *respectively, embedded using Veronese and Segre embeddings.*

*Example* 2.5. Let $P \subset \mathbb{R}^2$ be the trapezium with the vertices $(1,0)$, $(2,0)$, $(0,1)$ and $(0,2)$. There are 5 integer points in $P$: vertices and the point $(1,1)$. Hence,

$$\varphi_P : (\mathbb{C}^*)^2 \to \mathbb{P}^4; \quad \varphi_P : (x_1, x_2) \mapsto (x_1 : x_1^2 : x_2 : x_2^2 : x_1 x_2).$$

The variety $X_P$ coincides with the *blow-up* $\hat{\mathbb{P}}^2_O$ of the projective plane at the point $O = (1 : 0 : 0)$. Recall that $\hat{\mathbb{P}}^2_O$ can be defined as the subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ where the second factor is identified with the variety of lines passing through $O$ in $\mathbb{P}^2$. Namely,

$$\hat{\mathbb{P}}^2_O := \{(a, l) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid a \in l\}.$$

Note that the natural projection

$$\pi : \hat{\mathbb{P}}^2_O \to \mathbb{P}^2; \quad (a, l) \mapsto a$$

is one to one unless $a = O$ while $\pi^{-1}(O) \simeq \mathbb{P}^1$ (the curve $\pi^{-1}(O)$ is called the *exceptional divisor*).

**Exercise 2.6.** *Consider the embedding*

$$(\mathbb{C}^*)^2 \subset \hat{\mathbb{P}}^2_O; \quad (x_1, x_2) \mapsto p^{-1}(1 : x_1 : x_2).$$

*Check that the Segre embedding* $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ *restricted to* $(\mathbb{C}^*)^2 \subset \hat{\mathbb{P}}^2_O \subset \mathbb{P}^2 \times \mathbb{P}^1$ *coinsides with the map* $\varphi_P$ *up to a linear change of coordinates in* $\mathbb{P}^5$.

It is easy to reformulate Kushnirenko's theorem in terms of the toric variety $X_P$.

**Theorem 2.7** (Kushnirenko's theorem)**.** *The degree of the projective toric variety* $X_P$ *is equal to* $n!$ *times the volume of* $P$.

Applying Hilbert's theorem to the variety $X_P$, one can prove Kushnirenko's theorem. The main steps of the proof are as follows.

**Exercise 2.8.** *Show that* $h_{X_P}(k)$ *is equal to the number of integer points inside and at the boundary of the polytope* $kP$. *In other words, the Hilbert function of the projective toric variety* $X_P$ *coincides with the Ehrhart function of the polytope* $P$.

This exercise combined with an elementary convex-geometric fact stated below allows one to relate the asymptotic behavior of the Hilbert function of $X_P$ to the volume of $P$.

**Exercise 2.9.** *Show that the number of integer points inside $kP$ is asymptotically equal to $k^n$ times the volume of $P$.*

Similarly to the Bézout theoerem, Kushnirenko's theorem can be extended to Laurent polynomials with different Newton polytopes (Bernstein's theorem). The volume gets replaced by *mixed volume* (see [1]).

2.2. **Divisors and line bundles.** So far we defined the self-intersection index of a hyperplane section (and even computed it in many cases using Hilbert's theorem). We used that hyperplane sections of a variety $X$ that come from the same projective embedding $X^d \hookrightarrow \mathbb{P}^n$ form a big family of "equivalent" hypersurfaces (parameterized by hyperplanes in $\mathbb{P}^n$). Moreover, one can choose $d$ transverse hyperplane sections from this family. However, there are hypersurfaces that can not be realized as hyperplane sections and, moreover, that do not have any "equivalent" transverse hypersurfaces.

*Example* 2.10. Let $X = \hat{\mathbb{P}}^2_O$ be the blow-up of the projective plane at the point $O = (1 : 0 : 0)$, and $E \subset X$ the exceptional divisor (see Example 2.5). Then there are no complex curves in $X$ that lie in a small neighborhood of $E$ (with respect to a Hermitian metric on $X$) and differ from $E$. Indeed, if there were such a curve then it would intersect $E$ in a finite number of points with nonnegative multiplicities. This would imply that the self-intersection index of $E$ is nonnegative. However, we will soon see that $E^2 = -1$.

This example indicates that apart from hypersurfaces we should consider more general objects, namely, *divisors* (=formal linear combinations of hypersurfaces with integer coefficients) together with a suitable equivalence relation. These notions arise naturally in the context of *line bundles*. For more details on the connection between line bundles and divisors see Section 3.1 and [7], Section *Divisors and line bundles*.

**Definition 2.11.** *Let $X$ be an algebraic variety. A* Weil divisor *$D$ on $X$ is a formal finite linear combination*

$$\sum_i k_i H_i,$$

*where $H_i$ is an irreducible algebraic hypersurface in $X$, and $k_i$ is an integer. The hypersurface $|D| = \cup_i H_i$ is called the* support *of $D$.*

Let $f$ be a rational function on $X$, and $H \subset X$ a hypersurface locally defined by the equation $g = 0$, where $g$ is a regular function in the neighborhood of a point $x \in H$. Define the *order* $\mathrm{ord}_H f$ of $f$ along the hypersurface $H$ to be the maximal integer $k$ such that there is a decomposition $f = g^k h$ for some function $h$ that is regular near $x$. It is easy to check that the order does not depend on the choice of the point $x \in H$. Define the *divisor* $(f)$ of the function $f$ by the formula

$$(f) = \sum_H \mathrm{ord}_H f,$$

where the sum is taken over hypersurfaces for which $\mathrm{ord}_H f \neq 0$ (there are only finitely many of them). Such divisors are called *principal divisors*.

**Definition 2.12.** *A* Weil divisor *$D$ on $X$ is called* Cartier divisor *if it is locally principal, i.e. there is a covering of $X$ by subvarieties $U_\alpha$ such that $D \cap U_\alpha$ is principal on $U_\alpha$.*

*Remark* 2.13. If $X$ is smooth, then every Weil divisor is Cartier. For non-smooth varieties these two notions may be different. E.g. if $X = \{xy = z^2\} \subset \mathbb{C}^3$ is a cone, then the Weil divisor $D = \{x = z = 0\} \subset X$ is not the divisor of any rational function in the neighborhood of the origin (though $2D$ is).

Weil divisors form an abelian group, and Cartier divisors form a subgroup in this group which includes the subgroup of principal divisors. Define the *Picard group* $\mathrm{Pic}(X)$ of $X$ as the quotient group of all divisors modulo principal divisors. Two divisors are *linearly* (or *rationally*) equivalent if their difference is a principal divisor, i.e. they represent the same class in the Picard group. One can notice that definition of Cartier divisors is very similar to the definition of a line bundle over $X$ (see Definition 3.3 and Exercise 3.5). Comparing these definitions it is not hard to show that $\mathrm{Pic}(X)$ is isomorphic to the group of isomorphism classes of all line bundles on $X$ (the operation is tensor product).

*Example* 2.14. (1) Let $X$ be a compact smooth curve. Then a divisor $D$ is a linear combination of points in $X$:

$$D = \sum_{a_i \in X} k_i a_i.$$

Define the *degree* $\deg D$ of $D$ as the sum $\sum_i k_i$. The divisor of a function $f$ on $X$ is the sum of all zeroes of $f$ counted with multiplicities minus the sum of all poles of $f$ counted with their orders:

$$(f) = \sum_{a_i \in f^{-1}(0)} (\mathrm{mult}_{a_i} f) a_i - \sum_{b_i \in f^{-1}(\infty)} (\mathrm{ord}_{b_i} f) b_i.$$

Note that the degree of a principal divisor is always zero. This follows from Lemma 9.3.

- If $X = \mathbb{P}^1$ then every degree zero divisor is principal. Indeed, every point $a \in \mathbb{P}^1$ is equivalent to any other point, because $a - b$ is the divisor of a fractional linear function $(x - a)/(x - b)$. Hence, $\mathrm{Pic}(\mathbb{P}^1)$ is isomorphic to $\mathbb{Z}$. The isomorphism sends a divisor to the degree of the divisor.

- If $X = E$ is an elliptic curve, i.e. a curve of genus one (one can think of a generic cubic plane curve). Then not all degree zero divisors are principal. E.g. for any two distinct points $a$ and $b$ in $E$, the divisor $a-b$ is not principal. Indeed, if it were principal, i.e. $a-b=(f)$ for some $f : E \to \mathbb{P}^1$ then $f$ would be a one-to-one holomorphic map (this again follows from Lemma 9.3). But elliptic curve is not homeomorphic to projective line since their genera are different.

