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Principles of Algebraic Geometry

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PRINCIPLES OF ALGEBRAIC GEOMETRY

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PREFACE

Algebraic geometry is among the oldest and most highly developed subjects in mathematics. It is intimately connected with projective geometry, complex analysis, topology, number theory, and many other areas of current mathematical activity. Moreover, in recent years algebraic geometry has undergone vast changes in style and language. For these reasons there has arisen about the subject a reputation of inaccessibility. This book gives a presentation of some of the main general results of the theory accompanied by—and indeed with special emphasis on—the applications to the study of interesting examples and the development of computational tools.

A number of principles guided the preparation of the book. One was to develop only that general machinery necessary to study the concrete geometric questions and special classes of algebraic varieties around which the presentation was centered.

A second was that there should be an alternation between the general theory and study of examples, as illustrated by the table of contents. The subject of algebraic geometry is especially notable for the balance provided on the one hand by the intricacy of its examples and on the other by the symmetry of its general patterns; we have tried to reflect this relationship in our choice of topics and order of presentation.

A third general principle was that this volume should be self-contained. In particular any "hard" result that would be utilized should be fully proved. A difficulty a student often faces in a subject as diverse as algebraic geometry is the profusion of cross-references, and this is one reason for attempting to be self-contained. Similarly, we have attempted to avoid allusions to, or statements without proofs of, related results. This book is in no way meant to be a survey of algebraic geometry, but rather is designed to develop a working facility with specific geometric questions. Our approach to the subject is initially analytic: Chapters 0 and 1 treat the basic techniques and results of complex manifold theory, with some emphasis on results applicable to projective varieties. Beginning in Chapter 2 with the theory of Riemann surfaces and algebraic curves, and continuing in Chapters 4 and 6 on algebraic surfaces and the quadric line complex, our treatment becomes increasingly geometric along classical lines. Chapters 3 and 5 continue the analytic approach, progressing to more special topics in complex manifolds.

Several important topics have been entirely omitted. The most glaring are the arithmetic theory of algebraic varieties, moduli questions, and singularities. In these cases the necessary techniques are not fully developed here. Other topics, such as uniformization and automorphic forms or monodromy and mixed Hodge structures have been omitted, although the necessary techniques are for the most part available.

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Cambridge, Massachusetts

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FOUNDATIONAL MATERIAL

In this chapter we sketch the foundational material from several complex variables, complex manifold theory, topology, and differential geometry that will be used in our study of algebraic geometry. While our treatment is for the most part self-contained, it is tacitly assumed that the reader has some familiarity with the basic objects discussed. The primary purpose of this chapter is to establish our viewpoint and to present those results needed in the form in which they will be used later on. There are, broadly speaking, four main points:

1. *The Weierstrass theorems and corollaries*, discussed in Sections 1 and 2. These give us our basic picture of the local character of analytic varieties. The theorems themselves will not be quoted directly later, but the picture—for example, the local representation of an analytic variety as a branched covering of a polydisc—is fundamental. The foundations of local analytic geometry are further discussed in Chapter 5.
2. *Sheaf theory*, discussed in Section 3, is an important tool for relating the analytic, topological, and geometric aspects of an algebraic variety. A good example is the *exponential sheaf sequence*, whose individual terms $\mathbb{Z}$, $\mathcal{O}$, and $\mathcal{O}^*$ reflect the topological, analytic, and geometric structures of the underlying variety, respectively.
3. *Intersection theory*, discussed in Section 4, is a cornerstone of classical algebraic geometry. It allows us to treat the incidence properties of algebraic varieties, a priori a geometric question, in topological terms.
4. *Hodge theory*, discussed in Sections 6 and 7. By far the most sophisticated technique introduced in this chapter, Hodge theory has, in the present context, two principal applications: first, it gives us the *Hodge decomposition* of the cohomology of a Kähler manifold; then, together with the formalism introduced in Section 5, it gives the vanishing theorems of the next chapter.

1. RUDIMENTS OF SEVERAL COMPLEX VARIABLES

Cauchy's Formula and Applications

NOTATION. We will write $z=(z_1,\dots,z_n)$ for a point in $\mathbb{C}^n$, with

$$z_i = x_i + \sqrt{-1}\, y_i;$$
$$\|z\|^2 = (z,z) = \sum_{i=1}^{n} |z_i|^2.$$

For U an open set in $\mathbb{C}^n$, write $C^\infty(U)$ for the set of C^∞ functions defined on U; $C^\infty(\overline{U})$ for the set of C^∞ functions defined in some neighborhood of the closure $\overline{U}$ of U.

The cotangent space to a point in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is spanned by $\{dx_i, dy_i\}$; it will often be more convenient, however, to work with the complex basis

$$dz_i = dx_i + \sqrt{-1}\, dy_i, \qquad d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i$$

and the dual basis in the tangent space

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

With this notation, the formula for the total differential is

$$df = \sum_i \frac{\partial f}{\partial z_i} dz_i + \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

In one variable, we say a C^∞ function f on an open set $U \subset \mathbb{C}$ is *holomorphic* if f satisfies the Cauchy-Riemann equations $\partial f/\partial \bar{z} = 0$. Writing $f(z) = u(z) + \sqrt{-1}\, v(z)$, this amounts to

$$\operatorname{Re}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$
$$\operatorname{Im}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

We say f is *analytic* if, for all $z_0 \in U$, f has a local series expansion in $z - z_0$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in some disc $\Delta(z_0, \varepsilon) = \{z : |z - z_0| < \varepsilon\}$, where the sum converges absolutely and uniformly. The first result is that f is analytic if and only if it is holomorphic; to show this, we use the

Cauchy Integral Formula. *For* Δ *a disc in* $\mathbb{C}$, $f \in C^\infty(\overline{\Delta})$, $z \in \Delta$,

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)\,dw}{w - z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z},$$

where the line integrals are taken in the counterclockwise direction (the fact that the last integral is defined will come out in the proof).

Proof. The proof is based on Stokes' formula for a differential form with singularities, a method which will be formalized in Chapter 3. Consider the differential form

$$\eta = \frac{1}{2\pi\sqrt{-1}} \frac{f(w)\,dw}{w-z};$$

we have for $z \neq w$

$$\frac{\partial}{\partial \bar{w}}\left(\frac{1}{w-z}\right) = 0$$

and so

$$d\eta = -\frac{1}{2\pi\sqrt{-1}} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}.$$

Let $\Delta_\varepsilon = \Delta(z,\varepsilon)$ be the disc of radius ε around z. The form η is C^∞ in $\Delta - \Delta_\varepsilon$, and applying Stokes' theorem we obtain

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)\,dw}{w-z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta-\Delta_\varepsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}.$$

Setting $w - z = re^{i\theta}$,

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(w)\,dw}{w-z} = \frac{1}{2\pi} \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta,$$

which tends to $f(z)$ as $\varepsilon \to 0$; moreover,

$$dw \wedge d\bar{w} = -2\sqrt{-1}\; dx \wedge dy = -2\sqrt{-1}\; r\,dr \wedge d\theta$$

so

$$\left|\frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z}\right| = 2\left|\frac{\partial f}{\partial \bar{w}}\, dr \wedge d\theta\right| \leqslant c\,|dr \wedge d\theta|.$$

Thus $(\partial f/\partial \bar{w})(dw \wedge d\bar{w})/(w-z)$ is absolutely integrable over Δ, and

$$\int_{\Delta_\varepsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} \to 0$$

as $\varepsilon \to 0$; the result follows. Q.E.D.

Now we can prove the

Proposition. *For* U *an open set in* $\mathbb{C}$ *and* $f \in C^\infty(U)$, f *is holomorphic if and only if* f *is analytic.*

Proof. Suppose first that $\partial f/\partial\bar{z}=0$. Then for $z_0\in U$, ε sufficiently small, and z in the disc $\Delta=\Delta(z_0,\varepsilon)$ of radius ε around z_0,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z}\\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)-(z-z_0)}\\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)\left(1-\dfrac{z-z_0}{w-z_0}\right)}\\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}}\right)(z-z_0)^n;
\end{aligned}$$

so, setting

$$a_n=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}},$$

we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta$, where the sum converges absolutely and uniformly in any smaller disc.

Suppose conversely that $f(z)$ has a power series expansion

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta=\Delta(z_0,\varepsilon)$. Since $(\partial/\partial\bar{z})(z-z_0)^n=0$, the partial sums of the expansion satisfy Cauchy's formula without the area integral, and by the uniform convergence of the sum in a neighborhood of z_0 the same is true of f, i.e.,

$$f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z}.$$

We can then differentiate under the integral sign to obtain

$$\frac{\partial}{\partial\bar{z}}f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{\partial}{\partial\bar{z}}\left(\frac{f(w)}{w-z}\right)dw=0,$$

since for $z\neq w$

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{w-z}\right)=0.$$

Q.E.D.

We prove a final result in one variable, that given a C^∞ function g on a disc Δ the equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

can always be solved on a slightly smaller disc; this is the

$\bar{\partial}$-Poincaré Lemma in One Variable. *Given* $g(z) \in C^\infty(\bar{\Delta})$, *the function*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{g(w)}{w-z}\, dw \wedge d\bar{w}$$

is defined and C^∞ *in* Δ *and satisfies*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

Proof. For $z_0 \in \Delta$ choose ε such that the disc $\Delta(z_0, 2\varepsilon) \subset \Delta$ and write

$$g(z) = g_1(z) + g_2(z),$$

where $g_1(z)$ vanishes outside $\Delta(z_0, 2\varepsilon)$ and $g_2(z)$ vanishes inside $\Delta(z_0, \varepsilon)$. The integral

$$f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_2(w) \frac{dw \wedge d\bar{w}}{w-z}$$

is well-defined and C^∞ for $z \in \Delta(z_0, \varepsilon)$; there we have

$$\frac{\partial}{\partial \bar{z}} f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial}{\partial \bar{z}} \left(\frac{g_2(w)}{w-z} \right) dw \wedge d\bar{w} = 0.$$

Since $g_1(z)$ has compact support, we can write

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_1(w) \frac{dw \wedge d\bar{w}}{w-z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(w) \frac{dw \wedge d\bar{w}}{w-z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(u+z) \frac{du \wedge d\bar{u}}{u}, \end{aligned}$$

where $u = w - z$. Changing to polar coordinates $u = re^{i\theta}$ this integral becomes

$$f_1(z) = -\frac{1}{\pi} \int_{\mathbf{C}} g_1(z + re^{i\theta}) e^{-i\theta}\, dr \wedge d\theta,$$

which is clearly defined and C^∞ in z. Then

$$\begin{aligned} \frac{\partial f_1(z)}{\partial \bar{z}} &= -\frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial g_1}{\partial \bar{z}} (z + re^{i\theta}) e^{-i\theta}\, dr \wedge d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial g_1}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}; \end{aligned}$$

but g_1 vanishes on $\partial\Delta$, and so by the Cauchy formula

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} f_1(z) = g_1(z) = g(z). \qquad \text{Q.E.D.}$$

Several Variables

In the formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}\, dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i}\, d\bar{z}_i$$

for the total differential of a function f on $\mathbb{C}^n$, we denote the first term ∂f and the second term $\bar{\partial} f$; ∂ and $\bar{\partial}$ are differential operators invariant under a complex linear change of coordinates. A C^∞ function f on an open set $U \subset \mathbb{C}^n$ is called *holomorphic* if $\bar{\partial} f = 0$; this is equivalent to $f(z_1, \ldots, z_n)$ being holomorphic in each variable z_i separately.

As in the one-variable case, a function f is holomorphic if and only if it has local power series expansions in the variables z_i. This is clear in one direction: by the same argument as before, a convergent power series defines a holomorphic function. We check the converse in the case $n = 2$; the computation for general n is only notationally more difficult. For f holomorphic in the open set $U \subset \mathbb{C}^2$, $z_0 \in U$, we can fix Δ the disc of radius r around $z_0 \in U$ and apply the one-variable Cauchy formula twice to obtain, for $(z_1, z_2) \in \Delta$,

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \frac{f(z_1, w_2)\, dw_2}{w_2 - z_2} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \left[\frac{1}{2\pi\sqrt{-1}} \int_{|w_1 - z_{0_1}| = r} \frac{f(w_1, w_2)\, dw_1}{w_1 - z_1} \right] \frac{dw_2}{w_2 - z_2} \\ &= \left(\frac{1}{2\pi\sqrt{-1}} \right)^2 \int\!\!\int_{|w_i - z_{0_i}| = r} \frac{f(w_1, w_2)\, dw_1\, dw_2}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned}$$

Using the series expansion

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \sum_{m,n=0}^{\infty} \frac{(z_1 - z_{0_1})^m (z_2 - z_{0_2})^n}{(w_1 - z_{0_1})^{m+1} (w_2 - z_{0_2})^{n+1}},$$

we find that f has a local series expansion

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} (z_1 - z_{0_1})^m (z_2 - z_{0_2})^n. \qquad \text{Q.E.D.}$$

Many results in several variables carry directly over from the one-variable theory, such as the identity theorem: *If* f *and* g *are holomorphic on a connected open set* U *and* f = g *on a nonempty open subset of* U, *then* f = g, and the maximum principle: *the absolute value of a holomorphic function* f *on an open set* U *has no maximum in* U. There are, however, some striking differences between the one- and many-variable cases. For example, let U be the polydisc $\Delta(r)=\{(z_1,z_2):|z_1|,|z_2|<r\}$, and let $V\subset U$ be the smaller polydisc $\Delta(r')$ for any $r'<r$. Then we have

Hartogs' Theorem. *Any holomorphic function* f *in a neighborhood of* U − V *extends to a holomorphic function on* U.

Proof. In each vertical slice $z_1=$constant, the region $U-V$ looks either like the annulus $r'<|z_2|<r$ or like the disc $|z_2|<r$. We try to extend f in each slice by Cauchy's formula, setting

$$F(z_1,z_2)=\frac{1}{2\pi\sqrt{-1}}\int_{|w_2|=r}\frac{f(z_1,w_2)\,dw_2}{w_2-z_2}.$$

F is defined throughout U; it is clearly holomorphic in z_2, and since $(\partial/\partial\bar{z}_1)f=0$, it is holomorphic in z_1 as well. Moreover, in the open subset $|z_1|>r'$ of $U-V$, $F(z_1,z_2)=f(z_1,z_2)$ by Cauchy's formula; thus $F|_{U-V}=f$.

Q.E.D.

Hartogs' theorem applies to many pairs of sets $V\subset U\subset\mathbb{C}^n$; it is commonly applied in the form

A holomorphic function on the complement of a point in an open set $U\subset\mathbb{C}^n$ $(n>1)$ *extends to a holomorphic function in all of* U.

Weierstrass Theorems and Corollaries

In one variable, every analytic function has a unique local representation

$$f(z)=(z-z_0)^n u(z),\qquad u(z_0)\neq 0,$$

from which we see in particular that the zero locus of f is discrete. Similarly, the Weierstrass theorems give local representations of holomorphic functions in several variables, from which we get a picture of the local geometry of their zero sets.

Suppose we are given a function $f(z_1,\ldots,z_{n-1},w)$ holomorphic in some neighborhood of the origin in $\mathbb{C}^n$, with $f(0,\ldots,0)=0$. Assume that f does not vanish identically on the w-axis, i.e., the power series expansion for f around the origin contains a term $a\cdot w^d$ with $a\neq 0$ and $d\geqslant 1$; clearly this will be the case for most choices of coordinate system.

For suitable r, δ, and $\varepsilon>0$, then, $|f(0,w)|\geqslant\delta>0$ for $|w|=r$, and consequently $|f(z,w)|\geqslant\delta/2$ for $|w|=r$, $\|z\|\leqslant\varepsilon$. Now if $w=b_1,\dots,b_d$ are the roots of $f(z,w)=0$ for $|w|<r$, by the residue theorem

$$b_1^q+b_2^q+\cdots+b_d^q=\frac{1}{2\pi\sqrt{-1}}\int_{|w|=r}\frac{w^q(\partial f/\partial w)(z,w)}{f(z,w)}\,dw;$$

so the power sums $\Sigma b_i(z)^q$ are analytic functions of z for $\|z\|<\varepsilon$. Let $\sigma_1(z),\dots,\sigma_d(z)$ be the elementary symmetric polynomials in $b_1,\dots,b_d$; $\sigma_1,\dots,\sigma_d$ can be expressed as polynomials in the power sums $\Sigma b_i(z)^q$. Thus the function

$$g(z,w)=w^d-\sigma_1(z)w^{d-1}+\cdots+(-1)^d\sigma_d(z)$$

is holomorphic in $\|z\|<\varepsilon,|w|<r$, and vanishes on exactly the same set as f. The quotient

$$h(z,w)=\frac{f(z,w)}{g(z,w)}$$

is defined and holomorphic in $\|z\|<\varepsilon$, $|w|<r$, at least outside the zero set of f and g. Moreover, for fixed z, $h(z,w)$ has only removable singularities in the disc $|w|<r$, so h can be extended to a function in all of $\|z\|<\varepsilon$, $|w|<r$ and analytic in w for each fixed z, as well as in the complement of the zero locus. Writing

$$h(z,w)=\frac{1}{2\pi\sqrt{-1}}\int_{|u|=r}\frac{h(z,u)\,du}{u-w},$$

we see that h is holomorphic in z as well.

DEFINITION. A *Weierstrass polynomial* in w is a polynomial of the form

$$w^d+a_1(z)w^{d-1}+\cdots+a_d(z),\qquad a_i(0)=0.$$

We have proved the existence part of the

Weierstrass Preparation Theorem. *If* f *is holomorphic around the origin in* $\mathbb{C}^n$ *and is not identically zero on the* w*-axis, then in some neighborhood of the origin* f *can be written uniquely as*

$$f=g\cdot h,$$

where g *is a Weierstrass polynomial of degree* d *in* w *and* h(0)≠0.

The uniqueness is clear, since the coefficients of any Weierstrass polynomial g vanishing exactly where f does are given as polynomials in the integrals

$$\int_{|w|=r}\frac{w^q(\partial f/\partial w)(z,w)\,dw}{f(z,w)}.$$

We see from the Weierstrass theorem that the zero locus of a function f, holomorphic in a neighborhood of the origin in $\mathbb{C}^n$, is for most choices of coordinate system $z_1,\ldots,z_{n-1},w$ the zero locus of a Weierstrass polynomial

$$g(z,w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z).$$

Now, the roots $b_i(z)$ of the polynomial $g(z,\cdot)$ are, away from those values of z for which $g(z,\cdot)$ has a multiple root, locally single-valued holomorphic functions of z. Since the discriminant of $g(z,\cdot)$ is an analytic function of z,

The zero locus of an analytic function $f(z_1,\ldots,z_{n-1},w)$, *not vanishing identically on the* w*-axis, projects locally onto the hyperplane* $(w=0)$ *as a finite-sheeted cover branched over the zero locus of an analytic function.*

As a corollary of the preparation theorem, we have the

Riemann Extension Theorem. *Suppose* $f(z,w)$ *is holomorphic in a disc* $\Delta\subset\mathbb{C}^n$ *and* $g(z,w)$ *is holomorphic in* $\overline{\Delta}-\{f=0\}$ *and bounded. Then* g *extends to a holomorphic function on* Δ.

Proof (in a neighborhood of 0). Assume that the line $z=0$ is not contained in $\{f=0\}$. As before, we can find r, ε, and $\delta>0$ such that $|f(0,w)|\geqslant\delta>0$ for $|w|=r$ and ε such that $|f(z,w)|>\delta/2$ for $\|z\|<\varepsilon$, $|w|=r$; f then has zeros only in the interior of the discs $z=z_0$, $|w|\leqslant r$. By the one-variable Riemann extension theorem, we can extend g to a function $\tilde{g}$ in $|z|<\varepsilon$, $|w|<r$, holomorphic away from $\{f=0\}$ and holomorphic in w everywhere. As before, we write

$$\tilde{g}(z,w) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=r}\frac{\tilde{g}(z,u)\,du}{u-w}$$

to see that $\tilde{g}$ is holomorphic in z as well. Q.E.D.

We recall some facts and definitions from elementary algebra:

Let R be an *integral domain*, i.e., a ring such that for $u,v\in R$, $u\cdot v=0\Rightarrow u=0$ or $v=0$. An element $u\in R$ is a *unit* if there exists $v\in R$ such that $u\cdot v=1$; u is *irreducible* if for $v,w\in R$, $u=v\cdot w$ implies v is a unit or w is a unit. R is a *unique factorization domain* (*UFD*) if every $u\in R$ can be written as a product of irreducible elements $u_1,\ldots,u_l$, the u_i's unique up to multiplication by units. The main facts we shall use are

1. R is a UFD$\Rightarrow R[t]$ is a UFD (Gauss' lemma).
2. If R is a UFD and $u,v\in R[t]$ are relatively prime, then there exist relatively prime elements $\alpha,\beta\in R[t]$, $\gamma\neq 0\in R$, such that

$$\alpha u + \beta v = \gamma.$$

γ is called the *resultant* of u and v.

Let $\mathcal{O}_{n,z}$ denote the ring of holomorphic functions defined in some neighborhood of $z \in \mathbb{C}^n$; write $\mathcal{O}_n$ for $\mathcal{O}_{n,0}$. $\mathcal{O}_n$ is an integral domain by the identity theorem, and moreover is a *local ring* whose *maximal ideal m* is $\{f: f(0)=0\}$. $f \in \mathcal{O}_n$ is a unit if and only if $f(0) \neq 0$. The first result is

Proposition. $\mathcal{O}_n$ *is a UFD.*

Proof. We proceed by induction. Assume $\mathcal{O}_{n-1}$ is a UFD and let $f \in \mathcal{O}_n$. We may assume f is regular with respect to $w = z_n$; i.e., $f(0,\ldots,0,w) \not\equiv 0$. Write

$$f = g \cdot u,$$

where u is a unit in $\mathcal{O}_n$ and $g \in \mathcal{O}_{n-1}[w]$ is a Weierstrass polynomial. $\mathcal{O}_{n-1}[w]$ is a UFD by Gauss' lemma, and so we can write g as a product of irreducible elements $g_1,\ldots,g_m \in \mathcal{O}_{n-1}[w]$

$$(*) \qquad f = g_1 \cdots g_m \cdot u,$$

where the factors g_i are uniquely determined up to multiplication by units. Now suppose we write f as a product of irreducible elements $f_1,\ldots,f_k \in \mathcal{O}_n$. Each f_i must be regular with respect to w, and we can write

$$f_i = g_i' \cdot u_i$$

with u_i a unit, g_i' a Weierstrass polynomial, necessarily irreducible in $\mathcal{O}_{n-1}[w]$. We have

$$f = g \cdot u = \prod g_i' \cdot \prod u_i',$$

with g and $\prod g_i'$ both Weierstrass polynomials; by the Weierstrass preparation theorem

$$g = \prod g_i',$$

and since $\mathcal{O}_{n-1}[w]$ is a UFD, it follows that the g_i' are the same, up to units, as the g_i. Thus the expression $(*)$ represents a unique factorization of f in $\mathcal{O}_n$. Q.E.D.

Proposition. *If* f *and* g *are relatively prime in* $\mathcal{O}_{n,0}$, *then for* $\|z\| < \varepsilon$, f *and* g *are relatively prime in* $\mathcal{O}_{n,z}$.

Proof. We may assume that f and g are regular with respect to z_n and are both Weierstrass polynomials; for each fixed $z' \in \mathbb{C}^{n-1}$ sufficiently small we have $f(z',z_n) \not\equiv 0$ in z_n. Now we can write

$$\alpha f + \beta g = \gamma$$

with $\alpha, \beta \in \mathcal{O}_{n-1}[w]$, $\gamma \in \mathcal{O}_{n-1}$; the equation holds in some neighborhood of $0 \in \mathbb{C}^n$.

If for some small $z_0 \in \mathbb{C}^n$, $f(z_0) = g(z_0) = 0$ and f and g have a common factor $h(z',z_n)$ in $\mathcal{O}_{n,z_0}$ with $h(z_0)=0$, then

$$\begin{aligned} h|f, h|g &\Rightarrow h|\gamma \\ &\Rightarrow h \in \mathcal{O}_{n-1}. \end{aligned}$$

But then $h(z_{0_1},\ldots,z_{0_{n-1}},z_n)$ vanishes identically in z_n, contradicting our assumption that $f(z_{0_1},\ldots,z_{0_{n-1}},z_n)\not\equiv 0$. Q.E.D.

We now prove the

Weierstrass Division Theorem. *Let* $g(z,w)\in\mathcal{O}_{n-1}[w]$ *be a Weierstrass polynomial of degree* k *in* w. *Then for any* $f\in\mathcal{O}_n$, *we can write*

$$f = g\cdot h + r$$

with $r(z,w)$ *a polynomial of degree* $<k$ *in* w.

Proof. For $\varepsilon,\delta>0$ sufficiently small, define for $\|z\|<\varepsilon$, $|w|<\delta$,

$$h(z,w) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}\frac{du}{u-w}.$$

h is clearly holomorphic, and hence so is $r=f-gh$. We have

$$\begin{aligned} r(z,w) &= f(z,w)-g(z,w)\cdot h(z,w)\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\left[f(z,u)-g(z,w)\frac{f(z,u)}{g(z,u)}\right]\frac{du}{u-w}\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}\frac{g(z,u)-g(z,w)}{u-w}\,du. \end{aligned}$$

But $(u-w)$ divides $[g(z,u)-g(z,w)]$ as polynomials in w; thus

$$p(z,u,w) = \frac{g(z,u)-g(z,w)}{u-w}$$

is a polynomial in w *of degree* $<k$. Since the factor w appears only in p in the expression for $r(z,w)$, we see that $r(z,w)$ is a polynomial of degree $<k$ in w. Explicitly, if

$$p(z,u,w) = p_1(z,u)\cdot w^{k-1}+\cdots+p_k(z,u),$$

then

$$r(z,w) = a_1(z)\cdot w^{k-1}+\cdots+a_k(z),$$

where

$$a_i(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}p_i(z,u)\,du.$$ Q.E.D.

Corollary (Weak Nullstellensatz). *If* $f(z,w)\in\mathcal{O}_n$ *is irreducible and* $h\in\mathcal{O}_n$ *vanishes on the set* $f(z,w)=0$, *then* f *divides* h *in* $\mathcal{O}_n$.

Proof. First, we may assume f is a Weierstrass polynomial of degree k in w. Since f is irreducible, f and $\partial f/\partial w$ are relatively prime in $\mathcal{O}_{n-1}[w]$ ($\deg_w f>\deg_w \partial f/\partial w$); thus we can write

$$\alpha\cdot f+\beta\cdot\frac{\partial f}{\partial w} = \gamma,\qquad \gamma\in\mathcal{O}_{n-1},\quad \gamma\not\equiv 0.$$

If, for a given z_0, $f(z_0, w) \in \mathbb{C}[w]$ has a multiple root u, we have

$$f(z_0, u) = \frac{\partial f}{\partial w}(z_0, u) = 0$$
$$\Rightarrow \gamma(z_0) = 0;$$

thus: f(z, w) *has* k *distinct roots in* w *for* $\gamma(z) \neq 0$.

Now by the division theorem, we can write

$$h = f \cdot g + r, \qquad r \in \mathcal{O}_{n-1}[w], \quad \deg r < k.$$

But for any z_0 outside the locus $(\gamma = 0)$, $f(z_0, w)$ and hence $h(z_0, w)$ have at least k distinct roots in w. Since degree $r < k$, this implies $r(z_0, w) = 0 \in \mathbb{C}[w]$; it follows that $r \equiv 0$ and $h = f \cdot g$. Q.E.D.

Analytic Varieties

The main purpose of the results given above is to describe the basic local properties of analytic varieties in $\mathbb{C}^n$. We say a subset V of an open set $U \subset \mathbb{C}^n$ is an *analytic variety* in U if, for any $p \in U$, there exists a neighborhood U' of p in U such that $V \cap U'$ is the common zero locus of a finite collection of holomorphic functions $f_1, \ldots, f_k$ on U'. In particular, V is called an *analytic hypersurface* if V is locally the zero locus of a single nonzero holomorphic function f.

An analytic variety $V \subset U \subset \mathbb{C}^n$ is said to be *irreducible* if V cannot be written as the union of two analytic varieties $V_1, V_2 \subset U$ with $V_1, V_2 \neq V$; it is said to be *irreducible at* $p \in V$ if $V \cap U'$ is irreducible for small neighborhoods U' of p in U. Note first that if $f \in \mathcal{O}_n$ is irreducible in the ring $\mathcal{O}_n$, then the analytic hypersurface $V = \{f(z) = 0\}$ given by f in a neighborhood of 0 is irreducible at 0: if $V = V_1 \cup V_2$, with V_1, V_2 analytic varieties $\neq V$, then there exist $f_1, f_2 \in \mathcal{O}_n$ with f_1 (respectively f_2) vanishing identically on V_1 (respectively V_2) but not on V_2 (respectively V_1). By the Nullstellensatz, f must divide the product $f_1 \cdot f_2$; since f is irreducible, it follows that f must divide either f_1 or f_2, i.e., either $V_1 \supset V$ or $V_2 \supset V$, a contradiction. In addition to the basic picture of an analytic hypersurface (p. 9) we see that

1. Suppose $V \subset U \subset \mathbb{C}^n$ is an analytic hypersurface, given by $V = \{f(z) = 0\}$ in a neighborhood of $0 \in V$. Since $\mathcal{O}_n$ is a UFD, we can write

$$f = f_1 \cdots f_n$$

with f_i irreducible in $\mathcal{O}_n$; if we set $V_i = \{f_i(z) = 0\}$ then we have

$$V = V_1 \cup \cdots \cup V_k$$

with V_i irreducible at 0. Thus if p is any point on any analytic hypersurface $V \subset U \subset \mathbb{C}^n$, V *can be expressed uniquely in some neighborhood* U′ *of* p *as the union of a finite number of analytic hypersurfaces irreducible at* p.

2. Let $W \subset U \subset \mathbb{C}^n$ be an analytic variety given in a neighborhood Δ of $0 \in W$ as the zero locus of two functions $f, g \in \mathcal{O}_n$. If W contains no analytic hypersurface through 0, then f and g are necessarily relatively prime in $\mathcal{O}_n$; if W does not contain the line $\{z' = 0\}$, then by taking linear combinations we may assume that neither $\{f(z) = 0\}$ or $\{g(z) = 0\}$ contains $\{z' = 0\}$, and hence that f and g are Weierstrass polynomials in z_n. Let

$$\gamma = \alpha f + \beta g \neq 0 \in \mathcal{O}_{n-1}$$

be the resultant of f and g. We claim that the image of W under the projection map $\pi: \mathbb{C}^n \to \mathbb{C}^{n-1}$ is just the locus of γ. To see this, write

$$\alpha = hg + r$$

with the degree of r strictly less than the degree of g. Then

$$\gamma = rf + (\beta + hf)g.$$

Now, if for some z in $\mathbb{C}^{n-1}$, γ vanishes at z but f and g have no common zeros along the line $\pi^{-1}(z)$, it follows that r vanishes at all the zeros of g in $\pi^{-1}(z)$; since $\deg(r) < \deg(g)$, this implies that r, and hence $\beta + hf$, vanish identically on $\pi^{-1}(z)$. Thus r and $\beta + hf$ both are zero on the inverse image of any component of the zero locus of γ other than $\pi(W)$; but r and $\beta + hf$ are relatively prime and so have no common components. We see then that *$\pi(W)$ is an analytic hypersurface in a neighborhood of the origin in* $\mathbb{C}^{n-1}$, and, reiterating our basic description of analytic hypersurfaces, that *projection of* W *onto a suitably chosen* (n−2)*-plane* $\mathbb{C}^{n-2} \subset \mathbb{C}^n$ *expresses* W *locally as a finite-sheeted branched cover of a neighborhood of the origin in* $\mathbb{C}^{n-2}$.

3. Last, let $V \subset U \subset \mathbb{C}^n$ be an analytic variety irreducible at $0 \in V$ such that for arbitrarily small neighborhoods Δ of 0 in $\mathbb{C}^n$, $\pi(V \cap \Delta)$ contains a neighborhood of 0 in $\mathbb{C}^{n-1}$. Write

$$V = \{f_1(z) = \cdots = f_k(z) = 0\}$$

near 0. Then the functions $f_i \in \mathcal{O}_n$ must all have a common factor in $\mathcal{O}_n$, since otherwise V would be contained in the common locus of two relatively prime functions, and by assertion 2, $\pi(V \cap \Delta)$ would be a proper analytic subvariety of $\mathbb{C}^{n-1}$. If we let $g(z)$ be the greatest common divisor of the f_i's, then we can write

$$V = \{g(z) = 0\} \cup \left\{ \frac{f_1(z)}{g(z)} = \cdots = \frac{f_k(z)}{g(z)} = 0 \right\}.$$

Since V is irreducible at 0 and since the locus $\{f_i(z)/g(z) = 0, \text{ all } i\}$ cannot contain $\{g(z) = 0\}$, we must have

$$V = \{g(z) = 0\},$$

i.e., V is an analytic hypersurface near 0.

The results 1, 2, and 3 above, together with our basic picture of an analytic hypersurface, give us a picture of the local behavior of those analytic varieties cut out locally by one or two holomorphic functions. In fact, the same picture is in almost all respects valid for general analytic varieties, but to prove this requires some relatively sophisticated techniques from the theory of several complex variables. Since the primary focus of the material in this book is on the codimension 1 case, we will for the time being simply state here without proof the analogous results for general analytic varieties:

1. If $V \subset U \subset \mathbb{C}^n$ is any analytic variety and $p \in V$, then in some neighborhood of p, V can be uniquely written as the union of analytic varieties V_i irreducible at p with $V_i \not\subset V_j$.
2. Any analytic variety can be expressed locally by a projection map as a finite-sheeted cover of a polydisc Δ branched over an analytic hypersurface of Δ.
3. If $V \subset \mathbb{C}^n$ does not contain the line $z_1 = \cdots = z_{n-1} = 0$, then the image of a neighborhood of 0 in V under the projection map $\pi:(z_1,\ldots,z_n) \to (z_1,\ldots,z_{n-1})$ is an analytic subvariety in a neighborhood of $0 \in \mathbb{C}^{n-1}$.

The difficulties in proving these results are more technical than conceptual. For example, to prove assertion 3, note that if V is given near $0 \in \mathbb{C}^n$ by functions $f_1,\ldots,f_k$, then $\pi(V)$ is defined in a neighborhood of $0 \in \mathbb{C}^{n-1}$ by the resultants of all pairs of relatively prime linear combinations of the f_i. The problem then is to show that the zero locus of an arbitrary collection of holomorphic functions in a polydisc is in fact given by a finite number of holomorphic functions in a slightly smaller polydisc. Granted assertions 3 and 1, 2 is not hard to prove by a sequence of projections.

All of these facts will follow from the proper mapping theorem, which we shall state in the next section and prove in Chapter 3.

Finally, several more foundational results in several complex variables will be proved by the method of residues in Chapter 5.

2. COMPLEX MANIFOLDS

Complex Manifolds

DEFINITION. A *complex manifold* M is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.

A function on an open set $U \subset M$ is *holomorphic* if, for all α, $f \cdot \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$. Likewise, a collection $z = (z_1,\ldots,z_n)$ of functions on $U \subset M$ is said to be a *holomorphic coordinate system* if $\varphi_\alpha \circ z^{-1}$ and $z \circ \varphi_\alpha^{-1}$ are holomorphic on $z(U \cap U_\alpha)$ and $\varphi_\alpha(U \cap U_\alpha)$, respectively, for each α; a map $f: M \to N$ of complex manifolds is *holomorphic* if it is given in terms of local holomorphic coordinates on N by holomorphic functions.

Examples

1. A one-dimensional complex manifold is called a *Riemann surface.*

2. Let $\mathbb{P}^n$ denote the set of lines through the origin in $\mathbb{C}^{n+1}$. A line $l \subset \mathbb{C}^{n+1}$ is determined by any $Z \neq 0 \in l$, so we can write

$$\mathbb{P}^n = \frac{\{[Z] \neq 0 \in \mathbb{C}^{n+1}\}}{[Z] \sim [\lambda Z]}.$$

On the subset $U_i = \{[Z] : Z_i \neq 0\} \subset \mathbb{P}^n$ of lines not contained in the hyperplane $(Z_i = 0)$, there is a bijective map φ_i to $\mathbb{C}^n$ given by

$$\varphi_i([Z_0, \ldots, Z_n]) = \left(\frac{Z_0}{Z_i}, \ldots, \frac{\hat{Z}_i}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right).$$

On $(z_j \neq 0) = \varphi_i(U_j \cap U_i) \subset \mathbb{C}^n$,

$$\varphi_j \circ \varphi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{\hat{z}_j}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

is clearly holomorphic; thus $\mathbb{P}^n$ has the structure of a complex manifold, called *complex projective space*. The "coordinates" $Z = [Z_0, \ldots, Z_n]$ are called *homogeneous coordinates* on $\mathbb{P}^n$; the coordinates given by the maps φ_i are called *Euclidean coordinates*. $\mathbb{P}^n$ is compact, since we have a continuous surjective map from the unit sphere in $\mathbb{C}^{n+1}$ to $\mathbb{P}^n$. Note that $\mathbb{P}^1$ is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Any inclusion $\mathbb{C}^{k+1} \to \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{P}^k \to \mathbb{P}^n$; the image of such a map is called a *linear subspace* of $\mathbb{P}^n$. The image of a hyperplane in $\mathbb{C}^{n+1}$ is again called a *hyperplane*, the image of a 2-plane $\mathbb{C}^2 \subset \mathbb{C}^{n+1}$ is a line, and in general the image of a $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ is called a k-plane. We may speak of linear relations among points in $\mathbb{P}^n$ in these terms: for example, the *span* of a collection $\{p_i\}$ of points in $\mathbb{P}^n$ is taken to be the image in $\mathbb{P}^n$ of the subspace in $\mathbb{C}^{n+1}$ spanned by the lines $\pi^{-1}(p_i)$; k points are said to be *linearly independent* if their corresponding lines in $\mathbb{C}^{n+1}$ are, that is, if their span in $\mathbb{P}^n$ is a $(k-1)$-plane.

Note that the set of hyperplanes in $\mathbb{P}^n$ corresponds to the set $\mathbb{C}^{n+1*} - \{0\}$ of nonzero linear functionals on $\mathbb{C}^{n+1}$ modulo scalar multiplication; it is thus itself a projective space, called the *dual projective* space and denoted $\mathbb{P}^{n*}$.

It is sometimes convenient to picture $\mathbb{P}^n$ as the compactification of $\mathbb{C}^n$ obtained by adding on the hyperplane H at infinity. In coordinates the inclusion $\mathbb{C}^n \to \mathbb{P}^n$ is $(z_1, \ldots, z_n) \to [1, z_1, \ldots, z_n]$; H has equation $(Z_0 = 0)$, and the identification $H \cong \mathbb{P}^{n-1}$ comes by considering the hyperplane at infinity as the directions in which we can go to infinity in $\mathbb{C}^n$.

3. Let $\Lambda = \mathbb{Z}^k \subset \mathbb{C}^n$ be a discrete lattice. Then the quotient group $\mathbb{C}^n/\Lambda$ has the structure of a complex manifold induced by the projection map $\pi: \mathbb{C}^n \to \mathbb{C}^n/\Lambda$. It is compact if and only if $k=2n$; in this case $\mathbb{C}^n/\Lambda$ is called a *complex torus*.

In general, if $\pi: M \to N$ is a topological covering space and N is a complex manifold, then π gives M the structure of a complex manifold as well; if M is a complex manifold and the deck transformations of M are holomorphic, then N inherits the structure of a complex manifold from M.

Another example of this construction is the *Hopf surface*, defined to be the quotient of $\mathbb{C}^2 - \{0\}$ by the group of automorphisms generated by $z \mapsto 2z$. The Hopf surface is the simplest example of a compact complex manifold that cannot be imbedded in projective space of any dimension.

Let M be a complex manifold, $p \in M$ any point, and $z = (z_1, \ldots, z_n)$ a holomorphic coordinate system around p. There are three different notions of a tangent space to M at p, which we now describe:

1. $T_{\mathbb{R},p}(M)$ is the usual *real tangent space* to M at p, where we consider M as a real manifold of dimension $2n$. $T_{\mathbb{R},p}(M)$ can be realized as the space of $\mathbb{R}$-linear derivations on the ring of real-valued C^∞ functions in a neighborhood of p; if we write $z_i = x_i + iy_i$,

$$T_{\mathbb{R},p}(M) = \mathbb{R}\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}.$$

2. $T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexified tangent space* to M at p. It can be realized as the space of $\mathbb{C}$-linear derivations in the ring of complex-valued C^∞ functions on M around p. We can write

$$\begin{aligned} T_{\mathbb{C},p}(M) &= \mathbb{C}\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\} \\ &= \mathbb{C}\left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\} \end{aligned}$$

where, as before,

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\, \frac{\partial}{\partial y_i} \right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\, \frac{\partial}{\partial y_i} \right).$$

3. $T'_p(M) = \mathbb{C}\{\partial/\partial z_i\} \subset T_{\mathbb{C},p}(M)$ is called the *holomorphic tangent space* to M at p. It can be realized as the subspace of $T_{\mathbb{C},p}(M)$ consisting of derivations that vanish on antiholomorphic functions (i.e., f such that $\bar{f}$ is holomorphic), and so is independent of the holomorphic coordinate system

$(z_1,\dots,z_n)$ chosen. The subspace $T_p''(M)=\mathbb{C}\{\partial/\partial\bar{z}_i\}$ is called the *antiholomorphic tangent space* to M at p; clearly

$$T_{\mathbb{C},p}(M) = T_p'(M)\oplus T_p''(M).$$

Observe that for M,N complex manifolds any C^∞ map $f:M\to N$ induces a linear map

$$f_*:\ T_{\mathbb{R},p}(M)\to T_{\mathbb{R},f(p)}(N)$$

and hence a map

$$f_*:\ T_{\mathbb{C},p}(M)\to T_{\mathbb{C},f(p)}(N),$$

but does not in general induce a map from $T_p'(M)$ to $T_{f(p)}'(N)$. In fact, *a map* f : M→N *is holomorphic if and only if*

$$f_*(T_p'(M))\subset T_{f(p)}'(N)$$

for all $p\in M$.

Note also that since $T_{\mathbb{C},p}(M)$ is given naturally as the real vector space $T_{\mathbb{R},p}(M)$ tensored with $\mathbb{C}$, the operation of conjugation sending $\partial/\partial z_i$ to $\partial/\partial\bar{z}_i$ is well-defined and

$$T_p''(M) = \overline{T_p'(M)}\,.$$

It follows that the projection

$$T_{\mathbb{R},p}(M)\ \to T_{\mathbb{C},p}(M)\to T_p'(M)$$

is an $\mathbb{R}$-linear isomorphism. This last feature allows us to "do geometry" purely in the holomorphic tangent space. For example, let $z(t)$ $(0\leqslant t\leqslant 1)$ be a smooth arc in the complex z-plane. Then $z(t)=x(t)+\sqrt{-1}\,y(t)$, and the tangent to the arc may be taken either as

$$x'(t)\frac{\partial}{\partial x}+y'(t)\frac{\partial}{\partial y}\qquad \text{in } T_{\mathbb{R}}(\mathbb{C})$$

or

$$z'(t)\frac{\partial}{\partial z}\qquad \text{in } T'(\mathbb{C}),$$

and these two correspond under the projection $T_{\mathbb{R}}(\mathbb{C})\to T'(\mathbb{C})$.

Now let M,N be complex manifolds, $z=(z_1,\dots,z_n)$ be holomorphic coordinates centered at $p\in M$, $w=(w_1,\dots,w_m)$ holomorphic coordinates centered at $q\in N$ and $f:M\to N$ a holomorphic map with $f(p)=q$. Corresponding to the various tangent spaces to M and N at p and q, we have different notions of the *Jacobian* of f, as follows:

1. If we write $z_i=x_i+\sqrt{-1}\,y_i$ and $w_\alpha=u_\alpha+\sqrt{-1}\,v_\alpha$, then in terms of the bases $\{\partial/\partial x_i,\partial/\partial y_i\}$ and $\{\partial/\partial u_\alpha,\partial/\partial v_\alpha\}$ for $T_{\mathbb{R},p}(M)$ and $T_{\mathbb{R},q}(N)$, the

linear map f_* is given by the $2m\times 2n$ matrix

$$\mathcal{J}_{\mathbf{R}}(f) = \left[\begin{array}{c|c} \dfrac{\partial u_\alpha}{\partial x_j} & \dfrac{\partial u_\alpha}{\partial y_j} \\ \hline \dfrac{\partial v_\alpha}{\partial x_j} & \dfrac{\partial v_\alpha}{\partial y_j} \end{array}\right].$$

In terms of the bases $\{\partial/\partial z_i, \partial/\partial \bar{z}_i\}$ and $\{\partial/\partial w_\alpha, \partial/\partial \bar{w}_\alpha\}$ for $T_{\mathbf{C},p}(M)$ and $T_{\mathbf{C},q}(N)$, f_* is given by

$$\mathcal{J}_{\mathbf{C}}(f) = \begin{pmatrix} \mathcal{J}(f) & 0 \\ 0 & \overline{\mathcal{J}(f)} \end{pmatrix}$$

where

$$\mathcal{J}(f) = \left(\frac{\partial w_\alpha}{\partial z_j}\right).$$

Note in particular that rank $\mathcal{J}_{\mathbf{R}}(f) = 2\cdot$rank $\mathcal{J}(f)$ and that if $m=n$, then

$$\begin{aligned} \det \mathcal{J}_{\mathbf{R}}(f) &= \det \mathcal{J}(f)\cdot \det \overline{\mathcal{J}(f)} \\ &= |\det \mathcal{J}(f)|^2 \geqslant 0, \end{aligned}$$

i.e., *holomorphic maps are orientation preserving*. We take the *natural orientation* on $\mathbb{C}^n$ to be given by the $2n$-form

$$\left(\frac{\sqrt{-1}}{2}\right)^n (dz_1\wedge d\bar{z}_1)\wedge(dz_2\wedge d\bar{z}_2)\wedge\cdots\wedge(dz_n\wedge d\bar{z}_n) = dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n;$$

it is clear that if $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$, $\varphi_\beta : U_\beta \to \mathbb{C}^n$ are holomorphic coordinate maps on the complex manifold M, the pullbacks via φ_α and φ_β of the natural orientation on $\mathbb{C}^n$ agree on $U_\alpha \cap U_\beta$. Thus *any complex manifold has a natural orientation* which is preserved under holomorphic maps.

Submanifolds and Subvarieties

Now that we have established the relations among the various Jacobians of a holomorphic map, it is not hard to prove the

Inverse Function Theorem. *Let* U,V *be open sets in* $\mathbb{C}^n$ *with* $0\in\mathrm{U}$ *and* $\mathrm{f}:\mathrm{U}\to\mathrm{V}$ *a holomorphic map with* $\mathcal{J}(\mathrm{f})=(\partial \mathrm{f}_i/\partial \mathrm{z}_j)$ *nonsingular at* 0. *Then* f *is one-to-one in a neighborhood of* 0, *and* f^{-1} *is holomorphic at* f(0).

Proof. First, since $\det|\mathcal{J}_{\mathbf{R}}(f)| = |\det(\mathcal{J}(f))|^2 \neq 0$ at 0, by the ordinary inverse function theorem f has a C^∞ inverse f^{-1} near 0. Now we have

$$f^{-1}(f(z)) = z$$

so

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \bar{z}_i}(f^{-1}(f(z))) \\
&= \sum_k \frac{\partial f_j^{-1}}{\partial z_k}\cdot\frac{\partial f_k}{\partial \bar{z}_i} + \sum_k \frac{\partial f_j^{-1}}{\partial \bar{z}_k}\cdot\frac{\partial \bar{f}_k}{\partial \bar{z}_i} \\
&= \sum_k \frac{\partial f_j^{-1}}{\partial \bar{z}_k}\cdot\left(\overline{\frac{\partial f_k}{\partial z_i}}\right) \qquad \text{for all } i,j.
\end{aligned}
$$

Since $(\partial f_k/\partial z_i)$ is nonsingular, this implies $\partial f_j^{-1}/\partial \bar{z}_k = 0$ for all j,k, so f^{-1} is holomorphic. Q.E.D.

Similarly, we have the

Implicit Function Theorem. *Given* $\mathrm{f}_1,\ldots,\mathrm{f}_k \in \mathcal{O}_n$ *with*

$$\det\left(\frac{\partial f_i}{\partial z_j}(0)\right)_{1 \leqslant i,j \leqslant k} \neq 0,$$

there exist functions $\mathrm{w}_1,\ldots,\mathrm{w}_k \in \mathcal{O}_{n-k}$ *such that in a neighborhood of* 0 *in* $\mathbb{C}^n$,

$$f_1(z) = \cdots = f_k(z) = 0 \Leftrightarrow z_i = w_i(z_{k+1},\ldots,z_n), \qquad 1 \leqslant i \leqslant k.$$

Proof. Again, by the C^∞ implicit function theorem we can find C^∞ functions $w_1,\ldots,w_k$ with the required property; to see that they are holomorphic we write, for $z=(z_{k+1},\ldots,z_n)$, $k+1 \leqslant \alpha \leqslant n$,

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \bar{z}_\alpha}(f_j(w(z),z)) \\
&= \frac{\partial f_j}{\partial \bar{z}_\alpha}(w(z),z) + \sum \frac{\partial w_l}{\partial \bar{z}_\alpha}\cdot\frac{\partial f_j}{\partial w_l}(w(z),z) + \sum \frac{\partial \bar{w}_l}{\partial \bar{z}_\alpha}\cdot\frac{\partial f_j}{\partial \bar{w}_l}(w(z),z) \\
&= \sum \frac{\partial w_l}{\partial \bar{z}_\alpha}\cdot\frac{\partial f_j}{\partial w_l}(w(z),z) \\
&\Rightarrow \frac{\partial w_l}{\partial \bar{z}_\alpha} = 0 \qquad \text{for all } \alpha, l.
\end{aligned}
$$

Q.E.D.

One special feature of the holomorphic case is the following:

Proposition. *If* $\mathrm{f}: \mathrm{U} \to \mathrm{V}$ *is a one-to-one holomorphic map of open sets in* $\mathbb{C}^n$ *then* $|\mathcal{J}(\mathrm{f})| \neq 0$, *i.e.,* f^{-1} *is holomorphic.*

Proof. We prove this by induction on n; the case $n=1$ is clear. Let $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$ be coordinates on U and V, respectively, and suppose $\mathcal{J}(f)$ has rank k at $0 \in U$; we may assume then that the matrix

$((\partial f_i/\partial z_j)(0))_{0\leqslant i,j\leqslant k}$ is nonsingular. Set

$$z_i' = f_i(z), \qquad 1 \leqslant i \leqslant k,$$
$$z_\alpha' = z_\alpha, \qquad k+1 \leqslant \alpha \leqslant n;$$

by the inverse function theorem, $z'=(z_1',\ldots,z_n')$ is a holomorphic coordinate system for U near 0. But now f maps the locus $(z_1'=\cdots=z_k'=0)$ one-to-one onto the locus $(w_1=\cdots=w_k=0)$ and the Jacobian $(\partial f_\alpha/\partial z_\beta')$ of $f|_{(z_1'=\cdots=z_k'=0)}$ is singular at 0; by the induction hypothesis, either $k=0$ or $k=n$. We see then that the Jacobian matrix of f vanishes identically wherever its determinant is zero, i.e., that f maps every connected component of the locus $|\mathcal{J}(f)|=0$ to a single point in V. Since f is one-to-one and the zero locus of the holomorphic function $|\mathcal{J}(f)|$ is positive-dimensional if nonempty, it follows that $|\mathcal{J}(f)|\neq 0$. Q.E.D.

Note that this proposition is in contrast to the real case, where the map $t\mapsto t^3$ on $\mathbb{R}$ is one-to-one but does not have a C^∞ inverse.

Now we can make the

DEFINITION. A *complex submanifold* S of a complex manifold M is a subset $S\subset M$ given locally either as the zeros of a collection $f_1,\ldots,f_k$ of holomorphic functions with rank $\mathcal{J}(f)=k$, or as the image of an open set U in $\mathbb{C}^{n-k}$ under a map $f\colon U\to M$ with rank $\mathcal{J}(f)=n-k$.

The implicit function theorem assures us that the two alternate conditions of the definition are in fact equivalent, and that the submanifold S has naturally the structure of a complex manifold of dimension $n-k$.

DEFINITION. An *analytic subvariety* V of a complex manifold M is a subset given locally as the zeros of a finite collection of holomorphic functions. A point $p\in V$ is called a *smooth point* of V if V is a submanifold of M near p, that is, if V is given in some neighborhood of p by holomorphic functions $f_1,\ldots,f_k$ with rank $\mathcal{J}(f)=k$; the locus of smooth points of V is denoted V^*. A point $p\in V-V^*$ is called a *singular point* of V; the *singular locus* $V-V^*$ of V is denoted V_s. V is called *smooth* or *nonsingular* if $V=V^*$, i.e., if V is a submanifold of M.

In particular, if p is a point of an analytic hypersurface $V\subset M$ given in terms of local coordinates z by the function f, we define the *multiplicity* $\text{mult}_p(V)$ to be the order of vanishing of f at p, that is, the greatest integer m such that all partial derivatives

$$\frac{\partial^k f}{\partial z_{i_1}\cdots\partial z_{i_k}}(p)=0, \qquad k\leqslant m-1.$$

We should mention here a piece of terminology that is pervasive in algebraic geometry: the word *generic*. When we are dealing with a family of objects parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement that "a (or the) generic member of the family has a certain property" means exactly that "the set of objects in the family that do not have that property is contained in a subvariety of strictly smaller dimension".

In general, it will be clear how the objects in our family are to be parametrized. One exception will be a reference to "the generic k-plane in $\mathbb{P}^n$": until the section on Grassmannians, we have—at least officially—no way of parametrizing linear subspaces of projective space. The fastidious reader may substitute "the linear span of the generic $(k+1)$-tuple of points in $\mathbb{P}^n$."

A basic fact about analytic subvarieties is the

Proposition. V_s *is contained in an analytic subvariety of* M *not equal to* V.

Proof. For $p \in V$ let k be the largest integer such that there exist k functions $f_1, \ldots, f_k$ in a neighborhood U of p vanishing on V and such that $\mathcal{J}(f)$ has a $k \times k$ minor not everywhere singular on V; we may assume that $|(\partial f_i/\partial z_j)_{1 \leqslant i,j \leqslant k}| \not\equiv 0$ on V. Let $U' \subset U$ be the locus of $|(\partial f_i/\partial z_j)_{1 \leqslant i,j \leqslant k}| \neq 0$ and V' the locus $f_1 = \cdots = f_k = 0$. Then $V' = V \cap U'$ is a complex submanifold of U', and for any holomorphic function f vanishing on V the differential $df \equiv 0$ on V', i.e., f is constant on V'. It follows that for $q \in V'$ near p, $V = V'$ is a manifold in a neighborhood of q and so $V_s \subset (|(\partial f_i/\partial z_j)_{1 \leqslant i,j \leqslant k}| = 0)$. Q.E.D.

It is in fact the case that V_s is an analytic subvariety of M—if we choose local defining functions $f_1, \ldots, f_l$ for V carefully, V_s will be the common zero locus of the determinants of the $k \times k$ minors of $\mathcal{J}(f)$. For our purposes, however, we simply need to know that the singular locus of an analytic variety is comparatively small, and so we will not prove this stronger assertion.

We state one more result on analytic varieties:

Proposition. *An analytic variety* V *is irreducible if and only if* V* *is connected.*

Proof. One direction is clear: if $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneqq V$ analytic varieties, then $(V_1 \cap V_2) \subset V_s$, so V^* is disconnected.

The converse is harder to prove in general; since we will use it only for analytic hypersurfaces, we will prove it in this case. Suppose V^* is disconnected, and let $\{V_i\}$ denote the connected components of V^*; we want to show that $\overline{V}_i$ is an analytic variety. Let $p \in \overline{V}_i$ be any point, f a defining function for V near p, and $z = (z_1, \ldots, z_n)$ local coordinates around p; we may assume that f is a Weierstrass polynomial of degree k in z_n.

Write

$$g=\alpha\cdot f+\beta\cdot\frac{\partial f}{\partial z_n},\qquad g\neq 0\in\mathcal{O}_{n-1};$$

then for Δ some polydisc around p and Δ' a polydisc in $\mathbb{C}^{n-1}$, the projection map π: $(z_1,\dots,z_n)\mapsto(z_1,\dots,z_{n-1})$ expresses $V_i\cap(\Delta-(g=0))$ as a covering space of $\Delta'-(g=0)$. Let $\{w_\nu(z')\}$ denote the z_n-coordinates of the points in $\pi^{-1}(z')$ for $z'=(z_1,\dots,z_{n-1})\in\Delta'-(g=0)$ and let $\sigma_1(z'),\dots,\sigma_k(z')$ denote the elementary symmetric functions of the w_ν. The functions σ_i are well-defined and bounded on $\Delta'-(g=0)$, and so extend to Δ'; the function

$$f_i(z)=z_n^k+\sigma_1(z')z_n^{k-1}+\cdots+\sigma_k(z')$$

is thus holomorphic in a neighborhood of p and vanishes exactly on $\bar{V}_i$.

Q.E.D.

We take the *dimension* of an irreducible analytic variety V to be the dimension of the complex manifold V^*; we say that a general analytic variety is of dimension k if all of its irreducible components are.

A note: if $V\subset M$ is an analytic subvariety of a complex manifold M, then we may define the *tangent cone* $T_p(V)\subset T_p'(M)$ to V at any point $p\in V$ as follows: if $V=(f=0)$ is an analytic hypersurface, and in terms of holomorphic coordinates $z_1,\dots,z_n$ on M centered around p we write

$$f(z_1,\dots,z_n)=f_m(z_1,\dots,z_n)+f_{m+1}(z_1,\dots,z_n)+\cdots$$

with $f_k(z_1,\dots,z_n)$ a homogeneous polynomial of degree k in $z_1,\dots,z_n$, then the tangent cone to V at p is taken to be the subvariety of $T_p'(M)=\mathbb{C}\{\partial/\partial z_i\}$ defined by

$$\left\{\sum\alpha_i\frac{\partial}{\partial z_i}:f_m(\alpha_1,\dots,\alpha_n)=0\right\}.$$

In general, then, the tangent cone to an analytic variety $V\subset M$ at $p\in V$ is taken to be the intersection of the tangent cones at p to all local analytic hypersurfaces in M containing V. In case V is smooth at p, of course, this is just the tangent space to V at p.

More geometrically, the tangent cone $T_p(V)\subset T_p'(M)$ may be realized as the union of the tangent lines at p to all analytic arcs $\gamma:\Delta\to V\subset M$.

The *multiplicity* of a subvariety V of dimension k in M at a point p, denoted $\operatorname{mult}_p(V)$, is taken to be the number of sheets in the projection, in a small coordinate polydisc on M around p, of V onto a generic k-dimensional polydisc; note that p is a smooth point of V if and only if $\operatorname{mult}_p(V)=1$. In general, if $W\subset M$ is an irreducible subvariety, we define the *multiplicity* $\operatorname{mult}_W(V)$ *of* V *along* W to be simply the multiplicity of V at a generic point of W.

De Rham and Dolbeault Cohomology

Let M be a differentiable manifold. Let $A^p(M,\mathbb{R})$ denote the space of differential forms of degree p on M, and $Z^p(M,\mathbb{R})$ the subspace of closed p-forms. Since $d^2=0$, $d(A^{p-1}(M,\mathbb{R}))\subset Z^p(M,\mathbb{R})$; the quotient groups

$$H^p_{\mathrm{DR}}(M,\mathbb{R}) = \frac{Z^p(M,\mathbb{R})}{dA^{p-1}(M,\mathbb{R})}$$

of closed forms modulo exact forms are called the *de Rham cohomology groups of M*.

In the same way, we can let $A^p(M)$ and $Z^p(M)$ denote the spaces of complex-valued p-forms and closed complex-valued p-forms on M, respectively, and let

$$H^p_{\mathrm{DR}}(M) = \frac{Z^p(M)}{dA^{p-1}(M)}$$

be the corresponding quotient; clearly

$$H^p_{\mathrm{DR}}(M) = H^p_{\mathrm{DR}}(M,\mathbb{R})\otimes\mathbb{C}.$$

Now let M be a complex manifold. By linear algebra, the decomposition

$$T^*_{\mathbb{C},z}(M) = T^{*\prime}_z(M)\oplus T^{*\prime\prime}_z(M)$$

of the cotangent space to M at each point $z\in M$ gives a decomposition

$$\wedge^n T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=n}\left(\wedge^p T^{*\prime}_z(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\right).$$

Correspondingly, we can write

$$A^n(M) = \bigoplus_{p+q=n} A^{p,q}(M),$$

where

$$A^{p,q}(M) = \left\{\varphi\in A^n(M)\colon\ \varphi(z)\in\wedge^p T^{*\prime}_z(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\text{ for all } z\in M\right\}.$$

A form $\varphi\in A^{p,q}(M)$ is said to be of *type* (p,q). By way of notation, we denote by $\pi^{(p,q)}$ the projection maps

$$A^*(M)\to A^{p,q}(M),$$

so that for $\varphi\in A^*(M)$,

$$\varphi = \sum \pi^{(p,q)}\varphi;$$

we usually write $\varphi^{(p,q)}$ for $\pi^{(p,q)}\varphi$.

If $\varphi\in A^{p,q}(M)$, then for each $z\in M$,

$$d\varphi(z)\in\left(\wedge^p T^{*\prime}(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\right)\wedge T^*_{\mathbb{C},z}(M),$$

i.e.,

$$d\varphi\in A^{p+1,q}(M)\oplus A^{p,q+1}(M).$$

We define the operators

$$\bar{\partial}: A^{p,q}(M) \to A^{p,q+1}(M)$$
$$\partial: A^{p,q}(M) \to A^{p+1,q}(M)$$

by

$$\bar{\partial} = \pi^{(p,q+1)} \circ d, \qquad \partial = \pi^{(p+1,q)} \circ d;$$

accordingly, we have

$$d = \partial + \bar{\partial}.$$

In terms of local coordinates $z = (z_1, \ldots, z_m)$, a form $\varphi \in A^n(M)$ is of type (p,q) if we can write

$$\varphi(z) = \sum_{\substack{\#I=p\\ \#J=q}} \varphi_{IJ}(z)\, dz_I \wedge d\bar{z}_J,$$

where for each multiindex $I = \{i_1, \ldots, i_p\}$,

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

The operators ∂ and $\bar{\partial}$ are then given by

$$\bar{\partial}\varphi(z) = \sum_{I,J,j} \frac{\partial}{\partial \bar{z}_j} \varphi_{IJ}(z)\, d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J,$$
$$\partial\varphi(z) = \sum_{I,J,i} \frac{\partial}{\partial z_i} \varphi_{IJ}(z)\, dz_i \wedge dz_I \wedge d\bar{z}_J.$$

In particular, we say that a form φ of type $(q,0)$ is *holomorphic* if $\bar{\partial}\varphi = 0$; clearly this is the case if and only if

$$\varphi(z) = \sum_{\#I=q} \varphi_I(z)\, dz_I$$

with $\varphi_I(z)$ holomorphic.

Note that since the decomposition $T^*_{\mathbb{C},z} = T^{*'}_z \oplus T^{*''}_z$ is preserved under holomorphic maps, so is the decomposition $A^* = \oplus A^{p,q}$. For f: $M \to N$ a holomorphic map of complex manifolds,

$$f^*(A^{p,q}(N)) \subset A^{p,q}(M)$$

and

$$\bar{\partial} \circ f^* = f^* \circ \bar{\partial} \qquad \text{on } A^{p,q}(N).$$

Let $Z^{p,q}_{\bar{\partial}}(M)$ denote the space of $\bar{\partial}$-closed forms of type (p,q). Since $\partial^2/\partial z_i \partial \bar{z}_j = \partial^2/\partial \bar{z}_j \partial \bar{z}_i$

$$\bar{\partial}^2 = 0$$

on $A^{p,q}(M)$, and we have

$$\bar{\partial}(A^{p,q}(M)) \subset Z^{p,q+1}_{\bar{\partial}}(M);$$

accordingly, we define the *Dolbeault cohomology groups* to be

$$H_{\bar\partial}^{p,q}(M) = \frac{Z_{\bar\partial}^{p,q}(M)}{\bar\partial(A^{p,q-1}(M))}.$$

Note in particular that if $f: M \to N$ is a holomorphic map of complex manifolds, f induces a map

$$f^*: H_{\bar\partial}^{p,q}(N) \to H_{\bar\partial}^{p,q}(M).$$

The ordinary Poincaré lemma that every closed form on $\mathbb{R}^n$ is exact assures us that the de Rham groups are locally trivial. Analogously, a fundamental fact about the Dolbeault groups is the

$\bar\partial$-Poincaré Lemma. *For $\Delta = \Delta(r)$ a polycylinder in $\mathbb{C}^n$,*

$$H_{\bar\partial}^{p,q}(\Delta) = 0, \qquad q \geqslant 1.$$

Proof. First note that if

$$\varphi = \sum_{\substack{{}^\# I = p \\ {}^\# J = q}} \varphi_{IJ} \cdot dz_I \wedge d\bar z_J$$

is a $\bar\partial$-closed form, then the forms

$$\varphi_I = \sum_{{}^\# J = q} \varphi_{IJ} \cdot d\bar z_J \in A^{0,q}(\Delta)$$

are again closed, and that if

$$\varphi_I = \bar\partial \eta_I$$

then

$$\varphi = \pm \bar\partial \Big(\sum_I dz_I \wedge \eta_I \Big);$$

thus it is sufficient to prove that the groups $H_{\bar\partial}^{0,q}(\Delta)$ vanish.

We first show that if φ is a $\bar\partial$-closed $(0,q)$-form on $\Delta = \Delta(r)$, then for any $s < r$, we can find $\psi \in A^{0,q-1}(\Delta(s))$ with $\bar\partial\psi = \varphi$ in $\Delta(s)$. To see this, write

$$\varphi = \sum \varphi_I \, d\bar z_I;$$

we claim that if $\varphi \equiv 0$ modulo $(d\bar z_1, \ldots, d\bar z_k)$—that is, if $\varphi_I \equiv 0$ for $I \not\subset \{1, \ldots, k\}$—then we can find $\eta \in A^{0,q-1}(\Delta(s))$ such that

$$\varphi - \bar\partial\eta \equiv 0 \text{ modulo } (d\bar z_1, \ldots, d\bar z_{k-1});$$

this will clearly be sufficient. So assume $\varphi \equiv 0$ modulo $(d\bar z_1, \ldots, d\bar z_k)$ and set

$$\varphi_1 = \sum_{I:\ k \in I} \varphi_I \cdot d\bar z_{I - \{k\}},$$

$$\varphi_2 = \sum_{I:\ k \notin I} \varphi_I \cdot d\bar z_I,$$

so that $\varphi = \varphi_1 \wedge d\bar{z}_k + \varphi_2$, with $\varphi_2 \equiv 0$ modulo $(d\bar{z}_1, \ldots, d\bar{z}_{k-1})$. If $l > k$, $\bar{\partial}\varphi_2$ contains no terms with a factor $d\bar{z}_k \wedge d\bar{z}_l$; since $\bar{\partial}\varphi = \bar{\partial}\varphi_1 + \bar{\partial}\varphi_2 = 0$, it follows that

$$\frac{\partial}{\partial \bar{z}_l} \varphi_I = 0$$

for $l > k$ and I such that $k \in I$.

Now set

$$\eta = \sum_{I:\ k \in I} \eta_I \, d\bar{z}_{I - \{k\}}$$

where

$$\eta_I(z) = \frac{1}{2\pi\sqrt{-1}} \int_{|w_k| \leq s_k} \varphi_I(z_1, \ldots, w_k, \ldots, z_n) \frac{dw_k \wedge d\bar{w}_k}{w_k - z_k}.$$

By the proposition on p. 5, we have

$$\frac{\partial}{\partial \bar{z}_k} \eta_I(z) = \varphi_I(z),$$

and for $l > k$,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_l} \eta_I(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_k| \leq s_k} \frac{\partial}{\partial \bar{z}_l} \varphi_I(z_1, \ldots, w_k, \ldots, z_n) \frac{dw_k \wedge d\bar{w}_k}{w_k - z_k} \\ &= 0 \end{aligned}$$

Thus

$$\varphi - \bar{\partial}\eta \equiv 0 \text{ modulo } (d\bar{z}_1, \ldots, d\bar{z}_{k-1})$$

in $\Delta(s)$ as was desired.

To prove the full $\bar{\partial}$-Poincaré lemma let $\{r_i\}$ be a monotone increasing sequence tending to r. By the first step, we can find $\psi_k \in A^{0,q-1}(\Delta)$ such that $\bar{\partial}\psi_k = \varphi$ in $\Delta(r_k)$—take $\psi_k' \in A^{0,q-1}(\Delta(r_{k+1}))$ with $\bar{\partial}\psi_k' = \varphi$, ρ_k a C^∞ bump function $\equiv 1$ on $\Delta(r_k)$ and having compact support in $\Delta(r_{k+1})$, and set $\psi_k = \rho_k \cdot \psi_k'$—the problem is to show that we can choose $\{\psi_k\}$ so that they converge suitably on compact sets. We do this by induction on q. Suppose we have ψ_k as above. Take $\alpha \in A^{0,q-1}(\Delta)$ with $\bar{\partial}\alpha = \varphi$ in $\Delta(r_{k+1})$; then

$$\bar{\partial}(\psi_k - \alpha) = 0 \qquad \text{in } \Delta(r_k),$$

and, if $q \geq 2$, then by the induction hypothesis we can find $\beta \in A^{0,q-2}(\Delta)$ with

$$\bar{\partial}\beta = \psi_k - \alpha \qquad \text{in } \Delta(r_{k-1}).$$

Set

$$\psi_{k+1} = \alpha + \bar{\partial}\beta;$$

then $\bar\partial\psi_{k+1}=\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$ and

$$\psi_{k+1} = \psi_k \qquad \text{in } \Delta(r_{k-1}).$$

Thus the sequence $\{\psi_k\}$ chosen in this way converges uniformly on compact sets.

It remains to consider the case $q=1$. Again, say $\psi_k \in C^\infty(\Delta)$ with $\bar\partial\psi_k=\varphi$ in $\Delta(r_k)$, $\alpha\in C^\infty(\Delta)$ with $\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$; then $\psi_k-\alpha$ is a holomorphic function in $\Delta(r_k)$ and hence has a power series expansion around the origin in $\mathbb{C}^n$. Truncate this series expansion to obtain a polynomial β with

$$\sup_{\Delta(r_{k-1})} |(\psi_k-\alpha)-\beta| < \frac{1}{2^k},$$

and set

$$\psi_{k+1} = \alpha+\beta.$$

Then $\bar\partial\psi_{k+1}=\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$, $\psi_{k+1}-\psi_k$ is holomorphic in $\Delta(r_k)$, and

$$\sup_{\Delta(r_{k-1})} |\psi_{k+1}-\psi_k| < \frac{1}{2^k},$$

so $\psi=\lim\psi_k$ exists, and $\bar\partial\psi=\varphi$. Q.E.D.

Note that the proof works for $r=\infty$.

We leave it as an exercise for the reader to prove, using a similar argument with annuli and Laurent expansions, that

$$H_{\bar\partial}^{p,q}(\Delta^{*k}\times\Delta^l)=0 \qquad \text{for } q\geqslant 1,$$

where Δ^* is the punctured disc $\Delta-\{0\}$.

Calculus on Complex Manifolds

Let M be a complex manifold of dimension n. A *hermitian metric* on M is given by a positive definite hermitian inner product

$$(\ ,\ )_z\colon\ T_z'(M)\otimes\overline{T_z'(M)}\to\mathbb{C}$$

on the holomorphic tangent space at z for each $z\in M$, depending smoothly on z—that is, such that for local coordinates z on M the functions

$$h_{ij}(z)=\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)_z$$

are C^∞. Writing $(\ ,\ )_z$ in terms of the basis $\{dz_i\otimes d\bar z_j\}$ for

$$\left(T_z'(M)\otimes\overline{T_z'(M)}\right)^*=T_z^{*\prime}(M)\otimes T_z^{*\prime\prime}(M),$$

the hermitian metric is given by

$$ds^2=\sum_{i,j}h_{ij}(z)\,dz_i\otimes d\bar z_j.$$

A *coframe* for the hermitian metric is an n-tuple of forms $(\varphi_1,\ldots,\varphi_n)$ of type $(1,0)$ such that

$$ds^2 = \sum_i \varphi_i \otimes \bar{\varphi}_i,$$

i.e., such that, in terms of the inner product induced on $T_z^{*\prime}(M)$ by $(\ ,\ )_z$ on $T_z'(M)$, $(\varphi_1(z),\ldots,\varphi_n(z))$ is an orthonormal basis for $T_z^{*\prime}(M)$. From this description it is clear that coframes always exist locally: we can construct one by applying the Gram-Schmidt process to the basis $(dz_1,\ldots,dz_n)$ for $T_z^{*\prime}(M)$ at each z.

The real and imaginary parts of a hermitian inner product on a complex vector space give an ordinary inner product and an alternating quadratic form, respectively, on the underlying real vector space. Since we have a natural $\mathbb{R}$-linear isomorphism

$$T_{\mathbb{R},z}(M) \longrightarrow T_z'(M),$$

we see that for a hermitian metric ds^2 on M,

$$\operatorname{Re} ds^2 \colon\ T_{\mathbb{R},z}(M) \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

is a *Riemannian metric* on M, called the induced Riemannian metric of the hermitian metric. When we speak of distance, area, or volume on a complex manifold with hermitian metric, we always refer to the induced Riemannian metric.

We also see that since the quadratic form

$$\operatorname{Im} ds^2 \colon\ T_{\mathbb{R},p}(M) \otimes T_{\mathbb{R},p}(M) \to \mathbb{R}$$

is alternating, it represents a real differential form of degree 2; $\omega = -\frac{1}{2}\operatorname{Im} ds^2$ is called the *associated (1,1)-form* of the metric.

Explicitly, if $(\varphi_1,\ldots,\varphi_n)$ is a coframe for ds^2, we write

$$\varphi_i = \alpha_i + \sqrt{-1}\,\beta_i,$$

where α_i, β_i are real differential forms; then

$$\begin{aligned} ds^2 &= \left(\sum(\alpha_i + \sqrt{-1}\,\beta_i)\right) \otimes \left(\sum(\alpha_i - \sqrt{-1}\,\beta_i)\right)\\ &= \sum_i (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i) + \sqrt{-1}\,\sum_i(-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i). \end{aligned}$$

The induced Riemannian metric is given by

$$\operatorname{Re} ds^2 = \sum(\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i),$$

and the associated $(1,1)$-form of the metric is given by

$$\begin{aligned} \omega &= -\tfrac{1}{2}\operatorname{Im} ds^2\\ &= \sum \alpha_i \wedge \beta_i\\ &= \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \bar{\varphi}_i. \end{aligned}$$

It follows from this last representation that the metric $ds^2=\Sigma\varphi_i\otimes\bar{\varphi}_i$ may be directly recovered from its associated (1,1)-form $\omega=\frac{1}{2}\sqrt{-1}\,\Sigma\varphi_i\wedge\bar{\varphi}_i$. Indeed, any real differential form ω of type (1,1) on M gives a hermitian form $H(\ ,\ )$ on each tangent space $T'_z(M)$. The form H will be positive definite—i.e., will induce a hermitian metric on M—if and only if for every $z\in M$ and holomorphic tangent vector $v\in T'_z(M)$,

$$\sqrt{-1}\cdot\langle\omega(z),v\wedge\bar{v}\rangle>0.$$

Such a differential form ω is called a *positive (1,1)-form*; in terms of local holomorphic coordinates $z=(z_1,\dots,z_n)$ on M, a form ω is positive if

$$\omega(z)=\frac{\sqrt{-1}}{2}\sum_{i,j}h_{ij}(z)\,dz_i\wedge d\bar{z}_j$$

with $H(z)=(h_{ij}(z))$ a positive definite hermitian matrix for each z.

If $S\subset M$ is a complex submanifold, then for $z\in S$ we have a natural inclusion

$$T'_z(S)\subset T'_z(M);$$

consequently a hermitian metric on M induces the same on S by restriction. More generally, if $f\colon N\to M$ is any holomorphic map such that

$$f_*\colon\ T'_z(N)\to T'_{f(z)}(M)$$

is injective for all $z\in N$, a metric on M induces a metric on N by setting

$$\left(\frac{\partial}{\partial w_\alpha},\frac{\partial}{\partial w_\beta}\right)_z=\left(f_*\frac{\partial}{\partial w_\alpha},f_*\frac{\partial}{\partial w_\beta}\right)_{f(z)}.$$

Note that in this case we can always find, for $U\subset N$ small, a coframe $(\varphi_1,\dots,\varphi_n)$ on $f(U)\subset M$ with $\varphi_{k+1},\dots,\varphi_n\in\operatorname{Ker}f^*\colon T^{*\prime}_{f(z)}(M)\to T^{*\prime}_z(N)$; then $f^*\varphi_1,\dots,f^*\varphi_k$ form a coframe on U for the induced metric on N. The associated (1,1)-form ω_N on N is thus given by

$$\begin{aligned}\omega_N&=\frac{\sqrt{-1}}{2}\sum_{i=1}^{k}f^*\varphi_i\wedge f^*\bar{\varphi}_i\\&=f^*\left(\frac{\sqrt{-1}}{2}\sum_{i=1}^{k}\varphi_i\wedge\bar{\varphi}_i\right)\\&=f^*\left(\frac{\sqrt{-1}}{2}\sum_{i=1}^{n}\varphi_i\wedge\bar{\varphi}_i\right)\\&=f^*\omega_M,\end{aligned}$$

i.e., *the associated (1,1)-form of the induced metric on* N *is the pullback of the associated (1,1)-form of the metric on* M.

Examples

1. The hermitian metric on $\mathbb{C}^n$ given by

$$ds^2 = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i$$

is called the *Euclidean* or *standard* metric; the induced Riemannian metric is, of course, the standard metric on $\mathbb{C}^n = \mathbb{R}^{2n}$.

2. If $\Lambda \subset \mathbb{C}^n$ is a full lattice, then the metric given on the complex torus $\mathbb{C}^n/\Lambda$ by

$$ds^2 = \sum dz_i \otimes d\bar{z}_i$$

is again called the *Euclidean metric* on $\mathbb{C}^n/\Lambda$.

3. Let $Z_0, \dots, Z_n$ be coordinates on $\mathbb{C}^{n+1}$ and denote by $\pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ the standard projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z: U \to \mathbb{C}^{n+1} - \{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ Z = id$; consider the differential form

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2.$$

If $Z': U \to \mathbb{C}^{n+1} - \{0\}$ is another lifting, then

$$Z' = f \cdot Z$$

with f a nonzero holomorphic function, so that

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z'\|^2 &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\left(\log \|Z\|^2 + \log f + \log \bar{f}\right) \\ &= \omega + \frac{\sqrt{-1}}{2\pi}\left(\partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f}\right) \\ &= \omega.\end{aligned}$$

Therefore ω is independent of the lifting chosen; since liftings always exist locally, ω is a globally defined differential form in $\mathbb{P}^n$. Clearly ω is of type $(1,1)$. To see that ω is positive, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form ω invariant, so that ω is positive everywhere if it is positive at one point. Now let $\{w_i = Z_i/Z_0\}$ be coordinates on the open set $U_0 = (Z_0 \neq 0)$ in $\mathbb{P}^n$ and use the lifting $Z = (1, w_1, \dots, w_n)$ on U_0; we have

$$\begin{aligned}\omega &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(1 + \sum w_i \bar{w}_i\right) \\ &= \frac{\sqrt{-1}}{2\pi} \partial\left[\frac{\sum w_i \, d\bar{w}_i}{1 + \sum w_i \bar{w}_i}\right] \\ &= \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dw_i \wedge d\bar{w}_i}{1 + \sum w_i \bar{w}_i} - \frac{\left(\sum \bar{w}_i \, dw_i\right) \wedge \left(\sum w_i \, d\bar{w}_i\right)}{\left(1 + \sum w_i \bar{w}_i\right)^2}\right].\end{aligned}$$

At the point $[1,0,\dots,0]$,

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum dw_i \wedge d\bar{w}_i > 0.$$

Thus ω defines a hermitian metric on $\mathbb{P}^n$, called the *Fubini-Study metric.*

The Wirtinger Theorem. The interplay between the real and imaginary parts of a hermitian metric now gives us the Wirtinger theorem, which expresses another fundamental difference between Riemannian and hermitian differential geometry. Let M be a complex manifold, $z=(z_1,\dots,z_n)$ local coordinates on M, and

$$ds^2 = \sum \varphi_i \otimes \bar{\varphi}_i$$

a hermitian metric on M with associated $(1,1)$-form ω. Write $\varphi_i = \alpha_i + \sqrt{-1}\,\beta_i$; then the associated Riemannian metric on M is

$$\operatorname{Re}(ds^2) = \sum_{i,j} \alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i,$$

and the volume element associated to $\operatorname{Re}(ds^2)$ is given by

$$d\mu = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n.$$

On the other hand, we have

$$\omega = \sum \alpha_i \wedge \beta_i,$$

so that the n^{th} exterior power

$$\begin{aligned} \omega^n &= n!\cdot\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n \\ &= n!\cdot d\mu. \end{aligned}$$

Now let $S \subset M$ be a complex submanifold of dimension d. As we have observed, the $(1,1)$-form associated to the metric induced on S by ds^2 is just $\omega|_S$, and applying the above to the induced metric on S, we have the

Wirtinger Theorem

$$\operatorname{vol}(S) = \frac{1}{d!}\int_S \omega^d.$$

The fact that the volume of a complex submanifold S of the complex manifold M is expressed as the integral over S of a globally defined differential form on M is quite different from the real case. For a C^∞ arc

$$t \mapsto (x(t), y(t))$$

in $\mathbb{R}^2$, for example, the element of arc length is given by

$$\left(x'(t)^2 + y'(t)^2\right)^{1/2} dt,$$

which is not, in general, the pullback of any differential form in $\mathbb{R}^2$.

To close this section, we discuss integration over analytic subvarieties of a complex manifold M. To begin with, we define the integral of a differential form φ on M over a possibly singular subvariety V to be the integral of φ over the smooth locus V^* of V. The first thing to prove is the

Proposition. V* *has finite volume in bounded regions.*

Proof. Since the question is local and the volume increases by increasing the metric, it is sufficient to prove it for $V\subset\mathbb{C}^n$ with the Euclidean metric. Suppose V is of dimension k and choose coordinates on $\mathbb{C}^n$ so that, in a neighborhood of 0, V meets each of the coordinate $(n-k)$-planes $(z_{i_1}=z_{i_2}=\cdots=z_{i_k}=0)$ only in discrete points. The (1,1)-form associated to the Euclidean metric on $\mathbb{C}^n$ is

$$\omega=\frac{\sqrt{-1}}{2}\sum dz_i\wedge d\bar{z}_i,$$

and so for $c=(\sqrt{-1}\,/2)^k(-1)^{k(k-1)/2}\cdot k!$

$$\omega^k=c\cdot\sum_{\#I=k}dz_I\wedge d\bar{z}_I.$$

Thus it will suffice to prove that

$$c\int_{V^*\cap\Delta}dz_I\wedge d\bar{z}_I<\infty$$

for $I=\{1,\dots,k\}$, Δ a small polydisc around the origin. But the projection map

$$\begin{aligned}\pi:\ & V^*\to\mathbb{C}^k\\ :\ & (z_1,\dots,z_n)\mapsto(z_1,\dots,z_k)\end{aligned}$$

expresses V^* as a d-sheeted branched cover of $\Delta'=\pi(\Delta)$ and consequently

$$c\int_{V^*\cap\Delta}dz_I\wedge d\bar{z}_I\leqslant d\cdot c\int_{\Delta'}dz_I\wedge d\bar{z}_I<\infty. \qquad \text{Q.E.D.}$$

Note again the contrast to the C^∞ case, where the set of manifold points of the zero locus of a smooth function—e.g., $f(y)=(e^{-y^{-2}}-1)\sin(1/y)$—need not have locally finite area.

As a corollary of the proof, we see that for any region $U\subset M$ with $\bar{U}$ compact and $\varphi\in A^*(\bar{U})$,

$$\int_{V^*\cap U}\varphi<\infty.$$

An obvious but fundamental observation is that if V^* has dimension k, $A^{p,q}(V^*)=0$ for p or $q>k$; consequently for any form φ,

$$\int_V\varphi=\int_V\varphi^{(k,k)}.$$

We can now prove

Stokes' Theorem for Analytic Varieties. *For* M *a complex manifold,* $V \subset M$ *an analytic subvariety of dimension* k, *and* φ *a differential form of degree* $2k-1$ *with compact support in* M,

$$\int_V d\varphi = 0.$$

Proof. The question is local, i.e., it will be sufficient to show that for every $p \in V$, there exists a neighborhood U of p such that for any $\varphi \in A_c^{2k-1}(U)$

$$\int_V d\varphi = 0.$$

For any $p \in V$, we can find a coordinate system $z = (z_1, \ldots, z_n)$ and a polycylinder Δ around p such that the projection map π: $(z_1, \ldots, z_n) \to (z_1, \ldots, z_k)$ expresses $V \cap \Delta$ as a branched cover of $\Delta' = \pi(\Delta)$, branched over an analytic hypersurface $D \subset \Delta'$. Let T_ε be the ε-neighborhood of D in Δ' and

$$V_\varepsilon = (V \cap \Delta) - \pi^{-1}(T_\varepsilon).$$

For $\varphi \in A_c^{2k-1}(\Delta)$,

$$\begin{aligned}\int_V d\varphi &= \lim_{\varepsilon \to 0} \int_{V_\varepsilon} d\varphi \\ &= \lim_{\varepsilon \to 0} \int_{\partial V_\varepsilon} \varphi \\ &= \lim_{\varepsilon \to 0} \int_{\partial \pi^{-1}(T_\varepsilon)} \varphi.\end{aligned}$$

Thus to prove the result, we simply have to prove that the volume of $\partial\pi^{-1}(T_\varepsilon) \to 0$ as $\varepsilon \to 0$. But $\partial\pi^{-1}(T_\varepsilon)$ is a finite cover of ∂T_ε; so we need prove only that $\mathrm{vol}(\partial T_\varepsilon) \to 0$ as $\varepsilon \to 0$. To see this, let D_1 be the singular locus of D, D_2 the singular locus of D_1, and so on; let T_ε^i be the ε-neighborhood of $D_i^* = D_i - D_{i+1}$ in $\Delta - D_{i+1}$. Then D_i^* is a submanifold of real dimension $\leqslant 2k-2$ having finite volume in $\Delta - D_{i+1}$, and so the volume of ∂T_ε^i goes to 0 as $\varepsilon \to 0$. But $\partial T_\varepsilon \subset \cup(\partial T_\varepsilon^i)$, and so $\mathrm{vol}(\partial T_\varepsilon) \to 0$ as $\varepsilon \to 0$. Q.E.D.

This result has to do with the fact that singularities of complex-analytic subvarieties occur only in real codimension 2. It assures us that integration over analytic varieties is much the same as integration over submanifolds; perhaps most importantly, it allows us to show (p. 61) that an analytic subvariety of a compact complex manifold always defines a homology class in $H_*(M, \mathbb{R})$.

Finally, we can state the

Proper Mapping Theorem. *If* M, N *are complex manifolds*, f: M→N *a holomorphic map, and* V ⊂ M *an analytic variety such that* $f|_V$ *is proper, then* f(V) *is an analytic subvariety of* N.

The proof will be given in Section 2 of Chapter 3.

3. SHEAVES AND COHOMOLOGY

Origins: The Mittag-Leffler Problem

Let S be a Riemann surface, not necessarily compact, p a point of S with local coordinate z centered at p. A *principal part* at p is the polar part $\sum_{k=1}^{n} a_k z^{-k}$ of a Laurent series. If $\mathcal{O}_p$ is the local ring of holomorphic functions around p, $\mathfrak{M}_p$ the field of meromorphic functions around p, a principal part is just an element of the quotient group $\mathfrak{M}_p/\mathcal{O}_p$. The *Mittag-Leffler* question is, given a discrete set $\{p_n\}$ of points in S and a principal part at p_n for each n, does there exist a meromorphic function f on S, holomorphic outside $\{p_n\}$, whose principal part at each p_n is the one specified? The question is clearly trivial locally, and so the problem is one of passage from local to global data. Here are two approaches, both of which lead to cohomology theories.

Čech. Take a covering $\underline{U}=\{U_\alpha\}$ of S by open sets such that each U_α contains at most one point p_n, and let f_α be a meromorphic function on U_α solving the problem in U_α. Set

$$f_{\alpha\beta} = f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta).$$

In $U_\alpha \cap U_\beta \cap U_\gamma$, we have

$$f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0.$$

Solving the problem globally is equivalent to finding $\{g_\alpha \in \mathcal{O}(U_\alpha)\}$ such that

$$f_{\alpha\beta} = g_\beta - g_\alpha \quad \text{in} \quad U_\alpha \cap U_\beta:$$

given such g_α, $f = f_\alpha + g_\alpha$ is a globally defined function satisfying the conditions, and conversely. In the Čech theory,

$$\{\{f_{\alpha\beta}\} : f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0\} = Z^1(\underline{U}, \mathcal{O})$$
$$\{\{f_{\alpha\beta}\} : f_{\alpha\beta} = g_\beta - g_\alpha, \ \text{some } \{g_\alpha\}\} = \delta C^0(\underline{U}, \mathcal{O})$$

and the *first Čech cohomology group*

$$H^1(\underline{U},\mathcal{O}) = \frac{Z^1(\underline{U},\mathcal{O})}{B^1(\underline{U},\mathcal{O})}$$

is the obstruction to solving the problem in general.

Dolbeault. As before, take f_α to be a local solution in U_α and let ρ_α be a bump function, 1 in a neighborhood of $p_n \in U_\alpha$ and having compact support contained in U_α. Then

$$\varphi = \sum_\alpha \bar\partial(\rho_\alpha f_\alpha)$$

is a $\bar\partial$-closed C^∞ (0,1)-form on S ($\varphi \equiv 0$ in a neighborhood of p_n). If $\varphi = \bar\partial\eta$ for $\eta \in C^\infty(S)$, then the function

$$f = \sum_\alpha \rho_\alpha f_\alpha - \eta$$

satisfies the conditions of the problem; thus the obstruction to solving the problem is in $H^{0,1}_{\bar\partial}(S)$.

Sheaves

Given X a topological space, a *sheaf* $\mathcal{F}$ on X associates to each open set $U \subset X$ a group $\mathcal{F}(U)$, called the sections of $\mathcal{F}$ over U, and to each pair $U \subset V$ of open sets a map $r_{V,U}: \mathcal{F}(V) \to \mathcal{F}(U)$, called the restriction map, satisfying

1. For any triple $U \subset V \subset W$ of open sets,

$$r_{W,U} = r_{V,U} \cdot r_{W,V}.$$

By virtue of this relation, we may write $\sigma|_U$ for $r_{V,U}(\sigma)$ without loss of information.

2. For any pair of open sets $U, V \subset M$ and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that

$$\sigma|_{U\cap V} = \tau|_{U \cap V}$$

there exists a section $\rho \in \mathcal{F}(U \cup V)$ with

$$\rho|_U = \sigma, \qquad \rho|_V = \tau.$$

3. If $\sigma \in \mathcal{F}(U \cup V)$ and

$$\sigma|_U = \sigma|_V = 0$$

then $\sigma = 0$.

Notation. The following are the sheaves we will be dealing with most often. In every case the restriction maps are the obvious ones, and the groups are additive unless otherwise stated.

1. On any C^∞ manifold M, we define sheaves C^∞, C^*, $\mathcal{A}^p$, $\mathcal{Z}^p$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ by

$C^\infty(U) = C^\infty$ functions on U
$C^*(U)$ = multiplicative group of nonzero C^∞ functions on U,
$\mathcal{A}^p(U) = C^\infty$ p-forms on U,
$\mathcal{Z}^p(U)$ = closed C^∞ p-forms on U,
$\mathbb{Z}(U)$, $\mathbb{Q}(U)$, $\mathbb{R}(U)$, $\mathbb{C}(U)$ = locally constant $\mathbb{Z}$-, $\mathbb{Q}$-, $\mathbb{R}$-, or $\mathbb{C}$-valued functions on U.

2. If M is a complex manifold, $V \subset M$ an analytic subvariety of M, and $E \to M$ a holomorphic vector bundle (defined below), we define the sheaves $\mathcal{O}$, $\mathcal{O}^*$, Ω^p, $\mathcal{A}^{p,q}$, $\mathcal{Z}^{p,q}_{\bar\partial}$, $\mathcal{I}_V$, $\mathcal{O}(E)$, and $\mathcal{A}^{p,q}(E)$ by

$\mathcal{O}(U)$ = holomorphic functions on U,
$\mathcal{O}^*(U)$ = multiplicative group of nonzero holomorphic functions on U,
$\Omega^p(U)$ = holomorphic p-forms on U,
$\mathcal{A}^{p,q}(U) = C^\infty$ forms of type (p,q) on U,
$\mathcal{Z}^{p,q}_{\bar\partial}(U) = \bar\partial$-closed C^∞ forms of type (p,q) on U,
$\mathcal{I}_V(U)$ = holomorphic functions on U vanishing on $V \cap U$,
$\mathcal{O}(E)(U)$ = holomorphic sections of E over U,
$\mathcal{A}^{p,q}(E)(U) = C^\infty$ E-valued (p,q)-forms over U.

3. If M is again a complex manifold, a *meromorphic function* f on an open set $U \subset M$ is given locally as the quotient of two holomorphic functions—i.e., for some covering $\{U_i\}$ of U, $f|_{U_i} = g_i/h_i$, where g_i, h_i are relatively prime in $\mathcal{O}(U_i)$ and $g_i h_j = g_j h_i$ in $\mathcal{O}(U_i \cap U_j)$. This definition makes implicit use of the proposition on p. 10. A meromorphic function f is not, strictly speaking, a function even if we consider ∞ a value: at points where $g_i = h_i = 0$, it is not defined. The sheaf of meromorphic functions on M is denoted $\mathcal{M}$; the multiplicative sheaf of meromorphic functions not identically zero is denoted $\mathcal{M}^*$.

A *map of sheaves* $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ on M is given by a collection of homomorphisms $\{\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)\}_{U \subset M}$ such that for $U \subset V \subset M$, α_U and α_V commute with the restriction maps. The *kernel* of the map $\alpha : \mathcal{F} \to \mathcal{G}$ is just the sheaf $\mathrm{Ker}(\alpha)$ given by $\mathrm{Ker}(\alpha)(U) = \mathrm{Ker}(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U))$; it is easy to check that this assignment does in fact define a sheaf. The *cokernel* of α is harder to define: if we set $\mathrm{Coker}(\alpha)(U) = \mathcal{G}(U)/\alpha_U \mathcal{F}(U)$, Coker may not satisfy the conditions on p. 35. [The basic example of this is the sheaf map

$$\exp : \mathcal{O} \to \mathcal{O}^*$$

on $\mathbb{C} - \{0\}$ given by sending $f \in \mathcal{O}(U)$ to $e^{2\pi\sqrt{-1}\,f} \in \mathcal{O}^*(U)$. The section $z \in \mathcal{O}^*(\mathbb{C} - \{0\})$ is not in the image of $\mathcal{O}(\mathbb{C} - \{0\})$ under exp, but its restriction to any contractible open set $U \subset \mathbb{C}-\{0\}$ is in the image of $\mathcal{O}(U)$.] Instead, we take a section of the cokernel sheaf $\mathrm{Coker}(\alpha)$ over U to be given by an open cover $\{U_\alpha\}$ of U together with sections $\sigma_\alpha \in \mathcal{G}(U_\alpha)$ such that for all α, β,

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in \alpha_{U_\alpha \cap U_\beta}(\mathcal{F}(U_\alpha \cap U_\beta));$$

we identify two such collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(U'_\alpha, \sigma'_\alpha)\}$ if for all $p \in U$ and $U_\alpha, U'_\beta \ni p$, there exists V with $p \in V \subset (U_\alpha \cap U'_\beta)$ such that $\sigma'_\alpha|_V - \sigma'_\beta|_V \in \alpha_V(\mathcal{F}(V))$.

We say that a sequence of sheaf maps

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

is *exact* if $\mathcal{E} = \mathrm{Ker}(\beta)$ and $\mathcal{G} = \mathrm{Coker}(\alpha)$; in this case we also say that $\mathcal{E}$ is a *subsheaf* of $\mathcal{F}$ and $\mathcal{G}$ the *quotient sheaf* of $\mathcal{F}$ by $\mathcal{E}$, written $\mathcal{F}/\mathcal{E}$. More generally, we say a sequence

$$\cdots \to \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \xrightarrow{\alpha_{n+1}} \mathcal{F}_{n+2} \to \cdots$$

is exact if $\alpha_{n+1} \circ \alpha_n = 0$ and

$$0 \to \mathrm{Ker}(\alpha_n) \to \mathcal{F}_n \to \mathrm{Ker}(\alpha_{n+1}) \to 0$$

is exact for each n. Note that by our definition of Coker, this does not imply that

$$0 \longrightarrow \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \longrightarrow 0$$

is exact for all U; it does imply that this sequence is exact at the first two stages for all U, and that for any section $\sigma \in \mathcal{G}(U)$ and any point $p \in U$ there exists a neighborhood V of p in U such that $\sigma|_V$ is in the image of β_V.

A note: if $M \subset N$ is a subspace, $\mathcal{F}$ a sheaf on M, we can "*extend* $\mathcal{F}$ *by zero*" to obtain a sheaf $\tilde{\mathcal{F}}$ on N, setting

$$\tilde{\mathcal{F}}(U) = \mathcal{F}(U \cap M)$$

and letting the restriction maps be the obvious ones. Thus we may consider $\mathcal{F}$ as a sheaf on either M or N.

Examples

1. On any complex manifold, the sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

is exact, where i is the obvious inclusion and exp the exponential map $\exp(f) = e^{2\pi\sqrt{-1}\, f}$. This fundamental sequence is called the *exponential sheaf sequence*.

2. If M is a complex manifold, $V \subset M$ a complex submanifold, the sheaf $\mathcal{O}_V$ may, by extension by zero, be considered a sheaf on M. The sequence

$$0 \to \mathcal{I}_V \xrightarrow{i} \mathcal{O}_M \xrightarrow{r} \mathcal{O}_V \to 0,$$

where i is inclusion and r restriction, is then exact.

3. By the ordinary Poincaré lemma, the sequence

$$0 \to \mathbb{R} \to \mathcal{C}^\infty \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to \cdots$$

is exact on any real manifold.

4. By the $\bar\partial$-Poincaré lemma, the sequence

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \to \cdots$$

is exact on any complex manifold.

5. If M is a Riemann surface and we let $\mathcal{P}\mathcal{P}$ be the quotient sheaf of the sheaf $\mathcal{M}$ by the subsheaf $\mathcal{O} \xrightarrow{i} \mathcal{M}$, then for $U \subset M$ open,

$$\mathcal{P}\mathcal{P}(U) = \{(p_n, f_n)\}: \quad \begin{cases} \{p_n\} \subset U & \text{discrete,} \\ f_n \in \mathcal{M}_{p_n} / \mathcal{O}_{p_n}; \end{cases}$$

i.e., giving a section of $\mathcal{P}\mathcal{P}$ over U is the same as specifying the data of a Mittag-Leffler problem for U.

Cohomology of Sheaves

Let $\mathcal{F}$ be a sheaf on M, and $\underline{U} = \{U_\alpha\}$ a locally finite open cover. We define

$$C^0(\underline{U}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha),$$

$$C^1(\underline{U}, \mathcal{F}) = \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta),$$

$$\vdots$$

$$C^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}).$$

An element $\sigma = \{\sigma_I \in \mathcal{F}(\cap U_{i_k})\}_{\# I = p+1}$ of $C^p(\underline{U}, \mathcal{F})$ is called a p-*cochain* of $\mathcal{F}$. We define a *coboundary operator*

$$\delta: C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_0, \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \ldots, \hat{i_j}, \ldots, i_{p+1}} \Big|_{U_{i_0} \cap \cdots \cap U_{i_p}}$$

In particular, if $\sigma=\{\sigma_U\}\in C^0(\underline{U},\mathfrak{F})$,

$$(\delta\sigma)_{U,V} = -\sigma_U+\sigma_V;$$

and if $\sigma=\{\sigma_{U,V}\}\in C^1(\underline{U},\mathfrak{F})$,

$$(\delta\sigma)_{U,V,W} = \sigma_{UV}+\sigma_{VW}-\sigma_{UW}$$

(omitting the restriction).

A p-cochain $\sigma\in C^p(\underline{U},\mathfrak{F})$ is called a *cocycle* if $\delta\sigma=0$. Note that any cocycle σ must satisfy the skew-symmetry condition

$$\sigma_{i_0,\dots,i_p} = -\sigma_{i_0,\dots,i_{q-1},i_{q+1},i_q,i_{q+2},\dots,i_p}.$$

σ is called a *coboundary* if $\sigma=\delta\tau$ for some $\tau\in C^{p-1}(\underline{U},\mathfrak{F})$. It is easy to see that $\delta^2=0$—i.e., a coboundary is a cocycle—and we set

$$Z^p(\underline{U},\mathfrak{F}) = \operatorname{Ker}\delta \subset C^p(\underline{U},\mathfrak{F})$$

and

$$H^p(\underline{U},\mathfrak{F}) = \frac{Z^p(\underline{U},\mathfrak{F})}{\delta C^{p-1}(\underline{U},\mathfrak{F})}.$$

Now, given two coverings $\underline{U}=\{U_\alpha\}_{\alpha\in I}$ and $\underline{U}'=\{U'_\beta\}_{\beta\in I'}$ of M, we say that $\underline{U}'$ is a *refinement* of $\underline{U}$ if for every $\beta\in I'$ there exists $\alpha\in I$ such that $U'_\beta\subset U_\alpha$; we write $U'<U$. If $\underline{U}'<\underline{U}$, we can choose a map $\varphi: I'\to I$ such that $U'_\beta\subset U_{\varphi\beta}$ for all β; then we have a map

$$\rho_\varphi:\ C^p(\underline{U},\mathfrak{F})\to C^p(\underline{U}',\mathfrak{F})$$

given by

$$(\rho_\varphi\sigma)_{\beta_0\cdots\beta_p} = \sigma_{\varphi\beta_0\cdots\varphi\beta_p}|_{U_{\beta_0}\cap\cdots\cap U_{\beta_p}}$$

Evidently $\delta\circ\rho_\varphi=\rho_\varphi\circ\delta$, and so ρ_φ induces a homomorphism

$$\rho:\ H^p(\underline{U},\mathfrak{F})\to H^p(\underline{U}',\mathfrak{F}),$$

which is independent of the choice of φ. (The reader may wish to check that the chain maps ρ_φ and ρ_ψ associated to two inclusion associations φ and ψ are *chain homotopic* and thus induce the same map on cohomology.) We define the pth *Čech cohomology group* of $\mathfrak{F}$ on M to be the direct limit of the $H^p(\underline{U},\mathfrak{F})$'s as $\underline{U}$ becomes finer and finer:

$$H^p(M,\mathfrak{F}) = \varinjlim_{\underline{U}} H^p(\underline{U},\mathfrak{F}).$$

Where there is a possibility of confusion, we will denote Čech cohomology groups by $\check{H}$. Clearly, for any covering $\underline{U}$

$$H^0(M,\mathfrak{F}) = H^0(\underline{U},\mathfrak{F}) = \mathfrak{F}(M).$$

Note that if $M\subset N$ is a closed subspace, $\mathfrak{F}$ any sheaf on M, then extending $\mathfrak{F}$ by zero to a sheaf on N, we have

$$H^*(M,\mathfrak{F}) = H^*(N,\mathfrak{F}).$$

The definition of $H^*(M, \mathcal{F})$ as a direct limit is, in practice, more or less impossible to work with. What is needed is a simple sufficient condition on a cover $\underline{U}$ for

$$H^*(\underline{U}, \mathcal{F}) = H^*(M, \mathcal{F}),$$

and this is provided by the

Leray Theorem. *If the covering* $\underline{\mathrm{U}}$ *is acyclic for the sheaf* $\mathcal{F}$ *in the sense that*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0, \qquad q > 0, \quad \text{any } i_1 \cdots i_p,$$

then $\mathrm{H}^*(\underline{\mathrm{U}}, \mathcal{F}) \cong \mathrm{H}^*(\mathrm{M}, \mathcal{F})$.

We will prove the Leray theorem in those cases where it will be used.

The most basic property of sheaf cohomology is: Given an exact sequence

$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$

of sheaves on M, we have maps

$$C^p(\underline{U}, \mathcal{E}) \xrightarrow{\alpha} C^p(\underline{U}, \mathcal{F}), \qquad C^p(\underline{U}, \mathcal{F}) \xrightarrow{\beta} C^p(\underline{U}, \mathcal{G})$$

that commute with δ and hence induce maps

$$H^p(M, \mathcal{E}) \xrightarrow{\alpha^*} H^p(M, \mathcal{F}), \qquad H^p(M, \mathcal{F}) \xrightarrow{\beta^*} H^p(M, \mathcal{G}).$$

We next define the coboundary map $\delta^* : H^p(M, \mathcal{G}) \to H^{p+1}(M, \mathcal{E})$: given $\sigma \in C^p(\underline{U}, \mathcal{G})$ with $\delta\sigma = 0$, we can always pass to a refinement $\underline{U}'$ of $\underline{U}$ and find $\tau \in C^p(\underline{U}', \mathcal{F})$ such that $\beta(\tau) = \rho\sigma$. Then $\beta\delta\tau = \delta\beta\tau = \delta\rho\sigma = 0$, so by passing to a further refinement $\underline{U}''$ we can find $\mu \in C^{p+1}(\underline{U}'', \mathcal{E})$ such that $\alpha\mu = \delta\tau$; $\alpha\delta\mu = \delta\alpha\mu = \delta^2\tau = 0$ and since α is injective this means $\delta\mu = 0$. Thus $\mu \in Z^{p+1}(\underline{U}'', \mathcal{E})$ and we take $\delta^*\sigma = M \in H^{p+1}(M, \mathcal{E})$.

Basic Fact. *The sequence*

$$\begin{aligned} 0 \to H^0(M, \mathcal{E}) &\to H^0(M, \mathcal{F}) \to H^0(M, \mathcal{G}) \\ \to H^1(M, \mathcal{E}) &\to H^1(M, \mathcal{F}) \to H^1(M, \mathcal{G}) \to \cdots \\ &\vdots \\ \to H^p(M, \mathcal{E}) &\to H^p(M, \mathcal{F}) \to H^p(M, \mathcal{G}) \to \cdots \end{aligned}$$

is exact.

For most exact sequences $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ that actually arise naturally—and certainly for all sheaves with which we shall deal in this book—it is the case that there exist arbitrarily fine coverings $\underline{U}$ such that for every open set $U=U_{i_0}\cap\ldots\cap U_{i_p}$ the sequence

$$0\to\mathcal{E}(U)\to\mathcal{F}(U)\to\mathcal{G}(U)\to 0$$

is exact. Thus, we can find arbitrarily fine coverings $\underline{U}$ of M for which the cochain groups form an exact sequence

$$0\to C^p(\underline{U},\mathcal{E})\to C^p(\underline{U},\mathcal{F})\to C^p(\underline{U},\mathcal{G})\to 0.$$

In this case, our basic fact is easy to verify: for example, to see that

$$H^p(\underline{U},\mathcal{F})\xrightarrow{\beta^*}H^p(\underline{U},\mathcal{G})\xrightarrow{\delta^*}H^{p+1}(\underline{U},\mathcal{E})$$

is exact, let $\sigma\in C^p(\underline{U},\mathcal{G})$ with $\delta\sigma=0$ and $\delta^*\sigma=0$ in $H^{p+1}(\underline{U},\mathcal{E})$. Then there exists $\tau\in C^p(\underline{U},\mathcal{F})$ such that $\beta\tau=\sigma$ and $\mu\in C^{p+1}(\underline{U},\mathcal{F})$ such that $\alpha\mu=\delta\tau$; by definition $\mu=\delta^*\sigma$ in $H^{p+1}(\underline{U},\mathcal{E})$, so $\mu=\delta\nu$ for some $\nu\in C^p(\underline{U},\mathcal{E})$. Then $\tau-\alpha\nu$ is a cocycle in $C^p(\underline{U},\mathcal{F})$ with $\beta(\tau-\alpha\nu)=\beta\tau=\sigma$, showing $\sigma\in\beta^*(H^p(\underline{U},\mathcal{F}))$. Conversely, it is clear that $\delta^*\beta^*=0$. The remaining stages are similar but easier.

The most common application of the *exact cohomology sequence* associated to a sheaf sequence

$$0\to\mathcal{E}\xrightarrow{\alpha}\mathcal{F}\xrightarrow{\beta}\mathcal{G}\to 0$$

is to answer the question: given a global section σ of $\mathcal{G}$, when is σ the image under β of a global section of $\mathcal{F}$? The answer, according to the exact cohomology sequence, is that this is the case exactly when $\delta^*\sigma=0$ in $H^1(M,\mathcal{E})$.

For example, we consider again the exact sequence

$$0\to\mathcal{O}\xrightarrow{\alpha}\mathcal{M}\xrightarrow{\beta}\mathcal{PP}\to 0$$

on a Riemann surface M. The data of the Mittag-Leffler problem consist of a global section $g\in\mathcal{PP}(M)=H^0(M,\mathcal{PP})$; the question is whether $g=\beta^*f$ for some global meromorphic function f. If $\{f_U\}$ are the local solutions of the problem, we have seen that

$$(\delta^*g)_{U,V}=f_V-f_U$$

and that $g=\alpha^*f$ if and only if $\delta^*g=0$ in $H^1(M,\mathcal{O})$.

There are, roughly speaking, three kinds of sheaves we will encounter:

1. *Holomorphic sheaves*—such as $\mathcal{O}$, $\mathcal{I}_V$, $\mathcal{O}(E)$, and Ω^p—whose sections are given locally by n-tuples of holomorphic functions. These contain for us the most information and are the principal objects of interest.
2. C^∞ *sheaves*, such as $\mathcal{A}^{p,q}$, whose local sections can be expressed as n-tuples of C^∞ functions. These are generally used in an auxiliary manner.
3. *Constant sheaves*, such as $\mathbb{Z},\mathbb{R},\mathbb{C}$. These, as we will see, contain topological information about the underlying manifold.

There are a couple of observations to be made about the latter two classes of sheaves:

1. $H^p(M, \mathcal{Q}^{r,s}) = 0$ for $p > 0$.

Proof. Given any locally finite cover $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ of M, we can find a *partition of unity subordinate* to $\underline{U}$, i.e., C^∞ functions ρ_α on M such that $\Sigma\rho_\alpha \equiv 1$ and support$(\rho_\alpha) \subset U_\alpha$. Now given $\sigma \in Z^p(U, \mathcal{Q}^{r,s})$, we define $\tau \in C^{p-1}(\underline{U}, \mathcal{Q}^{r,s})$ by setting

$$\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sum_{\beta \in I} \rho_\beta \sigma_{\beta, \alpha_0, \ldots, \alpha_{p-1}},$$

where the section $\rho_\beta \cdot \sigma_{\beta, \alpha_0, \ldots, \alpha_{p-1}}$ extends to $U_{\alpha_0} \cap \cdots \cap U_{\alpha_{p-1}}$ by zero; one verifies that $\delta\tau = \sigma$. In the case $p = 1$, explicitly:

$$\sigma = \{\sigma_{UV} \in \mathcal{Q}^{r,s}(U \cap V)\};$$
$$\sigma_{UV} + \sigma_{VW} + \sigma_{WU} = 0 \qquad \text{in} \quad U \cap V \cap W.$$

Set $\tau_U = \Sigma_V \rho_V \sigma_{VU}$; then

$$\begin{aligned}(\delta\tau)_{UV} &= -\tau_U + \tau_V \\ &= -\sum_W \rho_W \sigma_{WU} + \sum_W \rho_W \sigma_{WV} \\ &= \sum_W \rho_W \sigma_{UV} = \sigma_{UV}.\end{aligned}$$

In general, sheaves that admit partitions of unity [more precisely, for any $U = \cup U_\alpha$, maps $\eta_\alpha : \mathcal{F}(U_\alpha) \to \mathcal{F}(U)$ such that the support of $(\eta_\alpha \sigma)$ is contained in U_α and $\Sigma \eta_\alpha(\sigma|_{U_\alpha}) = \sigma$ for $\sigma \in \mathcal{F}(U)$] are called *fine*, and the same argument shows that their higher cohomology groups vanish.

2. For K a simplicial complex with underlying topological space M,

$$H^*(K, \mathbb{Z}) \cong \check{H}^*(M, \mathbb{Z}),$$

that is, *the Čech cohomology of the constant sheaf* $\mathbb{Z}$ *on* M *is isomorphic to the simplicial cohomology of the complex* K. To see this, we associate to every vertex v_α in K the open set St(v_α), called the *star* of v_α, which is the interior of the union of all simplices in K having v_α as a vertex. $\underline{U} = \{U_\alpha = \text{St}(v_\alpha)\}$ is an open covering of M. $\cap_{i=0}^p \text{St}(v_{\alpha_i})$ is nonempty and connected if $v_{\alpha_0} \cdots v_{\alpha_p}$ are the vertices of a p-simplex in our decomposition; otherwise it is empty. Thus a p-cochain σ of the sheaf $\mathbb{Z}$ associates to every $(\alpha_0 \cdots \alpha_p)$ an element

$$\sigma_{\alpha_0 \cdots \alpha_p} \in \mathbb{Z}\left(\cap \text{St}(v_{\alpha_i})\right) = \begin{cases} \mathbb{Z} & \text{if } v_{\alpha_i} \text{ span a } p\text{-simplex,} \\ 0 & \text{otherwise.} \end{cases}$$

Given $\sigma \in C^p(\underline{U}, \mathbb{Z})$, we are led to define a simplicial p-cochain σ' by setting, for $\Delta = \langle v_{\alpha_0} \cdots v_{\alpha_p} \rangle$ a p-simplex with vertices $v_{\alpha_0} \cdots v_{\alpha_p}$,

$$\sigma'(\Delta) = \sigma_{\alpha_0 \cdots \alpha_p}.$$

$\sigma \mapsto \sigma'$ gives an isomorphism of Abelian groups

$$C^p(\underline{U}, \mathbb{Z}) \longrightarrow C^p(K, \mathbb{Z}),$$

and

$$\begin{aligned}\delta\sigma'(\langle\alpha_0 \cdots \alpha_{p+1}\rangle) &= \sum_i (-1)^{i+1} \sigma'(\langle\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}\rangle)\\ &= (\delta\sigma)',\end{aligned}$$

so that we have an isomorphism of chain complexes $C^*(\underline{U}, \mathbb{Z}) \to C^*(K, \mathbb{Z})$, hence an isomorphism $H^*(\underline{U}, \mathbb{Z}) \to H^*(K, \mathbb{Z})$. Since we can subdivide the complex K to make the cover $\underline{U}$ of M arbitrarily fine without changing $H^*(K, \mathbb{Z})$, we finally obtain

$$\check{H}^*(M, \mathbb{Z}) \cong H^*(\underline{U}, \mathbb{Z}) \cong H^*(K, \mathbb{Z}).$$

The de Rham Theorem

Let M be a real C^∞ manifold. We say that a singular p-chain σ on M, given as a formal linear combination $\Sigma a_i f_i$ of maps $\Delta \xrightarrow{f_i} M$ of the standard p-simplex $\Delta \subset \mathbb{R}^p$ to M, is *piecewise smooth* if the maps f_i extend to C^∞ maps of a neighborhood of Δ to M. Let $C_p^{\mathrm{ps}}(M, \mathbb{Z})$ denote the space of piecewise smooth integral p-chains. Clearly the boundary of a piecewise smooth chain is again piecewise smooth, so $C_*^{\mathrm{ps}}(M, \mathbb{Z})$ forms a subcomplex of $C_*(M, \mathbb{Z})$ and we can set

$$\begin{aligned} Z_p^{\mathrm{ps}}(M, \mathbb{Z}) &= \operatorname{Ker} \partial : \ C_p^{\mathrm{ps}}(M, \mathbb{Z}) \to C_{p-1}^{\mathrm{ps}}(M, \mathbb{Z})\\ H_p^{\mathrm{ps}}(M, \mathbb{Z}) &= \frac{Z_p^{\mathrm{ps}}(M, \mathbb{Z})}{\partial C_{p+1}^{\mathrm{ps}}(M, \mathbb{Z})}.\end{aligned}$$

By a foundational result from differential topology, the inclusion map $C_*^{\mathrm{ps}}(M, \mathbb{Z}) \to C_*(M, \mathbb{Z})$ induces an isomorphism

$$H_p^{\mathrm{ps}}(M, \mathbb{Z}) \cong H_p(M, \mathbb{Z});$$

in other words, every homology class in $H_p(M, \mathbb{Z})$ can be represented by a piecewise smooth p-cycle, and if a piecewise smooth p-cycle σ is homologous to 0 in the usual sense, there exists a piecewise smooth $(p+1)$-chain τ with $\partial\tau = \sigma$.

Now let $\varphi \in A^p(M)$ be a C^∞ p-form and $\sigma = \Sigma a_i f_i$ a piecewise smooth p-chain; we set

$$\begin{aligned}\langle \varphi, \sigma \rangle &= \int_\sigma \varphi\\ &= \sum_i a_i \int_\Delta f_i^* \varphi.\end{aligned}$$

If φ is a closed form, then for σ the boundary of a $(p+1)$-chain τ, by Stokes' theorem

$$\int_\sigma \varphi = \int_\tau d\varphi = 0,$$

so that φ defines a real-valued singular p-cocycle. Again by Stokes' theorem, we have for σ a cycle

$$\int_\sigma \varphi = \int_\sigma \varphi + d\eta$$

for any $\eta \in A^{p-1}(M)$; thus there is a map

$$H^*_{\mathrm{DR}}(M) \to H^*_{\mathrm{sing}}(M, \mathbb{R}).$$

The *de Rham theorem* says that this map is in fact an isomorphism.

De Rham's theorem was originally proved essentially by defining relative de Rham groups and showing that the resulting homology theory satisfied the axioms of Eilenberg and Steenrod. We will give here the shorter sheaf-theoretic argument that, while not so geometric, can be merely rephrased to give a proof of the Dolbeault theorem later.

First, since any differentiable manifold M can be realized as the underlying topological space of a simplicial complex K, we have

$$H^*_{\mathrm{sing}}(M, \mathbb{R}) \cong H^*(K, \mathbb{R}) \cong \check{H}^*(M, \mathbb{R}).$$

Next by the ordinary Poincaré lemma, the sequence of sheaves

$$0 \to \mathbb{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to \cdots$$

on M is exact; in other words, the sequences

$$0 \to \mathbb{R} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \to 0$$

$$\vdots$$

$$0 \to \mathcal{Z}^p_d \to \mathcal{A}^p \xrightarrow{d} \mathcal{Z}^{p+1} \to 0$$

are all exact. Now we have seen that

$$H^q(M, \mathcal{A}^p) = 0$$

for $q > 0$ and all p; by the exact cohomology sequences associated to the short exact sheaf sequences above,

$$\begin{aligned}
\check{H}^p(M, \mathbb{R}) &\cong H^{p-1}(M, \mathcal{Z}^1) \\
&\cong H^{p-2}(M, \mathcal{Z}^2) \\
&\ \ \vdots \\
&\cong H^1(M, \mathcal{Z}^{p-1}) \\
&\cong \frac{H^0(M, \mathcal{Z}^p)}{\delta H^0(M, \mathcal{A}^{p-1})} \\
&= \frac{Z^p(M)}{dA^{p-1}(M)} \\
&= H^p_{\mathrm{DR}}(M).
\end{aligned}$$

Q.E.D.

Note that the de Rham isomorphism is functorial: if $f: M \to N$ is a differentiable map of C^∞ manifolds, φ a closed p-form on N representing $[\varphi] \in H^p_{\text{sing}}(N, \mathbb{R})$ under the de Rham map and $\sigma = \Sigma a_i f_i$ a piecewise smooth p-cycle on M,

$$\begin{aligned}\langle f^*\varphi, \sigma \rangle &= \sum_i a_i \int_\Delta f_i^* f^* \varphi \\ &= \langle \varphi, f_* \sigma \rangle\end{aligned}$$

i.e., $f^*[\varphi] = [f^*\varphi]$.

The Dolbeault Theorem

We saw in the beginning of this section that the obstruction to solving the Mittag-Leffler problem on a Riemann surface S can be taken to lie in either $H^1(S, \mathcal{O})$ or $H^{0,1}_{\bar{\partial}}(S)$. In fact, this represents a special case of the

Dolbeault Theorem. *For* M *a complex manifold,*

$$H^q(M, \Omega^p) \cong H^{p,q}_{\bar{\partial}}(M).$$

Proof. By the $\bar{\partial}$-Poincaré lemma the sequences

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,1}_{\bar{\partial}} \to 0$$

$$\vdots$$

$$0 \to \mathcal{Z}^{p,q}_{\bar{\partial}} \to \mathcal{A}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{Z}^{p,q+1}_{\bar{\partial}} \to 0$$

are exact for all p, q. Since

$$H^r(M, \mathcal{A}^{p,q}) = 0$$

for $r > 0$, all p, q, the long exact cohomology sequences associated to these sheaf sequences give us

$$\begin{aligned}H^q(M, \Omega^p) &\cong H^{q-1}(M, \mathcal{Z}^{p,1}_{\bar{\partial}}) \\ &\cong H^{q-2}(M, \mathcal{Z}^{p,2}_{\bar{\partial}}) \\ &\ \ \vdots \\ &\cong H^1(M, \mathcal{Z}^{p,q-1}_{\bar{\partial}}) \\ &\cong \frac{H^0(M, \mathcal{Z}^{p,q}_{\bar{\partial}})}{\bar{\partial} H^0(M, \mathcal{A}^{p,q-1})} \\ &= H^{p,q}_{\bar{\partial}}(M).\end{aligned}$$

Q.E.D.

As an application we will prove a special case of Leray's theorem: for a locally finite cover $\underline{U} = \{U_\alpha\}$ of M that is *acyclic* for the structure sheaf $\mathcal{O}$, i.e., has the property

$$H^p(U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}, \mathcal{O}) = 0 \qquad \text{for} \quad p > 0,$$

we have

$$H^*(\underline{U},\mathcal{O}) \cong H^*(M,\mathcal{O}).$$

Proof. We have, by hypothesis,

$$\mathcal{Z}_{\bar\partial}^{0,r}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}) = \bar\partial\mathcal{Q}^{0,r-1}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_p});$$

i.e., we have exact sequences of cochain groups

$$0\to C^p(\underline{U},\mathcal{Z}_{\bar\partial}^{0,r-1})\to C^p(\underline{U},\mathcal{Q}^{0,r-1})\to C^p(\underline{U},\mathcal{Z}_{\bar\partial}^{0,r})\to 0,$$

which by the usual algebraic reasoning gives exact sequences

$$\cdots\to H^p(\underline{U},\mathcal{Q}^{0,r-1})\to H^p(\underline{U},\mathcal{Z}_{\bar\partial}^{0,r})\to H^{p+1}(\underline{U},\mathcal{Z}_{\bar\partial}^{0,r-1})$$
$$\to H^{p+1}(\underline{U},\mathcal{Q}^{0,r-1})\to\cdots.$$

Since $H^p(\underline{U},\mathcal{Q}^{0,r})=0$ for $p>0$ by the partition of unity argument, we find

$$\begin{aligned}H^q(\underline{U},\mathcal{O}) &\cong H^{q-1}(\underline{U},\mathcal{Z}_{\bar\partial}^{0,1})\\ &\cong H^{q-2}(\underline{U},\mathcal{Z}_{\bar\partial}^{0,2})\\ &\ \ \vdots\\ &\cong H^1(\underline{U},\mathcal{Z}_{\bar\partial}^{0,q-1})\\ &\cong \frac{H^0(\underline{U},\mathcal{Z}_{\bar\partial}^{0,q})}{\bar\partial H^0(\underline{U},\mathcal{Q}^{0,q-1})}\\ &= H_{\bar\partial}^{0,q}(M)\cong H^q(M,\mathcal{O}).\end{aligned}$$

Q.E.D.

The same argument works as well for the sheaves Ω^p.