  What is true for elliptic curve is that for every three points $a$, $b$ and $O$ there exists a unique point $c$ such that the divisor $a + b - c - O$ is principal (it is easy to show this using that each elliptic curve is isomorphic to some cubic plane curve in $\mathbb{P}^2$). This allows to define the addition on $E$ by fixing $O$ (zero element) and putting $a + b = c$. This turns $E$ into an Abelian group. In fact, as a complex manifold $E$ is isomorphic to $\mathbb{C}/\mathbb{Z}^2$ for some integral lattice $\mathbb{Z}^2 \subset \mathbb{C}$, and this isomorphism is also a group isomorphism.

  It follows that $E$ is isomorphic to the subgroup $\mathrm{Pic}^0(E)$ of degree zero divisors. The isomorphism sends $a \in E$ to the divisor $a - O$. Then $\mathrm{Pic}(E)$ is isomorphic to $E \oplus \mathbb{Z}$.

(2) If $X = \mathbb{C}^n$ is an affine space, then every divisor is principal and $\mathrm{Pic}(\mathbb{C}^n) = 0$. This shows that the notion of Picard group is more suited for study of the intersection indices on compact varieties. For instance, to study intersection indices of hypersurfaces in $\mathbb{C}^n$ one can consider everything in the compactification $\mathbb{CP}^n$. There are also other ways to define intersection theory on non-compact varieties, in particular in $\mathbb{C}^n$. We will discuss them later.

(3) If $X = \mathbb{P}^n$ is a projective space then $\mathrm{Pic}(X) = \mathbb{Z}$. Indeed, if $D$ is a hypersurface given by the equation $\{f = 0\}$, then it is linearly equivalent to the degree of $f$ times the class of hyperplane $H = \{x_0 = 0\}$ in $\mathbb{P}^n$ (since $(D - \deg f \cdot H)$ is the divisor of the rational function $f/x_0^{\deg f}$).

(4) Let $X \subset \mathbb{P}^n$. Then all hyperplane sections of $X$ are linearly equivalent divisors. Their class in the Picard group of $X$ is called a *divisor of hyperplane section*.

**Exercise 2.15.** *Show that a divisor of hyperplane section is Cartier.*

**Definition 2.16.** *A divisor on $X$ is called* very ample *if it is linearly equivalent to the divisor of hyperplane section for an embedding $X \hookrightarrow \mathbb{P}^n$.*

Let $d$ denote the dimension of a projective variety $X$. We now define a symmetric $d$-linear map

$$(\mathrm{Pic}(X))^d \to \mathbb{Z}, \quad (D_1, \ldots, D_d) \to D_1 \cdot \ldots \cdot D_d,$$

whose values yield the intersection indices of divisors on $X$. In particular, we want $D_1 \cdot \ldots \cdot D_d$ be equal to $|D_1 \cap \ldots \cap D_d|$ whenever $D_1, \ldots, D_n$ are honest hypersurfaces that intersect transversally. The classes of very ample divisors form a semigroup in the Picard group of $X$ (i.e. the sum of two very ample divisors is also very ample), and this semigroup generates $\mathrm{Pic}(X)$ (see [7], Subsection 1.4, Corollary from Kodaira

Embedding Theorem). Very ample divisors can be freely moved into transversal position, hence, we can define their intersection indices simply by counting intersection points. Since any divisor is the difference of two very ample divisors we can extend the definition of the intersection index by linearity to all divisors.

A useful fact from linear algebra is that any $\mathbb{Z}$-valued symmetric $d$-linear map is uniquely defined by the restriction to the diagonal $\{(D,\dots,D), D \in \mathrm{Pic}(X)\}$ (this is usually formulated for vector spaces but is true for abelian groups as well). So to compute the intersection index of any collection of divisors on $X$ it is enough to compute the self-intersection index of every divisor. Moreover, it is enough to compute the self-intersection index of every very ample divisor on $X$ (that is, the degree of $X$ in different projective embedding). Hence, these computations can be in principle reduced to Hilbert's theorem.

*Example* 2.17.

(1) **Projective spaces.** Let $H$ be the divisor of a hyperplane in $\mathbb{P}^n$. Then $H^n = \deg(\mathbb{P}^n) = 1$. For any $D \in \mathrm{Pic}(\mathbb{P}^n)$ we have $D = kH$ for $k \in \mathbb{Z}$, hence, $D^n = k^n$ by $n$-linearity of the intersection index. It is easy to see that the unique symmetric $n$-linear form $\mathbb{Z}^n \to \mathbb{Z}$ (*polarization* of $k^n$) in this case is $(k_1,\dots,k_n) \mapsto k_1 \cdots k_n$. Together with Example 2.14(3) this proves Bézout theorem.

(2) **Blow-up of the projective plane.** Consider again the blow-up $X = \hat{\mathbb{P}}^2_O$ and the exceptional divisor $E \subset X$ (see Example 2.5). Let us compute $E^2$. Consider the function $f = x_1/x_0$ on $\mathbb{P}^2$. Then the composition $f\pi$ (where $\pi : X \to \mathbb{P}^2$ is the projection) is a function on $X$. It is easy to see that the divisor of $f\pi$ is equal to $E + H_1 - H_0$, where $H_0$ and $H_1$ are the pull-backs to $X$ of the hyperplanes $\{x_0 = 0\}$ and $\{x_1 = 0\}$, respectively (i.e. $H_i$ is an irreducible hypersurface in $X$ such that $p(H_i) = \{x_i = 0\}$). Hence, $E$ is linearly equivalent to $H_0 - H_1$. Then
$$E^2 = E(H_0 - H_1) = -1$$
since $E$ and $H_0$ do not intersect and $E$ and $H_1$ intersect transversally at one point. We see that the self-intersection index of a hypersurface might be negative. In particular, it is impossible to find an honest curve $E' \neq E$ that is linearly equivalent to $E$. We will necessarily have a negative component as well.

(3) **Product of projective spaces.** It is easy to show that $\mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z} \oplus \mathbb{Z}$, and the basis is given by the divisors of hyperplane section $H_1$ and $H_2$ corresponding to the projections of $\mathbb{P}^n \times \mathbb{P}^m$ onto the first and second factors, respectively. A straightforward calculation gives that $H_1^i H_2^j = 0$ unless $i = n$ and $j = m$, and $H_1^n H_2^m = 1$. In particular, we get that the self-intersection index of the divisor $D = dH_1 + eH_2$ is equal to $\binom{m+n}{n} d^n e^m$.

## 3. Lectures 5-7: vector bundles, Chow ring, Chern classes

3.1. **Vector bundles.** A connection between line bundles and divisors can be extended to a more general setting: with every vector bundle $E$ on $X$ one can associate a collection of subvarieties in $X$ (Chern classes) considered up to rational equivalence.

Recall the definition of a vector bundle. A variety $X \times \mathbb{C}^r$ is called a *trivial vector bundle* of rank $r$. A variety $E$ together with a projection $\pi : E \to X$ is called a *vector bundle* of rank $r$ over $X$ if there exists a covering of $X$ by subvarieties $U_\alpha$ such that $E$ is trivial over $U_\alpha$. More precisely, there exists an isomorphism $f_\alpha : U_\alpha \times \mathbb{C}^r \simeq \pi^{-1}(U_\alpha)$ such that $\pi_0 \circ f_\alpha = \pi$ (where $\pi_0 : U_\alpha \times \mathbb{C}^r \to U_\alpha$ is the projection to the first factor) and the resulting transition maps

$$g_{\alpha\beta} := f_\beta^{-1} \circ f_\alpha : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to \pi^{-1}(U_\alpha \cap U_\beta) \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

are linear on every vector space $x \times \mathbb{C}^r$, that is, $g_{\alpha\beta}(x) \in GL_r(\mathbb{C})$. In particular, if $E$ is a line bundle (that is, $r = 1$), then $E$ is encoded by a collection of rational functions $g_{\alpha\beta}$ such that $g_{\alpha\beta}$ is regular and non-zero everywhere on $U_\alpha \cap U_\beta$.

The usual operations on vector spaces (such as taking dual, tensor products, quotients, symmetric powers) can be easily extended to the setting of vector bundles.

*Example* 3.1. Very important examples of line bundles are the tautological and quotient tautological line bundles on the projective space $\mathbb{P}^n$. The fiber of the *tautological line bundle* $\mathcal{O}(-1)$ over a point $x \in \mathbb{P}^n$ is the line $l_x \subset \mathbb{C}^{n+1}$ represented by $x$. The trivializing covering is given by standard affine charts $U_i = \{x_i \neq 0\} \subset \mathbb{P}^n$ and the gluing functions $g_{ij} := \frac{x_j}{x_i}$.