Computations

1. The first observation is that if M is an n-dimensional complex manifold, then

$$H^q(M,\mathcal{O}) \cong H_{\bar\partial}^{0,q}(M) = 0 \qquad \text{for } q>n.$$

2. By the $\bar\partial$-Poincaré lemma,

$$H^q(\mathbb{C}^n,\mathcal{O}) = 0 \qquad \text{for } q>0$$

and more generally

$$H^q((\mathbb{C})^k\times(\mathbb{C}^*)^l,\mathcal{O}) = 0 \qquad \text{for } q>0.$$

Since $\mathbb{C}^n$ is contractible, moreover, we see that

$$H^q(\mathbb{C}^n,\mathbb{Z}) = 0 \qquad \text{for } q>0.$$

Now, from the long exact cohomology sequence associated to the exponential sheaf sequence on $\mathbb{C}^n$,

$$H^q(\mathbb{C}^n,\mathcal{O})\to H^q(\mathbb{C}^n,\mathcal{O}^*)\to H^{q+1}(\mathbb{C}^n,\mathbb{Z})$$

is exact, and it follows that

$$H^q(\mathbb{C}^n, \mathcal{O}^*) = 0 \qquad \text{for } q > 0.$$

As an immediate consequence, we have the answer to the *Cousin problem*:

Any analytic hypersurface in $\mathbb{C}^n$ *is the zero locus of an entire function.*

Proof. We have seen that in a neighborhood of any point p in $\mathbb{C}^n$ an analytic hypersurface $V \subset \mathbb{C}^n$ may be given as the zero locus of a holomorphic function $f \in \mathcal{O}_p$, and if we choose f not divisible by the square of any nonunit in $\mathcal{O}_p$ then f is unique up to multiplication by a unit. We can thus find a cover $\underline{U} = \{U_\alpha\}$ of $\mathbb{C}^n$ and functions $f_\alpha \in \mathcal{O}(U_\alpha)$ such that the locus $(f_\alpha = 0) = V \cap U_\alpha$, and such that for any α, β,

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

But since $H^1(\mathbb{C}^n, \mathcal{O}^*) = 0$, the cocycle

$$\{g_{\alpha\beta}\} \in C^1(\underline{U}, \mathcal{O}^*)$$

is a coboundary, i.e., after refinement of the covering if necessary there exists a cochain

$$\{h_\alpha\} \in C^0(\underline{U}, \mathcal{O}^*)$$

such that

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\beta}{h_\alpha}.$$

The entire function

$$f = f_\alpha h_\alpha = f_\beta h_\beta$$

then has zero locus exactly V. Q.E.D.

Another application of the vanishing $H^q((\mathbb{C})^k \times (\mathbb{C}^*)^l, \mathcal{O}) = 0$ is that a covering of a complex manifold by products of planes and punctured planes is acyclic, a fact we will use in the following two computations.

3. To compute the cohomology groups $H^q(\mathbb{P}^1, \mathcal{O})$, take u and $v = 1/u$ Euclidean coordinates on $\mathbb{P}^1$, and set $U = (v \neq 0)$, $V = (u \neq 0)$. U and V are biholomorphic to $\mathbb{C}$ via the coordinates u and v, respectively, while $U \cap V = \mathbb{C}^*$; thus the cover $\{U, V\}$ of $\mathbb{P}^1$ is acyclic. Now

$$C^0(\{U, V\}, \mathcal{O}) = \{(f, g) : f \in \mathcal{O}(U), g \in \mathcal{O}(V)\}$$

and

$$C^1(\{U, V\}, \mathcal{O}) = \{h \in \mathcal{O}(U \cap V)\}.$$

Given $(f,g)\in C^0(\{U,V\},\mathcal{O})$ we can write

$$f=\sum_{n=0}^{\infty}a_n u^n,\qquad g=\sum_{n=0}^{\infty}b_n v^n=\sum_{n=0}^{\infty}b_n u^{-n}.$$

Thus $\delta((f,g))=-f+g\in\mathcal{O}(U\cap V)$ is zero if and only if $a_n=b_n=0$ for n positive and $a_0=b_0$, i.e.,

$$H^0(\mathbb{P}^1,\mathcal{O})\cong\mathbb{C},$$

or in other words the only global holomorphic functions on $\mathbb{P}^1$ are constants.

In general it is clear from the maximum principle that $H^0(M,\mathcal{O})\cong\mathbb{C}$ for any compact, connected complex manifold.

On the other hand, given any

$$h=\sum_{n=-\infty}^{\infty}a_n u^n=\sum_{n=-\infty}^{\infty}a_n v^{-n}\in C^1(\{U,V\},\mathcal{O})$$

we can write

$$h=\delta((f,g)),$$

where

$$f=-\sum_{n=0}^{\infty}a_n u^n,\qquad g=\sum_{n=1}^{\infty}a_{-n}v^n,$$

and it follows that

$$H^1(\mathbb{P}^1,\mathcal{O})=0.$$

Similarly, any element (ω,η) of

$$C^0(\{U,V\},\Omega^1)=\{(\omega,\eta):\ \omega\in\Omega^1(U),\eta\in\Omega^1(V)\}$$

may be written as

$$\omega=\left(\sum_{n=0}^{\infty}a_n u^n\right)du;\qquad \eta=\left(\sum_{n=0}^{\infty}b_n v^n\right)dv=\left(-\sum_{n=0}^{\infty}b_n u^{-n-2}\right)du,$$

since $dv=d(u^{-1})=-u^{-2}du$. We see from this that $\delta((\omega,\eta))=0$ if and only if $\omega=\eta=0$, that is,

$$H^0(\mathbb{P}^1,\Omega^1)=0.$$

By the same token, an element

$$\nu=\left(\sum_{n=-\infty}^{\infty}a_n u^n\right)du\in C^1(\{U,V\},\Omega^1)=\Omega^1(U\cap V)$$

is expressible as $\delta((\omega,\eta))=-\omega+\eta$ if and only if $a_{-1}=0$; thus

$$H^1(\mathbb{P}^1,\Omega^1)\cong\mathbb{C}.$$

The reader may, in the same manner, verify that in general

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & \text{if } p=q\leqslant n, \\ 0 & \text{otherwise,} \end{cases}$$

a fact which we will prove later by means of Hodge theory.

4. Let $M=\mathbb{C}^2-\{0\}$. By Hartogs' theorem we have $\mathcal{O}(\mathbb{C}^2-\{0\})=\mathcal{O}(\mathbb{C}^2)$. Take the covering $U_1=\{z_1\neq 0\}, U_2=\{z_2\neq 0\}$; this is again an acyclic cover ($U_1\cong U_2\cong \mathbb{C}\times\mathbb{C}^*$; $U_1\cap U_2\cong\mathbb{C}^*\times\mathbb{C}^*$). Now $C^1(\{U_1,U_2\},\mathcal{O})=\mathcal{O}(U_1\cap U_2)$ consists of Laurent series

$$f(z_1,z_2)=\sum_{m,n=-\infty}^{\infty} a_{mn}z_1^m z_2^n;$$

$\mathcal{O}(U_1)$ consists of series

$$f(z_1,z_2)=\sum_{m\geqslant 0} b_{mn}z_1^m z_2^n$$

and $\mathcal{O}(U_2)$ of series

$$f(z_1,z_2)=\sum_{n\geqslant 0} c_{mn}z_1^m z_2^n.$$

Thus $\delta C^0(\{U_1,U_2\},\mathcal{O})=\mathcal{O}(U_1)+\mathcal{O}(U_2)$ contains no Laurent series with terms $z_1^m z_2^n$, $m,n<0$; we see that $\dim H^1(\mathbb{C}^2-\{0\},\mathcal{O})=\infty$.

4. TOPOLOGY OF MANIFOLDS

Intersection of Cycles

Consider the standard torus T and the two 1-cycles A and B drawn in Figure 1. It is intuitively reasonable that any 1-cycle homologous to B must intersect any 1-cycle homologous to A, while a cycle holomologous to A—for example, A'—may well be disjoint from A. This is an invariant of the classes $\alpha=(A)$ and $\beta=(B)$ in $H_1(T,\mathbb{Z})$, which we would like to formalize. The problem is that the number of points of intersection of cycles representing α and β is indeterminate: we can have, for example, either of the situations shown in Figure 2. What is needed is a way of counting up the points of intersection of two cycles on T such that "extraneous" intersections cancel each other out. We may do this as follows: first choose an orientation on T. Then if two cycles A and B on T intersect transversely at a point p, we define the *intersection index* $\iota_p(A\cdot B)$ of A and B at p to be $+1$ if the tangent vectors to A and B in turn form an oriented basis for $T_p(M)$, -1 if not; we define the *intersection number* $^{\#}(A\cdot B)$ of cycles A and B meeting transversely in smooth points to be the

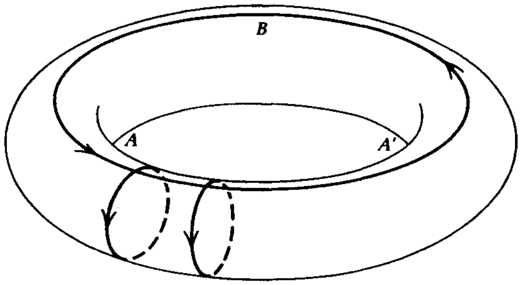


Figure 1

sum

$$^{\#}(A \cdot B) = \sum_{p \in A \cap B} \iota_p(A \cdot B).$$

It is easy to see that $^{\#}(A \cdot B)$ depends only on the homology classes of A and B: if A is homologous to zero—that is, if A is the boundary of regions $C_i \subset T$ with the tangent vector to A and the inward normal vector to ∂C_i always forming an oriented basis for $T(M)$—then the path B will intersect A positively every time it enters a region C_i and negatively every time it leaves; thus

$$^{\#}(A \cdot B) = 0$$

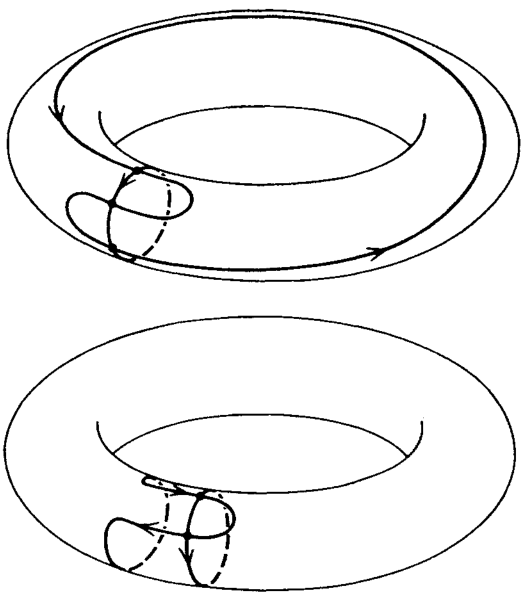


Figure 2

and, in general, since the intersection number is linear in either factor, if $A \sim A'$ then

$$^{\#}(A' \cdot B) = {}^{\#}(A \cdot B).$$

Finally, since for any two homology classes $\alpha, \beta \in H_1(T, \mathbb{Z})$ we can find cycles A and B on T representing α and β and intersecting transversely, we have defined a bilinear pairing

$$H_1(T,\mathbb{Z}) \times H_1(T,\mathbb{Z}) \longrightarrow \mathbb{Z}.$$

The definition of intersection of cycles on a general oriented manifold differs from this special case only in the difficulty of verifying the transversality statements made. Suppose M is an oriented n-manifold, A and B two piecewise smooth cycles on M of dimensions k and $n-k$, respectively, and $p \in A \cap B$ a point of transverse intersection of A and B. Let $v_1, \ldots, v_k \in T_p(A) \subset T_p(M)$ be an oriented basis for $T_p(A)$, $w_1, \ldots, w_{n-k}$ an oriented basis for $T_p(B) \subset T_p(M)$; we define the *intersection index* $\iota_p(A \cdot B)$ of A with B at p to be $+1$ if $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$ is an oriented basis for $T_p(M) = T_p(A) \oplus T_p(B)$, and -1 if not. If A and B intersect transversely everywhere, we define the *intersection number* $^{\#}(A \cdot B)$ to be

$$^{\#}(A \cdot B) = \sum_{p \in A \cap B} \iota_p(A \cdot B).$$

Note that this sum is finite, since A and B are compactly supported and by hypothesis $A \cap B$ is discrete.

We now have to show that the intersection number $^{\#}(A \cdot B)$ depends only on the homology class of A and B; i.e., that

$$A \sim 0 \Rightarrow {}^{\#}(A \cdot B) = 0.$$

In this case we may take $A = \partial C$ to be the sum of boundaries of piecewise smooth $(k+1)$-manifolds C_i, so that at each smooth point $p \in A$ an oriented basis $v_1, \ldots, v_k$ for $T_p(A)$ together with an inward normal vector to C_i gives the orientation on C_i. By a standard transversality argument, we may take the chain C to meet B transversely almost everywhere, so that the intersection $C \cap B$ will consist of a collection $\{\gamma_\alpha\}$ of piecewise smooth arcs. The endpoints of these arcs will, of course, constitute the points of intersection of A with B; we claim that for each γ, the two endpoints $\gamma(0), \gamma(1) \in A \cap B$ will have opposite intersection index for A and B. (See Figure 3.) This is not hard to see: we can find C^∞ vector fields $\{v_i(t) \in T_{\gamma(t)}(C)\}_{i=1,\ldots,k}$ to C along γ and $\{v_i(t) \in T_{\gamma(t)}(B)\}_{i=k+2,\ldots,n}$ to B along γ, such that for all t

1. $v_1(t), \ldots, v_k(t), \gamma'(t)$ is an oriented basis for $T_{\gamma(t)}(C)$,
2. $\gamma'(t), v_{k+2}(t), \ldots, v_n(t)$ is an oriented basis for $T_{\gamma(t)}(B)$,
3. $v_1(t), \ldots, v_k(t), \gamma'(t), v_{k+2}(t), \ldots, v_n(t)$ is an oriented basis for $T_{\gamma(t)}(M)$,

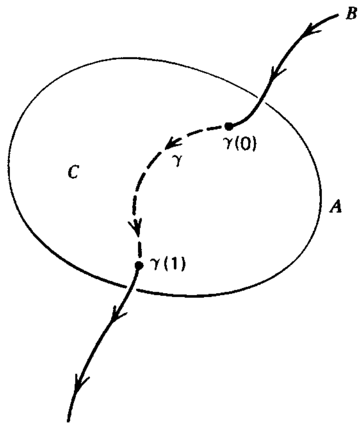


Figure 3

and such that $v_1(0),\ldots,v_k(0)$ is an oriented basis for $T_{\gamma(0)}(A)$, $v_1(1),\ldots,v_k(1)$ a basis for $T_{\gamma(1)}(A)$. (Satisfying all these conditions may require reversing the assigned direction of γ.) Then since $\gamma'(1)$ is outward normal to C and $v_1(1),\ldots,v_k(1),\gamma'(1)$ is positively oriented for C, the basis $v_1(1),\ldots,v_1(1)$ for $T_{\gamma(1)}(A)$ must be negatively oriented. Thus

$$\iota_{\gamma(0)}(A\cdot B) = +1 \quad \text{and} \quad \iota_{\gamma(1)}(A\cdot B) = -1,$$

and we are done.

Now if $\alpha\in H_k(M,\mathbb{Z})$ and $\beta\in H_{n-k}(M,\mathbb{Z})$ are any two homology classes, we may find C^∞ piecewise smooth cycles A and B on M representing α and β and intersecting transversely. The intersection number ${}^{\#}(A\cdot B)$ is determined by the classes α,β, and so we have defined a bilinear pairing

$$H_k(M,\mathbb{Z})\times H_{n-k}(M,\mathbb{Z})\longrightarrow\mathbb{Z},$$

called the *intersection pairing*, and denoted by ${}^{\#}(\alpha\cdot\beta)$. Note that from the definition of the intersection index,

$${}^{\#}(\beta\cdot\alpha) = (-1)^{k(n-k)}{}^{\#}(\alpha\cdot\beta).$$

We can also define a product

$$H_{n-k_1}(M,\mathbb{Z})\times H_{n-k_2}(M,\mathbb{Z})\longrightarrow H_{n-k_1-k_2}(M,\mathbb{Z})$$

on the homology of M in arbitrary dimensions: if $\alpha\in H_{n-k_1}(M)$ and $\beta\in H_{n-k_2}(M)$ are classes, we can find cycles A and B representing them and intersecting transversely almost everywhere. The intersection C is given the orientation such that if $v_1,\ldots,v_{n-k_1-k_2}$ is an oriented basis for $T_p(C)$ at a smooth point of C and we complete it to bases

$$w_1,\ldots,w_{k_2},v_1,\ldots,v_{n-k_1-k_2}$$

and

$$v_1, \ldots, v_{n-k_1-k_2}, u_1, \ldots, u_{k_1}$$

for $T_p(A)$ and $T_p(B)$, respectively, the full basis

$$w_1, \ldots, w_{k_2}, v_1, \ldots, v_{n-k_1-k_2}, u_1, \ldots, u_{k_1}$$

is positively oriented for $T_p(M)$. C, with this orientation, is called the *intersection cycle* $A \cdot B$ of A and B. Again, to show that intersection is well-defined on homology—that is, that the cycle $A \cdot B$ is homologous to zero if A is—we have to show first that we can find a chain C with

$$\partial C = A$$

intersecting B transversely almost everywhere, and then that the set-theoretic relation

$$A \cdot B = \partial(C \cdot B)$$

holds as well on the level of oriented cycles. The techniques used to prove these assertions are similar but more complicated than those used in the case of complementary dimension.

A point of terminology: when we speak of the intersection number or "topological intersection" of two cycles A and B on a manifold M, we shall always refer to the intersection number of the classes $\alpha, \beta \in H_*(M, \mathbb{Z})$ they represent. Thus the expression $^{\#}(A \cdot B)$ willl have meaning even when A and B fail to meet transversely.

Poincaré Duality

The fundamental result on intersection of cycles is the

Theorem (Poincaré Duality). *If* M *is a compact, oriented* n*-manifold, the intersection pairing*

$$H_k(M, \mathbb{Z}) \times H_{n-k}(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

is unimodular; i.e., any linear functional $H_{n-k}(M, \mathbb{Z}) \longrightarrow \mathbb{Z}$ *is expressible as intersection with some class* $\alpha \in H_k(M, \mathbb{Z})$, *and any class* $\alpha \in H_k(M, \mathbb{Z})$ *having intersection number* 0 *with all classes in* $H_{n-k}(M, \mathbb{Z})$ *is a torsion class.*

Proof. As in the previous section, we may assume that M is the underlying manifold of a simplicial complex $K = \{\sigma_\alpha^k, \partial\}_{\alpha,k}$. The essential step in the proof is the construction of the *dual cell decomposition* of M, as follows. (See Figure 4.) First let $\{\tau_\alpha^k, \partial\}$ be the first barycentric subdivision of the complex K. For each vertex σ_α^0 in the original triangulation, let

$$*\sigma_\alpha^0 = \bigcup_{\tau_\beta^n \ni \sigma_\alpha^0} \tau_\beta^n$$

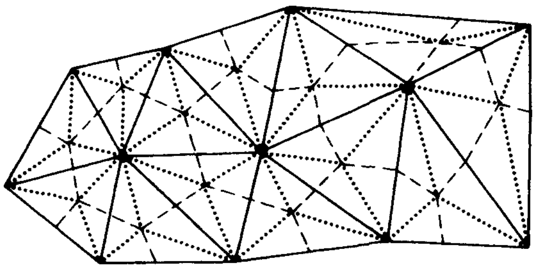


Figure 4

be the n-cell given as the union of the n-simplices τ_β^n in the subdivision having σ_α^0 as a vertex. Then for each k-simplex σ_α^k in the original decomposition, let

$$*\sigma_\alpha^k = \bigcap_{\sigma_\beta^0 \in \sigma_\alpha^k} *\sigma_\beta^0$$

be the intersection of the n-cells $*\sigma_\beta^0$ associated to the $k+1$ vertices of σ_α^k. The cells $\{\Delta_\alpha^{n-k} = *\sigma_\alpha^k\}$ then give a decomposition of M, called the dual cell decomposition to $\{\sigma_\alpha\}$.

Note that since the only point of a k-complex σ_α^k of our original complex held in common by $k+1$ cells of the dual decomposition is its barycenter, *the dual cell* $\Delta_\alpha^{n-k} = *\sigma_\alpha^k$ *of* σ_α *is the only* $(n-k)$*-cell of the dual decomposition meeting* σ_α^k; Δ_α^{n-k} will intersect σ_α^k transversely. Given an orientation on σ_α^k, we may take the dual orientation on Δ_α^{n-k} to be the one such that at $p = \sigma_\alpha^* \cap \sigma_\alpha$,

$$\iota_p(\sigma_\alpha, \Delta_\alpha) = +1.$$

Hereafter, if σ_α is considered an oriented simplex, $*\sigma_\alpha$ will denote the oriented cell Δ_α with the dual orientation; we will also write $*\Delta_\alpha$ to denote the original oriented simplex σ_α.

We now relate the boundary operator ∂ on the complex $\{\sigma_\alpha^k\}$ to the coboundary operator δ on $\{\Delta_\alpha^{n-k}\}$. Note first that if σ_α^k has vertices $\sigma_0^0, \ldots, \sigma_k^0$, then the dual cell $\Delta_\alpha^{n-k} = *\sigma_\alpha^k$ is given as the $(k+1)$-fold intersection $\cap_i \Delta_i^n = \cap_i *\sigma_i^0$ of the dual n-cells, and so the cells appearing in the coboundary $\delta\Delta_\alpha^{n-k}$ of Δ_α^{n-k} will be just the k-fold intersections $\Delta_j^{n-k+1} = \cap_{i \neq j} \Delta_i^n$ of the cells Δ_i^n, that is, the dual cells of the faces σ_j^{k-1} of σ^k. We claim now that the basic relation

$$\delta(\Delta_\alpha^{n-k}) = (-1)^{n-k+1} *(\partial\sigma_\alpha^k)$$

holds on the level of oriented cells, i.e., that if σ_j and Δ_j are oriented as the boundary and coboundary of σ_α and Δ_α, respectively, then at $p' = \sigma_j \cap \Delta_j$,

$$\iota_{p'}(\sigma_j \cdot \Delta_j) = (-1)^{n-k+1}.$$

(See Figure 5.) This is the same sort of argument as made in the verification of homology-invariance of intersection number. The simplex σ_α^k intersects the cell Δ_j^{n-k+1} in an arc γ running from the barycenter $p=\gamma(0)$ of σ_α to the barycenter $p'=\gamma(1)$ of the face σ_j of σ_α. Let $v_1,\ldots,v_{k-1}$ then be vector fields to σ_α along γ and $v_{k+1},\ldots,v_n$ vector fields to Δ_j along γ such that $v_1(0),\ldots,v_{k-1}(0)$, $\gamma'(0)$ is an oriented basis for $T_{\gamma(0)}(\sigma_\alpha)$ and $v_{k+1}(0),\ldots,v_n(0)$ an oriented basis for $T_{\gamma(0)}(\Delta_\alpha)$, and such that $v_1(1),\ldots,v_{k-1}(1)\in T_{\gamma(1)}(\sigma_\alpha)$, $v_{k+1}(1),\ldots,v_n(1)\in T_{\gamma(1)}(\Delta_j)$. By the hypothesis

$$\iota_{\gamma(0)}(\Delta_\alpha\cdot\sigma_\alpha) = +1,$$

the basis $v_1(0),\ldots,v_{k-1}(0),\gamma'(0),v_{k+1}(0),\ldots,v_n(0)$ is positive for the given orientation on M. Moreover, since $\gamma'(0)$ is inward normal to Δ_j at $\gamma(0)$, and since $v_{k+1}(0),\ldots,v_n(0)$ is positively oriented for $T_{\gamma(0)}(\Delta_\alpha)$, the basis

$$\gamma'(0),v_{k+1}(0),\ldots,v_n(0)$$

will have sign $(-1)^{n-k}$ with respect to the orientation on Δ_j. By continuity, these last two assertions will hold as well as $\gamma(1)$. There, since $\gamma'(0)$ is outward normal to Δ_α at $\gamma(1)$ and since $v_1(1),\ldots,v_{k-1}(1)$, $\gamma'(1)$ is positively oriented for $T_{\gamma(1)}(\sigma_\alpha)$, the basis $v_1(1),\ldots,v_{k-1}(1)$ will be negatively oriented for σ_j. Thus

$$\iota_{\gamma(1)}(\sigma_j\cdot\Delta_j) = (-1)^{n-k+1}$$

as desired.

We see from this that the map

$$\sigma_\alpha^p \longrightarrow \tilde{\Delta}_\alpha^{n-p}$$

induces an isomorphism between the complex (C_*,∂) of chains in the original simplicial decomposition of M and the complex $(\tilde{C}^*,\delta)$ of cochains in the dual cell decomposition. The resulting isomorphisms

$$D: H_k(M,\mathbb{Z}) \to H^{n-k}(M,\mathbb{Z})$$

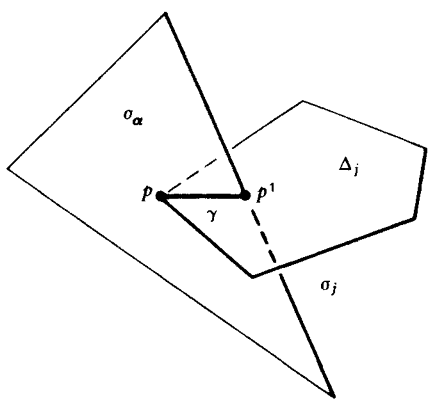


Figure 5

have the property that

$$^{\#}(\gamma\cdot\lambda) = D\gamma(\lambda)$$

for any $\gamma \in H_k(M,\mathbb{Z})$ and $\lambda \in H_{n-k}(M,\mathbb{Z})$; and the theorem follows. Q.E.D.

A somewhat weaker version of Poincaré duality is the statement that the map

$$H_k(M,\mathbb{Q}) \xrightarrow{P} H_{n-k}(M,\mathbb{Q})^* \cong H^{n-k}(M,\mathbb{Q})$$

given by

$$P(A)(B) = {}^{\#}(A\cdot B)$$

is an isomorphism, omitting the fact that the intersection pairing is unimodular. Via the de Rham isomorphism

$$H_{\mathrm{DR}}^{n-k}(M) \longrightarrow H^{n-k}(M,\mathbb{C})$$

this is equivalent to the assertion that for any k-cycle A on M there exists a closed $(n-k)$-form φ such that for any $(n-k)$ cycle B on M,

$$\int_B \varphi = {}^{\#}(A\cdot B).$$

Suppose φ and ψ are two closed forms on the oriented manifold M. Then the wedge product $\varphi\wedge\psi$ is closed, and by virtue of the relation

$$\varphi\wedge(\psi + d\eta) = \varphi\wedge\psi + (-1)^{\deg\varphi} d(\varphi\wedge\eta)$$

we see that the de Rham class of $\varphi\wedge\psi$ depends only on the de Rham classes of φ and ψ. Thus we have bilinear maps

$$H_{\mathrm{DR}}^{k}(M) \otimes H_{\mathrm{DR}}^{k'} \longrightarrow H_{\mathrm{DR}}^{k+k'}(M)$$

and in particular a pairing

$$H_{\mathrm{DR}}^{k}(M) \otimes H_{\mathrm{DR}}^{n-k}(M) \longrightarrow H_{\mathrm{DR}}^{n}(M) \cong \mathbb{C}.$$

We will now relate this pairing in de Rham cohomology to the intersection of cycles via Poincaré duality; to do this we must first establish the Künneth formula.

Suppose $M = \{\sigma_\alpha^k\}_{\alpha,k}$ and $N = \{\sigma_\alpha'^k\}_{\alpha,k}$ are two simplicial complexes. The products $\sigma_\alpha^k \times \sigma_\beta'^l$ give a cell decomposition of the product space $M \times N$, with boundary operator

$$\partial\left(\sigma_\alpha^k \times \sigma_\beta'^l\right) = \partial\sigma_\alpha^k \times \sigma_\beta'^l + (-1)^k \sigma_\alpha^k \times \partial\sigma_\beta'^l.$$

The product

$$A \times B = \sum a_\alpha b_\beta \sigma_\alpha^k \times \sigma_\beta'^l$$

of two cycles

$$A = \sum a_\alpha \sigma_\alpha^k \quad \text{and} \quad B = \sum b_\beta \sigma_\beta'^{\,l}$$

in M and N is a cycle, and the homology class of $A \times B$ depends only on the homology classes of A and B, since

$$(A + \partial C) \times B = A \times B + \partial(C \times B).$$

We have thus a map

$$H_*(M,\mathbb{Z}) \otimes H_*(N,\mathbb{Z}) \longrightarrow H_*(M \times N,\mathbb{Z});$$

we claim that it is, modulo torsion, an isomorphism. This is readily seen once we express the chains of the complexes M and N in terms of canonical bases, that is, ones in terms of which the boundary operators are diagonal. We may construct such a basis for the chains in M as follows. Suppose M has dimension m; let $\{\tau_\alpha^m\}$ be a rational basis for the m-cycles in M. Complete $\{\tau_\alpha^m\}$ to a rational basis for the m-chains of M; call the additional basis elements $\{\mu_\alpha^m\}$. Set

$$\sigma_\alpha^{m-1} = \partial \mu_\alpha^m;$$

so that $\{\sigma_\alpha^{m-1}\}$ is a basis for the boundaries of M in dimension $m-1$; complete $\{\sigma_\alpha^{m-1}\}$ to a rational basis $\{\sigma_\alpha^{m-1}, \tau_\beta^{m-1}\}$ for the $(m-1)$-cycles of M and complete $\{\sigma_\alpha^{m-1}, \tau_\beta^{m-1}\}$ to a rational basis $\{\sigma_\alpha^{m-1}, \tau_\beta^{m-1}, \mu_\gamma^{m-1}\}$ for all $(m-1)$-chains on M. Set $\sigma_\alpha^{m-2} = \partial \mu_\alpha^{m-1}$; continuing in this way, we obtain a rational basis $\{\sigma_\alpha^k, \tau_\alpha^k, \mu_\alpha^k\}$ for the chains of M, with $\{\sigma_\alpha^k\}$ a basis for the boundaries, $\{\sigma_\alpha^k, \tau_\alpha^k\}$ a basis for the cycles, and $\partial \mu_\alpha^k = \sigma_\alpha^{k-1}$.

Now let $\{\sigma_\alpha'^k, \tau_\alpha'^k, \mu_\alpha'^k\}$ be a similarly constructed basis for the chains of N, and let A be a cycle in $M \times N$, expressed as a linear combination of the products of the basis elements in M and N. Since the products $\sigma_\alpha^k \times \sigma_\beta'^l$, $\sigma_\alpha^k \times \tau_\beta'^l$, and $\tau_\alpha^k \times \sigma_\beta'^l$ are the boundaries of $\mu_\alpha^{k+1} \times \sigma_\beta'^l$, $\mu_\alpha^{k+1} \times \tau_\beta'^l$, and $(-1)^k \tau_\alpha^k \times \mu_\beta'^{l+1}$, respectively, we may, after replacing A with a homologous cycle, assume that no such terms appear in the expression for A. Also, if a term

$$\sigma_\alpha^k \times \mu_\beta'^l$$

appears in A, we may remove it by subtracting from A the boundary

$$\partial\left(\mu_\alpha^{k+1} \times \mu_\beta'^l\right) = \sigma^k \times \mu_\beta'^l + (-1)^{k+1} \mu_\alpha^{k+1} \times \sigma_\beta'^{l-1}.$$

Thus we can write

$$\begin{aligned} A &= \sum a_{\alpha\beta kl} \tau_\alpha^k \times \tau_\beta'^l + \sum b_{\alpha\beta kl} \tau_\alpha^k \times \mu_\beta'^l \\ &\quad + \sum c_{\alpha\beta kl} \mu_\alpha^k \times \tau_\beta'^l + \sum d_{\alpha\beta kl} \mu_\alpha^k \times \mu_\beta'^l \\ &\quad + \sum e_{\alpha\beta kl} \mu_\alpha^k \times \sigma_\beta'^l. \end{aligned}$$

Taking the boundary, we have

$$\begin{aligned}0 = \partial A &= \sum b_{\alpha\beta kl}(-1)^k \tau_\alpha^k \times \sigma_\beta^{\prime\, l-1} \\ &+ \sum c_{\alpha\beta kl}\sigma_\alpha^{k-1} \times \tau_\beta^{\prime\, l} + \sum d_{\alpha\beta kl}\sigma_\alpha^{k-1} \times \mu_\beta^{\prime\, l} \\ &+ \sum (-1)^k d_{\alpha\beta kl}\mu_\alpha^k \times \sigma_\beta^{l-1} \\ &+ \sum e_{\alpha\beta kl}\sigma_\alpha^{k-1} \times \sigma_\beta^{\prime\, l}.\end{aligned}$$

But now all the terms in the sum are linearly independent, and so each must be zero; thus $b_{\alpha\beta kl} = c_{\alpha\beta kl} = d_{\alpha\beta kl} = e_{\alpha\beta kl} = 0$ for each α, β, k, l and

$$A = \sum a_{\alpha\beta kl}\tau_\alpha^k \times \tau_\beta^{\prime\, l}$$

is a linear combination of products of cycles of M and N. Similarly, we see that A is homologous to 0 in $M \times N$ only if $a_{\alpha\beta kl} = 0$ for each α, β, k, l, and so we have established the *Künneth formula*:

$$H_*(M \times N, \mathbb{Q}) \cong H_*(M, \mathbb{Q}) \otimes H_*(N, \mathbb{Q}).$$

We now relate intersections of cycles to wedge products of forms on a compact oriented n-manifold M. Suppose σ is a k-cycle on M and τ an $(n-k)$-cycle and let $\varphi \in A^{n-k}(M), \psi \in A^k(M)$ be closed forms on M representing the cohomology classes Poincaré dual to the classes of σ and τ, i.e., such that for any $(n-k)$-cycle μ,

$$\int_\mu \varphi = {}^{\#}(\sigma \cdot \mu)$$

and for any k-cycle ν

$$\int_\nu \psi = {}^{\#}(\tau \cdot \nu).$$

On the product $M \times M$, with projection maps π_1, π_2, we have

$$\begin{aligned}\int_{\mu \times \nu} \pi_1^*\varphi \wedge \pi_2^*\psi &= \int_\mu \varphi \cdot \int_\nu \psi \\ &= {}^{\#}(\sigma \cdot \mu) \cdot {}^{\#}(\tau \cdot \nu).\end{aligned}$$

On the other hand, if (p_1, p_2) is a point of intersection of $\sigma \times \tau$ with $\mu \times \nu$, then writing out the orientations we see that

$$\iota_{(p_1, p_2)}(\sigma \times \tau, \mu \times \nu) = (-1)^{n-k}\iota_{p_1}(\sigma \cdot \mu) \cdot \iota_{p_2}(\tau \cdot \nu).$$

Thus

$$\begin{aligned}{}^{\#}(\sigma \times \tau, \mu \times \nu) &= (-1)^{n-k}\,{}^{\#}(\sigma \cdot \mu) \cdot {}^{\#}(\tau \cdot \nu) \\ &= (-1)^{n-k} \int_{\mu \times \nu} \pi_1^*\varphi \wedge \pi_2^*\psi;\end{aligned}$$

note that this formula holds for any $(n-k')$-cycle μ and k-cycle ν: if $k \neq k'$, both sides are zero. By Künneth such products of cycles generate

$H_n(M\times M,\mathbb{Q})$, and so it follows that the form $\pi_1^*\varphi\wedge\pi_2^*\psi$ is Poincaré dual to the cycle $(-1)^{n-k}\sigma\times\tau$, i.e., for any n-cycle η in $M\times M$,

$$(-1)^{n-k}\int_\eta \pi_1^*\varphi\wedge\pi_2^*\psi = {}^{\#}(\sigma\times\tau\cdot\eta).$$

We apply this in particular to the diagonal $\Delta\subset M\times M$. On the one hand,

$$\int_\Delta \pi_1^*\varphi\wedge\pi_2^*\psi = \int_M \varphi\wedge\psi.$$

On the other hand, a point (p,p) of intersection of $\sigma\times\tau$ with Δ corresponds to a point p of intersection of σ with τ, and examining the orientations we find that for such a point p,

$$\iota_{(p,p)}(\sigma\times\tau\cdot\Delta) = (-1)^{n-k}\iota_p(\sigma\cdot\tau).$$

Thus

$${}^{\#}(\sigma\cdot\tau) = (-1)^{n-k}\,{}^{\#}(\sigma\times\tau\cdot\Delta) = \int_M \varphi\wedge\psi,$$

i.e., *intersection of cycles in homology is Poincaré dual to wedge product in cohomology*.

Note that we can identify the Poincaré dual of the pullback map on cohomology. Explicitly, if $f: M\to N$ is a C^∞ map of manifolds nonsingular over the cycle $A\subset N$, then with the proper orientation *the cycle* $f^{-1}(A)$ *is Poincaré dual to the pullback via* f *of the Poincaré dual of* A. This is not hard to see: if $B\subset M$ is any cycle on M meeting $f^{-1}(A)$ transversely, then $f(B)$ will meet A transversely at $f(B\cap f^{-1}(A))$. If φ is a closed form on N Poincaré dual to A, then,

$$\int_B f^*\varphi = \int_{f(B)}\varphi = {}^{\#}(f(B)\cdot A) = {}^{\#}(B\cdot f^{-1}(A)),$$

so $f^{-1}(A)$ is Poincaré dual to $f^*\varphi$.

In this context, the weaker form of duality may be restated once again as the assertion that the pairing

$$H^k_{\mathrm{DR}}(M)\otimes H^{n-k}_{\mathrm{DR}}(M)\longrightarrow\mathbb{R}$$

given by

$$([\varphi],[\psi])\longrightarrow\int_M\varphi\wedge\psi$$

is nondegenerate, or that for any closed k-form φ on M there exists an $(n-k)$-cycle A, unique up to homology, such that for any closed $(n-k)$-form ψ,

$$\int_M\varphi\wedge\psi = \int_A\psi.$$

A note: The ordinary cup product $\alpha \cup \beta$ of two cohomology classes $\alpha \in H^k(M,\mathbb{Q})$ and $\beta \in H^{k'}(M,\mathbb{Q})$ may be defined as the pullback

$$\alpha \cup \beta = \Delta^*(\alpha \otimes \beta)$$

via the diagonal map $\Delta : M \to M \times M$ of the class $\alpha \otimes \beta$ on $M \times M$ defined by

$$\alpha \otimes \beta(\sigma \times \tau) = \alpha(\sigma) \cdot \beta(\tau)$$

for all cycles σ, τ on M. With this definition, it is clear that if φ and ψ are closed forms on M representing α and β, the form $\pi_1^*\varphi \wedge \pi_2^*\psi$ on $M \times M$ represents $\alpha \otimes \beta$, and hence that $\varphi \wedge \psi$ represents the class $\alpha \cup \beta$. Thus wedge product of forms corresponds, via the de Rham isomorphism, to cup product of cocycles.

As an example, let us compute the homology algebra of $\mathbb{P}^n$. To do this denote by $X = (X_0, \ldots, X_n)$ Euclidean coordinates on $\mathbb{C}^{n+1}$ and $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ the *flag* in $\mathbb{C}^{n+1}$ given by

$$V_i = (X_n = \cdots = X_{i+1} = 0);$$

let $\mathbb{P}^k \subset \mathbb{P}^n$ be the image of V^{k+1}. As we have seen, the complement $\mathbb{P}^n - \mathbb{P}^{n-1}$ of the hyperplane $\mathbb{P}^{n-1}$ in $\mathbb{P}^n$ is $\mathbb{C}^n$ with Euclidean coordinates $X_0/X_n, \ldots, X_{n-1}/X_n$; similarly, the complement of $\mathbb{P}^{k-1}$ in $\mathbb{P}^k$ is $\mathbb{C}^k$ with coordinates $X_0/X_k, \ldots, X_{k-1}/X_k$. We have therefore a cell-decomposition of $\mathbb{P}^n$,

$$\mathbb{P}^n = (\mathbb{P}^n - \mathbb{P}^{n-1}) \cup (\mathbb{P}^{n-1} - \mathbb{P}^{n-2}) \cup \cdots \cup (\mathbb{P}^1 - \mathbb{P}^0) \cup \mathbb{P}^0,$$

as a union of $2k$-cells $\mathbb{P}^k - \mathbb{P}^{k-1} \cong \mathbb{C}^k$, one for each $k = 0, \ldots, n$, generalizing the familiar picture of the Riemann sphere. Since there are cells only in even dimension, all boundary maps are zero, and so the homology of $\mathbb{P}^n$ is freely generated by the classes of the closures $\mathbb{P}^k$ of the cells, i.e., by the homology classes of its linear subspaces given the natural orientation.

Inasmuch as a k-plane $\mathbb{P}^k$ and an $(n-k)$-plane $\mathbb{P}^{n-k}$ in $\mathbb{P}^n$ will generically intersect transversely in one point, Poincaré duality is clear in this case. Indeed, since an $(n-k_1)$-plane will generically intersect an $(n-k_2)$-plane transversely in an $(n-k_1-k_2)$-plane,

$$((\mathbb{P}^{n-k_1}) \cdot (\mathbb{P}^{n-k_2})) = \pm (\mathbb{P}^{n-k_1-k_2}).$$

Intersection of Analytic Cycles

Suppose now that M is a compact complex manifold of dimension n, $V \subset M$ a possibly singular analytic subvariety of dimension k. As we have seen, Stokes' theorem

$$\int_V d\psi = 0$$

holds for any $(2k-1)$-form φ on M. We may thus define a linear functional on $H^{2k}_{\mathrm{DR}}(M)$ by

$$[\varphi] \longrightarrow \int_V \varphi,$$

where V is given the natural orientation. By Poincaré duality this linear functional determines a cohomology class $\eta_V \in H^{2n-2k}_{\mathrm{DR}}(M)$, called the *fundamental class* of V.

We may also define the fundamental class of V by means of the intersection pairing. For any homology class $\alpha \in H_{2n-2k}(M,\mathbb{Z})$ we may find a cycle A representing α and intersecting V transversely in smooth points. In fact, the intersection number

$$^{\#}(V\cdot A) = \sum_{p\in A\cap V} \iota_p(V,A)$$

—where V again is given the natural orientation—depends only on the homology class α: if $A' \sim A$, then *since the singular locus of* V *has real codimension* $\geqslant 2$ we can find a $(2n-2k+1)$-chain C on M avoiding the singular set of V, meeting V transversely almost everywhere and such that

$$\partial C = A - A'.$$

The proof that $^{\#}(A\cdot V) = {}^{\#}(A'\cdot V)$ then proceeds exactly as at the beginning of this section. Consequently V defines a linear functional

$$H_{2n-2k}(M,\mathbb{Z}) \longrightarrow \mathbb{Z};$$

the corresponding cohomology class $\eta_V \in H^{2n-2k}(M)$ is the fundamental class of V.

Note: When we speak of the fundamental class of a variety $V \subset M$ we may also refer to its Poincaré dual—that is, the element of homology given by the linear functional

$$H^{2n-2k}_{\mathrm{DR}}(M) \longrightarrow \mathbb{C}$$

$$[\varphi] \longmapsto \int_V \varphi$$

Usually it will either be clear from the context or unimportant which of these we are referring to.

We now make a very simple observation. Suppose V and W are analytic varieties of dimension k and $n-k$ intersecting transversely at a point p on the complex manifold M. We may take holomorphic coordinates $z = (z_1,\ldots,z_n)$ on M near p such that V and W are given by

$$V = (z_{k+1} = \cdots = z_{n-k} = 0)$$

and

$$W = (z_1 = \cdots = z_k = 0).$$

Writing $z_i = x_i + \sqrt{-1}\, y_i$, the natural orientation on M is given by the basis

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right)$$

for $T_p(V)$, while the natural orientations for V and W are given by

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_k}\right)$$

and

$$\left(\frac{\partial}{\partial x_{k+1}}, \frac{\partial}{\partial y_{k+1}}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right).$$

We see, then, that if V, W, and M are all given the natural orientations,

$$\iota_p(V \cdot W) = +1.$$

This trivial observation, that *the intersection index of two analytic subvarieties meeting transversely is always positive*, is in fact one of the cornerstones of algebraic geometry. It relates the set-theoretic intersection of two varieties—a priori a geometric invariant—to the intersection number—a topological invariant—and so provides a basic link. Before we can fully utilize this bond, however, we have to extend it to varieties that may not intersect transversely. This goes as follows (alternate discussion of intersections of analytic varieties will be given in Section 2 of Chapter 3 and in Section 2 of Chapter 5).

Let V and W be two analytic varieties of dimension k and $n-k$ in the polycylinder Δ of radius 1 in $\mathbb{C}^n$ having the origin as their only point of intersection. Consider in the product $\Delta' \times \Delta'$ of the polycylinder of radius $\frac{1}{2}$ with itself the two varieties

$$\tilde{V} = \pi_1^{-1}(V) = \{(z,w) : z \in V\}$$

and

$$\tilde{W} = \{(z,w) : z - w \in W\}.$$

For each ε, of course, the varieties $\tilde{V}$ and $\tilde{W}$ meet the fiber $\pi_2^{-1}(\varepsilon) = \Delta' \times \{\varepsilon\} \cong \Delta'$ in the analytic variety V and the analytic variety $W + \varepsilon$—that is, W translated by ε—respectively; moreover, $\pi_2^{-1}(\varepsilon)$ will meet the intersection $\tilde{V} \cap \tilde{W}$ transversely at a point (p,ε) exactly when V and $W+\varepsilon$ meet transversely at p. The intersection $\tilde{V} \cap \tilde{W} \subset \Delta' \times \Delta'$ is an analytic variety of dimension n, and so the projection $\pi_2 : \tilde{V} \cap \tilde{W} \longrightarrow \Delta'$ expresses $\tilde{V} \cap \tilde{W}$ as a branched μ-sheeted cover of Δ'; accordingly, we see that *for* $\varepsilon \in \Delta'$ *lying outside an analytic subvariety of* Δ', *the varieties* V *and* W $+ \varepsilon$ *will meet transversely in* μ *points in* Δ'. The number μ is called the *intersection multiplicity* of V and W at 0 and is written

$$\mu = m_0(V \cdot W).$$

By the construction the intersection multiplicity is always positive, and by the implicit function theorem will be 1 if and only if $\tilde{V} \cap \tilde{W}$ meets the fiber $\pi_2^{-1}(0)$ transversely—that is, if and only if V and W meet transversely at the origin. Note that the definition does not depend on the choice of coordinates z, so that it applies as well to two analytic subvarieties of a complex manifold. We now check that if V and W are analytic subvarieties of complementary dimension on a compact complex manifold M, then

$$^{\#}(V \cdot W) = \sum_{p \in V \cap W} m_p(V \cdot W).$$

To do this let z, w be local coordinates around a point $p \in V \cap W$, with $p = (0,0)$ the only point of intersection of V with W in the ball Δ of radius 1. Let $\rho(r)$ be a C^∞ bump function, identically 1 on the ball Δ'' of radius $\frac{1}{4}$ and identically zero outside the ball Δ' of radius $\frac{1}{2}$. Then for ε generic and sufficiently small, the locus

$$W_\varepsilon = \{(z) : z - \rho(\|z\|) \cdot \varepsilon \in W\} \subset \Delta$$

will

1. agree with W outside Δ',
2. be disjoint from V in $\Delta' - \Delta''$,
3. be an analytic variety in Δ'', meeting V transversely in $\mu = m_p(V \cdot W)$ points.

Now let $\{p_i\} = V \cap W$. Choose coordinate balls Δ_i around p_i and values ε_i as above; set

$$W' = (W - \cup \Delta_i) \cup (\cup W_{\varepsilon_i}).$$

W' is then smooth manifold outside a locus of codimension 2 or more, and so by our general method represents a cohomology class $\eta_{W'}$ in M; indeed, $\eta_{W'} = \eta_W$, since in each Δ_i

$$W - W' = \partial(\{(z : z - t \cdot \varepsilon_i \in W, 0 \leqslant t \leqslant \rho(\|z\|)\}).$$

Finally, since W' meets V transversely in $m_{p_i}(V \cdot W)$ points in Δ_i''—where W' and V are both analytic varieties with the natural orientation—and nowhere else,

$$^{\#}(W \cdot V) = {}^{\#}(W' \cdot V) = \sum m_{p_i}(V \cdot W)$$

as desired.

Summarizing,

The topological intersection number $^{\#}(V \cdot W)$ *of two analytic subvarieties of complementary dimension meeting in a finite set of points on a compact complex manifold is given by*

$$^{\#}(V \cdot W) = \sum_{p \in V \cap W} m_p(V \cdot W).$$

The intersection multiplicity $m_p(V \cdot W)$ *satisfies*

$$m_p(V \cdot W) \geqslant 1$$

with equality holding if and only if V *and* W *meet transversely at* p.

One important corollary of this is that if V and W meet in isolated points, their topological intersection number $^{\#}(V \cdot W)$ is greater than or equal to their set-theoretic intersection $^{\#}\{V \cap W\}$. Thus, for instance, if the intersection $V \cap W$ of two analytic varieties in M contains more than $^{\#}(V \cdot W)$ points, it follows that $V \cap W$ must contain a curve.

As a simple consequence of this assertion, note that

If M *is any complex submanifold of projective space* $\mathbb{P}^n$, $V \subset M$ *an analytic subvariety, then the fundamental class of* V *is nonzero in the homology of* M.

This is easy: if M has dimension m and V dimension k, we can find a linear subspace $\mathbb{P}^{n-k}$ of $\mathbb{P}^n$ meeting V in isolated points, and setting $W = M \cap \mathbb{P}^{n-k}$,

$$^{\#}(W \cdot V) > 0,$$

which implies $\eta_V \neq 0 \in H^{2n-2k}(M)$.

As a corollary, we see that

The even Betti numbers of M *are positive,*

since by the above the intersection V of M with a linear subspace $\mathbb{P}^{n-m+k}$ in $\mathbb{P}^n$ is an analytic subvariety of dimension k in M, and so represents a nonzero element of $H_{2k}(M)$.

Similarly,

Any analytic subvariety of $\mathbb{P}^n$ *homologous to a hyperplane is a hyperplane.*

To see this, we note that if V is homologous to a hyperplane it has intersection number 1 with a line. Then if p_1, p_2 are any two points of V, the line $L = \overline{p_1 p_2}$, having two points in common with V, must have a curve in common with V; that is, L must be contained in V. V thus contains the line joining any two of its points, and so is a linear subspace of $\mathbb{P}^n$.

From this it follows that

Any holomorphic automorphism of $\mathbb{P}^n$ *is induced by a linear transformation of* $\mathbb{C}^{n+1}$.

Let $X_0,\ldots,X_n$ be homogeneous coordinates on $\mathbb{P}^n$, $x_i = X_i/X_0$ the corresponding Euclidean coordinates on the complement of the hyperplane $H=(X_0=0)$. Since the fundamental class of a hyperplane in $\mathbb{P}^n$ generates $H^2(\mathbb{P}^n,\mathbb{Z})\cong\mathbb{Z}$, any holomorphic automorphism φ of $\mathbb{P}^n$ must take a hyperplane into a complex submanifold of $\mathbb{P}^n$ homologous to a hyperplane, hence to a hyperplane. Consequently, after composing φ with a linear transformation of $\mathbb{P}^n$, we may assume that $\varphi(H)=H$. Similarly, φ must carry the coordinate hyperplanes $H_i=(x_i=0)$ into hyperplanes other than H, and so we can write

$$\varphi(H_i) = (a_{1,i}x_1 + \cdots + a_{n,i}x_n + a_{0,i} = 0).$$

The pullback $\varphi^*(x_i)$ of the Euclidean coordinate x_i is then a meromorphic function on $\mathbb{P}^n$ with a simple pole along H and a zero along $\varphi(H_i)$; it follows that the function

$$\frac{\varphi^*(x_i)}{a_0 + a_1x_1 + \cdots + a_nx_n}$$

is holomorphic on all of $\mathbb{P}^n$, hence constant. Thus

$$\varphi^*(x_i) = a'_{0,i} + a'_{1,i}x_1 + \cdots + a'_{n,i}x_n,$$

and so φ is linear.

Note that the group of automorphisms of $\mathbb{P}^n$ is thus the quotient PGL_{n+1} of the general linear group GL_{n+1} by the one-dimensional subgroup of scalar matrices $\{\lambda I\}$.

A final remark: when two analytic subvarieties V and W of a complex manifold M—not necessarily of complementary dimension—intersect transversely, they likewise intersect positively in the sense that the variety $V\cap W$ is counted with the natural orientation in the topological intersection of V and W. More generally, if we define the *intersection multiplicity* $m_Z(V\cdot W)$ of V and W along an irreducible variety $Z\subset V\cap W$ to be the multiplicity

$$\mathrm{mult}_p((V\cap H)\cdot(W\cap H))_H,$$

where p is a generic smooth point of Z and H a submanifold in a neighborhood of p intersecting Z transversely at p, then the topological intersection of V and W is given by

$$(V\cdot W) = \sum_{Z_{\mathrm{irr}}\subset V\cap W} \mathrm{mult}_{Z_i}(V\cdot W)\cdot Z_i.$$

5. VECTOR BUNDLES, CONNECTIONS, AND CURVATURE

Complex and Holomorphic Vector Bundles

Let M be a differentiable manifold. A C^∞ *complex vector bundle* on M consists of a family $\{E_x\}_{x\in M}$ of complex vector spaces parametrized by M, together with a C^∞ manifold structure on $E=\cup_{x\in M}E_x$ such that

1. The projection map $\pi: E\to M$ taking E_x to x is C^∞, and
2. For every $x_0\in M$, there exists an open set U in M containing x_0 and a diffeomorphism

$$\varphi_U:\ \pi^{-1}(U)\longrightarrow U\times\mathbb{C}^k$$

taking the vector space E_x isomorphically onto $\{x\}\times\mathbb{C}^k$ for each $x\in U$; φ_U is called a *trivialization* of E over U.

The dimension of the *fibers* E_x of E is called the *rank* of E; in particular, a vector bundle of rank 1 is called a *line bundle.*

Note that for any pair of trivializations φ_U and φ_V the map

$$g_{UV}:\ U\cap V\to \mathrm{GL}_k$$

given by

$$g_{UV}(x)=\left(\varphi_U\circ\varphi_V^{-1}\right)|_{\{x\}\times\mathbb{C}^k}$$

is C^∞; the maps g_{UV} are called *transition functions* for E relative to the trivializations φ_U,φ_V. The transition functions of E necessarily satisfy the identities

$$g_{UV}(x)\cdot g_{VU}(x)=I \qquad \text{for all } x\in U\cap V$$

$$g_{UV}(x)\cdot g_{VW}(x)\cdot g_{WU}(x)=I \qquad \text{for all } x\in U\cap V\cap W.$$

Conversely, given an open cover $\underline{U}=\{U_\alpha\}$ of M and C^∞ maps $g_{\alpha\beta}: U_\alpha\cap U_\beta\to \mathrm{GL}_k$ satisfying these identities, there is a unique complex vector bundle $E\to M$ with transition functions $\{g_{\alpha\beta}\}$: it is not hard to check that E as a point set must be the union

$$\bigcup_\alpha U_\alpha\times\mathbb{C}^k$$

with points $(x,\lambda)\in U_\beta\times\mathbb{C}^k$ and $(x,g_{\alpha\beta}(x)\cdot\lambda)\in U_\alpha\times\mathbb{C}^k$ identified and with the manifold structure induced by the inclusions $U_\alpha\times\mathbb{C}^k\hookrightarrow E$.

As a general rule, operations on vector spaces induce operations on vector bundles. For example, if $E\to M$ is a complex vector bundle, we take the *dual bundle* $E^*\to M$ to be the complex vector bundle with fiber $E_x^*=(E_x)^*$; trivializations

$$\varphi_U:\ E_U\to U\times\mathbb{C}^k$$

(where $E_U = \pi^{-1}(U)$) then induce maps

$$\varphi_U^*: E_U^* \to U \times \mathbb{C}^{k^*} \cong U \times \mathbb{C}^k,$$

which give $E^* = \cup E_x^*$ the structure of a manifold. The construction is most easily expressed in terms of transition functions: if $E \to M$ has transition functions $\{g_{\alpha\beta}\}$, then $E^* \to M$ is just the complex vector bundle given by transition functions

$$j_{\alpha\beta}(x) = {}^t g_{\alpha\beta}(x)^{-1}.$$

Similarly, if $E \to M$, $F \to M$ are complex vector bundles of rank k and l with transition functions $\{g_{\alpha\beta}\}$ and $\{h_{\alpha\beta}\}$, respectively, then one can define bundles

1. $E \oplus F$, given by transition functions

$$j_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^k \oplus \mathbb{C}^l),$$

2. $E \otimes F$, given by transition functions

$$j_{\alpha\beta}(x) = g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) \in \mathrm{GL}(\mathbb{C}^k \otimes \mathbb{C}^l),$$

3. $\wedge^r E$, given by transition functions

$$j_{\alpha\beta}(x) = \wedge^r g_{\alpha\beta}(x) \in \mathrm{GL}(\wedge^r \mathbb{C}^k).$$

In particular, $\wedge^k E$ is a line bundle given by

$$j_{\alpha\beta}(x) = \det g_{\alpha\beta}(x) \in \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*,$$

called the *determinant* bundle of E.

A *subbundle* $F \subset E$ of a bundle E is a collection $\{F_x \subset E_x\}_{x \in M}$ of subspaces of the fibers E_x of E such that $F = \cup F_x \subset E$ is a submanifold of E; F is clearly a vector bundle itself. The condition that $F \subset E$ is a submanifold is equivalent to saying that for every $x \in M$, there exists a neighborhood U of x in M and a trivialization

$$\varphi_U: E_U \longrightarrow U \times \mathbb{C}^k$$

such that

$$\varphi_U|_{F_U}: F_U \longrightarrow U \times \mathbb{C}^l \subset U \times \mathbb{C}^k.$$

The transition functions g_{UV} of E relative to these trivializations will then look like

$$g_{UV}(x) = \left(\begin{array}{c|c} h_{UV}(x) & k_{UV}(x) \\ \hline 0 & j_{UV}(x) \end{array}\right).$$

The bundle F will have transition functions h_{UV}, and the maps j_{UV} are transition functions for the *quotient bundle* E/F given by $(E/F)_x = E_x/F_x$.

Given a C^∞ map $f: M \to N$ of differentiable manifolds M and N and a complex vector bundle $E \to N$, we can define the *pullback bundle* f^*E by setting

$$(f^*E)_x = E_{f(x)}.$$

If

$$\varphi: E_U \to U \times \mathbb{C}^n$$

is a trivialization of E in a neighborhood of $f(x)$, then the map

$$f^*\varphi: f^*E_{f^{-1}U} \to f^*U \times \mathbb{C}^n$$

gives f^*E its manifold structure over the open set $f^{-1}U$. Transition functions for the pullback f^*E will, of course, be the pullback of the transition functions for E.

A map between vector bundles E and F on M is given by a C^∞ map $f: E \to F$ such that $f(E_x) \subset F_x$ and $f_x = f|_{E_x}: E_x \to F_x$ is linear. Note that

$$Ker(f) = \cup\, Ker f_x \subset E$$

and

$$Im(f) = \cup\, Im f_x \subset F$$

are subbundles of E and F, respectively if and only if the maps f_x all have the same rank. Two bundles E and F on M are *isomorphic* if there exists a map $f: E \to F$ with $f_x: E_x \to F_x$ an isomorphism for all $x \in M$; a vector bundle on M is called *trivial* if it is isomorphic to the product bundle $M \times \mathbb{C}^k$.

Finally, a *section* σ of the vector bundle $E \xrightarrow{\pi} M$ over $U \subset M$ is a C^∞ map

$$\sigma: U \to E$$

such that $\sigma(x) \in E_x$ for all $x \in U$. A *frame* for E over $U \subset M$ is a collection $\sigma_1, \ldots, \sigma_k$ of sections of M over U such that $\{\sigma_1(x), \ldots, \sigma_k(x)\}$ is a basis for E_x for all $x \in U$. A frame for E over U is essentially the same thing as a trivialization of E over U: given

$$\varphi_U: E_U \to U \times \mathbb{C}^k$$

a trivialization, the sections

$$\sigma_i(x) = \varphi_U^{-1}(x, e_i)$$

form a frame, and conversely given $\sigma_1,\dots,\sigma_k$ a frame, we can define a trivialization φ_U by

$$\varphi_U(\lambda) = (x,(\lambda_1,\dots,\lambda_k))$$

for $\lambda=\Sigma\lambda_i\sigma_i(x)$ in E_x.

Note that given a trivialization φ_U of E over U, we can represent every section σ of E over U uniquely as a C^∞ vector-valued function $f=(f_1,\dots,f_k)$ by writing

$$\sigma(x) = \sum f_i(x)\cdot\varphi_U^{-1}(x,e_i);$$

if φ_V is a trivialization of E over V and $f'=(f_1',\dots,f_k')$ the corresponding representation of $\sigma|_{V\cap U}$, then

$$\sum f_i(x)\cdot\varphi_U^{-1}(x,e_i) = \sum f_i'(x)\cdot\varphi_V^{-1}(x,e_i)$$

so

$$\sum f_i(x)\cdot e_i = \sum f_i'(x)\cdot\varphi_U\varphi_V^{-1}(x,e_i)$$

i.e.

$$f = g_{UV}f'.$$

Thus, in terms of trivializations $\{\varphi_\alpha : E_{U_\alpha}\to U_\alpha\times\mathbb{C}^k\}$, sections of E over $\cup U_\alpha$ correspond exactly to collections $\{f_\alpha=(f_{\alpha_1},\dots,f_{\alpha_k})\}_\alpha$ of vector-valued C^∞ functions such that

$$f_\alpha = g_{\alpha\beta}\cdot f_\beta$$

for all α,β, where the $g_{\alpha\beta}$ are transition functions of E relative to $\{\varphi_\alpha\}$.

Now, let M be a complex manifold. A *holomorphic vector bundle* $E\xrightarrow{\pi}M$ is a complex vector bundle together with the structure of a complex manifold on E, such that for any $x\in M$ there exists $U\ni x$ in M and a trivialization

$$\varphi_U:\ E_U\to U\times\mathbb{C}^k$$

that is a biholomorphic map of complex manifolds. Such a trivialization is called a *holomorphic trivialization*. Note that if $\{\varphi_\alpha : E_{U_\alpha}\to U_\alpha\times\mathbb{C}^k\}$ are holomorphic trivializations, then the transition functions for E relative to $\{\varphi_\alpha\}$ are holomorphic maps, and that, conversely, given holomorphic maps $g_{\alpha\beta}: U_\alpha\cap U_\beta\to\mathrm{GL}_k$ satisfying the identities on p. 66, we can construct a holomorphic vector bundle $E\to M$ with transition functions $g_{\alpha\beta}$.

All the vector-bundle phenomena discussed so far carry over directly to the category of holomorphic vector bundles. We can define the dual bundle and direct, tensor, and alternating product bundles of holomorphic vector bundles to be holomorphic; likewise we observe that the pullback f^*E of a holomorphic vector bundle E under a holomorphic map $f: M\to N$ of complex manifolds has a natural holomorphic structure. A holomorphic map of holomorphic vector bundles E,F on M is a holomorphic map

$f: E \to F$ with $f: E_x \to F_x$ linear; a holomorphic subbundle of a holomorphic bundle E is a subbundle $F \subset E$ with F a complex submanifold of E, and the quotient bundle is again holomorphic. A section σ of the holomorphic bundle E over $U \subset M$ is said to be holomorphic if $\sigma: U \to E$ is a holomorphic map, a frame $\sigma = (\sigma_1, \ldots, \sigma_k)$ is called holomorphic if each σ_i is; and in terms of a holomorphic frame $\{\sigma_i\}$ any section

$$\sigma(x) = \sum f_i(x) \cdot \sigma_i(x)$$

is holomorphic if and only if the functions f_i are.

One important difference between C^∞ and holomorphic vector bundles is this: while there is no naturally defined exterior derivative d on the space of sections of a vector bundle, on a holomorphic vector bundle E the $\bar{\partial}$-operator

$$\bar{\partial}: A^{p,q}(E) \to A^{p,q+1}(E)$$

from E-valued (p,q)-forms to E-valued $(p,q+1)$-forms is well-defined: we take $\{e_1, \ldots, e_k\}$ any local holomorphic frame for E over U, write $\sigma \in A^{p,q}(E)$ as

$$\sigma = \sum \omega_i \otimes e_i, \qquad \omega \in A^{p,q}(U),$$

and set

$$\bar{\partial}\sigma = \sum \bar{\partial}\omega_i \otimes e_i.$$

If $\{e'_1, \ldots, e'_k\}$ is any other holomorphic frame for E over U, with

$$e_i = \sum g_{ij} e'_j.$$

then

$$\sigma = \sum g_{ij}\omega_i \otimes e'_j$$

and

$$\bar{\partial}\sigma = \sum \bar{\partial}(g_{ij}\omega_i) \otimes e'_j = \sum g_{ij} \cdot \bar{\partial}\omega_i \otimes e'_j = \sum \bar{\partial}\omega_i \otimes e_i,$$

so $\bar{\partial}\sigma$ does not depend on the frame.

Examples

Let M be a complex manifold, and let $T_x(M)$ be the complex tangent space to M at x (p. 16). For $x \in U \subset M$ and $\varphi_U: U \to \mathbb{C}^n$ a coordinate chart, we have maps

$$\varphi_{U_*}: T_x(M) \to T_{\varphi(x)}(U) \cong \mathbb{C}\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\} \cong \mathbb{C}^{2n}$$

for each $x \in U$, hence a map

$$\varphi_{U_*}: \bigcup_{x \in U} T_x(M) \to U \times \mathbb{C}^{2n}$$

giving $T(M)=\cup_{x\in M}T_x(M)$ the structure of a complex vector bundle, called the *complex tangent bundle*. Transition functions for $T(M)$ are given by

$$j_{U,V}=\mathcal{J}_{\mathbf{R}}(\varphi_U\varphi_V^{-1}).$$

Now for each $x\in M$

$$T_x(M)=T'_x(M)\oplus T''_x(M),$$

where $T'_x(M)$ and $T''_x(M)$ are as on p. 17. The subspaces $\{T'_x(M)\subset T_x(M)\}$ form a subbundle $T'(M)\subset T(M)$, called the *holomorphic tangent bundle*. Transition functions for $T'(M)$ are given by

$$j_{U,V}=\mathcal{J}_{\mathbf{C}}(\varphi_U\varphi_V^{-1}),$$

and so we see that $T'(M)$ has naturally the structure of a holomorphic vector bundle.

Similarly, we define:

$T^*(M)=T(M)^*$: the complex cotangent bundle,
$T^{*\prime}(M),T^{*\prime\prime}(M)$: the holomorphic and antiholomorphic cotangent bundles,
$T^{*(p,q)}(M)=\wedge^pT^{*\prime}(M)\otimes\wedge^qT^{*\prime\prime}(M)$.

The tensor, symmetric, and exterior products of the holomorphic and complexified tangent and cotangent bundles are called *tensor bundles*.

If $V\subset M$ is a complex submanifold, we define the *normal bundle* $N_{V/M}$ to V in M to be the quotient of the tangent bundle to M, restricted to V, by the subbundle

$$T'(V)\hookrightarrow T'(M)|_V.$$

The *conormal* bundle $N^*_{V/M}$ to V in M is the dual of the normal bundle.

Metrics, Connections, and Curvature

Let $E\to M$ be a complex vector bundle. A *hermitian metric* on E is a hermitian inner product on each fiber E_x of E, varying smoothly with $x\in M$—i.e., such that if $\zeta=\{\zeta_1,\ldots,\zeta_k\}$ is a frame for E, then the functions

$$h_{ij}(x)=(\zeta_i(x),\zeta_j(x))$$

are C^∞. A frame ζ for E is called *unitary* if $\zeta_1(x),\ldots,\zeta_k(x)$ is an orthonormal basis for E_x for each x; unitary frames always exist locally, since we can take any frame and apply the Gram-Schmidt process.

If E is a bundle with hermitian metric, $F\subset E$ a subbundle, then the subspaces $\{F_x^\perp\subset E_x\}$ form a subbundle of E, C^∞ isomorphic to the quotient bundle E/F.

A holomorphic vector bundle with a hermitian metric is called a *hermitian vector bundle.*

DEFINITION. A *connection* D on a complex vector bundle $E \to M$ is a map

$$D:\ \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying Leibnitz' rule

$$D(f\cdot\zeta) = df \otimes \zeta + f\cdot D(\zeta)$$

for all sections $\zeta \in \mathcal{A}^0(E)(U), f \in C^\infty(U)$.

A connection is essentially a way of differentiating sections: for $\xi \in \mathcal{A}^0(E)(U)$ the contraction of $D\xi$ with a tangent vector $\mathrm{v} \in T_x(M)$ may be thought of as the derivative of ξ in the direction v. It is, however, only a first-order approximation of differentiation, inasmuch as mixed partials will in general not be equal.

Let $e = e_1, \ldots, e_n$ be a frame for E over U. Given a connection D on E, we can decompose De_i into its components, writing

$$De_i = \sum \theta_{ij} e_j.$$

The matrix $\theta = (\theta_{ij})$ of 1-forms is called the *connection matrix* of D with respect to e. The data e and θ determine D: for a general section $\sigma \in \mathcal{A}^0(E)(U)$, writing

$$\sigma = \sum \sigma_i e_i,$$

we have

$$\begin{aligned} D\sigma &= \sum d\sigma_i \cdot e_i + \sum \sigma_i \cdot De_i \\ &= \sum_j \Big(d\sigma_j + \sum_i \sigma_i \theta_{ij} \Big) e_j. \end{aligned}$$

The connection matrix θ at a point $z_0 \in U$ depends on the choice of frame in a neighborhood of z_0: if $e' = e'_1, \ldots, e'_n$ is another frame with

$$e'_i(z) = \sum g_{ij}(z) e_j(z),$$

then

$$De'_i = \sum dg_{ij} \cdot e_j + \sum g_{ik} \theta_{kj} \cdot e_j,$$

so that

$$\theta_{e'} = dg \cdot g^{-1} + g \cdot \theta_e \cdot g^{-1} \qquad (g = (g_{ij})).$$

There is in general no "natural" connection on a vector bundle E. If M is complex and E hermitian, however, we can make two requirements that dictate a canonical choice of connection.

1. Using the decomposition $T^* = T^{*\prime} \oplus T^{*\prime\prime}$, we can write $D = D' + D''$, with $D' : \mathcal{Q}^0(E) \to \mathcal{Q}^{1,0}(E)$ and $D'' : \mathcal{Q}^0(E) \to \mathcal{Q}^{0,1}(E)$. Now we say that a connection D on E is *compatible with the complex structure* if $D'' = \bar{\partial}$.

2. If E is hermitian, D is said to be *compatible with the metric* if

$$d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta).$$

Lemma. *If* E *is a hermitian vector bundle, there is a unique connection* D *on* E *compatible with both the metric and the complex structure.*

Proof. Let $e = e_1, \ldots, e_n$ be a *holomorphic* frame for E, and let $h_{ij} = (e_i, e_j)$. If such a D exists, its matrix θ with respect to e must have type $(1,0)$, and consequently

$$\begin{aligned} dh_{ij} &= d(e_i, e_j) \\ &= \sum_k \theta_{ik} h_{kj} + \sum_k \bar{\theta}_{jk} h_{ik} \\ &= \text{type } (1,0) + \text{type } (0,1). \end{aligned}$$

Comparing types, we have

$$\partial h_{ij} = \sum \theta_{ik} h_{kj}, \qquad \text{i.e.,} \quad \partial h = \theta h,$$

$$\bar{\partial} h_{ij} = \sum \bar{\theta}_{jk} h_{ik}, \qquad \text{i.e.,} \quad \bar{\partial} h = h\,{}^t\bar{\theta},$$

and we see that $\theta = \partial h \cdot h^{-1}$ is the unique solution to both equations. Since θ is determined by the conditions of compatibility, θ is well-defined globally. Q.E.D.

The unique connection compatible with the complex and metric structures on E is called the associated, or *metric*, connection. As mentioned in the proof, its matrix with respect to a holomorphic frame is of type $(1,0)$; on the other hand if $e_1, \ldots, e_n$ is a unitary frame,

$$0 = d(e_i, e_j) = \theta_{ij} + \bar{\theta}_{ji},$$

so its matrix with respect to a unitary frame is skew-hermitian.

The metric connections of hermitian vector bundles behave well with respect to bundle operations, as we see in the next two lemmas.

Lemma. *Let* $E \to M$ *be a hermitian vector bundle and* $F \subset E$ *a holomorphic subbundle. Then* F *is itself a hermitian bundle with metric connection* D_F. *On the other hand, the metric connection* D_E *in* E *and direct-sum decomposition* $E = F \oplus F^\perp$ *induced by the metric give a connection* $\pi_F D_E$ *in* F, *and*

$$D_F = \pi_F \circ D_E,$$

where π_F *is the projection onto* F.

Proof. If ζ is a section of F, then $(\pi_F \circ D_E)''(\zeta) = \pi_F(D_E''\zeta) = \pi_F(\bar\partial\zeta) = \bar\partial\zeta$, so that $\pi_F \circ D_E$ is compatible with the complex structure. If ζ, ζ' are sections of F, then

$$\begin{aligned} d(\zeta,\zeta') &= (D_E\zeta,\zeta') + (\zeta, D_E\zeta') \\ &= (\pi_F \circ D_E\zeta,\zeta') + (\zeta,\pi_F \circ D_E\zeta'), \end{aligned}$$

so that $\pi_F \circ D_E$ is compatible with the metric. Q.E.D.

Similarly, if E, E' are hermitian vector bundles, there is a natural metric on $E \otimes E'$ given by

$$(\lambda\otimes\lambda', \delta\otimes\delta') = (\lambda,\delta)\cdot(\lambda',\delta')$$

for $\lambda,\delta \in E_x$, $\lambda',\delta' \in E'_x$. Let $D_E, D_{E'}, D_{E\otimes E'}$ denote the metric connections on E, E', and $E\otimes E'$, respectively, and let $D_E \otimes 1$ be the connection on $E\otimes E'$ given by

$$(D_E\otimes 1)(\zeta\otimes\xi) = D\zeta\otimes\xi;$$

define $1 \otimes D_{E'}$ analogously. Then we have

Lemma. $\mathrm{D}_{\mathrm{E}\otimes\mathrm{E}'} = \mathrm{D}_{\mathrm{E}}\otimes 1 + 1\otimes \mathrm{D}_{\mathrm{E}'}$.

Proof. Clearly $(D_E\otimes 1 + 1\otimes D_{E'})'' = \bar\partial$; thus we just have to check compatibility with the metric. Let ζ,ξ be sections of E, ζ',ξ' sections of E'. Then

$$\begin{aligned} d(\zeta\otimes\zeta',\xi\otimes\xi') &= (\zeta',\xi')((D_E\zeta,\xi) + (\zeta,D_E\xi)) + (\zeta,\xi)((D_{E'}\zeta',\xi') + (\zeta',D_{E'}\xi')) \\ &= ((D_E\otimes 1 + 1\otimes D_{E'})(\zeta\otimes\zeta',\xi\otimes\xi') \\ &\quad + (\zeta\otimes\zeta',(D_E\otimes 1 + 1\otimes D_{E'})(\xi\otimes\xi')). \end{aligned}$$

Q.E.D.

Finally, note that a hermitian metric on the holomorphic bundle E induces a metric on E^*—if e is a unitary frame for E, e^* the dual frame for E^*, set

$$(e_i^*, e_j^*) = \delta_{ij}$$

—and the metric connection D^* on E^* can be defined by the requirement

$$d\langle\sigma,\tau\rangle = \langle D\sigma,\tau\rangle + \langle\sigma, D^*\tau\rangle$$

for $\sigma\in \mathcal{A}^0(E)(U)$, $\tau \in \mathcal{A}^0(E^*)(U)$.

Now, returning to the general discussion, given a connection D on a complex vector bundle $E \to M$ we can define operators

$$D\colon\ \mathcal{A}^p(E) \to \mathcal{A}^{p+1}(E)$$

by forcing Leibnitz' rule

$$D(\psi\wedge\xi) = d\psi\otimes\xi + (-1)^p\psi\wedge D\xi$$

for $\psi \in \mathcal{A}^p(U)$, $\xi\in\mathcal{A}^0(E)(U)$. In particular we can discuss the operator

$$D^2\colon\ \mathcal{A}^0(E) \to \mathcal{A}^2(E).$$

The first fact about D^2 is that it is linear over $\mathcal{Q}^0$, i.e., for σ a section of E and f a C^∞ function,

$$\begin{aligned} D^2(f\cdot\sigma) &= D(df\otimes\sigma+f\cdot D\sigma) \\ &= -df\wedge D\sigma + df\wedge D\sigma + f\cdot D^2\sigma \\ &= f\cdot D^2\sigma. \end{aligned}$$

Consequently the map $D^2:\mathcal{Q}^0(E)\to\mathcal{Q}^2(E)$ is induced by a bundle map $E\to\wedge^2T^*\otimes E$, or in other words, D^2 corresponds to a global section Θ of the bundle

$$\wedge^2T^*\otimes\operatorname{Hom}(E,E) = \wedge^2T^*\otimes(E^*\otimes E).$$

If e is a frame for E, then in terms of the frame $\{e_i^*\otimes e_j\}$ for $E^*\otimes E$, we can represent $\Theta\in A^2(E^*\otimes E)$ by a matrix Θ_e of 2-forms—i.e., we can write

$$D^2e_i = \sum\Theta_{ij}\otimes e_j;$$

Θ_e is called the *curvature matrix* of D in terms of the frame e. If $\{e_i'=\sum g_{ij}e_j\}$ is another frame, then

$$\begin{aligned} D^2e_i' &= D^2\Big(\sum g_{ij}e_j\Big) \\ &= \sum g_{ij}\Theta_{jk}e_k \\ &= \sum g_{ij}\Theta_{jk}g_{kl}^{-1}e_l', \end{aligned}$$

that is,

$$\Theta_{e'} = g\cdot\Theta_e\cdot g^{-1}.$$

The curvature matrix is readily expressed in terms of the connection matrix: by definition

$$\begin{aligned} D^2e_i &= D\Big(\sum\theta_{ij}\otimes e_j\Big) \\ &= \sum\Big(d\theta_{ij}-\sum\theta_{ik}\wedge\theta_{kj}\Big)\otimes e_j. \end{aligned}$$

In matrix notation, therefore,

$$\Theta_e = d\theta_e - \theta_e\wedge\theta_e.$$

This is called the *Cartan structure equation.*

We can say more about Θ in the holomorphic case. If $E\to M$ is hermitian and the connection D on E is compatible with the complex structure, then $D''=\bar\partial$ implies $D''^2=0$ and hence $\Theta^{0,2}=0$. If, moreover, D is compatible with the metric, then in terms of a unitary frame e, the connection matrix θ_e is skew-hermitian and hence so is $\Theta=d\theta-\theta\wedge\theta$; thus $\Theta^{2,0}=-{}^t\overline{\Theta}^{0,2}=0$. Since the type of Θ is clearly invariant under change of frame, we see that *the curvature matrix of the metric connection on a hermitian bundle is a hermitian matrix of (1,1)-forms.*

To close this section, we give computations of the metric connection and curvature matrices of hermitian bundles in two special cases.

First, recall that for E a hermitian bundle with metric connection D, the metric connection D^* on E^* satisfies

$$d\langle\sigma,\tau\rangle = \langle D\sigma,\tau\rangle + \langle\sigma, D^*\tau\rangle$$

for all $\sigma\in\mathcal{A}^0(E)(U)$, $\tau\in\mathcal{A}^0(E^*)(U)$. In particular, if e is a frame for E and e^* the dual frame in E^*, θ and θ^* the corresponding connection matrices, we have

$$0 = d\langle e_i, e_j^*\rangle = \theta_{ij} + \theta_{ji}^*,$$

so that $\theta = -{}^t\theta^*$.

In view of this, a special situation holds when we consider the metric connection on the holomorphic tangent bundle of a hermitian manifold: we can compare the dual connection D^* on the holomorphic cotangent bundle with the ordinary exterior derivative. Thus

$$D^*: A^{1,0}\to A^{1,0}\otimes A^1 = (A^{1,0}\otimes A^{1,0})\oplus(A^{1,0}\otimes A^{0,1})$$
$$d: A^{1,0}\to A^{2,0}\oplus(A^{1,0}\otimes A^{0,1}).$$

Since D^* is compatible with the complex structure, we have $D^{*\prime\prime}=\bar\partial$; i.e., the two operators agree in the factor $A^{1,0}\otimes A^{0,1}$. As will now be seen, this gives us an effective means of computing the connection matrix of D. Let $ds^2=\sum h_{ij}\,dz_i\otimes d\bar z_j=\sum\varphi_i\otimes\bar\varphi_i$ be a hermitian metric on M.

Lemma. *There exists a unique matrix ψ_{ij} of 1-forms such that $\psi+{}^t\bar\psi=0$ and*

$$(*)\qquad d\varphi_i = \sum_j \psi_{ij}\wedge\varphi_j + \tau_i,$$

where τ_i is of type (2,0).

Proof. Write $\psi=\psi'+\psi''$ for the type decomposition of ψ. Then

$$\bar\partial\varphi_i = \sum \psi_{ij}''\wedge\varphi_j$$

determines ψ'', and $\psi+{}^t\bar\psi=0\Rightarrow\psi'=-{}^t\bar\psi''$. (Explicitly: if we write $\varphi_i=\sum a_{ij}\,dz_j$, where $a\,{}^t\bar a=h$, we have

$$\bar\partial\varphi_i = \sum_k \bar\partial a_{ik}\wedge dz_k$$
$$= \sum_{j,k}\bar\partial a_{ik}\wedge a_{kj}^{-1}\cdot\varphi_j,$$

so $\psi''=\bar\partial a a^{-1}$.) Q.E.D.

Let $v=v_1,\ldots,v_n$ be the frame for the tangent bundle $T'(M)$ dual to the frame $\varphi_1,\ldots,\varphi_n$; let θ be the connection matrix of D with respect to the frame v and θ^* the matrix for D^* in the frame $\varphi_1,\dots,\varphi_n$. Then

$$D^{*\prime\prime} = \bar\partial \Rightarrow \theta^{*\prime\prime} = \psi''$$
$$\Rightarrow \theta^* = \psi$$

since $\theta^* + {}^t\bar\theta^* = 0$ and $\psi + {}^t\bar\psi = 0$. Thus we have

$$\theta = -{}^t\theta^* = -{}^t\psi.$$

In summary, using the basic *structure equation* (*) we may determine the connection matrix $\theta = -{}^t\psi$ in the holomorphic tangent bundle $T'(M)$ by knowing the exterior derivatives $d\varphi_i$ of a unitary coframe. The vector $\tau = (\tau_1,\dots,\tau_n)$ is called the *torsion*; a metric is called *Kähler* if its torsion vanishes. Later on we shall give alternate definitions of the Kähler condition.

Examples

Let M be a Riemann surface with local coordinate z; a metric on M is given by

$$ds^2 = h^2 dz \otimes d\bar z = \varphi \otimes \bar\varphi,$$

where $\varphi = h\,dz$. Then

$$d\varphi = \bar\partial h \wedge dz = \frac{\bar\partial h}{h} \wedge \varphi,$$

so $\psi'' = \bar\partial \log h$ and $\psi = (\bar\partial - \partial)\log h$; by the structure equation the matrix for the metric connection on the tangent bundle is given by

$$\begin{aligned}\theta &= -\psi = (\partial - \bar\partial)\log h\\ &= \frac{\partial}{\partial z}\log h \cdot dz - \frac{\partial}{\partial \bar z}\log h \cdot d\bar z.\end{aligned}$$

Now $\theta \wedge \theta = 0$, so by the Cartan structure equation

$$\begin{aligned}\Theta = d\theta &= -2\left(\frac{\partial^2}{\partial z\,\partial \bar z}\log h\right) dz \wedge d\bar z\\ &= -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log h \cdot dz \wedge d\bar z\\ &= -\tfrac{1}{2}\Delta \log h \cdot dz \wedge d\bar z.\end{aligned}$$

Comparing the curvature "matrix" Θ with the associated (1,1)-form $\Phi = (\sqrt{-1}/2)\varphi \wedge \bar\varphi = (\sqrt{-1}/2)h^2 dz \wedge d\bar z$, we obtain

$$\sqrt{-1}\,\Theta = K \cdot \Phi,$$

where $K = (-\Delta \log h)/h^2$ is the usual *Gaussian curvature*.

Our second computation involves the curvature operator of sub- and quotient bundles of a hermitian bundle. While we cannot make a complete computation, we will run across a fundamental distinction between the C^∞ and holomorphic cases: the presence of a sign in the curvature of a hermitian bundle.

Let $E \to M$ be a hermitian bundle, $S \subset E$ a holomorphic subbundle, and $Q = E/S$ the quotient bundle. As mentioned earlier, Q is isomorphic, as a C^∞ vector bundle, to the orthogonal complement $S^\perp$ of S in E, and so both S and Q inherit hermitian structures from E; let D_E, D_S, and D_Q denote the corresponding metric connections. By the lemma on p. 73, D_S is equal to the composition of the operator

$$D_E|_{\mathcal{A}^0(S)}: \mathcal{A}^0(S) \to \mathcal{A}^1(E)$$

with the projection $\mathcal{A}^1(E) \to \mathcal{A}^1(S)$; thus the operator

$$A = D_E|_{\mathcal{A}^0(S)} - D_S$$

maps $\mathcal{A}^0(S)$ to $\mathcal{A}^1(Q)$. A is called the *second fundamental form* of S in E; clearly, it is of type $(1,0)$ and linear over C^∞ functions, i.e.,

$$A \in \mathcal{A}^{1,0}(\operatorname{Hom}(S,Q)).$$

To compute curvatures, we choose a unitary frame $e_1, \ldots, e_r$ for E such that $e_1, \ldots, e_s$ is a frame for S. Using this frame and our lemma, the connection matrix for E is

$$\theta_E = \begin{pmatrix} \theta_S & {}^t\bar{A} \\ A & \theta_Q \end{pmatrix},$$

where θ_S, θ_Q are the respective connection matrices for S and Q. Then

$$\begin{aligned} \Theta_E &= d\theta_E - \theta_E \wedge \theta_E \\ &= \begin{bmatrix} d\theta_S - \theta_S \wedge \theta_S - {}^t\bar{A} \wedge A & * \\ * & d\theta_Q - \theta_Q \wedge \theta_Q - A \wedge {}^t\bar{A} \end{bmatrix}, \end{aligned}$$

which implies that

$$\begin{aligned} \Theta_S &= \Theta_E|_S + {}^t\bar{A} \wedge A, \\ \Theta_Q &= \Theta_E|_Q + A \wedge {}^t\bar{A}. \end{aligned}$$

Now, we say that a curvature operator

$$\Theta \in A^2(\operatorname{Hom}(E,E))$$

is *positive* at $x \in M$ if for $\lambda \neq 0 \in E_x$, the multivector

$$(\lambda, \Theta\lambda) \in \Lambda^2 T_x^*(M)$$

is positive of type (1,1), or equivalently if for any holomorphic tangent vector $v \in T'_x(M)$, the hermitian matrix

$$-\sqrt{-1}\,\langle \Theta(x); v, \bar{v}\rangle \in \mathrm{Hom}(E_x, E_x)$$

is positive definite. We write $\Theta > 0$ if Θ is positive everywhere, $\Theta \geqslant 0$ if Θ is positive semidefinite, and $\Theta > \Theta'$ if $\Theta - \Theta' > 0$.

Let A be the second fundamental form of the subbundle $S \subset E$ above, and write

$$A = \sum_{\substack{1 \leqslant j \leqslant s \\ s < \lambda \leqslant r}} a^{\alpha}_{\lambda j}\, dz_\alpha \otimes e_\lambda \otimes e_j^*,$$

so

$${}^t\bar{A} = \sum \bar{a}^{\alpha}_{\lambda j}\, d\bar{z}_\alpha \otimes e_\lambda^* \otimes e_j$$

and

$$A \wedge {}^t\bar{A} = \sum a^{\alpha}_{ik} \bar{a}^{\beta}_{\mu k}\, dz_\alpha \wedge d\bar{z}_\beta \otimes e_i \otimes e_j^*.$$

Thus, if we let $A^\alpha = (a^\alpha_{ij})$,

$$\left\langle A \wedge {}^t\bar{A}; \frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\alpha} \right\rangle = \sum a^{\alpha}_{ik} \bar{a}^{\alpha}_{jk} e_i \otimes e_j^* = A^{\alpha\,t}\bar{A}^{\alpha} \geqslant 0,$$

which implies that

$$\Theta_S \leqslant \Theta_E|_S,$$

$$\Theta_Q \geqslant \Theta_E|_Q,$$

with equality holding if and only if $A \equiv 0$. The principle that *curvature decreases in holomorphic subbundles and increases in holomorphic quotient bundles* is in marked contrast to the real case.

For example, if $M \subset \mathbb{C}^n$ is a complex submanifold with the metric induced from the Euclidian metric an $\mathbb{C}^n$, we see that

$$T'(M) \subset T'(\mathbb{C}^n)|_M \Rightarrow \Theta_M \leqslant \Theta_{\mathbb{C}^n}|_M = 0.$$

If M is a Riemann surface, then by the calculations on p. 77, this just means that its Gaussian curvature $K \leqslant 0$.

Another basic fact that comes out of this calculation is the following: suppose $E \to M$ is a holomorphic bundle and that there exist global holomorphic sections $\sigma_1, \ldots, \sigma_n \in \Gamma(M, E)$ such that, for all $x \in M$, $\{\sigma_1(x), \ldots, \sigma_n(x)\}$ generate E_x. Then we have a surjective holomorphic bundle map

$$M \times \mathbb{C}^n \to E \to 0$$

given by

$$(x, \lambda) \to \sum \lambda_i \sigma_i(x) \in E_x$$

for $x \in M$, $\lambda \in \mathbb{C}^n$. It follows that, if we give E the metric induced from the Euclidean metric on $M \times \mathbb{C}^n$,

$$\Theta_E \geqslant 0;$$

i.e., *any holomorphic bundle with a finite number of global sections that generate each fiber has a metric with nonnegative curvature.*

The connection between the sign of the curvature of a vector bundle and the existence of global sections is fundamental in the theory of complex manifolds.

6. HARMONIC THEORY ON COMPACT COMPLEX MANIFOLDS

The Hodge Theorem

This section is devoted to the statement and proof of the Hodge theorem for the $\bar\partial$-operator together with some of its immediate corollaries.

M will be a connected, compact complex manifold of complex dimension n. We choose a hermitian metric ds^2 with associated (1,1)-form

$$\omega = \frac{\sqrt{-1}}{2} \sum_j \varphi_j \wedge \bar\varphi_j$$

in terms of a unitary coframe $\{\varphi_1, \ldots, \varphi_n\}$. The metric ds^2 induces a hermitian metric on all tensor bundles $T^{*(p,q)}(M)$; the inner product in $T_z^{*(p,q)}(M)$ is given by taking the basis $\{\varphi_I(z) \wedge \bar\varphi_J(z)\}_{\# I = p, \# J = q}$ to be orthogonal and of length $\|\varphi_I \wedge \bar\varphi_J\|^2 = 2^{p+q}$ (recall that $\|dz_i\|^2 = 2$ on $\mathbb{C}^n$). Let $C_n = (-1)^{n(n-1)/2}(\sqrt{-1}/2)^n$ and

$$\Phi = \frac{\omega^n}{n!} = C_n \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar\varphi_1 \wedge \cdots \wedge \bar\varphi_n$$

be the volume form on M associated to the metric. The global inner product

$$(\psi, \eta) = \int_M (\psi(z), \eta(z)) \Phi(z)$$

makes the space $A^{p,q}(M)$ into a pre-Hilbert space. We pose the question: *Given a $\bar\partial$-closed form $\psi \in Z^{p,q}_{\bar\partial}(M)$, among all the forms $\{\psi + \bar\partial\eta\}$ representing the Dolbeault cohomology class $[\psi] \in H^{p,q}_{\bar\partial}(M)$ of φ, can we find one of smallest norm*? To answer this we pretend for a moment that $A^{p,q}(M)$ is complete and $\bar\partial$ is bounded, and define the adjoint operator

$$\bar\partial^*: A^{p,q}(M) \to A^{p,q-1}(M)$$

by requiring that

$$(\bar\partial^* \psi, \eta) = (\psi, \bar\partial \eta)$$

for all $\eta \in A^{p,q-1}(M)$. This will be justified in a moment, but first we show

Lemma. *A $\bar{\partial}$-closed form $\psi \in Z_{\bar{\partial}}^{p,q}(M)$ is of minimal norm in $\psi + \bar{\partial}A^{p,q-1}(M)$ if and only if $\bar{\partial}^*\psi = 0$.*

Proof. If $\bar{\partial}^*\psi = 0$, then for any $\eta \in A^{p,q-1}(M)$ with $\bar{\partial}\eta = 0$

$$\begin{aligned}\|\psi + \bar{\partial}\eta\|^2 &= (\psi + \bar{\partial}\eta, \psi + \bar{\partial}\eta)\\ &= \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\operatorname{Re}(\psi, \bar{\partial}\eta)\\ &= \|\psi\|^2 + \|\bar{\partial}\eta\|^2 + 2\operatorname{Re}(\bar{\partial}^*\psi, \eta)\\ &= \|\psi\|^2 + \|\bar{\partial}\eta\|^2\\ &> \|\psi\|^2,\end{aligned}$$

so ψ has minimal norm. Conversely, if ψ is of smallest norm, then for any $\eta \in A^{p,q-1}(M)$

$$\frac{\partial}{\partial t}\|\psi + t\bar{\partial}\eta\|^2(0) = 0.$$

But at $t = 0$

$$\frac{\partial}{\partial t}(\psi + t\bar{\partial}\eta, \psi + t\bar{\partial}\eta) = 2\operatorname{Re}(\psi, \bar{\partial}\eta)$$

and

$$\frac{\partial}{\partial t}(\psi + t\bar{\partial}(i\eta), \psi + t\bar{\partial}(i\eta)) = 2\operatorname{Im}(\psi, \bar{\partial}\eta).$$

So

$$(\bar{\partial}^*\psi, \eta) = (\psi, \bar{\partial}\eta) = 0$$

for all $\eta \in A^{p,q-1}(M)$, and hence $\bar{\partial}^*\psi = 0$. Q.E.D.

So, at least formally, the Dolbeault cohomology group $H_{\bar{\partial}}^{p,q}(M) = Z_{\bar{\partial}}^{p,q}(M)/\bar{\partial}A^{p,q-1}(M)$ is represented exactly by the solutions of the two first-order equations

$$\bar{\partial}\psi = 0, \qquad \bar{\partial}^*\psi = 0.$$

These two may be replaced by the single second-order equation

$$\Delta_{\bar{\partial}}\psi = (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\psi = 0:$$

clearly $\bar{\partial}\psi = 0 = \bar{\partial}^*\psi \Rightarrow \Delta\psi = 0$, and conversely

$$\begin{aligned}(\Delta_{\bar{\partial}}\psi, \psi) &= (\bar{\partial}\bar{\partial}^*\psi, \psi) + (\bar{\partial}^*\bar{\partial}\psi, \psi)\\ &= \|\bar{\partial}^*\psi\|^2 + \|\bar{\partial}\psi\|^2,\end{aligned}$$

so $\Delta_{\bar{\partial}}\psi = 0 \Rightarrow \bar{\partial}\psi = \bar{\partial}^*\psi = 0$. The operator

$$\Delta_{\bar{\partial}} : A^{p,q}(M) \longrightarrow A^{p,q}(M)$$

is called the $\bar{\partial}$-*Laplacian*, or simply the *Laplacian* (written Δ) if no ambiguity is likely. Differential forms satisfying the *Laplace equation*

$$\Delta\psi = 0$$

are called *harmonic forms*; the space of harmonic forms of type (p,q) is denoted $\mathcal{H}^{p,q}(M)$ and called the *harmonic space*. What the above formal argument suggests is the isomorphism

$$(*) \qquad \mathcal{H}^{p,q}(M) \cong H^{p,q}_{\bar{\partial}}(M);$$

if this can be proved, then we will have a unique representative for each cohomology class, which should certainly be an advantage. The isomorphism $(*)$ is part of the Hodge theorem, whose proof together with the corollaries of $(*)$ will occupy this section.

We begin by giving an explicit formula for the adjoint $\bar{\partial}^*$, thereby proving its existence. First we define the *star*, or *duality operator*,

$$*: A^{p,q}(M) \to A^{n-p,n-q}(M)$$

by requiring

$$(\psi(z),\eta(z))\Phi(z) = \psi(z) \wedge *\eta(z)$$

for all $\psi \in A^{p,q}(M)$. This is an algebraic operator, which is given locally as follows: if we write

$$\eta = \sum_{I,J} \eta_{I\bar{J}} \varphi_I \wedge \bar{\varphi}_J$$

then

$$*\eta = 2^{p+q-n} \sum_{I,J} \varepsilon_{IJ} \bar{\eta}_{I\bar{J}} \varphi_{I^0} \wedge \bar{\varphi}_{J^0},$$

where $I^0 = \{1,\ldots,n\} - I$ and we write $\varepsilon_{I,J}$ for the sign of the permutation

$$(1,\ldots,n,1',\ldots,n') \to \left(i_1,\ldots,i_p,j_1,\ldots,j_q,i_1^0,\ldots, i_{n-p}^0,j_1^0,\ldots,j_{n-q}^0\right).$$

The signs work out so that

$$**\eta = (-1)^{p+q}\eta.$$

In terms of star, the adjoint operator is

$$\bar{\partial}^* = -*\bar{\partial}*.$$

Indeed, we have, for $\psi \in A^{p,q-1}(M)$ and $\eta \in A^{p,q}(M)$

$$\begin{aligned}(\bar{\partial}\psi,\eta) &= \int_M \bar{\partial}\psi \wedge *\eta \\ &= (-1)^{p+q}\int_M \psi \wedge \bar{\partial}*\eta + \int_M \bar{\partial}(\psi \wedge *\eta).\end{aligned}$$

Since $\bar\partial = d$ on forms of type $(n, n-1)$, the second term on the right is

$$\int_M d(\psi \wedge *\eta) = 0$$

by Stokes' theorem. Thus, for all ψ,

$$(\bar\partial\psi, \eta) = -\int_M \psi \wedge *(*\bar\partial * \eta)$$

so that $\bar\partial^*$ is defined by the above formula. Note that $\bar\partial^2 = 0 \Rightarrow \bar\partial^{*2} = 0$.

We now digress for a moment to explain the origins of the terminology Laplacian and harmonic. Provided we work with compactly supported forms, the above definitions are valid for any complex manifold. It is reasonable to expect the case of $\mathbb{C}^n$ with the Euclidean metric to provide a good local approximation to what is going on. Suppose we take $p = q = 0$ and write $dz = dz_1 \wedge \cdots \wedge dz_n$. Then, for $f \in C_c^\infty(\mathbb{C}^n)$,

$$\begin{aligned}
\Delta(f) &= \bar\partial^* \bar\partial f \\
&= \bar\partial^* \left(\sum_j \frac{\partial f}{\partial \bar z_j} d\bar z_j \right) \\
&= *\bar\partial \left(2^{1-n} \sum \pm \overline{\frac{\partial f}{\partial \bar z_j}}\, dz \wedge d\bar z_j{}^0 \right) \\
&= * \left(2^{1-n} \sum_j - \frac{\partial}{\partial \bar z_j} \left(\overline{\frac{\partial f}{\partial \bar z_j}} \right) \right) dz \wedge d\bar z \\
&= v * \left(2^{1-n} \sum_j - \frac{\partial^2 f}{\partial z_j \partial \bar z_j} \right) dz \wedge d\bar z \\
&= \left(2 \sum_j - \frac{\partial^2 f}{\partial z_j \partial \bar z_j} \right).
\end{aligned}$$

Since

$$2 \frac{\partial^2}{\partial z_j \partial \bar z_j} = \frac{1}{2} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right),$$

we find that, up to a constant, $\Delta(f)$ is the usual Laplacian on functions in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Later on, in the discussion of Kähler manifolds, this computation will be extended to show that

$$\Delta(f\, dz_I \wedge d\bar z_J) = \left(-2 \sum_j \frac{\partial^2 f}{\partial z_j \partial \bar z_j} \right) dz_I \wedge d\bar z_J,$$

which explains the terminology for compactly supported forms in $\mathbb{C}^n$.

Returning to our compact, complex manifold M, we are aiming for the famous

Hodge Theorem

1. $\dim \mathcal{H}^{p,q}(M) < \infty$; *and*
2. *because of this, the orthogonal projection*

$$\mathcal{H}: A^{p,q}(M) \to \mathcal{H}^{p,q}(M)$$

is well-defined, and there exists a unique operator, the Green's operator,

$$G: A^{p,q}(M) \longrightarrow A^{p,q}(M),$$

with $G(\mathcal{H}^{p,q}(M))=0$, $\bar{\partial}G=G\bar{\partial}$, $\bar{\partial}^*G=G\bar{\partial}^*$ *and*

$$(**) \qquad I = \mathcal{H} + \Delta G$$

on $A^{p,q}(M)$.

The equation $(**)$ in the form

$$\psi = \mathcal{H}(\psi) + \bar{\partial}(\bar{\partial}^* G\psi) + \bar{\partial}^*(\bar{\partial}G\psi)$$

is called the *Hodge decomposition on forms*, since it directly implies the orthogonal direct-sum decomposition

$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \bar{\partial}A^{p,q-1}(M) \oplus \bar{\partial}^*A^{p,q+1}(M).$$

The content of $(**)$ is sometimes expressed by saying that, given η, the equation

$$\Delta\psi = \eta$$

has a solution ψ if and only if $\mathcal{H}(\eta)=0$, and then

$$\psi = G(\eta)$$

is the unique solution satisfying $\mathcal{H}(\psi)=0$. So, in effect what we shall be doing is trying to solve the Laplace equation on a compact manifold. The idea is to first solve this equation in the *weak sense*—i.e., in the Hilbert-space completion $\mathcal{L}^{p,q}(M)$ of $A^{p,q}(M)$ to find a ψ such that

$$(\psi, \Delta\varphi) = (\eta, \varphi)$$

for all $\varphi \in A^{p,q}(M)$—and then to prove that this ψ is in fact C^∞. The first step is pretty much formal Hilbert-space theory, and the second—usually called the *regularity theorem*—is at least a local problem, since φ may be written as a sum of forms with compact support in coordinate patches.

Proof of the Hodge Theorem I: Local Theory

The proof of the Hodge theorem given here uses elementary Hilbert-space techniques. We are looking for the element of smallest norm in the affine subspace $\psi+\bar{\partial}A^{p,q-1}(M)\subset A^{p,q}(M)$. Clearly such an element can be found in the closure of $\psi+\bar{\partial}A^{p,q-1}(M)$ in the completion $\mathcal{L}^{p,q}(M)$ of the pre-Hilbert space $A^{p,q}(M)$, simply by orthogonal projection. The problem then is to show that the element found in this way in fact lies in $A^{p,q}(M)$. We start by discussing functions on the torus. This will provide a model for the formalism underlying the basic estimates; also, by rendering transparent the behavior of the Euclidean Laplacian on the torus, we will gain some idea of what to expect in general.

Let T be the real torus $(\mathbb{R}/(2\pi\mathbb{Z}))^n$ with coordinates $x=(x_1,\dots,x_n)$. Denote by $\mathcal{F}$ the space of formal Fourier series

$$u = \sum_{\xi\in\mathbf{Z}^n} u_\xi e^{i\langle\xi,x\rangle}.$$

The *Sobolev* s-*norm is given by*

$$\|u\|_s^2 = \sum_\xi (1+\|\xi\|^2)^s |u_\xi|^2,$$

and we define the *Sobolev spaces* H_s by

$$H_s = \{u\in\mathcal{F}:\ \|u\|_s<\infty\}.$$

These are Hilbert spaces; we have clearly a sequence of inclusions

$$\supset H_{-n}\supset H_{-n+1}\supset\cdots\supset H_{-1}\supset H_0\supset H_1\supset\cdots\supset H_n\supset\cdots,$$

and we let

$$H_\infty = \cap H_s,\qquad H_{-\infty} = \cup H_s.$$

Now let $C^s(T)$ be the functions of class s on T. A function $\varphi\in C^0(T)$ has a Fourier expansion $\sum\varphi_\xi e^{i\langle\xi,x\rangle}$, where

$$\varphi_\xi = \int_T \varphi(x)e^{-i\langle\xi,x\rangle}dx \qquad \left(dx=\frac{dx_1\wedge\cdots\wedge dx_n}{(2\pi)^n}\right).$$

We have *Parseval's identity*

$$\begin{aligned}\int_T|\varphi|^2 &= \int_T\Big(\sum\varphi_\xi e^{i\langle\xi,e\rangle}\Big)\Big(\sum\bar{\varphi}_{\xi'}e^{-i\langle\xi',x\rangle}\Big)\\ &= \int_T \varphi_\xi\bar{\varphi}_{\xi'}e^{i\langle\xi-\xi',x\rangle}dx\\ &= \int_T\sum_\xi|\varphi_\xi|^2dx\\ &= \sum_\xi|\varphi_\xi|^2,\end{aligned}$$

so that $C^0(T)$ maps into H_0 injectively with $\|\quad\|_0$ as L^2-norm on $C^0(T)$. The justification of this interchange of limits is done by using partial sums and the Cauchy-Schwarz inequality.

We set $D_j = (1/\sqrt{-1})(\partial/\partial x_j)$ and use the standard multiindex notations

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad \alpha = (\alpha_1, \ldots, \alpha_n),$$
$$[\alpha] = \alpha_1 + \cdots + \alpha_n,$$
$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

By integration by parts

$$\int_T D^\alpha \varphi \cdot \bar{\psi} = \int_T \varphi\, \overline{D^\alpha \psi}, \qquad \varphi, \psi \in C^\infty(T),$$

and so for $\varphi \in C^s(T)$ and $[\alpha] \leqslant s$

$$\begin{aligned}(D^\alpha \varphi)_\xi &= \int_T D^\alpha \varphi e^{-i\langle \xi, x\rangle} dx \\ &= \int_T \varphi \xi^\alpha e^{-i\langle \xi, x\rangle} dx \\ &= \xi^\alpha \varphi_\xi,\end{aligned}$$

i.e.,

$$\|D^\alpha \varphi\|_0^2 = \sum_\xi |\xi^\alpha|^2 |\varphi_\xi|^2.$$

Thus there is an inclusion

$$C^s(T) \subset H_s,$$

and from

$$\sum_{[\alpha] \leqslant s} |\xi^{2\alpha}| \leqslant (1 + \|\xi\|^2)^s \leqslant C_s \sum_{[\alpha] \leqslant s} |\xi^{2\alpha}|$$

we see that on $C^s(T) \subset H_s$ the Sobolev norm $\| \quad \|_s$ is equivalent to

$$\sum_{[\alpha] \leqslant s} \|D^\alpha \varphi\|_0^2,$$

which we may describe as *the* L^2*-norm of the function* φ *together with its derivatives up to order* s. Indeed, H_s is the completion of $C^\infty(T)$ in this norm.

There is a partial converse to this, the important

Sobolev Lemma. $H_{s+[n/2]+1} \subset C^s(T)$; *that is, every* $u \in H_{s+[n/2]+1}$ *is the Fourier series of a function* $\varphi \in C^s(T)$, *and this series converges uniformly to* φ.

Proof. First, consider the case $s = 0$; let

$$u = \sum_\xi u_\xi e^{i\langle \xi, x\rangle}$$

with

$$\sum_\xi (1 + \|\xi\|^2)^{[n/2]+1} |u_\xi|^2 < \infty.$$

The partial sums

$$S_R = \sum_{\|\xi\| < R} u_\xi e^{i\langle \xi, x\rangle}$$

are continuous, and for $R \leqslant R'$,

$$\begin{aligned}
|S_R(x) - S_{R'}(x)| &\leqslant \sum_{\|\xi\| > R} |u_\xi| \\
&= \sum_{\|\xi\| > R} \frac{\left((1+\|\xi\|^2)^{[n/2]+1}|u_\xi|^2\right)^{1/2}}{\left((1+\|\xi\|^2)^{[n/2]+1}\right)^{1/2}} \\
&\leqslant \|u\|_{[n/2]+1}\left[\sum_{\|\xi\| > R}\left(\frac{1}{(1+\|\xi\|^2)^{[n/2]+1}}\right)^{1/2}\right].
\end{aligned}$$

Now apply the integral test in $\mathbb{R}^n$ to conclude that

$$\sum_\xi \left(\frac{1}{(1+\|\xi\|^2)^{[n/2]+1}}\right)^{1/2} \leqslant \sum_{\xi \neq 0} \frac{1}{\|\xi\|^{n+1}}$$

converges, from which it follows that $S_R(x)$ converges uniformly to $\varphi \in C^0(T)$ with $\varphi_\xi = u_\xi$.

Now we proceed by induction on s. Since the proof for general n involves only inessential formalism beyond what we have just done together with the one-variable case, we shall complete the argument only when $n=1$.

So, we suppose $H_{s+1} \subset C^s(T)$ and

$$u = \sum_{\xi \in \mathbb{Z}} u_\xi e^{i\xi x}$$

satisfies $u \in H_{s+2}$, i.e.,

$$\sum_\xi |\xi|^{2s+4}|u_\xi|^2 < \infty.$$

Set

$$v = \sum_{\xi \neq 0} i\xi u_\xi e^{i\xi x}.$$

Then $v \in H_{s+1}$, and therefore is a function in $C^s(T)$ by induction hypothesis. The convergence being uniform, we may integrate term-by-term:

$$\int_0^x v(t)\,dt = \sum_\xi u_\xi e^{i\xi x}\,dx = u(x) - u_0,$$

so $u'(x) = v(x)$ and $u \in C^{s+1}(T)$. Q.E.D.

Summarizing, we have shown that the Fourier series mapping $C^0(T) \to \mathfrak{F}$ leads to inclusions

$$C^s(T) \subset H_s,$$
$$H_{s+[n/2]+1} \subset C^s(T),$$
$$C^\infty(T) = H_\infty.$$

A useful remark is that the proof of the Sobolev lemma gives an estimate

$$\sup_{x \in T} |D^\alpha \varphi(x)| \leqslant C_\alpha \|\varphi\|_{[n/2]+1+[\alpha]}.$$

Rellich Lemma. *For* s > r *the inclusion*

$$H_s \subset H_r$$

is compact.

Proof. Given a bounded sequence $\{u_k\}$ in H_s, we want to find a convergent subsequence in H_r. Since, for all k we have

$$\sum (1 + \|\xi\|^2)^r |u_{k,\xi}|^2 \leqslant \sum_\xi (1 + \|\xi\|^2)^s |u_{k,\xi}|^2 < C,$$

for fixed ξ the sequence $\left\{ (1 + \|\xi\|^2)^{r/2} u_{k,\xi} \right\}_k$ is bounded and hence has a Cauchy subsequence. By the standard diagonalization, then, we can find a subsequence $\{u_k\}$ such that $\left\{ (1 + \|\xi\|^2)^{r/2} u_{k,\xi} \right\}_k$ is Cauchy for every ξ.

Now we separate the terms with small ξ, of which there are only a finite number, from those with large ξ where the factor $(1 + \|\xi\|^2)^r$ will help: given $\varepsilon > 0$, choose R and m such that

$$\frac{4C}{(1 + \|\xi\|^2)^{s-r}} < \frac{\varepsilon}{2} \quad \text{for} \quad \|\xi\| > R,$$

$$\sum_{\|\xi\| < R} (1 + \|\xi\|^2)^r |u_{k,\xi} - u_{l,\xi}|^2 < \frac{\varepsilon}{2}$$

for $k, l \geqslant m$. Then

$$\begin{aligned} \|u_k - u_l\|_r^2 &= \sum_{\|\xi\| < R} (1 + \|\xi\|^2)^r |u_{k,\xi} - u_{l,\xi}|^2 \\ &\quad + \sum_{\|\xi\| > R} \frac{(1 + \|\xi\|^2)^s}{(1 + \|\xi\|^2)^{s-r}} |u_{k,\xi} - u_{l,\xi}|^2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Q.E.D.

We now wish to examine the Laplace equation on the torus T. Essentially we are going to prove in this case the Hodge theorem for 0-forms, or functions, with the standard Euclidean metric and relative to the exterior derivative d.

Although it is probably unnecessary, we remark that on a compact Riemannian manifold M we may define the adjoint d^* of d, form the Laplacian $\Delta_d = dd^* + d^*d$, and arrive at the exact same formalism as for $\bar\partial$ on complex manifolds. The Hodge theorem is, of course, also true, and the proof is the same as the one we shall give in the complex case.

For $\varphi \in C^\infty(T)$ the Laplacian is

$$\begin{aligned}\Delta_d\varphi &= \sum_i D_i^2\varphi\\ &= -\sum_i \frac{\partial^2\varphi}{\partial x_i^2}\\ &= -\sum_\xi \varphi_\xi \|\xi\|^2 e^{i\langle \xi, x\rangle}.\end{aligned}$$

We will discuss the equation

$$(*) \qquad \Delta_d\varphi = \psi$$

in a manner such that the conclusions carry over to a general compact manifold. A function $\varphi \in L^2(T) = H_0$ is said to be a *weak solution* to $(*)$ if

$$(\Delta_d\eta, \varphi) = (\eta, \psi)$$

for all $\eta \in C^\infty(T)$. In case the weak solution φ is also a C^∞ function, the Laplacian is self-adjoint, meaning that

$$(\eta, \Delta_d\varphi) = (\eta, \psi)$$

for all $\eta \in C^\infty(T)$, and so $\Delta_d\varphi = \psi$ in the usual sense. Weak solutions are easy to find by Hilbert-space methods, and the point is to prove regularity.

We first note that the weak solutions of the homogeneous equation

$$\Delta_d\varphi = 0$$

satisfy

$$(\|\xi\|^2 e^{i\langle \xi, x\rangle}, \varphi) = 0$$

for all ξ. Thus the weak harmonic space consists of the constant functions, defined by $\varphi_\xi = 0$ for $\xi \neq 0$.

Next, we observe that $(*)$ makes sense when $\psi \in L^2(T) = H_0$. A necessary condition for it to have a weak solution is that $\psi_0 = 0$, i.e., ψ should be orthogonal to the harmonic space.

Now, assuming this to be the case,

$$\varphi = -\sum_{\xi \neq 0} \frac{1}{\|\xi\|^2}\psi_\xi e^{i\langle \xi, x\rangle}$$

gives a formal Fourier series solution to $(*)$. Since clearly $\psi \in L^2(T) \Rightarrow \varphi \in L^2(T)$, it is a weak solution. In fact we can say more:

For $\psi \in L^2(T)$, if we define the Green's operator by

$$G(\psi) = -\sum_{\xi \neq 0} \frac{1}{\|\xi\|^2} \psi_\xi e^{i\langle \xi, x\rangle},$$

then

$$G: H_s \longrightarrow H_{s+2}$$

is a bounded linear operator. In case ψ is perpendicular to the harmonic space,

$$\varphi = G(\psi)$$

gives a weak solution to (*). By the Sobolev lemma, if $\psi \in C^\infty(T)$ then $\varphi \in C^\infty(T)$ and φ is a solution of (*) in the usual sense. Finally, by the Rellich lemma

$$G: L^2(T) \to L^2(T)$$

is a compact, self-adjoint operator. The spectral decomposition for G on $L^2(T)$ is just Fourier series.

At this juncture the observations of the preceding paragraph more than establish the Hodge theorem for zero-forms on a torus. The essential point is this: The operator

$$I + \Delta_d: H_s \longrightarrow H_{s-2}$$

is trivially bounded, since Δ is second order. More importantly, the identity

$$\|(I+\Delta_d)\varphi\|_{s-2}^2 = \|\varphi\|_s^2$$

allows us to invert $\mathrm{I}+\Delta_\mathrm{d}$ on $L^2(T)$ using the closed graph theorem. This inverse is a compact smoothing operator and contains the information of the Green's operator. If, on a general compact manifold M, we carry over the Sobolev-space formalism and can prove the *basic estimate*

$$\|(I+\Delta_d)\varphi\|_{s-2}^2 \geq C_s\|\varphi\|_s^2$$

by calculus, then we can hope to obtain the same sort of picture as on the torus.

We conclude the Fourier series discussion with some remarks concerning *distributions*, defined as the linear functions

$$\lambda: C^\infty(T) \longrightarrow \mathbb{C},$$

which are continuous in the sense that

$$|\lambda(\varphi)| \leq C_\lambda \sup_{\substack{[\alpha] \leq k \\ x \in T}} |D^\alpha \varphi(x)|$$

for some k. Each distribution generates a formal Fourier series $\sum\lambda_\xi e^{i\langle\xi,x\rangle}$ where

$$\lambda_\xi = \lambda(e^{-i\langle\xi,x\rangle}).$$

It follows from the definition of continuity of λ and the above estimate on $\sup_{x\in T}|D^\alpha\varphi(x)|$ that each distribution λ is a continuous linear function on H_s for some s. The pairing

$$(u,v) = \sum_\xi u_\xi v_\xi$$

identifies H_{-s} with the dual of H_s, so that $\lambda\in H_{-s}$ with its formal Fourier series given above. If we denote by $\mathfrak{D}(T)$ the space of distributions, then we conclude that

$$\mathfrak{D}(T) = H_{-\infty}.$$

The derivatives of a distribution are defined by

$$D^\alpha\lambda(\varphi) = \lambda(D^\alpha\varphi).$$

The Fourier coefficients of $D^\alpha\lambda$ are $(D^\alpha\lambda)_\xi=\xi^\alpha\lambda_\xi$. With this definition, a distribution is obtained by taking a finite number of derivatives of a continuous function.

A final piece of useful terminology is this: A distribution λ is said *to be in* L^2 in case $\lambda\in H_0\subset H_{-\infty}$. Then we may describe the Sobolev spaces by

H_s *consists of all distributions* λ *such that the distributional derivatives* $D^\alpha\lambda$ *are in* L^2 *for* $[\alpha]\leqslant s$.

An example of an interesting distribution is the *delta function* defined by

$$\delta(\varphi) = \varphi(0).$$

It has formal Fourier series

$$\delta = \sum_\xi e^{i\langle\xi,x\rangle}.$$

We shall not use distributions in proving the Hodge theorem, but they will be rather extensively discussed in Section 1 of Chapter 3. Note in passing that the equation

$$\Delta_d\varphi = \psi,$$

where ψ is a distribution, may be solved provided that $\psi(\eta)=0$ for any harmonic η. If $\psi\in H_s$ for any s, positive or negative, then $\varphi\in H_{s+2}$. In particular, regularity holds for distribution solutions as well as weak Hilbert-space solutions. We shall work in this latter setting in order to take advantage of the standard theory of compact, self-adjoint operators on Hilbert spaces.

Proof of the Hodge Theorem II: Global Theory

On a torus the Sobolev s-norm is given equivalently by a weighted Fourier series norm or by the L^2-norm

$$\sum_{[\alpha]<s} \int_T |D^\alpha \varphi|^2 dx.$$

This latter may be extended to vector bundles over manifolds so that the Sobolev lemma and Rellich lemma both remain valid. We now explain how this is done.

To begin with, suppose that $U \subset V \subset \mathbb{R}^n$ are open sets in $\mathbb{R}^n$ with each relatively compact in the next. Functions with compact support in U may be considered as functions on a torus T. Suppose that $v_1(x), \dots, v_n(x)$ are C^∞ vector fields in V that are everywhere linearly independent, and that $\rho(x)$ is a positive function on V. For $\varphi \in C_c^\infty(U)$ the Sobolev 0- and 1-norms are equivalent to

$$\int_V \rho(x)|\varphi(x)|^2 dx, \qquad \int_V \rho(x)\left\{|\varphi(x)|^2 + \sum_i |v_i(x)\cdot\varphi(x)|^2\right\} dx,$$

respectively. More generally, note that the commutator

$$[v_i, v_j]\varphi = v_i(v_j\varphi) - v_j(v_i\varphi)$$

is an operator of order 1, where an operator of *order s* is one involving at most s-derivatives and denoted by a generic symbol $A^s\varphi$. An expression

$$v^\alpha\varphi = v_1^{\alpha_1}(v_2^{\alpha_2}\cdots(v_n^{\alpha_n}\varphi)\cdots)$$

is independent of the ordering modulo operators of order $<[\alpha]$. It follows that the Sobolev s-norm of $\varphi \in C_c^\infty(U)$ is equivalent to

$$\sum_{[\alpha]<s} \int |v^\alpha\varphi(x)|^2 dx.$$

Suppose now that $E \to M$ is a vector bundle over a compact manifold M. Assume that we have connection ∇ in E and in the tangent bundle $T(M)$ of M. (It is more convenient to denote the connection operator by ∇, rather than D as in Section 5 of Chapter 0.) If $\{e_\alpha\}$ is a local frame for E and $\{v_i\}$ a local frame for $T(M)$ with dual coframe $\{\varphi_i\}$, then the covariant derivatives $\nabla_i f_\alpha = f_{\alpha,i}$ of a section $f = \sum_\alpha f_\alpha e_\alpha$ of $E \to M$ are defined by

$$\nabla f = \sum_{\alpha,i} f_{\alpha,i} e_\alpha \otimes \varphi_i.$$

We have

$$f_{\alpha,i} = v_i f_\alpha + A^0(f),$$

where A^0 is an operator of order zero involving the connection matrix.

Applying these considerations to $E\otimes T^*(M)$, we may define $f_{\alpha,i,j}=\nabla_j(\nabla_i f_\alpha)$, and so forth. The commutation rule

$$[\nabla_i,\nabla_j]f_\alpha = A^1(f)$$

follows from the above expression for $f_{\alpha,i}$.

Suppose now that E and $T(M)$ have metrics and that $\{e_\alpha\},\{v_i\}$ are orthonormal frames. The global *Sobolev* s*-norm* of sections $f\in C^\infty(M,E)$ is defined by

$$\|f\|_s^2=\sum_{k<s}\int_M\|\nabla^k f\|_0^2dx,$$

where

$$\nabla^k f=\underbrace{\nabla(\nabla(\cdots(\nabla f)\cdots))}_{k \text{ times}}.$$

Denote by $\mathcal{H}_s(M,E)$ the completion of $C^\infty(M,E)$ in this norm. Since, by our remarks at the beginning of this section, the global Sobolev norm induces a norm equivalent to the usual Sobolev norm on sections compactly supported in a neighborhood of a point, by using a partition of unity we may conclude the

Global Sobolev Lemma. $\mathcal{H}_{[n/2]+1+s}(M,E)\subset C^s(M,E)$, *the sections of differentiability class* s *on* M, *and*

$$\bigcap_s \mathcal{H}_s(M,E)=C^\infty(M,E).$$

Global Rellich Lemma. *For* $s>r$ *the inclusion*

$$\mathcal{H}_s(M,E)\to\mathcal{H}_r(M,E)$$

is a compact operator.

Now let M be a compact hermitian manifold with hermitian connection in the tangent bundle. Denote by $\mathcal{H}_s^{p,q}(M)$ the completion of $A^{p,q}(M)$ in the Sobolev s-norm, $\|\ \|=\|\ \|_0$, and define the *Dirichlet inner product* and *Dirichlet norm*, respectively, by

$$\begin{aligned}\mathfrak{D}(\varphi,\psi)&=(\varphi,\psi)+(\bar\partial\varphi,\bar\partial\psi)+(\bar\partial^*\varphi,\bar\partial^*\psi)\\&=(\varphi,(I+\Delta)\psi)\\ \mathfrak{D}(\varphi)&=\mathfrak{D}(\varphi,\varphi)=\|\varphi\|^2+\|\bar\partial\varphi\|^2+\|\bar\partial^*\varphi\|^2.\end{aligned}$$

The basic estimate in the theory is provided by

Gårding's Inequality. *For* $\varphi\in A^{p,q}(M)$

$$\|\varphi\|_1^2\leqslant C\mathfrak{D}(\varphi)\qquad(C>0).$$

We remark that the operator $I+\Delta$, rather than just the Laplacian Δ, is being used, since $\Delta \geqslant 0$ implies that $I+\Delta$ has no kernel and therefore we have a better chance of inverting it.

One use of the Gårding inequality will be to prove the

Regularity Lemma I. *Suppose that $\varphi \in \mathcal{H}_s^{p,q}(M)$, and that $\psi \in \mathcal{H}_0^{p,q}(M)$ is a weak solution of the equation*

$$\Delta\psi = \varphi$$

in the sense that

$$(\psi, \Delta\eta) = (\varphi, \eta)$$

for all $\eta \in A^{p,q}(M)$. Then $\psi \in \mathcal{H}_{s+2}^{p,q}(M)$.