If we regard $\mathbb{P}^n$ as the variety of hyperplanes in $(\mathbb{C}^{n+1})^*$ then the same construction yields the tautological bundle $\mathcal{H}$ of rank $n$ embedded into the trivial bundle $\mathcal{O}^{n+1} := \mathbb{P}^n \times \mathbb{C}^{n+1}$ of rank $n+1$. The *tautological quotient line bundle* $\mathcal{O}(1)$ is defined as the quotient vector bundle $\mathcal{O}^{n+1}/\mathcal{H}$.

Two vector bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ are *isomorphic* if there is an isomorphism $\varphi : E_1 \simeq E_2$ such that $\pi_2 \circ \varphi = \pi_1$.

**Exercise 3.2.** *Show that* $\mathcal{O}(1) \simeq \mathcal{O}(-1)^*$.

**Definition 3.3.** *Let $D$ be a Cartier divisor on $X$ (see Definition 2.12). Define the line bundle $\mathcal{O}(D)$ associated with $D$ using the gluing functions* $g_{\alpha\beta} := \frac{f_\alpha}{f_\beta}$.

*Example* 3.4. If $X = \mathbb{P}^n$ and $D$ is a hyperplane, then $\mathcal{O}(D) \simeq \mathcal{O}(1)$.

**Exercise 3.5.** *Check that $\mathcal{O}(D)$ is well-defined and that $\mathcal{O}(D_1) \simeq \mathcal{O}(D_2)$ if and only if $D_1$ and $D_2$ are linearly equivalent.*

Alternatively, vector bundles can be encoded by their sheaves of global sections. A rational map $s : X \to E$ is called a *section* of $E$ if $\pi \circ s = \mathrm{id}_X$. If $s$ is regular everywhere on $X$ then $s$ is called a *global section* of $E$. The space of global sections is denoted $H^0(X, \mathcal{O}(E))$. In particular, it is clear that a vector bundle of rank $r$ is trivial iff it has $r$ global sections $s_1, \ldots, s_r$ such that $s_1(x), \ldots, s_r(x)$ form a basis in $x \times \mathbb{C}^r$ for all $x \in X$.

**Exercise 3.6.** *Find* $H^0(\mathbb{P}^n, \mathcal{O}(k))$*, where* $\mathcal{O}(k) := \mathcal{O}(1)^{\otimes k}$*.*

Chern classes of a vector bundle measure non-triviality of the bundle, and in many important cases can be defined as *degeneracy loci* of global sections. Namely, assume that $E$ has is *globally generated*, that is, there exists a finite-dimensional subspace $\Gamma \subset H^0(X, \mathcal{O}(E))$ such that at every point $x \in X$ the sections in $\Gamma$ span the vector space $\pi^{-1}(x) \subset E$. Let $s_1, \ldots, s_r \in \Gamma$ be a generic collection of sections. Define their *i-th degeneracy locus* (for $i = 1, \ldots, r$) as the subvariety of all points $x \in X$ such that the vectors $s_1(x), \ldots, s_{r-i+1}(x)$ are linearly dependent. In particular, if $E = \mathcal{L}$ is a line bundle, then there is only one degeneracy locus, which is a hypersurface in $X$. The class of this hypersurface in $\mathrm{Pic}(X)$ is called the *first Chern class* of $\mathcal{L}$ and is denoted by $c_1(\mathcal{L})$. One can extend this definition of the first Chern class to all line bundles (not necessarily globally generated) using rational sections and counting multiplicities. Alternatively, one can use that $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$ and extend the definition by linearity from very ample line bundles to all line bundles (using again that any line bundle is the difference of two very ample line bundles).

*Example* 3.7. Let $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}(1)$. Then $\mathcal{O}(1)$ is globally generated by the sections $x_0, x_1, \ldots, x_n$. Let $s = a_0x_0 + \ldots + a_nx_n$ be a linear combination of these sections. The first degeneracy loci of $s$ is the hyperplane $\{a_0x_0 + \ldots + a_nx_n = 0\} \subset \mathbb{P}^n$. Hence, $c_1(\mathcal{O}(1))$ is the class of a hyperplane section in $\mathrm{Pic}(\mathbb{P}^n)$.

**Exercise 3.8.** *Let* $D \in \mathrm{Pic}(X)$ *be a very ample Cartier divisor. Show that* $c_1(\mathcal{O}(D)) = [D]$ *where* $[D] \in \mathrm{Pic}(X)$ *is the class of* $D$ *in the Picard group.*

Put $n = \dim \Gamma$. There is a useful map $\varphi_\Gamma$ from $X$ to the Grassmannian $G(n-r, n)$ of $(n-r)$-subspaces in $\mathbb{C}^n$. Assign to each point $x \in X$ the subspace $\varphi_\Gamma(x)$ of all sections from $\Gamma$ that vanish at $x$. By construction of the map $\varphi_\Gamma$ the vector bundle $E$ coincides with the pull-back of the tautological quotient vector bundle over the Grassmannian $\mathrm{G}(n-r, n)$. Recall that the tautological quotient vector bundle over $\mathrm{G}(n-d, n)$ is the quotient of two bundles. The first one is the trivial vector bundle whose fibers are isomorphic to $\Gamma$, and the second is the tautological vector bundle whose fiber at a point $\Lambda \in \mathrm{G}(N-d, N)$ is isomorphic to the corresponding subspace $\Lambda$ of dimension $n-r$ in $\Gamma$. The map $\varphi_\Gamma$ allows one to relate the Chern classes of $E$ with the Chern classes of the tautological quotient bundles on the Grassmannian.

**Exercise 3.9.** *Let* $X = \mathbb{P}^1$*,* $E = \mathcal{O}(n)$ *and* $\Gamma =$ *the space generated by all degree* $n$ *monomials in* $x_0$*,* $x_1$*. Show that* $\varphi_\Gamma$ *coincides with the Veronese embedding* $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$*.*

3.2. **Chow rings.** So far we only dealt with intersection indices of divisors. This is a partial case of *intersection product* of subvarieties of arbitrary dimension (in particular, their intersection is not necessarily a finite number of points). In order to be able to move divisors we used linear equivalence relation which led us to the notion of the Picard group. Below we use a similar equivalence relation on subvarieties of arbitrary codimension in order to define the Chow group (that contains the Picard group as a subgroup) and introduce the intersection product which turns the Chow group into a ring. For more details see [4].

Let $X$ be an algebraic variety of dimension $d$.

**Definition 3.10.** *Subvarieties $Z$ and $Z'$ in $X$ of codimension $i$ are* rationally *equivalent if there exists a subvariety $W$ of codimension $i-1$ such that $W$ contains both $Z$ and $Z'$ as rationally equivalent divisors.*

Note that the rational equivalence is finer than the homological equivalence, i.e. any two rationally equivalent subvarieties are also homologous (which is easy to show using the definition of rational equivalence).

The *$i$-th Chow group* $CH^i(X)$ consists of all formal integral linear combinations of subvarieties of codimension $i$ in $X$ quotiented by the rational equivalence relation. In particular, the Chow group $CH^1(X)$ coincides with the Picard group of $X$.

*Example* 3.11. (1) Let $X$ be the projective space $\mathbb{CP}^n$. Then $CH^i(X) = \mathbb{Z}$ if $0 \leq i \leq n$.

(2) Let $X$ be an elliptic curve over $\mathbb{C}$. Then $CH^1(X) = \mathrm{Pic}(X) = X \oplus \mathbb{Z}$, while $CH^0 = \mathbb{Z}$.

Let $Y$ and $Z$ be two irreducible subvarieties in $X$ of codimensions $d_1$ and $d_2$, respectively. They define classes in $CH^{d_1}(X)$ and $CH^{d_2}(X)$. It is possible to define the *intersection product* $[Y] \cdot [Z]$ as a class in $CH^{d_1+d_2}(X)$ in such a way that

$$[Y] \cdot [Z] = [Y \cap Z]$$

whenever $Y$ and $Z$ have transverse intersection. In what follows, we will always be able to move $Y$ and $Z$ into transverse position using rational equivalence. The ring

$$CH^*(X) := \bigoplus_{i=0}^{d} CH^i(X)$$

is called the *Chow ring* of $X$. This ring is a graded commutative ring with 1 (namely, $1 = [X]$), and the Picard group coincides with the degree one component of the Chow ring. The highest degree component $CH^d(X)$ is generated by classes of points in $X$, in particular, there is the *degree homomorphism*

$$\deg : CH^d[X] \to \mathbb{Z}, \quad \deg : \sum a_i[p_i] \mapsto \sum a_i.$$

Note that the intersection product of divisors that we defined in Section 2.2 can be defined by composing the degree homorphism with the $d$-linear map:

$$CH^1(X) \times \ldots \times CH^1(X) \to CH^d; \quad (D_1, \ldots, D_d) \mapsto D_1 \cdots D_d.$$

Chow ring is an important example of an *oriented algebraic cohomology theory*, that is, a functor $X \to A^*(X)$ from varieties to graded commutative ring with 1 that satisfies the following properties:

**(1) Pull-back.** For any morphism $f : X \to Y$ there exists a ring homomorphism $f^* : A^*(Y) \to A^*(X)$ (in particular, $A^*(X)$ is an $A^*(Y)$-module under multiplication by $f^*(A^*(Y)) \subset A^*(X)$).