For example, suppose that $\varphi \in \mathcal{H}_0^{p,q}(M)$ is an *eigenfunction* for the Laplacian, meaning that, for a constant λ, the equation

$$\Delta\varphi = \lambda\varphi$$

holds in the weak sense. Then by the regularity lemma, $\varphi \in \mathcal{H}_s^{p,q}(M)$ for all s, and by the global Sobolev lemma we conclude that any eigenfunction for Δ is smooth.

We note that any eigenvalue $\lambda \geqslant 0$, and $\lambda = 0 \Leftrightarrow \varphi$ is harmonic in the weak sense. By the regularity and Sobolev lemmas any such weakly harmonic form is C^∞ and harmonic in the usual sense.

We shall assume the Gårding inequality and regularity lemma and go ahead and complete the proof of the Hodge theorem. After this is done we shall prove the Gårding inequality. The regularity lemma will be proved when we discuss smoothing of distributions in general. The reader who wishes to have the complete argument at hand may find the proof at the end of the subsection entitled "Smoothing and Regularity" in Section 1 of Chapter 3.

The basic Hilbert-space tool is the spectral theorem for compact self-adjoint operators, together with the principle of representing bounded linear functions by taking the inner product with a fixed vector, in the form of the following

Lemma. *Given $\varphi \in \mathcal{H}_0^{p,q}(M)$, there exists a unique $\psi \in \mathcal{H}_1^{p,q}(M)$ such that*

$$(\varphi, \eta) = \mathfrak{D}(\psi, \eta) = (\psi, (I+\Delta)\eta)$$

for all $\eta \in A^{p,q}(M)$. The mapping

$$\psi = T(\varphi)$$

from $\mathcal{H}_0^{p,q}(M)$ to $\mathcal{H}_1^{p,q}(M)$ is bounded, and therefore the mapping

$$T: \mathcal{H}_0^{p,q}(M) \to \mathcal{H}_0^{p,q}(M)$$

is compact and self-adjoint.

Proof. The Gårding inequality says that the Dirichlet norm is equivalent to the Sobolev 1-norm on $\mathcal{H}_1^{p,q}(M)$. The linear functional

$$\eta \to (\varphi, \eta) \qquad (\eta \in A^{p,q}(M))$$

extends to a bounded linear form on $\mathcal{H}_1^{p,q}(M)$ with the Dirichlet norm, by virtue of

$$|(\varphi,\eta)| \leqslant \|\varphi\|_0 \|\eta\|_0 \leqslant \|\varphi\|_0 \mathfrak{D}(\eta).$$

Thus the equation

$$(\varphi,\eta) = \mathfrak{D}(\psi,\eta)$$

has a unique solution $\psi = T(\varphi)$ characterized by

$$(\varphi,\eta) = (T\varphi,(I+\Delta)\eta) \qquad (\eta \in A^{p,q}(M)).$$

T is self-adjoint, since this is true of I and Δ. From

$$2\alpha\beta \leqslant \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2$$

and

$$\begin{aligned} \|T\varphi\|_1^2 &\leqslant C\mathfrak{D}(T\varphi,T\varphi) \\ &= C(\varphi,T\varphi) \\ &\leqslant C\|\varphi\|_0\|T\varphi\|_0 \\ &\leqslant 2\varepsilon C\|T\varphi\|_0^2 + \frac{2C}{\varepsilon}\|\varphi\|_0^2 \end{aligned}$$

we deduce that

$$\|T\varphi\|_1^2 \leqslant C'\|\varphi\|_0^2.$$

This says that T is bounded as a map from $\mathcal{H}_0^{p,q}(M)$ to $\mathcal{H}_1^{p,q}(M)$, and by the global Rellich lemma it is compact as an operator on $\mathcal{H}_0^{p,q}(M)$. Q.E.D.

According to the spectral theorem for compact, self-adjoint operators there is a Hilbert-space decomposition

$$\mathcal{H}_0^{p,q}(M) = \bigoplus_m E(\rho_m),$$

where ρ_m are the eigenvalues of T and $E(\rho_m)$ are the *finite-dimensional* eigenspaces. Since T is one-to-one, all $\rho_m \neq 0$; moreover, the equation

$$T\varphi = \rho_m\varphi$$

is the same as

$$(\varphi,\eta) = (\rho_m\varphi,(I+\Delta)\eta) \qquad (\eta \in A^{p,q}(M)),$$

which implies that

$$\Delta\varphi = \left(\frac{1-\rho_m}{\rho_m}\right)\varphi$$

in the weak sense. It follows that the eigenspaces for T and Δ are the same and are finite-dimensional vector spaces consisting of C^∞ forms. The eigenvalues λ_m for Δ and ρ_m for T are related by

$$\lambda_m = \frac{1-\rho_m}{\rho_m}$$

$$\rho_m = \frac{1}{1+\lambda_m}.$$

We may assume that

$$0 = \lambda_0 < \lambda_1 < \cdots,$$

where $\lambda_m \uparrow \infty$, $\rho_m \downarrow 0$ as $m \to \infty$. The harmonic space $\mathcal{H}^{p,q}(M)$ corresponds to $\lambda_0 = 0$. For $\varphi \in \mathcal{H}^{p,q}(M)^\perp$

$$\|\Delta\varphi\|_0 \geqslant \lambda_1 \|\varphi\|_0 \qquad (\lambda_1 > 0),$$

and if we define the *Green's operator* by

$$\begin{cases} G = 0 & \text{on } \mathcal{H}^{p,q}(M), \\ G\varphi = \dfrac{1}{\lambda_m}\varphi, & \varphi \in E\left(\dfrac{1}{1+\lambda_m}\right), \end{cases}$$

then G is a compact, self-adjoint operator with spectral decomposition

$$\mathcal{H}_0^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \Big(\bigoplus_m E(\rho_m)\Big),$$

where

$$G\varphi = \left(\frac{\rho_m}{1-\rho_m}\right)\varphi, \qquad \varphi \in E(\rho_m).$$

At this point, we have proved the Hodge theorem. The essential idea is to produce the Green's operator by a Hilbert-space trick, and then to use the basic estimate to show that it is a compact smoothing operator. Actually, G is an integral operator of the form

$$(G\varphi)(x) = \int_M G(x,y)\varphi(y),$$

where $G(x,y)$ is a beautiful kernel on $M \times M$ with certain singularities along the diagonal Δ. The Hilbert-space method has the disadvantage of not giving us the Green's operator in this form. If we were working with distributions rather than just L^2-forms, then we could produce $G(x,y)$ by solving a distributional equation of the type

$$\Delta_x G(x,y) = \delta_y + S_y,$$

where δ_y is a delta function at y and S_y is an operator of order $-\infty$. Such equations will be discussed in Section 1 of Chapter 3.

Proof of the Gårding Inequality. We suppose that $\varphi_1,\ldots,\varphi_n$ is a local unitary coframe for the hermitian metric, so that

$$ds^2 = \sum_i \varphi_i \bar{\varphi}_i.$$

A form of type (p,q) is written locally as

$$\psi = \frac{1}{p!q!} \sum_{I,J} \psi_{I,\bar{J}} \varphi_I \wedge \bar{\varphi}_J,$$

where $\psi_{I,\bar{J}}$ is skew-symmetric in the indices i_α and $\bar{j}_\beta$. There is a famous formula for the Laplacian, the *Weitzenböck identity*, which we shall use in the crude form

$$\text{(W)} \qquad (\Delta\psi)_{I\bar{J}} = \left(- \sum_{k=1}^{n} \nabla_k \nabla_{\bar{k}} \psi_{I,\bar{J}} \right) + A^1(\psi).$$

In other words, modulo lower-order terms the global Laplacian on forms looks like the Euclidean Laplacian $-\Sigma_k \partial^2/(\partial z_k \partial \bar{z}_k)$ on vector-valued functions.

The precise Weitzenböck formula identifies the lower-order terms. For a general hermitian metric, $A^1(\psi)$ is a messy operator involving the torsion in its terms of first order. However, when the metric is Kähler, these drop out and $A^1(\psi)$ is the algebraic operator

$$A^1(\psi)_{I\bar{J}} = \sum_{k, j_\alpha \varepsilon J} R_{j_\alpha k} \psi_{I\bar{j}_1 \cdots \bar{k} \cdots \bar{j}_q} \qquad (k \text{ in } \alpha\text{th spot}),$$

where

$$R_{j\bar{k}} = \sum_i R^i_{ij\bar{k}}$$

is the *Ricci curvature*.

To prove the Weitzenböck formula we shall let $v_1,\ldots,v_n$ be the vector field frame dual to $\varphi_1,\ldots,\varphi_n$, and $v_{\bar{i}} = \bar{v}_i$. For a function f

$$\bar{\partial} f = \sum_i (v_{\bar{i}} \cdot f) \bar{\varphi}_i,$$

and for a tensor $\tau = \{\tau_I\}$ the components of the $\bar{z}$-covariant differential $\bar{\nabla}\tau$ are given by

$$(\bar{\nabla}\tau)_I = \bar{\partial}\tau_I + A^0(\tau).$$

It is convenient to use the symbol "$\equiv$" to denote "modulo lower-order terms," so that, e.g.,

$$(\bar{\nabla}\tau)_I \equiv \bar{\partial}\tau_I.$$

We set $\Phi' = \varphi_1 \wedge \cdots \wedge \varphi_n$.

It will suffice to prove (W) when $\psi = f\varphi_I \wedge \bar{\varphi}_J$ (no summation). Since the dz's act as, so to speak, vector bundle indices, we will assume $p=0$. Finally, by the symmetry in the formula we may take $J=(1,\ldots,q)$, so that

$$\psi = f\bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q.$$

Now we compute:

$$\bar{\partial}\psi \equiv (-1)^q \sum_{k>q} f_{\bar{k}} \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q \wedge \bar{\varphi}_k$$

$$*\bar{\partial}\psi \equiv (-1)^q 2^{q+1-n} \sum_{k>q} (-1)^{k-q-1} \bar{f}_{\bar{k}} \bar{\varphi}_{q+1} \wedge \cdots \wedge \hat{\bar{\varphi}}_k \wedge \cdots \wedge \bar{\varphi}_n \wedge \Phi'$$

$$\bar{\partial} * \bar{\partial}\psi \equiv \left(2^{q+1-n} \sum_{k>q} \bar{f}_{\bar{k},k}\right) \bar{\varphi}_{q+1} \wedge \cdots \wedge \bar{\varphi}_n \wedge \Phi'$$

$$+ \sum_{\substack{k>q \\ l \leq q}} 2^{q+1-n} (-1)^{k-1} \bar{f}_{\bar{k},l} \bar{\varphi}_l \wedge \bar{\varphi}_{q+1} \wedge \cdots \wedge \hat{\bar{\varphi}}_k \wedge \cdots \wedge \bar{\varphi}_n \wedge \Phi'$$

$$*\bar{\partial} * \bar{\partial}\psi \equiv \left(2 \sum_{k>q} f_{\bar{k},k}\right) \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q$$

$$+2 \sum_{\substack{k>q \\ l \leq q}} (-1)^{l-1+q} f_{\bar{k},l} \bar{\varphi}_1 \wedge \cdots \wedge \hat{\bar{\varphi}}_l \wedge \cdots \wedge \bar{\varphi}_q \wedge \bar{\varphi}_k.$$

This gives $*\bar{\partial} * \bar{\partial}\psi$, and the other term $\bar{\partial} * \bar{\partial} * \psi$ is similar but shorter:

$$*\psi = 2^{q-n} \bar{f} \bar{\varphi}_{q+1} \wedge \cdots \wedge \bar{\varphi}_n \wedge \Phi'$$

$$\bar{\partial} * \psi \equiv 2^{q-n} \sum_{l \leq q} \bar{f}_l \bar{\varphi}_l \wedge \bar{\varphi}_{q+1} \wedge \cdots \wedge \bar{\varphi}_n \wedge \Phi'$$

$$*\bar{\partial} * \psi \equiv 2 \sum_{l \leq q} (-1)^{l-1} f_l \bar{\varphi}_1 \wedge \cdots \wedge \hat{\bar{\varphi}}_l \wedge \cdots \wedge \bar{\varphi}_q$$

$$\bar{\partial} * \bar{\partial} * \psi \equiv \left(2 \sum_{l \leq q} f_{l,\bar{l}}\right) \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q$$

$$+ \left(2 \sum_{\substack{l \leq q \\ k>q}} (-1)^{q+l} f_{l,\bar{k}} \bar{\varphi}_1 \wedge \cdots \wedge \hat{\bar{\varphi}}_l \wedge \cdots \wedge \bar{\varphi}_q \wedge \bar{\varphi}_k\right).$$

Now $v_i(v_{\bar{j}} f) - v_{\bar{j}}(v_i f) \equiv A^1(f)$, so that modulo first-order terms

$$\Delta\psi = \left(-2 \sum_k f_{\bar{k},k}\right) \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q$$

$$+ \left(2 \sum_{\substack{l \leq q \\ k>q}} (-1)^{l-1+q} f_{\bar{k}} \bar{\varphi}_1 \wedge \cdots \wedge \hat{\bar{\varphi}}_l \wedge \cdots \wedge \bar{\varphi}_n \wedge \bar{\varphi}_k\right.$$

$$\left. +2 \sum_{\substack{l \leq q \\ k>q}} (-1)^{l+q} f_{\bar{k},l} \bar{\varphi}_1 \wedge \cdots \wedge \hat{\bar{\varphi}}_l \wedge \cdots \wedge \bar{\varphi}_n \wedge \bar{\varphi}_k\right).$$

The last two terms cancel to give

$$\Delta\psi \equiv \Big(-2\sum_k f_{\bar{k},k}\Big)\bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_q.$$

This proves the Weitzenböck formula.

We now come to the proof of the Gårding inequality, where we assume the Weitzenböck in the form

$$(\Delta\psi)_{I\bar{J}} = \Big(-2\sum_k \psi_{I\bar{J},\bar{k},k}\Big) + A^1(\psi).$$

Inequalities of the type

$$(*) \qquad 2\alpha\beta \leqslant \varepsilon\alpha^2 + \frac{1}{\varepsilon}\beta^2$$

will be used repeatedly, and $\Phi = C_n \Phi' \wedge \overline{\Phi'}$ denotes the volume form. Set

$$\eta = C_n\Big(-\sum_{I,J,k}(-1)^{k-1}\psi_{I\bar{J},\bar{k}}\overline{\psi_{I\bar{J}}}\,\varphi_1\wedge\cdots\wedge\hat{\varphi}_k\wedge\cdots\wedge\varphi_n\Big)\wedge\Phi'$$

$$= C_n'\big((\overline{\nabla}\psi,\psi)\wedge\omega^{n-1}\big).$$

The second expression shows that η is globally defined, and since it has type $(n-1,n)$, $d\eta = \partial\eta$. By Stokes' theorem

$$\int_M \partial\eta = 0.$$

On the other hand

$$\partial\eta = \Big(-2\sum_{I,J,k}\psi_{I\bar{J},\bar{k},k}\bar{\psi}_{I\bar{J}}\Big)\Phi - \Big(2\sum_{I,J,k}\psi_{I\bar{J},\bar{k}}\overline{\psi_{I\bar{J},\bar{k}}}\Big)\Phi + (A^1\psi,\psi)\Phi.$$

Thus, by the Weitzenböck formula

$$(\Delta\psi,\psi) = \|\overline{\nabla}\psi\|^2 + (A^1\psi,\psi),$$

where

$$\|\overline{\nabla}\psi\|^2 = \int_M (\overline{\nabla}\psi,\overline{\nabla}\psi)\Phi$$

is the L^2-norm of the $\bar{z}$-covariant differential of the tensor ψ, and $A^1(\psi)$ is a first-order operator involving $\bar{z}$-derivatives of ψ. Using $(*)$, we obtain

$$2|(A^1\psi,\psi)| \leqslant \varepsilon\|\overline{\nabla}\psi\|^2 + \frac{1}{\varepsilon}\|\psi\|^2,$$

which implies that

$$\|\overline{\nabla}\psi\|^2 \leqslant C'\{(\Delta\psi,\psi) + \|\psi\|^2\}, \qquad C' > 0.$$

We now repeat the argument applied this time to

$$\gamma = C_n\Big(-\sum_{I,J,k}(-1)^{k-1}\psi_{I\bar{J},k}\overline{\psi_{I\bar{J}}}\,\bar{\varphi}_1\wedge\cdots\wedge\hat{\bar{\varphi}}_k\wedge\cdots\wedge\bar{\varphi}_n\Big)\wedge\Phi'$$

and use $f_{k,\bar{k}}=f_{\bar{k},k}+A^1(f)$ to estimate the L^2-norm $\|\nabla\psi\|^2$ of the z-derivatives from below by the Dirichlet norm. Putting these together,

$$\|\nabla\psi\|^2+\|\bar{\nabla}\psi\|^2+\|\psi\|^2\geqslant C''((\Delta\psi,\psi)+\|\psi\|^2)=C''\mathfrak{D}(\psi),$$

which is the Gårding inequality.

Remark. In the Kähler case one may use the precise Weitzenböck formula and the above integration by parts calculation to prove the *Kodaira identity*

$$(\Delta\psi,\psi)=\|\bar{\nabla}\psi\|^2+(R\psi,\psi),$$

where, for $\psi\in A^{0,q}(M)$ and summing repeated indices,

$$(R\psi,\psi)=q\int_M\left(R_{i\bar{j}}\psi_{\bar{i}_1\cdots\bar{i}_{q-1}\bar{i}}\bar{\psi}_{\bar{i}_1\cdots\bar{i}_{q-1}\bar{j}}\right)\Phi.$$

If ψ is harmonic and the hermitian form

$$R_{i\bar{j}}\xi^i\bar{\xi}^j$$

is positive definite, then we deduce that $\psi\equiv 0$. By the Hodge theorem

$$0=\mathcal{H}^{0,q}(M)\cong H^{0,q}_{\bar{\partial}}(M),\qquad q>0.$$

This is a special case of the famous Kodaira vanishing theorem, for which the general argument will be given in Section 3 of Chapter 1.

Applications of the Hodge Theorem

We begin by noting the isomorphism

$$\mathcal{H}^{p,q}(M)\longrightarrow H^{p,q}_{\bar{\partial}}(M)$$

between the harmonic space and Dolbeault cohomology groups. In fact, by the Hodge decomposition every $\bar{\partial}$-closed form $\psi\in Z^{p,q}_{\bar{\partial}}(M)$ is

$$\psi=\mathcal{H}(\psi)+\bar{\partial}(\bar{\partial}^*G\psi),$$

since $\bar{\partial}G\psi=G\bar{\partial}\psi=0$. Combining this isomorphism with the Dolbeault isomorphism, we find

$$\mathcal{H}^{p,q}(M)\xrightarrow{\sim}H^q(M,\Omega^p).$$

By the first statement in the Hodge theorem, this implies

Finite Dimensionality

$$\dim H^q(M,\Omega^p)<\infty.$$

It is instructive to give a direct proof of finite dimensionality in the case $q=0$. Let $\{U_i\}$ be a finite coordinate covering of M with holomorphic coordinates $z_{i,1},\dots,z_{i,n}$ in U_i. We may find relatively compact open subsets $V_i \subset U_i$ that still constitute a covering of M. A global section

$$\begin{aligned}\varphi \in H^0(M,\Omega^q) &= H^0(\{U_i\},\Omega^q)\\ &= H^0(\{V_i\},\Omega^q)\end{aligned}$$

is given in U_i by

$$\varphi = \sum_J \varphi_{iJ}(z)\,dz_{i,J},$$

where $\phi_{iJ}(z) \in \mathcal{O}(U_i)$. We define the norm

$$\|\varphi\| = \sum_{i,J} \sup_{z\in V_i} |\varphi_{i,J}(z)|.$$

This norm is finite, and since (1) $H^0(\{V_i\},\Omega^q) \cong H^0(\{U_i\},\Omega^q)$ and (2) any sequence of analytic functions $\psi_\nu \in \mathcal{O}(U_i)$ satisfying $\sup_{z\in V_i}|\psi_\nu(z) - \psi_\mu(z)| \to 0$ has a subsequence converging uniformly to a holomorphic function $\psi \in \mathcal{O}(V_i)$, we deduce that with this norm $H^0(M,\Omega^q)$ is a complete Banach space. By the Montel theorem, given a sequence $\varphi_\nu \in H^0(M,\Omega^q)$ with $\|\varphi_\nu\| \leqslant 1$ we may extract a subsequence whose coefficient functions $\varphi_{\nu,i,J}(z) \in \mathcal{O}(U_i)$ converge uniformly to some $\varphi_{i,J}(z) \in \mathcal{O}(V_i)$. Thus the unit ball in this Banach space is compact, and by a result in Banach-space theory this implies that it is finite dimensional.

Actually, it is obvious that a Hilbert space whose unit ball is compact is finite dimensional, and we may make $H^0(M,\Omega^q)$ into a Hilbert space by defining

$$(\varphi,\psi) = \sum_{i,J} \int_{V_i} \varphi_{i,J}(z)\,\overline{\psi_{i,J}(z)}\,\Phi(z_i),$$

where $\Phi(z_i)$ is the Euclidean volume form in the coordinates z_i. Since (1) a sequence $\psi_\nu \in \mathcal{O}(U_i)$ that is Cauchy in $L^2(V_i)$ has a subsequence converging uniformly on compact subsets of V_i to $\psi \in \mathcal{O}(V_i)$, and (2) a sequence $\psi_\nu \in \mathcal{O}(U_i)$ that is bounded in $L^2(V_i)$ has a similarly convergent subsequence, we may adopt the previous argument to this Hilbert-space setting.

This argument may be modified to prove the finite dimensionality of all $H^q(M,\Omega^p)$, and indeed the finite dimensionality of $H^q(M,\mathcal{F})$ for any coherent analytic sheaf $\mathcal{F}$—these matters will be discussed further in Section 3 of Chapter 5, where it will emerge that the finite dimensionality is the central fact in the global theory of coherent sheaf cohomology on a compact manifold.

A second application of the Hodge theorem is to Kodaira-Serre duality. From the formula $\bar\partial^* = -*\bar\partial*$ we see that

$$*\Delta = \Delta *.$$

This implies that the star operator induces an isomorphism

$$*: \mathcal{H}^{p,q}(M) \longrightarrow \mathcal{H}^{n-p,n-q}(M).$$

In particular

$$\mathcal{H}^{n,n}(M) \cong \mathbb{C} \cdot \Phi,$$

where $\Phi = *1$ is the volume form of the metric.

To put this isomorphism in intrinsic form not depending on the choice of a metric, we remark in a general fashion that, given sheaves $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ over a space X and a sheaf mapping

$$\mathcal{F} \otimes \mathcal{G} \longrightarrow \mathcal{H},$$

there is an induced cup product

$$H^*(X, \mathcal{F}) \otimes H^*(X, \mathcal{G}) \longrightarrow H^*(X, \mathcal{H})$$

given by the cochain formula at the end of the discussion of de Rham's theorem. In particular, the pairings

$$\Omega^p \otimes \Omega^q \longrightarrow \Omega^{p+q}$$

induced by the exterior product of holomorphic differential forms induce

$$(*) \qquad H^*(M, \Omega^p) \otimes H^*(M, \Omega^q) \longrightarrow H^*(M, \Omega^{p+q}).$$

On the other hand, the pairing

$$\{\ ,\ \}: A^{p,r}(M) \otimes A^{q,s}(M) \longrightarrow A^{p+q,r++s}(M)$$

given by

$$\{\psi, \eta\} = \psi \wedge \eta$$

satisfies

$$\bar{\partial}\{\psi, \eta\} = \{\bar{\partial}\psi, \eta\}(-1)^{\deg \psi}\{\psi, \bar{\partial}\eta\}$$

and so induces

$$(**) \qquad H^{p,*}_{\bar{\partial}}(M) \otimes H^{q,*}_{\bar{\partial}}(M) \longrightarrow H^{p+q,*}_{\bar{\partial}}(M).$$

The pairings $(*)$ and $(**)$ correspond under the Dolbeault isomorphism, at least modulo signs, for the same reason as in the discussion at the end of de Rham's theorem. With this understood we have the

Kodaira-Serre Duality Theorem

1. $H^n(M, \Omega^n) \xrightarrow{\sim} \mathbf{C}$, *and*
2. *the pairing*

$$H^q(M, \Omega^p) \otimes H^{n-q}(M, \Omega^{n-p}) \longrightarrow H^n(M, \Omega^n)$$

is nondegenerate.

Proof. The mapping in 1. is given by composing

$$H^n(M,\Omega^n) \cong H^{n,n}_{\bar\partial}(M)$$

with the linear function

$$H^{n,n}_{\bar\partial}(M) \longrightarrow \mathbb{C}$$

defined by

$$\psi \longrightarrow \int_M \psi,$$

which is well-defined on account of Stokes' theorem and $d=\bar\partial$ on $A^{n,n-1}(M)$. The fact that 1. is an isomorphism results from

$$H^{n,n}_{\bar\partial}(M) \cong \mathcal{H}^{n,n}(M) \cong \mathbb{C}\cdot\Phi,$$

since

$$\int_M \Phi = \operatorname{vol}(M) > 0.$$

The pairing 2. is given by composing

$$H^q(M,\Omega^p) \cong H^{p,q}_{\bar\partial}(M)$$

with the pairing

$$H^{p,q}_{\bar\partial}(M) \otimes H^{n-p,n-q}_{\bar\partial}(M) \longrightarrow \mathbb{C}$$

defined by

$$\psi\otimes\eta \longrightarrow \int_M \psi\wedge\eta.$$

It is nondegenerate, since

$$H^{p,q}_{\bar\partial}(M) \cong \mathcal{H}^{p,q}(M),$$

and for a harmonic form $\psi \not\equiv 0$

$$\psi\otimes *\psi \longrightarrow \int_M \psi\wedge *\psi = \|\psi\|^2 > 0. \qquad \text{Q.E.D.}$$

We now come to the Künneth formula. Given compact, complex manifolds M and N, we consider the product $M\times N$. The projections onto the two factors induce maps

$$H^*(M,\Omega^p_M) \longrightarrow H^*(M\times N,\Omega^p_{M\times N}),$$
$$H^*(N,\Omega^p_N) \longrightarrow H^*(M\times N,\Omega^p_{M\times N}).$$

We will prove in a minute that these are injective, and will identify these groups with their images. This being understood, the cup product gives

$$(*) \qquad H^*(M,\Omega^*_M)\otimes H^*(M,\Omega^*_N) \longrightarrow H^*(M\times N,\Omega^*_{M\times N}).$$

The *Künneth formula* asserts that this is an isomorphism.

We will prove this using harmonic forms. Hermitian metrics on M and N induce the product metric on $M \times N$, and we will show that, with this choice of metrics,

$$(**) \qquad \mathcal{H}^{u,v}(M \times N) \cong \bigoplus_{\substack{p+r=u \\ q+s=v}} (\mathcal{H}^{p,q}(M) \otimes \mathcal{H}^{r,s}(N)).$$

This will establish the Künneth theorem.

To carry this out, we denote by z, w generic local coordinates on M and N. Given forms ψ, η on M, N, respectively, we will denote by $\psi \otimes \eta$ the induced form on $M \times N$ given by

$$(\psi \otimes \eta)(z,w) = \psi(z) \wedge \eta(w).$$

These forms will be said to be *decomposable*.

Lemma. *The decomposable forms are* L^2*-dense in all the forms on* $M \times N$.

Proof. We will do this in the case of functions; the modifications necessary to treat general forms will be clear.

It must be proved that a function $\varphi(z,w)$ that satisfies

$$\int_{M \times N} \varphi(z,w)\left(\overline{\psi(z)\eta(w)}\right) = 0$$

for all ψ and η is zero. Suppose $\operatorname{Re}\varphi(z_0, w_0) > 0$, choose $\psi(z), \eta(w)$ to have compact support near z_0, w_0, respectively, and satisfy

$$\operatorname{Re}(\varphi(\psi\eta)) \geqslant 0, \qquad \operatorname{Re}\left(\varphi(z_0,w_0)\,\overline{\psi(z_0)\eta(w_0)}\right) > 0.$$

This is easy to accomplish using a real nonnegative bump function. Then the above integral is nonzero. Q.E.D.

Forms on $M \times N$ are locally written

$$\varphi(z,w) = \sum \varphi_{II'JJ'}\, dz_I \wedge dw_{I'} \wedge d\bar{z}_J \wedge d\bar{w}_{J'},$$

and then

$$\bar{\partial}_{M \times N} = \bar{\partial}_M \pm \bar{\partial}_N$$

where $\bar{\partial}_M$ is exterior derivative with respect to the $\bar{z}_j$'s and similarly for $\bar{\partial}_N$. Since the metric is a product, we may choose an orthonormal coframe for $M \times N$ of the form

$$\{\varphi_1(z), \ldots, \varphi_m(z); \psi_1(w), \ldots, \psi_n(w)\},$$

where the $\varphi_i(z)$ are an orthonormal coframe for M and the $\psi_\alpha(w)$ are the same for N. Using the formula

$$\bar{\partial}^* = -*\bar{\partial}*,$$

we find that

$$\begin{cases} \bar\partial^*_{M\times N} = \bar\partial^*_M \pm \bar\partial^*_N, \\ \bar\partial_M \bar\partial^*_N + \bar\partial^*_N \bar\partial_M = 0 = \bar\partial^*_M \bar\partial_N + \bar\partial_N \bar\partial^*_M \end{cases}.$$

These relations imply that

$$\Delta_{M\times N} = \Delta_M + \Delta_N.$$

More precisely, on decomposable forms

$$\Delta_{M\times N}(\psi \otimes \eta) = (\Delta_M \psi) \otimes \eta + \psi \otimes (\Delta_N \eta),$$

and by the lemma this determines $\Delta_{M\times N}$ on all forms.

Now we come to the main point. If $\psi_1, \psi_2, \ldots$ are a complete set of eigenforms for Δ_M and $\eta_1, \eta_2, \ldots$ a complete set of eigenforms for Δ_N, then the forms

$$\psi_i \otimes \eta_\alpha$$

are eigenforms for $\Delta_{M\times N}$. By the lemma they form a complete set. If

$$\begin{aligned} \Delta_M \psi_i &= \lambda_i \psi_i, & \lambda_i &\geqslant 0, \\ \Delta_N \eta_\alpha &= \mu_\alpha \eta_\alpha, & \mu_\alpha &\geqslant 0, \end{aligned}$$

then

$$\Delta_{M\times N}(\psi_i \otimes \eta_\alpha) = (\lambda_i + \mu_\alpha)(\psi_i \otimes \eta_\alpha).$$

Since $\lambda_i + \mu_\alpha = 0 \Leftrightarrow \lambda_i = \mu_\alpha = 0$, the assertion (**) about the harmonic forms follows. Q.E.D. for Künneth.

If we define the *Hodge numbers*

$$h^{p,q}(M) = \dim H^q(M, \Omega^p),$$

then we have proved that

$$h^{p,q}(M) < \infty,$$

$$\begin{cases} h^{n,n}(M) = 1 \quad \text{and} \\ h^{p,q}(M) = h^{n-p,n-q}(M), \end{cases}$$

$$h^{u,v}(M\times N) = \sum_{\substack{p+r=u \\ q+s=v}} h^{p,q}(M) h^{r,s}(N).$$

In case M is Kähler, there will be additional deeper-lying relations among the Hodge numbers, such as

$$h^{p,q}(M) = h^{q,p}(M),$$

$$b_r(M) = \sum_{p+q=r} h^{p,q}(M),$$

$$h^{p,p}(M) \geqslant 1,$$

where $b_r(M) = \dim H^r(M, \mathbb{C})$ is the rth Betti number. These, and much more, will be derived in the next section.

One final comment. In general the exterior product of harmonic forms is not harmonic. Similarly, the restriction of a harmonic form to a submanifold is generally not harmonic for the induced metric. Otherwise the cohomology ring would have only those relations imposed by exterior algebra. Moreover, the two Laplacians on a hermitian manifold,

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial,$$
$$\Delta_d = dd^* + d^*d,$$

are generally unrelated. It is a miraculous fact that, when the metric is Kähler, both these general principles are violated and the theory of harmonic forms has an extraordinary amount of symmetry. More on this in the next section.

7. KÄHLER MANIFOLDS

The Kähler Condition

Let M be a compact complex manifold with Hermitian metric ds^2, and suppose that in some open set $U \subset M$, ds^2 is Euclidean; that is, there exist local holomorphic coordinates $z = (z_1, \ldots, z_n)$ such that

$$ds^2 = \sum dz_i \otimes d\bar z_i.$$

Write $z_i = x_i + \sqrt{-1}\, y_i$; one may directly verify that for a differential form

$$\varphi = \sum \varphi_{I\bar J} dz_I \wedge d\bar z_J$$

compactly supported in U,

$$\begin{aligned}\Delta_{\bar\partial}(\varphi) &= -2 \sum_{I,J,i} \frac{\partial^2}{\partial z_i \partial \bar z_i} \varphi_{I\bar J} \cdot dz_I \wedge d\bar z_J \\ &= -\frac{1}{2} \sum_{I,J,i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) \varphi_{I\bar J} \cdot dz_I \wedge d\bar z_J \\ &= \frac{1}{2} \cdot \Delta_d(\varphi),\end{aligned}$$

i.e., the $\bar\partial$-Laplacian is equal to the ordinary d-Laplacian in U, up to a constant (cf. Section 6 above). Of course, very few compact complex manifolds have everywhere Euclidean metrics, but as it turns out in order to insure the identity

$$\Delta_{\bar\partial} = \frac{1}{2} \cdot \Delta_d$$

on a complex manifold, it is sufficient that the metric approximate the Euclidean metric to order 2 at each point. This is the *Kähler condition*, and we will spend the greater part of this section discussing the condition and its consequences.

We start by giving three alternate forms of the Kähler condition. Again, let

$$ds^2 = \sum h_{ij}\, dz_i \otimes d\bar{z}_j = \sum \varphi_i \otimes \bar{\varphi}_i$$

be a Hermitian metric on the complex manifold M. We say that ds^2 is *Kähler* if its associated $(1,1)$-form

$$\omega = \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \bar{\varphi}_i$$

is d-closed. In Section 5 above we showed that there was a unique matrix ψ of 1-forms satisfying

$$\psi_{ij} + \bar{\psi}_{ji} = 0, \qquad d\varphi_i = \sum \psi_{ij} \wedge \varphi_j + \tau_i$$

where τ_i is of type (2,0); there we said that the metric was Kähler if the torsion $\tau = 0$. We now show that these conditions are equivalent. Write

$$\begin{aligned}\frac{2}{\sqrt{-1}}\, d\omega &= \sum d\varphi_i \wedge \bar{\varphi}_i - \sum \varphi_i \wedge d\bar{\varphi}_i \\ &= \sum \psi_{ij} \wedge \varphi_j \wedge \bar{\varphi}_i - \sum \varphi_i \wedge \bar{\psi}_{ij} \wedge \bar{\varphi}_j + \sum \tau_i \wedge \bar{\varphi}_i - \sum \varphi_i \wedge \bar{\tau}_i.\end{aligned}$$

We have

$$\sum \psi_{ij} \wedge \varphi_j \wedge \bar{\varphi}_i - \sum \varphi_i \wedge \bar{\psi}_{ij} \wedge \bar{\varphi}_j = \sum \psi_{ij} \wedge \varphi_j \wedge \bar{\varphi}_i + \sum \varphi_i \wedge \psi_{ji} \wedge \bar{\varphi}_j = 0$$

and so $(2/\sqrt{-1}\,)d\omega = \sum \tau_i \wedge \bar{\varphi}_i - \sum \varphi_i \wedge \bar{\tau}_i$. But τ_i is of type (2,0) and the $\bar{\varphi}_i$ are pointwise linearly independent $(0,1)$-forms, which implies that $d\omega = 0$ if and only if $\tau = 0$.

Another interpretation of the Kähler condition that gives some geometric insight is this: We say a metric ds^2 on M *osculates to order* k to the Euclidean metric on $\mathbb{C}^n$ if for every point $z_0 \in M$ we can find a holomorphic coordinate system (z) in a neighborhood of z_0 for which

$$ds^2 = \sum (\delta_{ij} + g_{ij})\, dz_i \otimes d\bar{z}_j,$$

where g_{ij} vanishes up to order k at z_0; we usually write

$$ds^2 = \sum (\delta_{ij} + [k])\, dz_i \otimes d\bar{z}_j.$$

Lemma. ds^2 *is Kähler if and only if it osculates to order* 2 *to the Euclidean metric everywhere.*

Proof. One direction is clear: if

$$\omega = \frac{\sqrt{-1}}{2} \sum (\delta_{ij} + [2]) dz_i \wedge d\bar{z}_j$$

in some coordinate system around z_0, then $d\omega(z_0) = 0$.

Conversely, we can always find coordinates (z) for which $h_{ij}(z_0) = \delta_{ij}$; i.e.,

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j,k} (\delta_{ij} + a_{ijk} z_k + a_{ij\bar{k}} \bar{z}_k + [2]) dz_i \wedge d\bar{z}_j;$$

note that

$$h_{ij} = \overline{h_{ji}} \Rightarrow a_{ji\bar{k}} = \overline{a_{ijk}}$$

and

$$d\omega = 0 \Rightarrow a_{ijk} = a_{kji}.$$

We want to find a change of coordinates

$$z_k = w_k + \frac{1}{2} \sum b_{klm} w_l w_m$$

such that

$$(*) \qquad \omega = \frac{\sqrt{-1}}{2} \sum (\delta_{ij} + [2]) dw_i \wedge d\bar{w}_j;$$

we normalize by requiring

$$b_{klm} = b_{kml}.$$

Then

$$dz_k = dw_k + \sum b_{klm} w_l dw_m,$$

so that

$$\frac{2}{\sqrt{-1}} \omega = \sum \left(dw_i + \sum b_{ilm} w_l dw_m \right) \wedge \sum \left(d\bar{w}_i + \sum \overline{b_{ipq}}\, \bar{w}_p d\bar{w}_q \right)$$
$$+ \sum \left(a_{ijk} w_k + a_{ij\bar{k}} \bar{w}_k \right) dw_i \wedge d\bar{w}_j + [2]$$
$$= \sum \left(\delta_{ij} + \sum_k \left(a_{ijk} w_k + a_{ij\bar{k}} \bar{w}_k + b_{jki} w_k + \overline{b_{ikj}}\, \bar{w}_k \right) \right) dw_i \wedge d\bar{w}_j + [2].$$

If we set

$$b_{jki} = -a_{ijk};$$

then

$$b_{jki} = -a_{ijk} = -a_{kji} = b_{jik}$$

and

$$\overline{b_{ikj}} = -\overline{a_{jik}} = -a_{ij\bar{k}},$$

so that the coordinate change does in fact satisfy the condition $(*)$. Q.E.D.

Another way of expressing this condition that is useful in computation is to say that for each point $z_0 \in M$ we can find a unitary coframe $\varphi_1, \ldots, \varphi_n$ for the metric in some neighborhood of z_0 such that $d\varphi_i(z_0) = 0$.

A manifold is called *Kähler* if it admits a Kähler metric; we now give some examples of Kähler manifolds.

Examples

Any metric on a compact Riemann surface is Kähler, since $d\omega$ is a 3-form, and hence zero.

If Λ is a lattice in $\mathbb{C}^n$, the complex torus $T = \mathbb{C}^n/\Lambda$ is Kähler with the Euclidean metric $ds^2 = \sum dz_i \otimes d\bar{z}_i$.

If M and N are Kähler then $M \times N$ is Kähler, with the product metric.

If $S \subset M$ is a submanifold, ω the associated (1,1)-form of a Kähler metric on M, we have already noted in Section 2 above that the associated (1,1)-form of the induced metric on S is just the pullback to S of ω; thus if M is Kähler then S is Kähler.

Recall that the Fubini-Study metric on $\mathbb{P}^n$ is given by its associated (1,1) form

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2$$

where Z is a local lifting of $U \subset \mathbb{P}^n$ to $\mathbb{C}^{n+1} - \{0\}$. Since $\partial\bar{\partial} = -\bar{\partial}\partial$,

$$\omega = \frac{\sqrt{-1}}{4\pi} (\partial + \bar{\partial})(\bar{\partial} - \partial) \log \|Z\|^2$$

$$= \frac{\sqrt{-1}}{4\pi} d((\bar{\partial} - \partial) \log \|Z\|^2),$$

so we see that ω is closed, and the Fubini-Study metric is Kähler.

(*Note*: It is convenient to define an operator d^c by

$$d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial).$$

d and d^c are both real differential operators, and

$$dd^c = -d^c d = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}.$$

We can consequently write

$$\omega = dd^c \log \|Z\|^2.)$$

Note by the above that *any compact complex manifold that can be embedded in projective space* $\mathbb{P}^n$ *is Kähler.*

We give some immediate topological consequences of the Kähler condition: For M a compact Kähler manifold,

1. *The even Betti numbers* $b_{2q}(M)$ *are positive*;

2. *The holomorphic* q-*forms* $H^0(M,\Omega^q)$ *inject into the cohomology* $H^q_{DR}(M)$, *i.e., every such* η *is closed, and is never exact*; *and*

3. *The fundamental class* η_V *of any analytic subvariety* $V \subset M$ *is nonzero.*

Proofs. 2. Let η be a holomorphic $(q,0)$-form; we want to show $d\eta=0$, and that $\eta=d\psi$ only if $\eta\equiv 0$. Let $\varphi_1,\ldots,\varphi_n$ be a local unitary coframe; if

$$\eta = \sum_I \eta_I \varphi_I,$$

then

$$\eta\wedge\bar{\eta} = \sum_{I,J} \eta_I \bar{\eta}_J \varphi_I \wedge \bar{\varphi}_J.$$

Now

$$\omega = \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \bar{\varphi}_i,$$

so

$$\omega^{n-q} = C_q (n-q)! \sum_{\#K=n-q} \varphi_K \wedge \bar{\varphi}_K;$$

thus, for suitable $C_q \neq 0$,

$$\eta\wedge\bar{\eta}\wedge\omega^{n-q} = C_q \sum_I |\eta_I|^2 \cdot \Phi$$

where Φ is the volume form. Consequently,

$$\int_M \eta\wedge\bar{\eta}\wedge\omega^{n-q} \neq 0 \qquad \text{if } \eta \neq 0.$$

Now suppose $\eta = d\psi$. Then $d\eta = d\bar{\eta} = 0$, and since $d\omega = 0$ we have

$$\int_M \eta\wedge\bar{\eta}\wedge\omega^{n-q} = \int_M d(\psi\wedge\bar{\eta}\wedge\omega^{n-q}) = 0.$$

Thus $\eta = d\psi$ implies that $\eta\equiv 0$. Finally, since $d\eta = \partial\eta$ is a holomorphic $(q+1)$-form and is exact, it follows that $d\eta = 0$.

1. To show $b_{2q}(M)>0$, we exhibit ω^q as a closed $2q$-form that is not exact: if $\omega^q = d\psi$, then we have

$$\int_M \omega^n = \int_M d(\psi\wedge\omega^{n-q}) = 0.$$

But $\omega^n/n!$ is the volume form on M, and so this cannot happen.

3. The proof of 3 is clear: for V of complex dimension d, by the Wirtinger theorem from Section 2 above

$$\operatorname{vol}(V) = \frac{1}{d!}\int_V \omega^d \neq 0;$$

so $(\eta_V) \neq 0$ in $H_{2d}(M)$. Q.E.D.

Note that 1 and 3 are extensions of the propositions proved on p. 64 for submanifolds of projective space.

The Hodge Identities and the Hodge Decomposition

Let M be a compact complex manifold with hermitian metric ds^2 and associated $(1,1)$-form ω. We have defined a number of operators on the space $A^*(M)$ of differential forms on M, such as ∂, $\bar{\partial}$, d, d^c, their respective adjoints and associated Laplacians, and the decompositions

$$\Pi^{p,q}: A^*(M) \to A^{p,q}(M)$$
$$\Pi^r = \bigoplus_{p+q=r} \Pi^{p,q}: A^*(M) \to A^r(M)$$

by type and degree. We define an additional operator

$$L: A^{p,q}(M) \to A^{p+1,q+1}(M)$$

by

$$L(\eta) = \eta \wedge \omega$$

and let

$$\Lambda = L^*: A^{p,q}(M) \to A^{p-1,q-1}(M)$$

be its adjoint. Now, for general M there are no nonobvious relationships among these various operators. If we assume that the metric on M is Kähler, however, we get a host of identities relating them, called the *Hodge identities*. Indeed, the Kähler condition is exactly that which insures a strong interplay between the real potential theory associated to the Riemannian metric and the underlying complex structure. The basic identity, from which all the others will easily follow, is

$$(*) \qquad [\Lambda, d] = -4\pi d^{c*},$$

where $[A,B]$ denotes the commutator $AB - BA$; or equivalently,

$$[L, d^*] = 4\pi d^c.$$

Proof. By decomposition into type, this identity is equivalent to

$$[\Lambda, \bar{\partial}] = -\sqrt{-1}\,\partial^* \quad \text{and} \quad [\Lambda, \partial] = \sqrt{-1}\,\bar{\partial}^*.$$

Since Λ, d, and d^c are real operators, either of these implies the other; we will prove $[\Lambda, \partial] = \sqrt{-1}\,\bar{\partial}^*$. We make the computation first on $\mathbb{C}^n$ with the Euclidean metric. Here it is messy but straightforward and will be facilitated by our breaking it up into component steps. To do this, we introduce some new operators on forms in $\mathbb{C}^n$: for each $k = 1, \ldots, n$, let $e_k: A_c^{p,q}(\mathbb{C}^n) \to A_c^{p+1,q}(\mathbb{C}^n)$ be the operator on compactly supported forms defined by

$$e_k(\varphi) = dz_k \wedge \varphi;$$

let $\bar{e}_k : A_c^{p,q}(\mathbb{C}^n) \to A_c^{p,q+1}(\mathbb{C}^n)$ similarly be given by

$$\bar{e}_k(\varphi) = d\bar{z}_k \wedge \varphi.$$

Let i_k and $\bar{i}_k$ be the adjoints of e_k and $\bar{e}_k$, respectively. Note that e_k, $\bar{e}_k$, i_k, and $\bar{i}_k$ are all linear over $C^\infty(\mathbb{C}^n)$. Now

$$i_k(dz_J \wedge d\bar{z}_K) = 0, \qquad \text{if } k \notin J,$$

and, recalling that the length $\|dz_k\| = 2$,

$$i_k(dz_k \wedge dz_J \wedge d\bar{z}_K) = 2\,dz_J \wedge d\bar{z}_K;$$

since in the former case, we have for any multiindexes L and M

$$\begin{aligned}(i_k(dz_J \wedge d\bar{z}_K), dz_L \wedge d\bar{z}_M) &= (dz_J \wedge d\bar{z}_K, dz_k \wedge dz_L \wedge d\bar{z}_M)\\ &= 0,\end{aligned}$$

so $i_k(dz_J \wedge d\bar{z}_K) = 0$, while in the latter case

$$\begin{aligned}(i_k(dz_k \wedge dz_J \wedge d\bar{z}_K), dz_L \wedge d\bar{z}_M) &= (dz_k \wedge dz_J \wedge d\bar{z}_K, dz_k \wedge dz_L \wedge d\bar{z}_M)\\ &= 2(dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M).\end{aligned}$$

Similarly, we see that

$$\bar{i}_k(dz_J \wedge d\bar{z}_K) = 0, \qquad \text{if } k \notin K,$$

and

$$\bar{i}_k(d\bar{z}_k \wedge dz_J \wedge d\bar{z}_K) = 2\,dz_J \wedge d\bar{z}_K.$$

Note also that for any monomial $dz_J \wedge d\bar{z}_K$,

$$i_k \cdot e_k(dz_J \wedge d\bar{z}_K) = \begin{cases} 0, & \text{if } k \in J,\\ 2\,dz_J \wedge d\bar{z}_K, & \text{if } k \notin J \end{cases}$$

while

$$e_k \cdot i_k(dz_J \wedge d\bar{z}_K) = \begin{cases} 2\,dz_J \wedge d\bar{z}_K, & \text{if } k \in J,\\ 0, & \text{if } k \notin J. \end{cases}$$

Thus

$$i_k e_k + e_k i_k = 2$$

and likewise $\bar{i}_k \bar{e}_k + \bar{e}_k \bar{i}_k = 2$. On the other hand, we have for $k \neq l$,

$$\begin{aligned} i_k \cdot e_l(dz_k \wedge dz_J \wedge d\bar{z}_K) &= i_k(dz_l \wedge dz_k \wedge dz_J \wedge d\bar{z}_K)\\ &= i_k(-dz_k \wedge dz_l \wedge dz_J \wedge d\bar{z}_K)\\ &= -2(dz_l \wedge dz_J \wedge d\bar{z}_K)\\ &= -2e_l(dz_J \wedge d\bar{z}_K)\\ &= -e_l \cdot i_k(dz_k \wedge dz_J \wedge d\bar{z}_K),\end{aligned}$$

while

$$i_k \cdot e_l(dz_J \wedge dz_K) = e_l \cdot i_k(dz_J \wedge d\bar{z}_K) = 0$$

in case $k \notin J$, so we have

$$e_k i_l + i_l e_k = 0.$$

We also define operators ∂_k and $\bar{\partial}_k$ on $A_c^{p,q}(\mathbb{C}^n)$ by

$$\partial_k\left(\sum \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_J\right) = \sum \frac{\partial \varphi_{I\bar{J}}}{\partial z_k}\, dz_I \wedge d\bar{z}_J$$

and

$$\bar{\partial}_k\left(\sum \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_J\right) = \sum \frac{\partial \varphi_{I\bar{J}}}{\partial \bar{z}_k}\, dz_I \wedge d\bar{z}_J.$$

Note that ∂_k and $\bar{\partial}_k$ commute with e_l, $\bar{e}_l$, i_l, and $\bar{i}_l$ and with each other. Finally, we see that the adjoint of ∂_k is $-\bar{\partial}_k$: we have for $\varphi = \sum \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_K$ any compactly supported form, L and M any multiindices and ψ any C^∞ function,

$$\begin{aligned}
\left(-\bar{\partial}_k \varphi, \psi\, dz_L \wedge d\bar{z}_M\right) &= \left(-\frac{\partial}{\partial \bar{z}_k}(\varphi_{L\bar{M}})\, dz_L \wedge d\bar{z}_M, \psi\, dz_L \wedge d\bar{z}_M\right) \\
&= 2^{\#L + \#M} \int_{\mathbb{C}^n} -\frac{\partial}{\partial \bar{z}_k}(\varphi_{L\bar{M}}) \cdot \bar{\psi} \\
&= 2^{\#L + \#M} \int_{\mathbb{C}^n} \varphi_{L\bar{M}} \cdot \frac{\partial}{\partial \bar{z}_k}(\bar{\psi}) \quad \text{(by integration by parts)} \\
&= 2^{\#L + \#M} \int_{\mathbb{C}^n} \varphi_{L\bar{M}} \cdot \overline{\frac{\partial}{\partial z_k}(\psi)} \\
&= \left(\varphi_{L\bar{M}}\, dz_L \wedge d\bar{z}_M, \partial_k(\psi \cdot dz_L \wedge d\bar{z}_M)\right) \\
&= \left(\varphi, \partial_k(\psi\, dz_L \wedge d\bar{z}_M)\right).
\end{aligned}$$

Likewise, the adjoint of $\bar{\partial}_k$ is $-\partial_k$.

We can express all of our operators on $A_c^{**}(\mathbb{C}^n)$ in terms of these elementary operators: clearly

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k,$$

$$\bar{\partial} = \sum \bar{\partial}_k \bar{e}_k = \sum \bar{e}_k \bar{\partial}_k,$$

and, taking adjoints,

$$\bar{\partial}^* = -\sum \partial_k \bar{i}_k,$$

$$\partial^* = -\sum \bar{\partial}_k i_k.$$

L is defined as exterior product with the standard Kähler form defined on $\mathbb{C}^n$, so

$$L = \frac{\sqrt{-1}}{2} \sum e_k \bar{e}_k$$

and, taking the adjoint,

$$\Lambda = -\frac{\sqrt{-1}}{2} \sum \bar{i}_k i_k .$$

so

$$\begin{aligned}
\Lambda\partial &= -\frac{\sqrt{-1}}{2} \sum_{k,l} \bar{i}_k i_k \partial_l e_l \\
&= -\frac{\sqrt{-1}}{2} \sum_{k,l} \partial_l \bar{i}_k i_k e_l \\
&= -\frac{\sqrt{-1}}{2} \Big(\sum_k \partial_k \bar{i}_k i_k e_k + \sum_{k \neq l} \partial_l \bar{i}_k i_k e_l \Big).
\end{aligned}$$

To evaluate the first term, write

$$\begin{aligned}
-\frac{\sqrt{-1}}{2} \sum_k \partial_k \bar{i}_k i_k e_k &= \frac{\sqrt{-1}}{2} \sum \partial_k \bar{i}_k e_k i_k - \frac{2\sqrt{-1}}{2} \sum \partial_k \bar{i}_k \\
&= -\frac{\sqrt{-1}}{2} \sum \partial_k e_k \bar{i}_k i_k - \sqrt{-1} \sum \partial_k \bar{i}_k .
\end{aligned}$$

For the second term

$$\begin{aligned}
-\frac{\sqrt{-1}}{2} \sum_{k \neq l} \partial_l \bar{i}_k i_k e_l &= \frac{\sqrt{-1}}{2} \sum_{l \neq k} \partial_l \bar{i}_k e_l i_k \\
&= -\frac{\sqrt{-1}}{2} \sum \partial_l e_l \bar{i}_k i_k .
\end{aligned}$$

Thus

$$\begin{aligned}
\Lambda\partial &= -\frac{\sqrt{-1}}{2} \sum_{k,l} \partial_l e_l \bar{i}_k i_k + \sqrt{-1} \sum \partial_k \bar{i}_k \\
&= \partial\Lambda + \sqrt{-1}\, \bar{\partial}^*,
\end{aligned}$$

so the identity is proved on $\mathbb{C}^n$.

To prove the result on a Kähler manifold M we use the condition of osculation to show that the identity holds at any point: for $z_0 \in M$, we can choose a coframe $\varphi_1, \ldots, \varphi_n$ for the metric such that $d\varphi_i(z_0) = 0$. The expression for Λ holds with dz_l replaced by φ_l; we can make essentially the same computation for $[\Lambda, \partial]\eta$ as on $\mathbb{C}^n$ except that we will get terms involving $\bar{\partial}\varphi_i$. Since $[\Lambda, \partial]$ involves only first derivatives, however, all the additional terms will have a factor $\bar{\partial}\varphi_i$ and hence will vanish at z_0. Likewise, we have computed $\bar{\partial}^* \eta = C_n * \partial * \eta$ on $\mathbb{C}^n$ where it agrees with $\sqrt{-1}\,[\Lambda, \partial]\eta$; the computation on M in terms of the φ_i will again be the same except for additional terms involving $\bar{\partial}\varphi_i$, which vanish at z_0. Thus we see that the identity holds at z_0, hence everywhere.

This argument is just one instance of a general principle: *any intrinsically defined identity that involves the metric together with its first derivatives and which is valid on $\mathbb{C}^n$ with the Euclidean metric, is also valid on a Kähler manifold.*

Now, some consequences: if $\Delta_d = dd^* + d^*d$ is the d-Laplacian, we have

$$[L, \Delta_d] = 0$$

or, equivalently,

$$[\Lambda, \Delta_d] = 0.$$

Proof. First note that since ω is closed,

$$d(\omega \wedge \eta) = \omega \wedge d\eta,$$

i.e.,

$$[L, d] = 0$$

and so by taking adjoints

$$[\Lambda, d^*] = 0.$$

Now

$$\begin{aligned} \Lambda(dd^* + d^*d) &= (d\Lambda d^* - 4\pi d^{c*}d^*) + d^*\Lambda d \\ &= d\Lambda d^* + (4\pi d^* d^{c*} + d^*\Lambda d) \\ &= (dd^* + d^*d)\Lambda. \end{aligned}$$

Q.E.D.

We also have, as mentioned earlier,

$$\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial .$$

Proof. First we show that $\partial\bar\partial^* + \bar\partial^*\partial = 0$: since $\Lambda\partial - \partial\Lambda = \sqrt{-1}\,\bar\partial^*$, we have

$$\begin{aligned} \sqrt{-1}\,(\partial\bar\partial^* + \bar\partial^*\partial) &= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial \\ &= \partial\Lambda\partial - \partial\Lambda\partial = 0. \end{aligned}$$

Now,

$$\begin{aligned} \Delta_d &= (\partial + \bar\partial)(\partial^* + \bar\partial^*) + (\partial^* + \bar\partial^*)(\partial + \bar\partial) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) + (\partial\bar\partial^* + \bar\partial\partial^* + \partial^*\bar\partial + \bar\partial^*\partial) \\ &= (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) \\ &= \Delta_\partial + \Delta_{\bar\partial}, \end{aligned}$$

so we have to show

$$\Delta_\partial = \Delta_{\bar\partial}.$$

For this

$$\begin{aligned} -\sqrt{-1}\,\Delta_\partial &= \partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial \\ &= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial \end{aligned}$$

and consequently

$$\begin{aligned}\sqrt{-1}\,\Delta_{\bar\partial} &= (\bar\partial(\Lambda\partial-\partial\Lambda)+(\Lambda\partial-\partial\Lambda)\bar\partial)\\ &= \bar\partial\Lambda\partial-\bar\partial\partial\Lambda+\Lambda\partial\bar\partial-\partial\Lambda\bar\partial\\ &= \sqrt{-1}\,\Delta_{\partial},\end{aligned}$$

since $\bar\partial\partial=-\partial\bar\partial$. Q.E.D.

As an immediate corollary we see that Δ_d preserves bidegree; i.e.,

$$[\Delta_d,\Pi^{pq}]=0.$$

There are two main applications of these identities, the *Hodge decomposition* and the *Lefschetz decomposition* and theorem. We do Hodge first: Set

$$H^{p,q}(M)=\frac{Z_d^{p,q}(M)}{dA^*(M)\cap Z_d^{p,q}(M)},$$

$$\mathcal{H}_d^{p,q}(M)=\{\eta\in A^{p,q}(M):\ \Delta_d\eta=0\},$$

$$\mathcal{H}_d^{r}(M)=\{\eta\in A^{r}(M):\ \Delta_d\eta=0\}.$$

Note that the first group is intrinsically defined by the complex structure, while the latter two depend on the particular metric. By the commutativity of Δ_d and $\Pi^{p,q}$ and the fact that Δ_d is real, the harmonic forms satisfy

$$(*)\qquad \begin{cases}\mathcal{H}^r(M)=\bigoplus_{p+q=r}\mathcal{H}^{p,q}(M),\\ \mathcal{H}^{p,q}(M)=\overline{\mathcal{H}^{q,p}(M)}.\end{cases}$$

On the other hand, for η a closed form of pure type (p,q),

$$\eta=\mathcal{H}(\eta)+dd^*G(\eta),$$

where the harmonic part $\mathcal{H}(\eta)$ also has pure type (p,q). Thus

$$H^{p,q}(M)\cong\mathcal{H}^{p,q}(M).$$

Combining this with $(*)$ and the Hodge theorem

$$H^*_{\mathrm{DR}}(M)\cong\mathcal{H}^*(M)$$

for the Laplacian Δ_d, we obtain the famous

Hodge Decomposition. *For a compact Kähler manifold* M, *the complex cohomology satisfies*

$$\begin{cases}H^r(M,\mathbb{C})\cong\bigoplus_{p+q=r}H^{p,q}(M),\\ H^{p,q}(M)=\overline{H^{q,p}(M)}.\end{cases}$$

Since $\Delta_d=2\Delta_{\bar\partial}$, we have $\mathcal{H}_d^{p,q}(M)=\mathcal{H}_{\bar\partial}^{p,q}(M)$ and consequently

$$H^{p,q}(M)\cong H_{\bar\partial}^{p,q}(M)\cong H^q(M,\Omega^p).$$

In particular, taking $q=0$,

$$H^{p,0}(M) = H^0(M,\Omega^p)$$

is the space of holomorphic p-forms. *The holomorphic forms are therefore harmonic for any Kähler metric on a compact manifold.*

We note also that

The Betti numbers $b_{2q+1}(M)$ *of odd degree are even.*

Proof. If we define the *Hodge numbers* by

$$h^{p,q}(M) = \dim H^{p,q}(M),$$

then the Hodge decomposition gives

$$b_r(M) = \sum_{p+q=r} h^{p,q}(M),$$
$$h^{p,q}(M) = h^{q,p}(M).$$

Taking $r=2q+1$, we find

$$b_{2q+1}(M) = 2\left[\sum_{p=0}^{q} h^{p,2q+1-p}(M)\right].$$

We can put the cohomology groups of a compact Kähler manifold diagrammatically in the *Hodge diamond* (Figure 6), so that the kth cohomology group of M can be read off as the sum of the groups in the kth horizontal row. The star operator gives a symmetry about the center of the diamond; conjugation gives a symmetry about the center vertical line.

As an immediate application of the Hodge decomposition, we have the

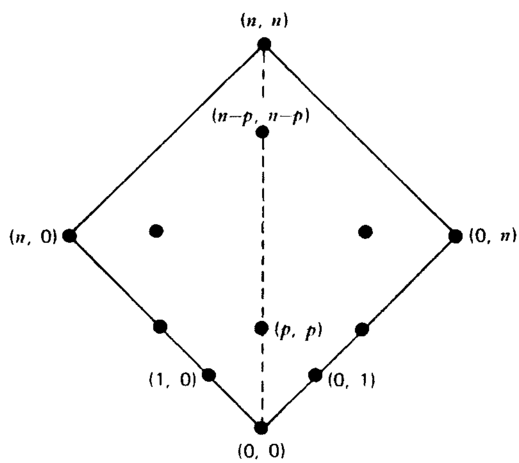


Figure 6

Corollary.

$$H^q(\mathbb{P}^n,\Omega^p) = H_{\bar\partial}^{p,q}(\mathbb{P}^n) = \begin{cases} 0, & \text{if } p \neq q, \\ \mathbb{C}, & \text{if } p = q. \end{cases}$$

Proof. This is clear: since $H^{2k+1}(\mathbb{P}^n,\mathbb{Z})=0$, we have $H_{\bar\partial}^{p,q}(\mathbb{P}^n)=0$ for $p+q$ odd; since $H^{2k}(\mathbb{P}^n,\mathbb{Z})=\mathbb{Z}$, we have for $p\neq k$,

$$\begin{aligned} 1 &= b_{2k}(\mathbb{P}^n) \geqslant h^{p,2k-p}(\mathbb{P}^n) + h^{2k-p,p}(\mathbb{P}^n) \\ &= 2\cdot h^{p,2k-p} \\ &\Rightarrow h^{p,2k-p}(\mathbb{P}^n) = 0 \end{aligned}$$

and hence

$$H_{\bar\partial}^{p,p}(\mathbb{P}^n) \cong H_{DR}^{2p}(\mathbb{P}^n) \cong \mathbb{C}. \qquad \text{Q.E.D.}$$

Note in particular that

There are no nonzero global holomorphic forms on $\mathbb{P}^n$.

The Lefschetz Decomposition

Another important application of the Hodge identities is the Lefschetz decomposition of the cohomology of a compact Kähler manifold. To put this in proper perspective, we must first digress for a moment and discuss representations of sl_2.

Representations of sl_2. sl_2 is the Lie algebra of the group SL_2; it is realized as the vector space of 2×2 complex matrices with trace 0, and with the bracket

$$[A,B] = AB - BA.$$

We take as standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the relations

$$[X,Y] = H, \qquad [H,X] = 2X, \qquad [H,Y] = -2Y.$$

Now, let V be a finite-dimensional complex vector space, $gl(V)$ its algebra of endomorphisms. We want to study Lie algebra maps

$$\rho\colon\ sl_2 \to gl(V),$$

i.e., linear maps ρ such that

$$\rho([A,B]) = \rho(A)\rho(B) - \rho(B)\rho(A).$$

Such a map is called a *representation of* sl_2 *in* V; V is called an sl_2*-module.*

A subspace of V fixed under $\rho(sl_2)$ is called a submodule; V (or ρ) is called *irreducible* if V has no nontrivial submodules. By a fundamental result, which we won't prove here, every submodule W of an sl_2-module V has a complementary submodule $W^\perp$; thus every sl_2-module is the direct sum of irreducible sl_2-modules, and to study representations of sl_2 we need only look at irreducible ones.

Suppose then that V is an irreducible sl_2-module. The key to analyzing the structure of V is to look at the eigenspaces for $\rho(H)$ (from now on, we will omit the ρ's). These are called *weight spaces*. First of all, note that if $v \in V$ is an eigenvector of H with eigenvalue λ, then Xv and Yv are also eigenvectors of H, with eigenvalues $\lambda+2$ and $\lambda-2$, respectively: this follows from

$$\begin{aligned} H(Xv) &= XHv + [H,X]v \\ &= X\lambda v + 2Xv \\ &= (\lambda+2)Xv, \end{aligned}$$

and similarly for Yv. Since H can have only a finite number of eigenvalues, we see from this that X and Y are nilpotent. We say that $v \in V$ is *primitive* if v is an eigenvector for H and $Xv=0$; clearly primitive elements exist.

Proposition. *If* $v \in V$ *is primitive, then* V *is generated as a vector space by*

$$v, Yv, Y^2v, \ldots .$$

Proof. Since V is irreducible, we need only show that the linear span V' of $\{Y^iv\}$ is fixed under sl_2. Clearly $HV' \subset V'$ and $YV' \subset V'$. We show $XV' \subset V'$ by an induction: $Xv=0$ trivially lies in V', and in general

$$XY^nv = YXY^{n-1}v + HY^{n-1}v;$$

so

$$XY^{n-1}v \in V' \Rightarrow XY^nv \in V'. \qquad \text{Q.E.D.}$$

Note that the elements $\{Y^nv\}_n$ that are nonzero are linearly independent, since they are all eigenvectors for H with different eigenvalues. Thus we have the picture of V: $V = \oplus V_\lambda$, where each V_λ is one-dimensional,

$$H(V_\lambda) = V_\lambda, \qquad X(V_\lambda) = V_{\lambda+2}, \qquad Y(V_\lambda) = V_{\lambda-2}.$$

Proposition. *All eigenvalues for* H *are integers, and we can write*

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_{-n}.$$

Proof. Let v be primitive, and suppose

$$Y^nv \neq 0, \qquad Y^{n+1}v = 0,$$

and $Hv=\lambda v$. Then

$$\begin{aligned} Xv &= 0, \\ XYv &= YXv + Hv = \lambda v, \\ XY^2v &= YXYv + HYv \\ &= Y\lambda v + (\lambda-2)Yv = (\lambda+(\lambda-2))Yv, \end{aligned}$$

and in general $XY^m v = YXY^{m-1}v + HY^{m-1}v$, so we have

$$\begin{aligned} XY^m v &= (\lambda+(\lambda-2)+(\lambda-4)+\cdots+(\lambda-2(m-1)))Y^{m-1}v \\ &= (m\lambda - m^2+m)Y^{m-1}v, \end{aligned}$$

and since $Y^n v \neq 0$, $Y^{n+1}v=0$,

$$(n+1)\lambda-(n+1)^2+n+1=0 \Rightarrow \lambda = n. \qquad \text{Q.E.D.}$$

In summary, *the irreducible* sl_2 *modules are indexed by nonnegative integers* n; *for each such* n *the corresponding* sl_2*-module* V(n) *has dimension* n+1. *Explicitly,*

$$V(n) \cong \mathrm{Sym}^n(\mathbb{C}^2)$$

is the n*th symmetric power of the vector space* $\mathbb{C}^2$. *The eigenvalues of* H *acting on* V(n) *are* $-n, -n+2, \ldots, n-2, n$, *each appearing with multiplicity* 1.

For any sl_2-module V, not necessarily irreducible, we define the *Lefschetz decomposition of* V as follows: let $PV = \operatorname{Ker}\rho(X)$; then

$$V = PV \oplus YPV \oplus Y^2PV \oplus \cdots,$$

and this decomposition is compatible with the decomposition of V into eigenspaces V_m for H. We also see that the maps

$$V_m \underset{X^m}{\overset{Y^m}{\rightleftarrows}} V_{-m}$$

are isomorphisms. Finally, in general,

$$(\operatorname{Ker} X) \cap V_k = \operatorname{Ker}\left(Y^{k+1}: V_k \to V_{-k-2}\right).$$

We return now to our compact complex manifold M with Kähler metric $ds^2 = \Sigma \varphi_i \otimes \bar{\varphi}_i$. First, we want to compute the commutator $[L,\Lambda]$ of the operators L and Λ; this may be done on $\mathbb{C}^n$ using the operators e_k, $\bar{e}_k$, i_k, and $\bar{i}_k$ defined earlier. Recall that

$$L = \frac{\sqrt{-1}}{2}\sum e_k\bar{e}_k \quad \text{and} \quad \Lambda = -\frac{\sqrt{-1}}{2}\sum \bar{i}_k i_k;$$

we have then

$$\begin{aligned} [L,\Lambda] &= \frac{1}{4}\left(\sum_{k,l} e_k\bar{e}_k\bar{i}_l i_l - \sum_{k,l}\bar{i}_l i_l e_k\bar{e}_k\right) \\ &= \frac{1}{4}\sum_{k\neq l}\left(e_k\bar{e}_k\bar{i}_l i_l - \bar{i}_l i_l e_k\bar{e}_k\right) \\ &\quad + \frac{1}{4}\sum_k\left(e_k\bar{e}_k\bar{i}_k i_k - \bar{i}_k i_k e_k\bar{e}_k\right). \end{aligned}$$

By our commutation relations, every term in the first sum is zero; for the second, we have

$$e_k\bar{e}_k\bar{i}_k i_k = 2e_k i_k - e_k\bar{i}_k\bar{e}_k i_k,$$
$$\bar{i}_k i_k e_k\bar{e}_k = 2\bar{i}_k\bar{e}_k - \bar{i}_k e_k i_k\bar{e}_k,$$

and, since $e_k\bar{i}_k\bar{e}_k i_k = \bar{i}_k e_k i_k\bar{e}_k$, this yields

$$\begin{aligned}[L,\Lambda] &= \frac{1}{2}\sum_k \left(e_k i_k - \bar{i}_k\bar{e}_k\right)\\ &= \frac{1}{2}\sum_k \left(2 - i_k e_k - \bar{i}_k\bar{e}_k\right)\\ &= n - \frac{1}{2}\sum \left(i_k e_k + \bar{i}_k\bar{e}_k\right).\end{aligned}$$

To evaluate this, note that $i_k e_k(dz_J \wedge d\bar{z}_K)$ is zero if $k \in J$, and $2dz_J \wedge d\bar{z}_K$ otherwise; $\bar{i}_k\bar{e}_k(dz_J \wedge d\bar{z}_K)$ is zero if $k \in K$ and $2dz_J \wedge d\bar{z}_K$ if not. Thus

$$\begin{aligned}\sum_k \left(i_k e_k + \bar{i}_k\bar{e}_k\right)(dz_J \wedge dz_K) &= 2\sum_{k \notin J} dz_J \wedge d\bar{z}_K + 2\sum_{k \notin K} dz_J \wedge d\bar{z}_K\\ &= (2(n - {}^{\#}J) + 2(n - {}^{\#}K))(dz_J \wedge d\bar{z}_K).\end{aligned}$$

and so on $A_c^{p,q}(\mathbb{C}^n)$

$$[L,\Lambda] = p + q - n.$$

Since L and Λ are both algebraic operators, this identity will hold on any Kähler manifold.

Now set

$$h = \sum_{p=0}^{2n} (n-p)\Pi^p;$$

since $L: A^p(M) \to A^{p+2}(M)$ and $\Lambda: A^p(M) \to A^{p-2}(M)$, we obtain

$$\begin{aligned}&[\Lambda, L] = h,\\ (*)\qquad &[h, L] = -2L,\\ &[h, \Lambda] = 2\Lambda.\end{aligned}$$

The operators L, Λ, and h all commute with Δ_d, and so act on the harmonic space $\mathcal{H}_d^*(M) \cong H^*(M)$ with relations $(*)$. We may therefore give a representation of sl_2 on $H^*(M)$ by sending

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \Lambda,$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to L,$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to h;$$

the eigenspace for h with eigenvalue $(n-p)$ will be $H^p(M)$. Applying our results on finite-dimensional representations of sl_2 to this representation yields the

Hard Lefschetz Theorem. *The map*

$$L^k:\ H^{n-k}(M) \longrightarrow H^{n+k}(M),$$

is an isomorphism; and if we define the primitive *cohomology*

$$\begin{aligned} P^{n-k}(M) &= \operatorname{Ker} L^{k+1}:\ H^{n-k}(M) \to H^{n+k+2}(M) \\ &= (\operatorname{Ker}\Lambda) \cap H^{n-k}(M), \end{aligned}$$

then we have

$$H^m(M) = \bigoplus_k L^k P^{m-2k}(M),$$

called the Lefschetz decomposition.

Note that the Lefschetz decomposition is compatible with the Hodge decomposition; i.e., if we set

$$P^{p,q}(M) = (\operatorname{Ker}\Lambda) \cap H^{p,q}(M),$$

then

$$P^l(M) = \bigoplus_{p+q=l} P^{p,q}(M).$$

We can give the following geometric interpretation of the Lefschetz theory in case the manifold M is embedded in projective space $\mathbb{P}^N$ with the induced metric. We have seen that the form

$$\omega = \frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log \|Z\|^2$$

is closed and not exact in $\mathbb{P}^N$. Since $H^2(\mathbb{P}^N)$ is one-dimensional, it follows that $[\omega] \in H^2_{\mathrm{DR}}(\mathbb{P}^N)$ is Poincaré dual to some nonzero multiple of the homology class of a hyperplane $H \subset \mathbb{P}^N$. In fact, $[\omega]$ is Poincaré dual to (H), as the reader may verify by integrating ω over a line $l \cong \mathbb{P}^1$ to obtain

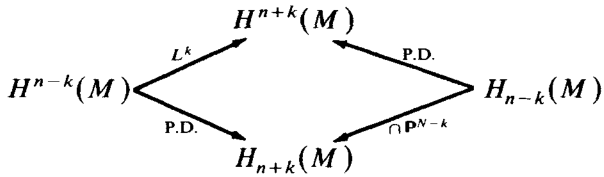
$$\int_l \omega = 1 = {}^{\#}(H \cdot l).$$

We see from this that for $M \subset \mathbb{P}^N$ a submanifold, the associated $(1,1)$ form $\omega|_M$ of the induced metric is Poincaré dual to the homology class (V) of the analytic subvariety $V = M \cap H \subset M$. The Poincaré-dualized version of the hard Lefschetz theorem says that the operation of intersection with an $(N-k)$-plane $\mathbb{P}^{N-k} \subset \mathbb{P}^N$ gives an isomorphism

$$H_{n+k}(M) \xrightarrow{\ \cap\,\mathbb{P}^{N-k}\ } H_{n-k}(M).$$

Note that in this interpretation, the primitive cohomology $P^{n-k}(M)$ of

M corresponds via the isomorphisms



to the subgroup of $(n-k)$-cycles that do not intersect a hyperplane, i.e., the image of the map

$$H_{n-k}(M-V) \to H_{n-k}(M).$$

Such cycles are called *finite cycles* since $M-V$ is the "finite part" $M \cap \mathbb{C}^N$ of M; their importance will be more apparent when we prove the Lefschetz theorem on hyperplane sections.

As another application of the Hodge and Lefschetz decompositions, we will now describe the Hodge-Riemann bilinear relations. We define a bilinear form

$$Q: H^{n-k}(M) \otimes H^{n-k}(M) \to \mathbb{C}$$

by setting

$$Q(\xi,\eta) = \int_M \xi \wedge \eta \wedge \omega^k.$$

Note that since ω is real, Q defines a real bilinear form on $H^{n-k}(M,\mathbb{R})$. By consideration of type, we see that

$$Q(H^{p,q}, H^{p',q'}) = 0 \qquad \text{unless}\, p = q', q = p'.$$

The *Hodge-Riemann bilinear relations* assert that for $\xi \in P^{p,q}(M)$ a primitive class and $k = p+q$,

$$\sqrt{-1}^{\,p-q}(-1)^{(n-k)(n-k-1)/2} Q(\xi,\bar{\xi}) > 0.$$

In the case $p+q$ even, this is the same as saying that on the real vector space

$$(P^{p,q} \oplus P^{q,p}) \cap H^{p+q}(M,\mathbb{R}) = \{\xi + \bar{\xi}, \xi \in P^{p+q}(M)\} \subset H^{p+q}(M),$$

the quadratic form $(\sqrt{-1}\,)^{p-q}(-1)^{(n-k)(n-k-1)/2} Q$ is positive definite; in the case $p+q$ odd, the bilinear relations tell us at least that Q is a nondegenerate skew-symmetric form on $P^{p+q}(M)$. In either case, since we have the Lefschetz decomposition

$$H^m = \oplus L^k P^{m-2k}$$

and $Q(L^k\xi, L^k\eta) = Q(\xi,\eta)$; the bilinear relations tell us that Q is nondegenerate on $H^{n-k}(M)$.

We will not prove the bilinear relations in full generality, but will verify them in some cases including all those to be used in our applications to geometry. (The general proof may be based on the following observations: In the full exterior algebra

$$V = \Lambda^*\mathbb{C}^n \otimes \Lambda^* \overline{\mathbb{C}^n}$$

corresponding to the differential forms at a point $x \in M$, there is an sl_2-action given by $\{L, \Lambda, h\}$ as above. Decomposing V into primitive spaces P^kV is the same as decomposing V under the action of the unitary group U_n, and thus by Schur's lemma any U_n-invariant quadratic form on P^kV is necessarily definite. The primitive harmonic forms on M are those which lie in P^kV (fixed k) for each $x \in M$, and this yields a proof. The result that decomposing V under sl_2 together with $\pi^{(p,q)}$ yields the same irreducible factors as under the action of U_n is proved in Herman Weyl's book *The Classical Groups*—it implies that, in general, there are no further Hodge identities.)

First, let M be a compact Riemann surface. By the Hodge decomposition,

$$\begin{aligned} H^1(M,\mathbb{C}) &= H^{1,0}(M) \oplus H^{0,1}(M) \\ &\cong H^0(M,\Omega^1) \oplus \overline{H^0(M,\Omega^1)}\,. \end{aligned}$$

The number of independent holomorphic 1-forms on M (classically called *differentials of the first kind*) is thus equal to $b_1(M)/2$; this in fact was one of the first links established between the topology of a complex manifold and its analytic structure. To verify the bilinear relations for M let $\xi = h(z)\,dz \in H^{1,0}(M)$; we have $(\sqrt{-1}\,)^{p-q}(-1)^{(n-k)(n-k-1)/2} = \sqrt{-1}\,$, and

$$\begin{aligned} Q(\xi,\bar{\xi}) &= \sqrt{-1}\int_M |h(z)|^2 dz \wedge d\bar{z} \\ &> 0. \end{aligned}$$

In general, for M of any dimension, $H^{p,0}(M)$ and $H^{0,p}(M)$ are primitive by consideration of type, and the same calculation works to verify the bilinear relations for them. In fact, it was in effect by deducing the Hodge-Riemann bilinear relations for holomorphic q-forms that we first proved that the holomorphic forms inject into the cohomology of a compact Kähler manifold.

Now let $\dim M = 2$; it remains only to verify the bilinear relations for $P^{1,1}$. Let ξ be a real, primitive harmonic (1,1)-form; in terms of a local unitary coframe φ_1, φ_2 we write

$$\xi = \sum \xi_{ij}\varphi_i \wedge \bar{\varphi}_j.$$

Since ξ is real, $\xi_{ij}=-\bar{\xi}_{ji}$; writing

$$\omega = \frac{\sqrt{-1}}{2}(\varphi_1\wedge\bar{\varphi}_1+\varphi_2\wedge\bar{\varphi}_2),$$

$$\xi\wedge\omega = \frac{\sqrt{-1}}{2}(\xi_{11}+\xi_{22})\varphi_1\wedge\bar{\varphi}_1\wedge\varphi_2\wedge\bar{\varphi}_2,$$

we see that ξ primitive implies $\xi_{11}+\xi_{22}=0$. For the bilinear relation

$$\begin{aligned}(\sqrt{-1})^{p-q}(-1)^{(n-k)(n-k-1)/2}Q(\xi,\bar{\xi}) &= -\int_M \xi\wedge\bar{\xi}\\ &= -\int_M\left(-\xi_{11}\bar{\xi}_{22}+2|\xi_{12}|^2-\xi_{22}\bar{\xi}_{11}\right)\\ &\qquad\times\varphi_1\wedge\bar{\varphi}_1\wedge\varphi_2\wedge\bar{\varphi}_2\\ &= -\left(\frac{2}{\sqrt{-1}}\right)^2\int_M(2|\xi_{11}|^2+2|\xi_{12}|^2)\Phi\\ &> 0.\end{aligned}$$

Recall that on a general oriented compact real manifold X of dimension $2k$, we have a bilinear form on $H^k(X,\mathbb{R})=H^k_{\mathrm{DR}}(X)$ defined by

$$\underline{Q}(\eta,\xi)=\int_M\eta\wedge\xi;$$

by Poincaré duality $\underline{Q}$ is nondegenerate. If k is even, $\underline{Q}$ is symmetric, and we can associate to X as a topological invariant the *signature* of $\underline{Q}$, defined as the number of positive eigenvalues minus the number of negative eigenvalues in a matrix representation of $\underline{Q}$. The signature of $\underline{Q}$ is called the *index* $I(X)$ of the manifold X. Of course, if M is a compact Kähler manifold of dimension $2n$, then $Q=\underline{Q}$ on $H^{2n}(M,\mathbb{R})$ and we may use the bilinear relations to compute the index of M:

$$\begin{aligned}H^{2n}(M) &= \oplus L^kP^{2(n-k)}(M)\\ &= \bigoplus_{\substack{p+q\equiv 0(2)\\ \leqslant 2n}} L^{n-(p+q)/2}P^{p,q}(M).\end{aligned}$$

We know that for $p+q\equiv 0(2)$, $(\sqrt{-1})^{p-q}(-1)^{(p+q)(p+q-1)/2}Q>0$ on the real space $(P^{p,q}\oplus P^{q,p})\cap \mathrm{H}^{p+q}(M,\mathbb{R})$; since $Q(L\eta,L\xi)=Q(\eta,\xi)$, we have

$$\begin{aligned}I(M) &= \sum_{\substack{p+q\equiv 0(2)\\ \leqslant 2n}}(\sqrt{-1})^{p-q}(-1)^{(p+q)(p+q-1)/2}\dim P^{p,q}(M)\\ &= \sum_{\substack{p+q\equiv 0(2)\\ \leqslant 2n}}(-1)^p\dim P^{p,q}(M).\end{aligned}$$

Now we have by the Lefschetz decomposition

$$h^{p,p+j} = \sum_{i=0}^{p} \dim P^{i,i+j};$$

thus, along a vertical line in the Hodge diamond,

$$\sum_{i=0}^{p-1} (-1)^i \dim P^{i,i+j} = (-1)^p h^{p,p+j} + 2\sum (-1)^i h^{i,i+j}$$

and we can write, finally,

$$I(M) = \sum_{p+q=2n} (-1)^p h^{p,q} + 2 \sum_{\substack{p+q \equiv 0(2) \\ <2n}} (-1)^p h^{p,q},$$

or

$$I(M) = \sum_{p+q \equiv 0(2)} (-1)^p h^{p,q},$$

the last equality holding by virtue of the duality $h^{p,q} = h^{n-p,n-q}$. Note in particular that on a Kähler surface M the cup product Q on $H^{1,1}(M)$ has exactly one positive eigenvalue; this fact is frequently called the *index theorem* for surfaces.

Note, finally, one distinction between the Hodge and Lefschetz theorems of this section: the Lefschetz theorems are essentially topological, while the Hodge decomposition reflects the analytic structure of the particular manifold M. For instance, if we take a real manifold and give it two different Kähler complex structures, the Hodge decomposition of H^* may vary—the rank of the groups $(H^{p,q}(M) \oplus H^{q,p}(M)) \cap H^{p+q}(M, \mathbb{Z})$ may even jump—but the Lefschetz isomorphism and decomposition remain the same.

REFERENCES

We give here a few references that should be helpful to the reader in supplementing the material in this chapter. This list is not meant to be a bibliography but is a small sampling of sources having a notation and point of view similar to that taken here. Moreover, many of these references have extensive bibliographies and so may be used as a guide to the literature.

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Section 4

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Note: Our proof of the Hodge theorem is a variant of one given by Joe Kohn in a course at Princeton University in 1961–62.

1
COMPLEX ALGEBRAIC VARIETIES

An *algebraic variety* is defined to be the set of complex zeros of homogeneous polynomials in projective space and may be viewed a priori as an analytic subvariety of $\mathbb{P}^n$. In case the variety is smooth, we may consider the associated abstract compact complex manifold, whose properties will be intrinsic to—i.e., not depending on the particular embedding of—the variety. Broadly speaking, we will approach algebraic geometry as the study of the interplay between the intrinsic and extrinsic or projective properties of algebraic varieties.

In Section 1 we introduce the notion of divisors and line bundles; the material here is central for all that follows. Since a compact complex manifold admits no global holomorphic functions, we might rather expect its structure to be reflected in the global meromorphic functions and related linear systems of divisors on the manifold; this notion is a basic one in classical algebraic geometry. Associated to a divisor is a holomorphic line bundle, to a meromorphic function a line bundle together with a holomorphic section, and to a line bundle its Chern class. The subsequent formalism, developed by Kodaira and Spencer and others in the early 1950s, gives an extremely useful technique for dealing with codimension-one subvarieties (points on a curve, curves on a surface, etc.) on an algebraic variety.

The basic question of constructing meromorphic functions with prescribed properties—e.g., the principal parts on a Riemann surface—is a problem admitting local solutions where the obstruction to patching these together globally may be measured by a sheaf cohomology group. The *Kodaira vanishing theorem* provides the most useful condition under which these higher groups are zero. It is a remarkable result, one which is proved by potential theory and differential geometry, but which in the end turns out to be equivalent to the Lefschetz theorem concerning the topological position of a hyperplane section of a complex algebraic variety. Explaining these matters occupies Section 2.

In Section 3 we began the transition

$$\left\{\begin{array}{c}\text{abstract compact}\\ \text{complex manifold}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{algebraic variety}\\ \text{in projective space}\end{array}\right\}$$

The intermediate step is an analytic variety in projective space; the *Chow theorem* asserts that this must be an algebraic variety. The essential philosophical point here is illustrated by the identity of the two objects "global meromorphic function on the Riemann sphere" and "rational function of one complex variable." The practical consequence is that we may work either locally complex analytically or globally algebraically with the same end result. Our approach at this stage is analytic, as this ties in more readily with the topological and metric properties of an algebraic variety, but the understanding that in the end we are talking about solutions of polynomial equations is fundamental.

In Section 4 we state and prove Kodaria's characterization of those compact complex manifolds which are derived from algebraic varieties, thus providing the essential link between the intrinsic and extrinsic properties of a variety. This embedding theorem and Chow's theorem are existence theorems—they do not by themselves provide a constructive method for finding the equations defining the image of a variety under a projective embedding—but together they form the philosophical cornerstone for our analytic treatment of algebraic geometry.

In the final section of this chapter we explain in some detail the Grassmannian, a variety whose points parametrize the linear subspaces of some fixed dimension in projective space and whose internal structure reflects the nongeneric intersections of a variable linear space with a fixed one. One reason for placing this discussion here is that the Grassmannian illustrates quite nicely the general structure theorems of this chapter. Another is that extensive use will be made in the following chapters of the Schubert calculus, a quantitative expression of the nongeneric incidence relations among linear spaces that is inherent in the structure of the Grassmannian.

1. DIVISORS AND LINE BUNDLES

Divisors

Let M be a complex manifold of dimension n, not necessarily compact. We recall from Section 1 of Chapter 0 some facts about analytic hypersurfaces in M:

Any analytic subvariety $V \subset M$ of dimension $n-1$ is an analytic hypersurface, i.e., for any point $p \in V \subset M$, V can be given in a neighborhood of p as the zeros of a single holomorphic function f. Moreover, any holomorphic function g defined at p and vanishing on V is divisible by f in a neighborhood of p. f is called a *local defining function* for V near p, and is unique up to multiplication by a function nonzero at p.

If V_1^* is a connected component of $V^* = V - V_s$, then $\overline{V_1^*}$ is an analytic subvariety in M. Thus V can be expressed uniquely as the union of irreducible analytic hypersurfaces

$$V = V_1 \cup \cdots \cup V_m,$$

where the V_i's are the closures of the connected components of V^*. In particular, V is irreducible if and only if V^* is connected.

Now we define:

DEFINITION. A *divisor* D on M is a locally finite formal linear combination

$$D = \sum a_i \cdot V_i$$

of irreducible analytic hypersurfaces of M.

"Locally finite" here means that for any $p \in M$, there exists a neighborhood of p meeting only a finite number of the V_i's appearing in D; of course, if M is compact, this just means the sum is finite. The set of divisors in M is naturally an additive group, denoted $\mathrm{Div}(M)$.

A divisor $D = \Sigma a_i V_i$ is called *effective* if $a_i \geqslant 0$ for all i; we write $D \geqslant 0$ for D effective. An analytic hypersurface V will usually be identified with the divisor ΣV_i where the V_i's are the irreducible components of V.

Let $V \subset M$ be an irreducible analytic hypersurface, $p \in V$ any point, and f a local defining function for V near p. For any holomorphic function g defined near p, we define the *order* $\mathrm{ord}_{V,p}(g)$ *of* g *along* V *at* p to be the largest integer a such that in the local ring $\mathcal{O}_{M,p}$,

$$g = f^a \cdot h.$$

By the result from p. 10 that relatively prime elements of $\mathcal{O}_{M,p}$ stay relatively prime in nearby local rings, we see that for g a holomorphic function on M, $\mathrm{ord}_{V,p}(g)$ *is independent of* p. Thus we can define the *order* $\mathrm{ord}_V(g)$ *of* g *along* V to be simply the order of g along V at any point $p \in V$. Note that for g, h any holomorphic functions, V any irreducible hypersurface,

$$\mathrm{ord}_V(gh) = \mathrm{ord}_V(g) + \mathrm{ord}_V(h).$$

Now let f be a meromorphic function on M, written locally as

$$f = \frac{g}{h}$$

with g,h holomorphic and relatively prime. For V an irreducible hypersurface, we define

$$\operatorname{ord}_V(f) = \operatorname{ord}_V(g) - \operatorname{ord}_V(h).$$

We usually say that f *has a zero of order* a *along* V if $\operatorname{ord}_V(f) = a > 0$, and that f has a *pole of order* a *along* V if $\operatorname{ord}_V(f) = -a < 0$.

We define the *divisor* (f) *of the meromorphic function* f by

$$(f) = \sum_V \operatorname{ord}_V(f) \cdot V.$$

If f is written locally as g/h, we take the *divisor of zeros* $(f)_0$ of f to be

$$(f)_0 = \sum_V \operatorname{ord}_V(g) \cdot V$$

and the *divisor of poles* $(f)_\infty$ to be

$$(f)_\infty = \sum_V \operatorname{ord}_V(h) \cdot V.$$

Clearly these are well-defined as long as we require g and h to be relatively prime, and

$$(f) = (f)_0 - (f)_\infty.$$

Divisors can also be described in sheaf-theoretic terms, as follows: Let $\mathcal{M}^*$ denote the multiplicative sheaf of meromorphic functions on M not identically 0, and $\mathcal{O}^*$ the subsheaf of nonzero holomorphic functions. Then *a divisor* D *on* M *is simply a global section of the quotient sheaf* $\mathcal{M}^*/\mathcal{O}^*$. On the one hand, a global section $\{f\}$ of $\mathcal{M}^*/\mathcal{O}^*$ is given by an open cover $\{U_\alpha\}$ of M and meromorphic functions $f_\alpha \not\equiv 0$ in U_α with

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta);$$

for any $V \subset M$, then,

$$\operatorname{ord}_V(f_\alpha) = \operatorname{ord}_V(f_\beta),$$

and we can associate to $\{f\}$ the divisor

$$D = \sum_V \operatorname{ord}_V(f_\alpha) \cdot V,$$

where for each V we choose α such that $V \cap U_\alpha \neq \varnothing$. On the other hand, given

$$D = \sum_{V_i} a_i V_i,$$

we can find an open cover $\{U_\alpha\}$ of M such that in each U_α, every V_i appearing in D has a local defining function $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. We can then set

$$f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathcal{M}^*(U_\alpha)$$

to obtain a global section of $\mathcal{M}^*/\mathcal{O}^*$. The f_α's are called *local defining functions* for D. It follows immediately from the definitions that the identification

$$H^0(M, \mathcal{M}^*/\mathcal{O}^*) = \text{Div}(M)$$

is in fact a homomorphism.

Given a holomorphic map $\pi: M \to N$ of complex manifolds, we define a map

$$\pi^*: \text{Div}(N) \to \text{Div}(M)$$

by associating to every divisor $D = (\{U_\alpha\}, \{f_\alpha\})$ on N the *pullback divisor* $\pi^* D = (\{\pi^{-1} U_\alpha\}, \{\pi^* f_\alpha\})$ on M; this is well-defined as long as $\pi(M) \not\subset D$. Note that for a divisor on N given by an analytic hypersurface $V \subset N$, the pullback divisor $\pi^* V$ on M lies over V but need not coincide with the analytic hypersurface $\pi^{-1}(V) \subset M$—multiplicities may occur.

We want to make one more remark before going on to consider line bundles. On a Riemann surface M, any point is an irreducible analytic hypersurface, and so clearly $\text{Div}(M)$ is always large. This is, in a sense, misleading: *a complex manifold* M *of dimension greater than one need not have any nonzero divisors on it at all.* If, however M is embedded in projective space $\mathbb{P}^N$, the intersections of M with hyperplanes in $\mathbb{P}^N$ generate a large number of divisors. In fact, among all compact complex manifolds those which are embeddable in projective space can be characterized by having "sufficiently many" divisors, in a sense that we shall make precise in later sections.

Line Bundles

All line bundles discussed in this section are taken to be holomorphic. Recall that for any holomorphic line bundle $L \xrightarrow{\pi} M$ on the complex manifold M, we can find an open cover $\{U_\alpha\}$ of M and trivializations

$$\varphi_\alpha: L_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}$$

of $L_{U_\alpha} = \pi^{-1}(U_\alpha)$. We define the transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ for L relative to the trivializations $\{\varphi_\alpha\}$ by

$$g_{\alpha\beta}(z) = (\varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z} \in \mathbb{C}^*.$$

The functions $g_{\alpha\beta}$ are clearly holomorphic, nonvanishing, and satisfy

$$(*) \qquad \begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} = 1, \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1; \end{cases}$$

conversely, given a collection of functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ satisfying these identities, we can construct a line bundle L with transition functions

$\{g_{\alpha\beta}\}$ by taking the union of $U_\alpha \times \mathbb{C}$ over all α and identifying $\{z\}\times\mathbb{C}$ in $U_\alpha\times\mathbb{C}$ and $U_\beta\times\mathbb{C}$ via multiplication by $g_{\alpha\beta}(z)$.

Now, given L as above, for any collection of nonzero holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ we can define alternate trivializations of L over $\{U_\alpha\}$ by

$$\varphi'_\alpha = f_\alpha \cdot \varphi_\alpha;$$

transition functions $g'_{\alpha\beta}$ for L relative to $\{\varphi'_\alpha\}$ will then be given by

$$(**) \qquad g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta}\cdot g_{\alpha\beta}.$$

On the other hand, any other trivialization of L over $\{U_\alpha\}$ can be obtained in this way, and so we see that collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying (**).

The description of line bundles by transition functions lends itself well to a sheaf-theoretic interpretation. First, the transition functions $\{g_{\alpha\beta}\in \mathcal{O}^*(U_\alpha\cap U_\beta)\}$ for a line bundle $L\to M$ represent a Čech 1-cochain on M with coefficients in $\mathcal{O}^*$; the relation (*) simply asserts that $\delta(\{g_{\alpha\beta}\})=0$, i.e., $\{g_{\alpha\beta}\}$ is a Čech cocycle. Moreover, by the last paragraph, two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle if and only if their difference $\{g_{\alpha\beta}\cdot g'^{-1}_{\alpha\beta}\}$ is a Čech coboundary; consequently *the set of line bundles on* M *is just* $H^1(M,\mathcal{O}^*)$.

We can give the set of line bundles on M the structure of a group, multiplication being given by tensor product and inverses by dual bundles. If L is given by data $\{g_{\alpha\beta}\}$, L' by $\{g'_{\alpha\beta}\}$, we have seen that

$$L\otimes L' \sim \{g_{\alpha\beta}g'_{\alpha\beta}\}, \qquad L^* \sim \{g^{-1}_{\alpha\beta}\},$$

and so the group structure on the set of line bundles is the same as the group structure on $H^1(M,\mathcal{O}^*)$. The group $H^1(M,\mathcal{O}^*)$ is called the *Picard group* of M, denoted $\operatorname{Pic}(M)$.

We now describe the basic correspondence between divisors and line bundles. Let D be a divisor on M, with local defining functions $f_\alpha \in \mathcal{M}^*(U_\alpha)$ over some open cover $\{U_\alpha\}$ of M. Then the functions

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta}$$

are holomorphic and nonzero in $U_\alpha\cap U_\beta$, and in $U_\alpha\cap U_\beta\cap U_\gamma$ we have

$$g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha} = \frac{f_\alpha}{f_\beta}\cdot\frac{f_\beta}{f_\gamma}\cdot\frac{f_\gamma}{f_\alpha} = 1.$$

The line bundle given by the transition functions $\{g_{\alpha\beta}=f_\alpha/f_\beta\}$ is called the *associated line bundle* of D, and written $[D]$. We check that it is well-defined: if $\{f'_\alpha\}$ are alternate local data for D, then $h_\alpha=f_\alpha/f'_\alpha\in\mathcal{O}^*(U_\alpha)$, and

$$g'_{\alpha\beta}=\frac{f'_\alpha}{f'_\beta}=g_{\alpha\beta}\cdot\frac{h_\beta}{h_\alpha}$$

for each α,β.

The correspondence $[\ \ ]$ has these immediate properties: First, if D and D' are two divisors given by local data $\{f_\alpha\}$ and $\{f'_\alpha\}$, respectively, then $D+D'$ is given by $\{f_\alpha\cdot f'_\alpha\}$; it follows that

$$[D+D']=[D]\otimes[D']$$

so the map

$$[\ \ ]:\ \mathrm{Div}(M)\to\mathrm{Pic}(M)$$

is a homomorphism. Second, if $D=(f)$ for some meromorphic function f on M, we may take as local data for D over any cover $\{U_\alpha\}$ the functions $f_\alpha=f|_{U_\alpha}$; then $f_\alpha/f_\beta=1$ and so $[D]$ is trivial. Conversely, if D is given by local data $\{f_\alpha\}$ and the line bundle $[D]$ is trivial, then there exist functions $h_\alpha\in\mathcal{O}^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta}=g_{\alpha\beta}=\frac{h_\alpha}{h_\beta};$$

$f=f_\alpha\cdot h_\alpha^{-1}=f_\beta\cdot h_\beta^{-1}$ is then a global meromorphic function on M with divisor D. Thus *the line bundle* [D] *associated to a divisor* D *on* M *is trivial if and only if* D *is the divisor of a meromorphic function.* We say that two divisors D,D' on M are *linearly equivalent* and write $D\sim D'$ if $D=D'+(f)$ for some $f\in\mathcal{M}^*(M)$, or equivalently if $[D]=[D']$.

Also, note that $[\ \ ]$ is functorial: that is, if $f:M\to N$ is a holomorphic map of complex manifolds, it is easy to check that for any $D\in\mathrm{Div}(N)$,

$$\pi^*([D])=[\pi^*(D)].$$

All these assertions are implicit in the following cohomological interpretation of the correspondence $[\ \ ]$. The exact sheaf sequence

$$0\to\mathcal{O}^*\xrightarrow{i}\mathcal{M}^*\xrightarrow{j}\mathcal{M}^*/\mathcal{O}^*\to 0$$

on M gives us, in part, the exact sequence

$$H^0(M,\mathcal{M}^*)\xrightarrow{j_*}H^0(M,\mathcal{M}^*/\mathcal{O}^*)\xrightarrow{\delta}H^1(M,\mathcal{O}^*)$$

of cohomology groups. The reader may easily verify that under the natural identifications

$$\mathrm{Div}(M)=H^0(M,\mathcal{M}^*/\mathcal{O}^*)\quad\text{and}\quad\mathrm{Pic}(M)=H^1(M,\mathcal{O}^*)$$

for any meromorphic function f on M,

$$j_*f = (f),$$

and for any divisor D on M,

$$\delta D = [D].$$

Indeed, we will generally violate the previous multiplicative notation and write $L+L'$ for the tensor product of two line bundles or mL for the mth tensor power $L^{\otimes m}$ of L.

We now wish to discuss holomorphic and meromorphic sections of line bundles. Let $L \to M$ be a holomorphic line bundle, with trivializations $\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$ over an open cover $\{U_\alpha\}$ of M and transition functions $\{g_{\alpha\beta}\}$ relative to $\{\varphi_\alpha\}$. As we have seen, the trivializations φ_α induce isomorphisms

$$\varphi_\alpha^*: \mathcal{O}(L)(U_\alpha) \longrightarrow \mathcal{O}(U_\alpha);$$

we see via the correspondence

$$s \in \mathcal{O}(L)(U) \to \{s_\alpha = \varphi_\alpha^*(s) \in \mathcal{O}(U \cap U_\alpha)\}$$

that a section of L over $U \subset M$ is given exactly by a collection of functions $s_\alpha \in \mathcal{O}(U \cap U_\alpha)$ satisfying

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta$$

in $U \cap U_\alpha \cap U_\beta$.

In the same way, a *meromorphic* section s of L over U—defined to be a section of the sheaf $\mathcal{O}(L) \otimes_{\mathcal{O}} \mathfrak{M}$—is given by a collection of meromorphic functions $s_\alpha \in \mathfrak{M}(U \cap U_\alpha)$ satisfying $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ in $U \cap U_\alpha \cap U_\beta$. Note that the quotient of two meromorphic sections $s, s' \not\equiv 0$ of L is a well-defined meromorphic function.

If s is a global meromorphic section of L, $s_\alpha / s_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, and so for any irreducible hypersurface $V \subset M$,

$$\operatorname{ord}_V(s_\alpha) = \operatorname{ord}_V(s_\beta).$$

Thus we can define the *order* of s along V by

$$\operatorname{ord}_V(s) = \operatorname{ord}_V(s_\alpha)$$

for any α such that $U_\alpha \cap V \neq \varnothing$; we take the *divisor* (s) *of the meromorphic section* s to be given by

$$(s) = \sum_V \operatorname{ord}_V(s) \cdot V.$$

With this convention s is holomorphic if and only if (s) is effective.

Now if $D \in \operatorname{Div}(M)$ is given by local data $f_\alpha \in \mathfrak{M}(U_\alpha)$, then the functions f_α clearly give a meromorphic section s_f of $[D]$ with $(s_f) = D$.

Conversely, if L is given by trivializations φ_α with transition functions $g_{\alpha\beta}$ and s is any global meromorphic section of L, we see that

$$\frac{s_\alpha}{s_\beta} = g_{\alpha\beta},$$

i.e., $L=[(s)]$. Thus if *D is any divisor such that* $[\mathrm{D}]=\mathrm{L}$, *there exists a meromorphic section* s *of* L *with* $(\mathrm{s})=\mathrm{D}$, *and for any meromorphic section* s *of* L, $\mathrm{L}=[(\mathrm{s})]$. In particular, we see that L is the line bundle associated to some divisor D on M if and only if it has a global meromorphic section not identically zero; it is the line bundle of an effective divisor if and only if it has a nontrivial global holomorphic section.

We can also view this correspondence as follows: Given a divisor

$$D = \sum a_i V_i$$

on M, let $\mathcal{L}(D)$ denote the space of meromorphic functions f on M such that

$$D + (f) \geqslant 0,$$

i.e., that are holomorphic on $M - \cup V_i$ with

$$\mathrm{ord}_{V_i}(f) \geqslant -a_i.$$

We denote by $|D| \subset \mathrm{Div}(M)$ the set of all effective divisors linearly equivalent to D; if $L=[D]$, we write $|L|$ for $|D|$. Let s_0 be a global meromorphic section of $[D]$ with $(s_0)=D$. Then for any global holomorphic section s of $[D]$, the quotient

$$f_s = \frac{s}{s_0}$$

is a meromorphic function on M with

$$(f_s) = (s) - (s_0) \geqslant -D,$$

i.e.,

$$f_s \in \mathcal{L}(D)$$

and

$$(s) = D + (f_s) \in |D|.$$

On the other hand, for any $f \in \mathcal{L}(D)$ the section $s = f \cdot s_0$ of $[D]$ is holomorphic. Thus *multiplication by* s_0 *gives an identification*

$$\mathcal{L}(D) \xrightarrow{\otimes s_0} H^0(M, \mathcal{O}([D])).$$

Now suppose M is compact. For every $D' \in |D|$, there exists $f \in \mathcal{L}(D)$ such that

$$D' = D + (f),$$

and conversely any two such functions f, f' differ by a nonzero constant. Thus we have the additional correspondence

$$|D| \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}\big(H^0(M, \mathcal{O}([D]))\big).$$

In general, the family of effective divisors on M corresponding to a linear subspace of $\mathbb{P}(H^0(M, \mathcal{O}(L)))$ for some $L \to M$ is called a *linear system* of divisors; a linear system is called *complete* if it is of the form $|D|$, i.e., if it contains every effective divisor linearly equivalent to any of its members. When we speak of the *dimension* of a linear system, we will refer to the dimension of the projective space parametrizing it; thus, when we write $\dim|D|$ for the dimension of the complete linear system associated to a divisor D, we have

$$\dim|D| = h^0(M, \mathcal{O}(D)) - 1.$$

A linear system of dimension 1 is called a *pencil*, of dimension 2 a *net*, and of dimension 3 a *web*.

We will mention here two special properties of linear systems. The first is elementary: if $E = \{D_\lambda\}_{\lambda \in \mathbb{P}^n}$ is a linear system, then for any $\lambda_0, \ldots, \lambda_n$ linearly independent in $\mathbb{P}^n$,

$$D_{\lambda_0} \cap \cdots \cap D_{\lambda_n} = \bigcap_{\lambda \in \mathbb{P}^n} D_\lambda.$$

The common intersection of the divisors in a linear system is called the *base locus* of the system; in particular, a divisor F in the base locus—that is, such that $D_\lambda - F \geqslant 0$ for all λ—is called a *fixed component* of E.

The second property is more remarkable; like the first, it is peculiar to linear systems and is not the case for general families of divisors, even general families of linearly equivalent divisors. This is

Bertini's Theorem. *The generic element of a linear system is smooth away from the base locus of the system.*

Proof. If the generic element of a linear system is singular away from the base locus of the system, then the same will be true for a generic pencil contained in the system; thus it suffices to prove Bertini for a pencil.

Suppose $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a pencil, given in a polydisc Δ contained in M by

$$D_\lambda = (f(z_1, \ldots, z_n) + \lambda \cdot g(z_1, \ldots, z_n) = 0)$$

and suppose P_λ is a singular point of the divisor D_λ ($\lambda \neq 0, \infty$) but not in the base locus B of the pencil. We have then

$$f(P_\lambda) + \lambda g(P_\lambda) = 0$$

and

$$\frac{\partial f}{\partial z_i}(P_\lambda) + \lambda \frac{\partial g}{\partial z_i}(P_\lambda) = 0, \qquad i = 1, \ldots, n.$$

Since P_λ is not a base point of $\{D_\lambda\}$, f and g cannot both vanish at P_λ and so neither one can; thus

$$\lambda = -\frac{f(P_\lambda)}{g(P_\lambda)}$$

and

$$\frac{\partial f}{\partial z_i}(P_\lambda) - \frac{f(P_\lambda)}{g(P_\lambda)} \cdot \frac{\partial g}{\partial z_i}(P_\lambda) = 0.$$

Then

$$\frac{\partial}{\partial z_i}\left(\frac{f}{g}\right)(P_\lambda) = \frac{(\partial f/\partial z_i)(P_\lambda) - [f(P_\lambda)/g(P_\lambda)]\cdot(\partial g/\partial z_i)(P_\lambda)}{g(P_\lambda)} = 0.$$

Now the locus V of singular points of the divisors D_λ, being locally the image in Δ of the variety $S \subset \Delta \times \mathbb{P}^1_\lambda$ cut out by the equations $\{f+\lambda g=0, \partial f/\partial z_i + \lambda \partial g/\partial z_i = 0\}$, is an analytic subvariety of Δ. But by the calculation above *the ratio* f/g *is constant on every connected component of* V − B and so V can meet only finitely many divisors D_λ away from the base locus of $\{D_\lambda\}$. Q.E.D.

The essential point here is that a pencil $\{D_\lambda\}_{\lambda \in \mathbb{P}^1}$ with base locus B gives a holomorphic mapping

$$M - B \to \mathbb{P}^1$$

since by linearity every $p \in M - B$ is on a *unique* D_λ. The Bertini theorem is a refinement of Sard's theorem for this mapping.

We make one final remark about sections of line bundles, which will be used repeatedly throughout the book. Recall that if $D = \sum a_i V_i$ is any effective divisor on the complex manifold M, $s_0 \in H^0(M, \mathcal{O}([D]))$ a section of $[D]$ with divisor D, then tensoring with s_0 gives an identification between the meromorphic functions on M with poles of order $\leqslant a_i$ on V_i and holomorphic sections of $[D]$. More generally, if E is any holomorphic vector bundle on M, $\mathcal{E}$ its sheaf of holomorphic sections, we write $\mathcal{E}(D)$ for the sheaf of meromorphic sections of E with poles of order $\leqslant a_i$ on V_i, $\mathcal{E}(-D)$ for the sheaf of sections of E vanishing to order $\geqslant a_i$ along V_i. Again, *tensoring with* s_0 *or* s_0^{-1} *gives identifications*

$$(*) \qquad \begin{aligned} \mathcal{E}(D) &\xrightarrow{\otimes s_0} \mathcal{O}(E \otimes [D]), \\ \mathcal{E}(-D) &\xrightarrow{\otimes s_0^{-1}} \mathcal{O}(E \otimes [-D]). \end{aligned}$$

Thus in particular if D is a smooth analytic hypersurface, the sequence of sheaves

$$O \to \mathcal{O}_M(E \otimes [-D]) \xrightarrow{\otimes s_0} \mathcal{O}_M(E) \xrightarrow{r} \mathcal{O}_D(E|_D) \to 0,$$

where r is the restriction map, is exact. Henceforth, *we shall make the identification* $(*)$ *implicitly and write* $\mathcal{O}(\mathrm{D})$ *for* $\mathcal{O}([\mathrm{D}])$.

Chern Classes of Line Bundles

Let M now be a compact complex manifold of dimension n. The exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

gives a boundary map in cohomology

$$H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}).$$

For a line bundle $L \in \mathrm{Pic}(M) = H^1(M, \mathcal{O}^*)$, we define the *first Chern class* $c_1(L)$ of L (or simply *Chern class*) to be $\delta(L) \in H^2(M, \mathbb{Z})$; for D a divisor on M, we define the Chern class of D to be $c_1([D])$. By a slight abuse of language, we will sometimes write $c_1(L) \in H^2_{\mathrm{DR}}(M)$ for the image of $c_1(L)$ under the natural map $H^2(M, \mathbb{Z}) \to H^2_{\mathrm{DR}}(M)$.

As an immediate consequence of the definition, note that

$$c_1(L \otimes L') = c_1(L) + c_1(L')$$

and

$$c_1(L^*) = -c_1(L).$$

Also, if $f: M \to N$ is a holomorphic map of complex manifolds, the diagram

$$\begin{array}{ccc} H^1(M, \mathcal{O}^*) & \longrightarrow & H^2(M, \mathbb{Z}) \\ \uparrow f^* & & \uparrow f^* \\ H^1(N, \mathcal{O}^*) & \longrightarrow & H^2(N, \mathbb{Z}) \end{array}$$

commutes, so that for $L \to N$ any line bundle,

$$c_1(f^*L) = f^*c_1(L).$$

We will be concerned in this subsection with giving two alternate interpretations of the Chern class; first, however, we want to make one observation:

Let $\mathcal{Q}$ and $\mathcal{Q}^*$ denote the sheaves of C^∞ functions and nonzero C^∞ functions, respectively. The transition functions of a C^∞ complex line bundle L then give a Čech cocycle

$$\{g_{\alpha\beta}\} \in C^1(M, \mathcal{Q}^*),$$

and by the same argument as for holomorphic bundles, the bundle L is determined, up to C^∞ isomorphism, by the cohomology class $[\{g_{\alpha\beta}\}] \in H^1(M, \mathcal{Q}^*)$. Now we have an exact sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{Q} \xrightarrow{\exp} \mathcal{Q}^* \to 0$$

and since the long exact sequence in Čech cohomology is functorial, the inclusion maps $\mathcal{O} \to \mathcal{Q}$ and $\mathcal{O}^* \to \mathcal{Q}^*$ give a commutative diagram

$$\begin{array}{ccccc} H^1(M, \mathcal{Q}) & \longrightarrow & H^1(M, \mathcal{Q}^*) & \xrightarrow{\delta'} & H^2(M, \mathbb{Z}) \\ \uparrow & & \uparrow & & \| \\ H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{O}^*) & \xrightarrow{\delta} & H^2(M, \mathbb{Z}) \end{array}$$

with both rows exact. Thus we can define the Chern class $c_1(L)$ of a C^∞ line bundle to be $\delta'(L)$, and this definition agrees with the one above for holomorphic bundles. But in the upper row we have $H^1(M, \mathcal{Q}) = 0$, since the sheaf $\mathcal{Q}$ is fine; the conclusion is that *a complex line bundle is determined up to* C^∞ *isomorphism by its Chern class.*

Recall now that for any vector bundle $E \to M$ of rank k and any connection D on E, the curvature operator D^2 is represented, in terms of a trivialization φ_α of E over U_α, by a $k \times k$ matrix Θ_α of 2-forms; if φ_β is another trivialization, we have

$$\Theta_\alpha = g_{\alpha\beta} \cdot \Theta_\beta \cdot g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k$ is the transition function relative to φ_α and φ_β. In particular, if E is a line bundle, since $\mathrm{GL}_1 = \mathbb{C}^*$ is commutative $\Theta = \Theta_\alpha = \Theta_\beta$ is a closed, globally defined differential form of degree 2, called the *curvature form* of E.

Recall also that for any analytic subvariety $V \subset M$ of dimension k, we have defined the *fundamental class* $(V) \in H_{2k}(M, \mathbb{R})$ to be given by the linear functional

$$\varphi \mapsto \int_V \varphi$$

on $H_{\mathrm{DR}}^{2k}(M)$; we denote its Poincaré dual by η_V. In particular, we take the fundamental class of a divisor $D = \sum a_i V_i$ on M to be $\sum a_i (V_i)$; we denote its Poincaré dual by

$$\eta_D = \sum a_i \cdot \eta_{V_i}.$$

This subsection will be devoted to proving the

Proposition. 1. *For any line bundle* L *with curvature form* Θ,

$$c_1(L)=\left[\frac{\sqrt{-1}}{2\pi}\Theta\right]\in H^2_{\mathrm{DR}}(M).$$

2. *If* L=[D] *for some* D∈Div(M),

$$c_1(L)=\eta_D\in H^2_{\mathrm{DR}}(M).$$

Proof. First, we unwind the definition of $c_1(L)$ for $L\to M$ a line bundle with trivializations φ_α and transition functions $g_{\alpha\beta}$ relative to a cover $\underline{U}=\{U_\alpha\}$ of M. We may assume the open sets U_α are simply connected and set

$$h_{\alpha\beta}=\frac{1}{2\pi\sqrt{-1}}\log g_{\alpha\beta}.$$

By the definition of δ, if we set

$$\begin{aligned} z_{\alpha\beta\gamma} &= h_{\alpha\beta}+h_{\beta\gamma}-h_{\alpha\gamma}\\ &=\frac{1}{2\pi\sqrt{-1}}(\log g_{\alpha\beta}+\log g_{\beta\gamma}-\log g_{\alpha\gamma}),\end{aligned}$$

then $\{z_{\alpha\beta\gamma}\}\in Z^2(\underline{U},\mathbb{Z})$ is a cocycle representing $c_1(L)$.

Now choose any connection D on L. In terms of the frame $e_\alpha(z)=\varphi_\alpha^{-1}(z,1)$ on U_α, D is given by its connection matrix, which in this case is a 1-form θ_α. As was worked out in Section 5 of Chapter 0, in $U_\alpha\cap U_\beta$

$$\theta_\alpha=g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1}+dg_{\alpha\beta}\cdot g_{\alpha\beta}^{-1},$$

i.e.,

$$\theta_\beta-\theta_\alpha=-g_{\alpha\beta}^{-1}dg_{\alpha\beta}=-d(\log g_{\alpha\beta}),$$

and the curvature matrix is the global 2-form

$$\Theta=d\theta_\alpha-\theta_\alpha\wedge\theta_\alpha=d\theta_\alpha=d\theta_\beta.$$

Since Θ is given as a closed 2-form and $c_1(L)$ is given as a Čech cocycle, we must now look at the explicit form of the de Rham isomorphism. From the proof of de Rham's theorem, we have exact sequences of sheaves

$$0\to\mathbb{R}\to\mathcal{Q}^0\to\mathcal{Z}_d^1\to 0,\qquad 0\to\mathcal{Z}_d^1\to\mathcal{Q}^1\to\mathcal{Z}_d^2\to 0,$$

giving us boundary isomorphisms

$$\frac{H^0(\mathcal{Z}_d^2)}{dH^0(\mathcal{Q}^1)}\xrightarrow{\delta_1}H^1(\mathcal{Z}_d^1),\qquad H^1(\mathcal{Z}_d^1)\xrightarrow{\delta_2}H^2(\mathbb{R}).$$

To calculate $\delta_1(\Theta)$, we write Θ locally as $d\theta_\alpha$; we see from the definition of δ_1 that

$$\delta_1(\Theta)=\{\theta_\beta-\theta_\alpha\}\in Z^1(\mathcal{Z}_d^1).$$

Now $\theta_\beta - \theta_\alpha = -d\log g_{\alpha\beta}$, so

$$\begin{aligned}\delta_2\delta_1(\Theta) &= \delta_2(\{\theta_\beta - \theta_\alpha\})\\ &= \{-(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma})\}\\ &= -2\pi\sqrt{-1}\cdot c_1(L).\end{aligned}$$

To prove assertion 2 we have to show that, for Θ a curvature matrix for the bundle $[D]$, the cohomology class $[(\sqrt{-1}/2\pi)\Theta]$ is the Poincaré dual of $(D)=\sum a_i(V_i)$—i.e., that for every real, closed form $\psi\in A^{2n-2}(M)$,

$$\frac{\sqrt{-1}}{2\pi}\int_M \Theta\wedge\psi = \sum a_i\int_{V_i}\psi.$$

Since both $D\mapsto c_1([D])$ and $D\mapsto\eta_D$ are homomorphisms from $\operatorname{Div}(M)$ to $H^2_{\mathrm{DR}}(M)$, we may take $D=V$ an irreducible subvariety.

First, we compute the curvature form of a metric connection on $[D]$. To do this, let e be a local nonzero holomorphic section of $[V]$ and write

$$|e(z)|^2 = h(z).$$

Then for any section $s=\lambda\cdot e$, the connection matrix θ for the metric connection D in terms of the frame e must satisfy

$$\theta = \theta^{1,0}$$

and

$$\begin{aligned}d(|s|^2) &= (Ds,s) + (s,Ds)\\ &= ((d\lambda+\theta\lambda)e,\lambda e) + (\lambda e,(d\lambda+\theta\lambda)e)\\ &= h\cdot\bar\lambda\cdot d\lambda + h\cdot\lambda\cdot d\bar\lambda + h\cdot|\lambda|^2(\theta+\bar\theta).\end{aligned}$$

Now

$$\begin{aligned}d(|s|^2) &= d(\lambda\cdot\bar\lambda\cdot h)\\ &= h\cdot\bar\lambda\cdot d\lambda + h\cdot\lambda\cdot d\bar\lambda + |\lambda|^2\cdot dh.\end{aligned}$$

So we have

$$\theta+\bar\theta = \frac{dh}{h},$$

i.e., $\theta = \partial\log h = \partial\log|e|^2$, and

$$\begin{aligned}\Theta &= d\theta - \theta\wedge\theta = d\theta\\ &= \bar\partial\partial\log|e|^2\\ &= 2\pi\sqrt{-1}\; dd^c\log|e|^2.\end{aligned}$$

Note that this holds for any nonzero holomorphic section e.

Now let $D=V$ be given by local data f_α and let s be a global section $\{f_\alpha\}$ of $[D]$ vanishing exactly on V. Set

$$D(\varepsilon) = (|s(z)|<\varepsilon)\subset M.$$

For small ε, $D(\varepsilon)$ is just a tubular neighborhood around V in M, and

$$\int_M \Theta\wedge\psi = \lim_{\varepsilon\to0} 2\pi\sqrt{-1}\int_{M-D(\varepsilon)} dd^c\log|s|^2\wedge\psi$$
$$= \lim_{\varepsilon\to0}\left(\frac{2\pi}{\sqrt{-1}}\right)\int_{\partial D(\varepsilon)} d^c\log|s|^2\wedge\psi$$

by Stokes' theorem. In $U_\alpha\cap D(\varepsilon)$, write

$$|s|^2 = |f_\alpha|^2\cdot h_\alpha = f_\alpha\cdot\bar f_\alpha\cdot h_\alpha$$

with $h_\alpha>0$; we have

$$d^c\log|s|^2 = d^c\log\big(f_\alpha\cdot\bar f_\alpha\cdot h_\alpha\big)$$
$$= \frac{\sqrt{-1}}{4\pi}\big(\bar\partial\log\bar f_\alpha - \partial\log f_\alpha + (\bar\partial-\partial)\log h_\alpha\big).$$

Since $d^c\log h_\alpha$ is bounded and, as we have seen in the proof of Stokes' theorem for analytic varieties, $\operatorname{vol}(\partial D(\varepsilon))\to0$ as $\varepsilon\to0$, we deduce that

$$\lim_{\varepsilon\to0}\int_{\partial D(\varepsilon)} d^c\log h_\alpha\wedge\psi = 0.$$

Moreover, $\bar\partial\log\bar f_\alpha = \overline{\partial\log f_\alpha}$ and, since ψ is real, this implies

$$\int_{\partial D(\varepsilon)}\bar\partial\log\bar f_\alpha\wedge\psi = \overline{\int_{\partial D(\varepsilon)}\partial\log f_\alpha\wedge\psi}\,.$$

Thus in U_α,

$$\lim_{\varepsilon\to0}\frac{2\pi}{\sqrt{-1}}\int_{\partial D(\varepsilon)} d^c\log|s|^2 = \lim_{\varepsilon\to0} -\sqrt{-1}\cdot\operatorname{Im}\int_{\partial D(\varepsilon)}\partial\log f_\alpha\wedge\psi.$$

Now in the neighborhood of any smooth point $z_0\in V\cap U_\alpha$, we can find a holomorphic coordinate system $w=(w_1,\dots,w_n)$ with $w_1=f_\alpha$. Write $\psi=\psi(w)dw'\wedge d\bar w'+\varphi$, where $w'=(w_2,\dots,w_n)$ and all terms of φ contain either dw_1 or $d\bar w_1$; then in any polydisc Δ around z_0,

$$\lim_{\varepsilon\to0}\int_{\partial D(\varepsilon)\cap\Delta}\partial\log f_\alpha\wedge\psi = \lim_{\varepsilon\to0}\int_{|w_1|=\varepsilon}\frac{dw_1}{w_1}\cdot\psi(w)\cdot dw'\wedge d\bar w'$$
$$= 2\pi\sqrt{-1}\int_{w'}\psi(0,w')\cdot dw'\wedge d\bar w'$$
$$= 2\pi\sqrt{-1}\int_{V\cap\Delta}\psi$$

and so

$$\int_M\Theta\wedge\psi = -\sqrt{-1}\cdot\operatorname{Im}\left(2\pi\sqrt{-1}\int_V\psi\right)$$
$$= \frac{2\pi}{\sqrt{-1}}\int_V\psi. \qquad \text{Q.E.D.}$$

The conclusion that *the Chern class* $c_1([D])$ *represents, on the one hand, the Poincaré dual of the fundamental homology cycle carried by a divisor* D, *and on the other hand is given in de Rham cohomology by* $(\sqrt{-1}\,/2\pi)$ *times the curvature of any connection in the line bundle* [D], is of fundamental importance for what follows. The method of proof of this proposition, i.e., applying Stokes' theorem to a differential form with singularities—is likewise ubiquitous, and will be systematized in Chapter 3.