**(2) Push-forward.** For any *projective* morphism $f : X \to Y$ such that $\dim Y = \dim X + k$ there exists a homomorphism of $A^*(Y)$-modules $f_* : A^*(Y) \to A^{*+k}(X)$.

These properties satisfy several axioms (see [16, Definition 1.1.2] for a full list). When computing Chow rings and intersection products we will mostly use the following two axioms.

**Projection formula.** For any projective morphism $f : X \to Y$ and any $\alpha \in A^*(X)$ and $\beta \in A^*(Y)$ we have

$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.$$

**Projective bundle formula.** Let $E \to X$ be a vector bundle of rank $r$ over $X$. Denote by $Y = \mathbb{P}(E^*)$ the variety of hyperplanes of $E$, or equivalently, of lines in $E^*$ (fibers of the natural projection $E \to X$ are isomorphic to $\mathbb{P}^{r-1}$). Consider the tautological quotient line bundle $\mathcal{O}_E(1)$ on $Y$ whose restriction on each fiber of $Y$ over $X$ coincides with $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Define $\xi = s^* s_*(1_Y) \in A^1(Y)$ where $s : Y \to \mathcal{O}_E(1)$. Then $A^*(Y)$ is a free $A^*(X)$-module with basis $(1, \xi, \ldots, \xi^{r-1})$.

For Chow rings, these properties and axioms are verified in [4]. Note that pull-backs and push-forwards for a morphism $f : X \to Y$ in this case can be defined by taking preimages and images, respectively, of subvarieties in $Y$ and in $X$, whenever the subvarieties are in general position with respect to the morphism $f$.

**Exercise 3.12.** *Let $\mathcal{L} \to X$ be a line bundle on $X$, and $s : X \to \mathcal{L}$ a global section, such that $s(X)$ is transverse to $s_0(X)$ (where $s_0 : X \to \mathcal{L}$ denotes a zero section). Show that $s_0^* s_*(1_X) = c_1(\mathcal{L})$.*

*Remark* 3.13. Note that the projective bundle formula gives a meaningful result for Chow rings already in the simplest case $X = pt$. In this case, $Y = \mathbb{P}^{r-1}$, in particular, $CH^r(Y) = 0$, hence, $\xi^r = 0$ by dimension reasons. The projective bundle formula implies the ring isomorphism

$$CH^*(\mathbb{P}^{r-1}) = \mathbb{Z}[\xi]/(\xi^r).$$

The formal variable $\xi$ under this isomorphism gets mapped to the class of a hyperplane section in $\mathbb{P}^n$.

3.3. **Chern classes.** We now assign to every vector bundle $\pi : E \to X$ of rank $r$ the elements $c_1(E) \in CH^1(X)$,..., $c_r(E) \in CH^r(X)$ called the *Chern classes* of $E$. It is convenient to encode the Chern classes by the single (non-homogeneous) element

$$c(E) := 1 + c_1(E) + \ldots + c_r(E) \in CH^*(X)$$

called the *total Chern class* of $X$.

If $E$ is globally generated, define $c_i(E)$ as the $i$-th degeneracy loci of $r$ generic global sections of $E$ (see Section 3.1). In particular, the total Chern class of a trivial vector bundle is 1. To compute the Chern classes of non-globally generated bundles it is often enough to use the *Whitney sum formula.*

**Theorem 3.14** (Whitney sum formula)**.** *If* $0 \to E_1 \to E \to E_2 \to 0$ *is a short exact sequence of vector bundles on* $X$*, then* $c(E) = c(E_1)c(E_2)$*.*

*Example* 3.15. Let us compute the Chern classes of the bundle $\mathcal{H}$ on $\mathbb{P}^n$ from Example 3.1. We have an exact sequence of vector bundles on $P^n$:

$$0 \to \mathcal{H} \to \mathcal{O}^{n+1} \to \mathcal{O}(1) \to 0.$$

By Example 3.7, we have $c(\mathcal{O}(1)) = 1 + \xi$, where $\xi \in CH^1(\mathbb{P}^n)$ is the class of a hyperplane. Hence, by the Whitney sum formula

$$c(\mathcal{H}) = \frac{1}{(1+\xi)} = 1 - \xi + \ldots + (-1)^{n-1}\xi^{n-1}.$$

We get that $c_i(\mathcal{H}) = (-1)^i \xi^i$.

**Exercise 3.16.** *Find the Chern classes of the tautological bundle and quotient bundle on the Grassmannian* $G(2,4)$ *in terms of Schubert cycles.*

The Chern classes are functorial, that is, for any morphism $f : Y \to X$ we have $c_i(f^*E) = f^*c_i(E)$. In fact, Chern classes are uniquely characterized by four properties: vanishing of $c_n(E)$ for $n > r$, functoriality, Whitney sum formula and normalization (that is, $c_1(\mathcal{O}_{\mathbb{P}^n})$= class of a hyperplane).

Note that the Whitney sum formula can be used to make the projective bundle formula more precise. Consider again the variety $\mathbb{P}(E^*)$ of hyperplanes in $E$, and denote by $p$ the natural projection $p : Y \to X$. Denote by $\xi = s^*s_*(1_Y)$ the first Chern class of the tautological quotient line bundle $\mathcal{O}_E(1)$ on $Y$.

**Theorem 3.17** (Projective bundle formula via Chern classes)**.** *There is a ring isomorphism:*

$$CH^*(Y) = CH^*(X)[\xi]/(\sum_{j=0}^{r}(-1)^j c_j(E)\xi^{r-j}).$$

*The isomorphism identifies a polynomial* $b_0 + b_1\xi + \ldots + b_{r-1}\xi^{r-1}$ *in* $CH^*(X)[\xi]$ *with the element* $p^*b_0 + (p^*b_1)\xi + \ldots + (p^*b_{r-1})\xi^{r-1}$ *in* $CH^*(Y)$*.*

*Proof.* The fact that $CH^*(Y)$ splits into the direct sum $p^*CH^*(X) \oplus \xi p^*CH^*(X) \oplus \ldots \oplus \xi^{r-1}p^*CH^*(X)$ follows from the projective bundle formula for Chow rings. It remains to express $\xi^r$ as a linear combination of $1, \ldots, \xi^{r-1}$ with coefficients in $CH^*(X)$.

Let us prove the relation

$$\sum_{j=0}^{r}(-1)^j c_j(p^*E)\xi^{r-j} = 0. \qquad (*)$$

Consider a short exact sequence of vector bundles on $Y$ (generalizing an analogous sequence on $\mathbb{P}^{r-1}$):

$$0 \to \mathcal{H}_E \to p^*E \to \mathcal{O}_E(1) \to 0,$$

where $\mathcal{H}_E$ is the tautological hyperplane bundle on $Y$. By the Whitney sum formula we have that the total Chern class $c(p^*E)$ is equal to the product $c(\mathcal{H}_E)c(\mathcal{O}_E(1))$. Since $c(\mathcal{O}_E(1)) = 1 + \xi$ we have $c(p^*E) = c(\mathcal{H}_E)(1 + \xi)$. We now divide this identity by $(1 + \xi)$ (that is, multiply by $\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j$) and get that $c(\mathcal{H}_E) = c(p^*E)(\sum_{j=0}^{r+\dim X-1}(-1)^j\xi^j)$. In particular,

$$c_r(\mathcal{H}_E) = (-1)^r \sum_{j=0}^{r}(-1)^j c_j(p^*E)\xi^{r-j},$$

so the relation $(*)$ is equivalent to the vanishing of the $r$-th Chern class of the bundle $\mathcal{H}_E$ (which has rank $r-1$). □

It is useful to define the *Chern roots* $x_1,\dots,x_r$ of $E$ as formal variables of degree 1 such that

$$c_i(E) = s_i(x_1, \dots, x_r),$$

where $s_i(x_1,\dots,x_r) = \sum_{1\le k_1<\ldots<k_i\le r} x_{k_1}\cdots x_{k_r}$ is the $i$-th elementary symmetric function. If $E$ is split, that is, $E = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_r$ is a direct sum of line bundles, then $x_i = c_1(\mathcal{L}_i)$. In particular, the total Chern class of the symmetric power $S^nE$ is equal to

$$\prod_{1\le k_1\le\ldots\le k_n\le r}(1 + x_{k_1} + \ldots + x_{k_n})$$

by the Whitney sum formula. Note that the right hand side is symmetric in $x_1,\dots,x_r$, that is, makes sense even if $E$ is not split. Using the *splitting principle* one can show that

$$c(S^nE) = \prod_{1\le k_1\le\ldots\le k_n\le r}(1 + x_{k_1} + \ldots + x_{k_n})$$

for any vector bundle $E$.