The simplest consequence of this proposition is the fact that the divisor (f) of a meromorphic function is homologous to zero. This is intuitively clear: drawing an arc γ from $\lambda_0 = \infty$ to $\lambda_1 = \infty$ on the Riemann sphere P^1_λ, the divisors

$$\{(\lambda_0 f + \lambda_1)\}_{[\lambda_0,\lambda_1]\in\gamma}$$

trace out a chain with boundary $(f)_0 - (f)_\infty$.

Examples

1. In case M is a compact connected Riemann surface, a divisor D on M is just a finite sum

$$D = \sum n_i p_i$$

of points $p_i \in M$ with multiplicities n_i. The *degree* of D is defined to be its fundamental class $(D) \in H_0(M, \mathbb{Z}) \cong \mathbb{Z}$; clearly

$$\deg D = \sum n_i.$$

By the above proposition, if Θ is the curvature form of a connection in the line bundle $[D]$,

$$\frac{\sqrt{-1}}{2\pi}\int_M \Theta = \langle c_1([D]), [M]\rangle = \deg D.$$

In general, we define the *degree* of a line bundle on M by

$$\deg(L) = \langle c_1(L), [M]\rangle,$$

or in other words $\deg(L) = c_1(L)$ under the isomorphism $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$ given by the natural orientation on M.

Note that by the relation proved on page 77 between the curvature form Θ of a metric connection on the tangent bundle of a Riemann surface and the ordinary Gaussian curvature K_M the classical Gauss-Bonnet theorem gives

$$\deg T'(M) = \frac{1}{4\pi}\int_M K_M \cdot \Phi = \chi(M).$$

2. By the exact cohomology sequence

$$H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathbb{P}^n, \mathbb{Z})$$

arising from the exponential sheaf sequence on $\mathbb{P}^n$ and by the vanishing of $H^1(\mathbb{P}^n, \mathcal{O})$ (Section 7 of Chapter 1), we see that *every line bundle on* $\mathbb{P}^n$ *is determined by its Chern class*, i.e.,

$$\operatorname{Pic}(\mathbb{P}^n) \cong H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}.$$

In other words, *every divisor on* $\mathbb{P}^n$ *is linearly equivalent to a multiple of the hyperplane divisor* $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. The bundle $[H]$ associated to a hyperplane in $\mathbb{P}^n$ is called *the hyperplane bundle*; its inverse, $J = [H]^* = [-H]$, is called *the universal bundle* on $\mathbb{P}^n$.

We can give a direct geometric construction of the universal bundle J on $\mathbb{P}^n$ as follows. Let $\mathbb{P}^n \times \mathbb{C}^{n+1}$ be the trivial bundle of rank $n+1$ on $\mathbb{P}^n$, with all fibers identified to $\mathbb{C}^{n+1}$. Then the universal bundle is just the subbundle J of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ whose fiber at each point $Z \in \mathbb{P}^n$ is the line in $\mathbb{C}^{n+1}$ represented by Z, i.e.,

$$J_Z = \{\lambda(Z_0, \ldots, Z_n), \lambda \in \mathbb{C}\}.$$

To see that in fact $J = [-H]$, consider the section e_0 of J over $U_0 = (Z_0 \neq 0) \subset \mathbb{P}^n$ given by

$$e_0(Z) = \left(1, \frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right).$$

e_0 is clearly holomorphic and nonzero in U_0 and extends to a global meromorphic section of J with a pole of order 1 along the hyperplane $(Z_0 = 0) \subset \mathbb{P}^n$. Thus $J = [(e_0)] = [-H]$.

If $M \subset \mathbb{P}^n$ is a submanifold of projective space, we usually call the restriction of $[H] \to \mathbb{P}^n$ to M simply the *hyperplane bundle on* M; by functoriality, it is the line bundle associated to a generic hyperplane section $\mathbb{P}^{n-1} \cap M$ of M.

3. Let M be a compact complex manifold, $V \subset M$ a smooth analytic hypersurface. Recall that we defined the *normal bundle* N_V on V to be the quotient line bundle

$$N_V = \frac{T'_M|_V}{T'_V}.$$

We defined the *conormal bundle* N_V^* to be the dual of N_V; it is the subbundle of $T_M^{*\prime}|_V$ consisting of cotangent vectors to M that are zero on $T'_V \subset T'_M|_V$.

There is an easy formula for the conormal bundle of a smooth hypersurface V, which we now derive: Suppose V is given locally by functions $f_\alpha \in \mathcal{O}(U_\alpha)$; the line bundle $[V]$ on M is then given by transition functions $\{g_{\alpha\beta} = f_\alpha / f_\beta\}$. Now since $f_\alpha \equiv 0$ on $V \cap U_\alpha$, the differential df_α is a section of the conormal bundle N_V^* of V; since V is smooth, df_α is everywhere nonzero. On $U_\alpha \cap U_\beta \cap V$, moreover, we have

$$\begin{aligned} df_\alpha &= d(g_{\alpha\beta} f_\beta) \\ &= dg_{\alpha\beta} \cdot f_\beta + g_{\alpha\beta} \cdot df_\beta \\ &= g_{\alpha\beta} \cdot df_\beta, \end{aligned}$$

i.e., *the sections* $df_\alpha \in \Gamma(U_\alpha, \mathcal{O}(N_V^*))$ *together give a nonzero global section of* $N_V^* \otimes [V]$. Thus $N_V^* \otimes [V]$ is the trivial line bundle; this is the

Adjunction Formula I

$$N_V^* = [-V]|_V,$$

4. One of the most important line bundles in general is the highest exterior power of the holomorphic cotangent bundle

$$K_M = \wedge^n T_M^{*\prime},$$

called the *canonical bundle* of the n-dimensional complex manifold M. Holomorphic sections of K_M are holomorphic n-forms, i.e., $\mathcal{O}(K_M) = \Omega_M^n$.

We will compute the canonical bundle $K_{\mathbb{P}^n}$ of projective space: Let $Z_0, \ldots, Z_n$ be homogeneous coordinates on $\mathbb{P}^n$, $w_i = Z_i / Z_0$ Euclidean coordinates on $U_0 = (Z_0 \neq 0)$, and consider the meromorphic n-form

$$\omega = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \cdots \wedge \frac{dw_n}{w_n}.$$

ω is clearly nonzero in U_0 with a single pole along each hyperplane $(Z_i = 0)$, $i = 1, \ldots, n$. Now if $w_i' = Z_i / Z_j$, $i = 0, \ldots, \hat{j}, \ldots, n$ are Euclidean coordinates on $U_j = (Z_j \neq 0)$, then

$$w_i = \frac{w_i'}{w_0'}, \quad i \neq j; \qquad w_j = \frac{1}{w_0'},$$

which gives

$$\frac{dw_i}{w_i} = \frac{dw_i'}{w_i'} - \frac{dw_0'}{w_0'}, \quad i \neq j; \qquad \frac{dw_j}{w_j} = \frac{-dw_0'}{w_0'},$$

and so in terms of $\{w_i'\}$,

$$\omega = (-1)^j \cdot \frac{dw_0'}{w_0'} \wedge \cdots \wedge \widehat{\frac{dw_j'}{w_j'}} \wedge \cdots \wedge \frac{dw_n'}{w_n'}.$$

Thus we see that ω has likewise a single pole along the hyperplane $(Z_0 = 0)$, and consequently

$$K_{\mathbb{P}^n} = [(\omega)] = [-(n+1)H].$$

In general, we can compute the canonical bundle K_V of a smooth analytic hypersurface V in a manifold M in terms of K_M as follows. We have an exact sequence of vector bundles on V

$$0 \to N_V^* \to T_M^{*\prime}|_V \to T_V^{*\prime} \to 0.$$

By simple linear algebra,

$$(\wedge^n T_M^{*\prime})|_V \cong \wedge^{n-1} T_V^{*\prime} \otimes N_V^*,$$

i.e.,

$$K_V = K_M|_V \otimes N_V.$$

Combining this with the adjunction formula I above, we have the

Adjunction Formula II

$$(*) \qquad K_V = (K_M \otimes [V])|_V.$$

We can give the corresponding map on sections

$$\Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1}$$

as follows: Considering a section ω of $\Omega_M^n(V)$ as a meromorphic n-form with a single pole along V and holomorphic elsewhere, we write

$$\omega = \frac{g(z)\,dz_1 \wedge \cdots \wedge dz_n}{f(z)},$$

where $z = (z_1, \ldots, z_n)$ are local coordinates on M and V is given locally by $f(z)$. Under the isomorphism $(*)$, then, ω corresponds to the form ω' such that

$$\omega = \frac{df}{f} \wedge \omega'$$

Explicitly,

$$df = \sum \frac{\partial f}{\partial z_i} \cdot dz_i,$$

and so we can take

$$\omega' = (-1)^{i-1} \frac{g(z)\,dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\partial f / \partial z_i}$$

for any i such that $\partial f / \partial z_i \neq 0$. The map

$$\frac{g(z)\,dz_1 \wedge \cdots \wedge dz_n}{f(z)} \to (-1)^{i-1} \frac{g(z)\,dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n}{\partial f / \partial z_i}\Bigg|_{f=0}$$

is called the *Poincaré residue map*, denoted P.R.

Note that the kernel of the Poincaré residue map consists simply of the holomorphic n-forms on M. The exact sheaf sequence

$$0 \to \Omega_M^n \to \Omega_M^n(V) \xrightarrow{\text{P.R.}} \Omega_V^{n-1} \to 0$$

then gives us, in part, the exact sequence

$$H^0(M,\Omega^n_M(V)) \xrightarrow{\text{P.R.}} H^0(V,\Omega^{n-1}_V) \xrightarrow{\delta} H^1(M,\Omega^n_M),$$

i.e., the Poincaré residue map is surjective on global sections if $H^1(M,\Omega^n_M) = H^{n,1}(M)=0$. For example, since $H^{n,1}(\mathbb{P}^n)=0$ for $n>1$, *every holomorphic form of top degree on a hypersurface* V *in* $\mathbb{P}^n$ *is the Poincaré residue of a meromorphic form on* $\mathbb{P}^n$. We will see later that the meromorphic n-forms on $\mathbb{P}^n$ are easy to describe, so that we can readily write down the holomorphic $(n-1)$-forms on V.

2. SOME VANISHING THEOREMS AND COROLLARIES

The Kodaira Vanishing Theorem

Let M be a compact Kähler manifold.

DEFINITION. A line bundle $L \rightarrow M$ is *positive* if there exists a metric on L with curvature form Θ such that $(\sqrt{-1}/2\pi)\Theta$ is a positive $(1,1)$-form; L is *negative* if L^* is positive. A divisor D on M is positive if the line bundle $[D]$ is.

The positivity of a line bundle is a topological property, as we see from the

Proposition. *If* ω *is any real, closed* $(1,1)$*-form with*

$$[\omega] = c_1(L) \in H^2_{\mathrm{DR}}(M),$$

then there exists a metric connection on L *with curvature form* $\Theta = (\sqrt{-1}/2\pi)\omega$. *Thus* L *is positive if and only if its Chern class may be represented by a positive form in* $H^2_{\mathrm{DR}}(M)$.

Proof. Let $|s|^2$ be a metric on L with curvature form Θ. If $\varphi: L_U \rightarrow U\times\mathbb{C}$ is a trivialization of L over an open set U, s a section of L over U and s_U the corresponding holomorphic function, then

$$|s|^2 = h(z)\cdot|s_U|^2$$

for some positive function $h(z)$. The curvature form and Chern class are given by

$$\Theta = -\partial\bar{\partial}\log h(z),$$
$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] \in H^2_{\mathrm{DR}}(M).$$

Now let $|s|'^2$ be another metric on L with curvature form Θ'. Then $|s|'^2/|s|^2 = e^{\rho}$ for some real C^∞ function ρ on M, and from the local formula

$$h'(z) = e^{\rho(z)}h(z)$$

it follows that

$$\Theta = \partial\bar{\partial}\rho + \Theta'.$$

In particular,

$$\left[\frac{\sqrt{-1}}{2\pi}\Theta\right] = \left[\frac{\sqrt{-1}}{2\pi}\Theta'\right].$$

Working in the other direction, suppose that $(\sqrt{-1}/2\pi)\varphi$ is a real, closed $(1,1)$-form representing $c_1(L)$ in $H^2_{\mathrm{DR}}(M)$. If we can solve the equation

$$\Theta = \partial\bar{\partial}\rho + \varphi$$

for a real C^∞ function ρ, then the metric $e^\rho|s|^2$ on L will have curvature form φ. Our proposition therefore follows from the

Lemma. *If η is any* (p,q)*-form on a compact Kähler manifold, and η is* d-, ∂-, *or* $\bar{\partial}$*-exact, then*

$$\eta = \partial\bar{\partial}\gamma$$

for some $(p-1,q-1)$*-form γ. If* $p=q$ *and η is real, then we may take $\sqrt{-1}\,\gamma$ also to be real.*

Proof. Let G_d denote the Green's operator associated to the Laplacian Δ_d, and similarly for G_∂ and $G_{\bar{\partial}}$. From the basic identity of page 115

$$\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar{\partial}}$$

it follows first that

$$2G_d = G_\partial = G_{\bar{\partial}},$$

and then that all the operators d, ∂, $\bar{\partial}$, d^*, ∂^*, and $\bar{\partial}^*$ commute with the Green's operators.

Now, since η is d-, ∂-, or $\bar{\partial}$-exact, its harmonic projection under any of the above Laplacians is zero. By the Hodge decomposition for $\bar{\partial}$,

$$\eta = \bar{\partial}\bar{\partial}^* G_{\bar{\partial}}\eta.$$

But $\bar{\partial}^* G_{\bar{\partial}}\eta$ has pure type $(p,q-1)$ and so

$$\partial\big(\bar{\partial}^* G_{\bar{\partial}}\eta\big) = \pm\bar{\partial}^* G_{\bar{\partial}}(\partial\eta) = 0.$$

Since the harmonic space for ∂ is the same as the harmonic space for $\bar{\partial}$ and hence is orthogonal to the range of $\bar{\partial}^*$, we deduce by the Hodge decomposition for ∂ that

$$\bar{\partial}^* G_{\bar{\partial}}\eta = \partial\partial^* G_\partial\big(\bar{\partial}^* G_{\bar{\partial}}\eta\big).$$

By commuting the various operators,

$$\eta = \pm \bar{\partial}\partial\left(\partial^*\bar{\partial}^* G_{\bar{\partial}}^2 \eta\right),$$

which implies the lemma. Q.E.D.

The basic example of a positive line bundle is the hyperplane bundle $[H]$ on $\mathbb{P}^n$. Recall that the dual of the hyperplane bundle is the bundle J whose fiber at $Z \in \mathbb{P}^n$ is the line $\{\lambda Z\} \subset \mathbb{C}^{n+1}$; we can put a metric on J by setting $|(Z_0,\ldots,Z_n)|^2 = \Sigma |Z_i|^2$. If Z is any nonzero section of J—i.e., a local lifting $U \subset \mathbb{P}^n \to \mathbb{C}^{n+1} - \{0\}$—then the curvature form in J is given by

$$\Theta^* = \bar{\partial}\partial \log\|Z\|^2 = 2\pi\sqrt{-1}\; dd^c \log\|Z\|^2.$$

The curvature form Θ for the dual metric in $[H]$ is then $-\Theta^*$, and consequently

$$\frac{\sqrt{-1}}{2\pi}\Theta = dd^c \log\|Z\|^2,$$

i.e., $(\sqrt{-1}/2\pi)\Theta$ is just the associated $(1,1)$-form ω of the Fubini-Study metric on $\mathbb{P}^n$, which we have seen is positive. As a corollary, we see again that the Poincaré dual of $[\omega] \in H^2_{\mathrm{DR}}(\mathbb{P}^n)$ is the fundamental class (H) of a hyperplane.

Note that since the restriction to a submanifold $V \subset M$ of a positive form is again positive, $L|_V \to V$ will be positive if $L \to M$ is. In particular, the hyperplane bundle on any complex submanifold of $\mathbb{P}^n$ is positive.

Our aim in this section is to prove that certain Čech cohomology groups $H^q(M, \Omega^p(L))$ associated to a positive line bundle $L \to M$ are zero. To begin with, we transpose the problem into one involving $\bar{\partial}$-cohomology and harmonic forms by a technique that will be familiar from the previous discussion.

Recall that for any holomorphic vector bundle $E \to M$, the $\bar{\partial}$-operator

$$\bar{\partial}: A^{p,q}(E) \to A^{p,q+1}(E)$$

is defined for global C^∞ E-valued differential forms, and satisifes $\bar{\partial}^2 = 0$. We let $Z_{\bar{\partial}}^{p,q}(E)$ denote the space of $\bar{\partial}$-closed E-valued differential forms of type (p,q), and we define the *Dolbeault cohomology groups* $H_{\bar{\partial}}^{p,q}(E)$ *of* E to be

$$H_{\bar{\partial}}^{p,q}(E) = \frac{Z_{\bar{\partial}}^{p,q}(E)}{\bar{\partial} A^{p,q-1}(E)}.$$

Let $\mathcal{Z}_{\bar{\partial}}^{p,q}(E)$ denote the sheaf of $\bar{\partial}$-closed E-valued (p,q)-forms. The exact sheaf sequences

$$0 \to \mathcal{Z}_{\bar{\partial}}^{p,q}(E) \to \mathcal{Q}^{p,q}(E) \xrightarrow{\bar{\partial}} \mathcal{Z}_{\bar{\partial}}^{p,q+1}(E) \to 0$$

give us isomorphisms

$$H^i\big(M,\mathcal{Z}_{\bar\partial}^{p,q+1}(E)\big) \xrightarrow{\delta} H^{i+1}\big(M,\mathcal{Z}_{\bar\partial}^{p,q}(E)\big),$$

since the sheaves $\mathcal{Q}^{p,q}(E)$ admit partitions of unity and hence have no Čech cohomology. Thus, repeating the reasoning from the proof of de Rham's theorem,

$$H^q(M,\Omega^p(E)) \cong H_{\bar\partial}^{p,q}(E).$$

Next we want to discuss harmonic theory in holomorphic vector bundles. Suppose we have metrics given on M and E; we have then induced metrics on all tangential tensor bundles of M tensored with E or E^*. In particular, if $\{\varphi_i\}$ is a local coframe for the metric on $T_M{}^{*\prime}$ and $\{e_\alpha\}$ a unitary frame for E, any section η of $A^{p,q}(E)$ can be written locally as

$$\eta(z) = \frac{1}{p!q!}\sum_{I,J,\alpha}\eta_{I,J,\alpha}(z)\varphi_I\wedge\bar\varphi_J\otimes e_\alpha;$$

for $\eta,\psi\in A^{p,q}(E)$,

$$(\eta(z),\psi(z)) = \frac{2^{p+q-n}}{p!q!}\sum_{I,J,\alpha}\eta_{I,J,\alpha}(z)\cdot\overline{\psi_{I,J,\alpha}(z)}\,.$$

Again, we define an inner product on $A^{p,q}(E)$ by setting

$$(\eta,\psi) = \int_M(\eta(z),\psi(z))\Phi,$$

where Φ is the volume form on M.

We have a "wedge product"

$$\wedge:\ A^{p,q}(E)\otimes A^{p',q'}(E^*)\mapsto A^{p+p',q+q'}(M)$$

defined by

$$(\eta\otimes s)\wedge(\eta'\otimes s') = \langle s,s'\rangle\cdot\eta\wedge\eta';$$

we define an operator

$$*_E:\ A^{p,q}(E)\to A^{n-p,n-q}(E^*)$$

by requiring, for $\eta,\psi\in A^{p,q}(E)$,

$$(\eta,\psi) = \int_M\eta\wedge *_E\psi.$$

Explicitly, if $\{e_\alpha\}$ and $\{e_\alpha^*\}$ are dual unitary frames for E and E^*, then for $\eta\in A^{p,q}(E)$ written as

$$\eta = \sum\eta_\alpha\otimes e_\alpha,\qquad \eta_\alpha\in A^{p,q}(M),$$
$$*_E\eta = \sum *\eta_\alpha\otimes e_\alpha^*,$$

where $*$ is the usual star operator on $A^{p,q}(M)$.

We take

$$\bar{\partial}^*: A^{p,q}(E) \to A^{p,q-1}(E)$$

to be given by

$$\bar{\partial}^* = -*_E \cdot \bar{\partial} \cdot *_E;$$

as before, $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$, i.e., for all $\varphi \in A^{p,q-1}(E)$ and $\psi \in A^{p,q}(E)$,

$$(\bar{\partial}\varphi, \psi) = (\varphi, \bar{\partial}^*\psi).$$

Finally, the $\bar{\partial}$-Laplacian on E is defined by

$$\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}: A^{p,q}(E) \to A^{p,q}(E).$$

An E-valued form φ is called *harmonic* if $\Delta\varphi = 0$. (Again, harmonic forms φ are exactly the forms of smallest norm in their Dolbeault cohomology class $\varphi + \bar{\partial}A^{p,q-1}(E)$.) We let

$$\mathcal{H}^{p,q}(E) = \operatorname{Ker} \Delta$$

be the *harmonic space*.

Now, the analytic part of the proof of the Hodge theorem for the $\bar{\partial}$-Laplacian on ordinary differential forms on M is essentially local: we can always find appropriate solutions of $\Delta\varphi = 0$ in the completion of $A^{p,q}(M)$ in the L_2-norm; the problem is to show that these solutions are in fact C^∞. Writing out E-valued forms in terms of a frame for E, all the local estimates used in the proof of the Hodge theorem for $A^*(M)$ go over to $A^{p,q}(E)$—the only difference is that in each estimate we will get lower-order terms involving the coefficient functions for the metric on E as well as the metric on T_M^*, and these can be estimated out as before. Thus the Hodge theorem holds for the $\bar{\partial}$-Laplacian on E, that is:

1. $\mathcal{H}^{p,q}(E)$ is finite dimensional, and
2. If $\mathcal{H}$ denotes the orthogonal projection $A^{p,q}(E) \to \mathcal{H}^{p,q}(E)$, there exists an operator

$$G: A^{p,q}(E) \to A^{p,q}(E)$$

such that

$$G(\mathcal{H}^{p,q}(E)) = 0,$$
$$[G, \bar{\partial}] = [G, \bar{\partial}^*] = 0,$$

and

$$I = \mathcal{H} + \Delta G.$$

3. Consequently, there is an isomorphism

$$\mathcal{H}^{p,q}(E) \longrightarrow H^{p,q}_{\bar{\partial}}(E),$$

and

4. The $*$-operator gives an isomorphism

$$H^q(M,\Omega^p(E)) \cong H^{n-q}(M,\Omega^{n-p}(E^*)).^*$$

For $p=0$, this last result reads

$$H^q(M,\mathcal{O}(E)) \cong H^{n-q}(M,\mathcal{O}(E^*\otimes K_M)).^*$$

This isomorphism is called *Kodaira-Serre duality*.

Now if M is Kähler with associated (1,1)-form ω, we define the operator

$$L:\ A^{p,q}(E)\to A^{p+1,q+1}(E)$$

by setting, for $\eta\in A^{p,q}(M)$ and $s\in A^0(E)$,

$$L(\eta\otimes s) = \omega\wedge\eta\otimes s;$$

let $\Lambda=L^*$ be the adjoint of L. If $D=D'+D''$ $(D''=\bar\partial)$ is the metric connection on E, then we have the *basic identity*

$$[\Lambda,\bar\partial] = -\frac{\sqrt{-1}}{2}D'^*.$$

This identity follows from the analogous identity $[\Lambda,\bar\partial]=-(\sqrt{-1}/2)\partial^*$ on scalar forms $A^{p,q}(M)$, which we have already proved. To see this, pick a local frame $\{e_\alpha\}$ for E; if $\theta=\theta'+\theta''$ is the connection matrix for D in terms of $\{e_\alpha\}$, we can write, for $\eta\in A^{p,q}(E)$,

$$\begin{aligned}
\eta &= \sum_\alpha \eta_\alpha\otimes e_\alpha, \qquad \eta_\alpha\in A^{p,q}(M),\\
\bar\partial\eta &= \sum_\alpha \bar\partial\eta_\alpha\otimes e_\alpha + \sum_{\alpha,\beta}(\eta_\alpha\wedge\theta''_{\alpha\beta})\otimes e_\beta,\\
\Lambda\eta &= \sum_\alpha \Lambda(\eta_\alpha)\otimes e_\alpha,
\end{aligned}$$

so

$$\begin{aligned}
[\Lambda,\bar\partial]\eta &= \sum[\Lambda,\bar\partial]\eta_\alpha\otimes e_\alpha + [\Lambda,\theta'']\eta\\
&= \sum -\frac{\sqrt{-1}}{2}\partial^*\eta_\alpha\otimes e_\alpha + [\Lambda,\theta'']\eta.
\end{aligned}$$

Similarly,

$$D'\eta = \sum_\alpha \partial\eta_\alpha\otimes e_\alpha + \sum_{\alpha,\beta}(\eta_\alpha\wedge\theta'_{\alpha\beta})\otimes e_\beta,$$

i.e.

$$D'^*\eta = \sum_i \partial^*\eta_\alpha\otimes e_\alpha + \theta'^*\eta.$$

The difference

$$[\Lambda,\bar\partial] + \frac{\sqrt{-1}}{2}D'^* = [\Lambda,\theta''] + \frac{\sqrt{-1}}{2}\theta'^*$$

is consequently an *intrinsically defined algebraic operator*; since we can choose at each $z_0 \in M$ a frame for E in a neighborhood of z_0 for which $\theta(z_0)$ vanishes, we see that $[\Lambda, \bar\partial] + (\sqrt{-1}/2)D'^* = 0$.

We will use the representation of Čech cohomology by harmonic forms to prove our first main result on the cohomology of vector bundles, the

Kodaira-Nakano Vanishing Theorem. *If* $L \to M$ *is a positive line bundle, then*

$$H^q(M, \Omega^p(L)) = 0 \qquad \textit{for } p+q>n.$$

Proof.* By hypothesis we can find a metric in L whose curvature form Θ is $2\pi/\sqrt{-1}$ times the associated (1,1)-form of a Kähler metric; let the metric on M be the one given by $\omega = (\sqrt{-1}/2\pi)\Theta$. Now by harmonic theory

$$H^q(M, \Omega^p(L)) \cong \mathcal{H}^{p,q}(L).$$

To prove the result, we will show that there are no nonzero harmonic L-valued forms of degree larger than n. We do this by interpreting the curvature operator $\Theta\eta = \Theta \wedge \eta$ alternately as $(2\pi/\sqrt{-1})L(\eta)$, and as $D^2\eta$, where D is the metric connection on L, and using the basic identity above.

Let $\eta \in \mathcal{H}^{p,q}(L)$ be a harmonic form. Then

$$\Theta = D^2 = \bar\partial D' + D'\bar\partial,$$

so from $\bar\partial\eta = 0$

$$\Theta\eta = \bar\partial D'\eta,$$

and

$$\begin{aligned} 2\sqrt{-1}\,(\Lambda\Theta\eta, \eta) &= 2\sqrt{-1}\,(\Lambda\bar\partial D'\eta, \eta) \\ &= 2\sqrt{-1}\left(\left(\bar\partial\Lambda - \frac{\sqrt{-1}}{2}D'^*\right)D'\eta, \eta\right) \\ &= (D'^*D'\eta, \eta) = (D'\eta, D'\eta) \geqslant 0, \end{aligned}$$

since $(\bar\partial\Lambda D'\eta, \eta) = (\Lambda D'\eta, \bar\partial^*\eta) = 0$. Similarly,

$$\begin{aligned} 2\sqrt{-1}\,(\Theta\Lambda\eta, \eta) &= 2\sqrt{-1}\,(D'\bar\partial\Lambda\eta, \eta) \\ &= 2\sqrt{-1}\left(D'\left(\Lambda\bar\partial + \frac{\sqrt{-1}}{2}D'^*\right)\eta, \eta\right) \\ &= -(D'D'^*\eta, \eta) = -(D'^*\eta, D'^*\eta) \leqslant 0. \end{aligned}$$

*This proof is due to Y. Akizuki and S. Nakano, Note on Kodaira-Spencer's proof of Lefschetz's theorems, *Proc. Japan Acad.*, Vol. 30 (1954).

Combining,

$$2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) \geqslant 0.$$

But $\Theta=(2\pi/\sqrt{-1}\,)L$, and so

$$\begin{aligned}2\sqrt{-1}\,([\Lambda,\Theta]\eta,\eta) &= 4\pi([\Lambda,L]\eta,\eta)\\ &= 4\pi(n-p-q)\|\eta\|^2 \geqslant 0.\end{aligned}$$

Thus $p+q>n \Rightarrow \eta=0$. Q.E.D.

As was suggested when we first introduced cohomology, the groups $H^q(M,\Omega^p(E))$ $(q\geqslant 1)$ most frequently arise as obstructions to globally solving analytic problems—this is especially true for $q=1$ as in the Mittag-Leffler problem, but once one admits H^1's, then all the rest become involved. The Kodaira vanishing theorem—together with its variants to be discussed later—is the best general method for eliminating cohomology.

Dualizing the Kodaira vanishing theorem, we obtain:

$H^q(M,\Omega^p(L)) = 0$ *for* $p+q<n$ *in case* $L\to M$ *is a negative line bundle.*

The special case when $p=q=0$ can be proved by elementary methods as follows: What we have to show is that

$$(*)\qquad H^0(M,\mathcal{O}(L))=0$$

in case $L\to M$ has a metric with curvature form equal to $2/\sqrt{-1}$ times a negative (1,1)-form. Suppose $s\neq 0\in H^0(M,\mathcal{O}(L))$, and let $x_0\in M$ be a point where $|s|^2$ attains a maximum. By hypothesis, if we write $z_i=x_i+\sqrt{-1}\,y_i$ the coefficient matrix for the curvature form

$$\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}\log\left(\frac{1}{|s|^2}\right)\right)=\frac{1}{4}\left(\left(\frac{\partial^2}{\partial x_i\partial x_j}+\frac{\partial^2}{\partial y_i\partial y_j}\right)+\sqrt{-1}\left(\frac{\partial^2}{\partial x_j\partial y_i}-\frac{\partial^2}{\partial y_j\partial x_i}\right)\right)\left(\log\frac{1}{|s|^2}\right)$$

is negative definite hermitian, and in particular the real symmetric matrix

$$\left(\frac{\partial^2}{\partial x_i\partial x_j}+\frac{\partial^2}{\partial y_i\partial y_j}\right)\log\frac{1}{|s|^2}$$

is negative definite. But $\log(1/|s|^2)$ attains a minimum at x_0, and by the maximum principle, the matrices

$$\left(\frac{\partial^2}{\partial x_i\partial x_j}\right)\log\frac{1}{|s|^2}\quad\text{and}\quad\left(\frac{\partial^2}{\partial y_i\partial y_j}\right)\log\frac{1}{|s|^2}$$

must both be positive semidefinite—a contradiction.

In case M is a Riemann surface, the special case (*) is the general case, since $p+q<1 \Rightarrow p=q=0$. The theorem then is even more elementary: if Θ is a curvature form for L with $(\sqrt{-1}/2\pi)\Theta$ negative, we have

$$c_1(L) = \int_M \frac{\sqrt{-1}}{2\pi}\Theta < 0.$$

But if $s \neq 0 \in H^0(M, \mathcal{O}(L))$, then L is the line bundle associated to the effective divisor $D=(s)$, and we have

$$c_1(L) = \deg D \geqslant 0,$$

a contradiction.

As an immediate consequence of the vanishing theorem, we see that

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = 0 \qquad \text{for } 1 \leqslant q \leqslant n-1, \quad \text{all } k.$$

This follows directly from the dualized version of the vanishing theorem in case k is negative; if k is nonnegative,

$$\begin{aligned} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) &= H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}(kH - K_{\mathbb{P}^n})) \\ &= H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((k+n+1)H)) \\ &= 0 \end{aligned}$$

by the original version of the theorem.

The Lefschetz Theorem on Hyperplane Sections

Using the Kodaira vanishing theorem, we can give a proof of the famous Lefschetz theorem relating the homology of a projective variety to that of its hyperplane sections.

Let M be an n-dimensional compact, complex manifold and $V \subset M$ a smooth hypersurface with $L=[V]$ positive—e.g., $M \subset \mathbb{P}^N$ a submanifold of projective space and $V = M \cap H$ a hyperplane section of M. Then we have the

Lefschetz Hyperplane Theorem. *The map*

$$H^q(M, \mathbb{Q}) \to H^q(V, \mathbb{Q})$$

induced by the inclusion $i: V \hookrightarrow M$ *is an isomorphism for* $q \leqslant n-2$ *and injective for* $q = n-1$.

Proof. It will suffice to prove the result over $\mathbb{C}$. By the Hodge decomposition

$$H^r(M, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M),$$

and by Dolbeault

$$H^{p,q}(M) \cong H^{p,q}_{\bar\partial}(M) \cong H^q(M, \Omega^p_M).$$

The same holding for V, it is sufficient to prove that the map

$$H^p(M,\Omega_M^q) \to H^p(V,\Omega_V^q)$$

is an isomorphism for $p+q \leq n-2$, and injective for $p+q=n-1$.

To see this, we factor the restriction map $\Omega_M^p \to \Omega_V^p$ by

$$\Omega_M^p \xrightarrow{r} \Omega_M^p|_V \xrightarrow{i} \Omega_V^p,$$

where $\Omega_M^p|_V$ is the sheaf of sections of $(\Lambda^p T_M^{*\prime})|_V$—considered either as a sheaf on V or, by extension, as a sheaf on M—r is the restriction map, and i is the pullback map induced by the natural projection $(\Lambda^p T_M^{*\prime})|_V \to \Lambda^p T_V^{*\prime}$.

The kernel of the restriction map r is clearly just the sheaf of holomorphic p-forms on M vanishing along V, so we have an exact sequence of sheaves on M

$$(*) \qquad 0 \to \Omega_M^p(-V) \to \Omega_M^p \xrightarrow{r} \Omega_M^p|_V \to 0.$$

We can likewise fit the map i into an exact sequence: for $p \in V$, the sequence

$$0 \to N_{V,p}^* \to T_p^{*\prime}(M) \to T_p^{*\prime}(V) \to 0,$$

yields, by linear algebra,

$$0 \to N_{V,p}^* \otimes \wedge^{p-1} T_p^{*\prime}(V) \to \wedge^p T_p^{*\prime}(M) \to \wedge^p T_p^{*\prime}(V) \to 0,$$

and consequently an exact sequence of sheaves on V

$$0 \to \Omega_V^{p-1}(N_V^*) \to \Omega_M^p|_V \xrightarrow{i} \Omega_V^p \to 0.$$

But by the adjunction formula I, $N_V^* = [-V]|_V$; we can thus rewrite this last sequence as

$$(**) \qquad 0 \to \Omega_V^{p-1}(-V) \to \Omega_M^p|_V \to \Omega_V^p \to 0.$$

Now $[-V]$ is negative on M, and likewise $[-V]|_V$ is negative on V. The Kodaira vanishing theorem accordingly gives

$$H^q(M,\Omega_M^p(-V)) = 0, \qquad p+q < n,$$
$$H^q(V,\Omega_V^{p-1}(-V)) = 0, \qquad p+q < n.$$

By the exact cohomology sequences associated to the sheaf sequences $(*)$ and $(**)$, recalling that $H^*(M,\Omega_M^p|_V) = H^*(V,\Omega_M^p|_V)$,

$$H^q(M,\Omega_M^p) \overset{r^*}{\cong} H^q(M,\Omega_M^p|_V) \overset{i^*}{\cong} H^q(V,\Omega_V^p)$$

for $p+q \leq n-2$, and with both maps injective for $p+q=n-1$. Q.E.D.

The Lefschetz theorem on hyperplane sections is, of course, purely topological. There is another proof using a little Morse theory; we will give here a sketch of the argument:*

*Due to R. Bott, On a theorem of Lefschetz, *Mich. Math. J.*, Vol. 6 (1959), pp. 211–216.

To begin with, suppose that A is a compact manifold, $B \subset A$ a smooth submanifold, and $\varphi: A \to \mathbb{R}^+$ a C^∞ function such that $\varphi^{-1}(0) = B$. A *critical point* $x_\nu \in A$ of φ is a point such that $d\varphi(x_\nu) = 0$; $\varphi(x_\nu)$ is called a *critical value* of φ. At each critical point the *Hessian* $\partial^2\varphi/(\partial u_i \partial u_j) = H(\varphi)$ is a well-defined quadratic form in the tangent space $T_{x_\nu}(A)$; the critical point is *nondegenerate* in case $H(\varphi)$ is nonsingular. The function φ is called a *Morse function* if all critical points of φ are nondegenerate; according to a standard approximation theorem, such functions are dense in the C^2-topology. By the main lemma of Morse theory, if φ is a Morse function and the Hessian $H(\varphi)$ is nonsingular in the normal bundle to B in A, then the homotopy type of

$$A_t = \{x \in A : \varphi(x) \leqslant t\},$$

remains the same as long as t does not cross a critical value (this is obvious; we just retract along the gradient vector field of φ), and changes by attaching a cell of dimension k when we cross a critical value whose Hessian has exactly k negative eigenvalues. (This requires a local analysis of the Morse function ψ around the critical point x_ν, and is the main step.)

Now let M be a compact, complex manifold, $L \to M$ a positive holomorphic line bundle, and $s \in H^0(M, \mathcal{O}(L))$ a holomorphic section whose zero divisor $V = (s)$ is a smooth hypersurface. Choose a metric for $L \to M$ such that $(\sqrt{-1}/2\pi)\Theta = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log|s|^{-2}$ is positive and set

$$\varphi(x) = \log|s|^2.$$

φ—or a function near φ in the C^2 topology—may be used as a Morse function (the fact that $\varphi: M \to [-\infty, \infty)$ with $\varphi^{-1}(-\infty) = V$ causes no essential difficulty; what is important is that $d(|s|) \neq 0$ along V). Now for any critical point $x \in M$ of φ, the matrix

$$\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_j}\right)\log\frac{1}{|s|^2} = \left(\frac{1}{4}\left(\frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial y_i \partial y_j}\right) + \frac{\sqrt{-1}}{4}\left(\frac{\partial^2}{\partial y_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial y_i}\right)\right)\log|s|^2$$

is negative definite hermitian, and consequently the Hessian

$$H(\varphi) = \begin{bmatrix} \dfrac{\partial^2}{\partial x_i \partial x_j} & \dfrac{\partial^2}{\partial x_i \partial y_j} \\ \dfrac{\partial^2}{\partial x_j \partial y_i} & \dfrac{\partial^2}{\partial y_i \partial y_j} \end{bmatrix} \log|s|^2$$

of φ has at least n negative eigenvalues. Clearly, this will also be true for functions ψ sufficiently close to φ in the C^2-topology. Thus, by Morse theory, as far as homotopy type is concerned M is obtained from V by attaching cells of dimension at least n, and this gives the Lefschetz theorem on the homotopy level and for homology with $\mathbb{Z}$-coefficients. Q.E.D.

When $n=1$, the theorem doesn't say anything. However, when $n=2$—i.e., M is a (connected and compact) complex surface—and $V \subset M$ is a Riemann surface embedded as a positive divisor, then the Lefschetz theorem gives

$$H_0(V,\mathbb{Z}) \cong H_0(M,\mathbb{Z}) = \mathbb{Z},$$
$$H_1(V,\mathbb{Z}) \to H_1(M,\mathbb{Z}) \to 0,$$

i.e., all of the first homology of the 4-manifold M lies on the irreducible embedded Riemann surface V.

We may also apply it to hypersurfaces of projective space: since any effective nonzero divisor on $\mathbb{P}^n$ is positive, the theorem tells us that if V is any smooth hypersurface in $\mathbb{P}^n$, then $H^{2k-1}(V)=0$ for $k \neq n/2$, while $H^{2k}(V)$ is generated by the class of a k-plane section of V for $k<n/2$. In particular any smooth hypersurface of dimension 2 or more is connected and simply connected. The same results apply, for an appropriate range of k, to any submanifold of projective space given as the transverse intersection of hypersurfaces.

A final remark on the Lefschetz theorem: Lefschetz's method was insofar as possible to study the topology of an algebraic variety M inductively, reducing questions about the homology of M to questions about the homology of a smaller-dimensional variety. His original proof of the last theorem asserted that for a hyperplane section V of M, the map $H_q(V,\mathbb{Z}) \to H_q(M,\mathbb{Z})$ is an isomorphism for $q<n-1$ and surjective in dimension $n-1$. By the hard Lefschetz theorem, the homology of M in dimension above n is mirrored in dimensions less than n, and by the Lefschetz decomposition, any nonprimitive cycle in dimension n can be obtained by intersecting a cycle in dimension greater than n with hyperplanes. Thus, the Lefschetz theorems together assert that the only "new" rational homology in varieties in each dimension is the primitive homology of the middle dimension.

Theorem B

Our second vanishing theorem for the cohomology of holomorphic vector bundles is less precise but broader in scope than the Kodaira Vanishing Theorem:

Theorem B. *Let* M *be a compact, complex manifold and* L→M *a positive line bundle. Then for any holomorphic vector bundle* E, *there exists* μ *such that*

$$H^q(M,\mathcal{O}(L^\mu \otimes E)) = 0 \qquad \text{for } q>0, \mu \geqslant \mu_0.$$

Proof. Before we prove this, note that in case E is a line bundle the result is already implied by the Kodaira theorem: just take μ_0 such that $L^\mu \otimes E \otimes$

K_M^* is positive for $\mu \geqslant \mu_0$; then since $c_1(L^\mu \otimes E) = \mu c_1(L) + c_1(E)$

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) = H^q(M, \Omega^n(L^\mu \otimes E \otimes K_M^*)) = 0$$

for $q > 0$, $\mu \geqslant \mu_0$. Indeed, the proof of Theorem B is essentially the same as that of Kodaira's theorem, the only difference being that now we must associate a definite sign to the curvature operator on a general vector bundle.

First, by Kodaira-Serre duality,

$$H^q(M, \mathcal{O}(L^\mu \otimes E)) \cong H^{n-q}(M, \mathcal{O}(L^{-\mu} \otimes E^* \otimes K_M)),$$

so it will be sufficient to prove that for any E, there exists μ_0 such that

$$H_{\bar\partial}^{0,p}(M, L^{-\mu} \otimes E) \cong H^p(M, \mathcal{O}(L^{-\mu} \otimes E)) = 0$$

for $\mu \geqslant \mu_0$, $p < n$.

Choose a metric in L such that $\omega = (\sqrt{-1}/2\pi)\Theta_L$ is positive, where Θ_L is the curvature form associated to the metric; let the metric on M be the one given by ω. Now we have seen that if E, E' are two hermitian vector bundles and if we give $E \otimes E'$ the induced metric, then

$$D_{E \otimes E'} = D_E \otimes 1 + 1 \otimes D_{E'}$$

and so

$$\Theta_{E \otimes E'} = \Theta_E \otimes 1 + 1 \otimes \Theta_{E'}$$

where D, Θ always refer to the metric connection and curvature. In particular, for L and E as above with any metric on E,

$$\Theta_{L^\mu \otimes E} = \frac{2\pi\mu}{\sqrt{-1}} \omega \otimes 1_E + \Theta_E.$$

Let $\eta \in \mathcal{H}^{0,p}(L^{-\mu} \otimes E)$ be harmonic. Writing Θ for $\Theta_{L^{-\mu} \otimes E}$, D for $D_{L^{-\mu} \otimes E}$, we have

$$\Theta = D^2 = D'\bar\partial + \bar\partial D',$$

so

$$\Theta\eta = \bar\partial D'\eta,$$

and by the Kähler identity

$$[\Lambda, \bar\partial] = -\frac{\sqrt{-1}}{2} D'^*$$

we see that

$$\begin{aligned} 2\sqrt{-1}\,(\Lambda\Theta\eta, \eta) &= 2\sqrt{-1}\,(\Lambda\bar\partial D'\eta, \eta) \\ &= 2\sqrt{-1}\left(\left(\bar\partial\Lambda + \frac{1}{2\sqrt{-1}} D'^*\right) D'\eta, \eta\right) \\ &= (D'^* D'\eta, \eta) = (D'\eta, D'\eta) \geqslant 0, \end{aligned}$$

since $(\bar\partial\Lambda D'\eta,\eta)=(\Lambda D'\eta,\bar\partial^*\eta)=0$. On the other hand,

$$\begin{aligned}2\sqrt{-1}\,(\Theta\Lambda\eta,\eta) &= 2\sqrt{-1}\,\big(D'\bar\partial\Lambda\eta,\eta\big)\\ &= 2\sqrt{-1}\left(\left(\Lambda\bar\partial-\frac{1}{2\sqrt{-1}}D'^*\right)\eta,D'^*\eta\right)\\ &= -\big(D'^*\eta,D'^*\eta\big)\leqslant 0.\end{aligned}$$

Thus we have

$$2\sqrt{-1}\,\big([\Lambda,\Theta]\eta,\eta\big)\geqslant 0.$$

But now

$$\Theta=\Theta_{L^{-\mu}\otimes E}=\Theta_E-\frac{2\pi}{\sqrt{-1}}\mu\omega,$$

and so

$$\begin{aligned}2\sqrt{-1}\,\big([\Lambda,\Theta]\eta,\eta\big) &= 2\sqrt{-1}\,\big([\Lambda,\Theta_E]\eta,\eta\big)-4\pi\mu\big([\Lambda,L]\eta,\eta\big)\\ &= 2\sqrt{-1}\,\big([\Lambda,\Theta_E]\eta,\eta\big)-4\pi\mu(n-p)\|\eta\|^2.\end{aligned}$$

Now $[\Lambda,\Theta_E]$ is bounded on $A^{0,*}(L^{-\mu}\otimes E)$, so we can write

$$|([\Lambda,\Theta_E]\eta,\eta)|\leqslant C\|\eta\|^2,$$

and consequently for $p<n$,

$$\mu>\frac{C}{2\pi}\Rightarrow\eta=0$$

i.e.,

$$\mathcal{H}^{0,p}(L^{-\mu}\otimes E)=0\qquad\text{for }\mu>\frac{C}{2\pi},\qquad p<n.\qquad\text{Q.E.D.}$$

The Lefschetz Theorem on (1,1)-classes

As an application of Theorem B, we will complete our picture of the correspondences among divisors, line bundles, and Chern classes on a complex submanifold of projective space. First, we have the

Proposition. *Let* $M\subset\mathbb{P}^N$ *be a submanifold. Then every line bundle on* M *is of the form* $L=[D]$ *for some divisor* D; *i.e.*,

$$\operatorname{Pic}(M)\cong\frac{\operatorname{Div}(M)}{\textit{linear equivalence}}.$$

Proof. To prove this, we have to show that every line bundle on M has a global meromorphic section. To find such a section, let H denote the restriction to M of the hyperplane bundle on $\mathbb{P}^N$. We will show that for $\mu \gg 0$, $L+\mu H$ has a nontrivial global holomorphic section s; then if t is any global holomorphic section of $[H]$ over M, s/t^μ will be a global meromorphic section of L as desired.

We proceed by induction on $n=\dim M$: assume that for every submanifold $V\subset\mathbb{P}^N$ of dimension less than n and every line bundle $L\to V$, $H^0(V,\mathcal{O}(L+\mu H))\neq 0$ for $\mu\gg 0$. By Bertini's theorem we can find a hyperplane $\mathbb{P}^{N-1}\subset\mathbb{P}^N$ with $V=\mathbb{P}^{N-1}\cap M$ smooth; we consider the exact sheaf sequence

$$0\to\mathcal{O}_M(L+(\mu-1)H)\xrightarrow{\otimes s}\mathcal{O}_M(L+\mu H)\xrightarrow{r}\mathcal{O}_V(L+\mu H)\to 0,$$

where s is the section of H vanishing exactly on H and r is the restriction map. For $\mu\gg 0$ we have both

$$H^0(V,\mathcal{O}(L+\mu H))\neq 0$$

by induction and

$$H^0(M,\mathcal{O}(L+\mu H))\to H^0(V,\mathcal{O}(L+\mu H))\to 0,$$

since

$$H^1(M,\mathcal{O}(L+(\mu-1)H))=0$$

by Theorem B. Thus $H^0(M,\mathcal{O}(L+\mu H))\neq 0$, and the result is proved. Q.E.D.

We now consider for a moment the general problem of analytic cycles. On a compact Kähler manifold M, the Hodge decomposition

$$H^n(M,\mathbb{C})=\bigoplus_{p+q=n}H^{p,q}(M)$$

on complex cohomology gives a slightly coarser decomposition of real cohomology

$$H^n(M,\mathbb{R})=\bigoplus_{\substack{p+q=n\\ p\leqslant q}}(H^{p,q}(M)\oplus H^{q,p}(M))\cap H^n(M,\mathbb{R}).$$

A natural question to ask is whether we can characterize geometrically the classes in homology that are Poincaré dual to classes in one of these factors. For example, we say a homology class $\gamma\in H_{2p}(M,\mathbb{Z})$ is *analytic* if it is a rational linear combination of fundamental classes of analytic subvarieties of M; dually, we say a cohomology class is *analytic* if its Poincaré dual is. Now, we have seen for purely local reasons that if $V\subset M$ is an analytic subvariety of dimension p and ψ any differential form on M,

$$\int_V\psi=\int_V\psi^{n-p,n-p}.$$

Thus if η is the harmonic form on M representing the cohomology class η_V

and ψ any harmonic form,

$$\int_M \psi\wedge\eta=\int_V\psi=\int_V\psi^{n-p,n-p}=\int_M\psi\wedge\eta^{p,p}$$

i.e., $\eta=\eta^{p,p}$, and so we see that *any analytic cohomology class of degree* 2p *is of pure type* (p,p). The famous *Hodge Conjecture* asserts that the converse is also true: On $M\subset\mathbb{P}^N$ a submanifold of projective space every rational cohomology class of type (p,p) is analytic. Whether the Hodge conjecture is true or false is at present unknown; it is a very beautiful and very difficult problem. The only case which has been proved in general is the case $p=1$; this is the

Lefschetz Theorem on (1,1)-Classes. *For* $M\subset\mathbb{P}^N$ *a submanifold, every cohomology class*

$$\gamma\in H^{1,1}(M)\cap H^2(M,\mathbb{Z})$$

is analytic; in fact

$$\gamma=\eta_D$$

for some divisor D *on* M.

Here, of course, we are writing $H^2(M,\mathbb{Z})$ for its image under the natural inclusion in $H^2(M,\mathbb{R})$.

Proof. Consider again the exact sequence

$$0\to\mathbb{Z}\to\mathcal{O}\to\mathcal{O}^*\to 0$$

and the associated cohomology sequence

$$H^1(M,\mathcal{O}^*)\xrightarrow{c_1}H^2(M,\mathbb{Z})\xrightarrow{i_*}H^2(M,\mathcal{O})\cong H^{0,2}(M).$$

We claim that the map i_* is given by first mapping $H^2(M,\mathbb{Z})\to H^2(M,\mathbb{C})$ and then projecting onto the (0,2)-factor of $H^2(M,\mathbb{C})$ in the Hodge decomposition; i.e., that the diagram

$$\begin{array}{ccc}
H^2(M,\mathbb{Z}) & \xrightarrow{i_*} & H^2(M,\mathcal{O}) \\
\downarrow & & \\
H^2(M,\mathbb{C}) & & \big\downarrow \wr\ \text{Dolbeault} \\
\text{de Rham}\ \downarrow\wr & & \\
H^2_{DR}(M,\mathbb{C}) & \xrightarrow{\pi^{0,2}} & H^{0,2}_{\bar\partial}(M)
\end{array}$$

commutes. (The map $\pi^{0,2}$ is defined on the form level, since for $\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}\in Z^2_d(M)$, $\bar\partial\omega^{0,2}=(d\omega)^{0,3}=0$). To see this, let $z=(z_{\alpha\beta\gamma})\in Z^2(M,\mathbb{Z})$; to find the image of z under the de Rham isomorphism, we take

$f_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$ such that

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} \qquad \text{in } U_\alpha \cap U_\beta \cap U_\gamma;$$

since $z_{\alpha\beta\gamma}$ is constant, $df_{\alpha\beta} + df_{\beta\gamma} - df_{\alpha\gamma} = 0$, so $(df_{\alpha\beta}) \in Z^1(M, \mathcal{Z}_k^1)$ and we can find $\omega_\alpha \in A^1(U_\alpha)$ such that

$$df_{\alpha\beta} = \omega_\beta - \omega_\alpha \qquad \text{in } U_\alpha \cap U_\beta.$$

The global 2-form $d\omega_\alpha = d\omega_\beta$ then represents the image of z in $H^2_{\mathrm{DR}}(M, \mathbb{C})$. On the other hand, take the image of $i_* z$ under the Dolbeault isomorphism: we write

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma},$$
$$\bar{\partial} f_{\alpha\beta} = \omega_\beta^{0,1} - \omega_\alpha^{0,1},$$

and we see that $\bar{\partial}\omega_\alpha^{0,1} = (d\omega_\alpha)^{0,2}$ represents z in $H^{0,2}_{\bar{\partial}}(M)$.

Now we are just about done: given $\gamma \in H^{1,1}(M) \cap H^*(M, \mathbb{Z})$, we have $i_*(\gamma) = 0$, and hence $\gamma = c_1(L)$ is the Chern class of some line bundle $L \in H^1(M, \mathcal{O}^*)$. Writing $L = [D]$ for some divisor $D = \Sigma n_i V_i$,

$$\gamma = c_1([D]) = \eta_D. \qquad \text{Q.E.D.}$$

Note that since the isomorphism

$$L^{n-1}\colon H^{1,1}(M, \mathbb{Q}) \longrightarrow H^{n-1,n-1}(M, \mathbb{Q})$$

of the hard Lefschetz theorem is given by intersection with $n-1$ hyperplanes, it takes analytic classes to analytic classes; thus the Lefschetz (1, 1) theorem also implies the Hodge conjecture for $H^{2n-2}(M, \mathbb{Q}) \cap H^{n-1,n-1}(M)$. In particular, we see that *the intersection pairing between divisors and curves on a submanifold of projective space is nondegenerate.*

3. ALGEBRAIC VARIETIES

Analytic and Algebraic Varieties

Let $X_0, \ldots, X_n$ denote Euclidean coordinates on $\mathbb{C}^{n+1}$ and also the corresponding homogeneous coordinates on $\mathbb{P}^n$. Recall that the *universal bundle* $J \to \mathbb{P}^n$ is the subbundle of the trivial bundle $\mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathbb{P}^n$ whose fiber over a point $X \in \mathbb{P}^n$ is simply the line $\{\lambda X\}_\lambda \subset \mathbb{C}^{n+1}$ corresponding to X. The *hyperplane bundle* $H \to \mathbb{P}^n$ is the dual of J, i.e., it is the bundle whose fiber over $X \in \mathbb{P}^n$ corresponds to the space of linear functionals on the line $\{\lambda X\}$. As we saw in Section 1 of this chapter, the Chern class of H is the fundamental class ω of a hyperplane in $\mathbb{P}^n$—that is, a generator of $H^2(\mathbb{P}^n, \mathbb{Z})$—and it follows from $H^1(\mathbb{P}^n, \mathcal{O}) = 0$ that every line bundle on $\mathbb{P}^n$ is a multiple H^d of H.

Consider now the global sections of the bundle H. First, we note that any linear functional L on $\mathbb{C}^{n+1}$ induces a section σ_L of H by restriction, i.e., by setting

$$\sigma_L(X) = L|_{\{\lambda X\}}.$$

Clearly σ_L is identically zero only if L is, so we have an injection

$$\mathbb{C}^{n+1*} \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(H)).$$

In fact, all of $H^0(\mathbb{P}^n, \mathcal{O}(H))$ is obtained in this way: if σ is any section of $H, D=(\sigma)$ its zero divisor, then the fundamental class η_D is given by

$$\eta_D = c_1(H) = \omega$$

and by the argument of Section 4, Chapter 0, it follows that D is a hyperplane in $\mathbb{P}^n$. If we let $L \in \mathbb{C}^{n+1*}$ be any linear functional vanishing on the hyperplane $\pi^{-1}D \subset \mathbb{C}^{n+1}$, then, the meromorphic function σ/σ_L will be holomorphic on all of $\mathbb{P}^n$, hence constant.

In general, the fiber of a power H^d of H over a point X corresponds to the space of d-linear forms on the line $\{\lambda X\} \subset \mathbb{C}^{n+1}$, and so as before any d-linear form F on $\mathbb{C}^{n+1}$ induces by restriction a global section

$$\sigma_F(X) = F|_{\{\lambda X\}}$$

of H^d. Since we are restricting F to one line at a time, we see that $\sigma_F = 0$ if F is alternating in any two factors, and so we have a map

$$\mathrm{Sym}^d(\mathbb{C}^{n+1*}) \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(H^d))$$

from the space of symmetric d-linear forms on $\mathbb{C}^{n+1}$—that is, homogeneous polynomials $F(X_0,\ldots,X_n)$ of degree d in $X_0,\ldots,X_n$—to the space of global sections of H^d. Again, the map is injective, and the zero divisor of the section σ_F is just the image in $\mathbb{P}^n$ of the zero locus of $F(X_0,\ldots,X_n)$ in $\mathbb{C}^{n+1}$.

We claim now that *these are all the global sections* of H^d. To show this, let σ be any global section of H^d, and denote by σ_F be the section of H^d corresponding to an arbitrary homogeneous polynomial $F(X_0,\ldots,X_n)$. The quotient σ/σ_F is then a meromorphic function on $\mathbb{P}^n$; let

$$G' = \pi^*\left(\frac{\sigma}{\sigma_F}\right)$$

be its pullback to $\mathbb{C}^{n+1} - \{0\}$. G' has a simple pole along the divisor $F=0$ in $\mathbb{C}^{n+1} - \{0\}$ and is holomorphic elsewhere, so the function

$$G = G' \cdot F$$

is holomorphic everywhere in $\mathbb{C}^{n+1} - \{0\}$ *and hence by Hartogs' theorem extends to an entire holomorphic function on* $\mathbb{C}^{n+1}$. Now since $G'(\lambda X) = G'(X)$ for all $X \in \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}$, and $F(\lambda X) = \lambda^d F(X)$,

$$G(\lambda X) = \lambda^d G(X),$$

i.e., G is homogeneous of degree d. Thus if $\iota: t \to (\mu_0 t,\ldots,\mu_n t)$ is any line through the origin in $\mathbb{C}^{n+1}$, the pullback $\iota^* G$ either is identically zero or has a zero of order d at $t=0$ and a pole of order d at $t=\infty$, i.e.,

$$\iota^* G = \mu \cdot t^d$$

for some μ. It follows that the power series expansion

$$G(X_0,\ldots,X_n) = \sum a_{i_0,\ldots,i_n} X_0^{i_0} \cdots X_n^{i_n}$$

for G around the origin in $\mathbb{C}^{n+1}$ contains no terms of degree other than d, i.e., that G *is a homogeneous polynomial of degree* d *in* $X_0,\ldots,X_n$. Thus $\sigma=\sigma_G$ is of the desired form, and we have shown that every global section of H^d is given by a homogeneous polynomial in $X_0,\ldots,X_n$.

We note in passing that there is a useful formula for the dimension $h^0(\mathbb{P}^n, \mathcal{O}(H^d))$ of the space of global sections of H^d, that is, the number of monomials $X_0^{i_0},\ldots,X_n^{i_n}$ of degree d in $(n+1)$ variables. We associate to any sequence $i_0,\ldots,i_n$ of integers with $\sum i_k = d$ the set of n integers

$$\{i_0+1, i_0+i_1+2,\ldots,i_0+\cdots+i_{n-1}+n\} \subset \{1,\ldots,d+n\}.$$

This subset of $\{1,\ldots,d+n\}$ determines the sequence i_k, and conversely any subset of n distinct numbers between 1 and $d+n$ corresponds to such a sequence. Thus the number of monomials of degree d in $X_0,\ldots,X_n$ is just the number $\binom{d+n}{n}$ of subsets of order n in a set of order $d+n$, and so

$$h^0(\mathbb{P}^n, \mathcal{O}(H^d)) = \binom{d+n}{n}.$$

Note that the locus of a homogeneous polynomial $F(X_0,\ldots,X_n)$ of degree d in the homogeneous coordinates X_i may also be given in terms of Euclidean coordinates $x_i = X_i/X_0$, $i=1,\ldots,n$ in $(X_0 \neq 0)$ by the inhomogeneous polynomial of degree $\leqslant d$

$$f(x_1,\ldots,x_n) = F(1,x_1,\ldots,x_n) = \frac{1}{X_0^d} F(X_0,\ldots,X_n),$$

and conversely any such polynomial

$$f(x_1,\ldots,x_n) = \sum a_{i_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

corresponds to a homogeneous polynomial

$$F(X_0,\ldots,X_n) = \sum a_{i_1,\ldots,i_n} X_0^{d-\Sigma i_k} \cdot X_1^{i_1} \cdots X_n^{i_n}.$$

f is called the *affine*, or *inhomogeneous* form of F.

We now make the

Definition. An *algebraic variety* $V \subset \mathbb{P}^n$ is the locus in $\mathbb{P}^n$ of a collection of homogeneous polynomials $\{F_\alpha(X_0,\ldots,X_n)\}$.

An algebraic variety is clearly an analytic subvariety of $\mathbb{P}^n$ and will be considered primarily as such (i.e., an algebraic variety $V \subset \mathbb{P}^n$ is called smooth, irreducible, connected, etc. if it has these properties as an analytic subvariety of $\mathbb{P}^n$). Conversely, we will show that any analytic subvariety of projective space is expressible as the locus of homogeneous polynomials. We have already done this in essence for hypersurfaces: if $V \subset \mathbb{P}^n$ is any divisor, the line bundle $[V]$ is of the form H^d for some d, and V is the zero locus of some section σ of $[V]$. But all sections σ of H^d are of the form σ_F, and so

$$V = (\sigma_F) = (F(X_0, \ldots, X_n) = 0)$$

is algebraic. In general, suppose $V \subset \mathbb{P}^n$ is a k-dimensional variety, $p \in \mathbb{P}^n$ any point not lying on V. We can find an $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^n$ through p and missing V; let $\mathbb{P}^{n-k-2}$ be an $(n-k-2)$-plane in $\mathbb{P}^{n-k-1}$ disjoint from p. Let π denote the projection from $\mathbb{P}^{n-k-2}$ onto a complementary $(k+1)$-plane $\mathbb{P}^{k+1}$; choose coordinates $X_0, \ldots, X_n$ on $\mathbb{P}^n$ so that

$$\mathbb{P}^{k+1} = (X_{k+2} = \cdots = X_n = 0)$$
$$\mathbb{P}^{n-k-2} = (X_0 = \cdots = X_{k+1} = 0)$$

and

$$\pi([X_0, \ldots, X_n]) = [X_0, \ldots, X_{k+1}].$$

By the proper mapping theorem the image $\pi(V)$ of V in $\mathbb{P}^{k+1}$ is an analytic hypersurface in $\mathbb{P}^{k+1}$, and by the hypothesis that $\mathbb{P}^{n-k-1} = \overline{\mathbb{P}^{n-k-2}, p}$ misses V, $\pi(p)$ will lie outside $\pi(V)$. By what we have seen, we can find a homogeneous polynomial $F(X_0, \ldots, X_{k+1})$ vanishing along $\pi(V)$ but not at $\pi(p)$; correspondingly, the polynomial

$$\tilde{F}(X_0, \ldots, X_n) = F(X_0, \ldots, X_{k+1})$$

vanishes on V but not at p. We can thus find, for any point $p \in V$, a polynomial vanishing identically on V but not at p, and so we have

Chow's Theorem. *Any analytic subvariety of projective space is algebraic.*

If $F(X_0, \ldots, X_n)$ and $G(X_0, \ldots, X_n) \neq 0$ are two homogeneous polynomials of the same degree d in the homogeneous coordinates X on $\mathbb{P}^n$, the quotient

$$\varphi(X) = \frac{F(X)}{G(X)}$$

is a well-defined meromorphic function on $\mathbb{P}^n$; such a mermorphic function is called a *rational function*. Note that after dividing top and bottom by powers of X_0, we may write the function φ as

$$\varphi(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)},$$

where f and g are polynomials (not necessarily both of degree d) in the Euclidean coordinates x_i. Thus the field $K(\mathbb{P}^n)$ of rational functions on $\mathbb{P}^n$ is isomorphic to $\mathbb{C}(x_1, \ldots, x_n)$.

It is not hard to see that any meromorphic function on $\mathbb{P}^n$ is rational. By Chow's theorem, both the zero-divisor $(\varphi)_0$ and the polar divisor $(\varphi)_\infty$ of φ are expressible as the loci of homogeneous polynomials $F(X)$ and $G(X)$. Since moreover the divisor (φ) is homologous to zero, F and G have the same degree, so F/G is a well-defined rational function on $\mathbb{P}^n$; then from

$$(F/G) = (\varphi)$$

it follows that

$$\varphi = \lambda F/G$$

for some $\lambda \in \mathbb{C}$.

Now if $V \subset \mathbb{P}^n$ is any smooth variety, a meromorphic function on V is called rational if it is the restriction to V of a rational function on $\mathbb{P}^n$. The rational functions of V a priori form a subfield of the field $\mathfrak{M}(V)$ of meromorphic functions; in fact,

Every meromorphic function on an algebraic variety $V \subset \mathbb{P}^n$ *is rational.*

The proof of this assertion is in two stages: first, we express V as a branched cover of a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ by projection, and deduce from this representation that the pullback $\pi^* K(\mathbb{P}^k)$ to V of the field of rational functions on $\mathbb{P}^k$ has index at most $d = \deg(V)$ in the field $\mathfrak{M}(V)$; we then show that the field $K(V)$ is an extension of degree at least d over $\pi^* K(\mathbb{P}^k)$.

For the first part, choose a generic $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^n$; at this stage we require only that $\mathbb{P}^{n-k-1}$ be disjoint from V. Let $\mathbb{P}^k$ be a complementary k-plane, and $\pi : V \to \mathbb{P}^k$ the projection from $\mathbb{P}^{n-k-1}$. For each point p of $\mathbb{P}^k$, the inverse image $\pi^{-1}(p)$ is just the intersection of V with the $(n-k)$-plane $\overline{\mathbb{P}^{n-k-1}, p}$; since $\overline{\mathbb{P}^{n-k-1}, p}$ will generically intersect V in $d = \deg(V)$ points, π expresses V as a d-sheeted branched cover of $\mathbb{P}^k$ almost everywhere. In fact, π must be everywhere finite: if for any point p in $\mathbb{P}^k$ the $(n-k)$-plane $\overline{\mathbb{P}^{n-k-1}, p}$ intersected V in a curve, that curve would necessarily meet the hyperplane $\mathbb{P}^{n-k-1} \subset \overline{\mathbb{P}^{n-k-1}, p}$, contrary to the hypothesis that $\mathbb{P}^{n-k-1}$ is disjoint from V.

Note that if we choose homogeneous coordinates $X = [X_0, \ldots, X_n]$ on $\mathbb{P}^n$ such that $\mathbb{P}^{n-k-1}$ is given as $(X_0 = \cdots = X_k = 0)$ and $\mathbb{P}^k$ as $(X_{k+1} = \cdots = X_n = 0)$, the map π is given by

$$\pi([X_0, \ldots, X_n]) = [X_0, \ldots, X_k].$$

In particular, the pullback $\pi^* f$ to V of any rational function f on $\mathbb{P}^k$ is clearly rational, so that on V we have inclusions

$$\pi^* K(\mathbb{P}^k) \subset K(V) \subset \mathfrak{M}(V).$$

Now, to see that $\pi^* K(\mathbb{P}^k)$ has index at most d in $M(V)$, let φ be any meromorphic function on V and let $D = (\varphi)_\infty$. Let $B \subset \mathbb{P}^k$ be the branch locus of π; on $\mathbb{P}^k - B$ we can define functions ψ_i by

$$\psi_1(p) = \sum_{q \in \pi^{-1}(p)} \varphi(q),$$

$$\psi_2(p) = \sum_{q \neq q' \in \pi^{-1}(p)} \varphi(q) \cdot \varphi(q'),$$

$$\vdots$$

$$\psi_d(p) = \prod_{q \in \pi^{-1}(p)} \varphi(q),$$

i.e., we let $\psi_i(p)$ be the ith symmetric polynomial in the values of φ at the d points of $\pi^{-1}(p)$ in V. ψ_i is then a holomorphic function on $\mathbb{P}^k - B - \pi(D)$, and being bounded away from $\pi(D)$ it extends by the Riemann extension theorem to a holomorphic function on $\mathbb{P}^k - \pi(D)$. We claim that ψ_i extends to a meromorphic function on all of $\mathbb{P}^k$. If $p \in \pi(D)$ is any point and $f(X)$ a local defining function for $\pi(D)$ in a neighborhood Δ of p, then for m sufficiently large, the function

$$\varphi' = \varphi \cdot \pi^* f^m$$

will be holomorphic in $\pi^{-1}(\Delta)$. For $q \in \Delta - B$, then, let

$$\psi_i'(q) = \sum_{\alpha_1, \ldots, \alpha_i} \left(\prod \varphi'(p_{\alpha_1}) \cdots \varphi'(p_{\alpha_i}) \right)$$

be the ith symmetric function of the values of φ' at the points of $\pi^{-1}(q)$; being bounded in any compact subset of Δ, ψ_i' likewise extends to a holomorphic function on Δ. Writing

$$\psi_i = \frac{\psi_i'}{f^{i-m}},$$

we see that ψ_i extends to a meromorphic function in Δ, and hence in all of $\mathbb{P}^k$. Thus the functions ψ_i are rational functions. But now on V we have

$$\varphi^d - \pi^* \psi_1 \cdot \varphi^{d-1} + \pi^* \psi_2 \cdot \varphi^{d-2} - \cdots + (-1)^d \pi^* \psi \equiv 0,$$

i.e., every meromorphic function $\varphi \in \mathfrak{M}(V)$ satisfies a polynomial relation of degree d over $\pi^* K(\mathbb{P}^k)$. By the primitive element theorem, then, the field extension $\mathfrak{M}(V) \supset \pi^* K(\mathbb{P}^k)$ is finite of degree at most d.

To complete the proof of our assertion, we want to exhibit a rational function on V which satisfies no polynomial relation of degree less than d over the field $\pi^* K(\mathbb{P}^k)$. To do this, we factor the projection map π: choose generic planes $\mathbb{P}^{n-k-2} \subset \mathbb{P}^{n-k-1}$ and $\mathbb{P}^{k+1} \subset \mathbb{P}^k$, and let $\pi': V \to \mathbb{P}^{k+1}$ be projection from $\mathbb{P}^{n-k-2}$. We may take homogeneous coordinates $X = [X_0, \ldots, X_n]$ on $\mathbb{P}^n$ such that

$$\mathbb{P}^{n-k-1} = (X_0 = \cdots = X_k = 0), \qquad \mathbb{P}^k = (X_{k+1} = \cdots = X_n = 0),$$

$$\mathbb{P}^{n-k-2} = (X_0 = \cdots = X_{k+1} = 0), \qquad \mathbb{P}^{k+1} = (X_{k+2} = \cdots = X_n = 0);$$

in terms of these coordinates, π is given as before and

$$\pi'([X_0,\ldots,X_n]) = [X_0,\ldots,X_{k+1}]$$

so that π is just the composition of π' with projection from the point $(X_0 = \cdots = X_k = X_{k+2} = \cdots = X_n = 0)$ in $\mathbb{P}^{k+1}$ onto $\mathbb{P}^k$. Note that, $\mathbb{P}^{n-k-2}$ having been chosen generically, the map π' will be one-to-one over an open set in its image: this will be the case as long as for some point p in V the $(n-k-1)$-plane $\overline{\mathbb{P}^{n-k-2},p}$ meets V only at p—but for any p in V, the generic $(n-k-1)$-plane through p meets V only at p.

Now, consider the rational function

$$x_{k+1} = \frac{X_{k+1}}{X_0}$$

on V. Suppose that x_{k+1} satisfied an equation of the form

$$x_{k+1}^{d'} + \psi_1(x_1,\ldots,x_k)\cdot x_{k+1}^{d'-1} + \cdots + \psi_{d'}(x_1,\ldots,x_k) \equiv 0, d' < d.$$

Then for a generic point $p=[\alpha_0,\ldots,\alpha_k]$ in $\mathbb{P}^k$, the inverse image of p in $\pi'(V) \subset \mathbb{P}^{k+1}$ would consist of at most of the d' points $\{[\alpha_0,\ldots,\alpha_k,\beta]\}$, where

$$\beta^{d'} + \psi_1\left(\frac{\alpha_1}{\alpha_0},\ldots,\frac{\alpha_k}{\alpha_0}\right)\beta^{d'-1} + \cdots + \psi_{d'}\left(\frac{\alpha_1}{\alpha_0},\ldots,\frac{\alpha_k}{\alpha_0}\right) = 0.$$

But since the projection $\pi' : V \to \mathbb{P}^{k+1}$ is generically one-to-one onto its image and the fibers of π generically consist of $d > d'$ points, this is impossible. Q.E.D.

Note, as a consequence, that the field of rational functions on an algebraic variety V is independent of the embedding. Thus, the *sheaf of germs of polynomial functions on* V, which associates to every open set U on V the ring of rational functions on V finite in U, is intrinsically associated to V. This sheaf, the basic structure sheaf in algebraic treatments of the subject, is also denoted by $\mathcal{O}_V$.

It is not hard to see by the same sort of argument that

1. Any meromorphic differential form on a smooth variety is algebraic, that is, expressible in terms of rational functions and their differentials.
2. Any holomorphic map between smooth varieties may be given by rational functions.
3. Any holomorphic vector bundle on a smooth variety is algebraic, that is, may be given by rational transition functions.

The first assertion we can prove now: clearly the differentials $d\varphi$ of the rational functions on V span the cotangent space to V at every point, and so a finite number of them do; any meromorphic form on V is then a linear combination of wedge products of these forms with meromorphic, hence rational, coefficient functions. The second assertion will follow once we see in the following section that the product $V\times W$ of two algebraic varieties is again a variety; by Chow's theorem the graph $\Gamma\subset V\times W\subset\mathbb{P}^n$ is then cut out by polynomials. The third assertion will be clear once we have discussed the Grassmannian manifold and proved an embedding theorem for vector bundles on algebraic varieties in Sections 5 and 6 of this chapter.

All these results are special instances of the general *G.A.G.A. principle** that *any global analytic object on an algebraic variety is algebraic*. The importance of Chow's theorem and the G.A.G.A. principle is, in this treatment, primarily philosophical rather than practical. While we shall not use them as tools in our study—most of our techniques apply uniformly to all analytic phenomena on a variety, so it will not be useful for us to know, for instance, that a given meromorphic function or map is rational—they assure us that, in treating varieties as analytic rather than algebraic entities, we are still dealing with the same class of objects.

Degree of a Variety

The fundamental projective invariant of an algebraic variety $V\subset\mathbb{P}^n$ is its degree, defined as follows: Taking the class of a k-plane $\mathbb{P}^k\subset\mathbb{P}^n$ as generator, we have an isomorphism

$$H_{2k}(\mathbb{P}^n,\mathbb{Z})\cong\mathbb{Z}.$$

The *degree* of a k-dimensional variety $V\subset\mathbb{P}^n$ is its fundamental class in $H_{2k}(\mathbb{P}^n,\mathbb{Z})$ via this identification.

Alternative definitions abound. First, by Bertini applied to the smooth locus of V the generic $(n-k)$-plane $\mathbb{P}^{n-k}\subset\mathbb{P}^n$ will intersect V transversely, and so will meet V in exactly

$$^{\#}(\mathbb{P}^{n-k}\cdot V)=\text{degree}(V)$$

points; thus we may define the degree of a variety to be the number of points of intersections of V with a generic linear subspace of complementary dimension. On the other hand, if ω is the standard Kähler form on $\mathbb{P}^n$,

$$\int_V\omega^k=\text{degree}(V)\cdot\int_{\mathbb{P}^k}\omega^k=\text{degree}(V),$$

so we may define the degree of V to be simply its volume divided by $k!$. (This is sometimes called the Wirtinger theorem.) In case $V\subset\mathbb{P}^n$ is a hypersurface, we have seen that it may be given in terms of homogeneous

*So named after J. P. Serre's paper, Geometrie Algebrique et Geometrie Analytique, *Annals of the Institute Fourier*, Vol. 6.

coordinates $X_0,\ldots,X_n$ on $\mathbb{P}^n$ as the locus

$$V = (F(X_0,\ldots,X_n)=0)$$

of a homogeneous polynomial F. If F has degree d, then the fundamental class of $V=(\sigma_F)$ is $\eta_V = c_1(H^d)$—that is, d times the class of a hyperplane—so V has degree d. Alternatively, if

$$[Y_0, Y_1] \xrightarrow{\mu} [a_0Y_0+b_0Y_1,\ldots,a_nY_0+b_nY_1]$$

is a generic line in $\mathbb{P}^n$, the pullback μ^*F of F to $\mathbb{P}^1$ will be homogeneous of degree d in Y_0 and Y_1, and so by the fundamental theorem of algebra will have exactly d roots. The degree of V is thus the degree d of the polynomial F.

A basic fact about degree is that it is multiplicative with respect to intersections. Since a $\mathbb{P}^{n-k_1}$ and a $\mathbb{P}^{n-k_2}$ intersect transversely in a $\mathbb{P}^{n-k_1-k_2}$, the degree of the intersection of two varieties meeting transversely almost everywhere is the product of their degrees. More generally, if V and W are varieties of degrees d_1 and d_2 in $\mathbb{P}^n$ intersecting in a variety of the appropriate dimension, $\{Z_i\}$ the irreducible components of $V\cap W$, then

$$d_1\cdot d_2 = \sum_i \operatorname{mult}_{Z_i}(V\cdot W)\cdot \operatorname{degree}(Z_i)$$

with $\operatorname{mult}_{Z_i}(V\cdot W)$ defined as in Section 4 of Chapter 0. This is of particular interest in the case of complementary dimension. For example, if C and D are two curves in $\mathbb{P}^2$ of degree d_1 and d_2 and having no component in common—that is, intersecting only in points—we see that they can have at most d_1d_2 points of intersection. This is a weak form of

Bezout's Theorem. *Two relatively prime polynomials* $f(x,y),\ g(x,y)\in \mathbb{C}[x,y]$ *of degrees* d_1 *and* d_2 *can have at most* d_1d_2 *simultaneous solutions.*

The degree also behaves well with respect to the geometric operations of projection and coning. Clearly, if $V\subset\mathbb{P}^n$ is any variety, $p\in\mathbb{P}^n$ any point not on V, and $\pi_p: V\to\mathbb{P}^{n-1}$ the projection onto a hyperplane, then

$$\deg(V) = \deg(\pi_p(V)):$$

the number of points of intersection of $\pi(V)$ with a generic $(n-k-1)$-plane $\mathbb{P}^{n-k-1}$ in $\mathbb{P}^{n-1}$ is just the number of points of intersection of V with the $(n-k)$-plane $\mathbb{P}^{n-k}=\overline{\mathbb{P}^{n-k-1},p}$ in $\mathbb{P}^n$; since by Bertini the generic $\mathbb{P}^{n-k}$ through p meets V transversely, this is just the degree of V.

Coning is an operation we have not previously encountered. If $V\subset\mathbb{P}^n$ is any variety, $p\in\mathbb{P}^n$ at any point lying off V, we take the *cone* $\overline{p,V}$ *through* p *over* V to be the union of the lines through p meeting V. That $\overline{p,V}$ is a variety is easy to see: it is the image under projection on the first factor of the incidence correspondence $I \subset \mathbb{P}^n \times \mathbb{P}^n$ defined by

$$I = \{(q,r):\ r \in V, p \wedge q \wedge r = 0\},$$

itself an analytic subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. (Alternatively, if in homogeneous coordinates $p = [0,\dots,0,1]$, let $\mathbb{P}^{n-1}$ be the hyperplane $X_n = 0$; if the image $\pi_p(V) \subset \mathbb{P}^{n-1}$ of V under projection from p is cut out in $\mathbb{P}^{n-1}$ by polynomials $\{F_\alpha(X_0,\dots,X_{n-1})\}$, then the cone $\overline{p,V}$ is cut out by the polynomials $\{\tilde{F}_\alpha(X_0,\dots,X_n) = F_\alpha(X_0,\dots,X_{n-1})\}$.) Now if $H \subset \mathbb{P}^n$ is a generic hyperplane, not containing p, then the intersection of H with the cone $\overline{p,V}$ will be simply the projection $\pi_p(V)$ of V from p into H; so

$$\deg(\overline{p,V}) = \deg(H \cap \overline{p,V})$$

$$= \deg(\pi_p(V)) = \deg(V).$$

Another variety we may associate with a variety $V \subset \mathbb{P}^n$ is its *chordal variety* $C(V)$, defined to be the union of all lines meeting V twice or, in the limiting case, tangent to V. $C(V)$ is the image under projection on the third factor of the closure of the incidence correspondence $I \subset \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$ defined by

$$I = \{(p,q,r):\ p \neq q \in V, p \wedge q \wedge r = 0\}.$$

$\bar{I}$ is an analytic subvariety of $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, and so $C(V)$ is an analytic variety in $\mathbb{P}^n$. Note that since projection on the first factor maps I onto V with $(\dim V + 1)$-dimensional fibers, I has dimension $2 \cdot \dim V + 1$. $C(V)$ will thus have dimension at most $2 \cdot \dim V + 1$; generally, this will be exact. In particular, since the projection π_p of a smooth variety into a hyperplane will be an embedding if and only if $p \notin C(V)$, we see that if $n > 2 \cdot \dim V + 1$, then V may be smoothly projected into a hyperplane. Thus

Any smooth algebraic variety of dimension k *may be embedded in* $\mathbb{P}^{2k+1}$.

As we shall see, the degree of the chordal variety $C(V)$ of a variety does not depend on the degree of V alone.

A variety $V \subset \mathbb{P}^n$ is called *nondegenerate* if it does not lie in a hyperplane. We have the following condition on the degree of a nondegenerate variety:

If $V \subset \mathbb{P}^n$ *is an irreducible, nondegenerate,* k*-dimensional variety, then*

$$\deg(V) \geqslant n - k + 1.$$

We prove this first for V a curve in $\mathbb{P}^n$. Any n points of V lie in a hyperplane H, and if the degree of V were less than n, then H, having n

points in common with V, would have a curve in common with V; being irreducible, V would then lie in H.

Turning to the general case, we have to show that the generic hyperplane section $H \cap V$ of an irreducible nondegenerate variety V of dimension $\geqslant 2$ is again irreducible and nondegenerate in H. The latter part is clear: the condition that $H \cap V$ be degenerate is a closed one on $H \in \mathbb{P}^{n*}$, and since V itself is nondegenerate, we can find n points of V spanning a hyperplane, so not every hyperplane section of V can be degenerate.

The former half of our assertion—that the generic hyperplane section of an irreducible variety is irreducible—is somewhat harder. We note first that in case V is smooth, this follows easily from the Bertini theorem and the Lefschetz theorem on hyperplane sections: by Bertini, the generic hyperplane section $H \cap V$ is smooth, and so by Lefschetz,

$$H_0(H \cap V, \mathbb{C}) \cong H_0(V, \mathbb{C}) \cong \mathbb{C};$$

i.e., $H \cap V$ is connected. Thus, if $H \cap V$ were reducible, the components of $H \cap V$ would have to meet each other; but their points of intersection would be singular points of $H \cap V$, and so this cannot happen.

To prove the assertion in the general case requires a different approach. We argue as follows: let $p \in V$ be any smooth point, and let $\mathbb{P}^{n-2} \subset \mathbb{P}^n$ be an $(n-2)$-plane meeting V transversely at p; let Z be the irreducible component of $V \cap \mathbb{P}^{n-2}$ containing p. Now consider the pencil $\{H_\lambda\}$ of hyperplanes in $\mathbb{P}^n$ containing $\mathbb{P}^{n-2}$. Each hyperplane section $H_\lambda \cap V$ of V contains Z, but since each H_λ intersects V transversely at p, p—being a smooth point of $H_\lambda \cap V$—can lie on at most one of the irreducible components of $H_\lambda \cap V$ for each λ. Let V' be the union of the irreducible components of the sections $H_\lambda \cap V$ that contain Z. Then V' is an open k-dimensional analytic variety contained in V, and hence its closure $\overline{V'}$ must be all of V; thus $H_\lambda \cap V = H_\lambda \cap V'$ is irreducible for generic λ.

Now the original lemma follows readily from the curve case: if $V \subset \mathbb{P}^n$ is any irreducible nondegenerate k-dimensional variety of degree d, then the generic intersection of V with $k-1$ hyperplanes is an irreducible, nondegenerate curve of degree d in $\mathbb{P}^{n-k+1}$, and so

$$d \geqslant n - k + 1.$$

We can restate the lemma as follows: any irreducible k-dimensional variety $V \subset \mathbb{P}^n$ of degree d must lie in a linear space of dimension $d + k - 1$; as a corollary, then, we see again that any variety of degree one in $\mathbb{P}^n$ is a linear subspace.

We shall see that varieties that realize this lower bound on the degree—e.g., curves of degree n in $\mathbb{P}^n$, surfaces of degree $n-1$ in $\mathbb{P}^n$, etc.—are of a special character.

Tangent Spaces to Algebraic Varieties

To a variety $V\subset\mathbb{P}^n$ and a smooth point $p\in V$ is associated a linear subspace of $\mathbb{P}^n$, the *tangent space to V at p*. This may be defined in several ways; we mention two here.

1. The complement of a hyperplane $H\subset\mathbb{P}^n$ is isomorphic to $\mathbb{C}^n$ via Euclidean coordinates; we may take the tangent space to $V\subset\mathbb{P}^n$ at p to be the closure in $\mathbb{P}^n$ of the usual tangent subspace $T_p(V)\subset T_p(\mathbb{C}^n)$. Explicitly, if $x_1,\ldots,x_n$ are Euclidean coordinates on $\mathbb{P}^n$ in a neighborhood of $p=(\alpha_1,\ldots,\alpha_n)$ and V is cut out by functions $\{f_\alpha(x_1,\ldots,x_n)\}$, this is just the linear subspace of $\mathbb{P}^n$ defined by

$$\sum_{i=1}\frac{\partial f_\alpha}{\partial x_i}(p)\cdot(x_i-\alpha_i)=0.$$

2. Alternatively, if V is given in terms of homogeneous coordinates $X_0,\ldots,X_n$ as the locus of polynomials $\{F_\alpha(X_1,\ldots,X_n)\}$, this is the linear subspace

$$\sum_{i=0}^{n}\frac{\partial F_\alpha}{\partial X_i}(p)\cdot X_i=0,$$

where the differentiation is formal: if f_α is the inhomogeneous form of F_α, then $\partial f_\alpha/\partial x_i$ is the inhomogeneous form of $\partial F_\alpha/\partial X_i$, and by virtue of the relation

$$\frac{1}{d}\cdot\sum_{i=0}^{n}\frac{\partial F}{\partial X_i}=F,$$

where $d=\deg(F)$, we can write

$$\begin{aligned}\sum_{i=0}^{n}\frac{\partial F_\alpha}{\partial X_i}(p)\cdot X_i&=X_0\sum_{i=0}^{n}\frac{\partial F_\alpha}{\partial X_i}(p)\cdot x_i\\&=X_0\sum_{i=0}^{n}\frac{\partial F_\alpha}{\partial X_i}(p)\cdot(x_i-\alpha_i)\\&=X_0\cdot\sum_{i=1}^{n}\frac{\partial f_\alpha}{\partial x_i}(p)\cdot(x_i-\alpha_i),\end{aligned}$$

so the homogeneous form describes the same subspace.

In a similar way we may define the *tangent cone* to a variety $V\subset\mathbb{P}^n$ at a (possibly singular) point $p\in V$. First, if V is a hypersurface cut out by the homogeneous polynomial F, and p a point of multiplicity k on V—so that all the partial derivatives of F of order $\leqslant k-1$ vanish—we take the tangent cone to V at p to be the locus

$$T_p(V) = \left(\sum \frac{\partial^k F}{\partial^{i_0} X_0 \cdots \partial^{i_n} X_n}(p) \cdot X_0^{i_0} \cdots X_n^{i_n} = 0 \right).$$

In general, we will take the tangent cone to a variety $V \subset \mathbb{P}^n$ at a point p to be the intersection of the tangent cones at p to all the hypersurfaces containing V near p.

This may be realized alternately as the union of the tangent lines at p to all curves lying on V and passing through p; or as the limiting position of chords $\lim_{\lambda\to 0} \overline{p, q(\lambda)}$ where $q(\lambda)$ is an arc in V with $q(0)=p$.

4. THE KODAIRA EMBEDDING THEOREM

Line Bundles and Maps to Projective Space

We will be concerned in this section with determining exactly when a compact complex manifold is an algebraic variety, i.e., when it can be embedded in projective space. We first establish a basic formalism for maps to $\mathbb{P}^N$.

Let M be a compact complex manifold, $L \to M$ a holomorphic line bundle. Recall that to any subspace E of the vector space $H^0(M, \mathcal{O}(L))$ is associated the linear system

$$|E| = \{(s)\}_{s \in E} \subset \operatorname{Div}(M)$$

of divisors on M. Since M is compact, $(s)=(s')$ only if $s = \lambda s'$ for some nonzero constant $\lambda \in \mathbb{C}$; thus $|E|$ is parametrized by the projective space $\mathbb{P}(E)$.

Suppose in addition that the linear system $|E|$ has no base points, i.e., that not all $s \in E$ vanish at any point $p \in M$. Then for each $p \in M$ the set of sections $s \in E$ vanishing at p forms a hyperplane $\tilde{H}_p \subset E$—or, equivalently, the set of divisors $D \in |E|$ containing p forms a hyperplane H_p in $\mathbb{P}(E)$—and so we can define a map

$$\iota_E : M \to \mathbb{P}(E)^*,$$

by sending $p \in M$ to $H_p \in \mathbb{P}(E)^*$.

We can describe the map ι_E more explicitly as follows. Choose a basis $s_0, \ldots, s_N$ for E. If we let $s_{i,\alpha} = \varphi_\alpha^*(s_i) \in \mathcal{O}(U)$ for any trivialization φ_α of L over an open set $U \in M$, it is clear that the point $[s_{0,\alpha}(p), \ldots, s_{N,\alpha}(p)] \in \mathbb{P}^N$ is independent of the trivialization φ_α chosen; we denote this point by $[s_0(p), \ldots, s_N(p)]$. In terms of the identifications $\mathbb{P}(E)^* \cong \mathbb{P}^N$ corresponding to the choice of basis $s_0, \ldots, s_N$, then, the map ι_E is given by

$$\iota_E(p) = [s_0(p), \ldots, s_N(p)].$$

We see from this representation that ι_E is holomorphic.

Now let H be the hyperplane bundle on $\mathbb{P}^N$. The pullback bundle $\iota_E^*(H)$ on M is given by the divisor (s_i)—that is,

$$L = \iota_E^*(H).$$

Moreover, any section $s=\Sigma a_i s_i \in E$ is the pullback of the section $\Sigma a_i Z_i$ of H on $\mathbb{P}^N$; i.e.,

$$E = \iota_E^*(H^0(\mathbb{P}^N, \mathcal{O}(H))) \subset H^0(M, \mathcal{O}(L)).$$

Thus $\iota_E : M \to \mathbb{P}^N$ determines both the line bundle L and the subspace $E \subset H^0(M, \mathcal{O}(L))$, and we have a basic dictionary

$$\left\{\begin{array}{l}\text{nondegenerate maps}\\ f\colon\ M \to \mathbb{P}^N, \text{ modulo}\\ \text{projective}\\ \text{transformations}\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{line bundles } L \to M\\ \text{with } E \subset H^0(M, \mathcal{O}(L))\\ \text{such that } |E| \text{ has no base points}\end{array}\right\}$$

where the choice of homogeneous coordinates on $\mathbb{P}^N$ corresponds to the choice of basis $s_0, \ldots, s_N$ for E.

We will often write ι_L for $\iota_{H^0(M, \mathcal{O}(L))}$ and ι_D for $\iota_{[D]}$.

Note that the degree of the image of M under ι_E—that is, the intersection of M with n general hyperplanes in $\mathbb{P}^n$—is just the n-fold self-intersection of a representative divisor $D \in |E|$, that is,

$$\deg(\iota_E M) = c_1(L)^n.$$

A variety $V \subset \mathbb{P}^n$ is called *normal* if the linear system on V giving the embedding $\iota\colon V \hookrightarrow \mathbb{P}^n$ is complete, that is, if the restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}(H)) \to H^0(V, \mathcal{O}(H))$$

is surjective. Note that *any hypersurface* $V \subset \mathbb{P}^n$ *is normal*: from the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(H - V) \to \mathcal{O}_{\mathbb{P}^n}(H) \xrightarrow{r} \mathcal{O}_V(H) \to 0$$

we have an exact sequence of cohomology groups

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H)) \xrightarrow{r} H^0(V, \mathcal{O}_V(H)) \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H - V)),$$

But

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(H - V)) = H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}((1-d)H)) = 0$$

so r must be surjective. Note that two normal varieties $V, V' \subset \mathbb{P}^n$ will be *projectively isomorphic*—that is, V may be carried into V' by an automorphism of $\mathbb{P}^n$—if V is biholomorphic to V' via a mapping carrying $H_{V'}$ to H_V. In particular, if V and V' are smooth hypersurfaces of dimension $\geqslant 3$

and degree $d \neq n+1$ in $\mathbb{P}^n$, then by the adjunction formula

$$K_V = (K_{\mathbb{P}^n} \otimes [V])|_V = [(d-n-1)H]$$

and likewise for V'. But by the Lefschetz theorem on hyperplane sections

$$H^1(V,\mathcal{O}) \cong H^1(\mathbb{P}^n,\mathcal{O}) = 0, \qquad H^2(V,\mathcal{O}) \cong H^2(\mathbb{P}^n,\mathcal{O}) = 0$$

so from the long exact cohomology sequence associated to the exponential sheaf sequence and the Lefschetz theorem again

$$\operatorname{Pic}(V) = H^1(V,\mathcal{O}^*) \cong H^2(V,\mathbb{Z}) \cong H^2(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$$

and likewise for V'. Thus if $\varphi: V \to V'$ is biholomorphic,

$$\varphi^* K_{V'} = K_V \Rightarrow \varphi^*(H|_{V'}) = H|_V,$$

so V and V' are projectively isomorphic. In conclusion

Two smooth hypersurfaces of dimension ≥ 3 *and degree* $\mathrm{d} \neq \mathrm{n}+1$ *in* $\mathbb{P}^{\mathrm{n}}$ *are isomorphic if and only if they are projectively isomorphic; or, equivalently,*

Any automorphism of a smooth hypersurface of dimension ≥ 3 *and degree* $\mathrm{d} \neq \mathrm{n}+1$ *in* $\mathbb{P}^{\mathrm{n}}$ *is induced by an automorphism of* $\mathbb{P}^{\mathrm{n}}$.

This result in fact holds for surfaces V of degree $d \neq 4$ in $\mathbb{P}^3$ as well: to apply the previous argument, we need to know only that $H^2(V,\mathbb{Z})$ contains no torsion; this follows from the fact that V is simply connected (Lefschetz theorem once more), and the statement of Poincaré duality for the torsion part $H_{*,\mathrm{tor}}$ of homology:

$$H_{i,\mathrm{tor}}(M,\mathbb{Z}) \cong H_{\mathrm{tor}}^{n-i-1}(M,\mathbb{Z}).$$

We may illustrate the correspondence between maps to projective space and base-point-free linear systems with a classical example: the *Veronese map* associated to the line bundle dH on $\mathbb{P}^n$. We have seen that the global sections of dH correspond to homogeneous polynomials of degree d in $Z = [Z_0, \ldots, Z_n]$, so that if $\{Z^\alpha = Z_0^{\alpha_0} \cdots Z_n^{\alpha_n}\}$ denotes the set of monomials of degree d in Z, then the Veronese map is given by

$$[Z_0, \ldots, Z_n] \mapsto [\ldots, Z^\alpha, \ldots].$$

It is easily verified that the Veronese map is a smooth embedding, with the property that every hypersurface of degree d in $\mathbb{P}^n$ becomes a hyperplane section of $\iota_{dH}(\mathbb{P}^n) \subset \mathbb{P}^N$. Here are a few cases:

1. The Veronese map

$$\iota_{nH}: \mathbb{P}^1 \to \mathbb{P}^n$$

is given, in terms of the Euclidean coordinate $t = Z_1/Z_0$ on $\mathbb{P}^1$, by

$$t \mapsto [1, t, t^2, \ldots, t^n].$$

Its image is a nondegenerate curve of degree n, called the *rational normal curve.*

Conversely, if $C \subset \mathbb{P}^n$ is an irreducible, nondegenerate curve of degree n, let $p_1,\ldots,p_{n-1}$ be any $n-1$ independent points of C, $V=\overline{p_1,\ldots,p_{n-1}} \cong \mathbb{P}^{n-2}$ their linear span, and $\{H_\lambda\}_{\lambda\in\mathbb{P}^1}$ the pencil of hyperplanes in $\mathbb{P}^n$ containing V. Each hyperplane H_λ will then intersect C in n points: $p_1,\ldots,p_{n-1}$, and an additional point we will call $q(\lambda)$. (In case H_λ is the hyperplane containing V and tangent to C at p_i, the point $q(\lambda)=p_i$.) Every point of C will lie on a unique hyperplane H_λ, and so the map $q:\mathbb{P}^1\to C$ is an isomorphism. Since moreover nH is the unique line bundle of degree n on $\mathbb{P}^1$, it follows that *every irreducible nondegenerate curve of degree* n *in* $\mathbb{P}^n$ *is projectively isomorphic to the rational normal curve.*

2. In terms of Euclidean coordinates $s=Z_1/Z_0$, $t=Z_2/Z_0$ on $\mathbb{P}^2$, the Veronese map $f=\iota_{2H}:\mathbb{P}^2\to\mathbb{P}^5$ is given by

$$(s,t)\mapsto[1,s,t,s^2,st,t^2].$$

The image $S=f(\mathbb{P}^2)$ is a nondegenerate surface of degree $c_1(f^*H_{\mathbb{P}^5})^2=c_1(2H_{\mathbb{P}^2})^2=4$; note that this degree is minimal in the sense of the last section.

We digress for a moment to discuss a curious feature of the Veronese surface $S\subset\mathbb{P}^5$: *it is the unique nondegenerate surface in* $\mathbb{P}^5$ *whose variety of chords* $C(S)=\bigcup_{p,q\in S}\overline{pq}$ *is a proper subvariety of* $\mathbb{P}^5$. To see this, note that for any point $p\in\mathbb{P}^5$ lying on the chord $\overline{f(u),f(u')}$ of S, the line $L=\overline{uu'}\subset\mathbb{P}^2$ is mapped into a curve of degree

$$^\#(H_{\mathbb{P}^5}\cdot f(L)) = {}^\#(2H_{\mathbb{P}^2}\cdot L) = 2$$

$\mathbb{P}^5$, hence by the result of p. 173 is a conic lying in a 2-plane $V_2\subset\mathbb{P}^5$. Now $p\in\overline{f(u),f(u')}\subset V_2$, and any line through p in V_2 must intersect $f(L)$ twice, so that *any point of* $\mathbb{P}^5$ *lying on a chord of* S *lies on infinitely many chords of* S. In particular, if we let L_0 be the line $(s=0)$ in $\mathbb{P}^2$, and let $u_0=L_0\cap L$, then the line $\overline{f(u_0),p}\subset\mathbb{P}^5$ is a chord of S. Thus

$$C(S) = \bigcup_{\substack{p\in L_0\\ q\in\mathbb{P}^2}} \overline{f(p),f(q)},$$

from which we see that $C(S)$ is of dimension at most four. Explicitly, we describe $C(S)$ as the locus

$$\{\alpha\cdot f(s,t)+(1-\alpha)\cdot f(0,t')\} = \{[1,\alpha s,\alpha t+(1-\alpha)t',\alpha s^2,\alpha st,\alpha t^2+(1-\alpha)t'^2]\}.$$

Now we solve for α, s, t, and t': given $X=[X_0,\ldots,X_5]\in C(S)$, X must be the point $\alpha\cdot f(s,t)+(1-\alpha)\cdot f(0,t')$ for the values

$$s = X_3/X_1,\quad t = X_4/X_1,\quad \alpha = X_1^2/X_0X_3,$$
$$t' = (X_2X_3-X_1X_4)/(X_0X_3-X_1^2).$$

Consequently the coordinates of $X \in C(S)$ must satisfy

$$\begin{aligned} X_5/X_0 &= \alpha t^2 + (1-\alpha)t'^2 \\ &= X_4^2/X_0X_3 + (X_2X_3 - X_1X_4)^2/(X_0X_3(X_0X_3 - X_1^2)), \end{aligned}$$

i.e.,

$$(X_0X_3 - X_1^2)X_5 = X_0X_4^2 + X_2^2X_3 - 2X_1X_2X_4,$$

and we see that the variety of chords of the Veronese surface in $\mathbb{P}^5$ is a cubic hypersurface.

We may state the original question of this section as: Given $L \to M$ a holomorphic line bundle, when is $\iota_L : M \to \mathbb{P}^N$ an embedding? First, in order for ι_L to be well-defined the linear system $|L|$ cannot have any base points, i.e., for each $x \in M$ the restriction map

$$H^0(M, \mathcal{O}(L)) \xrightarrow{r_x} L_x$$

must be surjective. Granted this, ι_L will be an embedding if

1. ι_L *is one-to-one*. Clearly this is the case if and only if for all x and y in M, there exists a section $s \in H^0(M, \mathcal{O}(L))$ vanishing at x but not at y, i.e., if and only if the restriction map

$$(*) \qquad H^0(M, \mathcal{O}(L)) \xrightarrow{r_{x,y}} L_x \otimes L_y$$

is surjective for all $x \neq y \in M$. Note that if L satisfies this condition, then $|L|$ must be base-point-free.

2. ι_L *has nonzero differential everywhere*. If φ_α is a trivialization of L near x, then this is the case if and only if for all $v^* \in T_x^*(M)$, there exists $s \in H^0(M, \mathcal{O}(L))$ with $s_\alpha(x) = 0$ and $ds_\alpha(x) = v^*$ where $s_\alpha = \varphi_\alpha^* s$. We can express this requirement more intrinsically as follows: let $\mathcal{I}_x \subset \mathcal{O}$ denote the sheaf of holomorphic functions on M vanishing at x, and let $\mathcal{I}_x(L)$ be the sheaf of sections of L vanishing at x. If s is any section of $\mathcal{I}_x(L)$ defined near x, and $\varphi_\alpha, \varphi_\beta$ are trivializations of L in a neighborhood U of x, then writing $s_\alpha = \varphi_\alpha^* s$, $s_\beta = \varphi_\beta^* s$, $s_\alpha = g_{\alpha\beta} s_\beta$, we have

$$\begin{aligned} d(s_\alpha) &= d(s_\beta) \cdot g_{\alpha\beta} + dg_{\alpha\beta} \cdot s_\beta \\ &= d(s_\beta) \cdot g_{\alpha\beta} \qquad \text{at } x. \end{aligned}$$

Thus we have a well-defined sheaf map

$$d_x : \mathcal{I}_x(L) \to T_x^{*\prime} \otimes L_x$$

and condition 2 can be stated as requiring that the map

$$(**) \qquad H^0(M, \mathcal{I}_x(L)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x$$

be surjective for all $x \in M$. Note that $(**)$ is the limiting case of $(*)$ when $y \to x$.

The result we are aiming for is the

Kodaira Embedding Theorem. *Let* M *be a compact complex manifold and* L→M *a positive line bundle. Then there exists* k_0 *such that for* $k \geqslant k_0$, *the map*

$$\iota_{L^k}: M \to \mathbb{P}^N$$

is well-defined and is an embedding of M.

Let us consider how one might go about proving this. The first thing to do is to fit the maps (*) and (**) into exact sequences and try to use our vanishing theorems directly. To this end, let $\mathcal{I}_{x,y}(L)$ denote the sheaf of sections of L vanishing at x and y, and $\mathcal{I}_x^2(L)$ the sheaf of sections of L vanishing to order 2 at x, i.e., sections s of $\mathcal{I}_x(L)$ such that $d_x(s)=0$. We have exact sheaf sequences

$$0 \to \mathcal{I}_{x,y}(L) \to \mathcal{O}(L) \xrightarrow{r_{x,y}} L_x \oplus L_y \to 0$$

and

$$0 \to \mathcal{I}_x^2(L) \to \mathcal{I}_x(L) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x \to 0;$$

so that to show that the maps (*) and (**) are surjective, it would suffice to prove that

$$H^1(M, \mathcal{I}_x^2(L)) = H^1(M, \mathcal{I}_{x,y}(L)) = 0;$$

indeed, replacing L by L^k and using $H^1(M, \mathcal{O}(L^k))=0$ for $k \geqslant k_1$, the reader may check that our theorem is equivalent to this vanishing theorem for high powers of L. The problem is that unless M is of dimension 1 neither of the sheaves $\mathcal{I}_{x,y}(L)$ and $\mathcal{I}_x^2(L)$ is the sheaf of sections of a holomorphic vector bundle—for $E \to M$ a holomorphic vector bundle and $V \subset M$ a subvariety, the kernel of the restriction map $\mathcal{O}_M(E) \to \mathcal{O}_V(E)$ is the sheaf of sections of a vector bundle if and only if V is of codimension 1 in M—and so we cannot get a direct grip on them using our technique of harmonic theory. $\mathcal{I}_x^2$ and $\mathcal{I}_{x,y}$ are examples of *coherent sheaves*, a class of sheaves broader than, but closely related to, sheaves of sections of holomorphic vector bundles. The theory of coherent sheaves will be discussed in Chapter 5.

Another approach to the problem might be to emulate the proof of the proposition on p. 161 and do an induction on the dimension of M—for example, if we could find a smooth hypersurface $V \subset M$ containing x and y, then to show the map (*) surjective, we would only have to prove it for $L|_V$ on V and show that the restriction map

$$H^0(M, \mathcal{O}_M(L)) \to H^0(V, \mathcal{O}_V(L))$$

was surjective, i.e., that

$$H^1(M, \mathcal{O}_M(L-V)) = 0.$$

But this is very nearly presupposing the result to be proved: a priori, M need not have any divisors on it at all.

It is clear by now that our difficulty lies in the simple fact that, unless M is a Riemann surface, a point on M is not a divisor. We can overcome this problem by means of a beautiful classical construction called *blowing up*, which transforms points on a complex manifold into divisors.

Blowing Up

We will first describe the blow-up of the origin in a disc Δ in $\mathbb{C}^n$. Let $z=(z_1,\ldots,z_n)$ be Euclidean coordinates in Δ and $l=[l_1,\ldots,l_n]$ corresponding homogeneous coordinates on $\mathbb{P}^{n-1}$. Let $\tilde{\Delta}\subset\Delta\times\mathbb{P}^{n-1}$ be the submanifold of $\Delta\times\mathbb{P}^{n-1}$ given by the quadratic relations

$$\tilde{\Delta} = \{(z,l): \ z_i l_j = z_j l_i \qquad \text{for all } i,j\}.$$

If we consider points $l\in\mathbb{P}^{n-1}$ as lines in $\mathbb{C}^n$, then writing these equations as $z\wedge l=0$ we see that this is just the *incidence correspondence* defined as $\{(z,l): z\in l\}$.

Now $\tilde{\Delta}$ maps onto Δ via projection on the first factor $\pi:(z,l)\mapsto z$; from the geometric interpretation it follows that the map is an isomorphism away from the origin in Δ, and $\pi^{-1}(0)$ is just the projective space of lines in Δ. In effect, $\tilde{\Delta}$ *consists of all the lines through the origin in* Δ *made disjoint.* $\tilde{\Delta}$, together with its projection map π to Δ, is called the *blow-up* of Δ at 0. The real points of the blow-up of $\Delta\subset\mathbb{C}^2$ are pictured in Figure 1.

Note that we have encountered the manifold $\tilde{\Delta}$ before: together with the projection $\pi':\tilde{\Delta}\to\mathbb{P}^{n-1}$ on the second factor it is the universal bundle J on $\mathbb{P}^{n-1}$.

Now let M be a complex manifold of dimension n, $x\in M$ any point, and $z: U\to\Delta$ a coordinate polydisc centered around $x\in M$. The restriction of the projection map

$$\pi: \ \tilde{\Delta}-E\longrightarrow U-\{x\}\subset M$$

gives an isomorphism between a neighborhood of $E=\pi^{-1}x$ in $\tilde{\Delta}$ and a neighborhood of x in M; we define the *blow-up* $\tilde{\mathrm{M}}_{\mathrm{x}}$ *of* M *at* x to be the complex manifold

$$\tilde{M}_x = M-\{x\}\cup_\pi\tilde{\Delta}$$

obtained by replacing $\Delta\subset M$ with $\tilde{\Delta}$, together with the natural projection map $\pi:\tilde{M}_x\to M$. Again, the projection $\pi:\tilde{M}_x-\{\pi^{-1}(x)\}\to M-\{x\}$ is an isomorphism; the inverse image $\pi^{-1}(x)$ in $\tilde{M}_x$ is called the *exceptional divisor* of the blow-up, and is usually denoted E or E_x.

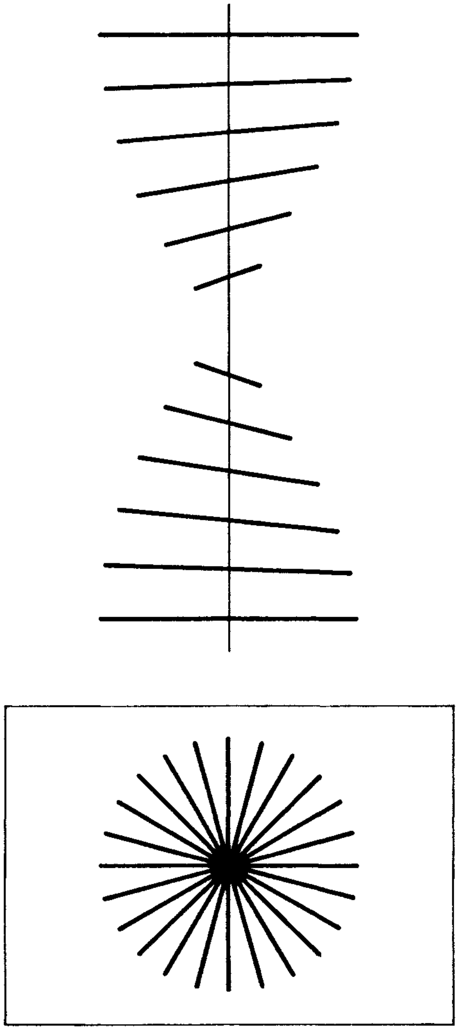


Figure 1

Note that the blow-up $\tilde{M} \to M$ is independent of the coordinates used in the disc Δ: if $\{z_i' = f_i(z)\}$ are other coordinates in Δ with $f_i(0) = 0$, $\tilde{\Delta}'$ the blow up of Δ in terms of these coordinates, then the isomorphism

$$f\colon \tilde{\Delta} - E \to \tilde{\Delta}' - E'$$

may be extended over E by setting $f(0, l) = (0, l')$, where

$$l_j' = \sum \frac{\partial f_j}{\partial z_i}(0) \cdot l_i.$$

Indeed, we see from this that the identification

$$E \longrightarrow \mathbb{P}(T_x(M))$$

given by

$$(0,l) \mapsto \left[\sum l_i \cdot \frac{\partial}{\partial z_i} \right]$$

is likewise independent of the coordinate system chosen.

Now we will describe the geometry of $\tilde{M}_x$ near E in more detail. First, we give local coordinates near E on $\tilde{M}_x$: let $z=(z_1,\ldots,z_n)$ be local coordinates on $U \ni x$ with center x. Then

$$\tilde{U} = \pi^{-1}(U) = \{(z,l) \in U \times \mathbb{P}^{n-1}: z_i l_j = z_j l_i\};$$

and we set

$$\tilde{U}_i = (l_i \neq 0) \subset \tilde{U}.$$

In this way we obtain an open cover of the neighborhood $\tilde{U}$ of E, and in each open set $\tilde{U}_i$ we have local coordinates $z(i)_j$:

$$z(i)_j = \frac{l_j}{l_i} = \frac{z_j}{z_i}, \qquad j \neq i,$$

and

$$z(i)_i = z_i.$$

The map $\pi: \tilde{M}_x \to M$ is given in $\tilde{U}_i$ by

$$(z(i)_1,\ldots,z(i)_i,\ldots,z(i)_n)) \mapsto / (z((i)_i \cdot z(i)_1,\ldots,z(i)_i,\ldots,z(i)_i \cdot z(i)_n)$$

and the divisor E is given by

$$E = (\tilde{z}(i)_i = 0)$$

in $\tilde{U}_i$. In $\tilde{U}_i \cap \tilde{U}_j$,

$$\begin{aligned} z(i)_k &= z(j)_i^{-1} \cdot z(j)_k, \\ z(i)_j &= z(j)_j^{-1}, \\ z_i &= z(j)_i \cdot z_j. \end{aligned}$$

Now, since $E=(z_i)$ in $\tilde{U}_i$, the line bundle $[E]$ is given in $\tilde{U}$ by transition functions

$$g_{ij} = z(j)_i = \frac{z_i}{z_j} = \frac{l_i}{l_j}, \text{ in } U_i \cap U_j$$

and so we can realize $[E]|_{\tilde{U}}$ by identifying the fiber

$$(*) \qquad [E]_{(z,l)} = \{\lambda(l_1,\ldots,l_n), \lambda \in \mathbb{C}\}.$$

In particular, we see that *the line bundle* $[E]|_E$ *is just the universal bundle* $J=-H$ *on* $E\cong\mathbb{P}^{n-1}$.

Dually, the line bundle $[-E]=[E]^*$ has as fiber over any point $(z,l)\in\tilde{U}$ the space of linear functionals on the line $l\subset\mathbb{C}^n$; $[-E]|_E$ is the hyperplane bundle on E.

Now we have seen that E is naturally identified with $\mathbb{P}(T_x'(M))$, so that the global sections of $[-E]$ over E correspond exactly to the linear functionals on the tangent space, i.e.,

$$(**)\qquad H^0(E,\mathcal{O}_E(-E)) = T_x^{*\prime}(M).$$

On the other hand, given a function f on U vanishing at x, the function $\pi^*f\in\mathcal{O}(\tilde{U})$ vanishes along E and so can be considered as a section of $[-E]$ over $\tilde{U}$. By explicit computation we check that for any $f\in\mathcal{I}_x(U)$ *the restriction to* E *of the section* $\pi^*f\in\mathcal{O}(-E)(\tilde{U})$ *corresponds, via the identification* $(**)$, *to the differential* $df(x)$ *of* f *at* x, i.e., the diagram

$$\begin{array}{ccc} H^0(\tilde{U},\mathcal{O}(-E)) & \xrightarrow{r_E} & H^0(E,\mathcal{O}(-E)) \\ \uparrow & & \| \\ H^0(U,\mathcal{I}_x) & \xrightarrow{d_x} & T_x^{*\prime}(U) \end{array}$$

commutes.

This correspondence reflects a basic aspect of the local analytic character of blow-ups: the infinitesimal behavior of functions, maps, or differential forms at the point x of M is transformed into global phenomena on $\tilde{M}$. Indeed, in classical terminology, a point in the exceptional divisor of the blow-up of M at x was called an "infinitely near point" of x; the exceptional divisor itself was called an "infinitesimal neighborhood" of x.

The next thing to do is to compute the curvature of the line bundles $[E]$ and $[-E]$ on $\tilde{M}$. We construct a metric on $[E]$ as follows: let h_1 be the metric on $[E]|_{\tilde{U}}$ given, in terms of the representation $(*)$ of E, by

$$|(l_1,\ldots,l_n)|^2 = \|l\|^2.$$

Let $\sigma\in H^0(\tilde{M},\mathcal{O}([E]))$ be the above global section of $[E]$ on $\tilde{M}$ with $(\sigma)=E$, so that σ is nonzero on $\tilde{M}-E$; let h_2 be the metric on $[E]|_{\tilde{M}-E}$ given by

$$|\sigma(z)|\equiv 1.$$

For $\varepsilon>0$, denote by U_ε the ball $(\|z\|<\varepsilon)$ around x in U and set $\tilde{U}_\varepsilon=\pi^{-1}(U_\varepsilon)$; let ρ_1,ρ_2 be a partition of unity for the cover $\{\tilde{U}_{2\varepsilon},\tilde{M}-\tilde{U}_\varepsilon\}$ of $\tilde{M}$, and let h be the global metric given by

$$h=\rho_1\cdot h_1+\rho_2\cdot h_2.$$

We will compute the curvature of $[E]$ with this metric. For notational convenience, let $\Omega_{[E]}$ denote $\sqrt{-1}\,/2$ times the curvature $\Theta_{[E]}$ of $[E]$. It is necessary to consider three cases:

1. On $\tilde{M}-\tilde{U}_{2\varepsilon}$, $\rho_2\equiv 1$ so $|\sigma|^2\equiv 1$; consequently

$$\Omega_{[E]} = dd^c \log \frac{1}{|\sigma|^2} \equiv 0.$$

2. On $\tilde{U}_\varepsilon - E \cong U_\varepsilon - \{x\}$, let σ be given in terms of the representation (*) by

$$\sigma(z,l) = z;$$

then

$$\Omega_{[E]} = dd^c \log \frac{1}{\|z\|^2} = -dd^c \log \|z\|^2,$$

i.e., $-\Omega_{[E]}$ is just the pullback $\pi'^*\omega$ of the associated (1,1)-form ω of the Fubini-Study metric on $\mathbb{P}^{n-1}$ under the map $\pi' : \tilde{U}_\varepsilon \to \mathbb{P}^{n-1}$ given by $(z,l)\mapsto l$. Thus

$$-\Omega_{[E]} \geqslant 0 \qquad \text{on } \tilde{U}_\varepsilon - E.$$

3. We have seen that $-\Omega_{[E]} = \pi'^*\omega$ on $\tilde{U}_\varepsilon - E$; by continuity it follows that $-\Omega_{[E]} = \pi'^*\omega$ throughout $\tilde{U}_\varepsilon$, and in particular

$$-\Omega_{[E]}|_E = \omega > 0$$

on E.

Summing up, if we let $\Omega_{[-E]}$ be $\sqrt{-1}\,/2$ times the curvature form of the dual metric in $[E]^* = [-E]$, we have

$$\Omega_{[-E]} = -\Omega_{[E]} = \begin{cases} 0 & \text{on } \tilde{M}-\tilde{U}_{2\varepsilon}, \\ \geqslant 0 & \text{on } \tilde{U}_\varepsilon, \\ >0 & \text{on } T_x'(E)\subset T_x'(\tilde{M}) \quad \text{for all } x\in E. \end{cases}$$

The point of this computation is the following: let $L\to M$ be a positive line bundle with a metric h_L whose curvature form Θ_L is $2/\sqrt{-1}$ times a positive form Ω_L. Then if Ω_{π^*L} is $\sqrt{-1}\,/2$ times the curvature form of the induced metric on the bundle $\pi^*L\to\tilde{M}$,

$$\Omega_{\pi^*L} = \pi^*\Omega_L,$$

hence $\Omega_{\pi^*L}>0$ on $\tilde{M}-E$. Moreover, for any $x\in E$ and tangent vector $v\in T_x(\tilde{M})$,

$$\langle \Omega_{\pi^*L}; v,\bar{v}\rangle = \langle \Omega_L; \pi_* v, \overline{\pi_* v}\rangle \geqslant 0$$

with equality holding if and only if $\pi_*(v)=0$, i.e., if and only if v is tangent to E. Thus

$$\Omega_{\pi^*L} = \begin{cases} \geqslant 0 & \text{everywhere,} \\ >0 & \text{on } \tilde{M}-E, \\ >0 & \text{on } T'_x(\tilde{M})/T'_x(E) \quad \text{for all } x \in E, \end{cases}$$

and the form

$$\begin{aligned} \Omega_{\pi^*L^k \otimes [-E]} &= \Omega_{\pi^*L^k} + \Omega_{[-E]} \\ &= k\Omega_{\pi^*L} + \Omega_{[-E]} \end{aligned}$$

is positive everywhere in $\tilde{U}_\varepsilon$ and $\tilde{M} - \tilde{U}_{2\varepsilon}$. Moreover, since the form $\Omega_{[-E]}$ is bounded below in $\tilde{U}_{2\varepsilon} - \tilde{U}_\varepsilon$ and Ω_{π^*L} is strictly positive there, we see that $\Omega_{\pi^*L^k\otimes[-E]}$ is everywhere positive for k sufficiently large; i.e., *there exists* k_0 *such that* $\pi^*\mathrm{L}^{\mathrm{k}} - \mathrm{E}$ *is a positive line bundle on* $\tilde{\mathrm{M}}$ *for* $\mathrm{k} \geqslant \mathrm{k}_0$.

Note that by the same argument, for any positive integer n the bundle $\pi^* L^k - nE$ will be positive for $k \gg 0$.

We need to establish one more relation between $\tilde{M}_x$ and M:

Lemma. $K_{\tilde{M}} = \pi^* K_M + (n-1)E$.

Proof. This is easy in case M has a nontrivial meromorphic n-form ω. In terms of local coordinates $z_1, \ldots, z_n$ in a neighborhood U of x, write

$$\omega(z) = \frac{f(z)}{g(z)} \cdot dz_1 \wedge \cdots \wedge dz_n.$$

Now let $z(i)_j$ be local coordinates in as before. The map π is given in $\tilde{U}_i$ by

$$(z(i)_1, \ldots, z(i)_n) \to (z(i)_1 z_i, \ldots, z_i, \ldots, z(i)_n z_i),$$

and so

$$\begin{aligned} \pi^*\omega &= \pi^*(f/g) \cdot d(z(i)_1 z_i) \wedge \cdots \wedge dz_i \wedge \cdots \wedge d(z(i)_n z_i) \\ &= \pi^*(f/g) \cdot z_i^{(n-1)} dz(i)_1 \wedge \cdots \wedge dz(i)_n. \end{aligned}$$

Thus we see that in a neighborhood of $E = \pi^{-1}(x_0)$, the divisor $(\pi^*\omega)$ is given by $\pi^*(\omega) + (n-1)E$. Since clearly $(\pi^*\omega) = \pi^*(\omega)$ away from E,

$$K_{\tilde{M}} = [(\pi^*\omega)] = \pi^* K_M + (n-1)E$$

as desired. Thus the formula is proved under the assumption that M has a meromorphic n-form; this is the easiest way to see the result.

To prove the lemma in general, we let $\underline{U} = \{U_0, U_\alpha\}_\alpha$ be an open coordinate cover of M with $x \in U_0$, $x \notin U_\alpha$ and all sets U_α having nonempty intersection with U_0 lying in one coordinate patch with coordinates $z_1, \ldots, z_n$. Let

$$\underline{\tilde{U}} = \{ \tilde{U}_\alpha = \pi^{-1} U_\alpha, \ \ \tilde{U}_i = \pi^{-1} U_0 \cap (l_i \neq 0) \}$$

be a corresponding cover for $\tilde{M}$; we compute the transition functions $\{g_{ij}, g_{i\alpha}, g_{\alpha\beta}\}$ for $K_{\tilde{M}}$ in terms of the coordinates $z(i)_j$ on $\tilde{U}_i$ and $w_{i,\alpha} = \pi^* w_{i,\alpha}$ on $\tilde{U}_\alpha$, where $\{w_{i,\alpha}\}_i$ are coordinates on U_α in M. First we have in $\tilde{U}_1 \cap \tilde{U}_2$

$$\begin{aligned} z(2)_1 &= z(1)_2^{-1}, \\ z_2 &= z(1)_2 \cdot z_1, \\ z(2)_i &= z(1)_i \cdot z(1)_2^{-1}, \qquad i \neq 1,2, \end{aligned}$$

and so the Jacobian matrix for the change of coordinates is

$$J_{12} = \begin{pmatrix} 0 & -z(1)_2^{-2} & 0 & \cdots & & 0 & & & \\ z(1)_2 & z_1 & 0 & \cdots & & 0 & & & \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & -z(1)_j \cdot z(1)_2^{-2} & 0 & \cdots & 0 & z(1)_2^{-1} & 0 & \cdots & 0 \\ \vdots & & & & & & & & \end{pmatrix};$$

in general

$$g_{ij} = \det J_{ij} = z(1)_j^{-n+1}.$$

Similarly, in $\tilde{U}_\alpha \cap \tilde{U}_1$

$$w_{1,\alpha} = z_1, \qquad w_{i,\alpha} = z_1 \cdot z(1)_i,$$

$$J_{1\alpha} = \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ z(1)_i & 0 & \cdots & z_1 & \cdots & 0 \\ \vdots & & & & & \end{pmatrix},$$

and in general

$$g_{i\alpha} = z_i^{(n-1)}.$$

Also

$$g_{\alpha\beta} = \pi^* g'_{\alpha\beta},$$

where $g'_{\alpha\beta}$ are the transition functions for K_M with respect to coordinates $w_{i,\alpha}$ in U_α, $w_{i,\beta}$ in U_β.