Similarly, the splitting principle implies the relation between the Chern classes of the dual vector bundles:

$$c_i(E) = (-1)^i c_i(E^*).$$

## 4. Lecture 8: Chow rings of complete flag varieties and Grassmannians, applications to enumerative geometry

We now apply projective bundle formula in order to describe Chow rings of special varieties that are important for enumerative geometry.

4.1. **Flag varieties.** The *complete flag variety* $\mathbb{F}_n$ is the variety of all complete flags of subspaces in $\mathbb{C}^n$:

$$\mathbb{F}_n = \{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n\},$$

where dim $V^i = i$. More generally, for any sequence $m^\bullet = (m_1, \ldots, m_k)$ such that $0 < m_1 < \ldots < m_k < n$, the *partial flag variety* $\mathbb{F}_{m^\bullet}$ is the variety of all partial flags of the form

$$\{0\} = V^0 \subset V^{m_1} \subset \ldots \subset V^{m_k} \subset V^n = \mathbb{C}^n.$$

*Remark* 4.1. If $m^\bullet = (k)$, then $\mathbb{F}_{m^\bullet} = G(k,n)$ is the Grassmannian of $k$-dimensional subspaces in an $n$-dimensional vector space.

Grassmannians and complete flag varieties are two extreme cases of partial flag varieties. At first glance it might seem that Grassmannians are simpler but we shall see soon that in some aspects complete flag varieties are easier to study.

*Example* 4.2. (1) For $n = 2$, the complete flag variety $\mathbb{F}_2 = \mathbb{CP}^1$ is the projective line.

(2) For $n = 3$, a point in $\mathbb{F}^3$ is a pair $(\Lambda, \Pi)$, where $\Lambda$ is a line in $\mathbb{C}^3$ and $\Pi$ is a plane. The pair must satisfy incidence condition $\Lambda \subset \Pi$. Hence, we can describe $\mathbb{F}_3$ as the following *incidence variety*

$$\mathbb{F}^3 = \{(\Lambda, \Pi) \in \mathbb{CP}^2 \times \mathbb{CP}^2 \mid \Lambda \subset \Pi\}.$$

Here we identified the set of all lines and the set of all planes in $\mathbb{C}^3$ with $\mathbb{CP}^2$. It is easy to check that in homogeneous coordinates $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_3)$ the flag variety in $\mathbb{CP}^2 \times \mathbb{CP}^2$ is given by the equation $x_0y_0 + x_1y_1 + x_2y_2 = 0$. In particular, dim $\mathbb{F}_3 = 3$. Note that this is less than the dimension of the Grassmannian $G(2, 4)$, which is the smallest example of a Grassmannian different from a projective space. The flag variety $\mathbb{F}_3$ will be our main example.

It is more convenient to represent the flag $(\Lambda, \Pi)$ by its projectivization $(\mathbb{P}(\Lambda), \mathbb{P}(\Pi))$, that is by the pair $(a, l)$ where $a$ is a point in $\mathbb{P}^2$ and $l \subset \mathbb{P}^2$ is a line.

So far we have not endowed the flag variety $\mathbb{F}_n$ with a structure of algebraic variety or smooth manifold (except for $\mathbb{F}_3$, which we realized as a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$). There are different approaches to define such a structure on $\mathbb{F}_n$. We will use the fact that $\mathbb{F}_n$ is a homogeneous space under the left action of the algebraic group $G = GL_n(\mathbb{C})$. Each operator $g \in GL_n(\mathbb{C})$ takes a flag $F = (V^1 \subset \ldots \subset V^{n-1})$ to the flag $gF := (gV^1 \subset \ldots \subset gV^{n-1})$. It is easy to check that this action is transitive. Fix a basis $e_1, \ldots, e_n$ in $\mathbb{C}^n$, and denote by $F^0$ the standard flag $F^0 := (\langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e \ldots, e_{n-1}\rangle)$. The stabilizer of $F^0$ in $G$ is the upper-triangular subgroup $B$, which is a closed algebraic subgroup of $G$. Hence, the quotient space $G/B$ has a structure of a smooth algebraic variety. We now identify $\mathbb{F}_n$ with $G/B$. It is easy to show that $\mathbb{F}_n$ is compact and has dimension $n(n-1)/2$.

**Definition 4.3.** Let $G$ be a complex reductive group, and $B \subset G$ a Borel subgroup. The variety $X = G/B$ is called the *generalized complete flag variety* $X = G/B$.

An alternative way to describe the generalized flag variety $G/B$ as a smooth manifold is to replace $G$ with a maximal compact Lie subgroup $K \subset G$. Let $T \subset K$ be a maximal torus (so $T$ is isomorphic to $S^1 \times \ldots \times S^1$). Then it is not hard to show that $G/B$ and $K/T$ (regarded as real differentiable manifolds) are diffeomorphic. This implies in particular that $G/B$ is compact. For instance, in the case $G = GL_n(\mathbb{C})$ we can take $U_n(\mathbb{C})$ as $K$ and the diagonal subgroup of $K$ as $T$. The desired diffeomorphism then follows from the fact that every flag can be realized as a standard flag for some unitary basis (by Gram-Schmidt orthogonalization process).

*Remark* 4.4 ($K/T$ versus $G/B$). Some books and papers (e.g. on algebraic geometry or algebraic groups) define the flag variety as $G/B$ and think of it being an algebraic variety, while others (e.g. on differential or symplectic geometry) rather use $K/T$ and regard it as a real analytic manifold. Note that some constructions allowed by the latter approach can not be translated to algebro-geometric setting (working over reals sometimes gives more freedom than working over complex numbers). Here is one example.

*Example* 4.5. *Recall that the Weyl group of $K$ (which is the same as the Weyl group of $G$) can be identified with $N(T)/T$, where $N(T)$ is the normalizer of the maximal torus $T$, and hence acts on $K/T$ as follows:*

$$n \cdot kT := (kn)T.$$

*This action is smooth but not holomorphic.*

**Exercise 4.6.** *Show that if $G = GL_2(\mathbb{C})$, then the nontrivial element of $N(T)/T \simeq S_2$ acts on $\mathbb{F}_2 \simeq \mathbb{CP}^1$ by complex conjugation $z \to \bar{z}$.*

In our lectures we work in the algebro-geometric setting. So for us the flag variety is $G/B$. Most of the results we will discuss are true for arbitrary generalized complete flag varieties but some are only true for $\mathbb{F}_n$. So we will distinguish between the flag variety for $GL_n$ and the flag variety $G/B$ for arbitrary reductive group $G$. The former will be denoted by $\mathbb{F}_n$ and the latter by $X$.

We now describe the Chow ring of $X$. First, note that the Chow ring $CH^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with rational coefficients is generated multiplicatively by the Picard group $\mathrm{Pic}(X)$. Moreover, if $X = \mathbb{F}_n$, this statement is already true over integers. Note that there are plenty of tautological vector bundles and various their quotients on $\mathbb{F}_n$. We will be particularly interested in the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ defined by the condition that the fiber of $\mathcal{L}_i$ at the flag

$$\{\{0\} = V^0 \subset V^1 \subset \ldots \subset V^{n-1} \subset V^n = \mathbb{C}^n\}$$

is equal to the quotient $V^i/V^{i-1}$.

**Theorem 4.7** (Borel presentation). *Let $X = \mathbb{F}_n$. Then there is an isomorphism of graded rings*

$$CH^*(X) = \mathbb{Z}[x_1, \ldots, x_n]/S,$$

*where* $x_1,\dots,\ x_n$ *have degree one, and* $S$ *is the ideal generated by homogeneous symmetric polynomials of positive degree. The isomorphism sends* $x_i$ *to* $c_1(\mathcal{L}_i)$*.*

*In particular, the Picard group of* $\mathbb{F}^n$ *is generated by the line bundles* $\mathcal{L}_1,\dots,\ \mathcal{L}_n$ *with the single relation* $\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_n = \mathcal{O}$*.*

*Remark* 4.8. One can show that the monomials $x_1^{k_1}\dots x_{n-1}^{k_{n-1}}$ for $k_1 \le n-1$, $k_2 \le n-2,\dots,\ k_{n-1} \le 1$ form a basis in $\mathbb{Z}[x_1,\dots,x_n]/S$. In particular, it is easy to check that $x_n = -x_1 - \dots - x_{n-1} \mod S$ and $x_i^n = 0 \mod S$.