Now E is given in $\tilde{U}_i$ by (z_i), in $\tilde{U}_\alpha$ by (1); so the transition functions for $[E]$ over $\underline{\tilde{U}}$ are

$$\begin{aligned} h_{ij} &= \frac{z_i}{z_j} = z(i)_j^{-1}, \\ h_{i\alpha} &= z_i, \\ h_{\alpha\beta} &= 1. \end{aligned}$$

Thus the transition functions for the bundle $K_{\tilde{M}} \otimes [E]^{-n+1}$ are

$$f_{ij} = z(i)_j^{-n+1} \cdot z(i)_j^{n-1} = 1,$$
$$f_{i\alpha} = {}_iz_i^{n-1} \cdot {}_iz_i^{-n+1} = 1,$$
$$f_{\alpha\beta} = \pi^* g_{\alpha\beta},$$

and we see that $K_{\tilde{M}} - (n-1)E$ is just the pullback via π of the bundle on M given by transition functions

$$e_{0\alpha} = 1, \qquad e_{\alpha\beta} = g_{\alpha\beta};$$

i.e., $K_{\tilde{M}} - (n-1)E = \pi^* K_M$. Q.E.D.

We will develop a much more complete picture of the geometry of blow-ups later on in the chapter on surfaces; for the time being, we have enough information to proceed to the proof of the embedding theorem.

Proof of the Kodaira Theorem

Again, let $L \to M$ be a positive line bundle on the compact complex manifold M. We want to prove that there exists k_0 such that

1. The restriction map

$$H^0(M, \mathcal{O}(L^k)) \xrightarrow{r_{x,y}} L_x^k \oplus L_y^k$$

is surjective for all $x \neq y \in M$, $k \geqslant k_0$; and

2. The differential map

$$H^0(M, \mathcal{I}_x(L^k)) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x^k$$

is surjective for all $x \in M$, $k \geqslant k_0$.

To prove assertion 1, let $\tilde{M} \xrightarrow{\pi} M$ denote the blow-up of M at both x and y, $E_x = \pi^{-1}(x)$ and $E_y = \pi^{-1}(y)$ the exceptional divisors of the blow-up; for notational convenience, let E denote the divisor $E_x + E_y$ and $\tilde{L} = \pi^* L$. (Here we are tacitly assuming that $n = \dim(M) \geqslant 2$; in case M is a Riemann surface, all the arguments that follow will be valid for $\tilde{M} = M$, $\pi = id$.)

Consider the pullback map on sections

$$\pi^*: H^0(M, \mathcal{O}_M(L^k)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)).$$

For any global section $\tilde{\sigma}$ of $\tilde{L}^k$, the section of L^k given by σ over $M - \{x,y\}$ extends by Hartogs' theorem to a global section $\sigma \in H^0(M, \mathcal{O}(L^k))$, and so we see that π^* is an isomorphism. Furthermore, by definition $\tilde{L}^k$ is trivial along E_x and E_y, i.e.,

$$(\tilde{L}^k)|_{E_x} = E_x \times L_x^k, \quad (\tilde{L}^k)|_{E_y} = E_y \times L_y^k,$$

so that

$$H^0(E, \mathcal{O}_E(\tilde{L}^k)) \cong L_x^k \oplus L_y^k,$$

and if r_E denotes the restriction map to E, the diagram

$$\begin{array}{ccc} H^0(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k)) & \xrightarrow{r_E} & H^0(E,\mathcal{O}_E(\tilde{L}^k)) \\ \uparrow & & \| \\ H^0(M,\mathcal{O}(L^k)) & \xrightarrow{r_{x,y}} & L_x^k \oplus L_y^k \end{array}$$

commutes. Thus to prove assertion 1 for x and y, we have to show the map r_E is surjective.

Now, on $\tilde{M}$ we have the exact sheaf sequence

$$0 \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k) \xrightarrow{r_E} \mathcal{O}_E(\tilde{L}^k) \to 0.$$

Choose k_1 such that $L^{k_1} + K_M^*$ is positive on M. By virtue of the computation on p. 186, we can choose k_2 such that $\tilde{L}^k - nE$ is positive on $\tilde{M}$ for $k \geqslant k_2$. By the previous lemma

$$K_{\tilde{M}} = \tilde{K}_M + (n-1)E,$$

where $\tilde{K}_M = \pi^* K_M$; and so for $k \geqslant k_0 = k_1 + k_2$,

$$\begin{aligned} \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) &= \Omega^n_{\tilde{M}}\left(\tilde{L}^k - E + K^*_{\tilde{M}}\right) \\ &= \Omega^n_{\tilde{M}}\left((\tilde{L}^{k_1} + \tilde{K}^*_M) \otimes (\tilde{L}^{k'} - nE)\right) \end{aligned}$$

with $k' \geqslant k_2$. Now by hypothesis, $\tilde{L}^{k'} - nE$ has a positive definite curvature form on $\tilde{M}$; $L^{k_1} + K_M^*$ has a positive curvature form on M, and so $(\tilde{L}^{k_1} + \tilde{K}_M^*)$ has a positive semidefinite one on $\tilde{M}$. Thus the line bundle $(\tilde{L}^{k_1} + \tilde{K}_M^*) + \tilde{L}^{k'} - nE$ is positive on $\tilde{M}$, and by the Kodaira vanishing theorem,

$$\begin{aligned} H^1\left(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)\right) &= H^1\left(\tilde{M}, \Omega^n_{\tilde{M}}\left((\tilde{L}^{k_1} + \tilde{K}^*_M) + (\tilde{L}^{k'} - nE)\right)\right) \\ &= 0 \qquad \text{for } k \geqslant k_0. \end{aligned}$$

Hence the map

$$r_E\colon\ H^0\left(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)\right) \to H^0\left(E, \mathcal{O}_E(\pi^* L^k)\right)$$

is surjective for $k \geqslant k_0$, and so assertion 1 is proved for x and y.

Assertion 2 is proved similarly. Let $\tilde{M} \xrightarrow{\pi} M$ now denote the blow-up of M at x, $E = \pi^{-1}(x)$ the exceptional divisor. Again, the pullback map

$$\pi^*\colon\ H^0\left(M, \mathcal{O}_M(L^k)\right) \to H^0\left(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)\right)$$

is an isomorphism. Further, if $\sigma \in H^0(M, \mathcal{O}_M(L^k))$, then $\sigma(x) = 0$ if and only if $\tilde{\sigma} = \pi^*\sigma$ vanishes on E; thus π^* restricts to give an isomorphism

$$\pi^*\colon\ H^0\left(M, \mathcal{I}_x(L^k)\right) \longrightarrow H^0\left(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)\right).$$

As before, we can identify

$$H^0\left(E, \mathcal{O}_E(\tilde{L}^k - E)\right) = L_x^k \otimes H^0\left(E, \mathcal{O}_E(-E)\right) \cong L_x^k \otimes T_x^{*\prime},$$

and the diagram

$$\begin{array}{ccc} H^0(\tilde{M},\mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)) & \xrightarrow{r_E} & H^0(E,\mathcal{O}_E(\tilde{L}^k - E)) \\ \wr\!\uparrow \pi^* & & \| \\ H^0(M,\mathcal{I}_x(L^k)) & \xrightarrow{d_x} & T_x^{*\prime}\otimes L_x^k \end{array}$$

commutes. Thus we must prove that r_E is surjective for $k \gg 0$.

On $\tilde{M}$, there is an exact sequence

$$O \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E) \to \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E) \xrightarrow{r_E} \mathcal{O}_E(\tilde{L}^k - E) \to 0.$$

Again, choose k_1 such that $L^{k_1} + K_M^*$ is positive on M and k_2 such that $\tilde{L}^{k'} - (n+1)E$ is positive on $\tilde{M}$ for $k' \geqslant k_2$. For $k \geqslant k_0 = k_1 + k_2$

$$\mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E) = \Omega^n_{\tilde{M}}\big((\tilde{L}^{k_1} + \tilde{K}_M^*)\otimes(\tilde{L}^{k'} - (n+1)E)\big)$$

with $k' \geqslant k_2$. It follows by the Kodaira vanishing theorem that

$$H^1\big(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - 2E)\big) = 0$$

for $k \geqslant k_0$; hence r_E is surjective on global sections and assertion 2 is proved for arbitrary fixed x.

All that remains now to be proved is that we can find one value of k_0 such that assertions 1 and 2 hold for all choices of x and y and all $k \geqslant k_0$. But clearly if ι_{L^k} is defined at x and y and $\iota_{L^k}(x) \neq \iota_{L^k}(y)$, the same will be true for x' near x and y' near y, and likewise if ι_{L^k} is smooth at x it will be smooth at x' near x and separate points $x' \neq x''$ near x. Since M is compact, then, the result follows. Q.E.D.

Before proceeding to some examples and corollaries, we give a somewhat more intrinsic restatement of the theorem:

Kodaira Embedding Theorem. *A compact complex manifold* M *is an algebraic variety—i.e., is embeddable in projective space—if and only if it has a closed, positive* (1,1)*-form* ω *whose cohomology class* $[\omega]$ *is rational.*

Proof. If $[\omega] \in H^2(M,\mathbb{Q})$, then for some k, $[k\omega] \in H^2(M,\mathbb{Z})$; in the exact sequence

$$H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z}) \xrightarrow{\iota_*} H^2(M,\mathcal{O})$$

$\iota_*([k\omega]) = 0$, and so there exists a holomorphic line bundle $L \to M$ with $c_1(L) = [k\omega]$. The line bundle L will then be positive. Q.E.D.

A metric whose (1,1)-form is rational is called a *Hodge metric.*

Corollary. *If* M, M′ *are algebraic varieties, then* M × M′ *is.*

Proof. If ω, ω' are closed, integral, positive $(1,1)$-forms on M, M', respectively, and $\pi : M \times M' \to M$, $\pi' : M \times M' \to M'$ are the projection maps, then $\pi^*\omega + \pi'^*\omega'$ is again closed, integral, and positive of type $(1,1)$. Q.E.D.

A classical example of this is the *Segré map* $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ given by the complete linear system of the line bundle $\pi_1^* H \otimes \pi_2^* H$ on $\mathbb{P}^n \times \mathbb{P}^m$. For example, the Segré map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given, in terms of homogeneous coordinates $[z_0, z_1]$ and $[w_0, w_1]$ on $\mathbb{P}^1$, by

$$([z_0, z_1], [w_0, w_1]) \to [z_0 w_0, z_0 w_1, z_1 w_0, z_1 w_1].$$

The image is just the quadric hypersurface $(X_0 X_3 = X_1 X_2)$ in $\mathbb{P}^3$.

Corollary. *If* M *is an algebraic variety,* $\tilde{\mathrm{M}} \overset{\pi}{\to} \mathrm{M}$ *the blow-up of* M *at a point* x, *then* $\tilde{\mathrm{M}}$ *is algebraic.*

Proof. We have seen in the course of the proof of the embedding theorem that if $L \to M$ is positive and $E = \pi^{-1}(x)$, then $\pi^* L^k - E$ is positive for $k \gg 0$.

Corollary. *If* $\tilde{\mathrm{M}} \overset{\pi}{\to} \mathrm{M}$ *is a finite unbranched covering of compact complex manifolds, then* M *is algebraic if and only if* $\tilde{\mathrm{M}}$ *is.*

Proof. Clearly, if $L \to M$ is positive, then $c_1(\pi^* L) = \pi^* c_1(L)$ implies that $\pi^* L$ is positive. Conversely, say ω is an integral, positive $(1,1)$-form on $\tilde{M}$. For any $p \in M$, we have isomorphisms of a neighborhood U of p in M with neighborhoods U_i of the points $q_i \in \pi^{-1}(p)$; we can define a $(1,1)$-form ω' on M by

$$\omega'(p) = \sum_{q \in \pi^{-1}(p)} \omega(q).$$

Then ω' is closed and of type $(1,1)$, and if $\eta \in H_{\mathrm{DR}}^{2n-2}(M)$ is any integral cohomology class, then

$$\int_M \omega' \wedge \eta = \frac{1}{m} \int_{\tilde{M}} \omega \wedge \pi^* \eta \in \mathbb{Q},$$

where m is the number of sheets of the cover. Thus $[\omega']$ is rational.

DEFINITION. We say that a line bundle $L \to M$ over an algebraic variety is *very ample* if $H^0(M, \mathcal{O}(L))$ gives an embedding $M \to \mathbb{P}^N$, i.e., if there exists an embedding $f : M \hookrightarrow \mathbb{P}^N$ such that $L = f^* H$.

Now from the proof of the Kodaira embedding theorem, we see

Corollary. *If* $\mathrm{E} \to \mathrm{M}$ *is any line bundle and* $\mathrm{L} \to \mathrm{M}$ *a positive line bundle, then for* $\mathrm{k} \gg 0$, *the bundle* $\mathrm{L}^{\mathrm{k}} + \mathrm{E}$ *is very ample.*

5. GRASSMANNIANS

Definitions; The Cell Decomposition and Schubert Cycles

In this section, we will construct and describe the Grassmannians, a fundamental family of compact complex manifolds. Grassmannians may be thought of as a generalization of projective space; the analogy will be apparent throughout.

Let V be a complex vector space of dimension n. The *Grassmannian* $G(k,V)$ is defined to be the set of k-dimensional linear subspaces of V; we write $G(k,n)$ for $G(k,\mathbb{C}^n)$. Given a k-plane Λ in $\mathbb{C}^n$, we may represent Λ by a set of k row vectors in $\mathbb{C}^n$ spanning Λ, i.e., by a $k\times n$ matrix

$$\begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{k1} & \cdots & v_{kn} \end{bmatrix}$$

of rank k. Clearly any such matrix represents an element of $G(k,n)$ and any two such matrices A,A' represent the same element of $G(k,n)$ if and only if $\Lambda = g\Lambda'$ for some $g\in \mathrm{GL}_k$.

For every multiindex $I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,n\}$ of cardinality k, let $V_{I^\circ}\subset \mathbb{C}^n$ be the $(n-k)$-plane in $\mathbb{C}^n$ spanned by the vectors $\{e_j : j\notin I\}$, and let

$$U_I = \{\Lambda\in G(k,n):\ \Lambda\cap V_{I^\circ}=\{0\}\};$$

U_I is just the set of $\Lambda\in G(k,n)$ such that the Ith $k\times k$ minor of one, and hence for any, matrix representation for Λ is nonsingular. Any $\Lambda\in U_I$ has a unique matrix representation Λ^I whose Ith $k\times k$ minor is the identity matrix, e.g., any $\Lambda\in U_{\{1,\ldots,k\}}$ can be represented uniquely by a matrix of the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & * & \cdots & * \\ 0 & 1 & & & \vdots & * & & \\ \vdots & & \ddots & & 0 & \vdots & \ddots & \vdots \\ 0 & & & \cdots & 1 & * & \cdots & * \end{bmatrix}.$$

(Note that the row vectors of such a matrix representative for $\Lambda\in U_I$ are just the points of intersection of Λ with the affine $(n-k)$-planes $\{V_{I^\circ}+e_j : j\in I\}$.) Conversely, any $k\times n$ matrix of the form above represents a k-plane $\Lambda\in U_I$; thus the $k(n-k)$ entries of the I°th $k\times(n-k)$ minor $\Lambda^I_{I^\circ}$ of Λ^I give a bijection of sets

$$\varphi_I:\ U_I\to\mathbb{C}^{k(n-k)}$$

for each I. Note that $\varphi_I(U_I \cap U_{I'})$ is open in $\mathbb{C}^{k(n-k)}$ for all I, I'; we claim that in fact the map $\varphi_I \circ \varphi_{I'}^{-1}$ is holomorphic on this open set and hence that the maps φ_I give $G(k,n)$ the structure of a complex manifold. But this is clear: if, for $\Lambda \in U_I \cap U_{I'}$, we let $\Lambda_{I'}^I$ be the I'th $k \times k$ minor of Λ^I, then

$$\Lambda^{I'} = \left(\Lambda_{I'}^I\right)^{-1} \cdot \Lambda^I,$$

and since the entries of $(\Lambda_{I'}^I)^{-1}$ vary holomorphically with the entries of Λ^I, $\varphi_I \circ \varphi_{I'}^{-1}$ is holomorphic.

With this topology $G(k,n)$ is compact and connected, since the unitary group U_n maps surjectively and continuously onto $G(k,n)$ by the map

$$g \mapsto g(V_k),$$

where $V_k = \{e_1, \ldots, e_k\} \subset \mathbb{C}^n$. The full linear group GL_n likewise acts transitively on $G(k,n)$.

Note in particular that $G(1,n)$ is biholomorphic to $\mathbb{P}^{n-1}$ as a complex manifold: the "matrix representative" $(v_1, \ldots, v_n)$ for a line $\Lambda \in G(1,n)$ corresponds, via the natural set-theoretic identification of $G(1,n)$ with $\mathbb{P}^{n-1}$, to the homogeneous coordinates of $\Lambda \in \mathbb{P}^{n-1}$, and

$$\Lambda^{\{i\}} = \left(\frac{v_1}{v_i}, \ldots, 1, \ldots, \frac{v_n}{v_i}\right),$$

so

$$\varphi_{\{i\}} = \Lambda \mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_n}{v_i}\right),$$

i.e., the coordinates on $G(1,n)$ given by $\varphi_{\{i\}}$ are just the Euclidean coordinates on $\mathbb{P}^{n-1}$. Dually, we have $G(n-1,n) \cong \mathbb{P}^{n-1*}$, the projective space of hyperplanes in $\mathbb{P}^{n-1}$.

Finally, we note that $G(k,n)$ can be considered either as the set of linear k-planes Λ in $\mathbb{C}^n$, or equivalently as the set of $(k-1)$-planes $\overline{\Lambda}$ in $\mathbb{P}^{n-1}$. Our viewpoint in this section will for the most part be the former, as it is easier to keep track of dimension and codimension of cycles, but when Grassmannians arise in geometric questions we will generally want to think of them in the latter way.

The Cell Decomposition

Recall that the cell decomposition

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0$$

of $\mathbb{P}^n = G(1, n+1)$ is obtained by choosing a *flag*

$$V = \left(V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n \subsetneq \mathbb{C}^{n+1}\right)$$

of linear subspaces of $\mathbb{C}^{n+1}$ and taking $W_i \cong \mathbb{C}^{i-1} = \{l \subset \mathbb{C}^{n+1} : l \subset V_i,\ l \not\subset V_{i-1}\}$. The same technique works to give a cell decomposition of the Grassmannian: if we set $V_i = \{e_1, \ldots, e_i\} \subset \mathbb{C}^n$, then the set of $\Lambda \in G(k,n)$ whose intersection with each V_i is of a specified dimension turns out, as we shall see, to be a simple cell. The set-up is as follows: for every $\Lambda \in G(k,n)$ consider the increasing sequence of subspaces

$$(*) \qquad 0 \subset \Lambda \cap V_1 \subset \Lambda \cap V_2 \subset \cdots \subset \Lambda \cap V_{n-1} \subset \Lambda \cap V_n = \Lambda.$$

For generic Λ, $\Lambda \cap V_i$ will be zero for $i \leqslant n-k$, and $(i+k-n)$-dimensional thereafter—indeed, we have seen that the set of such Λ is just the open set $U_{I^\circ} \cong \mathbb{C}^{k(n-k)} \subset G(k,n)$. Now, for any sequence of integers $a_1, \ldots, a_k$, set

$$W_{a_1, \ldots, a_k} = \left\{ \Lambda \in G(k,n) :\ \dim(\Lambda \cap V_{n-k+i-a_i}) = i \right\}.$$

We observe that $\dim(\Lambda + V_{n-k+i-a_i}) = n - a_i$, and consequently $W_{a_1,\ldots,a_k}$ will be empty unless $a_1, \ldots, a_k$ is a nonincreasing sequence of integers $\leqslant n-k$. Since $\dim(\Lambda \cap V_{n-k+i-a_i}) = i$ if and only if the rank of the *last* $k \times (k + a_i - i)$ minor of a matrix representative for Λ is exactly $k-i$ it follows that the closure

$$\overline{W_{a_1,\ldots,a_k}} = \left\{ \Lambda :\ \dim(\Lambda \cap V_{n-k+i-a_i}) \geqslant i \right\}$$

is an analytic subvariety of $G(k,n)$.

We can choose a special basis for a k-plane $\Lambda \in W_{a_1,\ldots,a_k}$ as follows: let v_1 be a generator for the line $\Lambda \cap V_{n-k+1-a_1}$, normalized so that $\langle v_1, e_{n-k+1-a_1} \rangle = 1$; i.e.,

$$v_1 = (*, *, \ldots, *, 1, 0, \ldots, 0).$$

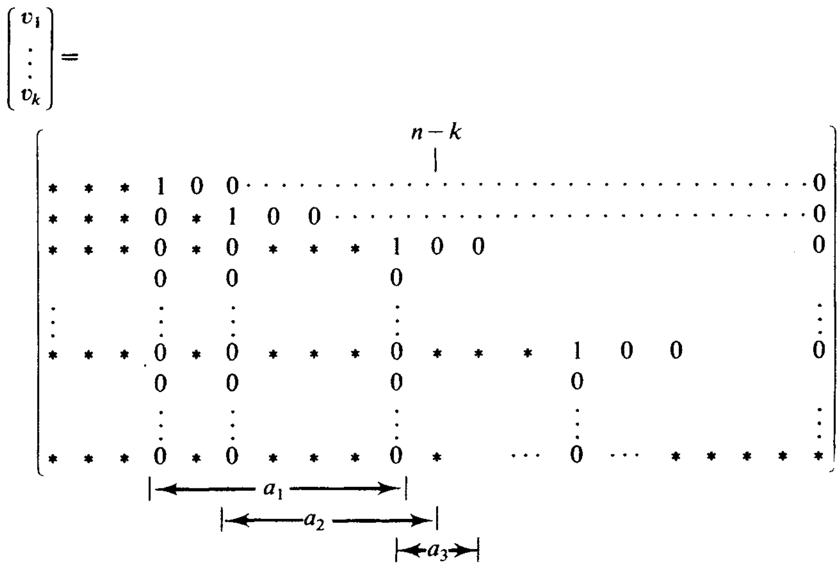
Now take v_2 so that v_1 and v_2 together span $\Lambda \cap V_{n-k+2-a_2}$, normalized so that

$$\langle v_2, e_{n-k+1-a_1} \rangle = 0, \qquad \langle v_2, e_{n-k+2-a_2} \rangle = 1.$$

Continue in this way, choosing v_i so that $v_1, \ldots, v_i$ span $\Lambda \cap V_{n-k+i-a_i}$ and such that

$$\langle v_i, e_{n-k+j-a_j} \rangle = \begin{cases} 0, & j < i, \\ 1, & j = i. \end{cases}$$

Clearly, the choice of v_i at each stage is completely specified by these conditions; thus the k-plane Λ has a unique matrix representative of the form



Conversely, any matrix of this form describes a k-plane $\Lambda \in W_{a_1,\ldots,a_k}$. Since $(k^2+\Sigma a_i)$ entries are specified in the diagram and the rest are completely free to vary, we have homeomorphisms

$$W_{a_1,\ldots,a_k} \cong \mathbb{C}^{k(n-k)-\Sigma a_i};$$

consequently, the sets $W_{a_1,\ldots,a_k}$ give a cell decomposition of $G(k,n)$. Since we have cells only in even dimensions, all boundary maps are zero, and we deduce the

Proposition. *The integral homology of the Grassmannian* $G(k,n)$ *has no torsion and is freely generated by the cycles* $\sigma_{a_1,\ldots,a_k} = [\overline{W_{a_1,\ldots,a_k}}]$ *in real codimension* $2\Sigma a_i$, *where* $\{(a_1,\ldots,a_k)\}$ *ranges over all nonincreasing sequences of integers between* 0 *and* $n-k$. *In particular, all cohomology in* $G(k,n)$ *is analytic.*

In general, for any flag $V=(V_1\subset V_2 \subset \cdots \subset V_{n-1}\subset V_n)$ in $\mathbb{C}^n$ we let

$$\sigma_a(V) = \{\Lambda : \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i\}.$$

Clearly the homology class of the subvariety $\sigma_a(V)$ is independent of the flag chosen, since we can find a continuous family of linear automorphisms of $\mathbb{C}^n$ taking any flag into any other. The subvarieties $\sigma_a(V)$ are called the *Schubert cycles* of the Grassmannian.

The simplest example of a Grassmannian different from projective space is the $G(2,4)$ of 2-planes in $\mathbb{C}^4$. The Schubert cycles on $G(2,4)$ are

$$\begin{aligned} \text{codim}\,1: &\quad \sigma_{1,0}(V_2)=\{\Lambda:\ \dim(\Lambda\cap V_2)\geq 1\},\\ \text{codim}\,2: &\quad \sigma_{1,1}(V_3)=\{\Lambda:\ \Lambda\subset V_3\},\\ &\quad \sigma_{2,0}(V_1)=\{\Lambda:\ \Lambda\supset V_1\},\\ \text{codim}\,3: &\quad \sigma_{2,1}(V_1,V_3)=\{\Lambda:\ V_1\subset\Lambda\subset V_3\}. \end{aligned}$$

Alternatively, if we think of $G(2,4)$ as the set of lines l in $\mathbb{P}^3$ and fix the projective flag $p\in l_0\subset h$ consisting of a point, line, and hyperplane in $\mathbb{P}^3$, then

$$\begin{aligned} \sigma_{1,0}(l_0) &= \{l:\ l\cap l_0\neq\varnothing\},\\ \sigma_{2,0}(p) &= \{l:\ p\in l\},\\ \sigma_{1,1}(h) &= \{l:\ l\in h\},\\ \sigma_{2,1}(p,h) &= \{l:\ p\in l\subset h\}. \end{aligned}$$

The Schubert Calculus

Now that we have determined the additive cohomology of $G(k,n)$, we would like to describe its multiplicative structure—that is, to express the intersection of general Schubert cycles σ_a,σ_b as a linear combination of other Schubert cycles in homology.

The first task is to write down the intersection pairing in complementary dimensions. To do this, let

$$\sigma_a(V) = \{\Lambda:\ \dim(\Lambda\cap V_{n-k+i-a_i})\geq i\}$$

and

$$\sigma_b(V') = \{\Lambda:\ \dim(\Lambda\cap V'_{n-k+i-b_i})\geq i\}$$

be general Schubert cycles. Then for each i and any $\Lambda\in\sigma_a(V)\cap\sigma_b(V')$,

$$\begin{aligned} &\dim(\Lambda\cap V_{n-k+i-a_i})\geq i,\\ &\dim(\Lambda\cap V'_{n-k+(k-i+1)-b_{k-i+1}})\geq k-i+1\\ &\Rightarrow \Lambda\cap V_{n-k+i-a_i}\cap V'_{n-i+1-b_{k-i+1}}\neq(0). \end{aligned}$$

But now if $a_i+b_{k-i+1}>n-k$, we have

$$\begin{aligned} (n-k+1-a_i)+(n-i+1-b_{k-i+1}) &= 2n-k+1-(a_i+b_{k-i+1})\\ &\leq n, \end{aligned}$$

and so we can choose our flags V and V' such that $V_{n-k+i-a_i}$ and $V'_{n-i+1-b_{k-i+1}}$ intersect only at the origin. Consequently the cycles $\sigma_a(V)$

and $\sigma_b(V')$ can be made disjoint, i.e.,

$$^{\#}(\sigma_a \cdot \sigma_b) = 0 \qquad \text{unless } a_i + b_{k-i+1} \leqslant n-k, \quad \text{for all } i.$$

Now suppose σ_a and σ_b are cycles of complementary dimension, so that

$$\sum a_i + \sum b_i = k(n-k);$$

then

$$a_i + b_{k-i+1} \leqslant n-k \ \text{ for all } i \Rightarrow b_{k-i+1} = n-k-a_i,$$

i.e., the cycle σ_a has intersection number zero with all Schubert cycles in complementary dimension except $\sigma_{n-k-a_n,\ldots,n-k-a_1}$. Since the Schubert cycles form an integral basis for $H_*(G(k,n),\mathbb{Z})$, it follows either by Poincaré duality and the fact that analytic cycles intersect positively or by direct examination that

$$^{\#}(\sigma_a, \sigma_{n-k-a_n,\ldots,n-k-a_1}) = 1.$$

Summing up, then, we have the formula

$$^{\#}(\sigma_a \cdot \sigma_b) = \delta_{(a_1,\ldots,a_k)}^{(n-k-b_k,\ldots,n-k-b_1)}.$$

This enables us to express an arbitrary cycle γ on $G(k,n)$ as a linear combination of Schubert cycles, by computing intersections, i.e.,

$$\gamma = \sum {}^{\#}(\gamma \cdot \sigma_{n-k-a_k,\ldots,n-k-a_1}) \cdot \sigma_a,$$

and in particular reduces the problem of computing the intersection of pairs of Schubert cycles in arbitrary dimension to the problem of computing triple intersections in complementary dimension:

$$(\sigma_a \cdot \sigma_b) = \sum {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1}) \cdot \sigma_c.$$

As an example, for any hypersurface $W \subset \mathbb{P}^n$ of degree 2, let $\tau(W) \subset G(2,n+1)$ denote the set of lines in $\mathbb{P}^n$ lying on W. $\tau(W)$ is clearly an analytic cycle in $G(2,n+1)$, and since a line $l \subset \mathbb{P}^n$ lies on W if and only if three points of l lie on W, $\tau(W)$ has complex codimension 3. $G(2,n+1)$ has only two Schubert cycles of codimension 3—$\sigma_{3,0}$ and $\sigma_{2,1}$—and so we can write

$$\tau(W) = {}^{\#}(\tau(W) \cdot \sigma_{n-1,n-4}) \cdot \sigma_{3,0} + {}^{\#}(\tau(W) \cdot \sigma_{n-2,n-3}) \cdot \sigma_{2,1}.$$

Now, $\sigma_{n-1,n-4}$ is the set of lines in $\mathbb{P}^n$ containing a point p and contained in a 4-plane $V_4 \subset \mathbb{P}^n$; if we choose our point p to lie off W, clearly $\tau(W)$ will be disjoint from $\sigma_{n-1,n-4}$. On the other hand, $\sigma_{n-2,n-3}$ is the cycle of lines meeting a line $l_0 \subset \mathbb{P}^n$ and contained in a 3-plane $S \subset \mathbb{P}^n$ containing l_0. Generically, $W' = W \cap S$ will be a smooth quadric surface in $S \cong \mathbb{P}^3$, with l_0 meeting it at two points p_1 and p_2; clearly any line $l \subset \tau(W) \cap \sigma_{n-2,n-3}$

will pass through either p_1 or p_2. But any line on W' through p_i must lie in the tangent plane $T_{p_i}(W')$; and $T_{p_i}(W')\cap W$ is a singular curve of degree 2, hence consists of two lines. Thus $\tau(W)$ meets $\sigma_{n-2,n-3}$ in four points generically, and so

$$\tau(W)\sim 4\cdot\sigma_{2,1}.$$

In particular, if W and W' are two generic quadric hypersurfaces in $\mathbf{P}^4$, meeting transversally in a smooth surface S, then by the above S will have

$$^{\#}(\tau(W)\cdot\tau(W'))_{G(2,5)} = {}^{\#}(4\sigma_{2,1}\cdot 4\sigma_{2,1})_{G(2,5)} = 16$$

lines in $\mathbb{P}^4$ lying on it. We will verify this in Section 4 of Chapter 4.

Similarly, we will be able to compute the homology class of $\tau(W)\subset G(2,n+1)$ for other hypersurfaces of low degree, once we know a few more things about special cases.

Before we go on to consider general intersections, we want to offer two general observations.

First, we will alter our formalism slightly, as follows: for *any* sequence $a=a_1,a_2,\ldots$ of nonnegative integers, we let $\sigma_a(V)$ denote the cycle

$$\sigma_a(V) = \{\Lambda:\ \dim(\Lambda\cap V_{n-k+i-a_i})\geqslant i\}\subset G(k,n)$$

so that the symbol σ_a can be used to refer to a Schubert cycle in any Grassmannian. Of course, σ_a will be null in $G(k,n)$ unless $a_i\leqslant n-k$ for all i, $a_i=0$ for all $i>k$, and a is nonincreasing.

Now, the inclusion $\mathbb{C}^n\to\mathbb{C}^{n+1}$ induces inclusions

$$\iota_1:\ G(k,n)\to G(k,n+1)$$

and

$$\iota_2:\ G(k,n)\to G(k+1,n+1)$$

obtained by sending $\Lambda\subset\mathbb{C}^n$ to $\Lambda\subset\mathbb{C}^{n+1}$ and $\Lambda\oplus\{e_{n+1}\}\subset\mathbb{C}^{n+1}$, respectively. Under these inclusions, it is not hard to see that for appropriate choices of flags V in $\mathbb{C}^n$ and V' in $\mathbb{C}^{n+1}$,

$$\sigma_a(V) = \iota_1^{-1}(\sigma_a(V')) = \iota_2^{-1}(\sigma_a(V')),$$

i.e., if we denote the Poincaré dual of σ_a by $\tilde{\sigma}_a$,

$$\iota_1^*\tilde{\sigma}_a = \iota_2^*\tilde{\sigma}_a = \tilde{\sigma}_a.$$

Thus any formula

$$(\sigma_a\cdot\sigma_b) = \sum n_c\cdot\sigma_c$$

for the intersection of Schubert cycles in $G(k,n+1)$ or $G(k+1,n+1)$ holds as well in $G(k,n)$, and we can define the *universal Schubert coefficients*

$\delta(a,b;c)$ to be such that the formula

$$(\sigma_a \cdot \sigma_b) = \sum \delta(a,b;c) \cdot \sigma_c$$

holds in all $G(k,n)$.

Note that by our first computation, we have

$$\delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{n-k-c_k,\ldots,n-k-c_1})_{G(k,n)}$$

for any k,n such that σ_c is nonnull in $G(k,n)$, i.e., such that $c_i \leqslant n-k$ for all i and $c_i = 0$ for all $i > k$. In particular, if we let $l(c)$ denote the length of the sequence c, that is, the number of nonzero entries, we may take $k = l(c)$, $n-k = c_1$ in the above to obtain

$$(*) \qquad \delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{c_1 - c_k,\ldots,c_1 - c_2}) \qquad \text{in } G(l(c), l(c)+c_1).$$

As an immediate consequence, we see that $\delta(a,b;c) = 0$ if σ_a or σ_b is null in $G(l(c), l(c)+c_1)$, i.e., $\delta(a,b;c) = 0$ if either

1. $c_1 < a_1$ or $c_1 < b_1$, or
2. $l(c) < l(a)$ or $l(c) < l(b)$.

Next, note that for any vector space W of dimension n, we have a natural isomorphism

$$*: \ G(k,W) \longrightarrow G(n-k, W^*)$$

defined by

$$*\Lambda = \mathrm{Ann}(\Lambda) = \{ l \in V^* : \ l(\Lambda) = 0 \}.$$

Let $V = \{ V_1 \subset V_2 \subset \cdots \subset V_n = W \}$ be a flag in W, and let $V^* = \{ V_1^* \subset V_2^* \subset \cdots \subset V_n^* = W^* \}$ be the dual flag in W^* given by

$$V_i^* = \mathrm{Ann}(V_{n-i}).$$

By linear algebra, for Λ any k-plane in W,

$$\dim(\Lambda \cap V_{n-k+i-a_i}) \geqslant i \Longleftrightarrow \dim(*\Lambda \cap V_{k-i+a_i}^*) \geqslant a_i,$$

Thus, *for any* a, *the image* $*\sigma_{\mathrm{a}} \subset \mathrm{G(n-k,n)}$ *of the Schubert cycle* $\sigma_{\mathrm{a}} \subset \mathrm{G(k,n)}$ *is the Schubert cycle* a^*, *where* a^* *is defined to be the smallest nonincreasing sequence such that*

$$a_{a_i}^* \geqslant i \qquad \text{for all } i.$$

For example,

$$*(\sigma_2) = \sigma_{1,1}, \qquad *(\sigma_{2,1,1}) = \sigma_{3,1}.$$

In general, we will have

$$\delta(a,b;c) = \delta(a^*, b^*; c^*),$$

and so we may expect that any formula for the intersection of Schubert cycles σ_a, σ_b gives a dual formula, when applied to $\sigma_{a^*}, \sigma_{b^*}$.

Note that

$$l(a^*) = a_1 \quad \text{and} \quad a_1^* = l(a)$$

so that the formulas 1 and 2 above are, as expected, equivalent under the * map.

We turn now to the original problem of computing $\delta(a,b;c)$ for general a, b, and c. We will first give a reduction that allows us to compute effectively in many cases.

Our basic technique is simply a linear algebra reduction to smaller Grassmannians. For example, consider a triple of indices α, β, γ such that $\alpha+\beta+\gamma=2k+1$. Then for any k-plane $\Lambda \in \sigma_a(V) \cap \sigma_b(V') \cap \sigma_c(V'')$,

$$\begin{aligned} &\dim\left(\Lambda \cap V_{n-k+\alpha-a_\alpha}\right) \geqslant \alpha, \\ &\dim\left(\Lambda \cap V'_{n-k+\beta-b_\beta}\right) \geqslant \beta, \\ &\dim\left(\Lambda \cap V''_{n-k+\gamma-c_\gamma}\right) \geqslant \gamma \\ \Rightarrow &\dim\left(\Lambda \cap V_{n-k+\alpha-a_\alpha} \cap V'_{n-k+\beta-b_\beta} \cap V''_{n-k+\gamma-c_\gamma}\right) \geqslant 1. \end{aligned}$$

Thus $^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_c) = 0$ in $G(k,n)$ if

$$(k-\alpha+a_\alpha)+(k-\beta+b_\beta)+(k-\gamma+c_\gamma) > n-1,$$

i.e., if

$$a_\alpha + b_\beta + c_\gamma > n-k.$$

Suppose on the other hand that $a_\alpha + b_\beta + c_\gamma = n-k$, i.e., that generically chosen subspaces $V_{n-k+\alpha-a_\alpha}$, $V'_{n-k+\beta-b_\beta}$, and $V''_{n-k+\gamma-c_\gamma}$ will intersect in a line $L \subset \mathbb{C}^n$. Then any $\Lambda \in \sigma_a(V) \cap \sigma_b(V') \cap \sigma_c(V'')$ must contain L. Let L^0 denote a subspace complementary to L in $\mathbb{C}^n$ and let π denote the projection of $\mathbb{C}^n$ onto L^0 with kernel L. Let

$$\begin{aligned} \bar{V}_1 &= \pi(V_1), \\ &\vdots \\ \bar{V}_{n-k+\alpha-a_\alpha-1} &= \pi(V_{n-k+\alpha-a_\alpha-1}) = \pi(V_{n-k+\alpha-a_\alpha}), \\ &\vdots \\ \bar{V}_{n-2} &= \pi(V_{n-1}), \\ \bar{V}_{n-1} &= \pi(V_n) = L^0, \end{aligned}$$

and define $\bar{V}_i'$ and $\bar{V}_i''$ similarly. Then $\bar{V} = \{\bar{V}_i\}$, $\bar{V}' = \{\bar{V}_i'\}$, and $\bar{V}'' = \{\bar{V}_i''\}$

are transverse flags in L^0, and for any $(k-1)$-plane $\bar{\Lambda}\subset L^0$, we see that

$$\Lambda = \overline{L,\bar{\Lambda}} \in \sigma_a(V)\cap\sigma_b(V')\cap\sigma_c(V'')$$
$$\Leftrightarrow \bar{\Lambda}\in\sigma_{a_1,\dots,\hat{a}_\alpha,\dots,a_k}(\bar{V})\cap\sigma_{b_1,\dots,\hat{b}_\beta,\dots,b_k}(\bar{V}')\cap\sigma_{c_1,\dots,\hat{c}_\gamma,\dots,c_k}(\bar{V}'').$$

Thus we have the

Reduction Formula I. *For any three indices* $0\leqslant\alpha,\beta,\gamma\leqslant k$ *with* $\alpha+\beta+\gamma=2k+1$,

$$^{\#}(\sigma_a\cdot\sigma_b\cdot\sigma_c)_{G(k,n)} = \begin{cases} 0 & \text{if } a_\alpha+b_\beta+c_\gamma>n-k, \\ ^{\#}(\sigma_{a-a_\alpha}\cdot\sigma_{b-b_\beta}\cdot\sigma_{c-c_\gamma})_{G(k-1,n-1)} & \text{if } a_\alpha+b_\beta+c_\gamma=n-k. \end{cases}$$

Note that in case we take $\beta=\gamma=k$, this reduction applies if $a_1=n-k$; in case we take $\gamma=k$, it applies if $a_i+b_{k+1-i}=n-k$ for any i.

As suggested, we can apply this first reduction to the intersection of cycles $^{\#}(\sigma_{a^*}\cdot\sigma_{b^*}\cdot\sigma_{c^*})$ in $G(n-k,n)$; we obtain

Reduction Formula II. *For any three coefficients* $a_\alpha, b_\beta, c_\gamma$ *with* $a_\alpha+b_\beta+c_\gamma\geqslant 2(n-k)+1$,

$$^{\#}(\sigma_a\cdot\sigma_b\cdot\sigma_c)_{G(k,n)}$$
$$=\begin{cases} 0 & \text{if } \alpha+\beta+\gamma>k \\ ^{\#}(\sigma_{a_1-1,\dots,a_\alpha-1,a_{\alpha+1},\dots,a_k}\cdot\sigma_{b_1-1,\dots,b_\beta-1,b_{\beta+1},\dots,b_k}\cdot\sigma_{c_1-1,\dots,c_\gamma-1,c_{\gamma+1},\dots,c_k})_{G(k,n-1)} & \\ \quad \text{if } \alpha+\beta+\gamma=k. & \end{cases}$$

For the purposes of this formula, we may set $a_0=b_0=c_0=n-k$ formally; thus in case we take $\gamma=\beta=0$, this reduction applies if $a_k\neq 0$, and if we take $\gamma=0$, it applies in case $a_i+b_{k-i}\geqslant n-k+1$ for some i.

Note also that if the sequence $b_1-1,\dots,b_{\beta-1},b_{\beta+1},\dots$ appearing in the formula is no longer nonincreasing—i.e., if $b_\beta=b_{\beta+1}$—then the intersection number is zero: just apply the formula to $\alpha,\beta+1,\gamma$. Thus we may use the formula in all circumstances, if we adopt the convention that σ_b is null for b not a nonincreasing sequence.

As a sample calculation, we compute the coefficient $\delta(311,21;521)$ of σ_{521} in the expression for $(\sigma_{311}\cdot\sigma_{21})$ as a linear combination of Schubert cycles. By (*) and the reductions we have

$$\begin{aligned} \delta(311,21;521) &= {}^{\#}(\sigma_{311}\cdot\sigma_{21}\cdot\sigma_{43}) && \text{in } G(3,8)\\ &= {}^{\#}(\sigma_2\cdot\sigma_{21}\cdot\sigma_{43}) && \text{in } G(3,7)\\ &= {}^{\#}(\sigma_2\cdot\sigma_{21}\cdot\sigma_3) && \text{in } G(2,6)\\ &= {}^{\#}(\sigma_2\cdot\sigma_1\cdot\sigma_3) && \text{in } G(2,5)\\ &= {}^{\#}(\sigma_2\cdot\sigma_1) && \text{in } G(1,4)=\mathbb{P}^3\\ &= 1. \end{aligned}$$

The two formulas given here will not apply every time, but in low codimension will yield the answer more often than not. They work especially well in case one of the factors σ_a is a *special Schubert cycle*, defined to be one of the form $\sigma_{a,0,0,\ldots}$. In this case, we can use the reductions to obtain the general

Pieri's Formula. *If* $a=a,0,0,\ldots$, *then for any* b,

$$(\sigma_a \cdot \sigma_b) = \sum_{\substack{b_i \leqslant c_i \leqslant b_{i-1} \\ \Sigma c_i = a + \Sigma b_i}} \sigma_c.$$

Proof. We want to show that, for σ_c of appropriate codimension,

$$\delta(a,b;c) = \begin{cases} 1, & \text{if } b_i \leqslant c_i \leqslant b_{i-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We have, setting $k = l(c)$,

$$\delta(a,b;c) = {}^{\#}(\sigma_a \cdot \sigma_b \cdot \sigma_{c_1 - c_k, \ldots, c_1 - c_2, 0}) \qquad \text{in } G(k, k + c_1).$$

To start, suppose that $c_i < b_{i-1}$ for some i. Then we have

$$c_1 + b_{i-1} + (c_1 - c_i) \geqslant 2c_1 + 1,$$

and applying the second reduction formula with $\sigma = 0$, $\beta = i - 1$, and $\gamma = k - i + 1$, we obtain

$$\begin{aligned} \delta(a,b;c) &= {}^{\#}(\sigma_a \cdot \sigma_{b_1 - 1, \ldots, b_{i-1} - 1, b_i, \ldots} \cdot \sigma_{c_1 - c_k - 1, \ldots, c_1 - c_i - 1, c_1 - c_{i-1}, \ldots}) \\ &\qquad\qquad\qquad\qquad \text{in } G(k, k + c_1 - 1) \\ &= \delta(a, b'; c') \end{aligned}$$

where

$$b' = b_1 - 1, \ldots, b_{i-1} - 1, b_i, \ldots$$

and

$$c' = c_1 - 1, \ldots, c_{i-1} - 1, c_i, \ldots.$$

Now

$$(b_i \leqslant c_i \leqslant b_{i-1} \text{ for all } i) \Leftrightarrow (b'_i \leqslant c'_i \leqslant b'_{i-1} \text{ for all } i)$$

and of course

$$b'_{i-1} - c'_i = b_{i-1} - c_i - 1 \geqslant 0.$$

Thus we may assume from the start that $c_i \geqslant b_{i-1}$ for all i. Since $\Sigma c_i = a + \Sigma b_i$, it follows that $a \geqslant c_1$; and so there are three cases:

1. If $c_i > b_{i-1}$ for some i, then $a > c_1$ and so $\delta(a,b;c) = 0$.
2. If $c_i < b_i$ for any i, then $c_i \geqslant b_{i-1}$ implies that $b_i > b_{i-1}$, i.e., the sequence b is not nonincreasing and σ_b is taken to be null; so $\delta(a,b;c) = 0$.
3. If $b_i \leqslant c_i \leqslant b_{i-1}$ for all i, it follows that $c_i = b_{i-1}$ for all i, hence $a = c_1, b_k = 0$, and applying the first reduction formula with $\alpha = 1$ and

$\beta=\gamma=k$ we have

$$\begin{aligned}\delta(a,b;c) &= {}^{\#}(\sigma_a\cdot\sigma_b\cdot\sigma_{c_1-c_k,\dots,c_1-c_2,0}) &&\text{in } G(k,k+c_1)\\ &= {}^{\#}(\sigma_b\cdot\sigma_{c_1-c_k,\dots,c_1-c_2}) &&\text{in } G(k-1,k+c_1-1)\\ &= {}^{\#}(\sigma_b\cdot\sigma_{c_1-b_{k-1},\dots,c_1-b_1})\\ &= 1.\end{aligned}$$

Q.E.D.

Our final result on Schubert cycles is a formula that expresses the general Schubert cycle as a polynomial in the special Schubert cycles $\sigma_{b,0,\dots}$.

We proceed as follows: for $\sigma_{a_1,\dots,a_d}$ any Schubert cycle, we consider the cycle

$$(*)\qquad \tilde{\sigma}_a = \sum_{j=1}^{d}(-1)^j\sigma_{a_1,\dots,a_{j-1},a_{j+1}-1,\dots,a_d-1}\cdot\sigma_{a_j+d-j}.$$

Note that $\tilde{\sigma}_a$ has the same dimension as σ_a. Now, we can by Pieri's formula write out each of the intersections in the sum $(*)$ as a sum of Schubert cycles. Let $\sigma_{c_1,\dots,c_d}$ be any Schubert cycle; if σ_c appears in this expression, consider the sequence of integers

$$c_1-1,c_2-2,\dots,c_d-d.$$

By Pieri, at most one of these numbers will lie in each of the $(d+1)$ closed intervals

$$\begin{gathered}[a_1-1,n-k],\\ [a_2-2,a_1-2],\\ \vdots\\ [a_d-d,a_{d-1}-d],\\ [-d-1,a_d-d-1],\end{gathered}$$

and so exactly one of these intervals will fail to contain an integer c_i-i. By cases, then:

1. If no integer c_i-i lies in the interval $[-d-1,a_d-d-1]$, then

$$c_i-i\in[a_i-i,a_{i-1}-i],$$

and σ_c can appear only in the last term of the sum $(*)$. But since

$$c_i\geqslant a_i \quad\text{and}\quad \sum c_i=\sum a_i,$$

it follows that $c=a$. The Schubert cycle $\sigma_{a_1,\dots,a_d}$ thus appears once in $(*)$, with coefficient $(-1)^d$.

2. If no integer $c_i - i$ appears in the interval $[a_k - k, a_{k-1} - k]$, then we have

$$\begin{aligned}
c_1 - 1 \quad & \in [a_1 - 1, n-k],\\
& \vdots\\
c_{k-1} - k + 1 & \in [a_{k-1} - k + 1, a_k - k + 1],\\
c_k - k \quad & \in [a_{k+1} - k - 1, a_k - k - 1],\\
& \vdots\\
c_d - d \quad & \in [-d-1, a_d - d - 1],
\end{aligned}$$

i.e.,

$$a_i \leqslant c_i \leqslant a_{i-1}, \qquad i = 1,\dots,k-1,$$

and

$$a_{i+1} - 1 \leqslant c_i \leqslant a_i - 1, \qquad i = k,\dots,d.$$

In this case, the Schubert cycle σ_c will appear twice in the expression for (*): once in the kth term, and once in the $(k-1)$st term. Since these two have opposite sign, σ_c will not appear in the final expression for $\tilde{\sigma}_a$.

3. If the interval $[a_1 - 1, n-k]$ is unoccupied, we have

$$c_i - i \in [a_{i+1} - i - 1, a_i - i - 1]$$

for each i—but then $c_i \leqslant a_i - 1$, and hence $\Sigma c_i < \Sigma a_i$, so σ_c cannot appear in (*).

We have, then, the formula

$$(**) \qquad (-1)^d \sigma_{a_1,\dots,a_d} = \sum_{j=1}^{d} (-1)^j \sigma_{a_1,\dots,a_{j-1},a_{j+1}-1,\dots,a_d-1} \cdot \sigma_{a_j+d-j}.$$

Note that since each factor on the right has length $<d$, this already implies that σ_a is expressible as a polynomial in the special Schubert cycles $\sigma_{b,0,\dots}$, i.e., that

The cohomology ring of the Grassmannian G(k, n) *is generated by the classes of the special Schubert cycles.*

Now, we will use the relation (**) to prove *Giambelli's formula*

$$\sigma_{a_1,\dots,a_d} = \begin{vmatrix}
\sigma_{a_1} & \sigma_{a_1+1} & \sigma_{a_1+2} & \cdots & \sigma_{a_1+d-1}\\
\sigma_{a_2-1} & \sigma_{a_2} & \sigma_{a_2+1} & \cdots & \sigma_{a_2+d-2}\\
\sigma_{a_3-2} & \sigma_{a_3-1} & \sigma_{a_3} & & \\
\vdots & & & & \vdots\\
\sigma_{a_d-d+1} & & \cdots & & \sigma_{a_d}
\end{vmatrix}.$$

We will prove this by induction; clearly it is true for $d=1$. Assume that it holds for $d-1$; expanding by cofactors along the left-hand row, the determinant is given by

$$\sum(-1)^j\sigma_{a_j+d-j}\cdot\begin{vmatrix}\sigma_{a_1} & \cdots & \sigma_{a_1+d-2}\\ \vdots & & \vdots\\ \sigma_{a_{j-1}-j} & \cdots & \sigma_{a_{j-1}+d-j}\\ \sigma_{a_{j+1}-j-2} & \cdots & \sigma_{a_{j+1}+d-j-2}\\ \vdots & & \vdots\\ \sigma_{a_d-d+1} & \cdots & \sigma_{a_d-1}\end{vmatrix}$$

$$\begin{aligned}&=\sum(-1)^j\sigma_{a_j+d-j}\cdot\sigma_{a_1,\ldots,a_{j-1},a_{j+1}-1,\ldots,a_d-1}\\ &=\sigma_{a_1,\ldots,a_d},\end{aligned}$$

and the formula is proved. Q.E.D.

Note that Pieri's formula together with the formula (∗∗) give an algorithm for evaluating an arbitrary intersection of Schubert cycles.

The Schubert calculus will appear frequently in the remainder of the book, in a variety of contexts; for the time being we give some applications of our formulas to elementary problems in enumerative geometry. Perhaps the simplest nontrivial such problem is the question: given four lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$ in general position, how many lines meet all four? The answer is easily obtained: since the set of lines meeting L_i is just the Schubert cycle $\sigma_1(L_i)$, the answer is just the fourfold self-intersection number of σ_1 in $G(2,4)$; this is

$$\begin{aligned}\sigma_1^4 &= \sigma_1^2\cdot(\sigma_{1,1}+\sigma_2)\\ &= \sigma_1\cdot(2\sigma_{2,1})\\ &= 2.\end{aligned}$$

In general, the number of lines meeting four $(n+1)$-planes in general position in $\mathbb{P}^{2n+1}$ is given by the fourfold self-intersection of σ_n in $G(2,2n+2)$; this is

$$\begin{aligned}(\sigma_n)^4 &= (\sigma_n^2)^2\\ &= \left(\sum_{i=0}^{n}\sigma_{2n-i,i}\right)^2\\ &= n+1.\end{aligned}$$

In a similar vein, the number of lines in $\mathbb{P}^4$ meeting six 2-planes in general position is given by σ_1^6 in $G(2,5)$; we have

$$\sigma_1^3 = \sigma_1(\sigma_{1,1}+\sigma_2) = 2\sigma_{2,1}+\sigma_3,$$

so

$$\sigma_1^6 = (2\sigma_{2,1}+\sigma_3)^2 = 4+1 = 5.$$

Universal Bundles

Let $\mathbb{C}^n \times G(k,n)$ denote the trivial vector bundle of rank n over $G(k,n)$. We define the *universal subbundle* $S \to G(k,n)$ to be the subbundle of $\mathbb{C}^n \times G(k,n)$ whose fiber at each point $\Lambda \in G(k,n)$ is just the subspace $\Lambda \subset \mathbb{C}^n$. S is clearly a holomorphic subbundle of $\mathbb{C}^n \times G(k,n)$—explicitly, in each open $U_I \subset G(k,n)$ the row vectors of the normalized matrix representatives for $\Lambda \in U_I$ give a frame for S over U_I; transition functions relative to these frames are given in $U_I \cap U_{I'}$ by $g_{U_I U_{I'}} = \Lambda_{I'} \cdot \Lambda_{I'}^{-1}$. The quotient bundle $Q = \mathbb{C}^n / S$ is called the *universal quotient bundle* on $G(k,n)$. Note that under the identification $*: G(k,n) \to G(n-k,n)$, the universal subbundle on $G(n-k,n)$ corresponds to the *dual* of the universal quotient bundle in $G(k,n)$, and likewise $Q \to G(n-k,n)$ pulls back to the dual $S^* \to G(k,n)$. Note in particular that the universal subbundle $S \to G(1,n) \cong \mathbb{P}^{n-1}$ is just the universal line bundle mentioned earlier.

Now let $E \to M$ be any holomorphic vector bundle of rank k on a complex manifold M, $V \subset H^0(M, \mathcal{O}(E))$ an n-dimensional vector space of global holomorphic sections, and suppose that the values $\{\sigma(x)\}_{\sigma \in V}$ of the sections σ in V span E_x for all $x \in M$. Then for each $x \in M$, the subspace $\Lambda_x \subset V$ of sections $\sigma \in V$ vanishing at x is an $(n-k)$-dimensional subspace; accordingly, we obtain a map

$$\iota_V : M \to G(n-k, V) = G(k, V^*)$$

with

$$E = \iota_V^* S^* \quad \text{and} \quad V = \iota_V^*(H^0(G(k,n), \mathcal{O}(S^*)))$$

just as for line bundles. Explicitly, if we choose a basis $\sigma_1, \ldots, \sigma_n$ for V and a frame $e_1, \ldots, e_k$ for E locally and write

$$\sigma_i = \sum a_{i\alpha} e_\alpha,$$

then in terms of the corresponding identification $G(n-k, V) \cong G(k, V^*)$ the map ι_V is given by

$$x \mapsto \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix},$$

so that ι_V is clearly holomorphic.

As in the case of line bundles, we have an embedding theorem:

Theorem. *For* M *any compact complex manifold,* L→M *a positive line bundle and* E→M *any holomorphic vector bundle, then for* m *sufficiently large, the map* $\iota_{E \otimes L^m}$ *is an embedding.*

Proof. Most of the work has been done for us already by the Kodaira embedding theorem: since M has a positive line bundle, we may take

$M \subset \mathbb{P}^N$ an algebraic variety and $L \to M$ the hyperplane bundle.

Now $\iota_{E\otimes L^m}$ will be 1-1 if for all $x, y \in M$, the restriction map

$$(*) \qquad H^0(M, \mathcal{O}(E\otimes L^m)) \to (E\otimes L^m)_x \oplus (E\otimes L^m)_y$$

is surjective. Similarly, we have a differential map

$$(**) \qquad H^0(M, \mathcal{I}_x(E\otimes L^m)) \to T_x^{*\prime} \otimes (E\otimes L^m)_x$$

defined as for line bundles; $\iota_{E\otimes L^m}$ will be smooth at x if this map is surjective. The compactness argument used in the proof of the Kodaira embedding theorem again assures us that to prove the result, it is sufficient to show that for any particular choice of x and y, the above two maps are surjective for m sufficiently large.

We proceed by induction on the dimension of M. For any $x,y \in M$, consider the linear system of hyperplane sections of $M \subset \mathbb{P}^N$ containing x and y: by Bertini's theorem, the generic element of this system is smooth outside the base locus $\{x,y\}$ of the system, and it is easy to see that, unless M is a curve with $T_x(M) = T_y(M) \subset \mathbb{P}^N$ (which circumstance we can always avoid by embedding M differently), the generic element of the system will be smooth at x and y as well. Thus we can find a smooth hyperplane section $V = H \cap M$ of M containing x and y. Consider the sequence

$$0 \to \mathcal{O}_M(E\otimes L^{m-1}) \to \mathcal{O}_M(E\otimes L^m) \to \mathcal{O}_V(E\otimes L^m) \to 0.$$

By Theorem B, there exists m_1 such that for $m > m_1$, $H^1(M, \mathcal{O}(E\otimes L^{m-1})) = 0$, so that the restriction map

$$H^0(M, \mathcal{O}(E\otimes L^m)) \to H^0(V, \mathcal{O}(E\otimes L^m))$$

will be surjective. On the other hand, by induction there exists m_2 such that for $m > m_2$,

$$H^0(V, \mathcal{O}_V(E\otimes L^m)) \to (E\otimes L^m)_x \oplus (E\otimes L^m)_y$$

is surjective. For $m > m_0 = \max(m_1, m_2)$, then, the map $(*)$ will be surjective.

Similarly, for each of a generating set of cotangent vectors $\{\omega_\alpha\}$ for $T_x^{*\prime}$ we can find a smooth hyperplane section V_α of M through x, such that ω_α is not in the kernel of the natural projection map $T_x^{*\prime}(M) \to T_x^{*\prime}(V_\alpha)$. Then by induction we can find m_α such that for $m > m_\alpha$, the differential map

$$H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m)) \to T_x^{*\prime}(V_\alpha) \otimes (E\otimes L^m)_x$$

is surjective. Likewise, from the sequence

$$0 \to \mathcal{O}_M(E\otimes L^{m-1}) \to \mathcal{I}_{x,M}(E\otimes L^m) \to \mathcal{I}_{x,V_\alpha}(E\otimes L^m) \to 0$$

we see that for $m > m_1$ as before,

$$H^0(M, \mathcal{I}_x(E\otimes L^m)) \to H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m))$$

is surjective. Thus for $m > m_0' = \max(m_1, m_\alpha)$, we have

$$\begin{array}{ccc} H^0(M, \mathcal{I}_x(E\otimes L^m)) & \xrightarrow{d_x} & T_x^{*\prime}(M)\otimes(E\otimes L^m)_x \\ \downarrow & & \downarrow \\ H^0(V_\alpha, \mathcal{I}_x(E\otimes L^m)) & \xrightarrow{d_x} & T_x^{*\prime}(V_\alpha)\otimes(E\otimes L^m)_x \end{array}$$

for all α, i.e., the map (**) is surjective. Q.E.D.

The Plücker Embedding

We close this section by describing the classical Plücker embedding of the Grassmannian $G(k,n)$ in projective space; this will illustrate both the Kodaira embedding theorem and Chow's theorem. The embedding line bundle over $G(k,n)$ will be $L = \det S^* = \det Q$. L may be seen to be positive by introducing a suitable metric with a positive curvature form in a similar manner to the Fubini-Study metric on projective space; rather than do this, however, we shall give the Plücker embedding directly. The *Plücker map*

$$p:\ G(k,n) \to \mathbb{P}(\wedge^k \mathbb{C}^n) = \mathbb{P}^{\binom{n}{k}-1}$$

simply sends a k-plane $\Lambda = \mathbb{C}\{v_1, \dots, v_k\} \subset \mathbb{C}^n$ to the multivector $v_1 \wedge \cdots \wedge v_k$. Explicitly, in terms of the basis $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}\}_{\# I = k}$ for $\wedge^k \mathbb{C}^n$, this map is given by

$$\Lambda \mapsto [\dots, |\Lambda_I|, \dots],$$

i.e., the homogeneous coordinates of the map are just the determinants $|\Lambda_I|$ of all the $k \times k$ minors Λ_I of a matrix representative of Λ. It follows that (1) p is holomorphic, (2) p takes every Schubert cycle of the form

$$\sigma_1(V) = \{\Lambda \in G(k,n):\ \dim(\Lambda \cap V_{n-k}) \geq 1\}$$

into a hyperplane section of $p(G(k,n)) \subset \mathbb{P}^{\binom{n}{k}-1}$. We can always find, for $\Lambda \neq \Lambda' \in G(k,n)$, an $(n-k)$-plane V_{n-k} such that $\Lambda \cap V_{n-k} \neq (0)$, $\Lambda' \cap V_{n-k} = (0)$, so p is 1-1; and since, in each open set $U_I = \{\Lambda : |\Lambda_I| \neq 0\}$ the Euclidean coordinates on $G(k,n)$ described above appear as

$$a_{jk} = \frac{|\Lambda_{I-j+k}|}{|\Lambda_I|},$$

the map p has nonzero differential. Thus the Plücker mapping is an embedding.

Now we shall determine equations which define the Plücker image of $G(k,V)$ in $\mathbb{P}(\wedge^k V)$. What we are asking for are the conditions that a multivector $\Lambda \in \bigwedge^k V$ be *decomposable*, i.e., of the form

$$\Lambda = v_1 \wedge \cdots \wedge v_k.$$

For this we pose the more general problem of determining the minimal linear subspace $W \subset V$ such that Λ is in the image of

$$\textstyle\bigwedge^k W \to \bigwedge^k V.$$

If $\dim W = l$, then $l \geq k$ with equality holding if and only if Λ is decomposable.

Recall the contraction operator

$$i(v^*): \textstyle\bigwedge^k V \to \bigwedge^{k-1} V$$

defined for $v^* \in V^*$ by

$$\langle i(v^*)\Lambda, \Xi \rangle = \langle \Lambda, v^* \wedge \Xi \rangle$$

for all $\Xi \in (\bigwedge^{k-1} V)^* \cong \bigwedge^{k-1} V^*$. We associate to Λ the linear spaces

$$\Lambda^\perp = \{v^* \in V^*: i(v^*)\Lambda = 0\} \subset V^*$$

and

$$W = \operatorname{Ann}(\Lambda^\perp) \subset V.$$

Lemma. W *is the minimal subspace of* V *such that* Λ *is in the image of* $\bigwedge^k W \to \bigwedge^k V$.

Proof. Let $w_1, \ldots, w_l$ be a basis for W, and complete it by $u_{l+1}, \ldots, u_n$ to a basis for V. Denote the dual basis of V^* by $\{w_i^*, u_\alpha^*\}$. Setting $U = \mathbb{C}\{u_{l+1}, \ldots, u_n\}$, the direct sum decomposition $V = W \oplus U$ induces

$$\textstyle\bigwedge^k V \cong \bigwedge^k W \oplus (\bigwedge^{k-1} W \otimes U) \oplus (\bigwedge^{k-2} W \otimes \bigwedge^2 U) \oplus \cdots.$$

We want to show that Λ lies in the first factor. Write the component of Λ in the second factor as $\sum_{\alpha=l+1}^n \Lambda_\alpha \otimes u_\alpha$, where $\Lambda_\alpha \in \bigwedge^{k-1} W$. Since

$$i(u_\alpha^*): \textstyle\bigwedge^{k-m} W \otimes \bigwedge^m U \to \bigwedge^{k-m} W \otimes \bigwedge^{m-1} U$$

and $i(u_\alpha^*)\Lambda = 0$, we deduce that all $\Lambda_\alpha = 0$. Similarly, the other factors of Λ in $\bigwedge^{k-m} W \otimes \bigwedge^m U$ $(m \geq 2)$ are zero, and consequently $\Lambda \in \bigwedge^k W$.

It is easy to see that W is the minimal such subspace. Q.E.D.

We now define

$$W' = \{w \in W: w \wedge \Lambda = 0\}.$$

If Λ is decomposable, then clearly $W' = W$. Conversely, if Λ is not decomposable so that $\dim W = l > k$, then since the pairing $\bigwedge^k W \otimes \bigwedge^{l-k} W \to \bigwedge^l W$ is nondegenerate we deduce that $W' \neq W$. So Λ is decomposable if and only if $W' = W$.

We now express this condition by duality, in two ways. For the first we use the operator

$$i(\Xi): \textstyle\bigwedge^k V \to V^*$$

defined for $\Xi \in \bigwedge^{k+1} V^*$ by

$$\langle i(\Xi)\Lambda, v \rangle = \langle \Xi, \Lambda \wedge v \rangle$$

for all $v \in V$. We observe that, by the definition of $\Lambda^\perp$, for $v \in W$ the left-hand side depends only on the image of Ξ under the natural projection

$$\bigwedge^{k+1} V^* \to \bigwedge^{k+1}\left(\frac{V^*}{\Lambda^\perp}\right) \cong \Lambda^{k+1} W^*.$$

Consequently, the condition $\Lambda \wedge w = 0$ for all $w \in W$ is equivalent to $i(\Xi)\Lambda \in \Lambda^\perp$ for all Ξ, which is in turn equivalent to

$$(*) \qquad i(i(\Xi)\Lambda)\Lambda = 0 \qquad \text{for all } \Xi \in \bigwedge^{k+1} V^*.$$

The left-hand side of $(*)$ gives $\binom{n}{k+1}$ quadratic forms in the homogeneous coordinates Λ_I of $p(G(k, V))$; setting them equal to zero gives the classical *Plücker relations*. In sum,

> *the image of the Grassmannian under the Plücker embedding* $p: G(k, V) \to \mathbb{P}(\Lambda^k V)$ *is cut out by the linear system of quadrics given by* $(*)$.

Alternatively, we may characterize W as being the image of

$$\bigwedge^{k-1} V^* \to V$$

under the map

$$\Xi \to i(\Xi)\Lambda, \qquad \Xi \in \bigwedge^{k-1} V^*.$$

Then the condition $W' = W$ is equivalent to

$$(**) \qquad (i(\Xi)\Lambda) \wedge \Lambda = 0 \qquad \text{for all } \Xi \in \bigwedge^{k-1} V^*.$$

For example, suppose that

$$\Lambda = \tfrac{1}{2} \sum_{i,j} \lambda_{ij} e_i \wedge e_j, \qquad \lambda_{ij} + \lambda_{ji} = 0,$$

is a bivector. Since for $v^* \in V^*$

$$(i(v^*)\Lambda) \wedge \Lambda = \tfrac{1}{2} i(v^*)(\Lambda \wedge \Lambda),$$

we may rewrite the conditions $(**)$ as

$$\Lambda \wedge \Lambda = 0.$$

When $n = 4$ we find the single equation

$$\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0$$

expressing the condition that $\Lambda \in \mathbb{P}(\bigwedge^2 \mathbb{C}^4) \cong \mathbb{P}^5$ be decomposable. In other words, $G(2,4)$ *is naturally realized as a nonsingular quadric hypersurface in* $\mathbb{P}^5$. We will see more of this in the final chapter.

2
RIEMANN SURFACES AND ALGEBRAIC CURVES

The dominant theme of this chapter is the interplay between the extrinsic projective geometry of algebraic curves and the intrinsic structure of Riemann surfaces. The subject, initially studied in extrinsic terms, underwent a basic shift in viewpoint with the introduction of the notion of abstract Riemann surface; nonetheless, the central aspects of the theory of algebraic curves as presented here are the same in either approach. Most of the results of this chapter were stated, if not proved, before the turn of the century.

We begin in Section 1 by refining the Kodaira embedding theorem in the case of dimension one. We then describe the local structure of maps between Riemann surfaces, and we use this to prove the Riemann-Hurwitz and genus formulas.We suggest the reader start with Section 2 and refer back to Section 1 as needed.

In Section 2 we introduce the theory of Abelian integrals and prove Abel's theorem and its converse. This theorem is perhaps most accessible in the case of elliptic curves—where indeed it was originally found—and we conclude with a discussion of this case.

We turn in Section 3 to the study of linear systems on curves. The fundamental result here is, of course, the Riemann-Roch formula. Next, we introduce the canonical curve, an intrinsically defined projective model of any nonhyperelliptic Riemann surface. The importance of the canonical curve is suggested by the geometric version of the Riemann-Roch; its full significance will continue to emerge through the remainder of the chapter. We initiate our study of special linear systems with Castelnuovo's bound on the genus of a curve of given degree in projective space; following a discussion of hyperelliptic curves and Riemann's count—which establishes our notion of the dependence of a Riemann surface on parameters—we start out on the road toward the solution of the complementary problem of Brill and Noether.

Sections 4 and 5 represent a shift of focus toward the extrinsic aspect of curves. In section 4 we prove the general Plücker formulas and the Plücker formulas for plane curves. There is a basic distinction between these results: the general Plücker formulas apply to curves in projective space of arbitrary dimension but deal only with the local character of the curve, while the formulas for plane curves describe such global phenomena as bitangents and double points, but apply only to curves in $\mathbb{P}^2$. The apparent gap is partially filled in the following section, where we introduce the powerful computational technique of correspondences and as an application derive formulas for the geometry of space curves. In both sections, the application of projective-geometric formulas to the canonical curve yields results about the intrinsic structure of Riemann surfaces: in Section 4 we obtain the count of Weierstrass points, and in Section 5 we solve some special cases of the Brill-Noether problem.

In the final two sections of the chapter we return to the study of the Jacobian variety associated to a compact Riemann surface. To begin with we give in Section 6 the rudiments of the theory of Abelian varieties; the dominant theme here is the working out of the Kodaira embedding theorem in the case of complex tori. In Section 7 we specialize to the case of the Jacobian of a curve. We see, by two lovely theorems of Riemann, how the geometry of the Jacobian is intimately connected to the special linear systems on the curve; following this we are finally able to prove some results on the Brill-Noether problem. The chapter concludes with Torelli's theorem, following Andreotti.

1. PRELIMINARIES

Embedding Riemann Surfaces

Let S be a compact Riemann surface. Throughout this chapter we assume that S is connected. If ds^2 is any metric on S with associated (1,1)-form ω, then $d\omega$ has degree 3 and so is trivially 0; thus any metric on S is Kähler. Indeed, since the $\bar{\partial}$-Laplacian of any metric commutes with the decomposition into type, we see that a form φ, written in terms of a local coordinate $z = x + \sqrt{-1}\, y$ as

$$\varphi = p\,dx + q\,dy = \alpha\,dz + \beta\,d\bar{z},$$

is harmonic if and only if $\varphi^{1,0} = \alpha\,dz$ is holomorphic and $\varphi^{0,1} = \beta\,dz$ is antiholomorphic. This will be the case if and only if

$$\partial\varphi = \bar{\partial}\varphi = 0,$$

or equivalently, if and only if

$$d\varphi = d^c\varphi = 0.$$

If $d\varphi = 0$, then locally $\varphi = df$ for some C^∞ function f; we have

$$d^c\varphi = d^c df = -\frac{1}{\pi}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy,$$

i.e., φ is harmonic if and only if f is harmonic in the usual sense of one complex variable. In particular, we see that the harmonic space $\mathcal{H}^1(S)$ does not depend on the choice of metric.

Now let ds^2 be a metric with (1, 1)-form ω, multiplied by a constant so that

$$\int_S \omega = 1.$$

$[\omega] \in H^2_{\mathrm{DR}}(S)$ is an integral cohomology class, and by the Kodaira embedding theorem S can be embedded in projective space $\mathbb{P}^N$. In fact, as suggested in the discussion of the embedding theorem, a sharper statement and a simpler proof of the theorem are possible for Riemann surfaces, and we give these here.

Let $L \to S$ be a holomorphic line bundle. Recall that the *degree* of L is defined to be its first Chern class $c_1(L) \in H^2(S, \mathbb{Z})$ under the identification $H^2(S, \mathbb{Z}) = \mathbb{Z}$ given by the natural orientation of S. If $L = [D]$ for

$$D = \sum a_i p_i \in \mathrm{Div}(S),$$

then

$$\deg L = \sum a_i.$$

As we have seen, L has a nontrivial global holomorphic section only if $c_1(L) \geq 0$, i.e.,

$$\deg L < 0 \Rightarrow H^0(S, \mathcal{O}(L)) = 0.$$

On the other hand, since the generator of $H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ corresponding to $+1$ is represented by a positive form,

$$L \text{ positive} \Leftrightarrow \deg L > 0.$$

Thus, if $\deg L > \deg K_S$, then $L \otimes K_S^*$ is positive, and by Kodaira vanishing

$$H^1(S, \mathcal{O}(L)) = H^1(S, \Omega^1(L + K_S^*)) = 0.$$

Alternatively, this fact follows from Kodaira-Serre duality:

$$\begin{aligned}\deg L > \deg K_S &\Rightarrow \deg(K_S \otimes L^*) < 0\\ &\Rightarrow H^1(S, \mathcal{O}(L)) \cong H^0(S, \mathcal{O}(K_S \otimes L^*)) = 0.\end{aligned}$$

Now for any $p \in S$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}(L - p) \longrightarrow \mathcal{O}(L) \xrightarrow{r_p} L_p \longrightarrow 0.$$

If $\deg(L-p)=\deg L-1>\deg K_S$, then $H^1(S,\mathcal{O}(L)-p))=0$ and it follows that

$$H^0(S,\mathcal{O}(L))\to L_p\to 0,$$

that is, the complete linear system of a line bundle of degree greater than $\deg K_S+1$ has no base points. Moreover, if $\deg L>\deg K_S+2$, then from the exact sequences

$$0\longrightarrow\mathcal{O}(L-p-q)\longrightarrow\mathcal{O}(L)\overset{r_{p,q}}{\longrightarrow}L_p\oplus L_q\longrightarrow 0,$$

$$0\longrightarrow\mathcal{O}(L-2p)\longrightarrow\mathcal{O}(L)\overset{d_p}{\longrightarrow}T_p'^*\otimes L_p\longrightarrow 0,$$

and the vanishing

$$H^1(S,\mathcal{O}(L-p-q))=H^1(S,\mathcal{O}(L-2p))=0,$$

it follows that the complete linear system of L gives an embedding $\iota_L:S\to\mathbb{P}^N$.

Summarizing, we have:

1. $\deg L<0\Rightarrow H^0(S,\mathcal{O}(L))=0.$
2. $\deg L>\deg K_S\Rightarrow H^1(S,\mathcal{O}(L))=0.$
3. $\deg L>\deg K_S+2\Rightarrow\iota_L:S\to\mathbb{P}^N$ is well-defined and an embedding.

The phrases *compact Riemann surface* and *smooth algebraic curve* or just *curve* will be used pretty much interchangeably from now on. This is somewhat imprecise, as a smooth algebraic curve may be thought of as carrying the additional structure of an embedding—i.e., as a Riemann surface S together with a line bundle $L\to S$ and subspace $E\subset H^0(S,\mathcal{O}(L))$ —but hopefully no confusion should arise. What is important is the ability to think alternately of the abstract analytic object—the compact Riemann surface—and the algebraic object—the zeros of polynomials in $\mathbb{P}^N$; this is implicit in the use of the two terminologies.

As we saw in Section 4 of Chapter 1, the variety $C(S)$ of chords of an algebraic curve $S\subset\mathbb{P}^N$ is a closed subvariety of dimension $\leqslant 3$ in $\mathbb{P}^N$. Projecting from a point $p\not\in C(S)$ to any hyperplane $H\subset\mathbb{P}^N$ gives an embedding of S in $H\cong\mathbb{P}^{N-1}$; thus any curve can be smoothly embedded in 3-space $\mathbb{P}^3$. We cannot, in general, embed a curve in $\mathbb{P}^2$. Given a smooth curve $S\subset\mathbb{P}^3$, however, we can find a point $p\in\mathbb{P}^3$ that does not lie on any tangent line to S in $\mathbb{P}^3$, or on any line meeting S in more than two points, or on any line meeting S in two points with intersecting tangent lines. The projection map $\pi_p|_S:S\to\mathbb{P}^2$ will then have everywhere nonzero differential and will be at most 2-1 at isolated points; the image $\pi_p(S)\subset\mathbb{P}^2$ will be a plane algebraic curve whose only singularities are *ordinary double points*, or *nodes*—i.e., near a singular point, $\pi_p(S)$ will look like the union of two smooth analytic arcs meeting at a point with distinct tangents. (This discussion will be sharpened considerably in Section 4 of this chapter.)

Note also that for a curve $S \subset \mathbb{P}^N$ and smooth point $p \in S$, the projection map $\pi_p : S - \{p\} \to \mathbb{P}^{N-1} \subset \mathbb{P}^N$ to a hyperplane can be extended continuously, hence holomorphically, over all of S by sending p to the point of intersection of its tangent line with $\mathbb{P}^{N-1}$. The intersection of a general hyperplane $H \subset \mathbb{P}^{N-1}$ with the image $\pi_p(S)$ will be just the intersection of the hyperplane $\overline{H,p} \subset \mathbb{P}^N$ with $S - \{p\}$, so that

$$\deg \pi_p(S) = \deg S - 1$$

for $p \in S$. The simplest case here is the stereographic projection of a plane conic C from a point of C onto a line. (See Figure 1.)

The Riemann-Hurwitz Formula

We know from elementary topology that a compact Riemann surface S has only one topological invariant, which we may take to the be the *genus*

$$g(S) = \frac{b_1(S)}{2} = \frac{-\chi(S)+2}{2},$$

or, commonly, the "number of handles."

We saw in Section 2 of Chapter 1 that the curvature form of a metric on the holomorphic tangent bundle $T'(S) = K_S^*$ is just the Gaussian curvature of the metric times the volume form divided by $\sqrt{-1}$. By the classical Gauss-Bonnet theorem, then,

$$\deg K_S = -\chi(S) = 2g - 2.$$

This is a form of the *Riemann-Hurwitz formula* and can be proved directly as follows: Let $f: S \to S'$ be a holomorphic map between compact Riemann

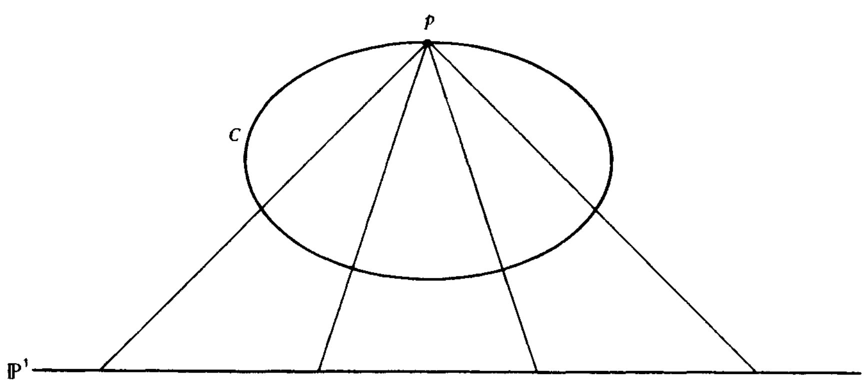


Figure 1

surfaces S and S'. For the induced map $f_*: H_2(S,\mathbb{Z}) \to H_2(S',\mathbb{Z})$, in homology

$$f_*([S]) = n \cdot [S'];$$

the integer n is called the *sheet number*, or *degree*, of the map. For any point $p \in S'$, let Θ be a curvature form for the line bundle $[(p)]$ associated to the divisor (p). Then $f^*\Theta$ is a curvature form for the line bundle $f^*[(p)] = [f^*(p)]$ on S, and we see from the proposition in Section 2 of Chapter 1 that

$$\deg f^*(p) = \int_S f^*\left(\frac{\sqrt{-1}}{2\pi}\Theta\right) = n\int_{S'}\left(\frac{\sqrt{-1}}{2\pi}\Theta\right) = n,$$

so the map f assumes all values $p \in S'$ exactly n times, counting multiplicity in the sense of divisors.

For any $p \in S$, we can find local coordinates z around p in S and w near $f(p)$ in S' such that the map f is given locally by

$$w = z^v.$$

The number v is called the *ramification index* of the map f at p; p is called a *branch point* if $v(p) > 1$. The *branch locus* of the map f is taken to be either the divisor

$$B = \sum_{p \in S} (v(p) - 1) \cdot p$$

on S or its image

$$B' = \sum_{p \in S} (v(p) - 1) \cdot f(p)$$

on S'. For any point $p \in S'$, we can write

$$f^*(p) = \sum_{q \in f^{-1}(p)} v(q) \cdot q,$$

$$\deg f^*(p) = n = \sum_{q \in f^{-1}(p)} v(q),$$

where the summation is over distinct points. This then gives us a picture of the map f: away from the branch locus of f in S', f is a covering map; at a branch point $p \in S$ of ramification index k, k sheets of the covering come together.

We can, in terms of the sheet number and ramification of f, relate the genus of S to the genus of S'. Take a triangulation of S' in which every point of the branch locus appears as a vertex. Because f is a covering map away from B, we can take a triangulation of S whose open cells are just the connected components of the inverse images of the open cells in our triangulation of S'. Then if c_0, c_1, c_2 denote the number of 0-, 1-, and 2-cells in S', respectively, we will have $n \cdot c_1$ 1-cells and $n \cdot c_2$ 2-cells in S. Since for any $p \in S'$,

$$\sum_{q \in f^{-1}(p)} v(q) = n,$$

we see also that the number of distinct points

$$^{\#}(f^{-1}(p)) = n - \sum_{q \in f^{-1}(p)} (v(q) - 1).$$

Consequently the number of vertices in our triangulation of S is

$$n \cdot c_0 - \sum_{q \in S} (v(q) - 1),$$

and the Euler characteristic

$$\begin{aligned}\chi(S) &= n \cdot c_2 - n \cdot c_1 + n \cdot c_0 - \sum_{q \in S} (v(q) - 1)\\ &= n \cdot \chi(S') - \sum_{q \in S} (v(q) - 1),\end{aligned}$$

so

$$g(S) = n \cdot (g(S') - 1) + 1 + \frac{1}{2} \sum_{q \in S} (v(q) - 1).$$

We can also relate the canonical bundle of S to that of S'. Let ω be a global meromorphic 1-form on S', written locally as

$$\omega = \frac{g(w)}{h(w)}\, dw.$$

For any point $p \in S$ of ramification index v we can find a coordinate z on S centered around p, with f given by

$$w = z^{v}.$$

Then

$$\begin{aligned}f^*\omega &= \frac{g(z^{v})}{h(z^{v})}\, dz^{v}\\ &= v \cdot z^{v-1} \cdot \frac{g(z^{v})}{h(z^{v})} \cdot dz,\end{aligned}$$

so

$$\operatorname{ord}_p(f^*\omega) = v \cdot \operatorname{ord}_{f(p)}(\omega) + (v - 1).$$

This implies the equation of divisors on S

$$(f^*\omega) = f^*(\omega) + \sum_{p \in S} (v(p) - 1) \cdot p$$

i.e.,

$$K_S = f^*K_{S'} + B,$$
$$\deg K_S = n\cdot \deg K_{S'} + \sum_{p\in S}(v(p)-1).$$

Now any compact Riemann surface S admits a holomorphic map to $\mathbb{P}^1$: if $f\in \mathfrak{M}(S)$ is any global meromorphic function written locally as g/h with g,h relatively prime, then f gives a map of S to $\mathbb{P}^1$ by $p\mapsto[g(p),h(p)]$. Let $f: S\to\mathbb{P}^1$ be such a map; on $\mathbb{P}^1$ we have

$$\chi(\mathbb{P}^1) = 2 = -\deg K_{\mathbb{P}^1},$$

and so

$$\begin{aligned}\chi(S) &= n\cdot\chi(\mathbb{P}^1) - \sum_{p\in S}(v(p)-1)\\ &= -n\cdot \deg K_{\mathbb{P}^1} - \sum_{p\in S}(v(p)-1)\\ &= -\deg K_S.\end{aligned}$$

Thus for any S,

$$\deg K_S = -\chi(S) = 2g-2,$$

and the Riemann-Hurwitz formula is established.

We will sometimes refer to the Riemann-Hurwitz formula as being any of the following:

$$\deg K_S = 2g-2,$$
$$\chi(S) = n\chi(S') - \sum_{q\in S}(v(q)-1),$$
$$K_S = f^*K_{S'} + B.$$

Note two things about maps $f: S\to S'$ between compact Riemann surfaces: first, that the number of branch points of f, counting multiplicity, is always even; and second, that unless f is constant, $g(S)\geqslant g(S')$. The latter follows also from the fact that a Riemann surface of genus g has exactly g linearly independent holomorphic 1-forms on it; if $f: S\to S'$ is nonconstant, it is easy to see that $f^*: H^0(S',\Omega^1_{S'})\to H^0(S,\Omega^1_S)$ is injective, and hence $g(S)\geqslant g(S')$.

The Genus Formula

We will give here three proofs of the *genus formula*, which gives the genus of a smooth plane curve in terms of its degree.

First, the topological argument. Suppose $S\subset\mathbb{P}^2$ is a smooth curve of degree d, given in $\mathbb{P}^2$ as the locus of zeros of a homogeneous polynomial $F(Z_0,Z_1,Z_2)$ of degree d. In terms of Euclidean coordinates $z_1=Z_1/Z_0$, $z_2=Z_2/Z_0$ on $C^2\subset\mathbb{P}^2$, the equation is

$$f(z_1,z_2) = F(1,z_1,z_2).$$

Choose a point $p \in \mathbb{P}^2$ not on S and a line H not containing p; after a linear change of coordinates we may take

$$p = [0,0,1], \qquad H = (Z_2 = 0);$$

we may also assume the line L at infinity $(Z_0=0)$ is not tangent to S.

Now consider the map $\pi_p : S \to \mathbb{P}^1$ given by projecting from p to H. Near a point $q \in S$ with $(\partial f/\partial z_2)(q) \neq 0$, z_1 will serve as local coordinate on S, so the map is unramified; if $(\partial f/\partial z_2)(q) = 0$, then $(\partial f/\partial z_1)(q) \neq 0$ and—taking z_2 as local coordinate on S near q, $z_1 = z_1(z_2)$ as a function of z_2—we can write

$$f(z_1(z_2), z_2) \equiv 0$$

so by the chain rule

$$\frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial z_1} \cdot \frac{\partial z_1}{\partial z_2} \equiv 0 \text{ on } S.$$

Consequently the order of vanishing of $\partial z_1/\partial z_2$ at q—that is, the ramification index $v(q)$ of the map π_p at q minus one—is equal to the order of zero of $\partial f/\partial z_2$ at $q \in S$—that is, the multiplicity of intersection of S with the curve $(\partial f/\partial z_2 = 0)$ at q. $(\partial f/\partial z_2 = 0)$ is a curve of degree $d-1$ in $\mathbb{P}^2$, and so its intersection number with S is $d(d-1)$; since all points of $S \cap (\partial f/\partial z_2 = 0)$ lie in the finite plane $(Z_0 \neq 0)$,

$$\sum (v(q) - 1) = d(d-1).$$

Now $[S] = d \cdot [H]$ in $H_2(\mathbb{P}^2, \mathbb{Z})$, so the sheet number of the projection map π_p is d; by the Riemann-Hurwitz formula,

$$\begin{aligned} \chi(S) &= d \cdot \chi(\mathbb{P}^1) - \sum_{q \in S} (v(q) - 1) \\ &= 2d - d(d-1) \end{aligned}$$

and so

$$\begin{aligned} g(S) &= \frac{2 - \chi(S)}{2} \\ &= \frac{(d-1)(d-2)}{2}. \end{aligned}$$

A second way to arrive at this formula is by the *adjunction formula* from Section 2 of Chapter 1. It gives

$$\begin{aligned} K_S &= K_{\mathbb{P}^2}|_S \otimes N_S \\ &= (K_{\mathbb{P}^2} + S)|_S. \end{aligned}$$

Now from that section $K_{\mathbb{P}^2} = -3H$ and $S = dH$, so $K_{\mathbb{P}^2} + S = (d-3)H$ on

$\mathbb{P}^2$. Thus

$$\chi(S) = -\deg K_S = -{}^{\#}(S\cdot(d-3)H) = -d(d-3)$$

and

$$g(S) = \frac{2-\chi(S)}{2} = \frac{(d-1)(d-2)}{2}.$$

The third way to compute $g(S)$ is by the Poincaré residue map. Recall (p. 147) that for a meromorphic 2-form ω on $\mathbb{P}^2$ holomorphic on $\mathbb{P}^2-S$ and with a single pole along S, and written locally as

$$\omega = g(z_1,z_2)\frac{dz_1\wedge dz_2}{f(z_1,z_2)},$$

the Poincaré residue $R(\omega)$ is given by

$$R(\omega) = -g(z_1,z_2)\frac{dz_1}{(\partial f/\partial z_2)(z_1,z_2)} = g(z_1,z_2)\frac{dz_2}{(\partial f/\partial z_1)(z_1,z_2)}.$$

Recall also that the Poincaré residue map gives in this case an isomorphism

$$H^0(\mathbb{P}^2,\Omega^2(S))\longrightarrow H^0(S,\Omega^1_S).$$

Now consider $\omega\in H^0(\mathbb{P}^2,\Omega^2(S))$, written as above. The form $dz_1\wedge dz_2$ extends to a meromorphic 2-form on $\mathbb{P}^2$; since $K_{\mathbb{P}^2}=-3H$ and $dz_1\wedge dz_2$ is nonzero holomorphic on $\mathbb{P}^2-L$, it follows that $dz_1\wedge dz_2$ must have a pole of order 3 along the line L. Similarly, f extends to a meromorphic function on $\mathbb{P}^2$, and since f is a polynomial with a single zero along a curve of degree d in $\mathbb{P}^2-L$, it must have a pole of order d along L. It follows that g must extend to a meromorphic function with a pole of order $\leqslant d-3$ along L, i.e., g must be a polynomial of degree $\leqslant d-3$ in z_1,z_2. Thus *the holomorphic 1-forms in* S *are exactly the differentials*

$$\omega = g(z_1,z_2)\frac{dz_1}{(\partial f/\partial z_2)(z_1,z_2)}$$

for g *a polynomial of degree* $\leqslant$ d$-$3. We have seen that the number of monomials of degree $\leqslant d$ in n variables is $\binom{n+d}{d}$, and so

$$g(S) = h^0(S,\Omega^1) = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2}.$$

Later on we will see how to extend this formula to certain singular curves.

Cases $g=0,1$

First, let S be any compact Riemann surface of genus 0. Then

$$h^1(S,\mathcal{O}) = h^0(S,\Omega^1) = 0$$

and so, for $L=[p]$ the line bundle associated to any point $p\in S$ we see from the long exact cohomology sequence associated to the sequence

$$0\to\mathcal{O}_S\to\mathcal{O}_S(L)\to L_p\to 0$$

that L has a global section nonzero at p, i.e., there exists a nonconstant meromorphic function f on S, holomorphic away from p and having only a simple pole at p. But such a function assumes the value ∞, and hence every value λ, exactly once, and so gives an isomorphism $f: S\to\mathbb{P}^1$. Thus,

any compact Riemann surface of genus 0 *is the Riemann sphere* $\mathbb{P}^1$

Next, we consider curves of genus 1. The full story on these curves will not be available to us until the next section; for the time being we will start by proving that *any compact Riemann surface S of genus 1 can be realized as a nonsingular cubic curve in* $\mathbb{P}^2$.

The proposition is easy to prove: we know that $\deg K_S=0$, and so, by the embedding theorem, for any $p\in S$ the complete linear system of the line bundle $L=[3p]$ gives an embedding of S as a cubic curve in $\mathbb{P}^N$ where $N=h^0(S,\mathcal{O}(L))-1\geq 2$. But $H^0(S,\mathcal{O}(L))$ corresponds to meromorphic functions on S holomorphic on $S-\{p\}$ and of order ≥ -3 at p; since any such function is uniquely determined by its principal part

$$\frac{a_{-3}}{z^3}+\frac{a_{-2}}{z^2}+\frac{a_{-1}}{z}+a_0+\cdots .$$

at p, and since there cannot exist a meromorphic function with only a single pole at p—as noted before, such a function would give a 1-1 map of S to $\mathbb{P}^1$—we see that $h^0(S,\mathcal{O}(L))\leq 3$, hence $h^0(S,\mathcal{O}(L))=3$ and we are done.

It is worthwhile, however, to go through the process explicitly in this case. First we shall establish a basic general fact:

Lemma (Residue Theorem). *For* φ *a meromorphic one-form on a compact Riemann surface* S *with polar divisor* $a_1+\cdots+a_d$,

$$\sum_i \operatorname{Res}_{a_i}(\varphi) = 0.$$

Proof. Letting $B_\varepsilon(a_i)$ be an ε-disc around a_i, we have by Stokes' theorem

$$0 = -\int_{S - \bigcup_i B_\varepsilon(a_i)} d\varphi = \int_{\partial(\bigcup_i B_\varepsilon(a_i))} \varphi = \sum_i \operatorname{Res}_{a_i}(\varphi). \qquad \text{Q.E.D.}$$

Applying this to $\varphi = df/f$ shows again that meromorphic function f on S has the same number of zeroes as poles.

We return to our Riemann surface S of genus 1. As noted before, there are no nonconstant meromorphic functions on S with only a single pole at p. On the other hand, by the vanishing theorem

$$H^1(S, \mathcal{O}(p)) = 0,$$

and so the exact sequence

$$0 \to \mathcal{O}(p) \to \mathcal{O}(2p) \to \mathbb{C}_p \to 0$$

tells us that there does indeed exist a meromorphic function F on S with a double pole at p, holomorphic elsewhere. Next,

$$h^0(S, \Omega^1) = g(S) = 1,$$

so S has a nonzero holomorphic 1-form ω; since $\deg K_S = \deg(\omega) = 0$, ω must be everywhere nonzero. Consider the meromorphic form $F \cdot \omega$; it is holomorphic on $S - \{p\}$, and by the residue theorem

$$\operatorname{Res}_p(F \cdot \omega) = 0.$$

Consequently if z is any local coordinate around p, after multiplying by a constant and adding a constant we can write the series expansion of F as

$$F(z) = \frac{1}{z^2} + [1].$$

Now consider the meromorphic function dF/ω on S. Since ω is nonzero everywhere, dF/ω is holomorphic on $S - \{p\}$ and has a triple pole at p; setting

$$F' = \lambda \frac{dF}{\omega} + \lambda' F + \lambda''$$

for suitable constants $\lambda, \lambda', \lambda''$, we can write

$$F'(z) = \frac{1}{z^3} + [1]$$

near p.

The map $\iota_L : S \to \mathbb{P}^2$ associated to the line bundle $L = [3p]$ can thus be given by

$$q \mapsto [1, F(q), F'(q)].$$

Writing out expansions around p, we have

$$F'(z)^2 = \frac{1}{z^6} + \frac{c}{z^2} + [-1]$$

and

$$F(z)^3 = \frac{1}{z^6} + \frac{c'}{z^3} + \frac{c''}{z^2} + [-1],$$

so that the meromorphic function

$$F'(z)^2 + c'F'(z) - F(z)^3 + (c''-c)F(z)$$

is holomorphic away from p, with at most a single pole at p, hence equal to a constant. The image of S under the embedding ι_L is accordingly the locus of the polynomial

$$y^2 + c'y = x^3 + ax + b,$$

where $x = Z_1/Z_0$, $y = Z_2/Z_0$ are Euclidean coordinates on $\mathbb{P}^2$. After a linear change of the coordinate y, we may take this polynomial of the form

$$(*) \qquad y^2 = x^3 + ax + b,$$

and finally, after a linear change in the x-coordinate, taking two of the roots of the polynomial $x^3 + ax + b$ to 0 and 1, we see that *any curve of genus 1 is the zero locus in* $\mathbb{P}^2$ *of a cubic polynomial*

$$y^2 = x \cdot (x-1) \cdot (x-\lambda)$$

for some $\lambda \in \mathbb{C}$.

Note that by the above a Riemann surface of genus 1 is determined by the one parameter λ in the polynomial (*) above; since the quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by any rank 2 lattice $\Lambda \subset \mathbb{C}$ is a Riemann surface of genus 1, and since one complex parameter is required to specify a lattice $\Lambda \subset \mathbb{C}$ of rank 2 up to an automorphism of $\mathbb{C}$, we might expect that in fact all curves of genus 1 may be realized as $\mathbb{C}/\Lambda$. This is in fact the case, as we shall see in the next section.

In closing we note that meromorphic functions on $S = \mathbb{C}/\Lambda$ are the same as entire meromorphic functions on $\mathbb{C}$, which are periodic for the lattice Λ.

2. ABEL'S THEOREM

Abel's Theorem—First Version

The indefinite integrals of the form

$$(*) \qquad \int \frac{dx}{\sqrt{x^2 + ax + b}}$$

are readily solved in closed form; more generally, any integral

$$\int R\left(x, \sqrt{x^2+ax+b}\,\right)dx,$$

for R a rational function, has a closed-form solution involving only elementary functions. The solutions to integrals of this type have been known since the early days of calculus. For a long time, however, mathematicians were unable to do much with the integrals

$$(**) \qquad \int \frac{dx}{\sqrt{x^3+ax^2+bx+c}}$$

or, more generally, the *Abelian integrals*

$$\int R(x,y)\,dx,$$

where R is a rational function, and x and y are related by a polynomial equation $f(x,y)=0$ of degree >2.

In view of the genus formula of the last section, one reason for the difficulty is easy to spot: the first integral $(*)$ can be thought of as the line integral

$$\int \frac{dx}{y}$$

of the meromorphic form dx/y on the curve C given in terms of Euclidean coordinates x,y in $\mathbb{P}^2$ by $y^2=x^2+ax+b$. Now C is a conic curve, hence isomorphic to $\mathbb{P}^1$ via a polynomial map; if $t=t(x,y)$ is a Euclidean coordinate on $\mathbb{P}^1$, the meromorphic form dx/y on C must be of the form $R(t)dt$ on $\mathbb{P}^1$ with R a rational function. Thus for $(x_0,y_0),(x,y)\in C$,

$$\int_{(x_0,y_0)}^{(x,y)} \frac{dx}{y} = \int_{t(x_0,y_0)}^{t(x,y)} R(t)\,dt,$$

and the latter integral is easy to solve. Note moreover that since $\mathbb{P}^1$ is simply connected and dx/y is closed, the only dependence of the integral on the choice of path arises from the residues of dx/y, which are readily calculated.

The integral $(**)$, on the other hand, is the integral of the form dx/y on the cubic curve $C=(y^2=x^3+ax^2+bx+c)$. Now, if C is smooth then by the genus formula it has genus 1, and hence cannot be parametrized by a single meromorphic function; thus no such simple expression as the one given above for $(*)$ is possible for $(**)$. Moreover, C is topologically a torus and therefore not simply connected; so the integral

$$\int_p^q \frac{dx}{y}$$

is well-defined only modulo the *periods* of dx/y, that is, the integrals of dx/y over closed loops $\gamma \in H_1(C,\mathbb{Z})$. More precisely, note that from the preceding section the form $\omega = dx/y$ is everywhere holomorphic on C and so is a generator of $H^0(C,\Omega^1)$. Let γ_1, γ_2 be closed loops on C generating $H_1(C,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, and denote by

$$a_1 = \int_{\gamma_1} \omega, \qquad a_2 = \int_{\gamma_2} \omega$$

the corresponding periods of ω. The general periods of ω on C will then be of the form $n \cdot a_1 + m \cdot a_2$, $n, m \in \mathbb{Z}$. If a_1 and a_2 were linearly dependent over $\mathbb{R}$, we could write

$$k_1 \int_{\gamma_1} \omega + k_2 \int_{\gamma_2} \omega = 0$$

for $k_1, k_2 \in \mathbb{R}$; we would then have

$$k_1 \int_{\gamma_1} \bar{\omega} + k_2 \int_{\gamma_2} \bar{\omega} = 0,$$

and since ω and $\bar{\omega}$ generate $H^{1,0}(C) \oplus H^{0,1}(C) = H^1_{\mathrm{DR}}(C)$, this would imply that

$$k_1[\gamma_1] + k_2[\gamma_2] = 0 \in H_1(C,\mathbb{R}),$$

which is impossible. Thus a_1 and a_2 are independent over $\mathbb{R}$, and so *the periods* $\Lambda = \{\mathrm{n} \cdot \mathrm{a}_1 + \mathrm{m} \cdot \mathrm{a}_2\}_{\mathrm{n,m} \in \mathbb{Z}} \subset \mathbb{C}$ *of* ω *in* C *form a lattice in* $\mathbb{C}$. Correspondingly, the value of the integral

$$\int_{p_0}^{p} \omega,$$

while not a well-defined number, is well-defined as a point of the complex torus $\mathbb{C}/\Lambda$.

The first major step toward understanding integrals of this type was made by Abel in 1826. Abel noted that, while the single integral above is a highly intractable function of the point $p=(x,y)$ on C, the qualitative behavior of the more general *Abelian sums*

$$\sum \int_{p_0}^{p_i} \omega$$

was in fact subject to easily expressed relations. A special case of what Abel proved is the following: for C and ω as above, and for any line $L \subset \mathbb{P}^2$, let $p_1(L)$, $p_2(L)$, and $p_3(L)$ denote the three points of intersection of L with C (the ordering of these points, of course, is not well-defined). Let $\psi(L)$ denote the Abelian sum

$$\psi(L) = \sum_{i=1}^{3} \int_{p_0}^{p_i} \omega;$$

as before, $\psi(L)$ is well-defined modulo the periods Λ of ω. Then we have

Abel's Theorem (First Version)

$$\psi(L) = \text{constant} \pmod{\Lambda}$$

Proof. A modern version of the proof is deceptively easy. We consider ψ as a map

$$\psi:\ \mathbb{P}^{2*} \to \mathbb{C}/\Lambda$$

from the space $\mathbb{P}^{2*}$ of lines in $\mathbb{P}^2$ to the complex torus $\mathbb{C}/\Lambda$; clearly it is holomorphic. Let z be a Euclidean coordinate on $\mathbb{C}/\Lambda$ and dz the corresponding global 1-form; then, since $H^{1,0}(\mathbb{P}^2) = H^0(\mathbb{P}^2, \Omega^1) = 0$,

$$\psi^* dz \equiv 0,$$

and hence ψ is constant. Q.E.D.

In a similar way, we prove a slight generalization: again let C be a curve of genus 1, $\omega \in H^0(C, \Omega^1)$ a holomorphic differential, $\Lambda \subset \mathbb{C}$ the period lattice of ω. Then, if $D = (g) = \Sigma p_i - \Sigma q_i$ is the divisor of a meromorphic function f on C, we have

$$\sum_i \int_{q_i}^{p_i} \omega \equiv 0 \ (\text{modulo } \Lambda),$$

i.e., *there exists a collection of paths* α_i *from* q_i *to* p_i *such that*

$$\Sigma \int_{\alpha_i} \omega = 0.$$

Proof. Write $D_\lambda = (\lambda_0 f - \lambda_1) = \Sigma p_i(\lambda) - \Sigma q_i(\lambda)$ for $\lambda = [\lambda_0, \lambda_1] \in \mathbb{P}^1$; set

$$\psi(\lambda) = \sum_i \int_{q_i(\lambda)}^{p_i(\lambda)} \omega \quad (\text{modulo } \Lambda).$$

ψ is thus a holomorphic map $\mathbb{P}^1 \to \mathbb{C}/\Lambda$; by the same argument as before we see

$$\psi^* dz \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$$

$\Rightarrow \psi$ constant and since, as $\lambda_0 \to 0$, $\{p_i(\lambda)\} \to \{q_i(\lambda)\}$, we have $\psi \equiv 0$ (modulo Λ). Q.E.D.

Following some preliminaries concerning the reciprocity formulas, we will give the converse to this version of Abel's theorem for Riemann surfaces of arbitrary genus. Together with the Riemann-Roch formula, these constitute the fundamental tools in the study of algebraic curves.

Let S now be a compact Riemann surface of genus g, and let $\delta_1, \ldots, \delta_{2g}$ be 1-cycles in S forming a basis for $H_1(X, \mathbb{Z})$. We may take $\delta_1, \ldots, \delta_{2g}$ to be a *canonical basis*, i.e., such that δ_i intersects δ_{i+g} once positively, and does not intersect any other δ_j. In such a canonical basis, the cycles $\delta_1,\ldots,\delta_g$ are called the *A-cycles*, $\delta_{g+1},\ldots,\delta_{2g}$ the *B-cycles*.

Now let $\omega_1,\ldots,\omega_g \in H^0(S,\Omega^1)$ be a basis for the space of holomorphic 1-forms on S. The *period matrix* of S is the $g\times 2g$ matrix

$$\Omega = \begin{bmatrix} \int_{\delta_1}\omega_1 & \cdots & \int_{\delta_{2g}}\omega_1 \\ \vdots & & \vdots \\ \int_{\delta_1}\omega_g & \cdots & \int_{\delta_{2g}}\omega_g \end{bmatrix}.$$

The (transposed) column vectors $\Pi_i = (\int_{\delta_i}\omega_1,\ldots,\int_{\delta_i}\omega_g) \in \mathbb{C}^q$ of the period matrix are called the *periods*; we first check that they are linearly independent over $\mathbb{R}$: If we have $\sum k_i\Pi_i = 0$, $k_i \in \mathbb{R}$, then

$$\sum k_i \int_{\delta_i}\omega_j = 0 \text{ for all } j \Rightarrow \sum k_i \int_{\delta_i}\bar{\omega}_j = 0 \text{ for all } j,$$

$$\Rightarrow \sum k_i[\delta_i] = 0 \in H_1(S,\mathbb{R}),$$

since $\{\omega_j,\bar{\omega}_j\}$ span $H^1_{\mathrm{DR}}(S)$; this is impossible, since $\{\delta_i\}$ is a basis for $H_1(S,\mathbb{Z})$.

The $2g$ periods $\Pi_i \in \mathbb{C}^g$ thus generate a lattice

$$\Lambda = \{m_1\Pi_1 + \cdots + m_{2g}\Delta_{2g}, m_i \in \mathbb{Z}\}$$

in $\mathbb{C}^g$; we define the *Jacobian variety* $\mathcal{J}(S)$ of S to be the complex torus $\mathbb{C}^g/\Lambda$. The Jacobian is a natural range for Abelian integrals: whereas the integral $\int_p^q\omega$ of a single holomorphic differential ω is defined only modulo the $2g$ periods of ω, which are usually dense in $\mathbb{C}$, the vector

$$\left(\int_p^q \omega_1,\ldots,\int_p^q \omega_g\right),$$

is well-defined as a vector in $\mathbb{C}^g$ modulo the discrete lattice $\Lambda \subset \mathbb{C}^g$. Picking a base point $p_0 \in S$, accordingly, we have a natural map

$$\mu: S \to \mathcal{J}(S)$$

given by

$$\mu(p) = \left(\int_{p_0}^p \omega_1,\ldots,\int_{p_0}^p \omega_g\right) \in \mathcal{J}(S).$$

More generally, if $\mathrm{Div}^0(S)$ denotes the group of divisors of degree 0 on S, we define $\mu: \mathrm{Div}^0(S) \to \mathcal{J}(S)$ by

$$\mu\left(\sum p_\lambda - \sum q_\lambda\right) = \left(\sum \int_{q_\lambda}^{p_\lambda} \omega_\lambda,\ldots,\sum \int_{q_\lambda}^{p_\lambda} \omega_g\right).$$

To study this map, we need to learn something about the relations among the periods of the ω_i. These are expressed in the reciprocity laws, one of which we now derive.

The First Reciprocity Law and Corollaries

To begin with, we may assume that all of the cycles δ_i on the Riemann surface S issue from a common point $s_0 \in S$. The complement of the δ_i's is then a simply connected region Δ on S; the boundary $\partial\Delta$ contains each δ_i twice with opposite orientation, and may be pictured as in Figure 2. What we are doing is making the familiar topological representation of a surface of genus g as a polygon with $4g$ sides, which are identified in pairs.

Now let ω be a holomorphic differential on S, η a meromorphic form whose only singularities are simple poles at points $s_\lambda \in S$. Assuming that η has no poles on the paths δ_i, let Π^i and N^i denote the periods of ω and η, respectively, along the path δ_i. Since the region Δ is simply connected and ω is holomorphic, we can set

$$\pi(s) = \int_{s_0}^{s} \omega$$

to obtain a holomorphic function π in $\bar{\Delta}$ with $\omega = d\pi$. (See Figure 3.) Note that for any pair of points $p \in \delta_i$, $p' \in \delta_i^{-1}$ on $\partial\Delta$ that are identified on S

$$\begin{aligned}\pi(p') - \pi(p) &= \int_p^{p'} \omega \\ &= \int_p^{\delta_i(1)} \omega + \int_{\delta_{g+i}} \omega + \int_{\delta_i(1)}^{p'} \omega \\ &= \int_{\delta_{g+i}} \omega \\ &= \Pi^{g+i}\end{aligned}$$

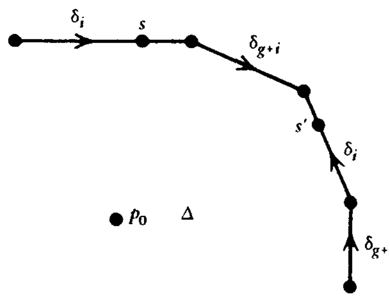


Figure 2

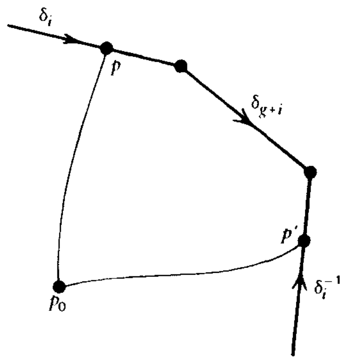


Figure 3

and similarly for $p \in \delta_{g+i}, p' \in \delta_{g+i}^{-1}$ identified on S,

$$\pi(p') - \pi(p) = -\Pi^i.$$

Consider the meromorphic 1-form $\pi \cdot \eta$ in $\bar{\Delta}$. By the residue theorem, since η has only first order poles,

$$\begin{aligned}\int_{\partial\Delta} \pi \cdot \eta &= 2\pi\sqrt{-1} \sum_{\lambda} \mathrm{Res}_{s_\lambda}(\pi \cdot \eta) \\ &= 2\pi\sqrt{-1} \sum_{\lambda} \mathrm{Res}_{s_\lambda}(\eta) \cdot \int_{s_0}^{s_\lambda} \omega.\end{aligned}$$

On the other hand, we can compute the integral of $\pi \cdot \eta$ around $\partial\Delta$ explicitly by considering together the contributions of the pair of sides of $\partial\Delta$ corresponding to δ_i and δ_i^{-1}: since, for points $p \in \delta_i$ and $p' \in \delta_i^{-1}$ identified on S, the difference $\Pi(p') - \Pi(p)$ is a constant Π^{g+i}, we see that

$$\int_{\delta_i + \delta_i^{-1}} \pi \cdot \eta = -\Pi^{g+i} \cdot \int_{\delta_i} \eta = -\Pi^{g+i} \cdot N^i.$$

Similarly

$$\int_{\delta_{g+i} + \delta_{g+i}^{-1}} \pi \cdot \eta = \Pi^i \cdot N^{g+i}.$$

Comparing the two expressions for $\int_{\partial\Delta} \pi \cdot \eta$, we find the

Reciprocity Law I

$$\sum_{i=1}^{g} (\Pi^i N^{g+i} - \Pi^{g+i} N^i) = 2\pi\sqrt{-1} \sum_{\lambda} \mathrm{Res}_{s_\lambda}(\eta) \cdot \int_{s_0}^{s_\lambda} \omega,$$

where the integrals on the right are taken in the interior of Δ.

This is classically known as the *reciprocity law for differentials of the first and third kinds*. In classical terminology, a *differential of the first kind* on a

Riemann surface S is a holomorphic 1-form; a *differential of the second kind* is a meromorphic 1-form with no residues, and a *differential of the third kind* is a meromorphic form with only single poles. Clearly a differential is of the first kind if and only if it is both of the second kind and of the third kind; we shall see shortly that any meromorphic 1-form is the sum of differentials of the second and third kinds. Later on we will prove a reciprocity law for differentials of the first and second kinds.

Before we can apply the reciprocity law, we need to prove a similar result, which will enable us to normalize our basis for $H^0(S,\Omega^1)$. Let ω,ω' be two holomorphic 1-forms on S, Π^i and Π'^i their respective periods around δ_i. Let Δ and π be as above, and consider the integral around $\partial\Delta$ of the form $\pi\cdot\bar{\omega}'$. The exterior derivative $d(\pi\cdot\bar{\omega}')=d\pi\wedge\bar{\omega}'=\omega\wedge\bar{\omega}'$, and so by Stokes' theorem

$$\int_{\partial\Delta}\pi\cdot\bar{\omega}' = \int_S \omega\wedge\bar{\omega}',$$

and, evaluating the line integral just as in the proof of the reciprocity law, we obtain

$$\int_S \omega\wedge\bar{\omega}' = \sum\left(\Pi^i\cdot\overline{\Pi'^{i+g}} - \Pi^{i+g}\,\overline{\Pi'^i}\right).$$

In particular if we take $\omega'=\omega$, then since $\omega\wedge\bar{\omega}$ is positive we find that

$$(**)\qquad 0<\sqrt{-1}\int_S \omega\wedge\bar{\omega} = \sqrt{-1}\sum_{i=1}^{g}\left(\Pi^i\,\overline{\Pi^{g+i}} - \Pi^{g+i}\,\overline{\Pi^i}\right)$$

for $\omega\not\equiv 0$. It follows from this that *any holomorphic 1-form ω whose* A-*periods all vanish must be identically zero*, i.e., the first $g\times g$ minor of the period matrix Ω is nonsingular. Once we know this, we can take our basis $\omega_1,\dots,\omega_g$ for $H^0(S,\Omega^1)$ so that

$$\int_{\delta_i}\omega_j = \delta_{ij}\qquad\text{for}\qquad 1\leqslant i,j\leqslant g,$$

i.e., so that the period matrix has the form

$$(I_g, Z).$$

Such a basis for $H^0(S,\Omega^1)$ is called *normalized.*

We now return to the reciprocity law and deduce some consequences. First, consider the case where $\eta=\omega'$ is a holomorphic 1-form; write Π'^i for the periods of ω'. Since ω' has no residues, the formula reads

$$\sum_{i=1}^{g}\left(\Pi^i\Pi'^{g+i} - \Pi^{g+i}\Pi'^i\right) = 0.$$

This is the *first Riemann bilinear relation* in the periods. In particular, if

$\omega = \omega_i, \omega' = \omega_j$ are elements of our normalized basis, all but two terms in the expression on the right vanish, and we have

$$\int_{\delta_{g+i}} \omega_j - \int_{\delta_{g+j}} \omega_i = 0,$$

i.e., the right-hand block Z in the period matrix above is symmetric. Note that since the quadratic form on $\overline{H^0(S,\Omega^1)}$ given by

$$(\omega_i, \omega_j) = \sqrt{-1} \int_S \omega_i \wedge \bar{\omega}_j = \sqrt{-1} \int_{\delta_{g+i}} \omega_j - \sqrt{-1} \int_{\delta_{g+j}} \omega_i = 2 \cdot \operatorname{Im} \int_{\delta_{g+i}} \omega_j$$

is positive definite, the imaginary part $\operatorname{Im}(Z)$ of Z is positive definite; this is the *second Riemann bilinear relation*. In sum, the two Riemann bilinear relations imply that for a normalized basis of $H^0(S,\Omega^1)$, the period matrix Ω of S has the form

$$\Omega = (I, Z) \quad \text{with} \quad Z = {}^t Z, \quad \operatorname{Im} Z > 0.$$

Abel's Theorem—Second Version

Let S as before be a Riemann surface of genus g, $D = \Sigma(p_\lambda - q_\lambda)$ a divisor of degree 0 on S, and consider the Abelian sum

$$\mu(D) = \left(\Sigma \int_{q_\lambda}^{p_\lambda} \omega_1, \ldots, \Sigma \int_{q_\lambda}^{p_\lambda} \omega_g \right) \in \mathcal{J}(S).$$

Abel's theorem in the case $g = 1$ tells us that if D is the divisor (f) of a meromorphic function f on S, then $\mu(D) = 0$. It is not hard to extend this statement to the case of genus g: If $D = (f)$, then the map

$$\psi: \ [\lambda_0, \lambda_1] \to \mu((\lambda_0 f - \lambda_1))$$

from $\mathbb{P}^1$ to $\mathcal{J}(S)$ is holomorphic, and since the holomorphic 1-forms dz_i on the complex torus $\mathcal{J}(S)$ span the cotangent space at each point,

$$\begin{aligned} \psi^*(dz_i) \equiv 0 &\Rightarrow \psi \quad \text{is constant} \\ &\Rightarrow \mu(D) = \psi(0) = \psi(\infty) = 0. \end{aligned}$$

Conversely, we will now show that *if* $D = \Sigma(p_\lambda - q_\lambda)$ *is any divisor on* S *of degree* 0 *and* $\mu(D) = 0$, *then* D *is the divisor of a meromorphic function.*

The problem, which may at first seem difficult, becomes straightforward once we transpose it from a question about the existence of a meromorphic function to one about the existence of a certain meromorphic form. Note that if f is a meromorphic function with $(f) = \Sigma(p_\lambda - q_\lambda)$, then the differential

$$\eta = \frac{1}{2\pi\sqrt{-1}} d\log f = \frac{1}{2\pi\sqrt{-1}} \frac{df}{f}$$

is a meromorphic form with polar divisor

$$(\eta)_\infty = -\Big(\sum_\lambda (p_\lambda + q_\lambda)\Big) \qquad 1)$$

$$\operatorname{Res}_{p_\lambda}(\eta) = \frac{a_\lambda}{2\pi\sqrt{-1}}, \qquad \operatorname{Res}_{q_\lambda}(\eta) = \frac{b_\lambda}{2\pi\sqrt{-1}}, \qquad 2)$$

where we are now writing

$$D = \sum a_i p_i + \sum b_i q_i$$

with the p_λ, q_λ distinct; and moreover

$$\int_\gamma \eta \in \mathbb{Z} \qquad 3)$$

for any closed loop γ on $S - \{p_i, q_i\}$. Conversely, if η is any meromorphic form with these three properties, we can set

$$f(p) = e^{2\pi i \int_{p_0}^{p} \eta}$$

to obtain a well-defined meromorphic function f with $(f) = D$. Thus, to prove the converse to Abel's theorem, we have to show that for $D = \Sigma(p_\lambda - q_\lambda)$ with $\mu(D) = 0$, there exists a differential of the third kind η, holomorphic on $S - \{p_\lambda, q_\lambda\}$ with residues a_λ at p_λ, b_λ at q_λ, and having all integral periods. First, we check that we can at least find a meromorphic differential with the requisite singularities:

Lemma. *Given a finite set of points* $\{p_\lambda\}$ *on* S *and complex numbers* a_λ *such that* $\Sigma a_\lambda = 0$, *there exists a differential of the third kind on* S, *holomorphic on* $S - \{p_\lambda\}$ *and having residue* a_λ *at* p_λ.

Proof. Consider the exact sheaf sequence on S

$$0 \longrightarrow \Omega^1 \longrightarrow \Omega^1(\Sigma p_\lambda) \xrightarrow{\text{res.}} \oplus \mathbb{C}_{p_\lambda} \longrightarrow 0.$$

By Kodaira-Serre duality

$$H^1(S, \Omega^1) \cong H^0(S, \mathcal{O}) \cong \mathbb{C},$$

so that the image of $H^0(S, \Omega^1([\Sigma p_\lambda]))$ in $\oplus \mathbb{C}_{p_\lambda}$ has codimension at most 1. But we have seen that the sum of the residues of any meromorphic 1-form on S is zero; so the image of $H^0(S, \Omega^1([\Sigma p_\lambda]))$ is contained in, hence equal to, the hyperplane $(\Sigma a_\lambda = 0) \subset \oplus \mathbb{C}_{p_\lambda}$. Q.E.D.

Now choose cycles $\delta_1, \ldots, \delta_{2g}$ representing a canonical basis for $H_1(S, \mathbb{Z})$ as on p. 227 such that no point p_λ, q_λ lies on one of the paths δ_j, and let $\omega_1, \ldots, \omega_g$ be a normalized basis for $H^0(S, \Omega^1)$ with respect to $\{\delta_1, \ldots, \delta_{2g}\}$. By the lemma, there exists a differential of the third kind with residues $a_\lambda/(2\pi\sqrt{-1})$ at p_λ, $b_\lambda/(2\pi\sqrt{-1})$ at q_λ; any two such forms differ by a holomorphic form on S, and hence there exists a unique such form η such that the A-periods

$$N^i = \int_{\delta_i} \eta = 0, \qquad i = 1,\dots,g.$$

The problem now is to alter η so as to make all its B-periods integral; clearly we can do this without disturbing the singularities of η or the integrality of its A-periods only by adding on an integral linear combination of the forms ω_i. To see if this is possible, we read off the B-periods of η by the reciprocity law: since $N^i = 0$ for $i = 1,\dots,g$, we have for each i,

$$\begin{aligned} N^{g+i} &= \sum_\lambda a_\lambda \int_{p_0}^{p_\lambda} \omega_i + \sum_\lambda b_\lambda \int_{p_0}^{q_\lambda} \omega_i \\ &= \sum_\lambda \int_{q_\lambda}^{p_\lambda} \omega_i \end{aligned}$$

for some choice of paths α_λ from q_λ to p_λ. Now we are essentially done. By hypothesis,

$$\mu(D) = \left(\sum_\lambda \int_{\alpha_\lambda} \omega_1, \dots, \sum_\lambda \int_{\alpha_\lambda} \omega_g \right) \in \Lambda,$$

i.e., there exists a cycle

$$\gamma \sim \sum_{k=1}^{2g} m_k \cdot \delta_k, \qquad m_k \in \mathbb{Z}$$

such that for each i,

$$\sum_\lambda \int_{\alpha_\lambda} \omega_i = \int_\gamma \omega_i,$$

and so

$$N^{g+i} = \int_\gamma \omega_i \qquad \textit{for all } i.$$

Set

$$\eta' = \eta - \sum_{k=1}^{g} m_{g+k} \omega_k.$$

The periods N'^i of η' are then given by

$$\begin{aligned} N'^i &= -m_{g+i}, \qquad i = 1,\dots,g, \\ N'^{g+i} &= N^{g+i} - \sum_{k=1}^{g} m_{g+k} \int_{\delta_{g+i}} \omega_k \\ &= \sum_{k=1}^{2g} m_k \int_{\delta_k} \omega_i - \sum_{k=1}^{g} m_{g+k} \int_{\delta_{g+i}} \omega_k \\ &= m_i + \sum_{k=1}^{g} m_{g+k} \left(\int_{\delta_{g+k}} \omega_i - \int_{\delta_{g+i}} \omega_k \right) \\ &= m_i, \end{aligned}$$

by the first bilinear relation of Riemann. Thus η' has all integral periods, and $D=(f)$ for $f(p)=\exp(2\pi\sqrt{-1}\int_{p_0}^{p}\eta')$.

Summarizing, we have proved

Abel's Theorem (Second Version). *Given* $D=\Sigma(p_\lambda-q_\lambda)\in \mathrm{Div}(S)$ *and* $\omega_1,\ldots,\omega_g$ *a basis for the space of holomorphic 1-forms on* S, *then* $D=(f)$ *for some meromorphic function* f *on* S *if and only if*

$$\varphi(D)=\left(\sum_\lambda\int_{q_\lambda}^{p_\lambda}\omega_1,\ldots,\sum_\lambda\int_{q_\lambda}^{p_\lambda}\omega_g\right)\equiv 0(\Lambda).$$

In fancier language: recalling that $\mathrm{Pic}^0(S)$ is the group of divisors of degree zero on S modulo linear equivalence, the map

$$\mu:\ \mathrm{Div}^0(S)\to\mathcal{J}(S)$$

factors

$$\begin{array}{ccc}\mathrm{Div}^0(S) & \xrightarrow{\mu} & \mathcal{J}(S)\\ & \searrow \qquad \nearrow{\scriptstyle\tilde{\mu}} & \\ & \mathrm{Pic}^0(S) & \end{array}$$

to give an injection $\tilde{\mu}:\mathrm{Pic}^0(S)\to\mathcal{J}(S)$.

Jacobi Inversion

The second statement of Abel's theorem above suggests our next question: Is the map $\mu:\mathrm{Div}^0(S)\to\mathcal{J}(S)$ given by Abelian sums surjective, or, in other words, is the induced map $\tilde{\mu}:\mathrm{Pic}^0(S)\to\mathcal{J}(S)$ an isomorphism? The Jacobi inversion theorem asserts that the answer to this question is yes, and in fact tells us that we obtain what is suggested by counting dimensions.

Theorem (Jacobi Inversion). *Given* S *a curve of genus* g, $p_0\in S$ *and* $\omega_1,\ldots,\omega_g$ *a basis for* $H^0(S,\Omega^1)$, *for any* $\lambda\in\mathcal{J}(S)$ *we can find* g *points* $p_1,\ldots,p_g\in S$ *such that*

$$(*)\qquad \mu\left(\sum_i(p_i-p_0)\right)=\lambda,$$

i.e., for any vector $\lambda\in\mathbb{C}^g$, *we can find* $p_1,\ldots,p_g\in S$ *and paths* α_i *from* p_0 *to* p_i *such that*

$$\sum_i\int_{\alpha_i}\omega_j=\lambda_j\qquad \textit{for all } j.$$

Moreover, for generic $\lambda\in\mathbb{C}^g$, *the divisor* Σp_i *is unique.*

Proof. For now, we will just prove the result; in Section 7 of this chapter, after introducing Riemann's theta function, we will see how to solve the equation (*) explicitly.

First let $S^{(d)}$ denote the set of effective divisors of degree d on S, i.e., the set of *unordered* d-tuples of points $\{p_1,\dots,p_d\}$ on S, not necessarily distinct. $S^{(d)}$ is the quotient of the d-fold product $S^d = S\times S\times\cdots\times S$ of S with itself d times by the action of the symmetric group Σ_d on d letters; as such it inherits from S^d the structure of a topological space. In fact, the projection map $\pi: S^d \to S^{(d)}$ gives $S^{(d)}$ the structure of a complex manifold: for a point $D=\Sigma p_i \in S^{(d)}$, let z_i be a local coordinate in a neighborhood U_i of p_i in S, where we take $U_i\cap U_j=\varnothing$ for $p_i\neq p_j$ and $z_i=z_j$ in $U_i=U_j$ for $p_i=p_j$. Then if we let $\sigma_1,\dots,\sigma_d$ denote the elementary symmetric functions, by the fundamental theorem of algebra the map

$$\sum q_i \mapsto (\sigma_1\{z_i(q_i)\},\dots,\sigma_d\{z_i(q_i)\})$$

gives a coordinate chart on $\pi(U_1\times\cdots\times U_d)\subset S^{(d)}$. Note that away from the branch locus the map π is a covering map and we can take coordinates $(z_1(p_1),\dots,z_d(p_d))$ on $S^{(d)}$. At the other extreme, around a point $d\cdot p$ local coordinates are

$$(z_1+\cdots+z_d,\ \dots,\ z_1\cdots z_d)$$

The compact complex manifold $S^{(d)}$ is called the dth *symmetric product* of S. (It is interesting to verify that $\mathbb{P}^{1(d)}=\mathbb{P}^d$.) Fixing a base point $p_0\in S$, there are inclusions

$$\iota:\ S^{(d)}\to \mathrm{Div}^0(S)$$

given by

$$\sum p_\lambda \mapsto \sum (p_\lambda - p_0)$$

and, correspondingly, holomorphic maps

$$\mu^{(d)}:\ S^{(d)}\to \mathcal{J}(S)$$

$$:\ \sum p_\lambda \mapsto \left(\sum_\lambda \int_{p_0}^{p_\lambda}\omega_1,\dots,\sum_i \int_{p_0}^{p_\lambda}\omega_g\right).$$

In this context, the Jacobi inversion theorem asserts that for S of genus g, the map $\mu^{(g)}$ is surjective and generically one-to-one.

Now let $D=\Sigma p_i$ be a point of $S^{(g)}$ with all p_i distinct, z_i a local coordinate on S centered at p_i, and $(z_1,\dots,z_g)$ corresponding coordinates on $S^{(g)}$ near D. For $D'=\Sigma z_i$ near D, by calculus

$$\frac{\partial}{\partial z_i}(\mu^{(g)}(D')) = \frac{\partial}{\partial z_i}\left(\int_{p_0}^{z_i}\omega_j\right)$$

$$= \omega_j/dz_i,$$

where we write ω/dz for the function $h(z)$ such that $\omega=h(z)dz$. Thus, and this is a fundamental observation, the Jacobian matrix of the map $\mu^{(d)}$ is given near D by

$$\mathcal{J}(\mu^{(d)}) = \begin{bmatrix} \omega_1/dz_1 & \cdots & \omega_1/dz_g \\ \vdots & & \vdots \\ \omega_g/dz_1 & \cdots & \omega_g/dz_g \end{bmatrix}.$$

We note that changing the local coordinate z_i multiplies the ith column by a nonzero factor but does not affect the rank of $\mathcal{J}(\mu^{(d)})$.

We may choose p_1 so that $\omega_1(p_1)\neq 0$, and then, subtracting a multiple of ω_1 from $\omega_2,\ldots,\omega_g$, we may arrange that $\omega_2(p_1)=\cdots=\omega_g(p_1)=0$. Next, we may choose p_2 so that $\omega_2(p_2)\neq 0$, and then arrange as before that $\omega_3(p_2)=\cdots=\omega_g(p_2)=0$. Continuing in this way, the Jácobian matrix at D will be triangular with zeros below the diagonal and nonzero on the diagonal, and so has maximal rank at D.

Thus the map $\mu^{(g)}$ is not everywhere singular, and the Jacobi inversion theorem follows from the fact that any holomorphic map $f: M \to N$ between compact connected equidimensional complex manifolds is surjective if $|\mathcal{J}(f)|\not\equiv 0$. This follows immediately from the proper mapping theorem: $f(M)\subset N$ is an analytic subvariety and contains an open set, hence $f(M)=N$. For a more elementary argument, let ψ_N be a volume form on N. Since f is orientation preserving and $|\mathcal{J}(f)|\not\equiv 0$,

$$\int_M f^*\psi_N > 0.$$

On the other hand, for any $q\in N$ we have

$$H^{2n}(N-\{q\},\mathbb{R}) = 0,$$

hence in $N-\{q\}$

$$\psi_N = d\varphi$$

for some $\varphi\in A^{2n-1}(N-\{q\})$. Then if $q\notin f(M)$,

$$\int_M f^*\psi_N = \int_{\partial M} df^*\varphi = 0,$$

a contradiction.

The only thing that remains to be proved is that $\mu^{(g)}$ is generically one-to-one. But this is clear: by Abel's theorem the fiber of $\mu^{(g)}$ over any point $\lambda\in\mathcal{J}(S)$ consists of the set $|D|$ of effective divisors linearly equivalent to any divisor $D\in{\mu^{(g)}}^{-1}(\lambda)$, which is a projective space. On the other hand, by dimension considerations the generic fiber of $\mu^{(g)}$ is 0-dimensional; it follows that the generic fiber of $\mu^{(g)}$ is one point. (The map $\mu^{(g)}$ is an example of a *birational map*; we shall discuss these in detail in Chapter 4.) Q.E.D.

Note that as a corollary to Jacobi inversion, we see that *every divisor of degree $\geq g$ on a Riemann surface of genus g is linearly equivalent to an effective divisor.*

Consider in particular the case of a Riemann surface S of genus 1. Then $\mathcal{J}(S)=\mathbb{C}/\Lambda$ and the map $\mu^{(1)}$ is given simply by

$$\mu: p\mapsto\int_{p_0}^{p}\omega,$$

where ω is a generator of $H^0(S,\Omega^1)$. By Abel's theorem, $\mu(p)=\mu(p')$ only if there exists a meromorphic function f on S with $(f)=(p-p_0)-(p'-p_0)=p-p'$; since we have seen that there are no meromorphic functions on S with only a single pole, it follows that the map $\mu^{(1)}$ is injective. By the Jacobi inversion theorem, the map is surjective as well, and so we have an isomorphism

$$\mu: S\longrightarrow\mathcal{J}(S),$$

i.e., *every Riemann surface of genus 1 is of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda\subset\mathbb{C}$.*

We have thus established the fundamental fact that the nonsingular cubic curves in $\mathbb{P}^2$ are the same as the compact Riemann surfaces $\mathbb{C}/\Lambda$ for a suitable lattice Λ in the complex plane. It follows that every such curve C has a group structure; we want to briefly discuss this.

First, recall that at the end of the previous section we constructed meromorphic functions F and F' on a Riemann surface C of genus 1, having a double and triple pole, respectively, at a base point $p_0\in C$ and holomorphic elsewhere. We chose F and F' so that in terms of a local coordinate w around p_0,

$$F(w)=\frac{1}{w^2}+[1]$$

and

$$F'(w)=\frac{1}{w^3}+[1]$$

and

$$dF=F'\cdot\omega,$$

where ω is a global nonzero holomorphic 1-form on C. Now, we may express C as the complex torus $\mathbb{C}/\Lambda$; let z be the Euclidean coordinate on C with $\omega=dz$. The function F is then the *Weierstrass* $\wp$-function; its derivative $(\partial/\partial z)\wp=-2F'$ is denoted $\wp'$. Note that the Laurent expansion for $\wp$ around p_0 can contain no terms of odd degree, since otherwise

$\wp(z)-\wp(-z)$ would be a nonconstant holomorphic function on C. Thus

$$\begin{aligned}\wp(z) &= \frac{1}{z^2}+az^2+bz^4+[6],\\ \wp'(z) &= -\frac{2}{z^3}+2az+4bz^3+[5],\\ \wp(z)^3 &= \frac{1}{z^6}+\frac{3a}{z^2}+3b+[2],\\ \wp'(z)^2 &= \frac{4}{z^6}-\frac{8a}{z^2}-16b+[1],\end{aligned}$$

from which we deduce that $\wp$ and $\wp'$ satisfy the relation

$$\wp'^2 = 4\cdot\wp^3-20a\cdot\wp-28b;$$

it is conventional to write g_2 for $20a$ and g_3 for $28b$.

Now the holomorphic map

$$\psi:\ \mathbb{C}/\Lambda\to\mathbb{P}^2$$

given by

$$z\mapsto[1,\wp(z),\wp'(z)]$$

embeds $\mathbb{C}/\Lambda$ as the locus of the polynomial $f(x,y)=y^2-4x^3+g_2\cdot x+g_3$. The differential $\omega=dz$ on $C=\mathbb{C}/\Lambda$ is the Poincaré residue

$$\omega = R\left(\frac{dx\wedge dy}{f(x,y)}\right)=\frac{dx}{\partial f/\partial y}=\frac{dx}{y},$$

and the inverse of ψ is the Abelian integral

$$\psi^{-1}(p)=\int_{p_0}^{p}\frac{dx}{y}\qquad\text{(mod periods)}$$

where we take in this case $p_0=\psi(0)=[0,0,1]\in\mathbb{P}^2$. If $p_1,p_2,p_3\in C$ and $z_1,z_2,z_3\in\mathbb{C}$ are the corresponding points in $\mathbb{C}/\Lambda$, then Abel's theorem is equivalent to the assertion

$$z_1+z_2+z_3\equiv 0(\Lambda)\Leftrightarrow(3p_0-p_1-p_2-p_3)\sim 0,$$

i.e., there exists a meromorphic function $f(z)$ on C with a triple pole at p_0 and zeros at p_1,p_2,p_3. To see this, let $A(x,y)=ax+by+c$ be the equation of the line L joining p_1 and p_2 in $\mathbb{P}^2$ and denote by p' the third point of intersection of L with C (Figure 4). Then since the line at infinity intersects C in the divisor $3p_0$ $A(\wp(z),\wp'(z))$ is a meromorphic function on $C=\mathbb{C}/\Lambda$ with divisor $p_1+p_2+p'-3p_0$. Thus $p'\sim p_3$ and so $p'=p_3$. In summary:

$$(*)\qquad z_1+z_2+z_3\equiv 0(\Lambda)\Leftrightarrow p_1,p_2,p_3\ \text{are collinear.}$$

Setting $p_i = [1, \wp(z_i), \wp'(z_i)]$, we may rewrite (*) in the form

$$(**)\qquad \begin{vmatrix} 1 & \wp(z_1) & \wp'(z_1) \\ 1 & \wp(z_2) & \wp'(z_2) \\ 1 & \wp(z_3) & \wp'(z_3) \end{vmatrix} = 0 \Leftrightarrow z_1 + z_2 + z_3 \equiv 0(\Lambda).$$

This beautiful relation may be interpreted in several—eventually equivalent—ways. One is as the famous *addition theorem* for elliptic functions expressing $\wp(-z_1-z_2)=\wp(z_1+z_2)$ and $\wp'(-z_1-z_2)=-\wp'(z_1+z_2)$ rationally in terms of $\wp(z_1)$, $\wp(z_2)$, $\wp'(z_1)$, and $\wp'(z_2)$. Alternately, we may give the group structure on the cubic curve C geometrically by making the construction with lines dictated by (**). In any case, the inversion of the elliptic integral via Abel's theorem and corresponding theory of cubic curves in the plane occupies a singular position of harmony and depth in the subject of algebraic geometry.

3. LINEAR SYSTEMS ON CURVES

Reciprocity Law II

Let S be a compact Riemann surface of genus g, ω a global holomorphic 1-form on S, and η a differential of the second kind, i.e., a global meromorphic 1-form with no residues. We want, as in the first reciprocity law, to relate the periods of ω and η to the singularities of η. Since these singularities are not described by the intrinsically defined residue, we choose a local coordinate z around each singular point p of η, and write

$$\eta(z) = (a_{-n}^p z^{-n} + \cdots + a_0^p + a_1^p z + \cdots)dz,$$
$$\omega(z) = (b_0^p + b_1^p z + \cdots)dz.$$

Note that $a_{-1}^p = \mathrm{Res}_p(\eta) = 0$, and that $b_0^p(p) = (\omega/dz)(p)$ as defined earlier.

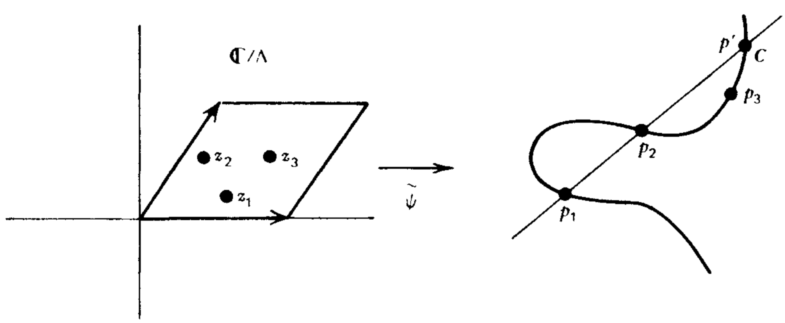


Figure 4

Now let $\delta_1,\ldots,\delta_{2g}$ be cycles on S representing a canonical basis for $H_1(S,\mathbb{Z})$, disjoint except for a common base point $s_0 \in S$ and not containing any singular points of η; let Π^i and N^i denote the periods of ω and η along δ_i. As before, $\Delta = S - \cup\,\delta_i$ is simply connected and we can set

$$\pi(s) = \int_{s_0}^{s} \omega$$

to obtain a holomorphic function π in Δ with $d\pi = \omega$. Consider the meromorphic differential $\pi\cdot\eta$ on Δ; since η is smooth on the arcs δ_i, the integral of $\pi\cdot\eta$ along the boundary of Δ is well-defined, and by the same argument as used in the first reciprocity law

$$\int_{\partial\Delta} \pi\cdot\eta = \sum_{i=1}^{g} \left(\Pi^i N^{g+i} - \Pi^{g+i} N^i\right).$$

On the other hand, near a singular point p of η with local coordinate z as above,

$$\pi(z) = \int_{s_0}^{p} \omega + b_0^p z + \tfrac{1}{2} b_1^p z^2 + \tfrac{1}{3} b_2^p z^3 + \cdots,$$

so that

$$\int_{\partial\Delta} \pi\cdot\eta = 2\pi\sqrt{-1}\,\sum_p \operatorname{Res}_p(\pi\cdot\eta) = 2\pi\sqrt{-1}\,\sum_p \left[\sum_{j=2}^{n} \frac{a_{-j}^p\cdot b_{j-2}^p}{j-1}\right].$$

Thus we have the *reciprocity law for differentials of the first and second kind*:

$$\sum_{i=1}^{g} \left(\Pi^i N^{g+i} - \Pi^{g+i} N^i\right) = 2\pi\sqrt{-1}\,\sum_{p,j} \frac{a_{-j}^p b_{j-2}^p}{j-1}.$$

The two reciprocity laws stated are the only ones we shall use in our discussion of curves. It should be pointed out, however, that more general laws can be obtained in the same way with little additional effort. For example, in either of the two formulas given, we may take ω to be a differential of the second kind: the function

$$\pi(s) = \int_{s_0}^{s} \omega$$

will then be meromorphic but still well-defined, and again we will have

$$\sum \left(\Pi^i N^{g+i} - \Pi^{g+i} N^i\right) = 2\pi\sqrt{-1}\,\sum_p \operatorname{Res}_p(\pi\cdot\eta).$$

Similarly, a reciprocity law for a pair of differentials of the third kind can be proved if we excise some additional arcs from our region Δ. We will not derive all these formulas—the general formalism should by now be evident—but we will mention one rather pretty result that is similarly obtained:

Theorem (Weil). *Let* f, g *be meromorphic functions on the compact Riemann surface* S, *with* (f) *disjoint from* (g). *Then*

$$\prod_{p\in S} f(p)^{\mathrm{ord}_p(g)} = \prod_{p\in S} g(p)^{\mathrm{ord}_p(f)}.$$

Proof. Let $\delta_1,\ldots,\delta_{2g}$ and Δ be as above. Let $\{p_i\}$ denote the support of (f), $\{q_i\}$ the support of (g), and draw smooth arcs α_i from s_0 to p_i disjoint except for their common base point s_0 and not containing any of the points $\{q_i\}$. Let Δ' be the complement of the arcs α_i in Δ; Δ' can again be considered as a polygon with sides $\ldots,\delta_i,\delta_{g+i},\delta_i^{-1},\delta_{g+i}^{-1},\ldots,\alpha_i,\alpha_i^{-1},\ldots$ as drawn in Figure 5. Since Δ' is simply connected and f is nonzero holomorphic in Δ', we can choose a single branch of the function $\log f$ in Δ'; we consider the meromorphic differential

$$\varphi = \log f\cdot d\log g = \log f\cdot \frac{dg}{g}$$

in Δ'. First, since dg/g has a single pole with residue $\mathrm{ord}_{q_i}(g)$ at each q_i, we have by the residue theorem

$$\begin{aligned}\int_{\partial\Delta'}\varphi &= 2\pi\sqrt{-1}\,\sum_{q_i}\mathrm{Res}_{q_i}(\varphi)\\ &= 2\pi\sqrt{-1}\,\sum_{q_i}\mathrm{ord}_{q_i}(g)\cdot\log f(q_i).\end{aligned}$$

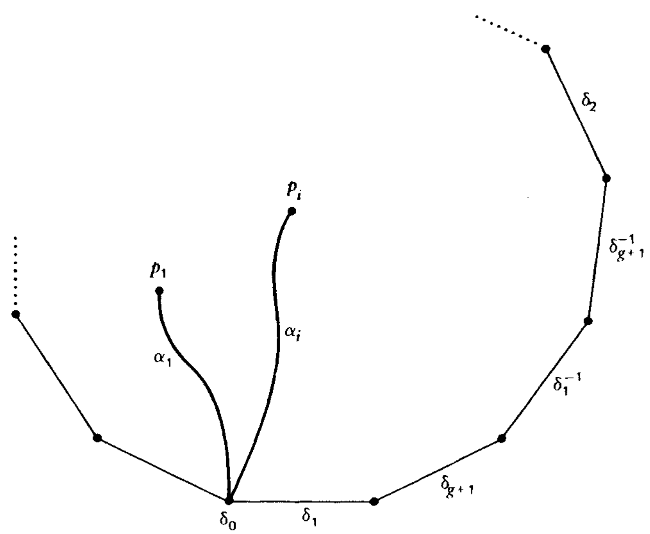


Figure 5

Now, for points $p \in \delta_i$, $p' \in \delta_i^{-1}$ on $\partial\Delta'$ identified on S,

$$\log f(p') = \log f(p) + \int_{\delta_{g+i}} d\log f,$$

and so

$$\int_{\delta_i + \delta_i^{-1}} \varphi = \left(\int_{\delta_i} d\log g\right)\left(-\int_{\delta_{g+i}} d\log f\right)$$

and similarly

$$\int_{\delta_{g+i} + \delta_{g+i}^{-1}} \varphi = \left(\int_{\delta_{g+i}} d\log g\right)\left(\int_{\delta_i} d\log f\right).$$

We also see that for points $p \in \alpha_i$, $p' \in \alpha_i^{-1}$ on $\partial\Delta'$ identified on S,

$$\log f(p') - \log f(p) = -2\pi\sqrt{-1} \cdot \operatorname{ord}_{p_i}(f),$$

and hence

$$\int_{\alpha_i + \alpha_i^{-1}} \varphi = 2\pi\sqrt{-1}\ \operatorname{ord}_{p_i}(f) \int_{s_0}^{p_i} d\log g.$$

Thus,

$$\begin{aligned}\sum_i \int_{\alpha_i + \alpha_i^{-1}} \varphi &= 2\pi\sqrt{-1}\left(\sum \operatorname{ord}_{p_i}(f)\cdot(\log g(p_i) - \log g(s_0))\right)\\ &= 2\pi\sqrt{-1}\ \sum \operatorname{ord}_{p_i}(f)\cdot \log g(p_i),\end{aligned}$$

since $\Sigma_{p_i} \operatorname{ord}_{p_i}(f) = 0$. In sum, we have

$$\begin{aligned}&2\pi\sqrt{-1}\left(\sum \operatorname{ord}_{q_i}(g)\cdot \log f(q_i) - \sum \operatorname{ord}_{p_i}(f)\cdot \log g(p_i)\right)\\ &\quad = \sum_{i=1}^{g}\left(\left(\int_{\delta_i} d\log f\right)\left(\int_{\delta_{g+i}} d\log g\right) - \left(\int_{\delta_i} d\log g\right)\left(\int_{\delta_{g+i}} d\log f\right)\right)\end{aligned}$$

But $\int_{\delta_i} d\log f$ is always an integral multiple of $2\pi\sqrt{-1}$; thus the right-hand term above is an integral multiple of $(2\pi\sqrt{-1})^2$ and we see that

$$\sum \operatorname{ord}_{q_i}(g)\cdot \log f(q_i) - \sum \operatorname{ord}_{p_i}(f)\cdot \log g(p_i) \in 2\pi\sqrt{-1}\ \mathbb{Z}.$$

Exponentiating, we obtain

$$\prod_i f(q_i)^{\operatorname{ord}_{q_i}(g)} = \prod_i g(p_i)^{\operatorname{ord}_{p_i}(f)}$$

as desired. Q.E.D.

The Riemann-Roch Formula

The starting point for our discussion of linear systems is the natural question: given a divisor D on a Riemann surface S of genus g, determine the dimension of $H^0(S,\mathcal{O}(D))$, i.e., the number of meromorphic functions f on S with

$$(f)+D \geqslant 0.$$

We will try to answer the question first for an effective divisor $D=\Sigma p_\lambda$ of degree d on S. We will assume moreover that the points p_λ are distinct —the only difference in the following computation if D has multiple points is a much more cumbersome notation.

As with Abel's theorem, the problem becomes tractable when expressed in terms of differentials. Now if $f\in\mathfrak{M}(S)$ with $(f)+D\geqslant 0$, then df is a meromorphic 1-form on S holomorphic on $S-\{p_\lambda\}$, with no periods, no residues, and a pole of order $\leqslant 2$ at each p_λ. Conversely, given any such differential η, the meromorphic function

$$f(p)=\int_{p_0}^{p}\eta$$

is well-defined and satisfies $(f)+D\geqslant 0$. Since $df=df' \Leftrightarrow f=f'+\lambda$, $\lambda\in\mathbb{C}$, we see that the dimension of $H^0(S,\mathcal{O}(D))$ is one more than the dimension of the vector space V of differentials of the second kind holomorphic on $S-\{p_\lambda\}$ with no periods and poles of order $\leqslant 2$ at p_λ.

By the Kodaira vanishing theorem, for any $p\in S$

$$H^1(S,\Omega^1(p))=0,$$

and so from the exact sequence

$$0\to\Omega^1(p)\to\Omega^1(2p)\to\mathbb{C}_p\to 0$$

we see that there exists a meromorphic form on S, holomorphic on $S-\{p\}$ and having a double pole at p; clearly this form cannot have any residues. It follows that if we let z_λ be a local coordinate around the point p_λ, for any sequence $a_1,\dots,a_d$ of complex numbers *there exists a meromorphic 1-form* η_a *on* S, *holomorphic on* $S-\{p_\lambda\}$ *and having principal part*

$$\eta_a(z)=\left(a_\lambda\cdot z_\lambda^{-2}+[0]\right)dz_\lambda$$

at p_λ. Since any two such forms differ by a holomorphic 1-form on S, we see moreover that *there exists a unique such differential* φ_a *with all* A*-periods zero*. Let $W\cong\mathbb{C}^d$ denote the vector space of such forms, and consider the linear map

$$\psi:\ W\to\mathbb{C}^g$$

obtained by integration over the B-cycles of S:

$$\psi:\ \varphi_a\mapsto\left(\int_{\delta_{g+1}}\varphi_a,\dots,\int_{\delta_{2g}}\varphi_a\right).$$

Clearly, the vector space V above is just the kernel of the map ψ.

To describe ψ explicitly, let $\omega_1,\ldots,\omega_g$ be a normalized basis for $H^0(S,\Omega^1)$. By the reciprocity law for differentials of the first and second kinds,

$$\int_{\delta_{g+j}} \varphi_a = 2\pi\sqrt{-1}\sum_{\lambda} a_\lambda\cdot(\omega_j/dz_\lambda)(p_\lambda),$$

i.e., the map ψ is given by the matrix

$$\begin{bmatrix} (\omega_1/dz_1)(p_1) & \cdots & (\omega_1/dz_d)(p_d) \\ \vdots & & \vdots \\ (\omega_g/dz_1)(p_1) & \cdots & (\omega_g/dz_d)(p_d) \end{bmatrix}.$$

Now the number of independent relations among the row vectors of this matrix is just the number of linearly independent holomorphic differentials vanishing at p_λ for all λ, that is, the dimension of $H^0(S,\Omega^1(-D))$. Thus

$$\begin{aligned} h^0(D) &= \dim(\ker\psi)+1 \\ &= d-\operatorname{rank}\psi+1 \\ &= d-g+h^0(K-D)+1. \end{aligned}$$

This is the classical *Riemann-Roch formula.*

We have proved the Riemann-Roch for effective divisors, and hence for all divisors of degree $\geqslant g$. For a general D of degree $\leqslant g-2$, we apply the formula to $K-D$ to obtain

$$h^0(K-D) = (2g-2-d)-g+1+h^0(D)$$

$$\Rightarrow h^0(D) = d-g+1+h^0(K-D).$$

Finally, if $\deg D = g-1$ and neither D nor $K-D$ is linearly equivalent to an effective divisor, then $h^0(D)=h^0(K-D)=0$, and the formula again holds.

The Riemann-Roch formula gives us immediately a picture of the behavior of generic linear systems: for generic effective divisors $D=\Sigma_{\lambda=1}^{d}p_\lambda$ of degree d the matrix $((\omega_i/dz_\lambda)(p_\lambda))$ has maximal rank, and so

$$h^0(D)=\begin{cases} 1, & d\leqslant g, \\ d-g+1, & d>g, \end{cases}$$

for D outside an analytic subvariety in $S^{(d)}$.

An effective divisor D such that $h^0(K-D)\neq 0$ is called *special*; a special divisor whose associated linear system is larger than that of the generic divisor of its degree—i.e., such that $h^0(K-D)>g-d$—is called *irregular.* A linear system is called special or irregular if its individual divisors are.

It should be mentioned at this point that the Riemann-Roch formula can be given a sheaf-theoretic proof. In general, if $E\to M$ is a holomorphic vector bundle on a compact complex manifold M, we define the *holomorphic Euler characteristic* of E to be

$$\chi(E) = \sum (-1)^p h^p(M, \mathcal{O}(E));$$

we usually write $\chi(\mathcal{O}_M)$ for the holomorphic Euler characteristic of the trivial line bundle, i.e.,

$$\chi(\mathcal{O}_M) = \sum (-1)^p h^{0,p}(M).$$

Now for a line bundle L on a Riemann surface S, by Kodaira-Serre duality we have

$$\begin{aligned} \chi(L) &= h^0(S, \mathcal{O}(L)) - h^1(S, \mathcal{O}(L)) \\ &= h^0(L) - h^0(K-L) \\ \chi(\mathcal{O}_S) &= h^{0,0}(S) - h^{0,1}(S) \\ &= 1 - g, \end{aligned}$$

and so the Riemann-Roch formula reads simply

$$\chi(L) = \chi(\mathcal{O}_S) + c_1(L).$$

To prove the Riemann-Roch in this form, we note that it is clear for the trivial bundle, and show that it holds for any $L=[D]$, if and only if it holds as well for $L'=[D+p]$ and $L''=[D-p]$, $p \in S$ any point. This is easy: the exact sheaf sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+p) \to \mathbb{C}_p \to 0$$

gives us the exact cohomology sequence

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}(D)) &\to H^0(S, \mathcal{O}(D+p)) \\ &\to \mathbb{C}_p \to H^1(S, \mathcal{O}(D)) \to H^1(S, \mathcal{O}(D+p)) \to 0, \end{aligned}$$

and since the alternating sum of the dimensions of the vector spaces in an exact sequence is zero, this implies that

$$\chi([D+p]) = \chi([D]) + 1. \qquad \text{Q.E.D.}$$

This version of the Riemann-Roch formula, while not as explicit as the first, points the way toward generalizations in higher dimensions. The principal fact that holds in general is this: the holomorphic Euler characteristic of a vector bundle $E \to M$ on a compact complex manifold is a topological invariant of E and M. In these terms, the essential point of the classical Riemann-Roch formula is the duality $h^1(D) = h^0(K-D)$.

Canonical Curves

Let S be a compact Riemann surface of genus $g \geqslant 2$, K the canonical bundle on S. We note immediately that the complete linear system $|K|$ has no base points: if $p \in S$ were in the base locus of $|K|$, we would have

$$h^0(K-p) = h^0(K) = g,$$

and hence by Riemann-Roch

$$h^0(p) = \deg(p) - g + 1 + h^0(K-p) = 1 - g + 1 + g = 2,$$

i.e., there would exist a nonconstant meromorphic function on S holomorphic on $S-\{p\}$ and having only a single pole at p, so S would be biholomorphic to $\mathbb{P}^1$. It follows that the line bundle K gives a map

$$\begin{aligned} \iota_K\colon\ & S \to \mathbb{P}^{g-1} \\ :\ & p \mapsto [\omega_1(p), \ldots, \omega_g(p)], \end{aligned}$$

where $\omega_1, \ldots, \omega_g$ are a basis for $H^0(S, \Omega^1)$. ι_K is called the *canonical mapping* of S, $\iota_K(S) \subset \mathbb{P}^{g-1}$ the *canonical curve* of S.

Now, the map ι_K is 1-1 if for any points $p, q \in S$, we can find a $\omega \in H^0(S, \Omega^1)$ with $\omega(p) = 0$, $\omega(q) \neq 0$; it is an immersion if for any $p \in S$ there exists ω vanishing exactly to order 1 at p. Thus, ι_K is an imbedding if and only if for any points p, q, not necessarily distinct,

$$h^0(K-p-q) < h^0(K-p) = g-1.$$

By Riemann-Roch,

$$h^0(K-p-q) = g - 3 + h^0(p+q),$$

and so

$$h^0(K-p-q) < h^0(K-p) \Leftrightarrow h^0(p+q) = 1.$$

Thus ι_K *fails to be an embedding if and only if there exists a meromorphic function on* S *having only two poles, i.e., if* S *can be expressed as a two-sheeted branched covering of* $\mathbb{P}^1$. Such a Riemann surface is called *hyperelliptic*. Hyperelliptic Riemann surfaces form an important subset of the set of all curves of genus g, with properties that often differ markedly from those of a general Riemann surface. We will discuss them in detail later on in this section; for the time being, we merely assure the reader that the "general" Riemann surface of genus $g \geqslant 3$ is indeed nonhyperelliptic.

Note that if $L \to S$ is any line bundle of degree $2g-2$, then

$$h^0(K-D) = \begin{cases} 0 & \text{if } D \neq K, \\ 1 & \text{if } D = K; \end{cases}$$

by Riemann-Roch, if $D \neq K$ we find $h^0(D) = g-1$. This implies that: *if* $S \subset \mathbb{P}^{g-1}$ *is any nondegenerate curve of genus* g *and degree* $2g-2$, *then* S *is a canonical curve.*

The canonical curve of a Riemann surface S derives much of its importance from the fact that it is intrinsically defined by S, and so as a general rule, *any projective invariant of the canonical curve reflects the* *intrinsic structure of* S. We will see this principle applied when we discuss Weierstrass points, and again in discussing the Torelli theorem.

We can rephrase the Riemann-Roch formula in terms of the geometry of the canonical curve: for any divisor $D=\Sigma p_i$ on the Riemann surface S, $h^0(K-D)$ is just the number of hyperplanes in $\mathbb{P}^{g-1}$ containing the points $\iota_K(p_i)$, and so $h^0(D)$ is equal to the degree of D minus the dimension of the linear space $\bar{D}$ spanned by the points p_i on the canonical curve. Here, of course, we take the "linear span" of a point p_i with multiplicity a_i in D to be the span of p_i together with the first a_i-1 derivatives of the canonical map. Finally, since the dimension of the linear span of d points on C is just $d-1$ less the number of independent linear relations on the points, we have the *geometric version of the Riemann-Roch*:

The dimension r *of the complete linear system containing a divisor* $\mathrm{D}=\Sigma\,\mathrm{p_i}$ *is equal to the number of independent linear relations on the points* $\mathrm{p_i}$ *on the canonical curve,*

i.e.,

The points of D *span exactly a* $(\mathrm{d}-\mathrm{r}-1)$*-plane.*

Indeed, the Riemann-Roch formula may be quite easily proved in this geometric form. To start, we prove the inequality

$$(*)\qquad \dim\bar{D} \leqslant (d-1)-\dim|D|.$$

Proof. Suppose that $D=\sum p_i$ moves in an r-dimensional linear system; that is, there exist $r+1$ independent meromorphic functions $f_0,\dots,f_r$ on S with

$$(f_\nu)+D \geqslant 0.$$

We may take $f_0=1$; then no nontrivial linear combination of the functions $f_1,\dots,f_r$ will be holomorphic. Equivalently, if z_i is a local coordinate on S centered around p_i and if we write

$$f_\nu(z_i) = a_{\nu,i}z_i^{-1}+\cdots$$

then the matrix

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,d} \\ \vdots & & \vdots \\ a_{r,1} & \cdots & a_{r,d} \end{bmatrix}$$

has maximal rank r. Now, if ω is any holomorphic 1-form on S, then by the residue theorem, for each ν,

$$\begin{aligned} 0 &= \sum_i \operatorname{Res}_{p_i}(f_\nu \omega) \\ &= \sum_i a_{\nu,i} \cdot \left(\frac{\omega(p_i)}{dz_i} \right). \end{aligned}$$

This gives r independent relations on the points p_i on the canonical curve, establishing the inequality (*).

We may now prove the opposite inequality by applying (*) to the residual series $K-D$ of D. Suppose that on the canonical curve

$$\dim \overline{D} = d - s - 1.$$

The hyperplanes in $\mathbb{P}^{g-1}$ containing D then cut out a linear subseries of $|K-D|$ of dimension

$$(g-1)-(d-s-1)-1 = g-d+s-1.$$

Applying (*) to a divisor $E \in |K-D|$, then, we see that

$$\begin{aligned} \dim \overline{E} &\leqslant \deg E - 1 - (g-d+s-1) \\ &= (2g-2-d) - 1 - (g-d+s-1) \\ &= g - s - 2. \end{aligned}$$

But now the hyperplanes in $\mathbb{P}^{g-1}$ containing E will cut out on S a linear subseries of $|D|$ having dimension at least

$$(g-1)-(g-s-2)-1 = s,$$

that is,

$$\dim |D| \geqslant s = d - 1 - \dim \overline{D}$$

and so the Riemann-Roch formula is proved. Q.E.D.

Special Linear Systems I

The Riemann-Roch formula describes exactly the behavior of the "generic" linear system on a Riemann surface, but it does not tell us much about irregular linear systems. We will now try to fill in the gap with some classical theorems relating the dimension, degree, and genus of special linear systems. Our basic lemma is

Lemma. *For* $C \subset \mathbb{P}^n$ *a nondegenerate curve, the points of a generic hyperplane section of* C *are in general position, i.e., no* n *of them are linearly dependent.*

Proof. Suppose C has degree d, and let $H_0 \subset \mathbb{P}^n$ be a hyperplane meeting C in d distinct points $p_1, \ldots, p_d$. Then for H in a sufficiently small neighborhood U of H_0 in $\mathbb{P}^{n*}$, the points $\{p_i(H)\}$ of intersection of H with C will vary holomorphically with $H \in \mathbb{P}^{n*}$. Accordingly, for every multiindex $I = \{i_1, \ldots, i_n\} \subset \{1, \ldots, d\}$, we get a map

$$\pi_I: U \to C^n = C \times C \times \cdots \times C,$$
$$: H \mapsto (p_{i_1}(H), \ldots, p_{i_n}(H));$$

moreover, since for any point $(q_1, \ldots, q_n) \in C^n$ sufficiently near $\pi_I(H_0)$, there is a hyperplane $H \in U$ containing $q_1, \ldots, q_n$, *the image of* U *under* π_I *contains an open set in* C^n.

Now let $D \subset C^n$ be the locus of points $(q_1, \ldots, q_n)$ such that $q_1, \ldots, q_n$ are linearly dependent. Since C is nondegenerate, D is a proper analytic subvariety of C^n, and so $\pi_I^{-1}(D)$ is likewise a proper subvariety of U. Thus, for $H \in U - \cup_I \pi_I^{-1}(D)$, the points of $H \cap C$ are in general position.

Q.E.D.

Now, we can characterize the dimension of a linear system $|D|$ as follows: $\dim |D| \geq t$ *if and only if for every* t *points* $p_1, \ldots, p_t \in S$ *there exists a divisor* $E \in |D|$ *containing* $p_1, \ldots, p_t$. Thus, if D and D' are two effective divisors on S, we can find a divisor, $E \sim D + D'$ containing any $h^0(D) - 1 + h^0(D') - 1$ points of S, and so we have

$$h^0(D + D') \geq h^0(D) + h^0(D') - 1$$

In particular, suppose D is special, so that $h^0(K - D) \neq 0$ and we can take $D' = K - D$. Then $h^0(D + D') = h^0(K) = g$, and we have

$$\begin{array}{ll} h^0(D) + h^0(K - D) & \leq g + 1 \\ h^0(D) - h^0(K - D) & = d - g + 1 \\ \hline 2h^0(D) & \leq d + 2 \end{array}$$

Note as well that equality holds in the last line if and only if every divisor in the canonical series $|D + D'| = |K|$ is the sum of a divisor in the linear system $|D|$ and a divisor from $|D'|$. But by our lemma the points of a generic hyperplane section of $\iota_K(S)$ are in general position, and so $2h^0(D)$ can equal $d + 2$ only if $D = 0$, $D = K$, or ι_K is not one-to-one. Summing up, then, we have

Clifford's Theorem. *For any two effective divisors on the compact Riemann surface* S,

$$\dim|D| + \dim|D'| \leqslant \dim|D + D'|$$

and for D *special*

$$\dim|D| \leqslant \frac{d}{2}$$

with equality holding only if D=0, D=K, *or* S *is hyperelliptic.*

Corollary. *If* $C \subset \mathbb{P}^n$ *is any curve of degree* $d < 2n$ *and genus* g,

$$g \leqslant d - n$$

with equality if and only if C *is normal.*

Proof. Let D be the hyperplane section of C. Then

$$\dim|D| = h^0(D) - 1 = n > \frac{d}{2},$$

and so by Clifford's theorem D is nonspecial. Thus $h^0(K - D) = 0$, and by Riemann-Roch

$$\begin{aligned} g &= d - h^0(D) + 1 \\ &\leqslant d - n. \end{aligned}$$

Q.E.D.

Of course, this bound can be realized by any Riemann surface of genus $g = d - n$, and any linear system of degree d.

It remains now to find the maximal genus of a curve of degree d in $\mathbb{P}^n$ for $d > 2n$, or equivalently to find a sharper bound on the dimension of a linear system than that provided by Clifford's theorem when $d \ll g$. We offer here an argument originally given by Castelnuovo in 1889.

Let $C \subset \mathbb{P}^n$ be a curve of degree d and genus g, with hyperplane section D. Consider the linear systems $|kD|$ for $k = 1, 2, \ldots$. By our basic lemma, we can take the points of D to be in general position in a hyperplane in $\mathbb{P}^n$.

Let $m = [(d-1)/(n-1)]$ be the greatest integer less than or equal to $(d-1)/(n-1)$, and for each integer $k \leqslant m$ choose a set Γ of $k(n-1)+1$ points of D. We claim that the hyperplanes in $H^0(C, \mathcal{O}(kD))$ corresponding to the points of Γ are all independent; to prove it we will exhibit, for any point $q \in \Gamma$, a hypersurface of degree k in $\mathbb{P}^n$ containing $\Gamma - \{q\}$ but not q. This is easy: if we partition the remaining points of Γ into k sets

$$\{p_1^1, p_2^1, \ldots, p_{n-1}^1\}, \{p_1^2, \ldots, p_{n-1}^2\}, \ldots, \{p_1^k, \ldots, p_{n-1}^k\}$$

of $(n-1)$ points each, then each set $\{p_\alpha^i\}_\alpha$ will be linearly independent, and its linear span will not contain q. We can thus find hyperplanes

$H_1,\ldots,H_k$ in $\mathbb{P}^n$ containing the points $\{p_\alpha^i\}_\alpha$ but not q; the sum $H_1 + \cdots + H_k$ is the desired hypersurface of degree k.

We see from this that the vector space of sections of $[kD]$ vanishing on all the points of D has codimension at least $k(n-1)+1$ in $H^0(C,\mathcal{O}(kD))$, i.e.,

$$h^0(kD) - h^0((k-1)D) \geqslant k(n-1)+1, \qquad \text{for } k \leqslant m.$$

The same argument likewise shows that for $k > \text{m}$ we can find k hyperplanes in $\mathbb{P}^n$ containing all but any one of the points of D, so that

$$h^0(kD) - h^0((k-1)D) = d \qquad \text{for } k > m.$$

Thus we have

$$\begin{aligned} h^0(D) &\geqslant n+1,\\ h^0(2D) &\geqslant n+1+2(n-1)+1\\ &= 3(n-1)+3,\\ h^0(3D) &\geqslant 6(n-1)+4,\\ &\vdots\\ h^0(mD) &\geqslant \frac{m(m+1)}{2}(n-1)+m+1,\\ &\vdots\\ h^0((l+m)D) &\geqslant \frac{m(m+1)}{2}(n-1)+m+1+ld. \end{aligned}$$

But now for m sufficiently large, the divisor $(l+m)D$ will be nonspecial. By Riemann-Roch, then,

$$h^0((l+m)D) = (l+m)d - g + 1,$$

so that

$$(*) \qquad \begin{aligned} g &\leqslant (l+m)d\,\frac{m(m+1)}{2}(n-1) - m - 1 - ld + 1\\ &= \frac{m(m-1)}{2}(n-1) + m(d-m(n-1)-1) \end{aligned}$$

Thus *the genus of a nondegenerate curve of degree* d *in* $\mathbb{P}^{\text{n}}$ *is at most*

$$\frac{m(m-1)}{2}(n-1)+m\varepsilon, \qquad \textit{where } m = \left[\frac{d-1}{n-1}\right], d-1 = m(n-1)+\varepsilon.$$

We will see in the section on ruled surfaces that in fact this bound is realized for each d and n, and give an explicit description of these curves of maximal genus. For the time being, let us summarize what we know in general about nondegenerate curves in $\mathbb{P}^n$: if C has degree d, then

$$d < n \Rightarrow C \text{ is degenerate,}$$
$$d = n \Rightarrow C \text{ is the rational normal curve,}$$
$$n < d < 2n \Rightarrow g \leqslant d - n, \text{ with equality if } C \text{ is normal,}$$
$$d = 2n \Rightarrow g \leqslant n + 1 \text{ with equality if and only if } C \text{ is a canonical curve,}$$
$$d \geqslant 2n \Rightarrow g \leqslant \frac{m(m-1)}{2}(n-1) + m\varepsilon,$$
$$\text{where } m = \left[\frac{d-1}{n-1}\right], d - 1 = m(n-1) + \varepsilon.$$

Note that if C achieves this bound, then equality must hold in the basic inequality (*) above, and it follows that the complete linear system $|kD|$ on C is cut out by hypersurfaces of degree k; or, in other words, the map

$$H^0(\mathbb{P}^n, \mathcal{O}(kH)) = \operatorname{Sym}^k H^0(\mathbb{P}^n, \mathcal{O}(H)) \to H^0(C, \mathcal{O}(kH))$$

must be surjective. Applying this in particular to the canonical curve, we have

Noether's Theorem. *For any curve nonhyperelliptic* C, *the map*

$$\operatorname{Sym}^l H^0(C, \mathcal{O}(K)) \to H^0(C, \mathcal{O}(lK))$$

is surjective for all l.

Castelnuovo's inequality can be inverted in two ways to give an upper bound on n in terms of d and g and a lower bound on d in terms of n and g. Without going through the manipulation, we have

$$n \leqslant \frac{2(l(d-1)-g)}{l(l+1)}, \qquad l = \left[\frac{2(g-1)}{d} + 1\right],$$
$$d \geqslant \frac{(j+1)}{2}(n-1) + \frac{g}{j} + 1, \qquad j(j-1) < \frac{2g}{n-1} \leqslant j(j+1).$$

Hyperelliptic Curves and Riemann's Count

Recall that a compact Riemann surface S of genus $g \geqslant 2$ is called hyperelliptic if there exists a meromorphic function f on S with only two poles, i.e., if S admits a 2-1 map $f: S \to \mathbb{P}^1$ to the Riemann sphere. By the Riemann-Hurwitz formula the number of branch points of such a map f is given by

$$\begin{aligned} b &= 2g - 2 + 2\chi(\mathbb{P}^1) \\ &= 2g + 2; \end{aligned}$$

of course, since f has only two sheets, it cannot have a multiple branch point. Let $z_1,\ldots,z_{2g+2}$ be the branch points, assumed finite, of f in $\mathbb{P}^1$, and consider the curve $S' = (w^2 = \prod_{i=1}^{2g+2}(z-z_i)) \subset \mathbb{C}^2$, together with the projection map π on the z-plane. Since the points z_i are distinct S' is smooth, and for $R > \max(|z_i|)$ we see that $\pi^{-1}(|z|>R)$ consists simply of two disjoint punctured discs; we can complete S' to a compact Riemann surface $\tilde{S}$ by replacing these punctured discs with full discs. The map $\pi: S' \to \mathbb{C}$ can be extended continuously, hence holomorphically, to a map $\tilde{\pi}: \tilde{S} \to \mathbb{P}^1$ by mapping the two added points to $z=\infty$. Thus $\tilde{S}$ will again be a double cover of $\mathbb{P}^1$ branched at the points $\{z_i\}$.

Now in general if two Riemann surfaces M, M' have maps $f: M \to \mathbb{P}^1$, $f': M' \to \mathbb{P}^1$ with the same branch locus $B \subset \mathbb{P}^1$, and if $f^{-1}(\mathbb{P}^1 - B)$ is isomorphic to $f'^{-1}(\mathbb{P}^1 - B)$ as topological covering spaces of $\mathbb{P}^1 - B$, then M and M' will be isomorphic: the isomorphism between $f^{-1}(\mathbb{P}^1 - B)$ and $f'^{-1}(\mathbb{P}^1 - B)$ will extend continuously, hence holomorphically, over the branch loci of f and f'. In the case at hand, it follows that the Riemann surfaces S and $\tilde{S}$ are isomorphic, i.e., that *any hyperelliptic Riemann surface of genus* g *can be realized as the smooth completion of the locus*

$$w^2 = g(z)$$

in $\mathbb{C}^2$, *for* g(z) *a polynomial of degree* 2g+2.

If S is a hyperelliptic Riemann surface given as the completion of $(w^2 = \prod_{i=1}^{2g+2}(z-z_i)) \subset \mathbb{C}^2$, we can compute explicitly a basis for $H^0(S, \Omega^1)$. First, note that we have an automorphism $j: S \to S$ of order 2 given by $j: (w,z) \mapsto (-w, z)$; j is called the *hyperelliptic involution* on S. The induced linear transformation

$$j^*: \ H^0(S,\Omega^1) \to H^0(S,\Omega^1)$$

is likewise of order 2, and so a priori we obtain a decomposition of $H^0(S,\Omega^1)$ into eigenspaces with eigenvalues $+1$ and -1. In fact, the $+1$ eigenspace is trivial, since a holomorphic 1-form ω on S with $j^*\omega = \omega$ would descend to give a holomorphic 1-form on $\mathbb{P}^1$, and none such exists. Thus we have $j^*\omega = -\omega$ for all $\omega \in H^0(S,\Omega^1)$.

Now consider the 1-form

$$\omega_0 = \frac{dz}{w}$$

on S. ω_0 is holomorphic and nonzero away from the points at ∞, since the points where w vanishes are exactly the zeros of dz. Since the total degree of ω_0 is $2g-2$ and ω_0 has the same order of zero or pole at the points of S lying over $z=\infty$, ω_0 must have a zero of order $g-1$ at each of these two points. If ω is any other holomorphic 1-form on S,

$$\omega = h \cdot \omega_0,$$

where h is a meromorphic function on S, holomorphic away from ∞. But we have $j^*\omega = -\omega$, $j^*\omega_0 = -\omega_0$, and so $j^*h = h$, i.e., h is a function of z alone, and so necessarily a polynomial in z. If h is of degree d, then h has $2d$ zeros on the finite part of S and hence a pole of order d at each of the points at ∞; since ω_0 has zeros of order $g-1$ at ∞ and $h\cdot\omega_0$ is holomorphic, we have $\deg h \leq g-1$. Thus we can write out a basis for $H^0(S,\Omega^1)$:

$$\left\{ \frac{dz}{w}, z\frac{dz}{w}, \ldots, z^{g-1}\frac{dz}{w} \right\}.$$

The canonical map ι_K of S is then given by

$$\iota_K(z,w) = [1, z, \ldots, z^{g-1}] \in \mathbb{P}^{g-1};$$

the image of S under ι_K is thus the rational normal curve in $\mathbb{P}^{g-1}$. Note, moreover that the canonical map factors through the projection f; since ι_K is intrinsically defined, it follows that *the map* f *is unique up to an automorphism of* $\mathbb{P}^1$.

To show that not all Riemann surfaces are hyperelliptic, we count the number of parameters needed to specify both a hyperelliptic curve and a general curve of genus g. First, we have seen that given any collection of $2g+2$ distinct points $z_i \in \mathbb{P}^1$ there is a unique hyperelliptic curve S with a 2-fold map $f: S \to \mathbb{P}^1$ having branch locus $B = \{z_i\}$. We can send any three points $z_1, z_2, z_3 \in B$ to 0, 1, and ∞ respectively by an automorphism of $\mathbb{P}^1$, and so we see that the general hyperelliptic Riemann surface of genus g can be described by specifying $(2g+2)-3 = 2g-1$ points on $\mathbb{P}^1$. Conversely, since f is unique up to an automorphism of $\mathbb{P}^1$, any hyperelliptic curve S corresponds to only finitely many such collections of $2g-1$ points; thus the family of such curves has $2g-1$ parameters locally.

We will now count the number of parameters needed to describe a general Riemann surface of genus g, following an argument of Riemann. Choose any integer n greater than $2g$. Any Riemann surface of genus g can be expressed as an n-sheeted branched cover of $\mathbb{P}^1$; the number of branch points of such a map is given by

$$\begin{aligned} b &= 2g - 2 + n\cdot\chi(\mathbb{P}^1) \\ &= 2n + 2g - 2. \end{aligned}$$

Conversely we claim that given any divisor B on $\mathbb{P}^1$ of degree $2n+2g-2$ and taking no point with multiplicity $> n-1$, there exist a finite number of Riemann surfaces S of genus g expressible as n-sheeted covers of $\mathbb{P}^1$ with branch locus B. We will construct these Riemann surfaces in case $B = \sum z_i$ consists of $2n+2g-2$ distinct points; the general case is more complicated but conceptually no more difficult.

Draw disjoint arcs γ_i in $\mathbb{P}^1$ from z_i to z_{i+1}; let $T_1,\ldots,T_n$ be n disjoint copies of $\mathbb{P}^1 - \cup\gamma_i$. (See Figure 6.) Choose a sequence of permutations $\sigma_0,\ldots,\sigma_{2n+2g-2} \in S_n$ with $\sigma_0 = \sigma_{2n+2g-2} = e$ and $\sigma_j \cdot \sigma_{j+1}^{-1}$ a nontrivial simple transposition for each j, such that $\{\sigma_j\}$ is transitive on $\{1,\ldots,n\}$; we can always find a finite number of such sequences. For each i, $1 \leqslant i \leqslant 2n+2g-3$, adjoin to $\cup_j T_j$ n copies $\{\gamma_i^j\}_j$ of the arc γ_i, identifying γ_i^j with the boundary of T_j along the upper edge of the cut γ_i and also with the boundary of $T_{\sigma_i(j)}$ along the lower edge of the cut γ_i. Let S be the resulting topological space, $f: S \to \mathbb{P}^1$ the obvious projection map. $f^{-1}(\mathbb{P}^1 - B) \subset S$ is a covering space of $\mathbb{P}^1 - B$, and so inherits uniquely a complex structure; this structure extends over all of S, taking as local coordinate at a point $p \in f^{-1}(z_i)$ either $(z - z_i)$ or $\sqrt{z - z_i}$ according to whether the map f is 1-1 or 2-1 in a neighborhood of p. Thus S is a compact Riemann surface that maps to $\mathbb{P}^1$ with branch locus B, and by the remark made earlier S is determined completely by the choices of permutations σ_i made in the construction.

We have seen that any divisor B of degree $2n+2g-2$ as above corresponds to a finite number of Riemann surfaces of genus g together with n-fold maps f to $\mathbb{P}^1$. It remains to see how many such divisors correspond to a single such Riemann surface S. Now the map f on S can be specified by giving first its polar divisor $D = (f)_\infty \in S^{(n)}$ and then the element of the linear system $H^0(S, \mathcal{O}([D]))$ corresponding to f. Clearly the choice of D depends on n parameters; since $n > 2g$, $h^0(K - D) = 0$, and by Riemann-Roch,

$$h^0(D) = n - g + 1,$$

so the choice of $f \in H^0(S, \mathcal{O}([D]))$ depends on $n - g + 1$ parameters. Thus the family of n-fold maps $f: S \to \mathbb{P}^1$ is

$$n + (n - g + 1) = 2n - g + 1$$

-dimensional. Since the family of divisors B as above is $(2n+2g-2)$-dimensional, it follows that the general Riemann surface of genus g depends locally on

$$2n + 2g - 2 - (2n - g + 1) = 3g - 3$$

parameters. In particular we note that for $g \geqslant 3$, the "generic" Riemann surface of genus g is nonhyperelliptic.

It is amusing to verify Riemann's count explicitly in cases $g = 3$, 4, and 5. First, as we have seen, the canonical curve of any Riemann surface of genus 3 is a quartic curve in $\mathbb{P}^2$, determined up to an automorphism of $\mathbb{P}^2$, and conversely if $C \subset \mathbb{P}^2$ is a smooth quartic curve, by the adjunction

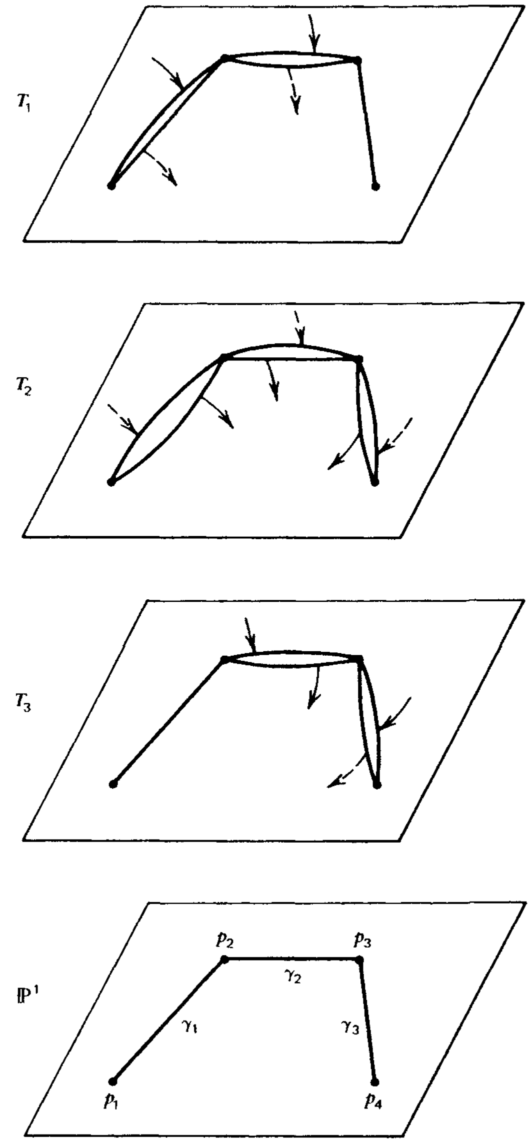


Figure 6. An example: $\sigma_1=(1,2)$, $\sigma_2=(1,3,2)$, $\sigma_3=(2,3)$.

formula

$$\begin{aligned} K_C &= (K_{\mathbb{P}^2} + C)|_C \\ &= (-3H + 4H)|_C \\ &= H|_C, \end{aligned}$$

so C is a canonical curve. Now the space of quartic curves in $\mathbb{P}^2$ has dimension

$$\binom{6}{2} - 1 = \frac{6 \cdot 5}{2} - 1 = 14,$$

and $\dim \mathrm{PGL}_3 = 9 - 1 = 8$; thus a curve of genus 3 depends on $14 - 8 = 6$ parameters, as predicted.

Next, let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. By Riemann-Roch, since $2K_C$ is nonspecial,

$$h^0(C, 2K_C) = 12 - 4 + 1 = 9.$$

But $h^0(\mathbb{P}^3, \mathcal{O}(2H)) = 10$, and so the restriction map

$$\begin{aligned} H^0(\mathbb{P}^3, \mathcal{O}(2H)) \to H^0&(C, \mathcal{O}(2H_C)) \\ &= H^0(C, 2K_C) \end{aligned}$$

has a kernel, i.e., C lies on a quadric surface Q; since a reducible quadric consists of two planes and so cannot contain C, Q is irreducible. Also, since

$$h^0(C, 3K_C) = 18 - 4 + 1 = 15$$

and

$$h^0(\mathbb{P}^3, \mathcal{O}(3H)) = 20,$$

C lies on a four-dimensional linear system of cubics in $\mathbb{P}^3$. The system of cubics containing Q is only $h^0(\mathbb{P}^3, \mathcal{O}(3H - Q)) - 1 = h^0(\mathbb{P}^3, \mathcal{O}(H)) - 1 = 3$-dimensional, and it follows that C also lies on a cubic Q' not containing Q. Since Q is irreducible, Q and Q' must then intersect in a curve of degree 6; but $C \subset Q \cap Q'$ and $\deg C = 6$, so

$$C = Q \cap Q'.$$

Conversely, by the adjunction formula, for any cubic Q' and quadric Q meeting in a smooth curve C, we have

$$\begin{aligned} K_{Q'} &= (K_{\mathbb{P}^3} + Q')|_{Q'} \\ &= (-4H + 3H)|_{Q'} = -H_{Q'} \end{aligned}$$

and

$$\begin{aligned} K_C &= (K_{Q'} + Q)|_C \\ &= (-H + 2H)|_C = H|_C, \end{aligned}$$

so C is a canonical curve of genus 4.

Now the quadric Q depends, as we said before, on 9 parameters. Two cubic polynomials will cut out the same curve on Q if their difference vanishes identically on Q; the vector space of cubics vanishing on Q has dimension

$$H^0(\mathbb{P}^3, \mathcal{O}(3H-2H)) = 4,$$

and so, Q having been chosen, the choice of Q' depends on

$$19-4=15$$

parameters. Bertini's theorem, moreover, assures us that the generic pair (Q,Q') do indeed meet transversely. Finally, PGL_4 has dimension 15, and so the number of parameters needed to describe a curve of genus 4 locally is

$$9+15-15=9,$$

as expected.

The case $g=5$ is somewhat easier. For $C\subset\mathbb{P}^4$ a canonical curve of genus 5, we have

$$H^0(C,2K_C) = 16-5+1 = 12.$$

But

$$H^0(\mathbb{P}^4, \mathcal{O}(2H)) = \binom{6}{4} = 15,$$

so the curve C must lie on three independent quadric surfaces Q, Q', and Q''; by the Enriques theorem of Section 3, Chapter 4, it is generically the intersection of these quadrics. Conversely, if Q, Q', and Q'' are any three quadrics in $\mathbb{P}^4$ meeting transversely, the adjunction formula applied three times tells us that C is a canonical curve of genus 5.

Now C is determined by specifying a three-dimensional subvector space of the vector space of polynomials of degree 2 on $\mathbb{P}^4$, i.e., by specifying a point in the Grassmannian $G=G(3,H^0(\mathbb{P}^4,\mathcal{O}(3H))$. G has dimension $3(15-3)=36$, and by Bertini's theorem applied three times we see that the linear system of quadrics corresponding to a generic point of G do in fact meet in a smooth curve. PGL_5 is 24-dimensional, and so we see that a curve of genus 5 depends locally on

$$36-24=12$$

parameters.

Special Linear Systems II

Earlier in this section we asked, what is the greatest possible genus of a nondegenerate curve of degree d in $\mathbb{P}^n$? Inverted, this is equivalent to the problem, what is the largest possible dimension of a linear system of degree d (or, the smallest possible degree of a linear system of dimension n) on a Riemann surface S of genus g, not counting those that factor through a quotient of S? We gave an answer to this question (which we shall later see is the correct one), but as we can now see, in case $d>2n$ or $n>g-1$ this bound cannot be realized by every Riemann surface of genus g. For example, we have seen that the greatest possible genus of a plane curve of degree d is $(d-1)(d-2)/2$; and of course this bound is sharp, being achieved by any smooth plane curve. The smallest degree of a two-dimensional linear system on a curve S of genus g which does not factor through a quotient of S is thus M, where

$$\frac{(M-1)(M-2)}{2} \geqslant g > \frac{(M-2)(M-3)}{2}.$$

We can see, however, that not every Riemann surface of genus g possesses such a linear system: if $g=(d-1)(d-2)/2$, then the family of Riemann surfaces of genus g with such a linear system—that is, plane curves of degree d—has dimension at most

$$h^0(\mathbb{P}^2, \mathcal{O}(dH)) - 1 - \dim \mathrm{PGL}_3 = \frac{(d+1)(d+2)}{2} - 9$$

while the family of all Riemann surfaces of genus g has dimension

$$3g-3 = \frac{3(d-1)(d-2)}{2} - 3.$$

For $d \geqslant 5$, then, the curves S of genus $g=(d-1)(d-2)/2$ having a net degree d are exceptional.

This example suggests another question, complementary to Castelnuovo's: what special linear systems exist on the *generic* Riemann surface of genus g? This is the *Brill-Noether problem*, which we will discuss further later in this chapter.

The presumed answer to—if not a proof of—the Brill-Noether problem is given by a simple-minded dimension count. Consider a canonical curve C of genus g in $\mathbb{P}^{g-1}$. By the geometric form of Riemann-Roch, an effective divisor $D=\Sigma p_i$ of degree d with $\dim|D|=r$ consists of d points on C spanning a $(d-1-r)$-plane in $\mathbb{P}^{g-1}$, that is, a d-secant $(d-1-r)$-plane to C. C will thus have a linear system of degree d and dimension r if and only if it has at least an r-dimensional family of d-secant $(d-r-1)$-planes.

Now the Grassmannian $G=G(d-r,g)$ of $(d-r-1)$-planes in $\mathbb{P}^{g-1}$ has dimension $(d-r)(g-d+r)$. The subvariety $\sigma_{g-d+r}(p)$ of $(d-r-1)$-planes passing through a point p has codimension $g-d+r$ in G, so the subvariety of $(d-r-1)$-planes meeting the curve C has codimension $g-d+r-1$. We may expect, then, that the subvariety of $(d-r-1)$-planes meeting C d times has codimension $d(g-d+r-1)$ in G, so that there will be an

r-dimensional family of such planes if

$$(d-r)(g-d+r)-d(g-d+r-1) \geqslant r.$$

Solving, we see that this will be the case when

$$(d-r)(r+1)-rg \geqslant 0.$$

Our count thus suggests that

The generic Riemann surface of genus g *will possess a linear system of degree* d *and dimension* r *if and only if*

$$d \geqslant \frac{rg}{r+1}+r$$

and there will in general be a $[(d-r)(r+1)-rg]$-*dimensional family of such linear systems.*

Clearly, our argument as it stands falls far short of a proof; a proof of one direction will be given in the final section of this chapter. Two cases we can check now are $r=1$ and 2. Since a linear system of degree d and dimension 1 without base points on a Riemann surface S gives a d-sheeted map $S \to \mathbb{P}^1$, the statement for $r=1$ amounts to

The generic Riemann surface of genus g *is expressible as a branched cover of* $\mathbb{P}^1$ *with*

$$d=\left[\frac{g+1}{2}\right]+1$$

sheets, but no fewer; in case g *is even it is so expressible in a finite number of ways (up to automorphisms of* $\mathbb{P}^1$*), while if* g *is odd there is a one-dimensional family of such representations.*

We can verify this in one direction by a count of parameters. A d-sheeted map of a curve of genus g to $\mathbb{P}^1$ has by Riemann-Hurwitz

$$b=2g-2+2d$$

branch points. By our general argument, then, the family of Riemann surfaces expressible as d-sheeted covers of $\mathbb{P}^1$ has dimension at most

$$b-3=2g+2d-5.$$

If the generic Riemann surface of genus g is so expressible, then, we have by Riemann's count

$$2g+2d-5 \geqslant 3g-3$$

i.e.,

$$d \geqslant \frac{g}{2}+1.$$

In case $r=2$, our result may be stated as

The generic Riemann surface may be represented as a plane curve of degree

$$d=\left[\frac{2g+2}{3}\right]+2$$

and no smaller.

Again we can check this in one direction. In the linear system of all plane curves of degree d, those that have a double point or worse form a subvariety of codimension 1, and the generic such curve has just one ordinary double point. Similarly if $\delta \leqslant (d-1)(d-2)/2$, the variety of curves of degree d having δ double points or worse has codimension δ, and the generic such curve has just δ ordinary double points. Now, as we shall see in the next section, the genus of a plane curve of degree d with δ ordinary double points is

$$g=\frac{(d-1)(d-2)}{2}-\delta;$$

there is, accordingly, an

$$\begin{aligned} h^0(\mathbb{P}^2,\mathcal{O}(dH))-1-\delta &= \frac{(d+1)(d+2)}{2}-1+g-\frac{(d-1)(d-2)}{2} \\ &= 3d+g-1 \end{aligned}$$

-dimensional family of plane curves of degree d and genus g. Since the group PGL_3 acts on the family of such curves, the number of Riemann surfaces so expressible has dimension

$$3d+g-1-8=3d+g-9.$$

Thus the generic Riemann surface of genus g may be represented in this way only if

$$3d+g-9 \geqslant 3g-3,$$

i.e., if

$$d \geqslant \tfrac{2}{3}g+2,$$

as predicted.

Some amusing enumerative problems arise from this discussion. For example, we have seen that the generic Riemann surface of genus $g=2k$ has a finite number of pencils of degree $k+1$; we may ask how many. This question will be answered in the cases $g=2$, 4, 6, and 8 in the discussion of correspondences in the next section, and in general in the final section.

4. PLÜCKER FORMULAS

Associated Curves

In this section we will concern ourselves with the *extrinsic geometry* of curves, i.e., the study of properties of curves $C \subset \mathbb{P}^n$ having to do with the embedding. To a certain extent, our study of associated curves is the complex analogue of the Frenet formalism in classical Euclidean differential geometry. Because of the complex analytic structure, however, the subject is much richer; we will obtain quantitative and qualitative results that could not be hoped for in the C^∞ case.

We make one remark before proceeding. Clearly, if $f: S \to \mathbb{P}^n$ is any map of a Riemann surface into projective space, then we can lift f locally to $\mathbb{C}^{n+1}$—that is, in a neighborhood of any point $p \in S$ we can find a holomorphic vector-valued function v to $\mathbb{C}^{n+1}$ such that $f(z) = [v_0(z), \ldots, v_n(z)]$. Conversely, for $v: S \to \mathbb{C}^{n+1}$ any vector-valued function, *the map* $\mathrm{f(z)=[v_0(z),\ldots,v_n(z)]}$ *is well-defined even if* $\mathrm{v}=0$ *at isolated points.* To see this, simply let z be a local coordinate centered around a zero p of v; then if $k = \min(\operatorname{ord}_p v_i)$, the map

$$\tilde{f}(z) = \left[z^{-k} v_0(z), \ldots, z^{-k} v_n(z) \right]$$

is well-defined and extends f.

Now suppose S is a compact Riemann surface, and $f: S \to \mathbb{P}^n$ a nondegenerate map to $\mathbb{P}^n$. Let f be given locally by the vector function $v(z) = [v_0(z), \ldots, v_n(z)]$. We define the kth *associated curve* of f:

$$f_k: \ S \to G(k+1, n+1) \subset \mathbb{P}(\Lambda^{k+1}\mathbb{C}^{n+1})$$

by

$$f_k(z) = \left[v(z) \wedge v'(z) \wedge \cdots \wedge v^{(k)}(z) \right].$$

We emphasize that the curve is the abstract Riemann surface together with the map.

In order to assure ourselves that f_k is well-defined, we have to check three things: that $v(z) \wedge \cdots \wedge v^{(k)}(z)$ cannot be identically zero, and that it is independent, up to multiplication by a scalar, of the choice of lifting v and local coordinate z. To show the first, suppose that for some $k \leqslant n$, $v(z) \wedge \cdots \wedge v^{(k)}(z) \equiv 0$ but $v(z) \wedge \cdots \wedge v^{(k-1)}(z) \not\equiv 0$. Then evidently

$$v^{(k)}(z) \equiv 0 \bmod (v(z), \ldots, v^{(k-1)}(z)),$$

i.e.,

$$\begin{aligned}(v(z) \wedge \cdots \wedge v^{(k-1)}(z))' &= v(z) \wedge \cdots \wedge v^{(k-2)}(z) \wedge v^{(k)}(z) \\ &= \lambda(z) \cdot v(z) \wedge \cdots \wedge v^{(k-1)}(z),\end{aligned}$$

so that $f_{k-1}(z)$ must be constant and $f(S)$ lies in a $(k-1)$-plane in $\mathbb{P}^n$, contradicting our assumption of nondegeneracy.

Now let $\tilde{v}(z) = \rho(z)\cdot v(z)$ be another lifting of f. Then

$$\tilde{v}' = \rho'\cdot v + \rho\cdot v'$$

so

$$\tilde{v}'\wedge\tilde{v} = \rho^2\cdot(v\wedge v')$$

and in general

$$\tilde{v}\wedge\cdots\wedge\tilde{v}^{(k)} = \rho^{k+1}\cdot v\wedge\cdots\wedge v^{(k)}.$$

Similarly, let w be another local coordinate on S. Then

$$\frac{\partial v}{\partial w} = \frac{\partial z}{\partial w}\cdot\frac{\partial v}{\partial z},$$

and so

$$v\wedge\frac{\partial v}{\partial w} = \frac{\partial z}{\partial w}\left(v\wedge\frac{\partial v}{\partial z}\right).$$

In general, we will have

$$v\wedge\cdots\wedge\frac{\partial^k v}{\partial w^k} = \left(\frac{\partial z}{\partial w}\right)^{k(k+1)/2} v\wedge\cdots\wedge\frac{\partial^k v}{\partial z^k},$$

and so f_k is well-defined.

Geometrically, for a point $z\in S$ with $v(z)\wedge\cdots\wedge v^{(k)}(z)\neq 0$, the k-plane $f_k(z)\subset\mathbb{P}^n$ is the unique k-plane having contact of order at least $k+1$ with $f(S)$ at z, called the *osculating* k*-plane*. In the case of a plane curve $f: S\to\mathbb{P}^2$, the map $f_1: S\to\mathbb{P}^{2*}$ is just the Gauss map sending $z\in S$ to the tangent line to $f(S)$ at $f(z)$; the curve $f_1(S)$, often written f^*, is called the *dual curve* of f. Note that even at a singular point z_0 of $f(S)$ the tangent line is well-defined by the remark at the beginning of this section. In practice, $f_1(z_0)$ corresponds to what would ordinarily be called the tangent line at z_0: the limiting position of the tangent lines at nearby points. (See Figure 7.)

Ramification

Let $f: S\to\mathbb{P}^n$ be any curve, given in terms of Euclidean coordinates in a neighborhood of $f(z_0)$ by $f_1(z),\ldots,f_n(z)$. We define the *ramification index* $\beta(z_0)$ of f at z_0 to be the order of vanishing of the Jacobian $(\partial f_1/\partial z,\ldots,\partial f_n/\partial z)$, i.e.,

$$\beta(z_0) = \min\left(\operatorname{ord}_{z_0}\left(\frac{\partial f_i}{\partial z}\right)\right).$$

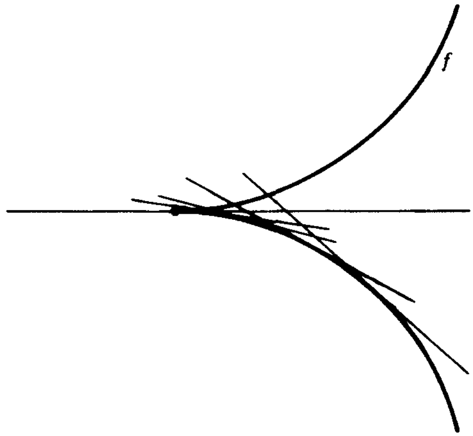


Figure 7

Clearly $\beta(z_0)=0$ if and only if the map f is smooth at z_0; in general, $\beta(z_0)$ is a measure of the singularity of f at z_0.

Another way to define the ramification index is as follows. Let ω be the associated (1,1)-form of the Fubini-Study metric on $\mathbb{P}^n$. Then $\beta(z_0)$ is the unique integer such that

$$f^*\omega = \frac{\sqrt{-1}}{2}|z-z_0|^{2\beta(z_0)}\cdot h(z)\cdot dz\wedge d\bar{z}$$

with $h(z)$ C^∞ and nonzero at z_0. To see that these two definitions are indeed equivalent, let $v(z)$ be any lifting of f near z_0. Then we have

$$\begin{aligned} f^*\omega &= \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\|v(z)\|^2 \\ &= \frac{\sqrt{-1}}{2}\partial\left(\frac{(v,v')}{(v,v)}d\bar{z}\right) \\ &= \frac{\sqrt{-1}}{2},\frac{(v,v)(v',v')-(v,v')(v',v)}{(v,v)^2}dz\wedge d\bar{z} \\ &= \frac{\sqrt{-1}}{2}\cdot\frac{1}{\|v\|^4}\cdot\sum_{i\neq j}|v_iv_j'-v_jv_i'|^2\cdot dz\wedge d\bar{z}. \end{aligned}$$

In particular we may take the lifting

$$v(z)=[1,f_1(z),\dots,f_n(z)];$$

then

$$f^*\omega = \frac{\sqrt{-1}\cdot dz\wedge d\bar{z}}{2\|v\|^4}\left(\sum_{i=1}^{n}|f_i'|^2+\sum_{i\neq j}|f_if_j'-f_i'f_j|^2\right)$$

and so clearly

$$f^*\omega = \frac{\sqrt{-1}}{2}|z-z_0|^{2\beta(z_0)}\cdot h(z)\cdot dz\wedge d\bar{z}$$

with $\beta(z_0)=\min(\operatorname{ord}_{z_0}(f_i'))$ as originally claimed.

Now we will be concerned not only with the ramification indices $\beta(z)$ of a curve $f: S\to\mathbb{P}^n$, but also with the ramification indices $\beta_k(z)$ of its associated curves. In order to make the numbers $\beta_k(z_0)$ computable for a given point $z_0\in S$, we may put the curve in *normal form* at z_0 as follows:

Write $f(z)=[v(z)]=[v_0(z),\ldots,v_n(z)]$ with $v(z_0)\neq 0$. Making a linear change of coordinates in $\mathbb{C}^{n+1}$, we may take

$$v(z_0) = (1,0,\ldots,0).$$

We have $v_1(z_0)=\cdots=v_n(z_0)=0$; write

$$(v_1(z),\ldots,v_n(z)) = (z-z_0)^{\alpha_1+1}(v_1^1(z),\ldots,v_n^1(z))$$

with $(v_1^1(z_0),\ldots,v_n^1(z_0))\neq 0$. Now make a linear change of the last n coordinates in $\mathbb{C}^{n+1}$ so that $(v_1^1(z_0),\ldots,v_n^1(z_0))=(1,0,\ldots,0)$; we write

$$(v_2^1(z),\ldots,v_n^1(z)) = (z-z_0)^{\alpha_2+1}(v_2^2(z),\ldots,v_n^2(z))$$

with $(v_2^2(z_0),\ldots,v_n^2(z_0))\neq 0$. We change the last $n-1$ coordinates on $\mathbb{C}^{n+1}$ so that $(v_2^2(z_0),\ldots,v_n^2(z_0))=(1,0,\ldots,0)$, and continuing in this way we end up with a system of coordinates for $\mathbb{C}^{n+1}$ in terms of which

$$v(z) = \big(1+\cdots,(z-z_0)^{\alpha_1+1}+\cdots,(z-z_0)^{2+\alpha_1+\alpha_2}+\cdots,$$
$$\ldots,(z-z_0)^{n+\alpha_1+\cdots+\alpha_n}+\cdots\big).$$

This is called the *normal form* of the curve f near z_0; from it we see that $f_k(z_0)$ is the $\mathbb{P}^k$ spanned by the first $k+1$ linearly independent vectors from the sequence $v(z_0), v'(z_0), v''(z_0),\ldots$. Putting a curve in normal form amounts to choosing a basis $e_0,\ldots,e_n$ for $\mathbb{C}^{n+1}$ such that $f_k(z_0)$ is spanned by $\{e_0,\ldots,e_k\}$.

Now we compute the ramification index $\beta_k(z_0)$ of the kth associated curve of f at z_0 in terms of the exponents $\alpha_1,\ldots,\alpha_n$ appearing in the normal form: assume $z_0=0$, normalize the homogeneous vector by making the first entry $v_0(z)\equiv 1$, and then write

$$v(z) = (1, z^{1+\alpha_1}+\cdots, z^{2+\alpha_1+\alpha_2}+\cdots, \ldots, z^{n+\alpha_1+\cdots+\alpha_n}+\cdots).$$

The homogeneous coordinates of $f_k(z)$ are then the determinants of the

$(k+1)\times(k+1)$ minors of the matrix

$$\begin{bmatrix} v(z) \\ v'(z) \\ \\ \vdots \\ v^{(k)}(z) \end{bmatrix} = \begin{bmatrix} 1+\cdots & z^{1+\alpha_1}+\cdots & z^{2+\alpha_1+\alpha_2}+\cdots & \cdots & z^{n+\alpha_1+\cdots+\alpha_n}+\cdots \\ 0 & (1+\alpha_1)z^{\alpha_1}+\cdots & & & \\ 0 & & \vdots & & \\ \vdots & \vdots & & & \\ 0 & & & & \end{bmatrix}$$

The minor whose determinant vanishes to least order at 0 is clearly the left-hand minor Λ_{I_0}, $I_0=\{1,\ldots,k+1\}$, and so we may take as Euclidean coordinates on $f_k(S)$ near z_0 the quotients $\{|\Lambda_I|/|\Lambda_{I_0}|\}_I$; the minor other than Λ_{I_0} whose determinant vanishes to smallest order at 0 is Λ_J, for $J=\{1,\ldots,k,k+2\}$, and so the index of ramification of f_k at z_0 is the order of vanishing of

$$\frac{\partial}{\partial z}\left(\frac{|\Lambda_J(z)|}{|\Lambda_{I_0}(z)|}\right).$$

Now we have

$$|\Lambda_{I_0}(z)| = z^{k\alpha_1+\cdots+\alpha_k}\cdot\begin{vmatrix} \alpha_1+1 & \cdots & k+\alpha_1+\cdots+\alpha_{k-1} \\ \alpha_1(\alpha_1+1) & & \vdots \\ \vdots & & \\ & \cdots & \end{vmatrix}+\cdots$$

and

$$|\Lambda_J(z)| = z^{k\alpha_1+\cdots+\alpha_k+\alpha_{k+1}+1}\cdot\begin{vmatrix} \alpha_1+1 & \cdots & k+1+\alpha_1+\cdots+\alpha_k \\ \alpha_1(\alpha_1+1) & & \vdots \\ \vdots & & \\ & \cdots & \end{vmatrix}+\cdots$$

and, since neither of the determinants appearing on the right is zero,

$$\operatorname{ord}_{z_0}\left(\frac{|\Lambda_J(z)|}{|\Lambda_{I_0}(z)|}\right) = \alpha_{k+1}+1,$$

hence

$$\beta_k(z_0) = \alpha_{k+1}.$$

The General Plücker Formulas I

Our aim now is to relate two invariants of the associated curves f_k of a curve $f: S \to \mathbb{P}^n$: the *degree* d_k of $f_k: S \to \mathbb{P}(\Lambda^{k+1}\mathbb{C}^{n+1})$ (or, if one prefers to think of f_k as a map to $G(k+1, n+1)$, the intersection number of $f_k(S)$ with the Schubert cycle σ_1, i.e., the number of osculating k-planes to $S \subset \mathbb{P}^n$ meeting a generic $(n-k-1)$-plane in $\mathbb{P}^n$) and the *total ramification* β_k of f_k, defined as the sum of $\beta_k(z)$ over all $z \in S$. We do this by considering the pullback $f_k^*(ds^2)$ to S of the standard metric on $\mathbb{P}(\Lambda^{k+1}\mathbb{C}^{n+1})$. On the one hand, $f_k^*(ds^2)$ is a metric on S away from the singular points of f_k, and so by a Gauss-Bonnet-type argument we can express the integral of its curvature form over S as a function of the genus of S and β_k; on the other hand, we can compute this curvature form directly and relate it to the degrees d_k of the various associated curves.

First, we say that a positive semidefinite inner product φ on the tangent bundle of a Riemann surface is a *pseudo-metric* if it is given locally as

$$\varphi = h(z) \cdot dz \otimes d\bar{z},$$

where

$$h(z) = |z|^{2\nu} \cdot h_0(z)$$

with

$$h_0(z) > 0.$$

We say that φ has a zero of order ν at $z=0$ and write $\operatorname{ord}_p(\varphi) = \nu$; the divisor

$$D_\varphi = \sum_{p \in S} \operatorname{ord}_p(\varphi) \cdot p$$

is called the *singular divisor* of the pseudo-metric φ. In fact, φ defines an honest metric on the line bundle $T' \otimes [D_\varphi]$: if we identify sections of $T' \otimes [D_\varphi]$ with meromorphic vector fields $\theta = f(z) \cdot (\partial/\partial z)$ having poles of order at most $\operatorname{ord}_p(\varphi)$ at p, then the inner product

$$(\theta, \theta) = |f(z)|^2 \cdot h(z)$$

will be a well-defined metric. Now the curvature form Θ of φ, considered as a metric on $T' \otimes [D_\varphi]$, will be given by

$$\Theta = -\partial\bar{\partial} \log h(z),$$

and so we have from the proposition on page 141

$$\begin{aligned} \frac{\sqrt{-1}}{2\pi} \int_S \Theta &= \deg(T' + D_\varphi) \\ &= 2 - 2g + \deg(D_\varphi). \end{aligned}$$

In particular, if we take $\varphi = f_k^*(ds^2)$ to be the pullback via f_k of the standard metric on $\mathbb{P}(\Lambda^{k+1}\mathbb{C}^{n+1})$,

$$D_\varphi = \sum_{p\in S} \beta_k(p)\cdot p,$$

and so if Θ is the curvature form of φ,

$$\frac{\sqrt{-1}}{2\pi}\int_S \Theta = 2-2g+\beta_k.$$

The problem now is to evaluate directly the curvature form of the pseudo-metric $f_k^*(ds^2)$. Let ω be the $(1,1)$-form associated to the Fubini-Study metric on $\mathbb{P}^n$; let $v(z)$ be as before a lifting of f, and set $\Lambda_k(z) = v(z)\wedge\cdots\wedge v^{(k)}(z)\in\Lambda^{k+1}\mathbb{C}^{n+1}$. Then we have the

Infinitesimal Plücker Formula

$$f_k^*(\omega) = \frac{\|\Lambda_{k-1}\|^2\cdot\|\Lambda_{k+1}\|^2}{\|\Lambda_k\|^4}\cdot\frac{\sqrt{-1}}{2}\,dz\wedge d\bar{z}.$$

Proof. First, note that the expression on the right is independent of the choice of lifting, as indeed it must be. Now if $\tilde{v}(z) = \rho(z)\cdot v(z)$ is another lifting, we have

$$\begin{aligned}
\tilde{v}' &= \rho'\cdot v + \rho\cdot v',\\
\tilde{v}'' &= \rho''\cdot v + 2\cdot\rho'\cdot v' + \rho\cdot v'',\\
&\ \ \vdots\\
\tilde{v}^{(k+1)} &= \rho^{(k+1)}\cdot v + \binom{k+1}{1}\rho^{(k)}v' + \cdots + \binom{k+1}{k}\rho' v^{(k)} + \rho\cdot v^{(k+1)}.
\end{aligned}$$

In particular, we see that we can find a function ρ with $\rho(z_0)\neq 0$ such that $\tilde{v}^{(k+1)}(z_0)$ is orthogonal to $v(z_0), v'(z_0),\ldots,v^{(k)}(z_0)$, and hence to $\tilde{v}(z_0),\tilde{v}'(z_0),\ldots,\tilde{v}^{(k)}(z_0)$; i.e., *at any point* z_0 *such that* $\Lambda_{k+1}(z_0)\neq 0$, *we can choose a lifting* v *of* f *with* $v^{(k+1)}(z_0)$ *orthogonal to* $v(z_0),\ldots,v^{(k)}(z_0)$.

Now write

$$\begin{aligned}
f_k^*(\omega) &= \frac{\sqrt{-1}}{2}\,\partial\bar{\partial}\log\|\Lambda_k\|^2\\
&= \frac{\sqrt{-1}}{2}\,\partial\left(\frac{(\Lambda_k,\Lambda_k')}{(\Lambda_k,\Lambda_k)}\,d\bar{z}\right)\\
&= \left(\frac{(\Lambda_k,\Lambda_k)(\Lambda_k',\Lambda_k')-(\Lambda_k,\Lambda_k')(\Lambda_k',\Lambda_k)}{(\Lambda_k,\Lambda_k)^2}\right)\cdot\frac{\sqrt{-1}}{2}\,dz\wedge d\bar{z}
\end{aligned}$$

with

$$\Lambda_k' = v\wedge v'\wedge\cdots\wedge v^{(k-1)}\wedge v^{(k+1)}.$$

Let $V_0 \subset \mathbb{C}^{n+1}$ be the linear span of $v(z_0),\ldots,v^{(k)}(z_0)$; let $V_0^\perp$ denote the orthogonal complement of V_0 in $\mathbb{C}^{n+1}$. Then the decomposition $\mathbb{C}^{n+1}=V_0 \oplus V_0^\perp$ gives a decomposition

$$\Lambda^{k+1}\mathbb{C}^{n+1} = \bigoplus_{p+q=k+1} (\Lambda^p V_0 \otimes \Lambda^q V_0^\perp)$$

of $\Lambda^{k+1}\mathbb{C}^{n+1}$ *as an inner product space*, with the induced metric on each factor $\Lambda^p V_0 \otimes \Lambda^q(V_0^\perp)$. If we assume that $v^{(k+1)}(z_0) \in V_0^\perp$, we have

$$\Lambda_k(z_0) \in \Lambda^{k+1}V_0, \qquad \Lambda_k'(z_0) \in \Lambda^k V_0 \otimes \Lambda^1 V_0^\perp;$$

hence

$$(\Lambda_k(z_0), \Lambda_k'(z_0)) = 0,$$
$$(\Lambda_k'(z_0), \Lambda_k'(z_0)) = \|\Lambda_{k-1}(z_0)\|^2 \cdot \|v^{(k+1)}(z_0)\|^2,$$

and

$$\begin{aligned}(\Lambda_k(z_0),\Lambda_k(z_0))\cdot(\Lambda_k'(z_0),\Lambda_k'(z_0)) &= \|\Lambda_{k-1}(z_0)\|^2 \cdot \|v^{(k+1)}(z_0)\|^2 \cdot \|\Lambda_k(z_0)\|^2 \\ &= \|\Lambda_{k-1}(z_0)\|^2 \cdot \|\Lambda_{k+1}(z_0)\|^2,\end{aligned}$$

proving the lemma.

The curvature form of pseudo-metric $f_k^*(ds^2)$ is then given by

$$\begin{aligned}\frac{\sqrt{-1}}{2}\Theta &= \frac{-\sqrt{-1}}{2}\partial\bar{\partial}\log\left(\frac{\|\Lambda_{k+1}\|^2 \cdot \|\Lambda_{k-1}\|^2}{\|\Lambda_k\|^4}\right) \\ &= -f_{k-1}^*(\omega) + 2f_k^*(\omega) - f_{k+1}^*(\omega)\end{aligned}$$

and so by the Wirtinger theorem,

$$\frac{\sqrt{-1}}{2\pi}\int_S \Theta = -d_{k-1} + 2d_k - d_{k+1}.$$

Comparing this with our first evaluation of $\int_S \Theta$, we have the

Global Plücker Formula

$$d_{k-1} - 2d_k + d_{k+1} = 2g - 2 - \beta_k.$$

As an immediate application of the Plücker formula, we show that we can characterize the rational normal curve by the absence of inflectionary behavior.

Proposition. *The only totally unramified curve* $f: S \to \mathbb{P}^n$ *is the rational normal curve.*

Proof. We can take a linear combination of the various Plücker formulas to eliminate d_k for $k > 0$:

$$\sum_{k=0}^{n-1}(n-k)(d_{k-1} - 2d_k + d_{k+1}) = \sum_{k=0}^{n-1}(n-k)(2g-2-\beta_k),$$

obtaining

$$\sum (n-k)\beta_k = (n+1)d + n(n+1)(g-1).$$

In particular, if $\beta_i = 0$ for all i

$$n(n+1)(g-1) < 0 \Rightarrow g = 0,$$

and so this formula reads

$$-(n+1)d = -n(n+1),$$

i.e., $d = n$ and the curve S is the rational normal curve. Q.E.D.

The General Plücker Formulas II

We now wish to give a second proof of the general Plücker formulas which, while it does not admit a local analogue, is of a more geometric character.

Let $f: C \to \mathbb{P}^n$ be a nondegenerate curve, $v(z)$ a local lifting of f to $\mathbb{C}^{n+1}$, and denote the cast of characters

$$\Lambda_k(z) = v(z) \wedge \cdots \wedge v^{(k)}(z) \in \wedge^{k+1}\mathbb{C}^{n+1},$$

$$f_k: C \to G(k+1, n+1) \subset \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1}),$$

$$d_k = \deg f_k(C) \subset \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$$

$$= {}^{\#}(f_k(C)\cdot \sigma_1)_{G(k+1,n+1)}$$

$$\beta_k = \sum_{z \in C} \beta_k(z)$$

as before; for convenience, set

$$m = \dim \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1}) = \binom{n+1}{k+1} - 1.$$

Let V_{m-2} be a generic $(m-2)$-plane in $\mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$, disjoint from C and consider the map

$$\pi_V: C \to \mathbb{P}^1$$

obtained by projecting $f_k(C)$ from V_{m-2} onto a line. The sheet number of π_V is clearly just the degree d_k of $f_k(C)$; by Riemann-Hurwitz, then,

$$2g - 2 = -2d_k + \tau_k,$$

where τ_k is the number of branch points of π_V.

To evaluate τ_k, put the map f_k in normal form at $z_0 \in C$:

$$f_k(z) = \left[1 + \cdots, (z-z_0)^{\gamma_1+1} + \cdots, (z-z_0)^{\gamma_1+\gamma_2+2} + \cdots, \ldots\right].$$

(Here the exponent γ_i is the ramification index of the $(i-1)$st associated curve of f_k at z_0; thus $\gamma_1 = \beta_k(z_0)$, while the remaining integers $\gamma_2, \ldots$ have no bearing on the proceedings.) From this normal form we see that z_0 will be a branch point of order $(\gamma_{l+1}+\cdots+\gamma_1+l-1)$ of π_V exactly when the hyperplane $\overline{V_{m-2},f_k(z_0)}\subset\mathbb{P}^m$ contains the osculating l-plane to $f_k(C)$ at z_0, but not the osculating $(l+1)$-plane. In particular, if we choose a sufficiently generic V_{m-2}—i.e., such that V_{m-2} does not meet the tangent line to $f_k(C)$ at any stationary point f_k, and does not meet the osculating 2-plane to any point of $f_k(C)$ in a line—then a singular point z_0 of $f_k(C)$ will be a branch point of order $\beta_k(z_0)$ of the map π_V, while a smooth point z_0 of $f_k(C)$ will be a simple branch point of π_V if the tangent line $T_{z_0}(f_k(C))$ to $f_k(C)$ at z_0 meets V_{m-2}, not a branch point otherwise. The number of branch points of π_V will thus be the total ramification index β_k of f_k, plus the number of times a tangent line to $f_k(C)$ meets a generic $(m-2)$-plane in $\mathbb{P}^m$—that is, the degree of the *tangential ruled surface*

$$T(f_k(C))=\bigcup_{z\in C}T_z(f_k(C))\subset\mathbb{P}^m$$

of $f_k(C)$.

Our computation of $\deg T(f_k(C))$ is based on one observation. The tangent line to $f_k(C)$ at a smooth point z is spanned by the vectors

$$\Lambda_k(z)=v(z)\wedge v'(z)\wedge\cdots\wedge v^{(k)}(z)$$

and

$$\Lambda_k'(z)=v(z)\wedge v'(z)\wedge\cdots\wedge v^{(k-1)}(z)\wedge v^{(k+1)}(z)$$

Thus the tangent line

$$T_z(f_k(C))=\left\{\left[v(z)\wedge\cdots\wedge v^{(k-1)}(z)\wedge\left(\lambda_0v^{(k)}(z)+\lambda_1v^{(k+1)}(z)\right)\right]\right\}_{[\lambda_0,\lambda_1]\in\mathbb{P}^1}$$

lies entirely in the Grassmannian $G(k+1,n+1)\subset\mathbb{P}^m$—in fact, *it is simply the Schubert cycle of* k*-planes in* $\mathbb{P}^n$ *containing the osculating* (k − 1)*-plane* $\Lambda_{k-1}(z)$ *to* f *at* z *and contained in the osculating* (k + 1)*-plane* $\Lambda_{k+1}(z)$ *to* f *at* z. Since the hyperplane section of $G(k+1,n+1)\subset\mathbb{P}^m$ is the Schubert cycle σ_1, we can then write

$$\begin{aligned}\deg T(f_k(C))&={}^{\#}\big(T(f_k(C))\cdot V_{m-2}\big)_{\mathbb{P}^m}\\&={}^{\#}\big(T(f_k(C))\cdot\sigma_1^2\big)_{G(k+1,n+1)}.\end{aligned}$$

Now by the Schubert calculus from Section 6 of Chapter 1, σ_1^2 is homologous to the Schubert cycle $\sigma_{1,1}(\Gamma_{n-k})$ of k-planes in $\mathbb{P}^n$ meeting an $(n-k)$-plane Γ_{n-k} in a line, plus the Schubert cycle $\sigma_2(\Gamma_{n-k-2})$ of k-planes meeting an $(n-k-2)$-plane Γ_{n-k-2}. We see, moreover, that the cycle $T_z(f_k(C))\subset G(k+1,n+1)$ of k-planes in $\mathbb{P}^n$ containing $\Lambda_{k-1}(z)$ and contained in $\Lambda_{k+1}(z)$ will meet the Schubert cycle $\sigma_2(\Gamma_{n-k-2})$ if and only if Γ_{n-k-2} has a point in common with $\Lambda_{k+1}(z)$, so that the intersection number of $T(f_k(C))$ with σ_2 in $G(k+1,n+1)$ is just the number of points

$z \in C$ whose $(k+1)$st osculating plane meets a generic $(n-k-2)$-plane $\Gamma_{n-k-2} \subset \mathbb{P}^n$, that is, *the degree* d_{k+1} *of the* $(k+1)$*st associated curve* $f_{k+1}(C)$. Similarly, $T_z(f_k(C))$ will meet the cycle $\sigma_{1,1}(\Gamma_{n-k})$ exactly when $\Lambda_{k-1}(z)$ has a point in common with Γ_{n-k}, so the intersection number of $T(f_k(C))$ with $\sigma_{1,1}$ is the number of points $z \in C$ whose $(k-1)$st osculating plane meets a generic $(n-k)$-plane $\Gamma_{n-k} \subset \mathbb{P}^n$, i.e., the degree d_{k-1} of the $(k-1)$st associated curve. We have thus

$$\begin{aligned}\deg T(f_k(C)) \subset \mathbb{P}^m &= {}^{\#}(T(f_k(C))\cdot(\sigma_{1,1}+\sigma_2))_{G(k+1,n+1)} \\ &= d_{k-1} + d_{k+1},\end{aligned}$$

and so the number of branch points of π_V is given by

$$\tau_k = \beta_k + d_{k-1} + d_{k+1}.$$

From Riemann-Hurwitz, then, we obtain the general Plücker formulas

$$\begin{aligned}2g-2 &= -2d_k + \tau_k \\ &= -2d_k + \beta_k + d_{k-1} + d_{k+1}.\end{aligned}$$

Weierstrass Points

In general, the Plücker formulas deal with extrinsic invariants of curves. Following the general principle that projective invariants of a canonical curve S correspond to intrinsic properties of S, however, we apply the Plücker formulas to the canonical curves and obtain a count of the number of Weierstrass points on a Riemann surface, as follows.

Let S be a Riemann surface of genus g, $p \in S$ any point, and consider the linear systems associated to the divisors $k \cdot p$, $k = 1, 2, \ldots$. We know by Riemann-Roch that $h^0(kp) = k - g + 1$ for $k \geq 2g-1$, and, in general,

$$h^0(kp) = \begin{cases} h^0((k-1)p)+1, & \text{if there exists } f \in \mathfrak{M}(S) \text{ such that } (f)_\infty = kp, \\ h^0((k-1)p), & \text{if there does not exist } f \in \mathfrak{M}(S) \text{ such that } (f)_\infty = kp. \end{cases}$$

It follows that *there exist exactly* g *positive integers* $a_1, \ldots, a_g$ *such that there does not exist a meromorphic function* f *on* S with $(f)_\infty = a_i p$. These integers $a_1 < a_2 < \cdots < a_g$ are called the *gap values* of the point $p \in S$.

Now we might expect that for a generic $p \in S$, all the divisors kp will be regular, i.e.,

$$h^0(kp) = \begin{cases} 1, & k \leq g, \\ k-g+1, & k \geq g, \end{cases}$$

so that

$$a_i = i, \qquad i = 1, \ldots, g.$$

We say a point p is a *Weierstrass point* of S if any of the divisors kp is irregular, or in other words if there exists a meromorphic function f on S holomorphic on $S-\{p\}$ and with a pole of order $\leqslant g$ at p. We take the *weight* of the Weierstrass point p to be

$$W(p) = \sum (a_i - i),$$

where the a_i are the gap values of $p \in S$. For example, if S is hyperelliptic with $h^0(2p)=2$, then the gap values of p are

$$a_i = 2i-1$$

and p is called a *hyperelliptic Weierstrass point*; at the other end of the scale, a point p with weight 1 has gap values

$$1,2,3,\ldots,g-1,g+1$$

—i.e., has minimal deviation from the expected pattern—and is called a *normal Weierstrass point* of S.

We can characterize Weierstrass points on a nonhyperelliptic Riemann surface S in another way. Let $C \subset \mathbb{P}^{g-1}$ be the canonical curve of S. Then by our geometric version of Riemann-Roch, for any $p \in C$, $h^0(gp)>1$ if and only if the point $p \in C$ and its first $g-1$ derivatives fail to span all of $\mathbb{P}^{g-1}$, i.e., *p is a Weierstrass point of* S *if and only if it is a singular point of one of the associated curves of* C. Precisely, if the canonical map ι_K is given, in terms of a local coordinate z centered around p, by

$$\iota_K(z) = [1, z^{1+\alpha_1}+\cdots, z^{2+\alpha_1+\alpha_2}+\cdots,\ldots, z^{g-1+\alpha_1+\cdots+\alpha_{g-1}}+\cdots],$$

then the gap values of $p \in S$ are

$$\begin{aligned} a_1 &= 1,\\ a_2 &= 2+\alpha_1,\\ a_3 &= 3+\alpha_1+\alpha_2,\\ &\vdots\\ a_g &= g+\alpha_1+\alpha_2+\cdots+\alpha_{g-1}, \end{aligned}$$

and the weight of p is

$$\begin{aligned} W(p) &= \sum_{k=1}^{g-1} (g-k)\alpha_k\\ &= \sum_{k=0}^{g-2} (g-k-1)\beta_k(p). \end{aligned}$$

Now we can count the number of Weierstrass points on S by applying the Plücker formulas obtained in the argument for the rational normal curve: setting $d=2g-2$ and $n=g-1$, we have

$$\sum (g-k-1)\beta_k = g(2g-2)+(g-1)g(g-1) = (g-1)g(g+1)$$

i.e., *the total weight of the Weierstrass points on a Riemann surface of genus* g *is exactly* $(g-1)\cdot g\cdot(g+1)$.

Weierstrass points are of interest because they are "marked" points on a Riemann surface, i.e., points intrinsically defined. For example, we can apply our last result to show:

Theorem. *Any Riemann surface* S *of genus* >1 *has only finitely many automorphisms.*

Proof. Any automorphism of S must permute its Weierstrass points; since there are only a finite number of these points, it will suffice to consider automorphisms of S fixing each of the Weierstrass points of S. Suppose now that S is nonhyperelliptic. First, note that by Clifford's theorem for any point $p \in S$ we have

$$h^0(kp) < \frac{k}{2} + 1$$

so

$$a_i \leqslant 2i - 2, \qquad i = 2,\dots,g,$$

and

$$\begin{aligned} W(p) &= \sum_{i=1}^{g} a_i - i \\ &\leqslant \sum_{i=2}^{g} i - 2 \\ &\leqslant \frac{(g-1)(g-2)}{2}, \end{aligned}$$

and so the number of distinct Weierstrass points on S is at least

$$\frac{(g-1)g(g+1)}{\frac{1}{2}(g-1)(g-2)} = \frac{2g(g+1)}{g-2} \geqslant 2g + 6.$$

Now suppose that S is nonhyperelliptic and let C be the canonical curve of S. Any automorphism of S is then induced by an automorphism of $\mathbb{P}^{g-1}$ fixing C; let $\tau: \mathbb{P}^{g-1} \to \mathbb{P}^{g-1}$ be such an automorphism fixing each of the Weierstrass points of S. It follows then that τ preserves all of the osculating planes at each Weierstrass point p_i; in particular, if we let V_i be the osculating $(g-3)$-plane to C at p_i, then τ preserves V_i and the pencil of hyperplanes $\{H^i_\lambda\}_{\lambda\in\mathbb{P}^1}$ containing V_i. Suppose V_i contains k points of C other than p_i. Then any hyperplane containing V_i has $k+g-2$ points of intersection with C lying inside V_i, *and hence contains at most* $g-k$ Weierstrass points outside V_i. But there are at least

$$2g + 2 - (k+1) = 2g - k + 1$$

Weierstrass points of C lying off V_i. Thus at least three of the hyperplanes $\{H^i_\lambda\}_{\lambda\in\mathbb{P}^1}$ contain a Weierstrass point outside V_i, and so are fixed by τ; it follows that τ fixes each of the hyperplanes H^i_λ, and hence, since the hyperplane sections of C are finite, that τ has finite order.

Suppose now that τ has order d and consider the quotient curve S' of S by the group $\{\tau^i\}$ of automorphisms. The projection map π expresses S as a d-sheeted cover of S', with each Weierstrass point a $(d-1)$-fold branch point; then we have

$$\begin{aligned} 2g-2 &\geqslant d(2g(S')-2)+(d-1)(2g+2) \\ &\geqslant (d-1)(2g-2)+2d\cdot g(S')+2(d-2), \end{aligned}$$

so $d \geqslant 2 \Rightarrow g(S')=0$ and $d=2$, i.e., S is hyperelliptic. Thus *if* S *is nonhyperelliptic, any automorphism fixing the Weierstrass points of* S *is the identity*, and so the theorem is proved in this case.

In case S is hyperelliptic, any automorphism of S is given, modulo the hyperelliptic involution, by an automorphism of $C \subset \mathbb{P}^{g-1}$; but C is rational, and so any automorphism of C fixing the $2g+2>3$ Weierstrass points of C is the identity. Q.E.D.

Now, let S be a Riemann surface of genus $g \geqslant 3$. By our last result, if S has any automorphisms at all, it has an automorphism φ of prime order p. Let S' be the quotient of S by the group of automorphisms $\{\varphi^i\}$, and g' the genus of S'. Since a fixed point of any power φ^i of φ is a fixed point of φ, the branch locus of the quotient map $\pi: S \rightarrow S'$ consists simply of a certain number k of $(p-1)$-fold branch points; and to specify the surface S up to a finite number of choices we simply have to specify the surface S', together with k points on S'. This is a total of $3g'+3+k$ parameters; but now by Riemann-Hurwitz,

$$2g-2=p(2g'-2)+k(p-1),$$

i.e.,

$$k=\frac{2g-2-p(2g'-2)}{p-1}=\frac{2g-2pg'}{p-1}+2.$$

Thus we have at most

$$3g'+\frac{2g-2pg'}{p-1}-1$$

parameters for S; and since we must have $g' \geqslant (1/p)(g-1)+1 \geqslant \frac{1}{2}(g+1)$ this number is less than $3g-3$. We conclude, then, that

The generic Riemann surface of genus $g \geqslant 3$ *has no automorphisms.*

The reader may check, by essentially the same techniques, that no Riemann surface of genus $g \geqslant 2$ can have more than $84(g-1)$ automorphisms.

A final note on Weierstrass points: we can see by a count of parameters that the generic Riemann surface of genus $g \geqslant 3$ has no Weierstrass points with gap value $a_i > i$ for $i < g$. To see this, suppose the contrary—i.e., that the generic Riemann surface S contains points p with $\dim|(g-1)p| \geqslant 1$. S is then expressible as a $(g-1)$-sheeted cover of $\mathbb{P}^1$, with p appearing as a branch point of order $g-2$; the branch locus B of this map consists of $(g-2)p$ plus

$$2g-2+2(g-1)-(g-2)=3g-2$$

other points, and so depends on $3g-1$ parameters. S thus depends on at most $3g-1-3=3g-4$ parameters, a contradiction. Likewise, the reader may verify that a generic Riemann surface of genus $g \geqslant 3$ contains no points p with $\dim |(g+1)p|) \geqslant 3$, by counting (as on p. 262) the number of parameters for plane curves of genus g and degree $g+1$ possessing a $(g+1)$-fold tangent line. Together, these two assertions imply that *the generic Riemann surface of genus* $g \geqslant 3$ *has only normal Weierstrass points.*

Plücker Formulas for Plane Curves

We want to consider now the projective invariants of plane curves. This calls for somewhat different techniques from those used previously: the Plücker formulas we have derived thus far deal only with singularities of a curve $f: S \rightarrow \mathbb{P}^n$ arising from the local character of the map f. We have seen, however, that plane curves $f: S \rightarrow \mathbb{P}^2$ are subject to singularities arising from the global behavior of f—e.g., nodes—that are not reflected thus far in our general formulas. To obtain a reasonably broad range of applicability, we will consider curves in $\mathbb{P}^2$ with *traditional singularities*, which we now define.

DEFINITION. We say that a curve $f: S \rightarrow \mathbb{P}^2$ has *traditional singularities* if every point $p \in S$ is one of the following:

1. A *regular point*, which is a smooth point of both f and the dual curve f^*. At such a point $\beta_0(p)=\beta_1(p)=0$, and f has the local normal form

$$f(z)=[1, z+\cdots, z^2+\cdots].$$

2. An *ordinary flex* of f; i e., a smooth point of f where the tangent line has contact of order three. In normal form

$$f(z)=[1, z+\cdots, z^3+\cdots],$$
$$f^*(z)=[1, z^2+\cdots, z^3+\cdots].$$

3. A *cusp* of f, i.e., a singular point of f that has normal form

$$f(z)=[1, z^2+\cdots, z^3+\cdots].$$

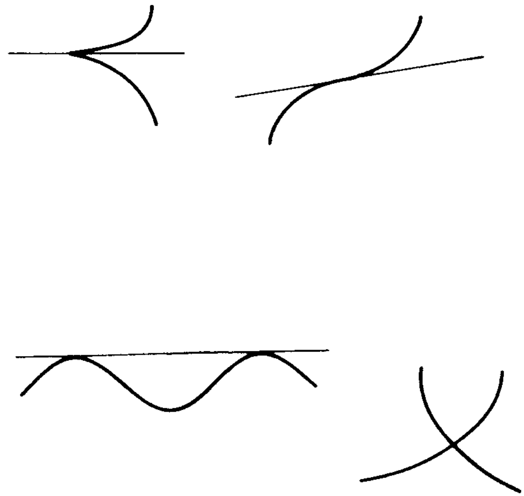


Figure 8

Thus, *p is a flex of* f⇔p *is a cusp of* f*.

4. A *bitangent* of f, i.e., a point p, not a flex, where the tangent line is also simply tangent at some other point $q \neq p$.

5. An *ordinary double point* of f, i.e., a point where two nonsingular branches of $f(S)$ cross transversely. Clearly p *is bitangent for* f ⇔ p *is an ordinary double point of* f*.

Note one important point: if $f: S \to \mathbb{P}^2$ is any plane curve, $f^*: S \to \mathbb{P}^{2*}$ its dual, then for a point $z_0 \in S$ the tangent line $f^*(z_0) \in \mathbb{P}^{2*}$ is the limiting position of the secant lines $\overline{f(z_0)f(z)}$ as $z \to z_0$. (See Figure 9.) Similarly, $(f^*)^*(z_0)$—that is, the point in $\mathbb{P}^2$ corresponding to the tangent line to $f^*(S) \subset \mathbb{P}^{2*}$ at $f^*(z_0)$—is the limiting position of the intersection of the tangent lines to $f(S)$ at z and z_0, as $z \to z_0$, which is of course z_0. We see then that *the dual of the dual is the original curve.*

Now suppose $f: S \to \mathbb{P}^2$ has traditional singularities and let $C = f(S)$, $C^* = f^*(S)$. With the notations

$g =$ genus of S,
$d = \deg C$, $d^* = \deg C^*$,
$b =$ number of bitangent lines of C, $b^* =$ number of bitangent lines of C^*,
$f =$ number of flexes of C, $f^* =$ number of flexes of C^*,
$\kappa =$ number of cusps of C, $\kappa^* =$ number of cusps of C^*,
$\delta =$ number of double points of C, $\delta^* =$ number of double points of C^*,

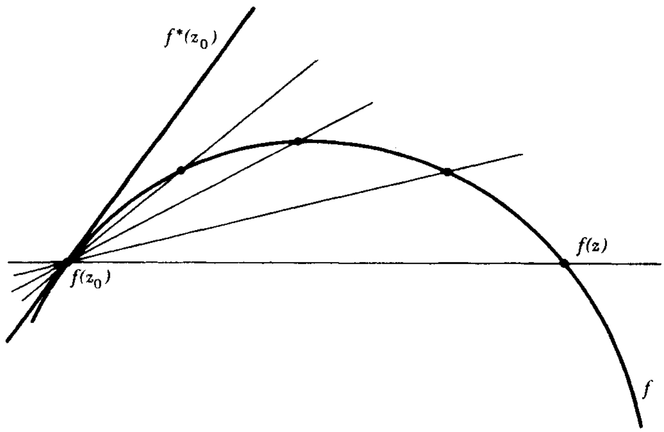


Figure 9

we have the relations

$$b = \delta^*, \qquad b^* = \delta,$$
$$f = \delta^*, \qquad f^* = \kappa.$$

The degree of the dual of C—usually called the *class* of C—is by definition the number of points of intersection of C^* with a generic line in $\mathbb{P}^{2*}$, that is, the number of tangent lines to C containing a generic point $p \in \mathbb{P}^2$. Let p be such a point, and assume moreover that p does not lie on any of the tangent lines to C at any of the singular points of C. Choose coordinates $[X_0, X_1, X_2]$ on $\mathbb{P}^2$ with $p = [0,0,1]$; if C is given in these coordinates as the locus of the polynomial $g(X_0, X_1, X_2) = 0$, then the tangent lines to C through p correspond exactly to the smooth points of C such that $(\partial g/\partial X_2)(q) = 0$. Now the curve $C' = (\partial g/\partial X_2 = 0)$ has degree $d-1$, and passes through each double point and cusp of C with intersection multiplicity 2 and 3, respectively. Thus the number of points of intersection of C' with the smooth points of C is $(C \cdot C') - 2\delta - 3\kappa$; i.e.,

$$(*) \qquad d^* = d(d-1) - 2\delta - 3\kappa.$$

Similarly, consider the projection map π of C from p onto a line. π expresses C as a d-sheeted cover of $\mathbb{P}^1$, and so we have

$$\chi(S) = 2 - 2g = 2d - b,$$

where b is the number of branch points of the map $\pi \circ f : S \to \mathbb{P}^1$. Now, as in the argument for the original genus formula, a smooth point q of C is a branch point of $\pi \circ f$ if and only if $(\partial g/\partial X_2)(q) = 0$; thus we have $d(d-1) - 2\delta - 3\kappa$ branch points of $\pi \circ f$ among the smooth points of C. In addition we see that, while neither of the points of S corresponding to an ordinary double point of C is a branch point of $\pi \circ f$, every cusp of C is a branch point of order 1 on S. Thus

$$\begin{aligned} b &= d(d-1) - 2\delta - 3\kappa + \kappa \\ &= d(d-1) - 2\delta - 2\kappa, \end{aligned}$$

and so

$$2 - 2g = 2d - d(d-1) + 2\delta + 2\kappa$$

$$(**) \qquad g = \frac{(d-1)(d-2)}{2} - \delta - \kappa.$$

Applying the relations (*) and (**) to the dual curve C^* as well, we obtain the *classical Plücker formulas*

$$d^* = d(d-1) - 2\delta - 3\kappa \qquad g = \frac{(d-1)(d-2)}{2} - \delta - \kappa$$

$$d = d^*(d^*-1) - 2b - 3f \qquad g = \frac{(d^*-1)(d^*-2)}{2} - b - f.$$

It will be useful to us later on to have a formula for the canonical bundle of a Riemann surface S expressed as a curve C of degree d in the plane with traditional singularities. To find this, let $f: S \to \mathbb{P}^2$ and $\pi: C \to \mathbb{P}^1$ be as above, and consider the pullback ω to S of the meromorphic 1-form $d(X_1/X_0)$. First, ω will have double poles over the points of intersection of C with the line $X_0 = 0$; thus

$$(\omega)_\infty = f^*(2H).$$

Now consider the section $\sigma \in H^0(\mathbb{P}^2, \mathcal{O}((d-1)H))$ given by the homogeneous polynomial $\partial g/\partial X_2$, g as above. Away from the singular locus of C, we have

$$(\omega)_0 = (f^*\sigma);$$

at an ordinary double point $p = f(q) = f(q')$ of C, on the other hand, ω will be nonzero while $f^*\sigma$ will vanish at both q and q'; at a cusp $p = f(q)$ of C, ω will have a simple zero while $f^*\sigma$ will vanish to order 3. If $D \subset S$ is the inverse image of the singular points of C (counting the inverse image of a cusp twice), then

$$(\omega)_0 = (f^*\sigma) - D,$$

and so finally

$$K_S = (\omega)_0 - (\omega)_\infty = f^*((d-3)H) - D.$$

We turn now to some special cases:

Plane Cubics. Let C be a nonsingular plane cubic. Then $d = 3$, $g = 1$, $K = \delta = 0$. Moreover, the number b of bitangents is zero, since no line can have four intersections with C. Similarly, any general flex with normal form

$$z \to [1, z + \cdots, z^{3+l} + \cdots] \qquad (l \geqslant 0)$$

must be an ordinary flex (i.e., $l=0$). The singularities are thus traditional, and the classical Plücker formulas give

$$d^* = 6, \qquad f = 9.$$

The nine flexes are distinct and can be found as follows: If $0 \in C$ is one flex, then according to the discussion of the inversion of the elliptic integral in Section 2 of this chapter we may describe C parametrically by

$$f:\ \mathbb{C} \to \mathbb{P}^2,$$

where

$$f(z) = [1, \wp(z), \wp'(z)],$$
$$z = \int_0^{f(z)} \omega$$

with $\omega \in H^0(C, \Omega^1)$ a generator. By the addition theorem the condition that points A, B, C be the points of intersection of a line is exactly

$$\int_0^A \omega + \int_0^B \omega + \int_0^C \omega \equiv 0(\Lambda),$$

where $\Lambda = f^{-1}(0)$ is the lattice in $\mathbb{C}$. The flexes are those lines for which $A = B = C$; i.e., they are just the nine points

$$[1, \wp(z), \wp'(z)], \qquad \text{where } 3z \in \Lambda.$$

From this we deduce the statement from classical geometry: *If a line* L *passes through two flexes of a nonsingular plane cubic, then it also passes through a third flex.*

Note that if C has an ordinary double point, then the number of flexes of C drops to three, while if C has a cusp, it has only one flex point.

Plane Quartics. In case C is a smooth plane quartic with ordinary singularities, the degree of C^* is

$$d^* = d(d-1) = 12.$$

We have, then,

$$2b + 3f = 12 \cdot 11 - 4 = 128.$$

On the other hand, C has genus 3, and so

$$3 = \frac{11 \cdot 10}{2} - b - f,$$

i.e., $b + f = 52$. Solving, we find

$$\begin{aligned} f &= (2b+3f) - (2b+2f) \\ &= 128 - 2 \cdot 52 \\ &= 24 \end{aligned}$$

and

$$\begin{aligned} b &= (3b+3f)-(2b+3f) \\ &= 3\cdot 52 - 128 \\ &= 28, \end{aligned}$$

i.e., C has 24 flexes and 28 bitangents. We will see the 28 bitangents to a smooth plane quartic reappear in Section 4 of Chapter 4, in another context.

In general, if C is a *smooth* plane curve of degree d having traditional singularities,

$$d^* = d(d-1)$$

and

$$g = \frac{(d-1)(d-2)}{2},$$

so

$$2b+3f = d(d-1)\cdot(d(d-1)-1)-d$$

and

$$b+f = \frac{(d(d-1)-1)(d(d-1)-2)}{2} - \frac{(d-1)(d-2)}{2}.$$

Thus

$$\begin{aligned} f &= (2b+3f)-(2b+2f) \\ &= 3d(d-2) \end{aligned}$$

and

$$\begin{aligned} b &= (3b+3f)-(2b+3f) \\ &= \tfrac{1}{2}d(d+1)(d-1)(d-2)-4d(d-2). \end{aligned}$$

5. CORRESPONDENCES

Definitions and Formulas

A *correspondence* $T: C \to C'$ of degree d between two curves C and C' associates to every point $p \in C$ a divisor $T(p)$ of degree d on C', varying holomorphically with p. It may be given either as a holomorphic map

$$C \to C'^{(d)}$$

from C to the dth symmetric product of C', or equivalently—and more usefully to us—by its *curve of correspondence* (intuitively, its graph)

$$D = \{(p,q) : q \in T(p)\} \subset C \times C';$$

conversely, given any curve $D \subset C \times C'$, we can define an associated correspondence by

$$T(p) = i_p^*(D) \in \mathrm{Div}(C'),$$

where $i_p : C' \to C \times C'$ sends q to (p,q). A correspondence will be called *irreducible* if its curve of correspondence is.

The *inverse* of a correspondence $T: C \to C'$ with curve of correspondence $D \subset C \times C^1$ is defined to be the correspondence given by the curve

$$D' = \{(q,p):\ (p,q) \in D\} \subset C' \times C,$$

i.e., by

$$T^{-1}(q) = \sum_{q \in T(p)} p.$$

Some basic correspondences are:

1. If $\{D_\lambda\}$ is a pencil on the curve C without base points, or equivalently a branched covering map $C \xrightarrow{\pi} \mathbb{P}^1$, then for each $p \in C$ there is a unique divisor $D(p) \in \{D_\lambda\}$ containing p; we may define a correspondence T by

$$T(p) = D(p) - p,$$

i.e., T is given by the curve

$$D = \{(p,q):\ D_\lambda - p - q \geqslant 0 \text{ for some } \lambda\} \subset C \times C.$$

Note that T is symmetric, that is, $T = T^{-1}$.

2. If $C \subset \mathbb{P}^2$ is a smooth plane curve, we define a correspondence $T: C \to C$ by

$$T(p) = T_p(C) \cdot C - 2p,$$

i.e., T is given by the closure D in $C \times C$ of the locus

$$\{(p,q):\ p \neq q,\ q \in T_p(C)\}.$$

Note that $p \in T(p)$ only if $T_p(C)$ meets C with multiplicity 3 or more at p, i.e., if p is a flex of C. T is called the *tangential* correspondence on C.

The phenomena associated to correspondences with which we will be concerned are these:

1. A *coincident point* of a correspondence $T: C \to C'$ is a pair $(p,q) \in C \times C'$ such that q appears in $T(p)$ with multiplicity 2 or more; we say that (p,q) is a coincident point of multiplicity m for T if q appears with multiplicity $(m+1)$ in $T(p)$. In example 1 above, a pair (p,q) will be a coincident point of T if q is a branch point of the map $C \xrightarrow{\pi} \mathbb{P}^1$ given by the pencil $\{D_\lambda\}$ and $p \neq q \in \pi^{-1}(\pi(q))$; in example 2 a coincident point corresponds to a bitangent line to C.

In general, if $T: C \to C'$ is a correspondence given by the curve $D \subset C \times C'$, a coincident point is either a branch point of the projection

$$\pi_1: D \to C$$

of D on the first factor, or a singular point of D.

2. A *united point* of a correspondence $T: C \to C$ from a curve to itself is a point $p \in C$ such that $p \in T(p)$; we say that p is a united point of multiplicity m for T if p appears with multiplicity m in $T(p)$. In example 1 above, the united points p of T are the branch points of the map $C \to \mathbb{P}^1$ given by $\{D_\lambda\}$; in example 2 they are the flexes of C. In general, if $T: C \to C$ is given by the curve $D \subset C \times C$, a united point is a point of intersection of D with the diagonal $\Delta \subset C \times C$.