*Example* 4.9. For $n = 3$, Theorem 4.7 gives

$$CH^*(\mathbb{F}_3) \simeq \mathbb{Z}[x_1,x_2,x_3]/(x_1+x_2+x_3, x_1x_2+x_2x_3+x_1x_3, x_1x_2x_3),$$

or if we use that $x_3 = -x_1 - x_2$,

$$CH^*(\mathbb{F}_3) \simeq \mathbb{Z}[x_1,x_2]/(x_1^2+x_1x_2+x_2^2, x_1^3),$$

and a basis in $CH^*(\mathbb{F}_3)$ consists of monomials 1, $x_1$, $x_2$, $x_1^2$, $x_1x_2$, $x_1^2x_2$. In particular, the dimensions $b_i := \dim CH^i(\mathbb{F}_3)$ of Chow groups are the following: $b_0 = b_3 = 1$, $b_1 = b_2 = 2$.

Below we outline a proof of Theorem 4.7 based on the successive applications of the projective bundle formula (see Theorem 3.17). For more details see the proof of [15, Theorem 3.6.15].

Here is a crucial observation: $\mathbb{F}_n$ can be constructed from a point by successive projective bundle constructions.

*Example* 4.10. (1) For $n = 2$, we have $\mathbb{F}_2 = \mathbb{P}(\mathbb{C}^2) \simeq \mathbb{CP}^1$ for a trivial vector bundle $\mathbb{C}^2$ of rank 2 over a point.

(2) For $n = 3$, define $P_1 = \mathbb{P}(\mathbb{C}^{3*})$ for a trivial vector bundle $\mathbb{C}^3$ of rank 3 over a point, that is, $P_1$ is the variety of planes in $\mathbb{C}^3$. Then it is easy to check that $\mathbb{F}_3 = \mathbb{P}(\mathcal{V}^*)$, where $\mathcal{V}$ is the tautological rank two vector bundle over $P_1$ defined as follows. The points in $P_1$ are planes in $\mathbb{C}^3$, and the fiber of $\mathcal{V}$ over such a point is the corresponding plane in $\mathbb{C}^3$. (Note that for any rank two vector bundle $V$ there is a canonical isomorphism $\mathbb{P}(V) \simeq \mathbb{P}(V^*)$ coming from the fact that lines in $\mathbb{C}^2$ are the same as hyperplanes. However, it is not true that $V$ is necessarily isomorphic to $V^*$.)

Similarly, for arbitrary $n$ we define $P_i$ for $0 \le i < n$ as the variety of partial flags in $\mathbb{C}^n$ of the form $(V^{n-i} \subset V^{n-i+1} \subset \dots \subset V^{n-1})$. Then every $P_i$ for $i \ge 1$ is the projective bundle over $P_{i-1}$, namely, $P_i = P_{i-1}(\mathcal{V}^*_{n-i+1})$ is the variety of hyperplanes in $\mathcal{V}_{n-i+1}$. Since $P_0 = pt$ and $P_{n-1} = \mathbb{F}_n$, we get a sequence of projective fibrations

$$\mathbb{F}_n \to P_{n-1} \to \dots \to P_1 \to pt.$$

Note that in the proof of [15, Theorem 3.6.15] the dual sequence of the partial flag varieties is used (e.g. $P_1$ becomes the variety of lines in $\mathbb{C}^n$ not hyperplanes). However, our choice of $P_i$ is more natural since it allows to extend the same proof to arbitrary oriented cohomology theories such as algebraic cobordism (see [6, Theorem

2.7] for more details). We can now apply the projective bundle formula $(n-1)$ times. Every time we recover the Chow ring of $P_i$ from that of $P_{i-1}$ using the Chern classes of $\mathcal{V}_{n-i+1}$ and in the end get the Chow ring of $P_{n-1} = \mathbb{F}_n$.

*Example* 4.11. For $n = 3$, there are two steps. First, we get $CH^*(P_1) = \mathbb{Z}[x_1]/(x_1^3)$ (since $P_1 \simeq P^3$). Then we get $CH^*(\mathbb{F}_3) = CH^*(P_2)[x_2]/(x_2^2 - c_1x_2 + c_2)$ where $c_1$ and $c_2$ are the Chern classes of $\mathcal{V}^2$. Applying results of Example 3.15 (since $\mathcal{V}^2 = \mathcal{H}$ on $\mathbb{P}^3$) we get $c_1 = -x_1$ and $c_2 = x_1^2$. Thus we get exactly the presentation of Example 4.9 for $CH^*(\mathbb{F}_3)$.

4.2. **Grassmannians.** Theorem 4.7 implies an analogous presentation for the Chow ring of the Grassmannian $G(k, n)$. Note that $CH^*(G(k, n))$ (for $k \neq 1,\ n-1$) is not multiplicatively generated by $\mathrm{Pic}(G(k, n)) \simeq \mathbb{Z}$. This is why the multiplicative structure of $CH^*(\mathbb{F}_n)$ has a more direct description than that of $CH^*(G(k, n))$.

**Corollary 4.12.** *Let $\mathbb{Z}[x_1, \ldots, x_n]^{S_k \times S_{n-k}} \subset \mathbb{Z}[x_1, \ldots, x_n]$ denote the subring of polynomials invariant under all permutations that preserve the partition of variables $x_1, \ldots, x_n$ into two sets $\{x_1, \ldots, x_k\}$ and $\{x_{k+1}, \ldots, x_n\}$. Then there is an isomorphism of graded rings*

$$CH^*(G(k, n)) \simeq \mathbb{Z}[x_1, \ldots, x_n]^{S_k \times S_{n-k}}/S.$$

*Here* deg $x_i = 1$.

*The isomorphism sends the $i$-th Chern class of the tautological and quotient tautological vector bundles on $G(k, n)$ to $s_i(x_1, \ldots, x_k)$ and $s_i(x_{k+1}, \ldots, x_n)$, respectively.*

In particular, $CH^*(G(k, n))$ is multiplicatevely generated by the Chern class of the tautological bundle (or quotient tautological bundle). To prove this corollary one can use the "forgetful" projection $\mathbb{F}_n \to G(k, n)$ (a complete flag gets mapped to its $k$-th subspace).

*Example* 4.13. For $G(2, 4)$ we get

$$CH^*(G(2, 4)) = \mathbb{Z}[x_1, x_2, x_3, x_4]^{S_2 \times S_2}/S =$$
$$= \mathbb{Z}[x_1 + x_2, x_1x_2]/((x_1 + x_2)^3 - 2(x_1 + x_2)x_1x_2, (x_1 + x_2)^2x_1x_2 - x_1^2x_2^2).$$

Put $c_1 := c_1(\tau) = x_1 + x_2$ and $c_2 := c_2(\tau) = x_1x_2$, where $\tau$ is the tautological vector bundle on $G(2, 4)$. Then $c_1$ and $c_2$ satisfy the identities

$$c_1^3 = 2c_1c_2, \quad c_1^2c_2 = c_2^2.$$

Let us give a geometric description of the Chern classes of the quotient tautological vector bundle $\bar{\tau}$ on the Grassmannian $G(k, n)$. This bundle is globally generated. Indeed, every vector $v \in \mathbb{C}^n$ defines a global section $s_v$ of $\bar{\tau}$ as follows:

$$s_v(\Lambda) = (v \bmod \Lambda) \in \mathbb{C}^n/\Lambda.$$

Here $\Lambda$ denotes a point of $G(k, n)$ as well as the corresponding $k$-space in $\mathbb{C}^n$. If $v_1, \ldots,\ v_n$ is a basis in $\mathbb{C}^n$, then the corresponding global sections $s_{v_1}, \ldots,\ s_{v_n}$ obviously generate $\bar{\tau}$. Hence, the $i$-th Chern class of $\bar{\tau}$ can be defined as the $i$-th degeneracy locus of $s_{v_1},\dots,\ s_{v_n}$ (see Section 3.1). This is equivalent to the following definition. Let $V^k \subset \mathbb{C}^n$ denote the subspace spanned by $v_1,\dots,\ v_n$. Then

$$c_i(\bar\tau) = \{\Lambda \in G(k,n) \mid \Lambda \cap V^{n-i+1} \neq 0\}.$$

This is one the Schubert cycles on $G(k,n)$.