3. A *common point* of two correspondences $T, S: C \to C'$ is, as the name suggests, a pair $(p,q) \in C \times C'$ such that q is in both $T(p)$ and $S(p)$. If T and S are given by curves D and F in $C \times C'$, a common point is just a point of intersection of D and F.

4. A correspondence $T: C \to C$ from a curve of genus $g \geqslant 1$ to itself is said to have *valence* k if the linear equivalence class of the divisor

$$T(p) + k \cdot p$$

is independent of p. The correspondence of examples 1 and 2 above have valence 1 and 2, respectively. A correspondence need not, in general, have any valence; if it does have a valence though, the valence is unique: if for $k > k'$ the linear equivalence classes of $T(p) + k \cdot p$ and $T(p) + k' \cdot p$ were both constant, it would follow that the divisors $(k - k') \cdot p$ all belonged to some linear system E, of dimension r. Since the generic point of the curve $\iota_E(C) \subset \mathbb{P}^r$ meets any hyperplane with multiplicity at most r, it follows that $r = k - k'$—but then $\iota_E(C)$ is the rational normal curve, contrary to the hypothesis that $g(C) \geqslant 1$.

In practice, the information about a correspondence $T: C \to C$ that will be most readily available to us is the *degree* of T—that is, the intersection number $^{\#}(D \cdot E)$ of the curve of correspondence $D \subset C \times C$ with the vertical fibers $E_p = \pi_1^{-1}(p) \subset C \times C$; the *degree of* T^{-1}—the intersection number of D with the horizontal fibers $F_p = \pi_2^{-1}(p) \subset C \times C$; and the valence of T if it has one. On the other hand, as we shall see, to compute the number of coincident, or united, points of T, we will want to know the *homology class* of the curve $D \subset C \times C$. This is, in general, impossible: the group

$$H^{1,1}(C \times C) \cap H^2(C \times C, \mathbb{Z})$$

of divisors on $C \times C$ modulo homology has highly unpredictable rank. What makes it possible to compute effectively with some correspondences is the fundamental

Lemma. *Let* $T: C \to C$ *be a correspondence,* $D \subset C \times C$ *its curve of correspondence. Then* T *has valence* k *if and only if* D *is homologous to a linear combination*

$$D \sim aE + bF - k\Delta$$

of the two fibers E, F *of* $C \times C$ *and the diagonal* $\Delta \subset C \times C$.

Proof. First, assume that $D \sim aE + bF - k\Delta$. We claim to begin with that D is then *linearly equivalent* to a sum

$$G = \sum a_i E_{p_i} + \sum b_i F_{q_i} - k\Delta,$$

where $E_p = \pi_1^{-1}(p)$, $F_q = \pi_2^{-1}(q)$. This follows from the Künneth formulas: since the first two vertical maps of the diagram

$$\begin{array}{ccccc}
H^1(C \times C, \mathbb{Z}) & \longrightarrow & H^1(C \times C, \mathcal{O}) & \longrightarrow & \operatorname{Pic}^0(C \times C) \to 0 \\
\uparrow \pi_1^* \times \pi_2^* & & \uparrow \pi_1^* \times \pi_2^* & & \uparrow \pi_1^* \times \pi_2^* \\
H^1(C, \mathbb{Z}) \oplus H^1(C, \mathbb{Z}) & \to & H^1(C, \mathcal{O}) \oplus H^1(C, \mathcal{O}) & \to & \operatorname{Pic}^0(C) \times \operatorname{Pic}^0(C) \to 0
\end{array}$$

are isomorphisms, the last one is also. Now if $D \subset C \times C$ is linearly equivalent to the divisor G written above, then for generic $p \in C$, the divisor $T(p) = i_p^*(D)$ is linearly equivalent to the divisor $i_p^*(G) = \Sigma b_i q_i - k \cdot p$; clearly, then, the linear equivalence class of $T(p) + k(p)$ is independent of p. Conversely, suppose that the correspondence T has valence k. Write

$$T(p) + k \cdot p = \sum b_i q_i;$$

and

$$T^{-1}(q_0) + k \cdot q_0 = \sum a_i p_i,$$

and let L be the line bundle

$$L = D - \sum a_i E_{p_i} - \sum b_i F_{q_i} + k\Delta.$$

Then by hypothesis the restriction of L to *any* fiber E_p of π_1, and to the fiber F_{q_0} of π_2 as well, is trivial. We claim now that under these circumstances L must be trivial; this will certainly suffice to prove the lemma. To see this, let s_0 be a global nonzero holomorphic section of the restriction of L to F_{q_0}. For each $p \in C$, then, there will be a unique global section $t(p)$ of $L|_{E_p}$ such that $t(p)(p,q) = s_0(p, q_0)$; set

$$t(p,q) = t(p)(q).$$

t is then a global nonzero holomorphic section of L, and consequently L is trivial. Q.E.D.

Note: It may seem, at first glance, that the notion of valence is an unlikely one, and that correspondences with valency will be relatively rare.

In fact, just the opposite is true: *on a generic Riemann surface there are no correspondences without valency*. (Here "generic" has a meaning slightly different from usual, as will be seen.) We will not prove this, but the reader may see why it should be true: by the Künneth formula,

$$H^{1,1}(C\times C) = (H^{1,1}(C)\otimes H^{0,0}(C))\oplus(H^{1,0}(C)\otimes H^{0,1}(C))$$
$$\oplus(H^{0,1}(C)\otimes H^{1,0}(C))\oplus(H^{0,0}(C)\otimes H^{1,1}(C)).$$

The first and last terms in this expression are one-dimensional and are generated by the classes of the fibers E and $F\subset C\times C$, respectively. Writing out a basis for $(H^{1,0}(C)\otimes H^{0,1}(C))\oplus(H^{0,1}(C)\otimes H^{1,0}(C))$, and integrating over a basis for $H_2(C\times C,\mathbb{Z})=H_1(C,\mathbb{Z})\otimes H_1(C,\mathbb{Z})$, the reader will see that the requirement that there exist an integral class in the middle factor other than that of the diagonal $\Delta\subset C\times C$ is that the period matrix of C satisfy certain rationality conditions (cf. Section 4 in Chapter 3 for similar computations); the set of curves of genus g possessing correspondence without valence is thus expected to be a countable union of proper subvarieties of the family of all curves of genus g.

For example, the reader may check that a curve of genus one has correspondences without valence if and only if it has *complex multiplication*, that is, writing

$$C=\frac{\mathbb{C}}{\Lambda},$$

where Λ is the lattice generated by 1 and τ, if and only if τ satisfies a quadratic polynomial over $\mathbb{Q}$.

Now, with our basic lemma, we can derive the three basic formulas for correspondences. The first thing to do is to determine the intersection pairing on the subspace of $H_2(C\times C,\mathbb{Z})$ spanned by the classes of E, F, and Δ. We have, clearly,

$$^{\#}(E\cdot F)=1,$$
$$^{\#}(E\cdot E)={}^{\#}(F\cdot F)=0,$$

and

$$^{\#}(\Delta\cdot E)={}^{\#}(\Delta\cdot F)=1;$$

it remains only to determine $\Delta\cdot\Delta$. To do this, let $\{D_\lambda\}$ be a pencil of degree d on C and let T be the correspondence defined by $\{D_\lambda\}$ as in example 1 above; let $D\subset C\times C$ be its curve of correspondence. Since T has valence 1, we can write

$$D\sim aE+bF-\Delta.$$

Since T and T^{-1} both have degree $d-1$, moreover, we have

$$d-1={}^{\#}(D\cdot E)=b-1$$

and

$$d-1={}^{\#}(D\cdot F)=a-1,$$

i.e.,

$$D\sim dE+dF-\Delta.$$

Now, the number ${}^{\#}(D\cdot\Delta)$ of united points of T is just the number b of branch points of the map $C\to\mathbb{P}^1$ given by $\{D_\lambda\}$; this being a d-sheeted cover, we have by Riemann-Hurwitz

$$2g-2=-2d+b,$$

i.e.,

$$b=2g-2+2d.$$

Thus

$$\begin{aligned}2g-2+2d&={}^{\#}(D\cdot\Delta)\\&=d\,{}^{\#}(E\cdot\Delta)+d\,{}^{\#}(F\cdot\Delta)-{}^{\#}(\Delta\cdot\Delta),\\&=2d-{}^{\#}(\Delta\cdot\Delta)\end{aligned}$$

and so we have

$$\Delta\cdot\Delta=2-2g.$$

The intersection pairing is therefore

#	E	F	Δ
E	0	1	1
F	1	0	1
Δ	1	1	$2-2g$

Now suppose T is any correspondence with $\deg(T)=d$, $\deg(T^{-1})=d'$, and valence k. If D is the curve of correspondence of T, we write

$$D\sim aE+bF-k\Delta.$$

Then, since

$$d=\deg T={}^{\#}(D\cdot E)=b-k$$

and

$$d'=\deg T^{-1}={}^{\#}(D\cdot F)=a-k,$$

we obtain

$$D\sim(d'+k)E+(d+k)F-k\Delta.$$

Consequently

$$\begin{aligned}D\cdot\Delta&=d'+k+d+k-k(2-2g)\\&=d+d'+2kg,\end{aligned}$$

i.e.,

(∗) T *has* d + d′ + 2kg *united points*,

this is known as the *Cayley-Brill* formula.

Similarly, if S is another correspondence with $\deg(S)=e$, $\deg(S^{-1})=e'$, and valence l, given by the curve $G\subset C\times C$,

$$G\sim(e'+l)E+(e+l)F-\Delta.$$

By an obvious computation using the intersection table,

$$^{\#}(D\cdot G)=ed'+e'd-2gkl,$$

i.e.,

(**) *The correspondences* T *and* S *have* ed′ + e′d − 2gkl *common points.*

The computation for the number of coincident points of a correspondence $T:C\to C$ is slightly more difficult. With T and D as above,

$$D\sim(d'+k)E+(d+k)F-k\cdot\Delta.$$

In case D is smooth and irreducible, we may apply the *adjunction formula*

$$K_D=(K_{C\times C}+D)|_D$$

for the canonical bundle of D to obtain

$$\deg K_D={}^{\#}(K_{C\times C}\cdot D)+{}^{\#}(D\cdot D).$$

Once we have evaluated these intersection numbers, we will be done; by the Hurwitz formula, the number b of branch points of the projection $\pi_1:D\to C$ on the first factor is given by

$$\deg K_D=d\cdot\deg K_C+b,$$

i.e.,

$$\begin{aligned}b&=\deg K_D-d\cdot\deg K_C\\&={}^{\#}(K_{C\times C}\cdot D)+{}^{\#}(D\cdot D)-d(2g-2).\end{aligned}$$

Now if ω,ω' are holomorphic 1-forms on C, then the divisor of the 2-form $\pi_1^*\omega\wedge\pi_2^*\omega'$ on $C\times C$ is

$$(\pi_1^*\omega\wedge\pi_2^*\omega')=\pi_1^*(\omega)+\pi_2^*(\omega').$$

Thus the homology class of $K_{C\times C}$ is

$$\begin{aligned}K_{C\times C}&\sim\pi_1^*K_C+\pi_2^*K_C\\&=(2g-2)E+(2g-2)F.\end{aligned}$$

We then obtain

$$\begin{aligned}{}^{\#}(D\cdot K_{C\times C})&=(2g-2)(d+k)-k(2g-2)+(2g-2)(d'+k)-k(2g-2)\\&=(2g-2)(d+d')\end{aligned}$$

and by another straightforward manipulation

$$^{\#}(D\cdot D)=2dd'-2gk^2.$$

Putting everything together,

$$\begin{aligned} b &= {}^{\#}(K_{C\times C}\cdot D) + {}^{\#}(D\cdot D) - d(2g-2) \\ &= (d+d')(2g-2) + 2dd' - 2gk^2 - d(2g-2) \\ &= 2dd' + (2g-2)d' - 2gk^2 \end{aligned}$$

i.e.,

(***) *The correspondence* T *has* $2dd' + (2g-2)d' - 2gk^2$ *coincident points.*

This computation may be readily extended to the case where the correspondence T is given by a sum of smooth irreducible curves $D_i \subset C \times C$—the coincident points of T will then consist of coincident points for the correspondence T_i defined by the curves D_i, plus common points of the correspondences D_i and D_j for $i \neq j$ (to be counted, as we shall see, with multiplicity 2), and it is easily checked that the formula holds. A more serious objection is that the formula as stated holds only for correspondences given by smooth curves D. Since this will not always be the case, we will borrow a couple of results from Section 2 of Chapter 4 to see how to handle at least the case where D has ordinary double points.

Suppose that $D \subset C \times C$ has δ ordinary double points (p_i, q_i) and is otherwise smooth. Assuming that neither branch of D at such a double point (p,q) is tangent to the fiber $E_p = \pi_1^{-1}(p)$, (p,q) will appear as a simple coincident point of T. From Section 2 of Chapter 4 we see that D is the image of a smooth curve $\tilde{D}$ via a map $\tilde{\pi}: \tilde{D} \to D \subset C \times C$ that is one-to-one and smooth away from the double points of D—we simply separate the two branches of D around the double points. The Riemann surface $\tilde{D}$, moreover, will have genus (cf. page 280)

$$g(\tilde{D}) = \frac{{}^{\#}(K_{C\times C}\cdot D) + {}^{\#}(D\cdot D)}{2} + 1 - \delta,$$

i.e.,

$$\deg K_{\tilde{D}} = {}^{\#}(K_{C\times C}\cdot D) + {}^{\#}(D\cdot D) - 2\delta.$$

The composite map $\pi_1 \cdot \tilde{\pi}: \tilde{D} \to C$ will thus have

$$\begin{aligned} b &= \deg K_{\tilde{D}} - d\cdot \deg K_C \\ &= 2dd' + (2g-2)d' - 2gk^2 - 2\delta \end{aligned}$$

branch points; i.e., the correspondence T will have

$$2dd' + (2g-2)d' - 2gk^2 - 2\delta$$

coincident points apart from the double points (p_i, q_i) of D. We see, then, that *the formula given above for the number of coincident points of* T *holds if we count with multiplicity* 2 *a coincident point arising from an ordinary double point of* D.

In practice, it will be easy to distinguish an ordinary coincident point of T from one corresponding to a double point of D: a smooth point $(p,q)\in D$ can be a branch point of only one of the projections $\pi_1: D\to C$ and $\pi_2: D\to C$, while a double point $(p,q)\in D$ will appear as a coincident point for both T and T^{-1}.

We apply our formulas to the correspondence of example 2 on a smooth plane curve C of degree d. As we saw, T has degree $d-2$ and valence 2; the degree of T^{-1} is the number of tangent lines to C other than $T_q(C)$ passing through a point $q\in C$, i.e., the number b of branch points of the projection π_q of C from q onto a line. The projection is $(d-1)$-sheeted, and so by Riemann-Hurwitz,

$$\begin{aligned}\deg T^{-1} &= b = 2g-2+2(d-1)\\ &= (d-1)(d-2)-2+2(d-1)\\ &= (d+1)(d-2).\end{aligned}$$

By the formula (*), the number of united points of T—that is, the number of flexes of C—is

$$\begin{aligned}f &= (d-2)+(d+1)(d-2)+2kg\\ &= (d-2)+(d+1)(d-2)+2(d-1)(d-2)\\ &= 3d(d-2),\end{aligned}$$

which agrees with our computation in the last subsection of Plücker formulas. To compute the number of bitangent lines to C we have to be careful: if q and p are distinct points of C with $T_p(C)=T_q(C)$, then both (p,q) and (q,p) are coincident points of T. The number b of bitangent lines to C is thus half the number of coincident points of T; by (***) this is

$$\begin{aligned}b &= \tfrac{1}{2}\left[2(d-2)(d-2)(d+1)+(2g-2)(d-2)(d+1)-2gk^2\right]\\ &= \tfrac{1}{2}d(d+1)(d-1)(d-2)-4d(d-2),\end{aligned}$$

as we found earlier.

Geometry of Space Curves

We will now illustrate the technique of correspondences by an application to the geometry of *space curves*, i.e., curves in $\mathbb{P}^3$. Our primary goal will be to find the number of quadrisecants to a space curve C of degree d and genus g; along the way we will come across several other invariants.

Before we begin, we want to make one observation. The family of lines in $\mathbb{P}^3$ meeting a curve C has codimension 1 in the four-dimensional Grassmannian $G(2,4)$. By a naive dimension count, then, we may expect that C will have finitely many quadrisecants but no lines meeting C five times. Similarly, we expect that there will be a finite number of points of C whose tangent lines meet a given tangent line to C, hence a finite number of trisecants $\overline{p,q,r}$ such that $T_p(C)$ and $T_q(C)$ meet, but no such quadrisecants. In general, we say that a curve has *nondegenerate behavior* if no such phenomena occur that are not predicted by a dimension count. We will be assuming, in the following discussion, that this is the case; in particular we will assume that

1. C has no quintisecant lines.
2. The tangent lines to the four points of intersection of each quadrisecant to C are disjoint.
3. No line has contact of order 3 with C, i.e., the first associated curve of C is smooth.
4. No osculating 2-plane to C contains another tangent line to C.

Now, the central object of our discussion will be the *trisecant correspondence* T on C, defined by the curve

$$D = \{(p,q): \ \overline{pq} \text{ is a trisecant of } C\}.$$

We first compute degree of T: if p is a generic point of C, then the image of C under projection of C from p will be a plane curve of degree $d-1$ having only ordinary double points, and the double points of this curve will correspond to the trisecants to C passing through p. By the Plücker formulas, the number of trisecants through p is

$$\delta = \frac{(d-2)(d-3)}{2} - g,$$

and since $T(p)$ will contain two points for each trisecant of C through p,

$$\deg T = (d-2)(d-3) - 2g.$$

(Note that by our bound on the genus of space curves (p. 252), this number will be positive unless $d=3$, $g=0$ or $d=4$, $g=1$.) Since T is symmetric, this is also the degree of T^{-1}.

Now T has valence, as we see from the following: if $\pi_p: C \to \mathbb{P}^2$ is the projection of C from a generic point $p \in C$ as above, then as we proved in the preceding section, since $\deg \pi_p C = d-1$

$$K_C = \pi_p^*((d-4)H_{\mathbb{P}^2}) - D,$$

where D is the inverse image in C of the double points of $\pi_p(C)$. But now on C we have

$$\pi_p^*(H_{\mathbb{P}^2}) = H_{\mathbb{P}^3} - p,$$

and of course

$$D = T(p),$$

so

$$K_C = (d-4)H_{\mathbb{P}^3} - (d-4)p - T(p),$$

i.e.,

$$T(p) + (d-4)p = (d-4)H_{\mathbb{P}^3} - K_C$$

is constant as a linear equivalence class, and so T has valence $k=(d-4)$.

Consider now the coincident points of the correspondence T. For $p \in C$, $T(p)$ will contain a multiple point if it fails to contain $(d-2)(d-3)-2g$ distinct points. This will be the case exactly when the projected curve has singularities other than ordinary double points; this may happen in the following three ways:

1. If a line through p is simply tangent to C at another point q, then the image of q in $\pi_p(C) \subset \mathbb{P}^2$ will be a cusp of $T_p(C)$. (See Figure 10.) By the Plücker formulas, in the absence of other special behavior $\pi_p(C)$ will have $\delta=(d-2)(d-3)/2-g-1$ double points apart from $\pi_p(q)$, and so $T(p)$ will contain $(d-2)(d-3)-2g-2$ points besides q; q is thus taken in $T(p)$ with multiplicity 2.

Note that (p,q) is not a coincident point of T^{-1}, so (p,q) is a simple point of the curve of correspondence D.

A line tangent to C and meeting C again elsewhere is called a *tangential trisecant* of C. We see that there will be one coincident point of T for every tangential trisecant to C.

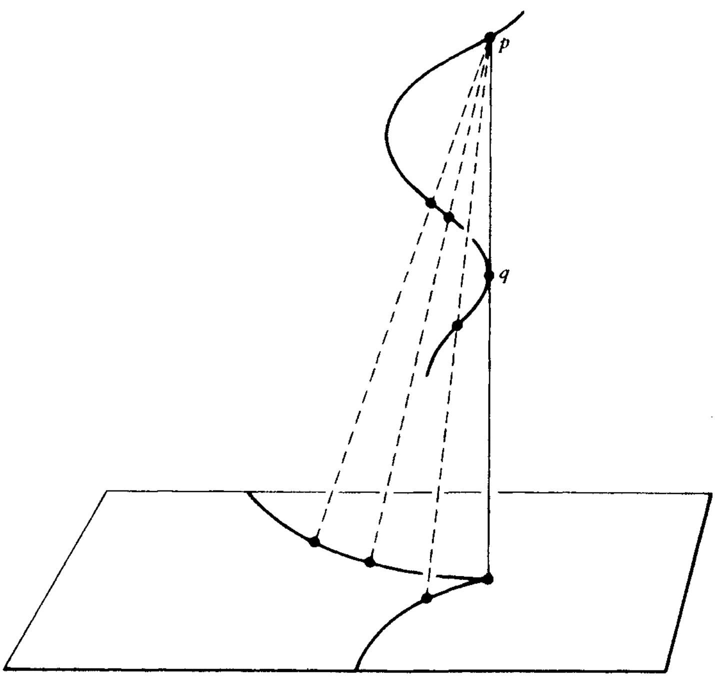


Figure 10

2. If a line through p meets C in two other points q and q', and the tangent lines to C at q and q' meet, then the image point $\pi_p(q)=\pi_p(q')\in\pi_p(C)$ will be a double point of $\pi_p(C)$, but not an ordinary one: the tangent lines to the two branches of $\pi_p(C)$ at $\pi_p(q)$ will coincide. (See Figure 11.) Such a double point is called a *tacnode*. Now, as we will see in Section 2 of Chapter 4, a tacnode drops the genus of a curve by 2, so if $\pi_p(C)$ has a tacnode, it will have only $(d-2)(d-3)/2-g-2$ other double points. $T(p)$ thus contains $(d-2)(d-3)-2g-4$ points other than q and q', and so each of the pairs (p,q) and (p,q') will be a coincident point of T. Again, since neither (p,q) nor (p,q') is a coincident point of T^{-1}, they will both be smooth points of D.

A trisecant $\overline{p,q,q'}$ such that $T_q(C)$ and $T_{q'}(C)$ meet is called a *stationary trisecant* of C; by the above, there will be two coincident points of T for every stationary trisecant to C.

3. If a line through p meets C in three other points q_1, q_2, and q_3, then the image point $\pi_p(q_1)=\pi_p(q_2)=\pi_p(q_3)$ will be an ordinary triple point of $\pi_p(C)$. (See Figure 12.) Looking ahead again to Section 2 of Chapter 4, we see that a triple point drops the genus of a curve by 3, so that $T(p)$ will have only $(d-2)(d-3)-2g-6$ coincident points other than q_1, q_2, and q_3;

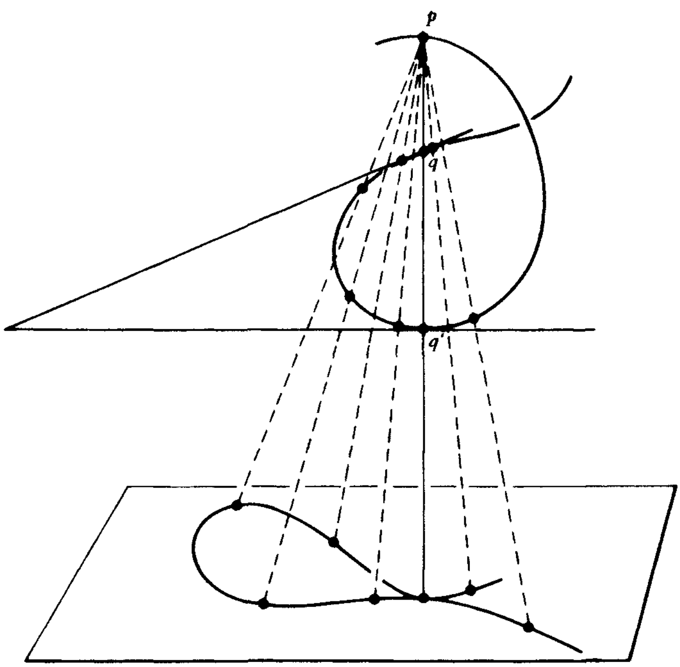


Figure 11

each of the pairs (p,q_1), (p,q_2), and (p,q_3) will thus be a coincident point of T. Moreover, (p,q_i) will by the same argument be a coincident point of T^{-1}, so (p,q_i) is in fact a double point of D. Thus, if $L=\overline{p_1p_2p_3p_4}$ is a quadrisecant of C, each of the 12 pairs (p_i,p_j) will be a double point of D, and so there are 24 coincident points of T for every quadrisecant to C.

Now, knowing the degree and valence of T, we can compute the total number of coincident points of T, so to find the number Q of quadrisecants to C we have to find the number of tangential and stationary trisecants. The number t of tangential trisecants is easy, since clearly the trisecants $\overline{p,q}$ with $q \in T_p(C)$ correspond exactly to the united points (p,p) of T. By the formula (*),

$$\begin{aligned} t &= (d-2)(d-3) - 2g + (d-2)(d-3) - 2g + 2(d-4)g \\ &= 2(d-2)(d-3) + 2(d-6)g. \end{aligned}$$

The number s of stationary trisecants is somewhat more difficult to calculate. To find it, we introduce the *bitangential correspondence* S on C,

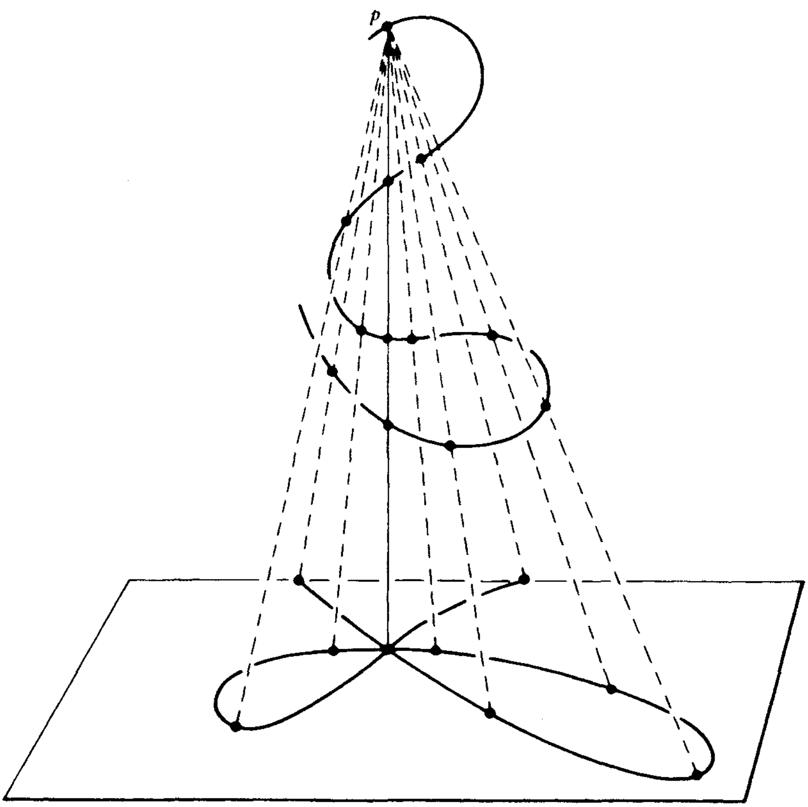


Figure 12

defined by the curve

$$G = \overline{\{(p,q):\ p\neq q, T_p(C)\cap T_q(C)\neq\varnothing\}} \subset C\times C.$$

For each $p\in C$, the divisor $S(p)$ is the branch locus of the projection map π_L of C from the tangent line $L=T_p(C)$ to C at p onto a line; since π_L has degree $d-2$, by the Riemann-Hurwitz formula

$$\begin{aligned}\deg S &= 2g-2+2(d-2)\\ &= 2g+2d-6.\end{aligned}$$

Since S is symmetric, this is also the degree of S^{-1}. We have, moreover,

$$K_C = \pi_L^*(-2H_{\mathbb{P}^1}) + S(p),$$

and because $\pi_L^*(H_{\mathbb{P}^1}) = H_{\mathbb{P}^3}-2p$, this yields

$$S(p)+4p = K_C + 2H_{\mathbb{P}^3},$$

$$S(p)+4p = K_C + 2H_{\mathbb{P}^3},$$

i.e., S has valence $l=4$.

Consider now the common points of the two correspondences T and S. (See Figure 13.) These can occur in two ways: if $\overline{p,q,q'}$ is a stationary trisecant of C with $T_q(C)\cap T_{q'}(C)\neq\varnothing$, then each of the pairs (q,q') and (q',q) will be a common point of S and T. Alternately, if $\overline{p,q}$ is a tangential trisecant with $q\in T_p(C)$, then (p,q) and (q,p) are both common points of S and T. The number of common points of S and T is therefore $2s+2t$; we have by the formula (**),

$$2s+2t = 2((d-2)(d-3)-2g)(2g+2d-6) - 2g(d-4)\cdot 4,$$

i.e.,

$$\begin{aligned}s &= ((d-2)(d-3)-2g)(2g+2d-6) - 4g(d-4)\\ &\quad -2(d-2)(d-3) - 2(d-6)g\\ &= 2d(d-2)(d-3) - 2(d-2)(d-3) - 6(d-2)(d-3)\\ &\quad +2g((d-2)(d-3)-2g-2d+6-2(d-4)-(d-6))\\ &= 2(d-2)(d-3)(d-4) + 2g(d^2-10d+26-2g).\end{aligned}$$

We now have enough to calculate the number Q of quadrisecants to C. As we have seen, the total number of coincident points of the correspondence T is $t+2s+24Q$; by our formula (***),

$$\begin{aligned}t+2s+24Q &= 2((d-2)(d-3)-2g)^2\\ &\quad + 2g - 2((d-2)(d-3)-2g) - 2g(d-4)^2,\end{aligned}$$

i.e.,

$$\begin{aligned}Q = \tfrac{1}{12}&((d-2)(d-3)-2g)^2 + (g-1)\big((d-2)(d-3)\\ &-2g-g(d-4)^2-(d-2)(d-3)-(d-6)g\\ &-2(d-2)(d-3)(d-4)-2g(d^2-10d+26-2g)\end{aligned}$$

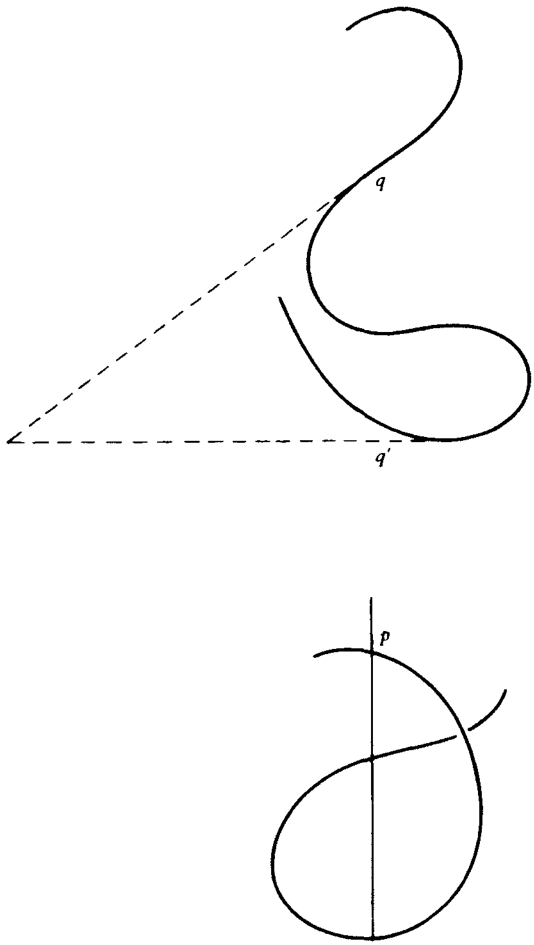


Figure 13

Omitting the explicit reduction, this gives

$$Q = \tfrac{1}{12}(d-2)(d-3)^2(d-4) - \tfrac{1}{2}g(d^2-7d+13-g).$$

One point: while the formulas derived in this discussion apply a priori to curves that do not exhibit degenerate behavior, it should be clear from the derivation how to account for such behavior. For example, if $L=\overline{p_1,\ldots,p_5}$ is a quintisecant to C, we can verify that each of the 20 pairs (p_i,p_j) is a triple point of the curve D and so drops the genus of D by 3; going back to the derivation of (∗∗∗), we see that each pair (p_i,p_j) counts as six coincident points. L thus contributes 120 coincident points to T, i.e., in terms of the last formula a quintisecant line to C is equivalent to $120/24=5$ quadrisecants.

Finally we take the opportunity here of remarking that some enumerative problems having to do with the extrinsic properties of more than one curve in $\mathbb{P}^3$ may also be solved by means of the Schubert calculus. For example, if C and C' are space curves, we may ask for the number of common chords of C and C'. To answer this, let $V(C)\subset G(2,4)$ be the algebraic surface of chords to C, in the Grassmannian of lines in $\mathbb{P}^3$. The fourth homology group of $G(2,4)$ is generated by the cycles

$$\sigma_2(p) = \{l\subset\mathbb{P}^3:\ p\in l\}$$

and

$$\sigma_{1,1}(H) = \{l\subset\mathbb{P}^3:\ l\subset H\};$$

we have clearly

$$^{\#}(\sigma_2\cdot\sigma_2) = {}^{\#}(\sigma_{1,1}\cdot\sigma_{1,1}) = 1; \qquad {}^{\#}(\sigma_2\cdot\sigma_{1,1}) = 0.$$

If we write

$$V(C)\sim a\cdot\sigma_{1,1}+b\cdot\sigma_2,$$

then

$$a = {}^{\#}(V(C)\cdot\sigma_{1,1}), \qquad b = {}^{\#}(V(C)\cdot\sigma_2).$$

These numbers are readily calculable. A generic hyperplane H will meet C in d distinct points $\{p_i\}$ and so contain exactly the $d(d-1)/2$ chords $\{\overline{p_ip_j}\}_{i\neq j}$; consequently

$$a = \frac{d(d-1)}{2}.$$

On the other hand, for p a generic point of $\mathbb{P}^3$, the number of chords to C passing through p will be just the number of double points of the image of C under projection from p into a hyperplane; by the Plücker formulas, this is

$$b = \frac{(d-1)(d-2)}{2} - g.$$

Combining,

$$V(C)\sim\frac{d(d-1)}{2}\cdot\sigma_{1,1}+\left(\frac{(d-1)(d-2)}{2}-g\right)\sigma_2.$$

If C' has degree d' and genus g', and if C and C' are in general position with respect to one another so that $V(C)$ and $V(C')$ meet transversely, then C and C' will have

$$\begin{aligned}{}^{\#}(V(C)\cdot V(C')) = {} & \frac{d(d-1)\cdot d'(d-1)}{4}+\frac{(d-1)(d-2)(d'-1)(d'-2)}{4}\\ & -(d-1)(d-2)g'-(d'-1)(d'-2)g+gg'\end{aligned}$$

common chords.

Special Linear Systems III

As promised earlier, we will use the results on correspondences to answer some of the enumerative questions arising from the Brill-Noether problem. We have seen that the generic Riemann surface of genus $g=2k$ has a finite number of pencils of degree $k+1$; the question is, how many? We will answer this in cases $g=4$, 6, and 8. (Note that in case $g=2$ the answer 1 has already been obtained.)

$g=4$. If S is a Riemann surface of genus 4, its canonical curve is a curve of degree 6 in $\mathbb{P}^3$. Now if $D=\sum p_i$ is a divisor of degree 3 on S, then by the geometric version of the Riemann-Roch, $l(D)$ will be 1 exactly when the points p_i are collinear; for each pencil of degree 3 there will be one such divisor through a generic point $p \in S$. The number of such pencils is thus the number of trisecants to S through a generic point $p \in S$—and this we have seen is

$$\begin{aligned} n &= \tfrac{1}{2}(d-2)(d-3) - g \\ &= \tfrac{1}{2}(4\cdot 3) - 4 = 2, \end{aligned}$$

i.e., there are two pencils of degree 3 on S. Thus, *the generic Riemann surface of genus* 4 *is expressible as a* 3-*sheeted cover of* $\mathbb{P}^1$ *in two ways.*

It is interesting to actually locate these two pencils. We saw in the last section that the canonical curve of S is the smooth intersection of a quadric Q and cubic Q' in $\mathbb{P}^3$; generically the quadric will be smooth. Now a smooth quadric surface in $\mathbb{P}^3$ (discussed on pages 478–480 below) contains two families of lines $\{L_t\},\{L_t'\}$, each parametrized by $t \in \mathbb{P}^1$; since S is cut out on Q by the cubic Q', each line L_t or L_t' will meet S in three points; the divisors

$$D_t = L \cap S \quad \text{and} \quad D_t' = L_t' \cap S$$

then form two pencils of degree 3 on S. Conversely, if $D=\sum_{i=1}^3 p_i$ is any divisor consisting of three collinear points, then the line $L=\overline{p_1p_2p_3}$, meeting Q in three points, must lie in Q; L is thus an L_t or an L_t' and D a D_t or a D_t'.

Note that in case Q is singular, S will contain only one pencil of degree 3; the projection of S from any point of S into a hyperplane will have not two ordinary double points but one tacnode.

$g=6$. If $D=\sum_{i=1}^4 p_i$ is a divisor of degree 4 and $\dim|D|=1$ on a curve S of genus 6, then the divisor $K-D$ will have degree $2g-2-4=6$, and by Riemann-Roch

$$\begin{aligned} h^0(K-D) &= \deg(K-D) - g + 1 + h^0(D) \\ &= 6-6+1+2 \\ &= 3, \end{aligned}$$

i.e., the complete linear system $|K-D|$ will have dimension 2. Consequently the number of pencils of degree 4 on S is the number of nets of degree 6; it is this number that we shall compute.

Now by the geometric Riemann-Roch formula a divisor D of degree 6 with $\dim|D|=2$ on S consists of six points on the canonical curve of S spanning a 3-plane in $\mathbb{P}^5$; and if $p,q\in S$ are generic points for every net of degree 6 on S, there will be one such divisor containing p and q. If $D=p+q+r_1+\cdots+r_4$ is such a divisor, moreover, then the images of the points r_i under the projection

$$\pi_L:\ S\to\mathbb{P}^3$$

of S from the line $L=\overline{pq}$ onto a 3-plane will be collinear, and conversely, if any four points $\pi_L(r_i)$ of $\pi_L(S)$ are collinear, then the points $p,q,r_1,\dots,r_4$ all lie in the 3-plane spanned by L and $\{\pi_L(r_i)\}$. *The number of nets of degree* 6 *on* S *will thus be the number of quadrisecants to* $\pi_L(S)$ *in* $\mathbb{P}^3$. Since $\pi_L(S)$ has degree $d=\deg(S)-2=8$ and genus $g=6$, by our previous formula this number is

$$\begin{aligned} n &= \tfrac{1}{12}\cdot 6\cdot 5\cdot 5\cdot 4-\tfrac{1}{2}6(64-56+13-6)\\ &= 50-3\cdot 15=5. \end{aligned}$$

We see that *the generic Riemann surface of genus* 6 *is expressible as a* 4-*sheeted cover of* $\mathbb{P}^1$ *in five ways.*

$g=8$. If D is a divisor of degree 5 with $\dim|D|=1$ on a Riemann surface S of genus 8, then the divisor $K-D$ has degree $14-5=9$ and

$$\begin{aligned} h^0(K-D) &= \deg(K-D)-g+1+h^0(D)\\ &= 9-8+1+2\\ &= 4, \end{aligned}$$

i.e., $\dim|K-D|=3$. By the Riemann-Roch, then, D will be represented by five points spanning a 3-plane on the canonical curve of S, while $K-D$ will be represented by nine points of S spanning a 5-plane.

To compute the number of pencils of degree 5 on S, we first prove the

Lemma. *If* D, D' *are two divisors of degree* 5 *on* S, $\dim|D|=\dim|D'|=1$, *then there will be a divisor* $E\in|K-D'|$ *containing* D *if and only if* D *and* D′ *are not linearly equivalent*; *if* E *exists, it is unique.*

Proof. We want to show that

$$h^0(K-D-D')=\begin{cases} 1, & \text{if } D\not\sim D',\\ 0, & \text{if } D\sim D'.\end{cases}$$

(Note that since C is generic, it has no pencil of degree 4; thus

$$h^0(K-D-D')\leqslant 1.)$$

By Riemann-Roch,

$$h^0(K-D-D') = \deg(K-D-D') - g + 1 + h^0(D+D')$$
$$= 4 - 8 + 1 + h^0(D+D'),$$

i.e., we have to show that

$$h^0(D+D') = \begin{cases} 3, & \text{if } D \sim D', \\ 4, & \text{if } D \nsim D'. \end{cases}$$

But now if $h^0(D+D')$ were 3—i.e., if $|D+D'|$ were parametrized by $\mathbb{P}^2$—then for any $D_0 \in |D|, D_0' \in |D'|$ the two lines $D_0 + |D'|$ and $D_0' + |D| \subset |D+D'|$ would meet; we would then have $D_0 \in |D'|$ and $D \sim D'$.

On the other hand, any divisor $G \in |K-D|$ consists of nine points spanning a 5-plane and so lies on a pencil $\{H_t\}$ of hyperplanes in $\mathbb{P}^7$; the divisors

$$D_t = (H_t \cdot C) - G$$

comprise the linear system $\{D_t\}$. Clearly, then, for $t \neq t'$, D_t and $D_{t'}$ will lie in no hyperplane in $\mathbb{P}^7$, i.e., $K - D_t - D_{t'} \sim K - 2D$ is not effective. Q.E.D.

Now suppose $D = \Sigma_{i=1}^5 p_i$ is any divisor on S of degree 5 and $\dim|D| = 1$. Then by our lemma the pencils of degree 5 in S *other than* D correspond exactly to 9-secant 5-planes to the canonical curve $S \subset \mathbb{P}^7$ containing D. If $\overline{p_1, \ldots, p_5, q_1, \ldots, q_4}$ is any such 9-secant 5-plane, then the images of the points q_i under the projection

$$\pi_V : S \to \mathbb{P}^3$$

of S from the 3-plane $V = \overline{p_1, \ldots, p_5}$ will be collinear, and conversely. Thus the number of pencils of degree 5 on S other than $|D|$ is just the number of quadrisecants to $\pi_V(S) \subset \mathbb{P}^3$. $\pi_V(S)$ has degree $d = \deg(S) - 5 = 9$, and so this number is

$$\tfrac{1}{12} 7 \cdot 6 \cdot 6 \cdot 5 - \tfrac{1}{2} 8 \cdot (81 - 63 + 13 - 8) = 105 - 4 \cdot 23 = 13.$$

Summarizing, we see that *the generic Riemann surface of genus* 8 *may be expressed as a* 5-*sheeted cover of* $\mathbb{P}^1$ *in* 14 *ways*.

6. COMPLEX TORI AND ABELIAN VARIETIES

The Riemann Conditions

En route to our analysis of the relationship between a compact Riemann surface and its Jacobian, we given here an introduction to the general theory of complex tori.

First, we make a definition: for V a complex vector space of dimension n, $\Lambda \subset V$ a discrete lattice of maximal rank $2n$, the complex torus $M = V/\Lambda$ is called an *Abelian variety* if it is a projective algebraic variety, i.e., if it admits an embedding in projective space.

Our first task will be to determine when a complex torus $M = V/\Lambda$ is an Abelian variety. Since the cohomology of M is easily expressed in terms of V and Λ, Kodaira's embedding theorem will give us necessary and sufficient conditions; later on, we will verify the sufficiency of these conditions by direct computation. To begin with, we make some general remarks about the cohomology of complex tori.

Let $M = V/\Lambda$ as above. Since Λ is a subgroup of V, M likewise has the structure of a group: for any $\mu \in M$, and any $x \in V$ over μ, the map

$$\begin{aligned} \tau_\mu :\ & V \to V \\ :\ & v \mapsto v + x \end{aligned}$$

induces a map $\tau_\mu : M \to M$, called *translation by* μ.

Now we have a natural identificaton

$$T'_\mu(M) \cong V$$

for each $\mu \in M$; accordingly, any hermitian inner product on the vector space V gives a Kähler metric on M, invariant under the automorphisms $\{\tau_\mu\}$. We claim first that with respect to such a metric, the harmonic forms are exactly the forms invariant under $\{\tau_\mu\}$. To see this, note first that since τ_μ preserves the metric, $\tau_\mu^* : A^*(M) \to A^*(M)$ sends harmonic forms to harmonic forms. Then, since τ_μ is homotopic to the identity map and by the Hodge theorem $\mathcal{H}^*(M)$ maps isomorphically to $H^*(M, \mathbb{C})$,

$$\tau_\mu^* :\ \mathcal{H}^*(M) \to \mathcal{H}^*(M)$$

is just the identity, i.e., *harmonic forms are invariant*. But now an invariant form on M is determined by its values on the tangent space $T_{p,\mathbb{C}}(M) = T'_p(M) \oplus T''_p(M)$ to M at a point p, and this tangent space is naturally identified with the vector space $V \oplus \overline{V}$. Letting $\mathcal{J}^*(M)$ denote the space of invariant forms on M, then,

$$\mathcal{J}^*(M) = \wedge^*(T_{p,\mathbb{C}}(M)^*) \cong \wedge^* V^* \otimes \wedge^* \overline{V}^*.$$

But we know that topologically $M \cong (S^1)^{2n}$, so the dimension of the space of harmonic forms of degree k is $\binom{2n}{k}$; since $\mathcal{H}^*(M) \subset \mathcal{J}^*(M)$, we count dimensions to obtain

$$H^*(M, \mathbb{C}) \cong \mathcal{H}^*(M) = \wedge^* V \otimes \wedge^* \overline{V}^*.$$

Thus if $z = (z_1, \ldots, z_n)$ are Euclidean coordinates on V, $\{dz_1, \ldots, dz_n\}$ and

$\{d\bar{z}_1,\dots,d\bar{z}_n\}$ the corresponding global 1-forms on M,

$$\mathcal{H}^*(M) = \mathbb{C}\{dz_I \wedge d\bar{z}_J\}_{I,J}$$

with

$$\mathcal{H}^{p,q}(M) = \mathbb{C}\{dz_I \wedge d\bar{z}_J\}_{^\#I=p,\,^\#J=q}.$$

On the other hand, note that any loop $\gamma \in H_1(M,\mathbb{Z})$ with base point $[0] \in M$ lifts to a path $\tilde{\gamma}$ in V starting at 0 and ending at a point $\lambda \in \Lambda \subset V$; since V is the universal covering space of M, we can make the identification

$$H_1(M,\mathbb{Z}) = \Lambda.$$

Let $\lambda_1,\dots,\lambda_{2n} \in \Lambda$ be lattice vectors forming an integral basis for Λ; $\lambda_1,\dots,\lambda_{2n}$ will also be a basis for the real vector space V. Let $x_1,\dots,x_{2n}$ be the dual real coordinates on V and $dx_1,\dots,dx_{2n}$ the corresponding 1-forms on M. Then

$$\int_{\lambda_i} dx_j = \delta_{ij},$$

i.e.,

$$H^1(M,\mathbb{Z}) = \mathbb{Z}\{dx_1,\dots,dx_{2n}\}$$

and in general

$$H^k(M,\mathbb{Z}) = \mathbb{Z}\{dx_I\}_{^\#I=k}.$$

Thus we have two alternate bases for the cohomology of M: the first, $\{dz_\alpha, d\bar{z}_\alpha\}$ reflecting the complex structure on $H^*(M)$ and the second, $\{dx_i\}$, reflecting the rational structure. Now the Kodaira embedding theorem says that M is algebraic if and only if there exists a *Hodge form* on M, i.e., a closed, positive form of type (1,1) representing a rational cohomology class. Moreover, if $\tilde{\omega}$ is any such form, written

$$\tilde{\omega} = \frac{\sqrt{-1}}{2} \sum \tilde{h}_{\alpha\beta}(z)\, dz_\alpha \wedge d\bar{z}_\beta,$$

and $d\mu$ is the invariant Euclidean measure on M with $\mu(M)=1$, we can set

$$h_{\alpha\beta} = \int_M \tilde{h}_{\alpha\beta}(z)\, d\mu$$

and

$$\omega = \frac{\sqrt{-1}}{2} \sum h_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$$

to obtain an *invariant* form that is again closed, positive of type (1,1), and integral. Thus M has a Hodge form if and only if it has an invariant Hodge form. Accordingly, to determine whether such a form exists, we have to relate our two bases for $H^*(M)$.

Let $\Pi=(\pi_{i\alpha})$ be the $2n\times n$ matrix such that

$$dx_i = \sum_\alpha \pi_{i\alpha}\, dz_\alpha + \sum_\alpha \overline{\pi_{i\alpha}}\, d\bar{z}_\alpha,$$

i.e., such that the $2n\times 2n$ matrix $\tilde{\Pi}=(\Pi,\overline{\Pi})$ gives the change of basis from $\{dz_\alpha, d\bar{z}_\alpha\}$ to $\{dx_i\}$. Then, if ω is an invariant, integral 2-form, we can write

$$\omega = \tfrac{1}{2}\sum q_{ij}\, dx_i \wedge dx_j$$

with $Q=(q_{ij})$ an integral skew-symmetric $2n\times 2n$ matrix. In terms of the $dz_\alpha, d\bar{z}_\alpha$, we have

$$\begin{aligned}\omega &= \tfrac{1}{2}\sum q_{ij}\left(\pi_{i\alpha}dz_\alpha + \bar{\pi}_{i\alpha}d\bar{z}_\alpha\right)\wedge\left(\pi_{j\beta}dz_\beta + \bar{\pi}_{j\beta}d\bar{z}_\beta\right)\\ &= \tfrac{1}{2}\sum q_{ij}\pi_{i\alpha}\pi_{j\beta}\,dz_\alpha\wedge dz_\beta + \tfrac{1}{2}\sum q_{ij}\bar{\pi}_{i\alpha}\bar{\pi}_{j\beta}\,d\bar{z}_\alpha\wedge d\bar{z}_\beta\\ &\quad + \tfrac{1}{2}\sum q_{ij}\left(\pi_{i\alpha}\bar{\pi}_{j\beta} - \bar{\pi}_{i\beta}\pi_{j\alpha}\right)dz_\alpha\wedge d\bar{z}_\beta.\end{aligned}$$

From this we see that ω is of type (1, 1) if and only if the coefficient matrix

$$\tfrac{1}{2}\left(\sum_{i,j} q_{ij}\pi_{i\alpha}\pi_{j\beta}\right) = {}^t\Pi\cdot Q\cdot\Pi$$

is zero, and that if this is the case, then ω is positive if and only if

$$\begin{aligned}&\frac{1}{2\sqrt{-1}}\left(\sum_{i,j} q_{ij}\left(\pi_{i\alpha}\bar{\pi}_{j\beta} - \bar{\pi}_{i\beta}\pi_{j\alpha}\right)\right)_{\alpha\beta}\\ &\quad = \frac{1}{2\sqrt{-1}}\left({}^t\Pi Q\overline{\Pi} - {}^t\Pi{}^tQ\overline{\Pi}\right) = \frac{1}{\sqrt{-1}}\,{}^t\Pi Q\overline{\Pi}\end{aligned}$$

is hermitian positive definite. Thus we have the

Riemann Conditions I. M *is an Abelian variety if and only if there exists an integral, skew-symmetric matrix* Q *such that*

$${}^t\Pi\cdot Q\cdot\Pi = 0$$

and

$$-\sqrt{-1}\;{}^t\Pi Q\overline{\Pi} > 0.$$

We can also express these conditions in terms of the square matrix $\tilde{\Pi}=(\Pi,\overline{\Pi})$:

$${}^t\tilde{\Pi}\cdot Q\cdot\overline{\tilde{\Pi}} = \begin{pmatrix}{}^t\Pi\\ {}^t\overline{\Pi}\end{pmatrix}\cdot Q\cdot(\overline{\Pi},\Pi) = \begin{pmatrix}{}^t\Pi\cdot Q\cdot\overline{\Pi} & {}^t\Pi\cdot Q\cdot\Pi\\ {}^t\overline{\Pi}\cdot Q\cdot\overline{\Pi} & {}^t\overline{\Pi}\cdot Q\cdot\Pi\end{pmatrix}$$

and ${}^t\overline{\Pi}Q\Pi = {}^t({}^t\Pi{}^tQ\overline{\Pi}) = -{}^t({}^t\Pi\cdot Q\cdot\overline{\Pi})$, so M is an Abelian variety if and only if there exists an integral, skew-symmetric matrix Q with

$$-\sqrt{-1}\;{}^t\tilde{\Pi}\cdot Q\cdot\overline{\tilde{\Pi}} = \begin{pmatrix}H & 0\\ 0 & -{}^tH\end{pmatrix}$$

with $H>0$.

The usual form of the Riemann conditions is in terms of the dual change-of-basis matrix. For $\lambda_1,\ldots,\lambda_{2n}$ an integral basis for Λ and $e_1,\ldots,e_n$ a complex basis for V, we take the *period matrix* of $\Lambda\subset V$ to be the $n\times 2n$ matrix $\Omega=(\omega_{\alpha i})$ such that

$$\lambda_i = \sum_\alpha \omega_{\alpha i} e_\alpha.$$

Then we have

$$dz_\alpha = \sum_i \omega_{\alpha i}\, dx_i,$$

$$d\bar{z}_\alpha = \sum_i \bar{\omega}_{\alpha i}\, dx_i,$$

so that the matrix $\tilde{\Omega}=\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}$ gives the change of basis from $\{dx_i\}$ to $\{dz_\alpha, d\bar{z}_\alpha\}$. Thus

$$\tilde{\Omega}\cdot\tilde{\Pi} = I_{2n} \qquad \text{or} \qquad \Omega\Pi = I_n,\quad \Omega\bar{\Pi} = 0.$$

Now

$$-\sqrt{-1}\;{}^t\tilde{\Pi}\cdot Q\cdot \bar{\tilde{\Pi}} = -\sqrt{-1}\;{}^t\tilde{\Omega}^{-1}\cdot Q\cdot \bar{\tilde{\Omega}}^{-1},$$

and so in terms of Ω we can write the Riemann conditions as

$$\sqrt{-1}\cdot\bar{\tilde{\Omega}}\cdot Q^{-1}\cdot{}^t\tilde{\Omega} = \begin{pmatrix} H^{-1} & 0\\ 0 & -{}^tH^{-1}\end{pmatrix},$$

where $H>0$. But $H>0 \Leftrightarrow H^{-1}>0$; thus

Riemann Conditions II. M *is an Abelian variety if and only if there exists an integral, skew-symmetric matrix* Q *satisfying*

$$\Omega\cdot Q^{-1}\cdot{}^t\Omega = 0, \qquad -\sqrt{-1}\,\Omega\cdot Q^{-1}\cdot{}^t\bar{\Omega} > 0.$$

It will be noticed that the period matrix Ω of $\Lambda\subset V$ depends on the choice of basis for both Λ and V. By normalizing our choice of both with respect to a given form, we can simplify the Riemann conditions somewhat. First, we prove the

Lemma. *If* Q(,) *is an integral, skew-symmetric quadratic form on* $\Lambda=\mathbb{Z}^{2n}$, *then there exists a basis* $\lambda_1,\ldots,\lambda_{2n}$ *for* Λ *in terms of which* Q *is given by the matrix*

$$Q = \begin{pmatrix} 0 & \Delta_\delta\\ -\Delta_\delta & 0\end{pmatrix},\qquad \Delta_\delta = \begin{bmatrix}\delta_1 & & 0\\ & \ddots & \\ 0 & & \delta_n\end{bmatrix},\qquad \delta_i\in\mathbb{Z}.$$

Proof. For each $\lambda\in\Lambda$ the set of values $\{Q(\lambda,\lambda'),\lambda'\in\Lambda\}$ forms a principal ideal $d_\lambda\mathbb{Z}$ in $\mathbb{Z}$, $d_\lambda\geqslant 0$. Let $\delta_1=\min(d_\lambda:\lambda\in\Lambda, d_\lambda\neq 0)$, and take λ_1 and λ_{n+1}

such that $Q(\lambda_1,\lambda_{n+1})=\delta_1$. Then for every $\lambda\in\Lambda$, δ_1 divides $Q(\lambda,\lambda_1)$ and $Q(\lambda,\lambda_{n+1})$, and we can write

$$\lambda+\frac{Q(\lambda,\lambda_1)}{\delta_1}\cdot\lambda_{n+1}-\frac{Q(\lambda,\lambda_{n+1})}{\delta_1}\cdot\lambda_1\in\mathbb{Z}\{\lambda_1,\lambda_{n+1}\}^{\perp},$$

i.e.,

$$\Lambda=\mathbb{Z}\{\lambda_1,\lambda_{n+1}\}\oplus\mathbb{Z}\{\lambda_1,\lambda_{n+1}\}^{\perp}.$$

Set $\Lambda'=\mathbb{Z}\{\lambda_1,\lambda_{n+1}\}^{\perp}$; we can repeat this process to obtain two elements $\lambda_2,\lambda_{n+2}\in\Lambda'$ with

$$\Lambda'=\mathbb{Z}\{\lambda_2,\lambda_{n+2}\}\oplus\mathbb{Z}\{\lambda_2,\lambda_{n+2}\}^{\perp}.$$

Continuing in this way, we obtain a basis $(\lambda_1,\ldots,\lambda_{2n})$ for Λ having the desired properties.

Note that the integers $\{\delta_i\}$ obtained satisfy $\delta_1|\delta_2$, $\delta_2|\delta_3$, and so on: if, for example, $\delta_1\nmid\delta_2$, then for some k we would have

$$0<Q(k\lambda_1+\lambda_2,\lambda_{n+1}+\lambda_{n+2})<\delta_1.$$

We observe that with the additional condition $\delta_i|\delta_{i+1}$ the integers δ_i are invariants of the quadratic form Q. Q.E.D.

We see from the lemma that if ω is any integral, invariant 2-form on $M=V/\Lambda$, we can find a basis $\lambda_1,\ldots,\lambda_{2n}$ for Λ such that in terms of the dual coordinates $x_1,\ldots,x_{2n}$ on V,

$$\omega=\sum_{i=1}^{n}\delta_i\,dx_i\wedge dx_{n+i},\qquad \delta_i\in\mathbb{Z}.$$

Now if ω is nondegenerate—that is, if $\omega^n\neq 0$, as will be the case if ω is positive—then $\delta_\alpha\neq 0$ for all α, and we can take as our basis for the complex vector space V the vectors

$$e_\alpha=\delta_\alpha^{-1}\lambda_\alpha,\qquad \alpha=1,\ldots,n.$$

The period matrix of $\Lambda\subset V$ will then be of the form

$$\Omega=(\Delta_\delta,Z);$$

such a period matrix is called *normalized*. As before, ω will be of type $(1,1)$ if

$$\Omega\cdot Q_\delta^{-1}\cdot{}^t\Omega=0,$$

i.e., if

$$(\Delta_\delta,Z)\begin{pmatrix}0 & -\Delta_{\delta^{-1}}\\ \Delta_{\delta^{-1}} & 0\end{pmatrix}\begin{pmatrix}\Delta_\delta\\ {}^tZ\end{pmatrix}=(\Delta_\delta,Z)\begin{pmatrix}-\Delta_\delta^{-1}\cdot{}^tZ\\ I\end{pmatrix}$$
$$=Z-{}^tZ=0,$$

i.e., if Z is symmetric; and ω will be positive as well if

$$-\sqrt{-1}\cdot\Omega\cdot Q_\delta^{-1}\cdot{}^t\overline{\Omega} > 0,$$

i.e., if

$$-\sqrt{-1}\,(\Delta_\delta, Z)\begin{pmatrix}0 & -\Delta_{\delta^{-1}}\\ \Delta_{\delta^{-1}} & 0\end{pmatrix}\begin{pmatrix}\Delta_\delta\\ {}^t\overline{Z}\end{pmatrix} = -\sqrt{-1}\,(Z-{}^t\overline{Z}) = 2\cdot\operatorname{Im}Z > 0.$$

Thus we have

Riemann Conditions III. $M = V/\Lambda$ *is an Abelian variety if and only if there exists an integral basis* $\lambda_1,\ldots,\lambda_{2n}$ *for* Λ *and complex basis* $e_1,\ldots,e_n$ *for* V *such that*

$$\Omega = (\Delta_\delta, Z)$$

with Z *symmetric and* $\operatorname{Im}Z$ *positive definite.*

Note that the matrix Π above likewise takes a relatively simple form in terms of the bases $\{\lambda_1,\ldots,\lambda_{2n}\}$ and $\{e_1,\ldots,e_n\}$: solving

$$(\Pi,\overline{\Pi})\cdot\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix} = I_{2n},$$

we see that

$$\Pi = \begin{pmatrix}\Pi_1\\ \Pi_2\end{pmatrix}$$

with

$$\Pi_2 = \frac{1}{2\sqrt{-1}}(\operatorname{Im}Z)^{-1},$$

$$\Pi_1 = \frac{\sqrt{-1}}{2}\Delta_\delta^{-1}\overline{Z}(\operatorname{Im}Z)^{-1}.$$

The cohomology class $[\omega]$ of a Hodge form ω on an Abelian variety $M = V/\Lambda$ is called a *polarization* of M. The integers δ_α appearing in the expression

$$\omega = \sum \delta_\alpha\, dx_\alpha \wedge dx_{n+\alpha}, \qquad \delta_\alpha | \delta_{\alpha+1}$$

for ω in terms of coordinates $\{x_i\}$ dual to an integral basis for Λ are invariants of the class $[\omega]$, and are called the *elementary divisors* of the polarization; $[\omega]$ is called a *principal polarization* if $\delta_\alpha = 1$ for all α.

Now if S is a compact Riemann surface of genus g with bases $\delta_1,\ldots,\delta_{2g}$ for $H_1(S,\mathbb{Z})$ and $\omega_1,\ldots,\omega_g$ for $H^0(S,\Omega^1)$, the Jacobian variety

$$\mathcal{J}(S) = \frac{\mathbb{C}^g}{\mathbb{Z}\{\lambda_1,\ldots,\lambda_{2g}\}},$$

where the λ_i are the column vectors.

$$\lambda_i = {}^t\left(\int_{\delta_i}\omega_1,\ldots,\int_{\delta_i}\omega_g\right)$$

of the period matrix Ω of S. We have seen in Section 2 of this chapter that if $\delta_1,\ldots,\delta_{2g}$ is a normalized basis for $H_1(S,\mathbb{Z})$, we can choose a basis $\omega_1,\ldots,\omega_g$ for $H^0(S,\Omega^1)$ such that

$$\int_{\delta_i}\omega_\alpha = \delta_{i\alpha}, \qquad 1 \leqslant i, \alpha \leqslant g;$$

the period matrix will then be of the form

$$\Omega = (I, Z),$$

and by the two Riemann bilinear relations also proved in Section 2, $Z = X + \sqrt{-1}\,Y$ is symmetric with $Y > 0$. Thus $\mathcal{J}(S)$ is an Abelian variety, and moreover has a principal polarization given in terms of the basis $\{dx_i\}$ for $H^1(\mathcal{J}(S),\mathbb{Z})$ dual to $\{\lambda_i\} \in H_1(\mathcal{J}(S),\mathbb{Z})$ by

$$\omega = \sum dx_\alpha \wedge dx_{n+\alpha}.$$

In intrinsic terms, the Jacobian variety $\mathcal{J}(S) = V(S)/\Lambda(S)$, where $V(S) = H^0(S,\Omega^1)^*$ and the lattice $\Lambda(S) \cong H_1(S,\mathbb{Z})$ is embedded in $V(S)$ by integration. The polarizing form $\omega \in H^2(\mathcal{J}(S),\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\wedge^2 H_1(S,\mathbb{Z}),\mathbb{Z})$ is the skew-symmetric bilinear form

$$Q\colon\ H_1(S,\mathbb{Z}) \otimes_{\mathbb{Z}} H_1(S,\mathbb{Z}) \to \mathbb{Z}$$

given by intersection of cycles; the fact that the polarization is principal is a reflection of Poincaré duality. Note that the polarizing class $[\omega]$ does not depend on the choice of basis $\delta_1,\ldots,\delta_{2g}$ for $H_1(S,\mathbb{Z})$.

A note: Up to now, we have indexed our complex basis $\{e_\alpha\}$ and dual complex coordinates $\{z_\alpha\}$ for V by $\alpha = 1,\ldots,n$; the integral basis $\{\lambda_i\}$ and dual real coordinates $\{x_i\}$ by $i = 1,\ldots,2n$. Once we have normalized our basis, however, we can no longer maintain the notational distinction; we will instead denote the integral basis by

$$\{\lambda_\alpha, \lambda_{n+\alpha}\}_{\alpha=1,\ldots,n} \quad \text{and} \quad \{x_\alpha, x_{n+\alpha}\}_{\alpha=1,\ldots,n}.$$

Line Bundles on Complex Tori

We will now give explicit descriptions of positive line bundles on a complex torus $M = V/\Lambda$. The fundamental observation, proved on p. 46, is simply that, since $H^1(\mathbb{C}^n,\mathcal{O}) = H^2(\mathbb{C}^n,\mathbb{Z}) = 0$, *any line bundle on* $V \cong \mathbb{C}^n$ *is trivial*. Thus if $L \to M$ is any line bundle, the pullback π^*L of L to V is trivial, and we can find a global trivialization

$$\varphi\colon\ \pi^*L \longrightarrow V \times \mathbb{C}.$$

Now for $z \in V$, $\lambda \in \Lambda$, the fibers of π^*L at z and $z+\lambda$ are by definition both identified with the fiber of L at $\pi(z)$, and comparing the trivialization φ at z and $z+\lambda$ yields a linear automorphism of $\mathbb{C}$:

$$\mathbb{C} \xleftarrow{\varphi_z} (\pi^*L)_z = L_{\pi(z)} = (\pi^*L)_{z+\lambda} \xrightarrow{\varphi_{z+\lambda}} \mathbb{C}.$$

Such an automorphism is given as multiplication by a nonzero complex number; if we denote this number by $e_\lambda(z)$, we obtain a collection of functions

$$\{e_\lambda \in \mathcal{O}^*(V)\}_{\lambda \in \Lambda}$$

called a set of *multipliers* for L. The functions e_λ necessarily satisfy the compatibility relation

$$e_{\lambda'}(z+\lambda)e_\lambda(z) = e_\lambda(z+\lambda')e_{\lambda'}(z) = e_{\lambda+\lambda'}(z)$$

for all $\lambda, \lambda' \in \Lambda$.

Conversely, given any collection of entire nonzero holomorphic functions $\{e_\lambda\}_{\lambda \in \Lambda}$ satisfying these relations, we can construct a line bundle $L \to M$ having multipliers $\{e_\lambda\}$: we take L to be the quotient space of $V \times \mathbb{C}$ under the identifications

$$(z,\xi) \sim (z+\lambda, e_\lambda(z)\cdot\xi).$$

Note that by the compatibility relations, we can give such a collection $\{e_\lambda\}$ by specifying e_{λ_α} for some basis $\{\lambda_\alpha\}$ for Λ so long as the functions $\{e_{\lambda_\alpha}\}$ satisfy

$$(*) \qquad e_{\lambda_\alpha}(z+\lambda_\beta)e_{\lambda_\beta}(z) = e_{\lambda_\beta}(z+\lambda_\alpha)e_{\lambda_\alpha}(z).$$

Our aim now is to show that any line bundle $L \to M$ can be given by multipliers $\{e_\lambda(z)\}$ of a very simple character. We will do this in two stages: first, we will construct line bundles having arbitrary positive Chern class, using elementary functions e_λ; then we will show that any positive line bundle $L \to M$ is determined, up to translation in M, by its Chern class.

One simplification is immediate: If $\{\lambda_1,\ldots,\lambda_{2n}\}$ is any basis for Λ over $\mathbb{Z}$ with $\lambda_1,\ldots,\lambda_n$ linearly independent over $\mathbb{C}$, then we have

$$\frac{V}{\mathbb{Z}\{\lambda_1,\ldots,\lambda_n\}} \cong (\mathbb{C}^*)^n,$$

and we can factor our projection map $\pi: V \to M$ by

$$V \to \frac{V}{\mathbb{Z}\{\lambda_1,\ldots,\lambda_n\}} \xrightarrow{\pi_1} M.$$

Now we have also seen on p. 27 that

$$H^1((\mathbb{C}^*)^n,\mathcal{O}) = H^2((\mathbb{C}^*)^n,\mathcal{O}) = 0,$$

and hence

$$H^1((\mathbb{C}^*)^n, \mathcal{O}^*) \xrightarrow[\sim]{c_1} H^2((\mathbb{C}^*)^n, \mathbb{Z}),$$

i.e., any line bundle on $(\mathbb{C}^*)^n$ is determined by its Chern class. For any L we can choose our basis $\lambda_1,\ldots,\lambda_{2n}$ for Λ such that in terms of the dual coordinates $x_1,\ldots,x_{2n}$ on V,

$$c_1(L) = \sum_{\alpha=1}^{n} \delta_\alpha \, dx_\alpha \wedge dx_{n+\alpha}.$$

But $x_{n+\alpha}$ is a well-defined function on $V/\mathbb{Z}\{\lambda_1,\ldots,\lambda_n\}$, so $[dx_{n+\alpha}]=0\in H^1_{\mathrm{DR}}(V/\mathbb{Z}\{\lambda_1,\ldots,\lambda_n\})$. Thus

$$c_1(\pi_1^* L) = \pi_1^*(c_1(L)) = 0,$$

and consequently $\pi_1^* L$ is trivial. If we then take a trivialization $\tilde{\varphi}:\pi_1^* L \to (\mathbb{C}^*)^n \times \mathbb{C}$ and choose our trivialization φ of $\pi^* L$ to extend $\tilde{\varphi}$, we have

$$e_{\lambda_\alpha}(z) \equiv 1, \qquad \alpha = 1,\ldots,n.$$

Now suppose ω is any invariant integral form, positive of type (1,1). Choose a basis $\lambda_1,\ldots,\lambda_{2n}$ for Λ over $\mathbb{Z}$ such that in terms of dual coordinates $x_1,\ldots,x_{2n}$ on V

$$\omega = \sum_{\alpha=1}^{n} \delta_\alpha \, dx_\alpha \wedge dx_{n+\alpha}, \qquad \delta_\alpha \in \mathbb{Z}.$$

Since ω is nondegenerate, $\delta_\alpha \neq 0$ for all α, and we can set

$$e_\alpha = \delta_\alpha^{-1} \lambda_\alpha, \qquad \alpha = 1,\ldots,n;$$

let $z_1,\ldots,z_n$ be linear coordinates on V dual to the basis $e_1,\ldots,e_n$. Then as before we can write

$$(\lambda_1,\ldots,\lambda_{2n}) = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \Omega,$$

i.e.,

$$\begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix} = {}^t\Omega(dx_1,\ldots,dx_{2n})$$

with

$$\Omega = (\Delta_\delta, Z);$$

and again, by the Riemann Conditions III, ω positive of type (1,1) implies that $Z={}^tZ$, $\operatorname{Im} Z>0$. Our fundamental calculation is the

Lemma. *The line bundle* $L\to M$ *given by multipliers*

$$e_{\lambda_\alpha}\equiv 1, \qquad e_{\lambda_{n+\alpha}}(z)=e^{-2\pi i z_\alpha}, \qquad \alpha=1,\ldots,n,$$

has Chern class $c_1(L)=[\omega]$.

Proof. We first check that the multipliers given do indeed satisfy the relations (*) above. Clearly (*) is satisfied for α or $\beta\leqslant n$; and writing $Z=(Z_{\alpha\beta})$, we have

$$\begin{aligned} e_{\lambda_{n+\beta}}(z+\lambda_{n+\alpha})\cdot e_{\lambda_{n+\alpha}}(z) &= e^{-2\pi i(z_\beta+Z_{\beta\alpha}+z_\alpha)} \\ &= e^{-2\pi i(z_\alpha+Z_{\alpha\beta}+z_\beta)} \\ &= e_{\lambda_{n+\alpha}}(z+\lambda_{n+\beta})\cdot e_{\lambda_{n+\beta}}(z) \end{aligned}$$

as required.

Now let $\varphi:\pi^*L\to V\times\mathbb{C}$ be a trivialization of π^*L inducing the multipliers given. Then for any section $\tilde\theta$ of L over $U\subset M$, $\theta=\varphi^*(\pi^*\tilde\theta)$ is an analytic function on $\pi^{-1}(U)$ satisfying

$$\begin{aligned} \theta(z+\lambda_\alpha) &= \theta(z), \\ \theta(z+\lambda_{n+\alpha}) &= e^{-2\pi i z_\alpha}\theta(z), \end{aligned}$$

and conversely any such function defines a section of L. Now if $\|\ \ \|$ is any metric on L, we can write

$$\|\tilde\theta(z)\|^2 = h(z)\cdot|\theta(z)|^2$$

for any section $\tilde\theta$ of L; evidently h will be a positive C^∞ function of z satisfying

$$h(z)|\theta(z)|^2=\|\tilde\theta(z)\|^2=h(z+\lambda)|\theta(z+\lambda)|^2$$

for any $\lambda\in\Lambda$; thus

$$\begin{aligned} h(z+\lambda_\alpha) &= h(z), \\ h(z+\lambda_{n+\alpha}) &= |e^{2\pi i z_\alpha}|^2 h(z). \end{aligned}$$

Conversely, any such function h defines a metric on L. Now write $Z=X+\sqrt{-1}\,Y$ as before; since $Y>0$, we can set $W=(W_{\alpha\beta})=Y^{-1}$. Then we claim that the function

$$h(z)=e^{(\pi/2)\sum W_{\alpha\beta}(z_\alpha-\bar z_\alpha)(z_\beta-\bar z_\beta-2iY_{\beta\beta})}$$

satisfies the functional equations above. Clearly $h(z+\lambda_\alpha)=h(z)$; for the others, write

$$\log h(z+\lambda_{n+\gamma}) = \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha + 2iY_{\alpha\gamma})(z_\beta - \bar{z}_\beta + 2i(Y_{\beta\gamma} - Y_{\beta\beta}))$$

$$= \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta - 2iY_{\beta\beta}) + \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)\cdot 2iY_{\beta\gamma}$$

$$+ \frac{\pi}{2}\sum_{\alpha,\beta} W_{\alpha\beta}\cdot 2iY_{\alpha\gamma}(z_\beta - \bar{z}_\beta + 2i(Y_{\beta\gamma} - Y_{\beta\beta}))$$

$$= \log h(z) + \frac{\pi}{2}\sum_{\alpha}\delta_{\alpha\gamma}\cdot 2i(z_\alpha - \bar{z}_\alpha) + \frac{\pi}{2}\sum_{\beta}\delta_{\beta\gamma}\cdot 2i(z_\beta - \bar{z}_\beta + 2i(Y_{\beta\gamma} - Y_{\beta\beta}))$$

(since $Y\cdot W = I$ and $W = {}^tW$)

$$= \log h(z) + \pi i(z_\gamma - \bar{z}_\gamma) + \pi i(z_\gamma - \bar{z}_\gamma)$$
$$= \log h(z) - 4\pi\,\mathrm{Im}(z_\gamma);$$

hence

$$h(z+\lambda_{n+\gamma}) = |e^{2\pi i z_\gamma}|^2 h(z).$$

Now we can compute the curvature form Θ_L associated to the metric in L given by h:

$$\Theta_L = \partial\bar{\partial}\log\frac{1}{h}$$
$$= -\frac{\pi}{2}\partial\bar{\partial}\Big(\sum_{\alpha,\beta} W_{\alpha\beta}(z_\alpha - \bar{z}_\alpha)(z_\beta - \bar{z}_\beta - 2iY_{\beta\beta})\Big)$$
$$= \frac{\pi}{2}\partial\sum_{\alpha,\beta} W_{\alpha\beta}((z_\alpha - \bar{z}_\alpha)\,d\bar{z}_\beta + (z_\beta - \bar{z}_\beta - 2iY_{\beta\beta})\,d\bar{z}_\alpha)$$
$$= \pi\sum_{\alpha,\beta} W_{\alpha\beta}\,dz_\alpha \wedge d\bar{z}_\beta.$$

We want to express this in terms of the basis $\{dx_\alpha, dx_{n+\alpha}\}$; we have

$$dz_\alpha = \delta_\alpha\,dx_\alpha + \sum_\beta z_{\alpha\beta}\,dx_{n+\beta},$$
$$d\bar{z}_\alpha = \delta_\alpha\,dx_\alpha + \sum_\beta \overline{z_{\alpha\beta}}\,dx_{n+\beta},$$

so

$$\Theta_L = \pi\sum W_{\alpha\beta}\,dz_\alpha \wedge d\bar{z}_\beta$$
$$= \pi\sum_{\alpha,\beta} W_{\alpha\beta}\delta_\alpha\delta_\beta\,dx_\alpha \wedge dx_\beta$$
$$+ \pi\sum_{\alpha,\beta,\gamma} W_{\alpha\beta}\delta_\alpha(\bar{Z}_{\beta\gamma} - Z_{\beta\gamma})\,dx_\alpha \wedge dx_{n+\gamma}$$
$$+ \pi\sum_{\alpha,\beta,\gamma,\varepsilon} W_{\alpha\beta}Z_{\alpha\gamma}\bar{Z}_{\beta\varepsilon}\,dx_{n+\gamma} \wedge dx_{n+\varepsilon}.$$

Since $W = {}^t W$ and $Z = {}^t Z$, the first and last of these three terms are zero, and hence

$$\begin{aligned}\Theta &= \pi \sum_{\alpha,\beta,\gamma} \delta_\alpha W_{\alpha\beta}\left(\bar{Z}_{\beta\gamma} - Z_{\beta\gamma}\right) dx_\alpha \wedge dx_{n+\gamma} \\ &= -2\pi\sqrt{-1} \sum_{\alpha,\beta,\gamma} \delta_\alpha W_{\alpha\beta} Y_{\beta\gamma}\, dx_\alpha \wedge dx_{n+\gamma} \\ &= -2\pi\sqrt{-1} \sum_\alpha \delta_\alpha\, dx_\alpha \wedge dx_{n+\alpha},\end{aligned}$$

and so finally

$$c_1(L) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right] = [\omega]. \qquad \text{Q.E.D.}$$

To continue our description of line bundles on M we want to consider the set of line bundles $L \to M$ having a given positive Chern class. We note that for any $\mu \in M$ the translation $\tau_\mu : M \to M$ is homotopic to the identity and hence for any line bundle $L \to M$,

$$c_1(\tau_\mu^* L) = c_1(L).$$

Note, moreover, that if L is given by multipliers

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i z_\alpha},$$

then $\tau_\mu^* L$ can be given by multipliers

$$\begin{aligned} e'_{\lambda_\alpha}(z) &= e_{\lambda_\alpha}(z+\mu) \equiv 1 \\ e'_{\lambda_{n+\alpha}}(z) &= e_{\lambda_{n+\alpha}}(z+\mu) \\ &= e^{-2\pi i (z_\alpha + \mu_\alpha)},\end{aligned}$$

i.e., e'_λ will differ from e_λ by multiplication by a constant $e^{-2\pi i \mu_\alpha}$. Conversely, if L' is any line bundle with multipliers $e'_{\lambda_\alpha} \equiv 1$ and $e'_{\lambda_{n+\alpha}} \equiv c_\alpha \cdot e_{\lambda_{n+\alpha}}$, $c_\alpha \in \mathbb{C}^*$, then, setting

$$\mu = \sum \frac{\sqrt{-1}}{2\pi} \log c_\alpha \cdot e_\alpha \in V,$$

we have

$$L' = \tau_\mu^* L.$$

Thus, to prove that any line bundle having the same Chern class as L must be a translate of L, it will suffice to show that any line bundle with Chern class 0 can be realized by constant multipliers. To this end we first note that the inclusion of exact sheaf sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\exp} & \mathcal{O}^* & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^* & \longrightarrow & 0 \end{array}$$

on any compact Kähler manifold X induces a commutative diagram

$$\begin{array}{ccccc} H^1(X,\mathcal{O}) & \longrightarrow & H^1(X,\mathcal{O}^*) & \xrightarrow{c_1} & H^2(X,\mathbb{Z}) \\ \uparrow \iota_1^* & & \uparrow \iota_2^* & & \| \\ H^1(X,\mathbb{C}) & \longrightarrow & H^1(X,\mathbb{C}^*) & \longrightarrow & H^2(X,\mathbb{Z}). \end{array}$$

The map ι_1^* represents projection of $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ on the second factor, and so is surjective. It follows that any cocycle $\gamma \in H^1(X,\mathcal{O}^*)$ in the kernel of c_1 is in the image of ι_2^*, i.e., is cohomologous to a cocycle with constant coefficients; thus *any line bundle on* X *with Chern class* 0 *can be given by constant transition functions.*

Now if $L \to M = V/\Lambda$ is any line bundle with trivial Chern class, we can find an open cover $\underline{U} = \{U_\alpha\}$ of M such that for each $\alpha, \pi^{-1}(U_\alpha) = \{U_{\alpha j}\}_j$ is a disjoint collection of open sets isomorphic via π to U_α, and a collection of trivializations $\varphi_\alpha : L_{U_\alpha} \to U_\alpha \times \mathbb{C}$ *having constant transition functions* $\{g_{\alpha j}\}$. *We can then define constants* $\{h_{\alpha j}\}_{\alpha,j}$ *by taking* $h_{\alpha_0, j_0} \equiv 1$ *for some* α_0, j_0 *and setting*

$$h_{\alpha j} = h_{\alpha' j'} \cdot g_{\alpha\alpha'} \qquad \text{for } \alpha, j, \alpha', j' \quad \text{such that} \quad U_{\alpha j} \cap U_{\alpha' j'} \neq \emptyset.$$

It is not hard to see that by the cocycle rule on $\{g_{\alpha\alpha'}\}$ this is well-defined, and the trivializations

$$\varphi_{\alpha j} : \pi^* L_{U_{\alpha j}} \to U_{\alpha j} \times \mathbb{C}$$

defined by

$$\varphi_{\alpha j} = h_{\alpha j} \cdot \pi^* \varphi_\alpha$$

patch together to give a trivialization of $\pi^* L$ having constant multipliers.

To shed some light on this argument, we will describe in more detail the geometry of the group of line bundles on M. Recall that by the exponential sheaf sequence

$$H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z})$$

the group $\operatorname{Pic}^0(M)$ of holomorphic line bundles on M with Chern class zero is given by

$$\operatorname{Pic}^0(M) = \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})}.$$

Now, $H^1(M,\mathcal{O}) = \mathcal{H}^{0,1}(M)$ is the space of invariant forms of type (0, 1) on M, i.e., the space $\overline{V}^* = \operatorname{Hom}_{\mathbb{C}}(\overline{V},\mathbb{C})$ of conjugate linear functionals on V. $H^1(M,\mathbb{Z})$, on the other hand, is the space of real invariant 1-forms ω on M having integral periods, that is, the space of real linear functionals on V taking integral values on $\Lambda \subset V$. The map

$$H^1(M,\mathbb{Z}) \to H^1(M,\mathcal{O})$$

is given simply by

$$\omega \mapsto \omega^{0,1};$$

since for ω real

$$\int_\lambda \omega = \int_\lambda \omega^{1,0} + \int_\lambda \omega^{0,1}$$
$$= \left(\int_\lambda \omega^{0,1}\right) + \overline{\left(\int_\lambda \omega^{0,1}\right)}$$

is integral for all $\lambda \in \Lambda$ if and only if

$$2\,\mathrm{Re} \int_\lambda \omega^{0,1} \in \mathbb{Z}$$

for all $\lambda \in \Lambda$, we see that *the image* $\overline{\Lambda}^*$ *of* $H^1(M,\mathbb{Z})$ *in* $H^1(M,\mathcal{O}) = \overline{V}^*$ *consists exactly of conjugate linear functionals on* V *whose real part is half-integral on* $\Lambda \subset V$. Thus $\mathrm{Pic}^0(M)$ is again a complex torus, often called the *dual Abelian varitey* of M and denoted $\hat{M}$.

Explicitly, if $x_1,\ldots,x_{2n}$ are, as above, real coordinates on V dual to the basis $\lambda_1,\ldots,\lambda_{2n}$, and we let x_i^* denote the conjugate linear part of the real functional x_i, then the x_i^* form a basis for Λ^*. Writing

$$x_i = \sum \pi_{i\alpha} z_\alpha + \sum \overline{\pi}_{i\alpha} \overline{z}_\alpha,$$

we have

$$x_i^* = \sum \overline{\pi}_{i\alpha} \overline{z}_\alpha,$$

where, as we found above

$$\Pi = \begin{pmatrix} \dfrac{\sqrt{-1}}{2} \Delta_\delta^{-1} \overline{Z} Y^{-1} \\ \dfrac{1}{2\sqrt{-1}} Y^{-1} \end{pmatrix}.$$

Reordering our basis $\{x^*\}$ for Λ^* by setting

$$y_\alpha^* = -x_{n+\alpha}^*, \qquad y_{n+\alpha}^* = x_\alpha^*,$$

we see that

$$(y_{n+1}^*,\ldots,y_{2n}^*) = -\frac{\sqrt{-1}}{2} \Delta_\delta^{-1} Z Y^{-1} (\overline{z}_1,\ldots,\overline{z}_n)$$
$$= \Delta_\delta^{-1} Z (y_1^*,\ldots,y_n^*).$$

Consequently, if we order the elementary divisors δ_α so that $\delta_1 | \delta_2 | \cdots | \delta_n$, we may set

$$e_\alpha^* = \frac{\delta_\alpha}{\delta_n} y_\alpha^*;$$

we then have

$$(y_1^*,\ldots,y_n^*) = \delta_n\Delta_\delta^{-1}(e_1^*,\ldots,e_n^*)$$

and

$$(y_{n+1}^*,\ldots,y_{2n}^*) = \delta_n\Delta_\delta^{-1}Z\Delta_\delta^{-1}(e_1^*,\ldots,e_n^*).$$

The period matrix of $\hat{M}$ in terms of the bases (y_α^*) for Λ^* and (e_α^*) for $\bar{V}^*$ is then

$$\Omega^* = \left(\delta_n\Delta_\delta^{-1},\delta_n\Delta_\delta^{-1}Z\Delta_\delta^{-1}\right).$$

Since $\delta_\alpha|\delta_n$ for all α, $\delta_n\Delta_\delta^{-1}$ is again diagonal and integral; and since

$$\begin{aligned}{}^t\left(\delta_n\Delta_\delta^{-1}Z\Delta_\delta^{-1}\right) &= \delta_n^t\Delta_\delta^{-1t}Z^t\Delta_\delta^{-1}\\ &= \delta_n\Delta_\delta^{-1}Z\Delta_\delta^{-1}\end{aligned}$$

and

$$\operatorname{Im}\left(\delta_n\Delta_\delta^{-1}Z\Delta_\delta^{-1}\right) = \delta_n\Delta_\delta^{-1}Y\Delta_\delta^{-1}$$

is again positive definite, we see that $\hat{M}$ is an Abelian variety; indeed, the original polarization on M induces a polarization on $\hat{M}$ with "dual" elementary divisors $\{\delta_n/\delta_\alpha\}$.

Now let L be a positive line bundle on M. We can define a map

$$\varphi_L : M \to \operatorname{Pic}^0(M)$$

by

$$\varphi_L(\mu) = L^{-1}\otimes\tau_\mu^*L.$$

We want to describe φ_L explicitly in terms of the bases $\{\lambda_\alpha\}$ for Λ and $\{e_\alpha\}$ for V normalized with respect to L, and the dual bases $\{y_\alpha^*\}$ for Λ^* and $\{e_\alpha^*\}$ for $\bar{V}^*$.

First, we trace out the map

$$H_{\bar\partial}^{0,1}(M)\xrightarrow{\delta}H^1(M,\mathcal{O})\to\operatorname{Pic}^0(M),$$

where δ is the Dolbeault isomorphism. If

$$\sigma = \sum\sigma_\alpha\,d\bar{z}_\alpha,\qquad \sigma_\alpha\in\mathbb{C},$$

is a constant $(0,1)$-form on M, then in each open set U_i of a sufficiently fine open cover we may write

$$\sigma = \bar\partial f_i(z),$$

where

$$f_i(z) = \sum\sigma_\alpha\bar{z}_\alpha$$

for a suitable choice of branch of z_α. The line bundle associated to σ thus has transition functions

$$g_{ij}(z) = e^{2\pi i(f_i(z)-f_j(z))}$$

and, correspondingly, in terms of a suitable trivialization, multipliers

$$e_{\lambda_\alpha}(z) = e^{-2\pi i \delta_\alpha \sigma_\alpha}, \qquad e_{\lambda_{n+\alpha}} = e^{-2\pi i \sum \sigma_\beta \bar{Z}_{\alpha\beta}}.$$

Multiplying the trivializations by the function

$$f(z) = e^{2\pi i \sum \sigma_\alpha z_\alpha},$$

yields the normalized multipliers

$$\begin{aligned} e_{\lambda_\alpha}(z) &= 1, \\ e_{\lambda_{n+\alpha}}(z) &= e^{-2\pi i \sum \sigma_\beta (\bar{Z}_{\alpha\beta} - Z_{\alpha\beta})} \\ &= e^{-4\pi \sum \sigma_\beta Y_{\alpha\beta}}, \end{aligned}$$

where $Y = \operatorname{Im} Z$ as above. In terms of the coordinates x_α^* on V, we see that the line bundle associated to

$$\sum_{\alpha=1}^{n} c_\alpha x_\alpha^* = \frac{\sqrt{-1}}{2} \sum \delta_\alpha^{-1} c_\alpha Z_{\alpha\beta} Y_{\beta\gamma}^{-1} dz_\gamma$$

has multipliers

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\beta}}(z) = e^{-2\pi i \sum \delta_\alpha^{-1} c_\alpha Z_{\alpha\beta}}$$

and, likewise, the bundle corresponding to

$$\sum c_\alpha x_{n+\alpha}^* = \frac{1}{2\sqrt{-1}} \sum c_\alpha Y_{\alpha\beta}^{-1} d\bar{z}_\beta$$

has multipliers

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\beta}}(z) = e^{2\pi i c_\alpha}.$$

On the other hand, since the line bundle L is given by

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\alpha}} = e^{-2\pi i z_\alpha},$$

for any $\mu = \sum \mu_\alpha e_\alpha \in V$, $\tau_\mu^* L$ is given by $e_{\lambda_\alpha} \equiv 1$ and

$$e_{\lambda_{n+\alpha}} = e^{-2\pi i (z_\alpha + \mu_\alpha)};$$

thus $\varphi(L) = L^{-1} \otimes \tau_\mu^* L$ has multipliers

$$e_{\lambda_\alpha} \equiv 1 \quad \text{and} \quad e_{\lambda_{n+\alpha}} = e^{-2\pi i \mu_\alpha}.$$

In particular, if $\mu = \Sigma c_\alpha \lambda_\alpha = \Sigma c_\alpha \delta_\alpha e_\alpha$, we see that $\varphi_L(\Sigma c_\alpha \lambda_\alpha)$ has multipliers

$$e_{\lambda_\alpha} \equiv 1 \quad \text{and} \quad e_{\lambda_{n+\beta}} = e^{-2\pi i \delta_\alpha c_\alpha},$$

i.e.,

$$\varphi_L\left(\sum c_\alpha \lambda_\alpha\right) = -\sum c_\alpha \delta_\alpha x_{n+\alpha}^* = \sum c_\alpha \delta_\alpha y_\alpha^*,$$

and if $v = \Sigma c_\alpha \lambda_{n+\alpha} = \Sigma c_\alpha Z_{\alpha\beta} e_\beta$, then $\varphi_L(\Sigma c_\alpha \lambda_{n+\alpha})$ has multipliers

$$e_{\lambda\alpha} \equiv 1 \quad \text{and} \quad e_{\lambda_{n+\beta}} = e^{-2\pi i Z_{\alpha\beta}},$$

i.e.,

$$\varphi_L\left(\sum c_\alpha \lambda_{n+\alpha}\right) = \sum \delta_\alpha c_\alpha x_\alpha^* = \sum c_\alpha \delta_\alpha y_{n+\alpha}^*.$$

(These last two assertions are equivalent, since φ_L is complex linear and so

$$\begin{aligned}\varphi_L\left(\sum c_\alpha \lambda_{n+\alpha}\right) &= \varphi_L\left(\sum c_\alpha Z_{\alpha\beta}\delta_\beta^{-1}\lambda_\beta\right)\\ &= \sum c_\alpha Z_{\alpha\beta} y_\beta^*\\ &= \sum c_\alpha Z_{\alpha\beta}(\delta_n/\delta_\beta)e_\beta^*\\ &= \sum c_\alpha \delta_\alpha \delta_n\left(\delta_\alpha^{-1} Z_{\alpha\beta}\delta_\beta^{-1} e_\beta^*\right)\\ &= \sum c_\alpha \delta_\alpha y_{n+\alpha}^*).\end{aligned}$$

In any event, we see clearly from this that the kernel of φ_L is exactly the subgroup of M generated by $\{\delta_\alpha^{-1}\lambda_\alpha, \delta_\alpha^{-1}\lambda_{n+\alpha}\}$; i.e., that *the line bundle* L *is fixed under exactly the* $\Pi\delta_\alpha^2$ *translations*

$$\left\{\tau_v:\ v \in \mathbb{Z}\left\{\delta_\alpha^{-1}\lambda_\alpha, \delta_\alpha^{-1}\lambda_{n+\alpha}\right\}\right\}.$$

Theta-Functions

Having described a positive line bundle $L \to M$ on an Abelian variety $M = V/\Lambda$ as a quotient of the trivial bundle $V \times \mathbb{C}$, we can accordingly realize global holomorphic sections of L as entire holomorphic functions on $V \cong \mathbb{C}^n$ satisfying certain functional equations. These functions are called *theta-functions*, and by examining them we shall prove the

Theorem. *Let* $\mathrm{L} \to \mathrm{M}$ *be any positive line bundle, and let* $\delta_1, \ldots, \delta_n$ *be the elementary divisors of the polarization* $c_1(\mathrm{L})$ *of* M. *Then*

1. $\dim H^0(M, \mathcal{O}(L)) = \Pi_\alpha \delta_\alpha$.
2. $H^0(M, \mathcal{O}(L^k))$ *has no base points for* $k \geq 2$ *and gives an embedding of* M *for* $k \geq 3$.

Before proving this theorem, we make a few remarks. First, since $K_M = 0$, we have by the Kodaira vanishing theorem

$$h^p(M, \mathcal{O}(L)) = h^p(M, \Omega^n(L)) = 0, \qquad p > 0,$$

and hence

$$h^0(M, \mathcal{O}(L)) = \chi(L).$$

On the other hand, we can find an integral basis $\{dx_1, \ldots, dx_{2n}\}$ for $H^1(M, \mathbb{Z})$ such that

$$c_1(L) = \sum \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}$$

and so

$$c_1(L)^n = n!\Pi\delta_\alpha \in H^{2n}(M, \mathbb{Z}) \cong \mathbb{Z}.$$

Thus assertion 1 may be thought of as a special case of the general Riemann-Roch formula, expressing the holomorphic Euler characteristic of a line bundle in terms of its topological invariants.

Assertion 2, due to Lefschetz, is of a deeper character, and will emerge later.

To prove the first statement, choose $\{\lambda_1,\ldots,\lambda_{2n}\}$ an integral basis for Λ such that in terms of dual coordinates $x_1,\ldots,x_{2n}$,

$$c_1(L) = \sum \delta_\alpha dx_\alpha \wedge dx_{n+\alpha}.$$

As before, set

$$e_\alpha = \delta_\alpha^{-1}\lambda_\alpha$$

and let $z_1,\ldots,z_n$ be the corresponding complex coordinates on V, so that the period matrix Ω of $\Lambda \subset V$ is of the form

$$\Omega = (\Delta_\delta, Z)$$

with $Z = X + \sqrt{-1}\ Y$ symmetric and $Y > 0$.

Now we have seen that the line bundle L is a translate of the bundle L_0 given by multipliers

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\alpha}}(z) = e^{-2\pi i z_\alpha}.$$

Since $h^0(L)$ is clearly invariant under translation, we will prove assertion 1 for $L = \tau_\mu^* L_0$, where

$$\mu = \tfrac{1}{2} \sum Z_{\alpha\alpha} \cdot e_\alpha$$

Multipliers for L are thus

$$e_{\lambda_\alpha} \equiv 1, \qquad e_{\lambda_{n+\alpha}} = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}},$$

and so global sections $\tilde{\theta}$ of L are given by entire holomorphic functions θ on V satisfying

$$\theta(z+\lambda_\alpha) = \theta(z), \qquad \theta(z+\lambda_{n+\alpha}) = e^{-2\pi i z_\alpha - \pi i Z_{\alpha\alpha}} \cdot \theta(z).$$

By the first condition, such a function θ must have a power series expansion in the variables $z_\alpha^* = e^{2\pi i \delta_\alpha^{-1} z_\alpha}$; we can write

$$\begin{aligned} \theta(z) &= \sum_{l \in \mathbb{Z}^n} a_l \cdot z_1^{*l_1} \cdots z_n^{*l_n} \\ (*) \qquad &= \sum_{l \in \mathbb{Z}^n} a_l \cdot e^{2\pi i \sum_a l_\alpha \delta_\alpha^{-1} z_\alpha} \\ &= \sum_{l \in \mathbb{Z}^n} a_l \cdot e^{2\pi i \langle l, \Delta_\delta^{-1} z \rangle}. \end{aligned}$$

Now the second set of conditions gives us recursive relations among the coefficients a_l of θ; to begin with

$$\begin{aligned}\theta(z+\lambda_{n+\alpha}) &= \sum_{l\in\mathbb{Z}^n} a_l\cdot e^{2\pi i\langle l,\Delta_\delta^{-1}(z+\lambda_{n+\alpha})\rangle}\\ &= \sum_{l\in\mathbb{Z}^n} a_l\cdot e^{2\pi i\langle l,\Delta_\delta^{-1}\lambda_{n+\alpha}\rangle}\cdot e^{2\pi i\langle l,\Delta_\delta^{-1}z\rangle}.\end{aligned}$$

But the second condition above asserts that

$$\begin{aligned}\theta(z+\lambda_{n+\alpha}) &= e^{-2\pi i z_\alpha-\pi i Z_{\alpha\alpha}}\cdot\theta(z)\\ &= e^{-2\pi i z_\alpha-\pi i Z_{\alpha\alpha}}\sum_{l\in\mathbb{Z}^n} a_l e^{2\pi i\langle l,\Delta_\delta^{-1}z\rangle}\\ &= \sum_{l\in\mathbb{Z}^n} a_{l+\Delta_\delta e_\alpha}\cdot e^{-\pi i Z_{\alpha\alpha}}\cdot e^{2\pi i\langle l,\Delta_\delta^{-1}z\rangle}.\end{aligned}$$

Comparing these two Fourier expansions for $\theta(z+\lambda_{n+\alpha})$, we obtain

$$a_{l+\delta_\alpha e_\alpha} = e^{2\pi i\langle l,\Delta_\delta^{-1}\lambda_{n+\alpha}\rangle+\pi i Z_{\alpha\alpha}}\cdot a_l.$$

Thus θ is completely determined by the choice of coefficients

$$\{a_l\}_{l:\ 0\leqslant l_\alpha<\delta_\alpha},$$

and accordingly we have

$$h^0(M,\mathcal{O}(L))\leqslant \Pi\delta_\alpha.$$

To prove equality, we have to show that the series (*) determined by an arbitrary choice of coefficients $\{a_l\}_{l:0\leqslant l_\alpha<\delta_\alpha}$ does in fact converge. Now we can write

$$\begin{aligned}\theta(z) &= \sum_{0\leqslant l_{0_\alpha}<\delta\alpha}\left(\sum_{l\in\mathbb{Z}^n} a_{l_0+\Delta_\delta l}\cdot e^{2\pi i\langle l_0+\Delta_\delta l,\Delta_\delta^{-1}z\rangle}\right)\\ &= \sum_{0\leqslant l_{0_\alpha}<\delta_\alpha} e^{2\pi i\langle l_0,\Delta_\delta^{-1}z\rangle}\cdot\left(\sum_{l\in\mathbb{Z}^n} a_{l_0+\Delta_\delta l}\cdot e^{2\pi i\langle l,z\rangle}\right).\end{aligned}$$

Let

$$(**)\qquad \theta_{l_0}(z) = e^{2\pi i\langle l_0,\Delta_\delta^{-1}z\rangle}\sum_{l\in\mathbb{Z}^n} a_{l_0+\Delta_\delta l}\cdot e^{2\pi i\langle l,z\rangle}$$

be the series determined by the choice $a_{l_0}=1$ and the recursion relations above; by the linearity of these relations we see that the general theta-function is of the form

$$\theta(z) = \sum_{0\leqslant l_{0_\alpha}<\delta_\alpha} a_{l_0}\theta_{l_0}(z),$$

and so it will suffice to prove the series (**) converges.

For convenience, set $b_l = a_{l_0+\Delta_\delta l}$; the recursion relations then read

$$\begin{aligned} b_{l+e_\alpha} &= a_{l_0+\Delta_\delta l+\Delta_\delta e_\alpha} \\ &= e^{2\pi i\langle (l_0+\Delta_\delta l),\Delta_\delta^{-1}\lambda_{n+\alpha}\rangle+\pi i Z_{\alpha\alpha}}\cdot a_{l_0+\Delta_\delta l} \\ &= e^{2\pi i\langle l,\lambda_{n+\alpha}\rangle+2\pi i\langle l_0,\Delta_\delta^{-1}\lambda_{n+\alpha}\rangle+\pi i Z_{\alpha\alpha}}. \end{aligned}$$

We can solve these relations by setting

$$b_l = e^{\pi i\langle l,Zl\rangle+2\pi i\langle \Delta_\delta^{-1}l_0,Zl\rangle};$$

to verify this, we have

$$\begin{aligned} b_{l+e_\alpha} &= e^{\pi i\langle (l+e_\alpha),Z(l+e_\alpha)\rangle+2\pi i\langle \Delta_\delta^{-1}l_0,Z(l+e_\alpha)\rangle} \\ &= e^{\pi i\langle l,Zl\rangle+2\pi i\langle l,Ze_\alpha\rangle+\pi i\langle e_\alpha,Ze_\alpha\rangle+2\pi i\langle\Delta_\delta^{-1}l_0,Zl\rangle+2\pi i\langle\Delta_\delta^{-1}l_0,Ze_\lambda\rangle} \\ &= e^{2\pi i\langle l,\lambda_{n+\alpha}\rangle+\pi i Z_{\alpha\alpha}+2\pi i\langle\Delta_\delta^{-1}l_0,\lambda_{n+\alpha}\rangle}b_l, \end{aligned}$$

since $Z={}^tZ$, and $Ze_\alpha=\lambda_{n+\alpha}$. Thus the b_l given are indeed the solutions to the recursion relations.

Now

$$|b_l| = e^{-\pi\langle l,Yl\rangle-2\pi\langle\Delta_\delta^{-1}l_0,Yl\rangle},$$

where $Y=\operatorname{Im} Z$ as above. But Y is positive definite, and so

$$\langle l,Yl\rangle > c'\cdot\|l\|^2$$

for some constant $c'>0$. Also, clearly

$$|\langle\Delta_\delta^{-1}l_0,Yl\rangle| < c''\cdot\|l\|$$

for some constant c'', and so for some constant $c>0$ we have

$$|b_l| < e^{-c\|l\|^2}$$

for l sufficiently large. Thus the series (**) converges uniformly on compact sets in $\mathbb{C}^n$, and we are done.

Note that in particular if $c_1(L)$ is a principal polarization of M, $H^0(M,\mathcal{O}(L))$ is one-dimensional and is generated by the section $\tilde{\theta}$ corresponding to the function

$$\theta(z) = \sum_{l\in\mathbb{Z}^n} e^{\pi i\langle l,Zl\rangle}\cdot e^{2\pi i\langle l,z\rangle},$$

which satisfies the functional equations

$$\begin{aligned} \theta(z+e_\alpha) &= \theta(z), \\ \theta(z+\lambda_{n+\alpha}) &= e^{-2\pi i(z_\alpha+Z_{\alpha\alpha}/2)}\cdot\theta(z), \end{aligned}$$

and

$$\theta(z) = \theta(-z).$$

This beautiful entire function is called the *Riemann θ-function* of the principally polarized Abelian variety $(M,[\omega])$.

Note also that since $h^0(M,\mathcal{O}(L))=1$, the divisor $\Theta=[\tilde{\theta}]$ is uniquely determined by L and hence determined up to translation by the cohomology class $[\omega]$; Θ is called the *Riemann theta-divisor* of the polarized Abelian variety $(M,[\omega])$.

It may help in understanding the last result to consider the following configuration: let Λ, λ, x, e, z, δ, and Z be as above. Let $\Lambda'\subset V$ be the lattice generated by the vectors

$$\lambda'_\alpha = \delta_\alpha^{-1}\lambda_\alpha, \qquad \lambda'_{n+\alpha} = \lambda_{n+\alpha};$$

set $M'=V/\Lambda'$. Since Λ is a sublattice of index $\Delta=\prod\delta_\alpha$ in Λ', the projection map

$$\pi': M \to M'$$

expresses M as a Δ-sheeted covering of M', the deck transformations being just the translations $\{\tau_\mu\}_{\mu\in\Lambda'/\Lambda}$.

But now the period matrix for $\Lambda'\subset V$ in terms of the bases $\{\lambda'_i\}$ and $\{e_\alpha\}$ is just

$$\Omega' = (I,Z).$$

Consequently if $x'_1,\ldots,x'_{2n}$ are real coordinates dual to $\{\lambda'_i\}$, the class

$$[\omega] = \left[\sum dx'_\alpha\wedge dx'_{n+\alpha}\right] = \left[\sum \delta_\alpha dx_\alpha\wedge dx_{n+\alpha}\right]$$

is a principal polarization of M. Since L is determined up to translation by its Chern class $c_1(L)=[\omega]$, it follows that we can find a line bundle $L'\to M'$ such that $\pi'^*L'=L$. Summarizing, *if* $\mathrm{L}\to\mathrm{M}=\mathrm{V}/\Lambda$ *is any positive line bundle on an Abelian variety, we can find an Abelian variety* M′ *with principally polarizing line bundle* $\mathrm{L}'\to\mathrm{M}'$ *and a finite map* $\pi':\mathrm{M}\to\mathrm{M}'$ *such that* $\pi'^*\mathrm{L}'=\mathrm{L}$.

It is fairly clear that the Δ sections $\tilde{\theta}_{l_0}$ corresponding to the theta-functions θ_{l_0} defined in the proof of assertion 1 are, up to multiplication, all translates of one another by the deck transformations of $\pi':M\to M'$, and that the generator $\tilde{\theta}$ of $H^0(M',\mathcal{O}(L'))$ is given by

$$\pi'^*\tilde{\theta} = \sum_{\lambda\in\Lambda'/\Lambda} \tau_\lambda^*\tilde{\theta}$$

We now prove assertion 2 for a line bundle L with $c_1(L)$ a principal polarization; the idea is to use the group law on the torus. Let $L\to M$ be principally polarized and normalize everything as in the last paragraph. We have

$$H^0(M,\mathcal{O}(L^k)) = \left\{\theta\in\mathcal{O}(V):\ \theta(z+e_\alpha)=\theta(z),\ \theta(z+\lambda_{n+\alpha})=e^{-2k\pi i(z_\alpha+Z_{\alpha\alpha}/2)}\theta(z)\right\}.$$

In particular, if θ is the Riemann theta-function for (M,L), then for any

$\mu \in M$

$$\Theta_\mu(z) = \theta(z+\mu)\theta(z-\mu) \in H^0(M, \mathcal{O}(L^2)).$$

Now if $z^* \in M$, we can find $\mu \in M$ such that $\theta(z^*+\mu) \neq 0$ and $\theta(z^*-\mu) \neq 0$; i.e., $\Theta_\mu(z^*) \neq 0$. Thus the linear system $|L^2|$ has no base points, and hence it gives a map $\iota_{L^2}: M \to \mathbb{P}^N$.

To see that the map $\iota_{L^3}: M \to \mathbb{P}^N$ given by the line bundle L^3 is an embedding, let $\theta_0, \dots, \theta_N$ be a basis for $H^0(M, \mathcal{O}(L^3))$, and set

$$\mathcal{J}(z) = \begin{bmatrix} \theta_0(z) & \cdots & \theta_N(z) \\ \dfrac{\partial \theta_0}{\partial z_1}(z) & \cdots & \dfrac{\partial \theta_N}{\partial z_1}(z) \\ \vdots & & \vdots \\ \dfrac{\partial \theta_0}{\partial z_n}(z) & \cdots & \dfrac{\partial \theta_N}{\partial z_n}(z) \end{bmatrix}.$$

We will show first that the rank of $\mathcal{J}(z)$ is $n+1$, and hence that ι_{L^3} is an immersion. Let $\theta(z)$ be the Riemann θ-function and set

$$\Theta(z, \mu, \nu) = \theta(z+\mu)\theta(z+\nu)\theta(z-\mu-\nu).$$

Θ is a holomorphic function of the three variables z, μ, and ν; for fixed μ and ν, $\Theta_{\mu,\nu}(z) = \Theta(z, \mu, \nu)$ is a global section of L^3. Thus we can write

$$\Theta(z, \mu, \nu) = c_0(\mu, \nu) \cdot \theta_0(z) + \cdots + c_N(\mu, \nu) \cdot \theta_N(z)$$

with c_i well-defined and holomorphic in μ, ν.

Now assume that $\mathcal{J}(z^*)$ has rank $< n+1$ for some $z^* \in M$, i.e., that

$$a_0 \theta_i(z^*) = a_1 \frac{\partial \theta_i}{\partial z_1}(z^*) + \cdots + a_n \frac{\partial \theta_i}{\partial z_n}(z^*)$$

for $0 \leqslant i \leqslant N$. Then

$$a_0 \Theta(z^*, \mu, \nu) = a_1 \frac{\partial \Theta}{\partial z_1}(z^*, \mu, \nu) + \cdots + a_n \frac{\partial \Theta}{\partial z_n}(z^*, \mu, \nu)$$

for all μ, ν. If we define the entire meromorphic function

$$\varphi(z) = a_1 \frac{\partial \log \theta}{\partial z_1}(z) + \cdots + a_n \frac{\partial \log \theta}{\partial z_n}(z),$$

then

$$\varphi(z^*+\mu) + \varphi(z^*+\nu) + \varphi(z^*-\mu-\nu) = \sum a_i \frac{\partial \log \Theta}{\partial z_i}(z^*, \mu, \nu)$$
$$= \frac{1}{\Theta(z^*, \mu, \nu)} \sum a_i \frac{\partial \Theta}{\partial z_i}(z^*, \mu, \nu) = a_0$$

for all μ,ν. Now for any μ, we can find a ν such that $\varphi(z^*+\nu)\neq\infty$ and $\varphi(z^*-\mu-\nu)\neq\infty$, i.e., such that both $z^*+\nu$ and $z^*-\mu-\nu$ are outside the polar divisor of φ; since $\varphi(z^*+\mu)+\varphi(z^*+\nu)+\varphi(z^*-\mu-\nu)=a_0$, it follows that $\varphi(z^*+\mu)$ is an entire *holomorphic* function of μ. Now clearly $\varphi(z+e_\alpha)=\varphi(z)$; and since

$$\theta(z+\lambda_{n+\alpha}) = e^{-2\pi i(z_\alpha+Z_{\alpha\alpha}/2)}\theta(z),$$

$$\log\theta(z+\lambda_{n+\alpha}) = -2\pi i\left(z_\alpha+\frac{Z_{\alpha\alpha}}{2}\right)+\log\theta(z),$$

i.e.,

$$\varphi(z+\lambda_{n+\alpha}) = \varphi(z)-2\pi i\cdot a_\alpha.$$

Thus each partial derivative $\partial\varphi/\partial z_i$ is periodic for the lattice Λ, hence bounded in $V\cong\mathbb{C}^n$ and therefore constant. Consequently φ must be linear; write

$$\varphi(z) = \sum_\alpha b_\alpha\cdot z_\alpha+c.$$

But $\varphi(z+e_\alpha)=\varphi(z)\Rightarrow b_\alpha=0$ for all α; hence $\varphi(z+\lambda_{n+\alpha})=\varphi(z)=c$. Then

$$\varphi(z+\lambda_{n+\alpha})-\varphi(z) = 2\pi i\cdot a_\alpha\Rightarrow a_\alpha=0 \qquad \text{for all } \alpha.$$

We deduce that the presumed linear relation

$$a_0\theta_j(z^*) = \sum a_i\frac{\partial\theta_j}{\partial z_i}(z^*) \qquad \text{for all } j$$

is trivial, and ι_{L^3} is an immersion.

It remains to show by a similar argument that ι_{L^3} is one-to-one. Suppose there exist $z_1,z_2\in\mathbb{C}^n$ with

$$\theta_i(z_1) = \rho\cdot\theta_i(z_2) \qquad \text{for all } i;$$

we will prove that z_1 and z_2 represent the same point on M. From the general relation

$$\Theta(z,\mu,\nu) = \sum_{i=0}^{N} c_i(\mu,\nu)\cdot\theta_i(z)$$

it follows that

$$\frac{\Theta(z_1,\mu,\nu)}{\Theta(z_2,\mu,\nu)} = \frac{\theta(z_1+\mu)\theta(z_1+\nu)\theta(z_1-\mu-\nu)}{\theta(z_2+\mu)\theta(z_2+\nu)\theta(z_1-\mu-\nu)} = \rho$$

identically in μ and ν. For any $\mu\in\mathbb{C}^n$, we can find ν such that

$$\theta(z_1+\nu),\theta(z_1-\mu-\nu),\theta(z_2+\nu),\theta(z_2-\mu-\nu)$$

are all nonzero; consequently $\theta(z_1+\mu)/\theta(z_2+\mu)$ is a nonzero entire function of μ. Then we can set

$$\psi(z) = \log\frac{\theta(z_1+z)}{\theta(z_2+z)}$$

and obtain an entire holomorphic function. By the functional equations of the θ-function,

$$\psi(z+e_\alpha) = \psi(z) + 2\pi i b_\alpha, \qquad b_\alpha \in \mathbb{Z},$$
$$\psi(z+\lambda_{n+\alpha}) = \psi(z) - 2\pi i(z_1 - z_2)_\alpha + 2\pi i c_\alpha, \qquad c_\alpha \in \mathbb{Z}.$$

As before this implies that $\partial\psi/\partial z_i$ is constant for all i, so we can write

$$\psi(z) = 2\pi i \sum a_\beta \cdot z + d.$$

Then $\psi(z+e_\alpha) = \psi(z) + 2\pi i b_\alpha \Rightarrow a_\beta = b_\beta \in \mathbb{Z}$; this in turn implies that

$$\psi(z+\lambda_{n+\alpha}) - \psi(z) = 2\pi i \sum a_\beta Z_{\alpha\beta}$$
$$\Rightarrow \qquad 2\pi i(z_1 - z_2)_\alpha = -2\pi i c_\alpha + 2\pi i \sum a_\beta Z_{\alpha\beta}$$
$$\Rightarrow \qquad z_1 - z_2 = -\sum c_\alpha \cdot e_\alpha + \sum a_\beta \lambda_{n+\beta},$$

i.e., $z_1 - z_2 \in \Lambda$.

Finally, note that for $L \to M$ an arbitrary positive line bundle, we can construct as before an Abelian variety M' with principally polarizing line bundle $L' \to M'$ and finite map

$$\pi' : M \to M'$$

such that

$$\pi'^*(L') = L.$$

Since the map π' is nowhere singular, the argument above applied to L' shows that ι_{L^3} is likewise an immersion, and that for p and q with $\pi'(p) \neq \pi'(q)$, $\iota_{L^3}(p) \neq \iota_{L^3}(q)$. That ι_{L^3} separates points in $\pi^{-1}(p)$ can be seen directly from the explicit form of the θ-functions given on p. 319.

Q.E.D.

The first case beyond curves is the embedding

$$\iota_{L^3}: M \hookrightarrow \mathbb{P}^8$$

of a principally polarized Abelian surface. As a special case, if $M = E_1 \times E_2$ is the product of two elliptic curves, $L_1 \to E_1$ and $L_2 \to E_2$ line bundles of degree 1, and $\iota_{L_i^3}: E_i \to \mathbb{P}^2$ the corresponding embeddings, then $L = \pi_1^* L_1 \otimes \pi_2^* L_2$ is principally polarizing, and ι_{L^3} is just the Segré map $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ applied to $M = E_1 \times E_2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$.

The Group Structure on an Abelian Variety

To close our discussion of complex tori, we want to make a few general remarks about the group structure on an Abelian variety.

Any complex torus $M = \mathbb{C}^n/\Lambda$ is a *complex Lie group*—that is, a complex manifold having a group structure in which the group operations are holomorphic. Conversely, we have

Proposition. *Any connected compact complex Lie group* M *is a complex torus.*

Proof. We first show that M must be commutative. For every $g \in M$, let $\mathrm{Ad}(g)$ denote the automorphism of M given by

$$\mathrm{Ad}(g): h \mapsto ghg^{-1}.$$

Clearly the identity e is a fixed point of $\mathrm{Ad}(g)$ for all $g \in M$.

Now let $z_1, \ldots, z_n$ be holomorphic coordinates around $e \in M$, and for each $g \in M$, write out the power series expansion of $\mathrm{Ad}(g)^* z_i$ as

$$\mathrm{Ad}(g)^*(z_i) = \sum a_{i_1,\ldots,i_n}(g) z_1^{i_1} \cdots z_n^{i_n}.$$

For each index $(i_1, \ldots, i_n)$ the function $a_{i_1,\ldots,i_n}(g)$ is clearly a holomorphic function of g ; since M is compact and connected, it follows that $a_{i_1,\ldots,i_n}(g)$ is constant. Thus

$$\mathrm{Ad}(g)^*(z_i) = \mathrm{Ad}(e)^*(z_i) = z_i,$$

and so

$$\mathrm{Ad}(g)^* \equiv I,$$

i.e., M is commutative.

Next, for any tangent vector $v \in T'_e(M)$ to M at e, let let $\tilde{v}$ be the vector field on M defined by

$$\tilde{v}(g) = (\iota_g)_*(v),$$

where $\iota_g : M \to M$ is multiplication by g; clearly $\tilde{v}$ is holomorphic. Let $\varphi_{t,v} : M \to M$ be the endomorphism of M obtained by integrating the vector field $\tilde{v}$ to time t, and let

$$\pi: \ T'_e(M) \to M$$

be the *exponential map*, defined by

$$\pi(v) = \varphi_{1,v}(e).$$

Since M is commutative, π is in fact a group homomorphism. Thus M is the quotient of $T'_e(M) \cong \mathbb{C}^n$ by a discrete subgroup, which must be a lattice Λ; since M is compact, Λ must have maximal rank $2n$, and hence $M = \mathbb{C}^n/\Lambda$ is a complex torus. Q.E.D.

Note that if $M = \mathbb{C}^n/\Lambda$, $M' = \mathbb{C}^m/\Lambda'$ are two complex tori and $f: M \to M'$ any holomorphic map, then f lifts to a map

$$\tilde{f}: \ \mathbb{C}^n \to \mathbb{C}^m.$$

We see then that in terms of Euclidean coordinates $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_m)$ on $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, *the Jacobian matrix*

$$\mathcal{J}_f = \left(\frac{\partial w_i}{\partial z_j}\right)$$

is a well-defined global holomorphic function on M, *hence constant.* It follows that $\tilde{f}$ is an affine linear transformation, and we have

Proposition. *Any holomorphic map between complex tori is a group homomorphism followed by a translation.*

An Abelian variety is a *homogeneous algebraic variety*—that is, it admits a transitive group of biholomorphic automorphisms. Other homogeneous varieties are Grassmannians, quadrics, etc. There is an important difference between these two types. In the latter examples the automorphisms may be taken to be projective transformations—i.e., for a suitable embedding $M\subset\mathbb{P}^N$, the automorphism group Aut(M) is just the group of linear automorphisms of $\mathbb{P}^N$ leaving M fixed. On the other hand:

Theorem. *If* $\mathrm{M}\subset\mathbb{P}^{\mathrm{N}}$ *is an Abelian variety, the group of automorphisms of* M *induced by linear transformations on* $\mathbb{P}^{\mathrm{N}}$ *is finite.*

Proof. Let $L\to M$ be the hyperplane bundle on M. Then if $\varphi: M\to M$ is any automorphism induced by a linear transformation of $\mathbb{P}^N$, clearly $\varphi^*L=L$. But L is positive, and so it is preserved by only a finite group of translations of M. Thus it will suffice to prove that the group of automorphisms φ of M fixing L and fixing the point $p=\pi(0)$ in M is finite. Now any such automorphism lifts to a linear transformation $\tilde{\varphi}:\mathbb{C}^n\to\mathbb{C}^n$ fixing the lattice $\Lambda\subset\mathbb{C}^n$; *since* φ *takes* L *to itself, moreover,* $\tilde{\varphi}$ *must be unitary with respect to the hermitian inner product given by* $\mathrm{c}_1(\mathrm{L})\in\mathrm{H}^{1,1}(\mathrm{M})=\mathrm{V}\otimes\overline{\mathrm{V}}$. In particular, $\tilde{\varphi}$ must take each lattice vector λ_i in a basis to a lattice vector of the same length. But there can be only a finite number of such lattice vectors for each i, and so the result is proved.

Intrinsic Formulations

It is frequently convenient to have the results on Abelian varieties expressed in a coordinate free manner, and we shall now give this together with a few applications.

Suppose that $V_{\mathbb{R}}$ is a real even-dimensional vector space containing a full lattice Λ, and with a decomposition of the complexification $V_{\mathbb{C}}=V_{\mathbb{R}}\otimes\mathbb{R}$

$$(*)\qquad V_{\mathbb{C}} = V\oplus\overline{V},$$

into conjugate subspaces being given. Then the image of Λ in $V_{\mathbb{C}}$ projects onto a full lattice in V which we still denote by Λ; and

$$M = V/\Lambda$$

is a complex torus. We will see in the next paragraph that every complex torus arises in this way with $V_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

The natural isomorphisms

$$\Lambda \cong H_1(M,\mathbb{Z}), \qquad V^* \cong H^0(M,\Omega^1)$$

have already been noted. If $\dim_{\mathbb{C}} V = n$, then by Kodaira-Serre and Poincaré dualities

$$V \cong H^{n-1,n}(M), \qquad \Lambda \cong H^{2n-1}(M,\mathbb{Z}).$$

It follows that $V_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is canonically isomorphic to $H^{2n-1}(M,\mathbb{R})$ with (*) being the Hodge decomposition

$$H^{2n-1}(M,\mathbb{C}) = H^{n-1,n}(M) \oplus \overline{H^{n-1,n}(M)}.$$

According to the proposition in the preceding section, an arbitrary holomorphic mapping

$$\varphi: \frac{V}{\Lambda} \to \frac{V'}{\Lambda'}$$

is given by an affine linear mapping $V \to V'$. To see explicitly what this mapping is, we compose φ with a translation so that $\varphi(e) = e'$ and let

$$\Phi: \Lambda \to \Lambda'$$

denote the induced map on homology. Since φ^* preserves the Hodge decomposition, we see that

$$\Phi: V \to V',$$

and this is the linear mapping inducing φ.

The Riemann conditions for the existence of a polarization may be formulated as follows: A class in $H^2(M,\mathbb{Z})$ is given by a bilinear form

$$Q: \Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}, \qquad Q(\lambda,\lambda') = -Q(\lambda',\lambda).$$

Identifying $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ with $V \oplus \overline{V}$, the bilinear relations are

$$Q(v,v') = 0, \qquad v,v' \in V,$$

$$-\sqrt{-1}\, Q(v,\bar{v}) > 0, \quad 0 \neq v \in V.$$

For example, if S is a compact Riemann surface with Jacobian variety

$$\mathcal{J}(S) = \frac{H^{0,1}(S)}{H^1(S,\mathbb{Z})},$$

the principal polarization given by the divisor Θ is that given by the cup product

$$H^1(S,\mathbb{Z})\otimes H^1(S,\mathbb{Z})\to\mathbb{Z},$$

which is unimodular by Poincaré duality. In general, if Q has elementary divisors $\delta_1,\ldots,\delta_n$, then $\Delta=\delta_1\cdot\ldots\cdot\delta_n$ is called the *Pfaffian* of Q and

$$\det Q = \Delta^2.$$

We shall use this intrinsic formulation to construct the *Poincaré line bundle*

$$P\to M\times\hat{M},$$

where $\hat{M}=\mathrm{Pic}^0(M)$ is the complex torus dual to M. Recall that $\mathrm{Pic}^0(M)$ is defined to be the group of holomorphic line bundles with first Chern class zero. Via the cohomology sequence of the exponential sheaf sequence we make the natural identifications

$$\begin{aligned}\mathrm{Pic}^0(M) &\cong \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})}\\ &\cong \frac{\bar{V}^*}{\Lambda^*}, \qquad \Lambda^* = \mathrm{Hom}(\Lambda,\mathbb{Z})\end{aligned}$$

and denote by $P_\xi\to M$ the line bundle corresponding to $\xi\in\mathrm{Pic}^0(M)$.