*Reminder.* Recall that every collection of integers $d_1,\ d_2,\ \dots d_k$, such that $1 \le d_1 < \dots < d_k \le n$ (that is, every *Young diagram* inscribed in the rectangle $k \times (n-k)$), defines the *Schubert cell* $C_V(d_1,\dots,d_k)$ as follows:

$$C_V(d_1,\dots,d_k) = \{\Lambda \in G(k,n) \mid \dim(\Lambda \cap V^j) = i \text{ iff } d_i \le j < d_{i+1}.\}$$

Equivalently, the intersection dimensions $\dim(\Lambda \cap V^j)$ form the sequence

$$(\underbrace{0,\dots,0}_{d_1-1},\underbrace{1,\dots,1}_{d_2-d_1},\dots,\underbrace{k-1,\dots,k-1}_{d_k-d_{k-1}},\underbrace{k,\dots,k}_{n-d_k+1}).$$

Sometimes this sequence is used instead of the sequence $d_1,\dots,\ d_k$ in order to label the Schubert cell.

The Zariski closure of the Schubert cell is called the *Schubert variety*, and its class in $CH^*(G(k,n))$ is called the *Schubert cycle*. The Schubert cell depends on the choice of a complete flag $F = (V^1 \subset \dots \subset V^{n-1})$, however, different choices of a flag yield different $GL_n$-translates of the same subvariety. In particular, the corresponding Schubert varieties are rationally equivalent and give the same Schubert cycle.

**Exercise 4.14.** *Show that $c_i(\bar\tau)$ is the Schubert cycle defined by the Schubert cell $C_F(n-k-i+1, n-k+2, n-k+3,\dots,n)$.*

*Example* 4.15. The Grassmannian $G(2,4)$ has 6 Schubert cells: $C_F(3,4)$, $C_F(2,4)$, $C_F(1,4)$, $C_F(2,3)$, $C_F(1,3)$, $C_F(1,2)$. We have

$$c_1(\bar\tau) = [\overline{C_F(2,4)}], \quad c_2(\bar\tau) = [\overline{C_F(1,4)}]$$

If we identify vector planes in $\mathbb{C}^4$ with projective lines in $\mathbb{P}^3$, then $\overline{C_F(2,4)}$ consists of all lines that intersect a given line, and $\overline{C_F(1,4)}$ consists of all lines that pass through a given point. In particular, the Schubert problem about 4 lines in $\mathbb{P}^3$ reduces to the computation of $c_1(\bar\tau)^4$ in $CH^*(G(2,4))$.

Note that the Chern classes of $\tau$ can be easily computed in terms of the Chern classes of $\bar\tau$ by the Whitney sum formula applied to the short exact sequence

$$0 \to \tau \to \mathcal{O}^n \to \bar\tau \to 0.$$

It is not hard to check that the ideal $S$ of relations from Corollary 4.12 is generated by the relations coming from this Whitney sum formula. In particular, Corollary 4.12 can be reformulated in more geometric terms as follows.

**Corollary 4.16.** *There is an isomorphism of graded rings*

$$CH^*(G(k,n)) \simeq \mathbb{Z}[c_1,\dots,c_k;\bar c_1,\dots,\bar c_{n-k}]/I,$$

*where* deg $c_i =$ deg $\bar{c}_i = i$*, and* $I$ *is the ideal generated by all homogeneous relations in the identity*

$$(1 + c_1 + \ldots + c_k)(1 + \bar{c}_1 + \ldots + \bar{c}_{n-k}) = 1.$$

*The isomorphism sends* $\bar{c}_i$ *to the Schubert cycle defined by the Schubert cell* $C_V(n - k - i + 1, n - k + 2, \ldots, n)$*.*

4.3. **Enumerative geometry.** We now formalize Schubert's solution to the problem:

*How many lines in* $\mathbb{P}^3$ *intersect 4 given lines in general position?*

By Example 4.15 the answer is equal to the coefficient $k$ in the identity $c_1(\bar{\tau})^4 = k[pt]$ in $CH^4(G(2,4))$. Let us compute $k$. By the Whitney sum formula applied to

$$0 \to \tau \to \mathcal{O}^n \to \bar{\tau} \to 0.$$

we have $c_1(\bar{\tau}) = -c_1(\tau)$ and $c_1(\bar{\tau})c_1(\tau) + c_2(\bar{\tau}) = -c_2(\tau)$. Hence,

$$c_1^2(\bar{\tau}) = c_2(\bar{\tau}) + c_2(\tau). \qquad (**)$$

Note that all terms in this identity have geometric meaning. Namely, $c_1^2(\bar{\tau})$ is represented by the variety of lines that intersect two given lines in general position, $c_2(\bar{\tau})$ — by the variety of lines that pass through a given point, and $c_2(\tau) = c_2(\tau^*)$ — by the variety of lines that are contained in a given plane.

*Remark* 4.17. In terms of Schubert's conditions, identity (**) says that

$$\sigma_1 \cdot \sigma_2 = \sigma_a + \sigma_\Pi,$$

where $\sigma_i$, $\sigma_a$, $\sigma_\Pi$ are the conditions that a line intersects a given line $l_i$, contains a given point $a$, lies in a given plane $\Pi$, respectively. Note that if $l_1$ and $l_2$ are not general position but intersect at exactly one point, then obviously $\sigma_1\sigma_2 = \sigma_a + \sigma_\Pi$, where $a = l_1 \cap l_2$, and $\Pi = l_1 + l_2$ (that is, a line intersects both $l_1$ and $l_2$ iff it either contains $a$ or lies in $\Pi$). Identity (**) implies that if lines $l_1$ and $l_2$ are in general position (that is, do not intersect), then it is still true that $\sigma_1\sigma_2 = \sigma_a + \sigma_\Pi$ (where $a$ is now any point and $\Pi$ is any plane) modulo a natural equivalence relation on conditions (coming from the rational equivalence).

It turns out that Schubert's idea of degenerating the intersection of Schubert varieties so that it is still multiplicity-free but splits into smaller Schubert varieties works for arbitrary Grassmannians (see [18]).

It is now easy to compute $c_1(\bar{\tau})^4$. By (**) we have

$$c_1(\bar{\tau})^4 = (c_2(\bar{\tau}) + c_2(\tau))^2 = c_2(\bar{\tau})^2 + 2c_2(\bar{\tau})c_2(\tau) + c_2(\tau)^2.$$

All terms on the right hand side of the second equality can be computed by simple geometric arguments. For instance, $c_2(\bar{\tau})^2$ is represented by the variety of lines in $\mathbb{P}^3$ that pass through two distinct points, hence, $c_2(\bar{\tau})^2 = [pt]$. Similarly, $c_2(\bar{\tau})c_2(\tau) = 0$ and $c_2(\tau)^2 = [pt]$. Hence, $c_1(\bar{\tau})^4 = 2[pt]$, and $k = 2$.

Here is another classical problem of enumerative geometry that reduces to a computation in $CH^*(G(2,4))$.

*How many lines lie on a smooth cubic surface in $\mathbb{P}^3$?*

A smooth cubic surface is given by an equation $f = 0$, where $f$ is a homogeneous polynomial on $\mathbb{P}^3$ of degree 3. Note that $f$ can be regarded as an element of the dual space to the symmetric power $S^3(\mathbb{C}^4)$. In particular, $f$ defines a global section $s_f$ of the vector bundle $S^3\tau^*$ on $G(2,4)$ as follows:

$$s_f(\Lambda) = f|_{S^3\Lambda}.$$

Note that a line $l = \mathbb{P}(\Lambda) \subset \mathbb{P}^3$ lies on the surface $\{f = 0\}$ if and only if $f$ vanishes everywhere on $S^3\Lambda$, or equivalently, the section $s_f$ vanishes at the point $\Lambda \in G(2,4)$. Hence, $c_3(S^3\tau^*) = k[pt]$, where $k$ is the number of lines on the surface $\{f = 0\}$.

Apply the formula for the Chern classes of symmetric powers (see Section 3.3):

$$c_3(S^3\tau^*) = (3x_1)(2x_1 + x_2)(x_1 + 2x_2)(3x_2) = 9x_1x_2(2(x_1 + x_2)^2 + x_1x_2),$$

where $x_1 + x_2 = c_1(\tau^*)$ and $x_1x_2 = c_2(\tau^*)$. Hence,

$$c_3(S^3\tau^*) = 9c_2(\tau^*)^2 + 18c_1(\tau^*)^2c_2(\tau^*).$$

Using geometric meaning of $c_1(\tau^*)$ and $c_2(\tau^*)$, it is easy to check that $c_2(\tau^*)^2 = [pt]$ (since the number of lines that lie in two distinct planes is equal to one) and $c_1(\tau^*)^2c_2(\tau^*) = [pt]$ (since the number of lines that intersect two given lines and lie in a given plane is equal to one). We get $c_3(S^3\tau^*) = 27[pt]$, hence, $k = 27$.

An interesting generalization of this problem leads to the string theory and concerns the number of rational curves of given degree that lie on a smooth quintic threefold in $\mathbb{P}^4$ (see [8]).

**Exercise 4.18.** *Compute the number of lines (=rational curves of degree one) on a smooth quintic threefold in $\mathbb{P}^4$.*

## 5. Lecture 9: topological applications, adjunction formula, Bernstein theorem

Content: normal and tangent bundles, Euler class, adjunction formula [7], Chern classes of $\mathbb{P}^n$ [7], Chern classes of toric varieties [3], Bernstein theorem for the Euler characteristic of a complete intersection in $(\mathbb{C}^*)^n$ [11], algebraically cellular varieties.

## 6. Lecture 10: Ring of conditions

Content: Kleiman's transversality theorem [14], group and ring of conditions [2], solution to the problem about 5 conics.

## 7. Lecture 11: Birationally invariant intersection theory

Intersection theory of Kaveh–Khovanskii [10], Newton–Okounkov convex bodies [9], Hodge inequality via isoperimeteric inequality [9].

## 8. Lecture 12: Intersection theory on spherical varieties

## 9. Appendix. Conservation of number principle and its applications

We will now consider several elementary examples in low dimension where the conservation of number principle arises naturally. In each case, an appropriate version of this principle will be formulated explicitly and then used to obtain some enumerative results. In particular, we will find the number of zeroes of a polynomial in one variable (Fundamental Theorem of Algebra), the genus of a generic plane curve and the number of common zeroes of two polynomials in two variables (Bezout Theorem).

9.1. **Fundamental Theorem of Algebra.** Let $f$ be a complex polynomial of degree $n$. The Fundamental Theorem of Algebra asserts that a *generic* $f$ has $n$ distinct complex roots. We will call a polynomial *generic* if it does not have multiple roots (i.e. all roots are simple). The space of all monic polynomials of degree $n$ can be identified with $\mathbb{C}^n$ (the polynomial $x^n + a_1x^{n-1} + \ldots + a_n$ goes to the point $(a_1, \ldots, a_n)$). Then it is easy to show that generic polynomials form a Zariski open dense subset in $\mathbb{C}^n$.
Remark. We will repeatedly use the notion of *generic* object. In each case, there will be a family of objects parameterized by the points of an algebraic variety, and *generic* objects will correspond to the points in some Zariski open dense subset of this variety. In particular, almost any object in the family is generic. In each case the subset of generic objects will be defined by an explicit condition (like the one above) and it will be left as an exercise to check that all generic objects indeed form a Zariski open dense subset.

To extend the Fundamental Theorem of Algebra to non-generic polynomials we need the notion of the *multiplicity* of a root. There are two equivalent definitions.

Algebraic definition of multiplicity. A root $a$ of $f$ has multiplicity $k$ iff

$$f(a) = f'(a) = \ldots = f^{(k-1)}(a) = 0, \text{and} f^{(k)}(a) \neq 0.$$

Geometric definition of multiplicity. A root $a$ of $f$ has multiplicity $k$ iff there is a neighborhood of $a$ (that does not contain other roots of $f$) such that all generic polynomials close enough to $f$ have exactly $k$ roots in this neighborhood.

In the second definition one needs to check that all generic polynomials close enough to $a$ have the same number of roots in some neighborhood of $a$. This follows easily from the Implicit Function Theorem.

**Example.** If $f = x^k$, then 0 is a root of multiplicity $k$. To check this using the geometric definition one can consider a generic polynomial $x^k - t$, which has $k$ distinct roots for all $t \neq 0$.

We will now prove the Fundamental Theorem of Algebra in the following form.

**Theorem 9.1.** *Any complex polynomial $f$ of degree $n$ has exactly $n$ complex roots counted with multiplicities.*

Note that this theorem holds over any algebraically closed field and can be proved in a purely algebraic way (by factoring out one root of $f$). However, the fact that $\mathbb{C}$ is algebraically closed is analytic and its proof must use some geometric arguments.

First, we will show that all generic polynomials have the same number of roots. Indeed, each generic polynomial has a neighborhood such that all polynomials in this neighborhood are generic and have the same number of roots (this again follows the Implicit Function Theorem). Note that this is also true over real numbers. The crucial observation is that any two generic polynomials (identified with the points in $\mathbb{C}^n$) can be connected by a path avoiding all non-generic polynomials (this is exactly what fails over real numbers). This follows from the simple but very important fact stated below.

**Lemma 9.2.** *Let $X$ be an irreducible complex algebraic variety, and $Y \subset X$ a subvariety of codimension one. Then the complement $X \setminus Y$ is connected.*

Hence, we proved that all generic polynomials have the same number of roots. To actually find this number we can consider a specific polynomial, say, $x(x-1)\cdots(x-n+1)$, which obviously has $n$ roots. The statement of the theorem for non-generic polynomials follows easily from the geometric definition of multiplicity.

There is the following generalization of the Fundamental Theorem of Algebra, which we will use in the sequel. Recall that a function $f$ has pole of order $k$ at a point $a$ if the function $1/f$ has zero of multiplicity $k$ at the point $a$.

**Lemma 9.3.** *Let $C$ be a compact smooth curve over $\mathbb{C}$, and $f : C \to \mathbb{CP}^1$ a non-constant meromorphic function on $C$. Then the number of zeroes of $f$ counted with multiplicities is equal to the number of poles of $f$ counted with orders.*

This lemma can also be proved using a conservation of number principle. Namely, using the Implicit Function Theorem and Lemma 9.2 one can show that all non-critical values of $f$ have the same number of preimages.

9.2. **Genus of plane curve.** Let $C$ be a curve in $\mathbb{CP}^2$ given as the zero set of a homogeneous polynomial of degree $d$. We say that $C$ is a *generic* plane curve of degree $d$ if $C$ is smooth. Recall that topologically each compact smooth complex curve is a 2-dimensional sphere with several handles. The number of handles is called the *genus* of a curve. E.g. $\mathbb{CP}^1$ is homeomorphic to a sphere, so its genus is zero. A curve of genus one is homeomorphic to a 2-dimensional tori (i.e. the direct product of two circles).

**Theorem 9.4.** *The genus of a generic plane curve of degree $d$ is equal to*

$$\frac{(d-1)(d-2)}{2}.$$

In particular, generic conic has genus zero and generic cubic curve has genus one. Note also that according to this theorem a generic plane curve can not have genus two.

To prove the theorem one can show that all generic curves of degree $d$ are homeomorphic and hence, have the same genus. Then it is enough to compute the genus of the easiest possible generic curve. At first glance, there is no particularly easy curve but one can do the following trick. Consider a non-generic curve which is just the union of $d$ lines, i.e. it is given by the equation $l_1 \cdot \ldots \cdot l_d = 0$ for some linear functions $l_1, \ldots, l_n$. Topologically it looks like $d$ spheres such that every two have one common point. Then we can perturb a little bit the coefficients of the equation $l_1 \cdot \ldots \cdot l_d = 0$ so that the curve becomes generic. Then it is easy to check that each common point of two spheres gets replaced by a tube. So the whole curve becomes the union of $d$ spheres such that every two are connected by a tube. The genus of such curve is exactly $\frac{(d-1)(d-2)}{2}$.

9.3. **Bezout Theorem.** Let $f$ and $g$ be two homogeneous polynomials on $\mathbb{CP}^2$. We say that the pair $(f, g)$ is *generic* if the intersection of the curves $\{f = 0\}$ and $\{g = 0\}$ in $\mathbb{CP}^2$ is transverse.

**Theorem 9.5.** *Two generic polynomials of degrees $m$ and $n$ on $\mathbb{CP}^2$ have exactly $mn$ common zeroes.*

Again one shows that for all generic pairs of polynomials the number of common zeroes is the same and then finds this number for, say, polynomials $f(x, y) = x(x - 1)\ldots(x - m + 1)$ and $g(x, y) = y(y - 1)\ldots(y - n + 1)$.

We now return to the definition of intersection indices in the case of two curves $M = \{f = 0\}$ and $N = \{g = 0\}$ in $\mathbb{CP}^2$. Bezout Theorem tells us that if the pair $(f, g)$ is generic then the number of intersection points $|M \cap N|$ depends only on the degrees of $f$ and of $g$. Hence, it is natural to require that the intersection index be preserved when we move each curve in the family of curves defined by the equation of the same degree. This gives the following definition of the intersection index. If the pair $(f, g)$ is generic then put $M \cdot N = |M \cap N|$. Otherwise, we perturb the coefficients of $f$ and $g$ so that they become generic and define $M \cdot N$ as the number of intersection points of the perturbed curves. The Bezout Theorem is then equivalent to the statement that the intersection index of $M$ and $N$ is always equal to the product of degrees of $f$ and $g$.

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