Lemma. *There is a unique holomorphic line bundle*

$$P\to M\times\hat{M},$$

called the Poincaré line bundle, *which is trivial on* $e\times\hat{M}$ *and which satisfies*

$$P|_{M\times\{\xi\}}\cong P_\xi.$$

Proof. Using $\hat{M}=H^{0,1}(M)/H^1(M,\mathbb{Z})$, the cohomology sequence of the exponential sheaf sequence and Künneth formula give

$$\begin{array}{ccccccc} H^1(M\times\hat{M},\mathcal{O}) & \to & H^1(M\times\hat{M},\mathcal{O}^*) & \to & H^2(M\times\hat{M},\mathbb{Z}) & \to & H^2(M\times\hat{M},\mathcal{O}). \\ \uparrow & & & & \uparrow & & \\ H^1(M,\mathcal{O})\oplus H^1(\hat{M},\mathcal{O}) & & & & H^1(M,\mathbb{Z})\otimes H^1(\hat{M},\mathbb{Z}) & & \\ & & & & \wr\!\!\parallel\uparrow & & \\ & & & & \mathrm{Hom}(H^1(M,\mathbb{Z}),H^1(M,\mathbb{Z})) & & \end{array}$$

Now the identity $I\in\mathrm{Hom}(H^1(M,\mathbb{Z}),H^1(M,\mathbb{Z}))$ has Hodge type $(1,1)$ since it preserves the Hodge decomposition on $H^1(M,\mathbb{C})$. By the Lefschetz theorem on $(1,1)$ classes, then, we obtain a holomorphic line bundle $P\to M\times\hat{M}$ with $c_1(P)=I$. The restriction $P|_{M\times\{\xi\}}$ has zero Chern class, and hence is $P_{\varphi(\xi)}$ for some holomorphic mapping

$$\varphi:\ \hat{M} \rightarrow \mathrm{Pic}^0(M).$$

Normalizing so that $\varphi(e)=\hat{e}$, which is achieved by multiplying P by $\pi_1^* P_{-\xi_0}$ where $\xi_0=\varphi(e)$, the induced homology mapping is, by construction, just the identity. Thus φ is also the identity, and this proves the existence of the Poincaré bundle.

If P,P' are two such line bundles, then $Q=P^*\otimes P'$ has the properties

$$Q|M\times\{\xi\}\cong M\times\mathbb{C}; \qquad Q|\{e\}\times\hat{M}\cong\hat{M}\times\mathbb{C}.$$

Denote the second trivialization by ψ and let $\sigma(\lambda,\xi)\in Q_{(\lambda,\xi)}$ be the unique section of $Q|_{M\times\{\xi\}}$ which has the value $\psi^{-1}(1)$ at (e,ξ). Then σ is a nonvanishing holomorphic section of Q, which must then be the trivial line bundle. Q.E.D.

For $L\rightarrow M$ a positive line bundle we set $L\otimes P=\pi_1^* L\otimes P$ on $M\times\hat{M}$; then

$$\begin{aligned} L\otimes P|_{M\times\{\xi\}} &\cong L\otimes P_\xi \\ &= L_\xi \end{aligned}$$

where the last step is a definition. For use in Section 5 of Chapter 3 on differentials of the second kind, we will prove the

Proposition. *There exist* Δ *sections* $\theta_j(\lambda,\xi)\in \mathrm{H}^0(\mathrm{M}\times\hat{\mathrm{M}},\mathcal{O}(\mathrm{L}\otimes\mathrm{P}))$ *inducing a basis of* $\mathrm{H}^0(\mathrm{M},\mathcal{O}(\mathrm{L}_\xi))$ *for each* $\xi\in\hat{\mathrm{M}}$.

The proof will follow some preliminary observations on the map

$$\varphi_L:\ M\rightarrow\hat{M}$$

defined by

$$\varphi_L(\lambda) = \tau_\lambda^* L\otimes L^*$$

which was discussed in the preceding section. Since $\varphi_L(e)=\hat{e}$, according to our general remarks φ_L is uniquely specified by the induced homology map

$$\begin{aligned} \Phi_L:&\ \Lambda\rightarrow\Lambda^* \\ :&\ H^{2n-1}(M,\mathbb{Z})\rightarrow \mathrm{Hom}(H^{2n-1}(M,\mathbb{Z}),\mathbb{Z}). \end{aligned}$$

We will compute Φ_L, thereby giving another proof of the fact that φ_L is an *isogeny*—i.e., a finitely sheeted covering mapping—of degree Δ^2.

For this we consider the group law

$$m:\ M\times M\rightarrow M$$

given by

$$m(\lambda',\lambda) = \lambda'+\lambda.$$

By construction

$$(*) \qquad \varphi_L(\lambda) = \pi_1^* L^* \otimes m^* L|_{M\times\{\lambda\}}.$$

If $x_1,\dots,x_{2n}$ are real coordinates on $V_{\mathbb{R}}$ such that the Chern class of $L\to M$ is

$$\omega = \sum_\alpha \delta_\alpha\, dx_\alpha \wedge dx_{n+\alpha},$$

then using (u_α, v_β) as corresponding coordinates on $M\times M$

$$m^*\omega = \sum_\alpha \delta_\alpha (du_\alpha + dv_\alpha)\wedge(du_{n+\alpha}+dv_{n+\alpha}).$$

If $\eta\in H^{2n-1}(M,\mathbb{Z})$, we let $\eta_1\in H^{2n-1}(M\times M,\mathbb{Z})$ be the class $\pi_1^*\eta$, and similarly for η_2. From $(*)$ we easily deduce that Φ_L is given by the bilinear form on $H^{2n-1}(M,\mathbb{Z})$ defined by

$$\Phi_L(\eta,\eta') = \int_{M\times M} m^*\omega\wedge\eta_1\wedge\eta_2'.$$

For the explicit computation we write

$$\eta = \sum(-1)^{i-1}\eta_i\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_{2n},$$
$$\eta' = \sum(-1)^{j-1}\eta_j\, dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_{2n},$$

and then by the formulas for $m^*\omega$ and Φ_L

$$\Phi_L(\eta,\eta') = \sum_\alpha \delta_\alpha(\eta_\alpha\eta'_{n+\alpha} - \eta_{n+\alpha}\eta'_\alpha).$$

This implies the previous assertion that

$\varphi_L: M\to\hat{M}$ *is an isogeny of degree* Δ^2, *or, equivalently, the line bundle* $L\to M$ *is fixed exactly under the group of translations*

$$\mathbb{Z}\{\delta_\alpha^{-1}x_\alpha, \delta_\alpha^{-1}x_{n+\alpha}\}$$

Returning to the proof of the proposition, the equation

$$\xi = \varphi_L(\lambda)$$

has Δ^2 solutions $\lambda_a(\xi)$. If $\theta(z)\in\mathcal{O}(V)$ is a θ-function giving a section of $L\to M$, then $\theta_\lambda(z)=\theta(z+\lambda)$ gives a section of $\tau_\lambda^* L = L\otimes L_{\varphi_L(\lambda)}$. Now if $\rho: M\times M\to M\times\hat{M}$ is the isogeny defined by $\rho(\lambda',\lambda)=(\lambda',\varphi_L(\lambda))$, then $\theta(z,\lambda)=\theta(z+\lambda)$ gives a section of

$$\pi_1^* L\otimes\rho^* P = \rho^*(L\otimes P)\to M\times\hat{M}.$$

Since $\theta(z+\lambda_a(\xi))=\theta(z+\lambda_b(\xi))$, this section is induced from a section $\theta(z,\xi)\in H^0(M\times\hat{M},\mathcal{O}(L\otimes P))$. In this way we may construct the sections required by the proposition. Q.E.D.

We conclude by discussing some complex tori which are intrinsically associated to an arbitrary compact Kähler manifold M. Recalling the Hodge decomposition

$$H^{2q-1}(V,\mathbb{C}) = \bigoplus_{r+s=2q-1} H^{r,s}(M)$$

from Section 1 of Chapter 2, we set

$$V_q = H^{q-1,q}(M)\oplus\cdots\oplus H^{0,2q-1}(M)$$

for $1 \leqslant q \leqslant n = \dim M$. Then

$$H^{2q+1}(M,\mathbb{C}) = V_q \oplus \overline{V}_q,$$

and, if we let Λ_q denote the image of $H^{2q+1}(M,\mathbb{Z}) \to V_q$, then the qth *intermediate Jacobian* is defined to be the complex torus

$$\mathcal{J}_q(M) = \frac{V_q}{\Lambda_q}.$$

We shall discuss briefly the extreme cases $q=1$ and $q=n$.

When $q=1$, we find the Picard variety

$$\begin{aligned}\mathcal{J}_1(M) &= \frac{H^{0,1}(M)}{H^1(M,\mathbb{Z})}\\ &= \operatorname{Pic}^0(M).\end{aligned}$$

For $q=n$ we obtain the *Albanese variety*

$$\begin{aligned}\mathcal{J}_n(M) &= \frac{H^{n-1,n}(M)}{H^{2n-1}(M,\mathbb{Z})}\\ &\cong \frac{H^0(M,\Omega^1)^*}{H_1(M,\mathbb{Z})}\\ &= \operatorname{Alb}(M),\end{aligned}$$

where the last step is a definition. Now, for the same reasons as in our discussion of Abel's theorem, choosing a base point $p_0 \in M$ and basis $\omega_1,\ldots,\omega_q \in H^0(M,\Omega^1)$ the map

$$\mu\colon\; M \to \operatorname{Alb}(M)$$

given by

$$\mu(p) = \left(\int_{p_0}^{p}\omega_1,\ldots,\int_{p_0}^{p}\omega_q\right)$$

is well-defined and holomorphic. The induced mappings

$$\begin{aligned}\mu_*\colon&\; \frac{H_1(M,\mathbb{Z})}{\text{torsion}} \to H_1(\operatorname{Alb}(M),\mathbb{Z}),\\ \mu^*\colon&\; H^0(\operatorname{Alb}(M),\Omega^1) \to H^0(M,\Omega^1)\end{aligned}$$

are, by construction, isomorphisms. Using our intrinsic formulations, we have

$$\begin{aligned}\operatorname{Pic}^0(\operatorname{Alb}(M)) &= \frac{H^{0,1}(M)}{H^1(M,\mathbb{Z})}\\ &= \operatorname{Pic}^0(M),\\ \operatorname{Alb}(\operatorname{Pic}^0(M)) &= \frac{H^{n-1,n}(M)}{H^{2n-1}(M,\mathbb{Z})}\\ &= \operatorname{Alb}(M).\end{aligned}$$

In particular, $\operatorname{Alb}(M)$ and $\operatorname{Pic}^0(M)$ are, in a natural way, dual complex tori.

Suppose now that $\omega \in H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ is the Chern class of a positive line bundle. Then, by the Hodge-Riemann bilinear relations, the bilinear form

$$Q:\ \Lambda_1 \otimes \Lambda_1 \to \mathbb{Z}$$

given by

$$Q(\eta,\eta') = \int_M \omega^{n-1} \wedge \eta \wedge \eta'$$

induces a polarization on $\operatorname{Pic}^0(M)$, which by the previous discussion induces a polarization on the dual torus $\operatorname{Alb}(M)$. For $L \to \operatorname{Alb}(M)$ a positive line bundle and $P \to \operatorname{Alb}(M) \times \operatorname{Pic}^0(\operatorname{Alb}(M))$ the Poincaré bundle we set $L_\xi = L \otimes P|_{\operatorname{Alb}(M)\times\{\xi\}}$, where $\xi \in \operatorname{Pic}^0(\operatorname{Alb}(M))$. Then for each section $\theta \in H^0(\operatorname{Alb}(M), L)$ we have constructed $\theta_\xi \in H^0(\operatorname{Alb}(M), L_\xi)$. Pulling this back under the canonical mapping

$$\mu:\ M \to \operatorname{Alb}(M)$$

and making the previous identification $\operatorname{Pic}^0(\operatorname{Alb}(M)) \cong \operatorname{Pic}^0(M)$, we deduce that:

There are holomorphic line bundles $L_\xi \to M$ *parametrized by* $\xi \in \operatorname{Pic}^0(M)$ *with* $L_\xi \otimes L_e^* = \xi$, *and holomorphic sections* $\theta_\xi \in H^0(M, \mathcal{O}(L_\xi))$ *depending holomorphically on* ξ.

If $q = \frac{1}{2} b_1(M)$, then setting $D_\xi = (\theta_\xi)$ this last assertion was classically stated as saying that on M there is a family D_ξ of ∞^q linearly *in*equivalent divisors.

7. CURVES AND THEIR JACOBIANS

Preliminaries

Let S now be a compact Riemann surface of genus g. Choose a canonical basis $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$ for $H_1(S,\mathbb{Z})$, so that

$$^{\#}(a_\alpha \cdot a_\beta) = {}^{\#}(b_\alpha \cdot b_\beta) = 0, \qquad {}^{\#}(a_\alpha \cdot b_\beta) = \delta_{\alpha\beta};$$

let $\omega_1,\ldots,\omega_g$ in turn be a basis for $H^0(S,\Omega^1)$ normalized with respect to $\{a_\alpha,b_\alpha\}$, i.e., such that

$$\int_{a_\alpha} \omega_\beta = \delta_{\alpha\beta}.$$

Recall that the Jacobian $\mathcal{J}(S)$ of S is given by $\mathcal{J}(S)=\mathbb{C}^g/\Lambda$, where Λ is the lattice generated by the vectors

$$e_\alpha = \lambda_\alpha = \left(\int_{a_\alpha}\omega_1,\ldots,\int_{a_\alpha}\omega_g\right)$$

$$\lambda_{g+\alpha} = \left(\int_{b_\alpha}\omega_1,\ldots,\int_{b_\alpha}\omega_g\right).$$

By the Riemann bilinear relations, the period matrix Ω of $\Lambda\subset\mathbb{C}^g$ is of the form

$$\Omega = (I,Z)$$

with $Z={}^tZ$ and $Y=\operatorname{Im}Z>0$; thus if we let $x_1,\ldots,x_{2g}$ be real coordinates on $\mathbb{C}^g$ dual to the real basis $\{\lambda_i\}$, the differential form

$$\omega = \sum_\alpha dx_\alpha \wedge dx_{n+\alpha}$$

represents a principal polarization of $\mathcal{J}(S)=\mathbb{C}^g/\Lambda$. In terms of standard complex coordinates $z=(z_1,\ldots,z_g)$ on $\mathbb{C}^g$, we can also write

$$\omega = \frac{\sqrt{-1}}{2}\sum Y_{\alpha\beta}^{-1}\, dz_\alpha \wedge d\bar{z}_\beta$$

with $Y=\operatorname{Im}Z$ as above.

(*Note*: Inasmuch as the Jacobian of a Riemann surface is always principally polarized, we will, after normalizing as above, write e_α for λ_α, and

$$Z_\alpha = (Z_{\alpha 1},\ldots,Z_{\alpha g}) = \left(\int_{b_\alpha}\omega_1,\ldots,\int_{b_\alpha}\omega_g\right)$$

for $\lambda_{n+\alpha}$.)

Let L be the line bundle on $\mathcal{J}(S)$ with Chern class $[\omega]$, translated so that a global section $\tilde{\theta}$ of L is represented by the Riemann theta function $\theta \in \mathcal{O}(\mathbb{C}^n)$ satisfying

$$\theta(z+e_\alpha) = \theta(z), \qquad \theta(z+Z_\alpha) = e^{-2\pi i(Z_\alpha + Z_{\alpha\alpha}/2)}\theta(z);$$

let $\Theta \subset \mathcal{J}(S)$ be the divisor of the section $\tilde{\theta}$.

Now, choose once and for all a base point $z_0 \in S$, and let $\mu : S \to \mathcal{J}(S)$ be the map given by

$$\mu(z) = \left(\int_{z_0}^{z} \omega_1, \ldots, \int_{z_0}^{z} \omega_g\right).$$

We compute first of all the intersection number of the curve $\mu(S) \subset \mathcal{J}(S)$ with the divisor Θ; to do this we simply count the zeros of the section $\mu^*\tilde{\theta}$ of μ^*L on S. Assume the cycles a_α, b_α are disjoint except for a common base point and, as in Section 2 of this chapter, represent S as a polygon Δ in the plane whose sides correspond in order to the cycles $a_1, b_1, a_1^{-1}, b_1^{-1}$, etc. (See Figure 14.) Then if $\tilde{\mu} : \Delta \to \mathbb{C}^g$ is the obvious lifting of μ given by integrating from z_0 to z in Δ and $\theta \in \mathcal{O}(\mathbb{C}^n)$ is the Riemann θ-function above, we see that

$$\text{number of zeros of } \tilde{\mu}^*\theta = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} d\log\theta(\mu(z)).$$

To evaluate this integral, we consider together the contributions of the sides a_β and a_β^{-1}, b_β and b_β^{-1} in $\partial\Delta$. If z, z^* are corresponding points on a_β

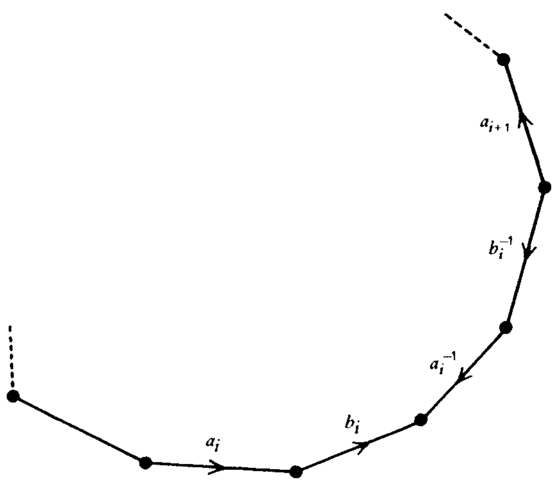


Figure 14

and a_β^{-1}, respectively, we have

$$\tilde{\mu}_\alpha(z^*) - \tilde{\mu}_\alpha(z) = \int_z^{z^*} \omega_\alpha = \int_z^{p_0} \omega_\alpha + \int_{b_\beta} \omega_\alpha + \int_{p_0}^{z^*} \omega_\alpha = \int_{b_\beta} \omega_\alpha = Z_{\beta\alpha},$$

i.e.,

$$\tilde{\mu}(z^*) = \tilde{\mu}(z) + Z_\beta.$$

Thus

$$\theta(\tilde{\mu}(z^*)) = e^{-2\pi i(\tilde{\mu}_\beta(z) + Z_{\beta\beta}/2)}\theta(\tilde{\mu}(z)),$$

so

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}} \int_{a_\beta} d\log\theta(\tilde{\mu}(z)) + \frac{1}{2\pi\sqrt{-1}} \int_{a_\beta^{-1}} d\log\theta(\tilde{\mu}(z))\\ &= \frac{1}{2\pi\sqrt{-1}} \int_{a_\beta^{-1}} d\log e^{-2\pi i(\tilde{\mu}(z)_\beta + Z_{\beta\beta}/2)}\\ &= \int_{a_\beta} d\tilde{\mu}(z)_\beta\\ &= \int_{a_\beta} \omega_\beta = 1.\end{aligned}$$

Similarly, we see that for z, z^* corresponding points on b_β, b_β^{-1},

$$\tilde{\mu}(z^*) = \tilde{\mu}(z) - e_\beta,$$

hence $\theta(\tilde{\mu}(z^*)) = \theta(\tilde{\mu}(z))$ and

$$(*) \qquad \frac{1}{2\pi i} \int_{b_\beta} d\log\theta(\tilde{\mu}(z)) + \frac{1}{2\pi i} \int_{b_\beta^{-1}} d\log\theta(\tilde{\mu}(z)) = 0.$$

Adding up the contributions from all the sides of Δ, we find

$$\deg \mu^* L = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} d\log\theta(\tilde{\mu}(z)) = g.$$

Note that we assume in the course of this computation that $\tilde{\mu}^*\theta \not\equiv 0$ on S; if this is not the case, we may take instead of L the translate $L_\lambda = \tau_\lambda^* L$ and corresponding section $\theta_\lambda(z) = \theta(z - \lambda) \in H^0(\mathcal{J}(S), \mathcal{O}(L_\lambda))$ for a suitable $\lambda \in \mathbb{C}^n$.

Another way to compute $\deg \mu^* L$ is topological:

$$\deg \mu^* L = \int_S c_1(\mu^* L) = \int_S \mu^* c_1(L) = \int_S \mu^* \Big(\sum_\alpha dx_\alpha \wedge dx_{n+\alpha} \Big).$$

Now $\mu_*(a_\alpha) = \lambda_\alpha$, $\mu_*(b_\alpha) = \lambda_{g+\alpha}$, and so

$$\int_{a_\beta} \mu^* dx_\alpha = \int_{\mu(a_\beta)} dx_\alpha = \delta_{\alpha\beta};$$

$$\int_{b_\beta} \mu^* dx_\alpha = \int_{\mu(b_\beta)} dx_\alpha = 0.$$

From this we see that $[\mu^* dx_\alpha]$ is Poincaré dual to the cycle $-b_\alpha$, and $[\mu^* dx_{n+\alpha}]$ is dual to $+a_\alpha$. Thus

$$\int_S \mu^*(dx_\alpha \wedge dx_{n+\alpha}) = {}^{\#}(-b_\alpha \cdot a_\alpha) = 1,$$

hence

$$\deg \mu^*(L) = \int_S \mu^*\Big(\sum dx_\alpha \wedge dx_{n+\alpha}\Big) = g.$$

Now let $\Theta = (\theta)$ denote the divisor of the line bundle L and $\Theta_\lambda = \Theta + \lambda = (\theta_\lambda)$, where $\theta_\lambda(z) = \theta(z-\lambda)$, denote the divisor of the translated bundle $L_\lambda = \tau_\lambda^* L$. Since $c_1(L_\lambda) = c_1(L)$, we have shown by the last computation that for any $\lambda \in \mathcal{J}(S)$, either

1. $\mu(S) \subset \Theta_\lambda$; or
2. $\mu(S)$ intersects Θ_λ in exactly g points, counting multiplicity.

For $\lambda \in \mathcal{J}(S)$ such that $\mu(S) \not\subset \Theta_\lambda$, write the divisor

$$(\mu^* \theta_\lambda) = z_1(\lambda) + \cdots + z_g(\lambda).$$

Now by the Abel and Jacobi theorems, the point $\lambda \in \mathcal{J}(S)$ represents a linear equivalence class of divisors of degree 0 on S. In fact, it turns out that up to a constant κ, λ is just the class of the divisor $\sum z_i(\lambda) - g \cdot z_0$. We express this as the

Lemma. *For a suitable constant* $\kappa \in \mathcal{J}(S)$,

$$\sum_{i=1}^{g} \mu(z_i(\lambda)) + \kappa = \lambda$$

for all $\lambda \in \mathcal{J}(S)$ *such that* $\mu(S) \not\subset \Theta_\lambda$.

Note that this lemma gives our promised explicit solution to the Jacobi inversion problem, at least for a general $\lambda \in \mathcal{J}(S)$ such that the curve $\mu(S)$ does not lie in Θ_λ.

Proof. Represent S again as a polygon Δ in the plane, with $\tilde{\mu} : \Delta \to \mathbb{C}^g$ the corresponding lifting of μ. $\tilde{\mu}^* \theta_\lambda$ vanishes exactly on the points $z_i(\lambda)$, and so by residues

$$\sum_i \tilde{\mu}_\alpha(z_i(\lambda)) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \tilde{\mu}_\alpha(z) \cdot d\log \theta_\lambda(\tilde{\mu}(z)).$$

We evaluate this integral as before by considering corresponding points z, z^* on sides a_β, a_β^{-1} of $\partial\Delta$. Since

$$\tilde{\mu}_\alpha(z^*) = \tilde{\mu}_\alpha(z) + Z_{\alpha\beta},$$

the functional equation for the θ-function gives

$$\theta_\lambda(\tilde{\mu}(z^*)) = e^{-2\pi i(\tilde{\mu}_\beta(z) + Z_{\beta\beta}/2 - \lambda_\beta)} \theta_\lambda(\tilde{\mu}(z))$$

hence

$$d\log\theta_\lambda(\tilde{\mu}(z^*)) = d\log\theta_\lambda(\tilde{\mu}(z)) - 2\pi\sqrt{-1}\cdot\omega_\beta(z).$$

Consequently

$$\frac{1}{2\pi\sqrt{-1}}\left(\int_{a_\beta}\tilde{\mu}_\alpha(z)\,d\log\theta_\lambda(\tilde{\mu}(z)) + \int_{a_\beta^{-1}}\tilde{\mu}_\alpha(z)\,d\log\theta_\lambda(\tilde{\mu}(z))\right)$$

$$= -\frac{Z_{\alpha\beta}}{2\pi\sqrt{-1}}\int_{a_\beta}d\log\theta_\lambda(\tilde{\mu}(z)) + Z_{\alpha\beta}\int_{a_\beta}\omega_\beta(z) + \int_{a_\beta}\tilde{\mu}_\alpha(z)\omega_\beta(z).$$

The last two terms of this expression are independent of λ and hence may be absorbed in the constant κ_α. As for the first term, if z_1 and z_2 are the endpoints of the arc a_β, then $\tilde{\mu}(z_2) = \tilde{\mu}(z_1) \pm e_\beta$; hence $\theta_\lambda(\tilde{\mu}(z_1)) = \theta_\lambda(\tilde{\mu}(z_2))$ and

$$\frac{1}{2\pi\sqrt{-1}}\int_{a_\beta}d\log\theta_\lambda(\tilde{\mu}(z)) \in \mathbb{Z}.$$

Thus the first term must likewise be constant and can be absorbed in κ_α. Now if z and z^* are corresponding points of b_β and b_β^{-1}, we have

$$\tilde{\mu}(z^*) = \tilde{\mu}(z) - e_\beta, \qquad \theta_\lambda(\tilde{\mu}(z^*)) = \theta_\lambda(\tilde{\mu}(z));$$

so

$$\frac{1}{2\pi\sqrt{-1}}\left(\int_{b_\beta}\tilde{\mu}_\alpha(z)\,d\log\theta_\lambda(\tilde{\mu}(z)) + \int_{b_\beta^{-1}}\tilde{\mu}_\alpha(z)\,d\log\theta_\lambda(\tilde{\mu}(z))\right)$$

$$= \frac{\delta_{\alpha\beta}}{2\pi\sqrt{-1}}\int_{b_\beta}d\log\theta_\lambda(\tilde{\mu}(z)).$$

Again, if z_1 and z_2 are the endpoints of b_β, we have $\tilde{\mu}(z_2) = \tilde{\mu}(z_1) + Z_\beta$; hence

$$\theta_\lambda(\tilde{\mu}(z_2)) = e^{-2\pi i(\tilde{\mu}_\beta(z) - \lambda_\beta + Z_{\beta\beta}/2)}\theta_\lambda(\tilde{\mu}(z_1)),$$

$$\frac{1}{2\pi\sqrt{-1}}\int_{b_\beta}d\log\theta_\lambda(\tilde{\mu}(z)) - \lambda_\beta \equiv \tilde{\mu}_\beta(z) + \frac{Z_{\beta\beta}}{2} \qquad (\mathrm{mod}\ \mathbb{Z}).$$

The expression on the right does not depend on λ; thus the expression on the left must be constant and can be absorbed into κ_α. Adding up the contributions from all the sides, we finally obtain

$$\sum_i \tilde{\mu}_\alpha(z_i(\lambda)) = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\tilde{\mu}_\alpha(z)\,d\log\theta_\lambda(\tilde{\mu}(z)) = \lambda_\alpha + \kappa_\alpha. \text{ Q.E.D.}$$

Following our discussion of Riemann's theorem, we will be able to identify the constant κ and determine exactly when $\mu(S) \subset \Theta_\lambda$.

Riemann's Theorem

We can use our last result to obtain a geometric description of the divisor Θ of θ. It will be convenient to change our notation slightly, and denote by

$$D = p_1 + \cdots + p_d \in S^{(d)}$$

an effective divisor of degree d, and by z_i a local coordinate around p_i. Once we have chosen z_i, we may define functions Ω_α around p_i by

$$\omega_\alpha(p) = \Omega_\alpha(p) \cdot dz_i;$$

Ω_α is the function we have previously written as $\omega_\alpha(p)/dz_i$.

As before, we define

$$\mu : S^{(d)} \to \mathcal{J}(S)$$

by

$$\begin{aligned} \mu(p_1 + \cdots + p_d) &= \mu(p_1) + \cdots + \mu(p_d) \\ &= \left(\sum_i \int_{p_0}^{p_i} \omega, \ldots, \sum_i \int_{p_0}^{p_i} \omega \right) \end{aligned}$$

At a point $D = p_1 + \cdots + p_d$ with the points p_i distinct the Jacobian matrix of the map μ is given, in terms of the coordinates $z_1, \ldots, z_d$ on $S^{(d)}$, by

$$\mathcal{J}(\mu) = \begin{bmatrix} \Omega_1(p_1) & \cdots & \Omega_g(p_1) \\ \vdots & & \vdots \\ \Omega_1(p_d) & \cdots & \Omega_g(p_d) \end{bmatrix}$$

This matrix has maximal rank exactly when the points p_i are linearly independent on the canonical curve of S; since this is generically the case as long as $d \leqslant g$, it follows that *for* d $\leqslant$ g *the image*

$$W_d = \mu(S^{(d)})$$

is an analytic subvariety of dimension d, and since the fibers of μ are linear spaces, that the *map* μ *is generically one-to-one*. The geometry of this mapping—especially its relation to the special linear systems on S—will be examined in the following two subsections.

We have intrinsically associated to the Jacobian $\mathcal{J}(S)$ of S two divisors, unique up to translation: the divisor Θ of the line bundle $L \to \mathcal{J}(S)$ with Chern class given by the intersection form on $H_1(S, \mathbb{Z}) \cong H_1(\mathcal{J}(S), \mathbb{Z})$, and the image W_{g-1} of $S^{(g-1)}$ under μ. The first of these divisors is defined purely in terms of the linear algebra of $\mathcal{J}(S)$ and $[\omega]$, while the second involves directly the geometry of S and μ. Of fundamental importance, accordingly, is

Riemann's Theorem

$$\Theta = W_{g-1} + \kappa,$$

where κ *is the constant appearing the last lemma.*

Proof. We first show that $W_{g-1}\subset\Theta_{-\kappa}$. To see this, let $D=p_1+\cdots+p_g\in S^{(g)}$ be a generic divisor, so that the points p_i are all distinct, $\mu:S^{(g)}\to\mathcal{J}(S)$ is one-to-one at D, and $\mu(S)\not\subset\Theta_{\kappa+\mu(D)}$. Set

$$\lambda=\mu(D)+\kappa,$$

so that by the preceding lemma,

$$\Theta_\lambda\cap\mu(S)=\mu(p_1)+\cdots+\mu(p_g).$$

Now—and this is the crucial step—we have seen that $\theta(\mu)=\theta(-\mu)$; therefore, using $\theta_\lambda(\mu(p_i))=0$ for $i=g$,

$$\theta(\mu(p_1)+\cdots+\mu(p_{g-1})+\kappa)=\theta(\lambda-\mu(p_g))=\theta_\lambda(\mu(p_g))=0,$$

i.e.,

$$\theta_{-\kappa}(\mu(p_1)+\cdots+\mu(p_{g-1}))=0.$$

Thus $\mu^*\theta_{-\kappa}$ vanishes in an open set in $S^{(g-1)}$, hence in all of $S^{(g-1)}$, and we see that $W_{g-1}\subset\Theta_{-\kappa}$.

Now from Section 1 of Chapter 1 we can write

$$\Theta_{-\kappa}=a\cdot W_{g-1}+\Theta'$$

with $a>0\in\mathbb{Z}$ and Θ' an effective divisor on $\mathcal{J}(S)$. We want to show first that $a=1$, and then that $\Theta'=0$; the first step will be to show that $^{\#}(\mu(S)\cdot W_{g-1})\geq g$. To prove this, note that the involution $\mu\mapsto-\mu$ acts as the identity on $H_2(\mathcal{J}(S))=H_1(\mathcal{J}(S))\wedge H_1(\mathcal{J}(S))$, and so the cycle $-\mu(S)$ is homologous to $\mu(S)$. Now take $\lambda=\mu(p_1)+\cdots+\mu(p_g)$ a generic point of $\mathcal{J}(S)$ so that $-\mu(S)\not\subset W_{g-1}-\lambda$; then $-\mu(S)$ and $W_{g-1}-\lambda$ meet in isolated points, and for each $i=1,\dots,g$ we have

$$-\mu(p_i)=\mu\Big(\sum_{j\neq i}p_j\Big)-\lambda\in W_{g-1}-\lambda.$$

Thus $^{\#}(\mu(S),W_{g-1})\geq g$.

Now we have proved that the intersection number of Θ with $\mu(S)$ is g, and consequently

$$a\cdot{}^{\#}(\mu(S)\cdot W_{g-1})+{}^{\#}(\mu(S)\cdot\Theta')=g.$$

But we can always find $\lambda\in\mathcal{J}(S)$ such that $\mu(S)\not\subset\Theta'+\lambda$; thus $^{\#}(\mu(S),\Theta')\geq0$, and it follows that $a=1$, $^{\#}(\mu(S)\cdot W_{g-1})=g$, and $^{\#}(\mu(S)\cdot\Theta')=0$.

It remains to show that $\Theta'=0$. We use the following argument: since $^{\#}(\mu(S)\cdot\Theta')=0$,

$$\mu(S)\cap\Theta'_\lambda\neq\varnothing\Rightarrow\mu(S)\subset\Theta'_\lambda\qquad\text{for any }\lambda\in\mathcal{J}(S);$$

it follows from this that

$$\Theta'_\lambda\cap W_2\neq\varnothing\Rightarrow W_2\subset\Theta'_\lambda\qquad\text{for any }\lambda\in\mathcal{J}(S),$$

since

$$\begin{aligned}
&\Theta'_\lambda \ni \mu(p_1)+\mu(p_2)\\
&\quad\Rightarrow \Theta'_{\lambda+\mu(p_1)} \ni \mu(p_2)\\
&\quad\Rightarrow \Theta'_{\lambda+\mu(p_1)} \ni \mu(p_2^*) \qquad \text{for all } p_2^* \in S\\
&\quad\Rightarrow \Theta'_\lambda \ni \mu(z_1)+\mu(p_2^*) \qquad \text{for all } p_2^* \in S\\
&\quad\Rightarrow \Theta'_\lambda \ni \mu(z_1^*)+\mu(p_2^*) \qquad \text{for all } p_1^*, p_2^* \in S
\end{aligned}$$

i.e., $W_2 \subset \Theta'_\lambda$. Repeating the argument gives

$$\Theta'_\lambda \cap W_n \neq \varnothing \Rightarrow \Theta'_\lambda \supset W_n$$

for any n. But by Jacobi's theorem $W_g = \mathcal{J}(S)$, and hence $\Theta'_\lambda \cap W_g = \varnothing$; thus $\Theta'_\lambda = 0$. Q.E.D.

Note that by Riemann-Roch, if D is any effective divisor of degree $g-1$, then $K-D$ is also; it follows that

$$W_{g-1} = \mu(K) - W_{g-1}.$$

It is now possible to identify the constant κ appearing in Riemann's theorem: we have

$$\begin{aligned}
W_{g-1}+\kappa &= \Theta\\
&= -\Theta\\
&= -W_{g-1}-\kappa\\
&= W_{g-1}-\kappa-\mu(K).
\end{aligned}$$

Since W_{g-1} is the theta-divisor of a principal polarization, by the result of p. 317 it cannot be fixed by any nonzero translation, and we find that

$$2\kappa = -\mu(K).$$

Similarly, we can determine exactly when $\mu(S) \subset \Theta_\lambda$: clearly, $\lambda - \mu(p) \in W_{g-1}$ if and only if $\lambda = \mu(D)$ for some $D \in S^{(g)}$ containing the point $p \in S$. Thus $\lambda - \mu(S) \subset W_{g-1}$ if and only if $\lambda = \mu(D)$ for D such that $h^0(D) > 1$. Now for any $\lambda = \mu(D)$, we have

$$\begin{aligned}
\lambda - \mu(S) \subset W_{g-1} &\Leftrightarrow \lambda - \mu(S) + \kappa \subset \Theta\\
&\Leftrightarrow \mu(S) - \lambda - \kappa \subset \Theta\\
&\Leftrightarrow \mu(S) \subset \Theta + \kappa + \lambda,
\end{aligned}$$

i.e., $\mu(S) \subset \Theta_\kappa + \lambda$ *if and only if* $\lambda = \mu(\mathrm{D})$ *for* $\mathrm{D} \in S^{(g)}$ *such that* $h^0(\mathrm{D}) > 0$. We can express this more intrinsically by noting that *the lemma above* (p. 336) *fails to give an explicit answer to the Jacobi inversion problem—finding* $\mathrm{D} \in S^{(g)}$ *such that* $\mu(\mathrm{D}) = \lambda$ *for a given* $\lambda \in \mathcal{J}(S)$*—exactly when that answer is not unique, i.e., when such a* D *varies in a nontrivial linear system.*

Riemann's Singularity Theorem

We turn our attention now to the subvariety $W_d = \mu(S^{(d)}) \subset \mathcal{J}(S)$ parametrizing linear equivalence classes of effective divisors of degree d on S. Our goal will be to prove a theorem, of which a special case was suggested by Riemann, relating the local geometry of the varieties W_d—specifically their tangent cones at various points $\mu(D)$—to the geometry of the corresponding linear systems $|D|$ on the canonical curve of S in $\mathbb{P}^{g-1}$.

To start, note that we have a natural identification

$$\mathbb{P}(T'_\mu(\mathcal{J}(S))) \cong \mathbb{P}(H^0(S,\Omega^1_S)^*)$$

between the projective space associated to the tangent space to the Jacobian $\mathcal{J}(S)$, and the ambient space of the canonical map

$$\iota_K: S \longrightarrow \mathbb{P}(H^0(S,\Omega^1_S)^*) \cong \mathbb{P}^{g-1}.$$

Hereafter, when we refer to $\mathbb{P}^{g-1}$ we will always mean specifically $\mathbb{P}(H^0(S,\Omega^1_S)^*)$. In particular, the projective tangent cones

$$P(T'_\mu(X)) \subset \mathbb{P}(T'_\mu(\mathcal{J}(S))) = \mathbb{P}^{g-1}$$

to any subvariety $X \subset \mathcal{J}(S)$ at any point $\mu \in X$ will be considered as subvarieties of the ambient space $\mathbb{P}^{g-1}$ of the canonical curve; we will denote this variety by $T_\mu(X)$.

To recall our notation, let $\omega_1,\dots,\omega_g$ be a basis for the holomorphic 1-forms on S, so that the map

$$\mu: S \longrightarrow \mathcal{J}(S)$$

is given by

$$\mu(p) = \left(\int_{p_0}^{p}\omega_1,\dots,\int_{p_0}^{p}\omega_g\right)$$

and the map $\mu: S^{(d)} \to \mathcal{J}(S)$ by

$$\mu(p_1 + \cdots + p_d) = \mu(p_1) + \cdots + \mu(p_d).$$

Whenever we have a local coordinate z on S we will define functions Ω_α by

$$\omega_\alpha(p) = \Omega_\alpha(p)\,dz$$

so that the vector

$$\Omega(p) = (\Omega_1(p),\dots,\Omega_g(p))$$

represents the point p on the canonical curve in $\mathbb{P}^{g-1}$.

Our first object is to describe geometrically the tangent cone to the variety W_d at a point $\mu(D)$. Suppose first that the divisor D is regular, i.e., that $\dim|D| = 0$. By the geometric version of the Reimann-Roch, the linear span $\overline{D}$ of the points of D on the canonical curve is a $(d-1)$-plane, and we claim that

W_d *is smooth at* $\mu(D)$, *with tangent space* $T_{\mu(D)}(W_d)=\overline{D}$.

Proof. Suppose first that $D=p_1+\cdots+p_d$ with the points p_i distinct, so that local coordinates z_i on S near p_i furnish local coordinates on $S^{(d)}$ near D. The map $\mu: S^{(d)}\to\mathcal{J}(S)$ is given by

$$\mu(p_1+\cdots+p_d)=\left(\sum_i\int_{p_0}^{p_i}\omega_1,\ldots,\sum_i\int_{p_0}^{p_i}\omega_g\right).$$

Differentiating, we find that the Jacobian $\mathcal{J}(\mu)$ of the map μ is

$$\mathcal{J}(\mu)=\begin{bmatrix}\Omega_1(p_1),\ldots,\Omega_g(p_1)\\ \vdots\\ \Omega_1(p_d),\ldots,\Omega_g(p_d)\end{bmatrix}.$$

By hypothesis, the row vectors $\Omega(p_i)=(\Omega_1(p_i),\ldots,\Omega_g(p_i))$, representing the point p_i on the canonical curve, are all independent. Thus $\mathcal{J}(\mu)$ has maximal rank d at D, so $W_d=\mu(S^{(d)})$ is smooth at $\mu(D)$, with tangent plane spanned by the points p_i.

We will illustrate what happens at the diagonals of $S^{(d)}$ by assuming that $D=2p_1+p_2+\cdots+p_{d-1}$ where the p_i are distinct; the general situation is only notationally more complicated. Let z be a coordinate of p varying in a neighborhood of p_1, and for $\Omega(p)=(\Omega_1(z),\ldots,\Omega_g(z))$ as above we define

$$\Omega'(p)=\left(\frac{d\Omega_1}{dz},\ldots,\frac{d\Omega_g}{dz}\right).$$

The line $\overline{\Omega(p)\Omega'(p)}$ in $\mathbb{P}^{g-1}$ determined by $\Omega(p)$ and $\Omega'(p)$ is the tangent line to the canonical curve at p. For

$$w_1=\frac{z_1+z_2}{2},\qquad w_2=z_1z_2$$

we set

$$F_\alpha(w_1,w_2)=\int_{p_0}^{z_1}\omega_\alpha+\int_{p_0}^{z_2}\omega_\alpha.$$

From

$$\frac{\partial F_\alpha}{\partial w_1}d\left(\frac{z_1+z_2}{2}\right)+\frac{\partial F_\alpha}{\partial w_2}d(z_1z_2)=\omega_\alpha(z_1)+\omega_\alpha(z_2)$$

we deduce that

$$\frac{\partial F_\alpha}{\partial w_1}=\frac{1}{2}(\Omega_\alpha(z_1)+\Omega_\alpha(z_2));\qquad \frac{\partial F_\alpha}{\partial w_2}=\frac{\Omega(z_1)-\Omega(z_2)}{z_2-z_1}.$$

Letting z_2 go to z_1 it follows that the Jacobian matrix evaluated at $D=2p_1+p_2+\cdots+p_{d-1}$ is

$$\begin{bmatrix} \Omega(p_1) \\ -\Omega'(p_1) \\ \Omega(p_2) \\ \vdots \\ \Omega(p_{d-1}) \end{bmatrix}$$

and the argument proceeds as before when the points were distinct. Q.E.D.

Suppose now that D moves in an r-dimensional linear system, and denote the divisors in $|D|$ by

$$D_\lambda = p_1(\lambda)+\cdots+p_d(\lambda), \qquad \lambda \in \mathbb{P}^r.$$

According to Riemann-Roch, the points $p_i(\lambda)$ of each D_λ will span a $(d-r-1)$-plane $\overline{D}_\lambda$ in $\mathbb{P}^{g-1}$; we claim that in this case

The projective tangent cone to W_d *at the point* $\mu(D)$ *is the union*

$$T_{\mu(D)}(W_d) = \bigcup_{\lambda\in\mathbb{P}^r} \overline{D}_\lambda$$

of the planes spanned by the divisors of the linear system $|D|$.

Proof. Recall that the tangent cone to W_d at $\mu(D)$ is the locus of all tangent lines at $\mu(D)$ to analytic arcs in W_d. Now, let

$$D(t) = q_1(t)+\cdots+q_d(t)$$

be any path in the symmetric product $S^{(d)}$ with

$$D(0) = D_\lambda = p_1(\lambda)+\cdots+p_d(\lambda)$$

for some $D_\lambda \in |D|$. The image arc

$$w(t) = \mu(D(t))$$

then lies in W_d with $w(0)=\mu(D)$; and conversely any arc in W_d may be given in this fashion. For simplicity of notation we assume that $p_i(\lambda)$ are distinct and let z_i be a local coordinate around $p_i(\lambda)$. Then if $q_i(t)$ has coordinate $z_i(t)$,

$$\begin{aligned} w(t) &= \mu(q_1(t))+\cdots+\mu(q_d(t)) \\ &= \left(\ldots,\sum_{i=1}^{d}\int_{p_0}^{z_i(t)}\Omega_\alpha(z_i)\,dz_i,\ldots\right) \end{aligned}$$

where $\omega_\alpha = \Omega_\alpha(z_i)dz_i$ near $p_i(\lambda)$ as before. Differentiating,

$$\frac{dw}{dt} = \left(\ldots, \sum_i \Omega_\alpha(z_i(t))z_i'(t), \ldots\right),$$

and, setting $t=0$, the tangent line to $w(t)$ at $\mu(D)$ is

$$\sum_{i=1}^{d} z_i'(0)\Omega(p_i(\lambda)).$$

Now the numbers $z_i'(0)$ and point $\lambda \in \mathbb{P}^r$ may be prescribed arbitrarily, which implies that as a set

$$T_{\mu(D)}(W_d) = \bigcup_\lambda \overline{p_1(\lambda), \ldots, p_d(\lambda)}$$

as desired. Q.E.D.

Note in particular that if $r>0$, then $T_{\mu(D)}(W_d)$ contains the r-secant variety of the canonical curve; since this does not lie in any linear subspace of $\mathbb{P}^{g-1}$, we conclude that

μ(D) *is a singular point of* $\mathrm{W_d}$ *if and only if* $\dim|\mathrm{D}|>0$.

We may interpret intrinsically the preceding computation as follows: With the identification $|D| = \mathbb{P}^r$, the linear system $|D| \subset S^{(d)}$ is a complex submanifold with normal bundle $N \to \mathbb{P}^r$. We denote by $\mathbb{P}(N)$ the associated projective bundle whose fibers are given by

$$\mathbb{P}(N)_\lambda = \mathbb{P}(N_\lambda).$$

Since $\mu: S^{(d)} \to \mathcal{J}(S)$ maps $\mathbb{P}^r$ to the point $\mu(D)$, the differential μ_* is zero on tangent vectors to $\mathbb{P}^r$ and hence $\mu_*(\xi) \in T'_{\mu(D)}(\mathcal{J}(S))$ is well-defined for any $\xi \in N_\lambda$. This induces a holomorphic mapping

$$\mu_*: \mathbb{P}(N) \longrightarrow \mathbb{P}^{g-1},$$

whose image is the tangent cone $T_{\mu(D)}(W_d)$. For each λ the fiber $\mathbb{P}(N)_\lambda$ is parametrized by arcs in $S^{(d)}$ passing through D_λ, and what the above computation shows is that μ_* maps $\mathbb{P}(N)_\lambda$ isomorphically to the subspace $\overline{D}_\lambda \subset \mathbb{P}^{g-1}$.

One aspect of the behavior of the planes $\overline{D}$ in a linear system which will be useful is the following:

Lemma. *If a point* $\mathrm{q} \in \mathbb{P}^{g-1}$ *lies on two secant planes* $\overline{\mathrm{D}}_\lambda, \overline{\mathrm{D}}_{\lambda'}$, *it lies on* $\overline{\mathrm{D}}_\mu$ *for every* D_μ *in the pencil spanned by* D_λ *and* $\mathrm{D}_{\lambda'}$; *or, in other words, the fibers of the map* $\mu_*: \mathbb{P}(\mathrm{N}) \to \mathrm{T}_{\mu(\mathrm{D})}(\mathrm{W_d})$ *are linear spaces.*

Proof. Suppose that the dimension of the complete linear system $|D|$ is r. By Riemann-Roch, the points of any divisor $F \in |K-D|$ span a

$(g-r-2)$-plane $\bar{F}$. *The linear system of hyperplanes in* $\mathbb{P}^{g-1}$ *containing* F *thus cuts out on the canonical curve the complete linear system* |D|; in particular, any pencil $\{D_\mu\}\subset|D|$ is cut out by a pencil of hyperplanes through F. Thus, if q lies on two secant planes $\bar{D}_\lambda$ and $\bar{D}_{\lambda'}$ of the pencil $\{D_\mu\}$, it lies in the hyperplane spanned by any plane $\bar{D}_\mu$ and any divisor $F\in|K-D|$. But of course the residual intersection of the canonical curve with any hyperplane containing D_μ is a divisor of the system $|K-D|$, so this implies q lies on any hyperplane containing D_μ, i.e., q lies on $\bar{D}_\mu$.
Q.E.D.

Next, we set $\mathbb{T}=T_{\mu(D)}(W_d)$ and will prove the

Proposition. *For* $\mathrm{D}\in\mathrm{S}^{(\mathrm{d})}$ *with* $\dim|\mathrm{D}|=\mathrm{r}$, *the degree of the projective tangent cone* $\mathbb{T}\subset\mathbb{P}^{g-1}$ *is* $\binom{g-d+r}{r}$.

Proof. Let $q_1,\ldots,q_{g-d+r}$ be generic points of S, in particular such that

$$(*)\qquad \dim|D+q_1+\cdots+q_{g-d+r}|=\dim|D|=r$$

and for any subset $q_1,\ldots,q_\alpha$ with $\alpha\leqslant r$,

$$(**)\qquad \dim|D-q_1\cdots-q_\alpha|=\dim|D|-\alpha=r-\alpha.$$

Note that by $(*)$ the points $q_1,\ldots,q_{g-d+r}$ are all independent on the canonical curve; denote by $\bar{E}$ the linear space $\mathbb{P}^{g-d+r-1}\subset\mathbb{P}^{g-1}$ they span.

To prove the proposition we will show that E intersects $\mathbb{T}$ transversely in a variety of degree $\binom{g-d+r}{r}$; specifically, we will prove that

1. The intersection $\mathbb{T}\cap\bar{E}$ is the union

$$\bigcup_{\substack{I\subset\{1,\ldots,g-d+r\}\\ {}^{\#}I=r}}\overline{q_{i_1},\ldots,q_{i_r}}$$

of the $\binom{g-d+r}{r}$ coordinate $(r-1)$-planes in $\mathbb{P}^{d-g+r-1}$; and

2. This intersection is transverse.

To prove the first statement, we note that for any multiindex $I=\{i_1,\ldots,i_r\}\subset\{1,\ldots,g-d+r\}$, we can find a divisor $D_\lambda\in|D|$ containing the points $q_{i_1},\ldots,q_{i_r}$; we then have

$$\overline{q_{i_1},\ldots,q_{i_r}}\subset\bar{D}_\lambda$$

and hence in general

$$\bigcup_I\overline{q_{i_1},\ldots,q_{i_r}}\subset\mathbb{T}\cap\bar{E}.$$

Conversely, suppose that D_λ is any divisor in $|D|$, and that D_λ contains exactly α of the points q_i, which we may take to be $q_1,\ldots,q_\alpha$ (by $(*)$, of course, $\alpha \leqslant r$). Then since by (**)

$$\dim|D + q_{\alpha+1} + \cdots + q_{g-d+r}| = \dim|D| = r$$

we have by the Riemann-Roch formula that

$$\dim(\overline{D_\lambda \cup E}) = \dim \overline{D_\lambda, q_{\alpha+1}, \ldots, q_{g-d+r}} = g - 1 - \alpha.$$

It follows from linear algebra that

$$\begin{aligned}\dim(\overline{D}_\lambda \cap \overline{E}) &= \dim \overline{D}_\lambda + \dim \overline{E} - \dim \overline{D_\lambda \cup E} \\ &= \alpha - 1\end{aligned}$$

i.e., that $\overline{D}_\lambda$ meets $\overline{E}$ *only* in the span $\overline{q_1, \ldots, q_\alpha}$. Thus

$$\overline{E} \cap \mathbb{T} \subset \bigcup_I \overline{q_{i_1}, \ldots, q_{i_r}}$$

and the first part of the lemma is proved.

Note that by this argument, for $q_1, \ldots, q_r$ generic points on S and q any point in $\overline{q_1, \ldots, q_r}$ not in the span of a proper subset of $q_1, \ldots, q_r$, there will be a unique plane $\overline{D}_\lambda$ containing q. Thus,

the map μ_* is generically one-to-one, i.e., the planes $\overline{D}_\lambda$ sweep out the variety $\mathbb{T}$ only once.

In particular, this assures us that $\mathbb{T}$ does indeed have dimension $d-1$.

The first—and principal—step in the proof of part 2 is to show that

(*) *For $q_1, \ldots, q_r \in S$ generic, the variety $\mathbb{T}$ is smooth in the complement*

$$\overline{q_1, \ldots, q_r} - \bigcup_{i=1}^{r} \overline{q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r}$$

of the coordinate hyperplanes in $\overline{q_1, \ldots, q_r}$.

We have already seen that μ_* is one-to-one over such a point q; in order to prove this, we have to show that the map $\mu_*: \mathbb{P}(N) \to \mathbb{T}$ has nonzero differential at q. For this we work in a neighborhood $U \subset \mathbb{P}^r$ with coordinates $\lambda = (\lambda_1, \ldots, \lambda_r)$ in which the $p_i(\lambda)$ are single-valued functions of λ; we choose a local coordinate z_i around $p_i(0)$ and consider $z_i(\lambda) = z_i(p_i(\lambda))$ as a function of λ. At any given point it is possible to choose the λ_a $(1 \leqslant a \leqslant r)$ such that $\partial z_a(\lambda)/\partial \lambda_b = \delta_b^a$.

We may assume that $p_1(\lambda), \ldots, p_{d-r}(\lambda)$ span $\overline{D}_\lambda$; then for $t = [t_1, \ldots, t_{d-r}]$ the corresponding homogeneous coordinates in the fibers of $\mathbb{P}(N)|_U$ the mapping μ_* has a lifting

$$\tilde{\mu}_*: U \times \mathbb{C}^{d-r} \longrightarrow \mathbb{C}^g$$

defined by

$$\tilde{\mu}_*(\lambda, t) = t_1 \Omega(p_1(\lambda)) + \cdots + t_{d-r} \Omega(p_{d-r}(\lambda)).$$

Noting that $\tilde{\mu}_*$ is linear on each fiber of $\mathbb{P}^n$, it is straightforward to check that the Jacobian matrix of μ_* has rank one less than that of $\tilde{\mu}_*$, and we shall compute the latter. Using the index range $1 \leqslant \sigma \leqslant d-r$, the Jacobian of $\tilde{\mu}_*$ is

$$\begin{bmatrix} \partial\tilde{\mu}_*/\partial t_1 \\ \vdots \\ \partial\tilde{\mu}_*/\partial t_{d-r} \\ \partial\tilde{\mu}_*/\partial\lambda_1 \\ \vdots \\ \partial\tilde{\mu}_*/\partial\lambda_r \end{bmatrix} = \begin{bmatrix} \Omega(p_1(\lambda)) \\ \vdots \\ \Omega(p_{d-r}(\lambda)) \\ \sum_\sigma t_\sigma(\partial z_\alpha/\partial\lambda_1)\Omega'(p_\sigma(\lambda)) \\ \vdots \\ \sum_\sigma t_\sigma(\partial z_\sigma/\partial\lambda_r)\Omega'(p_\sigma(\lambda)) \end{bmatrix}.$$

At a point where $\partial z_a(\lambda)/\partial\lambda_b = \delta_b^a$ this is

$$\begin{bmatrix} \Omega(p_1(\lambda)) \\ \vdots \\ \Omega(p_{d-r}(\lambda)) \\ t_1\Omega'(p_1(\lambda)) + \sum_{\sigma \geqslant r+1} t_\sigma(\partial z_\sigma/\partial\lambda_1)\Omega'(p_\sigma(\lambda)) \\ \vdots \\ t_r\Omega'(p_r(\lambda)) + \sum_{\sigma \geqslant r+1} t_\sigma(z_\sigma/\partial\lambda_r)\Omega'(p_\sigma(\lambda)) \end{bmatrix};$$

at a point $p \in \overline{p_1,\dots,p_r}$ in the span of the first r points p_i—so that $t_{r+1} = \cdots = t_{d-r} = 0$—but not in the span of any proper subset of them—so that $t_\alpha \neq 0$ for $\alpha \leqslant r$—the rank of this matrix is just the rank of

$$\begin{bmatrix} \Omega(p_1) \\ \vdots \\ \Omega(p_{d-r}) \\ \Omega'(p_1) \\ \vdots \\ \Omega'(p_r) \end{bmatrix}$$

But now if $p_1,\dots,p_r$ are generic, then

$$\dim|D + p_1 + \cdots + p_r| = \dim|D| = r$$

and so by Riemann-Roch the span

$$\begin{aligned} \overline{D_\lambda + p_1 + \cdots + p_r} &= \overline{2p_1 + \cdots + 2p_r + p_{r+1} + \cdots + p_d} \\ &= \overline{\Omega(p_1),\dots,\Omega(p_{d-r}),\Omega'(p_1),\dots,\Omega'(p_r)} \end{aligned}$$

has dimension $d-1$. Thus the Jacobian has maximal rank at a generic point of $\overline{p_1,\dots,p_r}$, and the first assertion is proved.

The remaining two steps in the proof of part 2 are much easier. The second step is to show that

For some $q_1,\dots,q_{g-d+r}$, *the intersection* $\overline{\mathbb{E}}\cap\mathbb{T}$ *is transverse at a generic point of* $\overline{q_1,\dots,q_r}$.

This is immediate: choose $q_1,\dots,q_r$ generically, take any $q\in\overline{q_1,\dots,q_r}$ lying away from the hyperplanes $\overline{q_1,\dots,q_{i-1},q_{i+1},\dots,q_r}$, and then choose $q_{r+1},\dots,q_{g-d+r}$ independent modulo the subspace $T_q(\mathbb{T})$.

Finally, we claim that for generic $q_1,\dots,q_{g-d+r}$ on S,

(*) *the intersection* $\overline{\mathbb{E}}\cap\mathbb{T}$ *is transverse at a generic point of each* $(r-1)$-*plane* $\overline{q_{i_1},\dots,q_{i_r}}$.

To see this, consider the map

$$\pi:\ S^{(r)}\times S^{(g-d)}\longrightarrow S^{(g-d+r)}$$

sending $(q_1+\cdots+q_r,q_{r+1}+\cdots+q_{g-d+r})$ to $(q_1+\cdots+q_{g-d+r})$, and let $B\subset S^{(g-d+r)}$ be the locus of $q_1,\dots,q_{g-d+r}$ for which (*) fails to hold. By the second step, $\pi^{-1}(B)\neq S^{(r)}\times S^{(g-d)}$, and since $S^{(r)}\times S^{(g-d)}$ is irreducible, it follows that the dimension of B is strictly less than $g-d+r$; thus the proposition is proved. Q.E.D.

In sum, we have proved the

Riemann-Kempf Singularity Theorem*. *For* $|\mathrm{D}|$ *a linear system of degree* d *and dimension* r, *the tangent cone*

$$T_{\mu(D)}(W_d)=\bigcup_{D_\lambda\in|D|}\overline{D}_\lambda$$

is the union of the planes $\overline{\mathrm{D}}_\lambda=\mathbb{P}^{\mathrm{d-r-1}}$ *spanned by the points of the divisors* $\mathrm{D}_\lambda\in|\mathrm{D}|$. *It has degree* $\binom{\mathrm{g-d+r}}{\mathrm{r}}$, *and is swept out once by the planes* $\overline{\mathrm{D}}_\lambda$.

In case $d=g-1$, we have seen that W_d is the translate $\Theta_{-\kappa}$ of the theta-divisor Θ on $\mathcal{J}(S)$, and this gives us the result originally stated by Riemann:

$$\mathrm{mult}_{\mu(D)}(\Theta_{-\kappa})=h^0(D);$$

in particular, the singular locus of the theta-divisor corresponds to those divisors of degree $g-1$ which move in a linear system. Using this, we see

*Cf. G. Kempf, On the geometry of a theorem of Riemann, *Annals of Math.*, Vol. 98 (1973), 178–185.

readily that

The singular locus of Θ *has dimension at least* $g-4$.

Proof. Assume, on the contrary, that the singular locus of $\Theta_{-\kappa}$ has dimension $\leqslant g-5$. A generic set of points $p_1,\dots,p_{g-3}$ on the canonical curves spans a $\mathbb{P}^{g-4}$. Let $\varphi: S \to \mathbb{P}^2$ be the projection of S from this $\mathbb{P}^{g-4}$. For points $p \neq q$ on S,

$$\varphi(p) = \varphi(q)$$

if, and only if,

$$\overline{p,p_1,\dots,p_{g-3}} = \overline{q,p_1,\dots,p_{g-3}}$$

which is equivalent to

$$\dim \overline{p,q,p_1,\dots,p_{g-3}} = q-3,$$

i.e.,

$$\dim |p+q+p_1+\cdots+p_{g-3}| = 1.$$

Counting dimensions, we deduce that if $\dim(\Theta_{-\kappa})_s$ were strictly less than $g-4$, there would be ∞^{g-4} divisors $D \in S^{(g-1)}$ with $\dim \overline{D} \leqslant g-3$, and hence for generic choice of $p_1,\dots,p_{g-3}$ the mapping φ would be one-to-one. But then the image curve would be a smooth plane curve of degree $(2g-2)-(g-3)=g+1$ and genus $g(g-1)/2$. Since $g<(g)(g-1)/2$ for $g \geqslant 4$, we have a contradiction. Q.E.D.

The reader may enjoy working out the Riemann singularity theorem in the special case of a linear system of degree 5 and dimension 2 on a Riemann surface S of genus 6. After checking that such a linear system $|D|$ always embeds S as a smooth plane quintic—so that $2D=K$—it is not hard to see that the tangent cone to W_5 at the point $\mu(D)$ is the chordal variety of the Veronese surface $\iota_{2H}(\mathbb{P}^2) \subset \mathbb{P}^5$, and that the singular locus of $\mathbb{T}$ is just the Veronese surface itself.

Special Linear Systems IV

We now have at our disposal the techniques necessary to answer in part the question of the existence of special linear systems on curves. For each pair of integers d and r with $1 \leqslant d \leqslant g-1$ we denote by W_d^r the image of the linear systems of degree d and dimension $\geqslant r$ under the map

$$\mu: S^{(d)} \longrightarrow \mathcal{J}(S).$$

By the proper mapping theorem, W_d^r is an analytic subvariety, and we will show that it has at least the dimension predicted by the naïve dimension count of Section 3 in this chapter.

We begin by computing the homology class of the subvariety $W_d = \mu(S^{(d)})$; the answer is *Poincaré's formula*

$$W_d \sim \frac{1}{(g-d)!}\Theta^{g-d}.$$

Proof. In terms of the real coordinates $x_1,\ldots,x_{2g}$ on $\mathcal{J}(S)$ corresponding to a choice of canonical basis $\delta_1,\ldots,\delta_{2g}$ for $H_1(S,\mathbb{Z})$, we have proved that the Poincaré dual of Θ is

$$\omega = \sum_{\alpha=1}^{g} dx_\alpha \wedge dx_{\alpha+g}.$$

Letting $A=(\alpha_1,\ldots,\alpha_k)$ run over index sets with $1 \leqslant \alpha_1 < \cdots < \alpha_k \leqslant g$ and setting $dx_A = dx_{\alpha_1}\wedge\cdots\wedge dx_{\alpha_k}$, $A+g=(\alpha_1+g,\ldots,\alpha_k+g)$,

$$\omega^{g-d} = (g-d)!(-1)^{g-d}\sum_{\#A=g-d} dx_A \wedge dx_{A+g}.$$

If $J=(j_1,\ldots,j_k)$ runs over index subsets from $(1,\ldots,2g)$, then since the dx_J give a basis for $H^k(\mathcal{J}(S))$ it will suffice to establish that

$$\int_{W_d} dx_J = \frac{1}{(g-d)!}\int_{\mathcal{J}(S)} \omega^{g-d}\wedge dx_J.$$

Using the formula for ω^{g-d}, the right-hand side is

$$\frac{1}{(g-d)!}\int_{\mathcal{J}(S)} \omega^{g-d}\wedge dx_J = \begin{cases} 1 & \text{if } J=(A,A+g) \text{ for} \\ & \text{some } A=(\alpha_1,\ldots,\alpha_d), \\ 0 & \text{otherwise.}\end{cases}$$

On the other hand, since the map $\mu^{(d)}: S^{(d)} \to W_d$ has degree one and $\pi: S^d \to S^{(d)}$ has degree $d!$,

$$\int_{W_d} dx_J = \frac{1}{d!}\int_{S^d} (\mu^d)^* dx_J$$

where $\mu^d = \mu^{(d)}\circ\pi: S^d \to W_d$ is the composition. Now

$$(\mu^d)^* dx_j = \sum_{k=1}^{d} \pi_k^* \mu^* dx_j, \qquad 1 \leqslant j \leqslant 2g,$$

where $\pi_k: S^d \to S$ is projection on the kth factor and $\mu = \mu^1$; since the $\mu^* dx_j \in H^1_{DR}(S)$ are Poincaré dual to cycles δ_j,

$$(*) \qquad \int_S \mu^* dx_\alpha \wedge \mu^* dx_{\alpha+g} = 1$$

and all other integrals of $\mu^* dx_i \wedge \mu^* dx_j$ for $i<j$ are zero. Thus, by

iteration,

$$\int_{S^d} (\mu^d)^* dx_J = \int_{S^d} \bigwedge_{j \in J} \left(\sum_{k=1}^{d} \pi_k^* \mu^* dx_j \right)$$

is zero unless $J = (B, B+g)$ for some $B = (\beta_1, \ldots, \beta_d)$, and in this case by (*)

$$\int_{S^d} (\mu^d)^* dx_{B,B+g} = d!. \qquad \text{Q.E.D.}$$

We now introduce a construction developed by Kempf and Kleiman-Laksov.* Recall from pp. 328–332 that the principal polarization on the Jacobian leads to an identification

$$\operatorname{Pic}^0(\mathcal{J}(S)) \cong \mathcal{J}(S),$$

and also to the Poincaré bundle

$$P \longrightarrow \mathcal{J}(S) \times \mathcal{J}(S)$$

with the properties:

1. under the above identification,

$$P|_{\mathcal{J}(S) \times \lambda} = P(\lambda)$$

is the line bundle corresponding to $\lambda \in \operatorname{Pic}^0(\mathcal{J}(S))$; and

2. if $p_0 \in S$ is a base point, then for any p

$$P(\lambda)_p \cong P(\lambda + \mu(p))_{p_0}.$$

Now we fix a divisor D_0 of degree $n > 2g-2$ on S and set

$$L(\lambda) = P(\lambda) + D_0$$

As λ varies over $\mathcal{J}(S)$ the $L(\lambda)$ vary over all line bundles of degree n on S. Choose $m > n$ generic points $p_1, \ldots, p_m$ on S. By property 2 there is a natural identification

$$L(\lambda)_{p_i} \cong P(\lambda + \mu(p_i))_{p_0} \otimes L_{p_i}.$$

Since, when restricted to $p_0 \times \mathcal{J}(S)$, the Poincaré bundle is topologically trivial, there is an isomorphism

$$L(\lambda)_{p_i} \longrightarrow L_{p_i}$$

which depends C^∞—but *not* holomorphically—on $\lambda \in \mathcal{J}(S)$.

By Riemann-Roch $h^0(\mathcal{O}(L(\lambda))) = n - g + 1$. Since no section of $L(\lambda)$ can vanish at m points, then, there is an injection

$$H^0(\mathcal{O}(L(\lambda))) \longrightarrow \bigoplus_{i=1}^{m} L(\lambda)_{p_i}$$

*S. Kleiman and D. Laksov, On the existence of special divisors, *Amer. J. Math.*, Vol. 93 (1972), 431–436.

whose image S_λ is an $(n-g+1)$-dimensional subspace S_λ varying holomorphically with λ. More precisely, there is a rank m holomorphic vector bundle $E \to \mathcal{J}(S)$ with fibers

$$E_\lambda = \bigoplus_{i=1}^{m} L(\lambda)_{p_i};$$

if $G(n-g+1, E)$ is the associated Grassmannian bundle with fibers

$$G(n-g+1, E)_\lambda = G(n-g+1, E_\lambda),$$

then the subspaces $\{S_\lambda \subset E_\lambda\}$ give a holomorphic section of

$$G(n-g+1, E).$$

The point of the construction is this: for each d we consider the holomorphic subbundle $V_{m-n+d} \subset E$ with fibers

$$V_{m-n+d,\lambda} = \bigoplus_{i=1}^{m-n+d} L(\lambda)_{p_i}.$$

and set

$$\begin{aligned} E_d(\lambda) &= L(\lambda) - p_{m-n+d+1} - \cdots - p_m \\ &= P(\lambda) + D_0 - p_{m-n+d+1} - \cdots - p_m. \end{aligned}$$

We have, then, that

$$h^0(E_d(\lambda)) = \dim S_\lambda \cap V_{m-n+d}.$$

This may be rephrased as follows: combining the C^∞ identifications $L(\lambda)_{p_i} \cong L_{p_i}$ with isomorphisms $L_{p_i} \cong \mathbb{C}$ gives a C^∞ trivialization

$$\varphi: E \longrightarrow \mathcal{J}(S) \times \mathbb{C}^m$$

taking the direct sum decomposition $E = \bigoplus_{i=1}^m L(\lambda)_{p_i}$ into the coordinate axes of $\mathbb{C}^m$. We have thus a C^∞ map

$$\alpha: \mathcal{J}(S) \to G(n-g+1, m)$$

given by $\alpha(\lambda) = \varphi(S_\lambda)$. If $e_1, \ldots, e_m$ is the standard basis of $\mathbb{C}^m$ and $V_{m-n+d} = \{e_1, \ldots, e_{m-n+d}\}$, then

The translate by $\mu(-\mathrm{D}_0 + \mathrm{p}_{m-n+d+1} + \cdots + \mathrm{p}_m)$ *of the variety* $\mathrm{W}_\mathrm{d}^\mathrm{r}$ *is set-theoretically the inverse image under* α *of the Schubert cycle*

$$\sigma_{g+r-d,\ldots,g+r-d}(V_{m-n+d}) = \{\Lambda \in G(n-g+1, m) : \dim(\Lambda \cap V_{m-n+d}) \geq r+1\}.$$

We note that, even though the mapping α is not holomorphic, the inverse image under α of any Schubert cycle corresponding to a flag whose subspaces are coordinate $\mathbb{C}^k$'s in $\mathbb{C}^m$ is a complex analytic subvariety of $\mathcal{J}(S)$. Alternatively, the Schubert conditions for these cycles makes sense in the fibers of the Grassmannian bundle.

The result we are aiming for is the lower bound on $\dim W_r^d$ given on p. 358 below. The crucial step in the proof is the

Lemma. $\alpha(\mathcal{J}(S))$ *meets the special Schubert cycles* $\sigma_{g-d}(V_{m-n+d})$ *transversely away from* $\sigma_{g-d,\,g-d}(V_{m-n+d})$.

Recall that set-theoretically $\alpha^{-1}(\sigma_{g-d}(V_{m-n+d}))$ is a translate of W_d; since W_d is irreducible, the lemma essentially amounts to showing that $\alpha(\mathcal{J}(S))$ is not everywhere tangent to $\sigma_{g-d}(V_{d+1})$ along the intersection.

Assuming for a moment the lemma, the idea behind the remainder of the proof is this: First, by the lemma the fundamental class of W_d is $\alpha^*(\sigma_{g-d})$. Then, by the Poincaré formula

$$\alpha^*(\sigma_{g-d}) \sim \frac{1}{(g-d)!}\Theta^{g-d}.$$

Finally, as proved in Section 6 of Chapter I, every Schubert cycle—in particular $\sigma_{g-d+r,\dots,g-d+r}$—can be expressed as a polynomial in the basic Schubert cycles σ_k; carrying this out explicitly will give

$$\alpha^*(\sigma_{g-d+r,\dots,g-d+r}) = c\Theta^{(r+1)(g+r-d)}, \qquad c \neq 0,$$

and this implies the bound $\dim W_d^r \geqslant g-(r+1)(g+r-d)$.

Proof of the Lemma. Take $\lambda_0 \in \alpha^{-1}\sigma_{g-d}(V_{m-n+d})$. Choose points $q_1,\dots,q_g$ independent on the canonical curve of S, and set

$$E_0 = q_1 + \cdots + q_g - P(\lambda_0).$$

Since $\mu\colon S^{(g)} \to \mathcal{J}(S)$ is one-to-one around $q_1+\cdots+q_g$, then, there will be for all λ near λ_0 uniquely determined points $q_1(\lambda),\dots,q_g(\lambda)$ such that

$$P(\lambda) = [q_1(\lambda)+\cdots+q_g(\lambda)-E_0]; \quad q_i(\lambda_0) = q_i.$$

Set

$$\begin{aligned}
\mathcal{L}(\lambda) &= \mathcal{L}\left(\sum_{i=1}^{g} q_i(\lambda) - E_0 + D_0\right) \\
&= \left\{ f \in \mathfrak{M}(S) : (f) + \sum q_\alpha(\lambda) - \sum r_k + D_0 \geqslant 0 \right\} \\
&\cong H^0(\mathcal{O}(L(\lambda))),
\end{aligned}$$

and consider the map

$$\mathcal{L}\left(\sum q_i(\lambda) - E_0 + D_0\right) \xrightarrow{R} \bigoplus_{i=1}^{m} \mathbb{C}_{p_i}$$

given by

$$R(f) = (f(p_i),\dots,f(p_m)).$$

This defines, for λ in a neighborhood U of λ_0, a map

$$\tilde{\alpha}\colon H^0(\mathcal{O}(L(\lambda))) \longrightarrow \mathbb{C}^m,$$

which is just the map α in a suitable local trivialization of E.

Now let

$$\beta\colon U \to G(m-n+g-1,m)$$

be the composition of $\tilde{\alpha}$ with the natural isomorphism

$$*\colon G(n-g+1,\mathbb{C}^m) \to G(m-n+g-1,\mathbb{C}^{m*}),$$

so that $\beta(\lambda)$ is just the $m-n+g-1$-dimensional vector space of relations on the values $R(f)$ for $f \in \mathfrak{L}(\lambda)$ at the points $p_1,\ldots,p_m$. We want to show that $\beta(U)$ meets the dual Schubert cycle

$$*(\sigma_{g-d}) = \{\Lambda^*\colon \dim(\Lambda^* \cap \operatorname{Ann} V_{m-n+d}) \geq g-d\}$$

transversely at $\beta(\lambda_0)$.

Now, if λ_0 corresponds to a divisor $D \in W_d$ not in W_d^1, that is, if

$$h^0(P(\lambda_0)+D_0-p_{m-n+d+1}-\cdots-p_m) = 1,$$

then for some choice of i between 1 and $m-n+d$, and $j_1,\ldots,j_{g-d}$ between $m-n+d+1$ and m, we will have

$$h^0\big(P(\lambda_0)+D_0-p_{m-n+d+1}-\cdots-p_m-p_i+p_{j_1}+\cdots+p_{j_{g-d}}\big) = 0.$$

For notational convenience we may take $i=1$, $j_1,\ldots,j_{g-d}=m-n+d+1,\ldots,m-n+g$. This means then that the subspace $\alpha(\lambda_0)$ in $\mathbb{C}^m$ is complementary to the subspace

$$\mathbb{C}_{p_2}\oplus\cdots\oplus\mathbb{C}_{p_{m-n+g}}.$$

As we saw in Section 5 of Chapter 2, then, any Λ in a neighborhood of $\beta(\lambda_0)$ is uniquely represented by a matrix of the form

$$\begin{bmatrix} a_{1,1} & 1 & 0 & \ldots & 0 & 0 & a_{1,m-n+g+1} & \cdots & a_{1,m} \\ & 0 & 1 & \ldots & 0 & 0 & . & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ & 0 & 0 & \ldots & 1 & 0 & . & & \\ a_{m-n+g-1,1} & 0 & 0 & \ldots & 0 & 1 & a_{m-n+g-1,m-n+g+1} & \cdots & a_{m-n+g-1,m} \end{bmatrix}$$

The unspecified entries a_{ij} in this matrix are Plucker coordinates on $G(m-n+g-1,m)$ near $\beta(\lambda_0)$; and the Schubert cycle $*\sigma_{g-d}(\operatorname{Ann} V_{m-n+d})$ is given, in these coordinates, by

$$a_{m-n+d,1} = \cdots = a_{m-n+g-1,1} = 0.$$

To give the map β around λ_0 in terms of these coordinates we first find the linear relations on the subspace $R(\mathfrak{L}(\lambda))$ as follows: by Riemann-Roch,

$$h^0\Big(S,\Omega^1\Big(E_0-\sum q_i(\lambda)+\sum p_i-D_0\Big)\Big) = m-n+g-1.$$

Let $\eta_1,\ldots,\eta_{m-n+g-1}$ be a basis of this space of meromorphic differentials. For any function $f \in \mathfrak{L}(\lambda)$ the meromorphic forms $f\cdot\eta_\alpha$ will have poles only at the points p_i, and adding up the residues we have

$$\sum_i \operatorname{Res}_{p_i}(f\cdot\eta_\alpha) = \sum_i f(p_i)\cdot \operatorname{Res}_{p_i}(\eta_\alpha) = 0$$

for each $\alpha = 1,\dots,m-n+g-1$. These are our desired relations, and the matrix $(\operatorname{Res}_{p_i}(\eta_\alpha))$ represents the space $\beta(\lambda)\subset C^{m^*}$. (Note that we do get all the relations on $\mathfrak{L}(\lambda)$ this way: since $h^0(S,\Omega^1(E_0-\sum q_i - D_0))=0$, the matrix $(\operatorname{Res}_{p_i}(\eta_\alpha))$ has maximal rank $m-n+g-1$.)

We can realize the coordinates of the map β geometrically: again by Riemann-Roch,

$$h^0\big(S,\Omega^1\big(E_0+\sum p_i - D_0\big)\big) = m-n+2g-1;$$

consider the corresponding embedding of S in $\mathbb{P}^{m-n+2g-2}$. By hypothesis the points $p_2,\dots,p_{m-n+g}$ and $q_1(\lambda),\dots,q_g(\lambda)$ are linearly independent; let V be the $(m-n+g-2)$-plane spanned by the points $p_2,\dots,p_{m-n+g}$ and $W(\lambda)$ the $(g-1)$-plane spanned by the points $q_i(\lambda)$. (See Figure 15.) Then, since

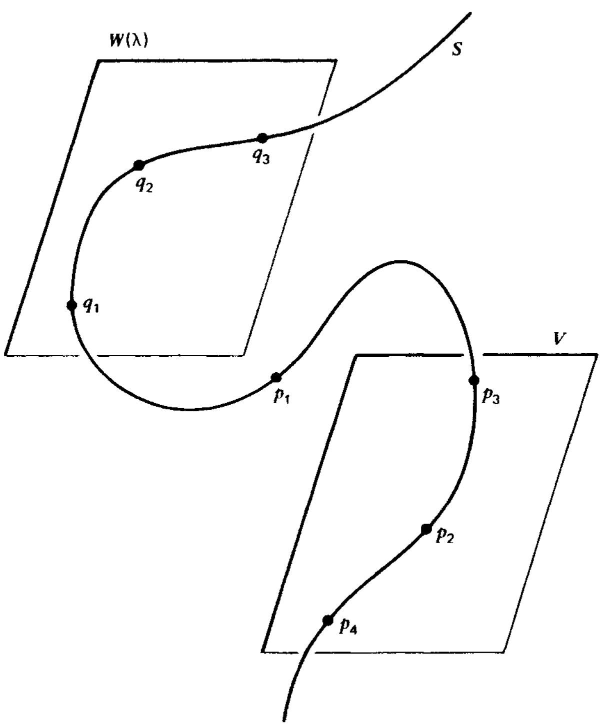


Figure 15

the forms

$$\eta_\alpha \in H^0\Big(S,\Omega^1\Big(E_0-\sum q_i(\lambda)+\sum p_i - D_0\Big)\Big)$$

correspond to the hyperplanes in $\mathbb{P}^{m-n+2g-2}$ containing the points $q_i(\lambda)$, we see that

The Plucker coordinates $a_{1,k},\ldots,a_{m-n+g-1,k}$ *of* $\beta(\lambda)$ *are the homogeneous coordinates of the image* $\pi_{W(\lambda)}(p_k)$ *of the point* p_k *under projection from* $W(\lambda)$ *to* V, *in a coordinate system on* V *in which the points* $p_2,\ldots,p_{m-n+g}$ *represent the coordinate axes.*

Thus, to prove the transversality of β at λ_0 we have to show that the map

$$\pi : U \to V \cong \mathbb{P}^{m-n+g-1}$$

given by

$$\pi(\lambda) = \pi_{W(\lambda)}(p_1)$$

is transverse to the subspace of V spanned by the points $p_2,\ldots,p_{m-n+d}$. To see this, let the point $q_i(\lambda)$ vary. Since $\pi(\lambda)$ is the intersection of V with the subspace $\overline{q_1(\lambda),\ldots,q_g(\lambda),p_1}$, the tangent line to the arc formed by $\pi(\lambda)$ as $q_i(\lambda)$ varies is just the intersection of V with the space spanned by $q_1(\lambda),\ldots,q_{i-1}(\lambda),q_{i+1}(\lambda),\ldots,q_g(\lambda),p_1$ and the tangent line to S at $q_i(\lambda)$. Thus π can fail to be transverse to $p_2,\ldots,p_{m-n+d}$ only if the tangent lines $\{T_{q_i(\lambda)}(S)\}_{i=1,\ldots,g}$, together with the points $p_1,\ldots,p_{m-n+d}$, fail to span $\mathbb{P}^{m-n+2g-2}$. But this is equivalent to the statement

$$\begin{aligned}h^0\Big(S,\Omega^1\Big(E_0+\sum p_i-D_0-2\cdot\sum q_i(\lambda_0)-p_1-\cdots-p_{m-n+d}\Big)\Big)\\ = h^0(S,\Omega^1(-P(\lambda_0)-D_0+p_{m-n+d}+\cdots+p_m-q_i(\lambda_0)))\\ = h^0\Big(S,\Omega^1\Big(-E_d(\lambda_0)-\sum q_i(\lambda_0)\Big)\Big)\neq 0;\end{aligned}$$

and since the points $q_i(\lambda)$ are independent on the canonical curve of S, this is not the case.

It follows from the lemma that

The fundamental class of the variety W_d *on* $\mathcal{J}(S)$ *is the pullback via* α *of the cohomology class of the Schubert cycle* σ_{g-d} *on* $G(n-g+1,m)$.

By the Poincaré formula

$$\alpha^*\sigma_{g-d}\sim\frac{1}{(g-d)!}\Theta^{g-d}.$$

Now, by Giambelli's formula the class of the cycle $\sigma_{g+r-d,\ldots,g+r-d}$ is given by the determinant

$$\sigma_{g+r-d,\ldots,g+r-d} = \begin{vmatrix} \sigma_{g+r-d} & \sigma_{g+r-d+1} & \cdots & \sigma_{g+2r-d} \\ \sigma_{g+r-d-1} & \sigma_{g+r-d} & & \\ \vdots & & & \vdots \\ \sigma_{g-d} & & & \sigma_{g+r-d} \end{vmatrix}$$

and so we can write

$$\alpha^*\sigma_{g+r-d,\ldots,g+r-d} \sim \Theta^{(r+1)(g+r-d)}$$

$$\cdot \begin{vmatrix} \frac{1}{(g+r-d)!} & \frac{1}{(g+r-d+1)!} & \cdots & \frac{1}{(g+2r-d)!} \\ \frac{1}{(g+r-d-1)!} & & & \\ \vdots & & & \vdots \\ \frac{1}{(g+d)!} & & \cdots & \frac{1}{(g+r-d)!} \end{vmatrix}$$

To evaluate in general the determinant $D(x,y)$ of the $(y+1)$-by-$(y+1)$ matrix

$$\left(a_{ij} = \frac{1}{(x-i+j)!} \right),$$

for each $k=2,\ldots,y+1$ in turn multiply the kth column by $(x+k-1)$ and subtract that quantity from the $(k-1)$st column. The new matrix A' will then be

$$\begin{aligned} a'_{ij} &= \frac{1}{(x-i+j)!} - \frac{(x+j)}{(x-i+j+1)!} \\ &= \frac{x-i+j+1}{(x-i+j+1)!} - \frac{x+j}{(x-i+j+1)!} \\ &= \frac{1-i}{(x-i+k+1)!}, \qquad j \neq y+1, \\ a'_{i,y+1} &= a_{i,y+1} = \frac{1}{(x-i+y+1)!}. \end{aligned}$$

In particular, all the entries of the top row will be zero, except $a'_{1,y+1} = 1/(x+y)!$; and its cofactor will be $y!\cdot D(x,y-1)$; thus

$$D(x,y) = \frac{y!}{(x+y)!} D(x,y-1),$$

and, since $D(x,1)=1/x$, this gives

$$D(x,y)=\frac{y!(y-1)!\cdots 0!}{(x+y)!\cdots x!}.$$

Applying this to the cycles $\alpha^*\sigma_{g-d+r,\dots,g-d+r}$, we have

$$\alpha^*\sigma_{g-d+r,\dots,g-d+r}\frac{r!(r-1)!\cdots 0!}{(g+2r-d)!\cdots(g+r-d)!}\cdot\Theta^{(r+1)(g+r-d)}.$$

Suppose now that the variety had dimension less than $g-(r+1)(g-d+r)$. Then we could find a cycle V on $\mathcal{J}(S)$ of dimension $(r+1)(g-d+r)$ missing W_d^r. We would then have

$$\begin{aligned}\alpha^*\sigma_{g-d+r,\dots,g-d+r}(V)&={}^{\#}\big(\alpha(V)\cdot\sigma_{g-d+r,\dots,g-d+r}\big)\\&=0.\end{aligned}$$

But on the other hand

$$\begin{aligned}\alpha^*\sigma_{g-d+r,\dots,g-d+r}(V)&=\frac{r!(r-1)!\cdots 0!}{(g+2r-d)!\cdots(g+r-d)!}\cdot{}^{\#}\big(V\cdot\Theta^{(r+1)(g+r-d)}\big)\\&>0,\end{aligned}$$

since Θ is positive. Thus we have, finally, the

Theorem. *The variety* $W_d^r\subset\mathcal{J}(S)$ *of linear systems of degree* d *and dimension* r *on* S *has dimension at least* $g-(r+1)(g+r-d)$. *In particular, if* $g\geqslant(r+1)(g+r-d)$, *then every Riemann surface of genus* g *has such a linear system.*

Note that it will not always be the case that $\alpha(\mathcal{J}(S))$ will meet the Schubert cycles $\sigma_{g+d+r,\dots,g-d+r}$ transversely—we have seen many cases where the variety $W_d^r=\alpha^{-1}\sigma_{g-d+r,\dots,g-d+r}$ has dimension greater than that expected. Thus we cannot say with certainty what the class of W_d^r will be. It is worth stating, however, the obvious fact that

In case $\alpha(\mathcal{J}(S))$ *is transverse to* $\sigma_{g-d+r,\dots,g-d+r}$, *the class of* W_d^r *is*

$$W_d^r\sim\frac{r!(r-1)!\cdots 0!}{(g+2r-d)!\cdots(g+r-d)!}\Theta^{(r+1)(g+r-d)}.$$

This gives us an "expected" answer to the enumerative questions raised in earlier discussions of special linear systems. For example, we have seen that a generic Riemann surface of genus $g=2k$ will have finitely many pencils of degree $k+1$, and asked how many; in case $\alpha(\mathcal{J}(S))$ is transverse to the Schubert cycle $\sigma_{k,k}$, this number will be

$$W^1_{k+1} = \frac{1}{(k+1)!k!}\cdot\Theta^g = \frac{(2k)!}{(k+1)!k!}$$

since $\Theta^g = g!$. Note that in cases $g=2$, 4, 6, and 8 this gives 1, 2, 5, and 14, which agrees with our previous computations.

Torelli's Theorem

Recall that a *polarized Abelian variety* is a pair $(M,[\omega])$, where M is an Abelian variety and $[\omega]\in H^2(M,\mathbb{Z})$ is a polarizing class on M. A mapping between polarized Abelian varieties $(M,[\omega])$ and $(M',[\omega'])$ is given by a holomorphic mapping $f: M\to M'$ with $f^*([\omega'])=[\omega]$. We have seen that a compact Riemann surface S of genus $g\geqslant 1$ gives a principally polarized Abelian variety $(\mathcal{J}(S),[\omega_S])$ where, in intrinsic terms, $\mathcal{J}(S)$ is the quotient of $(H^0(S,\Omega^1))^*$ by the lattice $\Lambda(S)\cong H_1(S,\mathbb{Z})$ of functionals on $H^0(S,\Omega^1)$ obtained by integration over the 1-cycles of S, and the class

$$[\omega_S]:\ H_2(\mathcal{J}(S),\mathbb{Z})\to\mathbb{Z}$$

is given, in terms of the natural identification

$$H_2(\mathcal{J}(S),\mathbb{Z}) = \Lambda^2(H_1(S,\mathbb{Z})),$$

by

$$[\omega_S](\alpha\wedge\beta) = {}^{\#}(\alpha\cdot\beta).$$

We will now prove that in fact the curve S can be reconstructed from the data $[\mathcal{J}(S),[\omega]]$: this is

Torelli's Theorem. *If* S *and* S′ *are compact Riemann surfaces such that*

$$(\mathcal{J}(S),[\omega_S])\cong(\mathcal{J}(S'),[\omega_{S'}])$$

as polarized Abelian varieties, then $S\cong S'$.

Proof.* We remark first that the essential transcendental step in the proof of Torelli's theorem—as stated above—consists of Riemann's theorem, which relates the divisor Θ as defined up to translation by $[\omega_S]$ to the divisor W_{g-1}; what follows now is a reconstruction of S from W_{g-1}.

We will prove Torelli's theorem first in the case S and S' are nonhyperelliptic. Recall that if $M=\mathbb{C}^g/\Lambda$ is any complex torus, then the tangent

*This proof is due to A. Andreotti, On Torelli's theorem, *Am. J. Math.*, Vol. 80 (1958), pp. 801–821.

spaces $\{T'_\lambda(M)\}_{\lambda\in M}$ are all naturally identified with $\mathbb{C}^g$. Thus if $X \subset M$ is any analytic subvariety of dimension k, $X^* = X - X_{\text{sing}}$ the smooth locus of X, we can define the *Gauss map*

$$\mathcal{G}_X : X^* \to G(k,g)$$

on X^* by

$$\mathcal{G}_X(\lambda) = T'_\lambda(X) \subset T'_\lambda(M) = \mathbb{C}^g.$$

We see immediately that $\mathcal{G}_X$ is intrinsically defined and that it does not vary if X is translated in M.

For example, consider the standard mapping $\mu : S \to \mathcal{J}(S)$ given by

$$\mu(z) = \left(\int_{z_0}^{z}\omega_1,\ldots,\int_{z_0}^{z}\omega_g\right).$$

The Gauss map

$$\mathcal{G}_\mu : S \to G(1,g) = \mathbb{P}^{g-1}$$

is then given by

$$\begin{aligned}\mathcal{G}_\mu(z) &= \left[\frac{\partial}{\partial z}\mu_1(z),\ldots,\frac{\partial}{\partial z}\mu_g(z)\right]\\ &= \left[\omega_1(z)/dz,\ldots,\omega_g(z)/dz\right].\end{aligned}$$

i.e., *The Gauss map of* $\mu(S)$ *is simply the canonical mapping* $\iota_K : S \to \mathbb{P}^{g-1}$.

Now consider the Gauss map

$$\mathcal{G} : \Theta^*_{-\kappa} = W^*_{g-1} \to G(g-1,g) = (\mathbb{P}^{g-1})^*$$

associated to the theta-divisor $\Theta_{-\kappa} \subset \mathcal{J}(S)$. We have seen that a point $\mu(D) \in \Theta_{-\kappa}$ is smooth if and only if the divisor $D = \Sigma p_i$ is regular, and that, if this is the case, the tangent plane to $\Theta_{-\kappa}$ at $\mu(D)$ is the hyperplane spanned by the points p_i on the canonical curve C of S. Since every hyperplane section of C contains only a finite number of points, it follows that the map $\mathcal{G} : \Theta^*_{-\kappa} \to \mathbb{P}^{g-1*}$ is everywhere finite; since the generic hyperplane section consists of $2g-2$ points in general position, we see that generically $\mathcal{G}$ has

$$\binom{2g-2}{g-1} = \frac{g\cdot(g+1)\cdots(2g-2)}{(g-1)!}$$

sheets.

Now let $B \subset (\mathbb{P}^{g-1})^*$ denote the branch locus of $\mathcal{G}$, that is, the image in $(\mathbb{P}^{g-1})^*$ of the set of points in $\Theta^*_{-\kappa}$ where the map $\mathcal{G}$ is singular. At a point $\mu(D) \in \Theta^*_{-\kappa}$, $D = \Sigma p_i \in S^{(g-1)}$, we may take as coordinates on $\Theta_{-\kappa}$ the local coordinates $z_1,\ldots,z_{g-1}$ around the points $p_1,\ldots,p_{g-1}$ on S. It is clear, then, that if the tangent line to any of the points p_i on C lies in the plane spanned by $p_1,\ldots,p_{g-1}$,

$$\frac{\partial}{\partial z_i}\mathcal{G}(\mu(D))=0,$$

i.e. $\mu(D)$ is a singular point of $\mathcal{G}$. Thus if we let $V\subset(\mathbb{P}^{g-1})^*$ denote the proper subvariety of hyperplanes in $\mathbb{P}^{g-1}$ whose intersections with C are not in general position, we see that *any tangent hyperplane* H *to* C *lying outside* V *is in the branch locus* B *of* $\mathcal{G}$.

Conversely, if H is not a tangent hyperplane to C, then H meets C in $2g-2$ distinct points $z_1(H),\ldots,z_{2g-2}(H)$ that vary analytically with H. For H' near H, the $\binom{2g-2}{g-1}$ branches of $\mathcal{G}$ are given by

$$\left\{D_I(H')=z_{i_1}(H')+\cdots+z_{i_{g-1}}(H')\right\}_{\#I=g-1},$$

and since no two of these branches come together at H, H cannot be a branch point. Denoting by $C^*\subset(\mathbb{P}^{g-1})^*$ the set of tangent hyperplanes to C, we have shown that

$$\begin{aligned} B&\subset C^* && \text{everywhere,}\\ B&=C^* && \text{in } (\mathbb{P}^{g-1})^*-V. \end{aligned}$$

But now we see that C^* is irreducible: it is the image of the incidence correspondence $I=\{p,H):H\supset T_p(C)\}\subset C\times\mathbb{P}^{g-1*}$, which is itself fibered over C with irreducible fibers, and hence irreducible.

It follows, then, that in $(\mathbb{P}^{g-1})^*$,

$$\bar{B}=C^*,$$

i.e., *the set of tangent hyperplanes to the canonical curve of* S *is the closure in* $(\mathbb{P}^{g-1})^*$ *of the branch locus of the Gauss map* $\mathcal{G}$ *on the theta-divisor* $\Theta\subset\mathcal{J}(S)$.

Now we are just about finished with the proof: since the data $(\mathcal{J}(S),[\omega])$ determine Θ and $\mathcal{G}$, and hence C^* up to an automorphism of $(\mathbb{P}^{g-1})^*$, all that remains is to show that if C and $C'\subset\mathbb{P}^{g-1}$ are two canonical curves with $C^*=C'^*$, then $C\cong C'$. This is not hard: just note that for every point $p\in C$, the $(g-3)$-plane

$$T_p(C)^*=\left\{H\in(\mathbb{P}^{g-1})^*:\ H\supset T_p(C)\right\}$$

is contained in $C^*=C'^*$. But by Bertini's theorem, the generic element of the linear system

$$\{H\cdot C'\}_{H\in T_p(C)^*}$$

is smooth outside the base locus $T_p(C)\cap C'$ of the system; since $T_p(C)\subset C'^*$, it follows that $T_p(C)$ *must be a tangent line to* C'. We see, moreover, that if $g>3$, no line can be tangent to C' at two points $q,q'\in C'$: if it were, by the geometric version of Riemann-Roch we would have $h^0(2q+2q')=3$, and by Clifford's theorem the curve C' would have to be hyperelliptic. Thus we can write $T_p(C)=T_{p'}(C')$ for a unique point $p'\in C'$, and the map $p\mapsto p'$ gives an isomorphism of C with C'. In case $g=3$, we have seen that there are only a finite number of bitangents to the quartic curves C and C' in $\mathbb{P}^2$; the map $p\mapsto p'$ will extend over these points to give an isomorphism $C\cong C'$.

Essentially the same proof will go over to the hyperelliptic case: again, the branch locus of $\mathcal{G}_\Theta$ in $(\mathbb{P}^{g-1})^*$ will consist of those hyperplanes H such that $\iota_K^{-1}(H\cap C)$ contains multiple points. In the hyperelliptic case, however, this can occur in two ways: if H is tangent to C, or if H passes through any of the points in the branch locus of ι_K. Thus $\overline{B}$ will consist of C^*, together with the hyperplane $p^*=\{H:H\ni p\}\subset(\mathbb{P}^{g-1})^*$ for each p in the branch locus of ι_K. In effect, then, $\overline{B}$ determines C and also determines $2g+2$ points $\{p_i\}$ on C such that S is expressible as a double cover of $C\cong\mathbb{P}^1$ branched exactly at $\{p_i\}$; as we saw in our discussion of hyperelliptic curves, these data determine S. Q.E.D.

The Torelli theorem assures us that theoretically all the behavior of a Riemann surface S is reflected in its polarized Jacobian $(\mathcal{J}(S),[\omega])$. In closing, we would like to make the remark that this is the case in practice as well as in theory—the reader may note that every result we have proved in this chapter can be readily expressed in terms of the geometry of the maps

$$\mu:\ S^{(d)}\to\mathcal{J}(S).$$

Indeed, as we have seen, some of the deeper properties of curves become tractable when expressed in terms of the essentially linear Jacobians. The relationship between curves and their Jacobians is, accordingly, an extraordinarily rich one. Unfortunately, no analogous technique for studying higher-dimensional varieties has been found, though analogous constructions may be made via the Hodge decomposition.

REFERENCES

The literature on Riemann surfaces and algebraic curves is incredibly vast. We do not attempt any sort of complete bibliography but simply list a few sources that best supplement the material in this chapter.

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3
FURTHER TECHNIQUES

We return now to the subject of general analytic varieties in order to develop some further techniques especially intended for higher-dimensional considerations. The motif of this chapter is *differential forms*; the theme is their wide variety of applications, cohomological and otherwise, to complex analytic geometry.

We begin in Sections 1 and 2 with the theory of currents, or differential forms with distribution coefficients. This theory, initiated by de Rham to include both the C^{∞} forms and piecewise smooth chains in the same framework, is especially fruitful in the complex analytic case. A pattern for the entire chapter is established, in that first the real or C^{∞} situation is discussed and then the theory in the richer complex-analytic case developed. The topics in Section 1 are pretty much standard and well described by the table of contents. Coming to Section 2, there has recently been a flurry of research into the remarkable properties of currents associated to complex-analytic varieties. We have taken advantage of this to illustrate how the theory of currents is useful in establishing many of the foundational results required in an analytic treatment of algebraic geometry. For example, there is now an elegant method for recognizing when a current is one defined by an analytic variety, and this affords a direct method for proving such results as Remmert's proper mapping theorem which, although intuitively plausible, were traditionally rather difficult to establish rigorously.

Next we turn to the theory of Chern classes. The definition by differential forms that are polynomials in the curvature matrix provides a quick and easy derivation of the functoriality properties, especially Whitney duality, of the Chern classes as well as exhibiting directly the type and positivity properties in the complex analytic case. This is carried out in the *beginning of Section 3; and in the second part we prove that the Chern* classes are Poincaré dual to the basic Schubert cycles in the Grassmannian. This identifies the differential-form Chern classes with the usual topological ones, at least modulo torsion, and establishes the basic link between the Chern classes and enumerative questions in algebraic geometry, a recurrent theme in the remainder of the book.

In Section 4 the currents and Chern classes are combined to establish two global formulas, the holomorphic Lefschetz fixed-point formula and Bott's residue formula. Although the external circumstances are different, in both cases we use the intersection-and-smoothing theory of currents to reduce the proof to an application of Stokes' theorem where the integrand is a singular differential form. This technique of Stokes' theorem with singularities is ubiquitous throughout the general theory presented in the book—e.g., it appears in Section 1 of Chapter 1, throughout Chapter 2, and again in the general residue theorem given in Section 1 of Chapter 5—and we have to some extent formalized it in Section 1 of this chapter. The "principal part" of the singular differential forms inevitably turns out to be the Bochner-Martinelli kernel—a glance at the index will attest to its presence. Here we wish to point out that what is important is not so much its specific formula but rather its role as a fundamental solution for the $\bar{\partial}$-equation on $\mathbb{C}^n$. This is brought out in Section 1 of Chapter 5, the upshot of which is that *any* such fundamental solution would do—what is essential is the implicit duality. On the other hand, the Bochner-Martinelli kernel is characterized among all fundamental solutions by unitary invariance, and this particular symmetry is manifest in the aforementioned two global formulas, as well as in the Todd polynomials, which appear in the Hirzebruch-Riemann-Roch formula.

This latter is briefly discussed at the end of Section 3 but is not proved in the book. One reason is that there are by now an abundance of proofs from many differing viewpoints, and we have nothing to add. A second reason is that our applications of the Riemann-Roch formula to specific geometric problems occur only for curves and surfaces, and we have given a direct argument establishing the result in these cases.

The final section of this chapter is about spectral sequences, together with a few of their applications to algebraic geometry, which we hope will give at least an idea of how they are utilized. Again the motif is differential forms, especially those with singularities. In our discussion of hypercohomology, the algebraic de Rham theorem, and differentials of the second kind it will be seen that the spectral sequence formalism distills out general patterns and yields sometimes deceptive derivations of classical results whose original proofs were responsible for introducing many of the techniques that have become second nature in the subject.

1. DISTRIBUTIONS AND CURRENTS

Let M be a compact, oriented n-manifold. We know from Poincaré duality and de Rham's theorem that if Γ is a p-cycle on M—for example, an oriented submanifold—then there exists a closed C^∞ $(n-p)$-form ω that is dual to Γ in the sense that for all closed p-forms φ

$$\int_\Gamma \varphi = \int_M \omega \wedge \varphi.$$

We also know that ω is unique up to an exact form.

A special case when all this has been made quite explicit is when M is complex manifold and Γ is the cycle carried by an analytic subvariety V of codimension 1. If V is given locally as the divisor of $f_\alpha \in \mathcal{O}(U_\alpha)$, and if we have chosen positive functions h_α in U_α with $h_\alpha/h_\beta = |f_\alpha/f_\beta|^2$ in $U_\alpha \cap U_\beta$, then $\omega = dd^c \log h_\alpha$ is the Chern class of the line bundle $[V]$. Especially noteworthy is the case when $[V]$ is positive in the sense that, with a suitable choice of the metric $\{h_\alpha\}$ in $[V]$, the real $(1,1)$-form ω is positive.

We shall introduce a formalism that includes both cycles and smooth forms. This will lead to a cohomology theory to which both the ordinary singular and de Rham's theories map, and both maps will be isomorphisms.

Definitions; Residue Formulas

We make our definitions first on $\mathbb{R}^n$. Let $C_c^\infty(\mathbb{R}^n)$ be the vector space of compactly supported smooth functions on $\mathbb{R}^n$. If $x=(x_1,\ldots,x_n)$ are coordinates on $\mathbb{R}^n$, we let $D_i = \partial/\partial x_i$ and $D^\alpha = D_1^{\alpha_1}\ldots D_1^{\alpha_n}$ for $\alpha=(\alpha_1,\ldots,\alpha_n) \in (\mathbb{Z}^+)^n$. The C^p-topology is defined on $C_c^\infty(\mathbb{R}^n)$ by saying that a sequence $\varphi_n \to 0$ in case there is a compact set K with all $\operatorname{supp}\varphi_n \subset K$ and with

$$D^\alpha \varphi_n(x) \longrightarrow 0$$

uniformly for $x \in K$ and all α satisfying $[\alpha] = \alpha_1 + \ldots + \alpha_n \leqslant p$. The C^∞ topology is defined by saying that $\varphi_n \to 0$ in case all $\operatorname{supp}\varphi_n \subset K$ and $\varphi_n \to 0$ in the C^p topology for each p.

Definition. A *distribution* on $\mathbb{R}^n$ is a linear map $T: C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ that is continuous in the C^∞ topology. The vector space of distributions on $\mathbb{R}^n$ is denoted $\mathfrak{D}(\mathbb{R}^n)$.

We say that a distribution is *of order* p if it is continuous in the C^p-topology. Now any linear map from a topological vector space V to $\mathbb{C}$ is continuous $\Leftrightarrow$ the inverse of the unit ball in $\mathbb{C}$ is open in the topology of V. Since for the space $C_c^\infty(K)$ of functions supported in a compact set K the C^∞ topology is the union of the C^p topologies, we see that any distribution is locally of finite order.

Examples

1. If $\psi(x)$ is a locally L^1 function on $\mathbb{R}^n$, then we may define a distribution T_ψ of order zero by

$$T_\psi(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx.$$

Here $dx = dx_1 \wedge \ldots \wedge dx_n$, and we always assume that $\mathbb{R}^n$ is oriented by this form.

2. The *δ-function* is the distribution defined by

$$\delta(\varphi) = \varphi(0).$$

Next we extend the operators D_i to the space of distributions by setting

$$(D_i T)(\varphi) = -T(D_i\varphi).$$

If $T = T_\psi$ is the distribution associated to a function ψ of class C^1, then for $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\begin{aligned}(D_i T_\psi)(\varphi) &= -T_\psi(D_i\varphi)\\ &= -\int_{\mathbb{R}^n} \psi(x)\left[(\partial\varphi/\partial x_i(x))\right]dx\\ &= \int_{\mathbb{R}^n} \frac{\partial\psi(x)}{\partial s_i}\varphi(x)\,dx\end{aligned}$$

(by Stokes' theorem)

$$= (T_{D_i\psi})(\varphi),$$

so that our extended notion of differentiating distributions makes sense.

An example that illustrates the principle underlying the various residue theorems we shall discuss is obtained by considering the locally L^1 function $\psi(x)$ on $\mathbb{R}$ defined by

$$\begin{cases} \psi(x) = 0, & x < 0, \\ \psi(x) = 1, & x \geqslant 1. \end{cases}$$

Formally—i.e., ignoring the singularity—$\psi'(x) = 0$. However, the distributional derivative is given by

$$\begin{aligned}(DT_\psi)(\varphi) &= -\int_{-\infty}^{+\infty} \varphi'(x)\psi(x)\,dx\\ &= -\int_0^\infty \varphi'(x)\,dx\\ &= \varphi(0),\end{aligned}$$

i.e.,

$$DT_\psi = \delta.$$

The general principle will be

$$DT_\psi - T_{D\psi} = \text{“residue,”}$$

where $D\psi$ is the derivative of ψ computed formally. We will expound this in greater detail in a little while.

Another picture of distributions is obtained by looking at the torus $T = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. In Section 6 of Chapter 0 on the proof of the Hodge theorem we defined the space $\mathfrak{D}(T)$ of distributions on the torus and showed that

$$\mathfrak{D}(T) = \bigcup_s H_s,$$

where H_s is the Sobolev space of formal Fourier series $T = \sum u_\xi e^{i(\xi, x)}$ satisfying $\sum(1+\|\xi\|^2)^s|u_\xi|^2 < \infty$. By the Sobolev lemma proved there

$$C^\infty(T) = \bigcap_s H_s,$$

and for $\varphi = \sum \varphi_\xi e^{i(\xi,x)} \in C^\infty(T)$

$$T(\varphi) = \sum_\xi u_\xi \varphi_\xi.$$

The δ-function is given by

$$\delta = \sum_\xi e^{i(\xi,x)}.$$

As usual, the torus will provide an excellent illustration of our general remarks.

Now let $A_c^q(\mathbb{R}^n)$ be the space of C^∞ q-forms on $\mathbb{R}^n$ with compact support. In the obvious way the topology on $C_c^\infty(\mathbb{R}^n)$ may be used componentwise to make $A_c^q(\mathbb{R}^n)$ into a complete topological vector space.

DEFINITION. The topological dual of $A_c^{n-q}(\mathbb{R}^n)$ is the space of *currents* of degree q, and is denoted by $\mathfrak{D}^q(\mathbb{R}^n)$.

Examples

1. In the following examples we will denote by $L^q(\mathbb{R}^n, \text{loc})$ the q-forms $\psi = \sum \psi_I(x)\,dx_I$ whose coefficient functions are locally L^1 functions on $\mathbb{R}^n$. For such a ψ there is an associated current $T_\psi \in \mathfrak{D}^q(\mathbb{R}^n)$ defined by

$$T_\psi(\varphi) = \int_{\mathbb{R}^n} \psi \wedge \varphi, \qquad \varphi \in A_c^{n-q}(\mathbb{R}^n).$$

2. If Γ is a piecewise smooth, oriented $(n-q)$ chain in $\mathbb{R}^n$, then Γ defines a current $T_\Gamma \in \mathfrak{D}^q(\mathbb{R}^n)$ by

$$T_\Gamma(\varphi) = \int_\Gamma \varphi, \qquad \varphi \in A_c^{n-q}(\mathbb{R}^n).$$

In general if we call the *support* of the current T the smallest closed set S such that $T(\varphi)=0$ for all $\varphi \in A_c^{n-q}(\mathbb{R}^n - S)$, then clearly $\operatorname{supp}(T_\Gamma)=\Gamma$.

The exterior derivative on smooth forms induces

$$d\colon\ \mathfrak{D}^q(\mathbb{R}^n) \longrightarrow \mathfrak{D}^{q+1}(\mathbb{R}^n)$$

defined by

$$(dT)(\varphi) = (-1)^{q+1} T(d\varphi), \qquad \varphi \in A_c^{n-q-1}(\mathbb{R}^n).$$

Then $d^2=0$. If $T=T_\psi$ for some smooth form $\psi \in A^q(\mathbb{R}^n)$, by Stokes' theorem

$$\begin{aligned} dT_\psi(\varphi) &= (-1)^{q+1}\int_{\mathbb{R}^n} \psi \wedge d\varphi \\ &= -\int_{\mathbb{R}^n} d(\psi \wedge \varphi) + \int_{\mathbb{R}^n} d\psi \wedge \varphi \\ &= T_{d\psi}(\varphi). \end{aligned}$$

Similarly, for T_Γ as in the second example,

$$\begin{aligned} dT_\Gamma(\varphi) &= (-1)^{q+1}\int_\Gamma d\varphi \\ &= (-1)^{q+1}\int_{\partial\Gamma} \varphi \\ &= (-1)^{q+1} T_{\partial\Gamma}(\varphi). \end{aligned}$$

Thus, d on the currents induces the usual exterior derivative on the smooth forms and $\pm\partial$ on the piecewise smooth chains.

Here is an example that interpolates between these two.

3. Suppose that $\psi \in L^q(\mathbb{R}^n, \mathrm{loc})$ is C^∞ outside a closed set S, and assume moreover that $d\psi$ on $\mathbb{R}^n - S$ extends to a locally L^1 form on $\mathbb{R}^n$. We define the *residue* $R(\psi)$ by the equation of currents

$$dT_\psi = T_{d\psi} + R(\psi). \tag{$*$}$$

It is clear that the support $\operatorname{supp} R(\psi) \subset S$.

For example, suppose that we consider the Cauchy kernel

$$\kappa = \frac{1}{2\pi\sqrt{-1}}\,\frac{dz}{z}$$

on $\mathbb{C}$. Then $\kappa \in L^{1,0}(\mathbb{C}, \mathrm{loc})$ and is C^∞ on $\mathbb{C} - \{0\}$; moreover, $d\kappa = \bar{\partial}\kappa = 0$

there. The general version of Cauchy's formula

$$\varphi(0) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial\varphi(z)}{\partial\bar{z}} \frac{dz \wedge d\bar{z}}{z}, \qquad \varphi \in C_c^\infty(\mathbb{C}),$$

given in Section 1 of Chapter 0 translates into the equation of currents

$$\bar{\partial}(T_\kappa) = \delta_{\{0\}}.$$

Equivalently, the residue

$$R\left(\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z}\right) = \delta_{\{0\}}$$

of the Cauchy kernel is the δ-function at the origin.

We will generalize this, first to $\mathbb{R}^n$ and then to $\mathbb{C}^n$. The notations

$$\begin{aligned} r^2 &= \sum_i x_i^2 = \|x\|^2, \\ r\,dr &= \sum_i x_i\,dx_i, \\ \Phi(x) &= dx_1 \wedge \cdots \wedge dx_n, \\ \Phi_i(x) &= (-1)^{i-1} x_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \end{aligned}$$

will be used. We will also let C_n stand for a generic constant depending only on n. Finally, the operators such as $*$ that depend on a metric will refer to $ds^2 = \sum_i (dx_i)^2$.

We note that the function r^{-s} is locally integrable for $s < n$ but not for $s = n$. Define

$$\begin{aligned} \sigma &= C_n \frac{\sum \Phi_i(x)}{\|x\|^n} \\ &= C_n \frac{*r\,dr}{r^n}. \end{aligned}$$

This form σ belongs to $L^{n-1}(\mathbb{R}^n, \text{loc})$, is invariant under the proper orthogonal group, and is smooth on $\mathbb{R}^n - \{0\}$. Since $d\Phi_i(x) = \Phi(x)$, it follows that in $\mathbb{R}^n - \{0\}$

$$\begin{aligned} d\sigma &= C_n\left(\frac{n\Phi(x)}{r^n} - \frac{n r\,dr \wedge *(r\,dr)}{r^{n+2}}\right) \\ &= 0. \end{aligned}$$

By Stokes' theorem, then, the integral

$$\int_{\|x\| = \varepsilon} \sigma$$

of σ over a sphere is independent of the radius $\varepsilon > 0$. Consequently, choosing C_n properly, σ is the unique form on $\mathbb{R}^n - \{0\}$ that is invariant under proper rotations, orthogonal to the normal dr to spheres, and that has integral 1 over a sphere of any radius. In $\mathbb{R}^2$ with coordinates $(x,y) = (r\cos\theta, r\sin\theta)$,

$$\sigma = \frac{1}{2\pi}\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{1}{2\pi}d\theta.$$

In general if

$$x = r\omega,$$

where $r = \|x\|$ and $\omega \in S^{n-1}$ are polar coordinates in $\mathbb{R}^n$, then we may write

$$\sigma = C_n\, d\omega.$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, by Stokes' theorem,

$$\begin{aligned} -\int_{\mathbb{R}^n} d\varphi \wedge \sigma &= \lim_{\varepsilon \to 0} -\int_{\mathbb{R}^n - \{\|x\| \leq \varepsilon\}} d\varphi \wedge \sigma \\ &= \lim_{\varepsilon \to 0} \int_{\|x\| = \varepsilon} \varphi\sigma \\ &= \varphi(0). \end{aligned}$$

Thus, the equation of currents

$$dT_\sigma = \delta_{\{0\}}$$

is valid, as is the residue relation

$$R(\sigma) = \delta_{\{0\}}.$$

On $\mathbb{C}^n \cong \mathbb{R}^{2n}$ the form σ decomposes into type, each component of which is invariant under the unitary group. Up to a constant to be specified in a moment, the component of type $(n, n-1)$ is

$$\begin{aligned} \beta &= C_n \frac{\left(\sum \overline{\Phi_i(z)} \wedge \Phi(z)\right)}{\|z\|^{2n}} \\ &= C_n \frac{*(r\bar{\partial}r)}{r^{2n}}. \end{aligned}$$

Clearly $\beta \in L^{n,n-1}(\mathbb{C}^n, \text{loc})$, and since $\overline{\partial\Phi_i(z)} = \overline{\Phi(z)}$, the same computation as for σ shows that

$$\bar{\partial}\beta = 0 \qquad \text{on } \mathbb{C}^n - \{0\}.$$

Since $d = \bar{\partial}$ on forms of type (n,q), we may repeat the previous argument to conclude that for a suitable choice of constant,

$$\bar{\partial} T_\beta = \delta_{\{0\}},$$

and therefore the residue

$$R(\beta) = \delta_{\{0\}}.$$

Explicitly, for $\varphi \in C_c^\infty(\mathbb{C}^n)$,

$$\varphi(0) = \int_{\mathbb{C}^n} \bar{\partial}\varphi \wedge \beta,$$

and just as in the one-variable case this formula may be extended to noncompactly supported forms to obtain

$$\varphi(0) = \int_{B[r]} \bar{\partial}\varphi \wedge \beta + \int_{\partial B[r]} \varphi\beta,$$

where $B[r] = \{z \in \mathbb{C}^n : \|z\| \leqslant r\}$ is the ball of radius r in $\mathbb{C}^n$. In case $\varphi \in \mathcal{O}(\mathbb{C}^n)$ is holomorphic, this reduces to the *Bochner-Martinelli formula*

$$\varphi(0) = \int_{\|z\| = r} \varphi(z)\beta(z, \bar{z}).$$

It is possible to prove these formulas by reducing to the one-variable case in a manner that sheds some additional light on the expression for β. First, we shall show that

$$\beta = C_n(\partial \log\|z\|^2) \wedge (\partial\bar{\partial} \log\|z\|^2)^{n-1}.$$

Proof. Denote by γ the form on the right-hand side of this equation. Since

$$\partial \log\|z\|^2 = \frac{(dz, z)}{(z, z)},$$

$$\partial\bar{\partial} \log\|z\|^2 = \partial\left(\frac{(z, dz)}{(z, z)}\right)$$

$$= \frac{(dz, dz)}{(z, z)} - \frac{(dz, z) \wedge (z, dz)}{(z, z)^2},$$

and since $(dz, z) \wedge (dz, z) = 0$,

$$\gamma = C_n' \frac{(dz, z) \wedge (dz, dz)^{n-1}}{\|z\|^{2n}}.$$

The numerator is

$$C_n'\left(\sum_j \bar{z}_i dz_i\right) \wedge \left(\sum_j dz_j \wedge d\bar{z}_j\right)^{n-1}$$

$$= C_n''\left(\sum (-1)^{i-1} \bar{z}_i d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n\right),$$

which implies the result.

Now we recall that under the projection

$$\mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}$$

the Kähler form of the Fubini-Study metric pulls back to

$$\Omega = dd^c \log\|z\|^2$$
$$= \frac{\sqrt{-1}}{4\pi} \partial\bar{\partial} \log\|z\|^2.$$

Let $\tilde{\mathbb{C}}^n$ be the blow-up of $\mathbb{C}^n$ at the origin and

$$\pi: \tilde{\mathbb{C}}^n \to \mathbb{P}^{n-1}$$

the extension to $\tilde{\mathbb{C}}^n$ of the projection. $\tilde{\mathbb{C}}^n$ is the total space of the universal line bundle over projective space, and $\pi^*\Omega$ is smooth up on $\tilde{\mathbb{C}}^n$. Thus, on $\tilde{\mathbb{C}}^n$

$$\pi^*\beta = C_n \theta \wedge (\pi^*\Omega)^{n-1},$$

where $\theta = \partial \log\|z\|^2$ is a (1,0)-form that on each fiber $\{\lambda z\}_{\lambda \in \mathbb{C}}$ of $\tilde{\mathbb{C}}^n \to \mathbb{P}^{n-1}$ reduces to $d\lambda/\lambda$. Summarizing, *$\pi^*\beta$ on $\tilde{\mathbb{C}}^n$ is the pullback of the standard volume form on $\mathbb{P}^{n-1}$ times a form θ that reduces to the Cauchy kernel in each fiber of $\tilde{\mathbb{C}}^n \to \mathbb{P}^{n-1}$*. Using this interpretation, the n-variable Bochner-Martinelli formulas may, by pulling forms back to $\tilde{\mathbb{C}}^n$ and making an obvious iteration of the integrals, be reduced to the one-variable Cauchy formula.

A final remark is that the definition of distributions and currents may be localized. Thus, for U open in $\mathbb{R}^n$ the space $\mathfrak{D}(U)$ of distributions on U is the dual of $C_c^\infty(U)$ with the obvious topology. Since a diffeomorphism $f: U \to V$ (U, V open in $\mathbb{R}^n$) induces a topological isomorphism $f^*: C_c^\infty(V) \to C_c^\infty(U)$, we may define the spaces $\mathfrak{D}(M)$ and currents $\mathfrak{D}^*(M)$ on a manifold M.

Smoothing and Regularity

A distribution $T \in \mathfrak{D}(\mathbb{R}^n)$ is said to be *smooth* in case $T = T_\psi$ for a C^∞ function $\psi(x)$ on $\mathbb{R}^n$. We shall now make precise the sense in which the smooth distributions are dense among all distributions.

Let $\chi(x) \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function supported in a neighborhood of the origin, with

$$\int_{\mathbb{R}^n} \chi(x)\, dx = 1.$$

In a little while we shall assume that χ is *radially symmetric*, i.e., in polar coordinates $x = r\omega$

$$\chi(x) = \chi(r).$$

We set

$$\chi_\varepsilon(x) = \frac{1}{\varepsilon^n}\chi\left(\frac{x}{\varepsilon}\right).$$

If $\operatorname{supp}\chi = K$, then $\operatorname{supp}\chi_\varepsilon = \varepsilon K$ and

$$\int_{\mathbb{R}^n}\chi_\varepsilon(x)\,dx = 1.$$

We remark that

$$T_{\chi_\varepsilon} \to \delta \qquad \text{as } \varepsilon \to 0$$

in the sense that for any test function $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n}\chi_\varepsilon(x)\varphi(x)\,dx = \varphi(0).$$

To see this, simply note that

$$\min_{x\in\varepsilon K}\varphi(x) \leqslant \int_{\mathbb{R}^n}\chi_\varepsilon(x)\varphi(x)\,dx \leqslant \max_{x\in\varepsilon K}\varphi(x),$$

which tends to $\varphi(0)$ as $\varepsilon \to 0$.

Having "smoothed" the δ-function, for a general distribution $T \in \mathfrak{D}(\mathbb{R}^n)$ we consider the function

$$T_\varepsilon(x) = T_y\big(\chi_\varepsilon(x-y)\big),$$

where we use the subscript y on T to indicate that we consider $\chi_\varepsilon(x-y)$ as a function of y and apply T accordingly. $T_\varepsilon(x)$ is a C^∞ function on $\mathbb{R}^n$ with derivatives

$$D^\alpha T_\varepsilon(x) = \pm T_y(D_x^\alpha \chi_\varepsilon(x-y)).$$

By an abuse of notation, we denote by T_ε the distribution on $\mathbb{R}^n$ defined by the function $T_\varepsilon(x)$.

The following formal properties of the T_ε's will be proved:

1. $(T_\varphi)_\varepsilon = T_{\varphi_\varepsilon}$ for $\varphi(x) \in C^\infty(\mathbb{R}^n)$.
2. $T_\varepsilon(\psi) = T(\psi_\varepsilon)$ for $\psi(x) \in C_c^\infty(\mathbb{R}^n)$.
3. $(DT)_\varepsilon = D(T_\varepsilon)$ for $D = \partial^\alpha/\partial x^\alpha$.

Proof of 1. For $\psi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}(T_\varphi)_\varepsilon(\psi) &= \int_{\mathbb{R}^n} T_{\varphi_y}(\chi_\varepsilon(x-y)\psi(x)\,dx)\\ &= \int\int \varphi(y)\chi_\varepsilon(x-y)\psi(x)\,dx\,dy\\ &= T_{\varphi_\varepsilon}(\psi)\end{aligned}$$

by interchanging the order of integration.

Proof of 2. Since T is linear,

$$
\begin{aligned}
T(\psi_\varepsilon) &= T_y\left(\int \psi(x)\chi_\varepsilon(x-y)\,dx\right)\\
&= \int \psi(x)\,T_y\chi_\varepsilon(x-y)\,dx\\
&= \int \psi(x)\,T_\varepsilon(x)\,dx\\
&= T_\varepsilon(\psi).
\end{aligned}
$$

Proof of 3. We may suppose that $D=\partial/\partial x_i$. If $T=T_\psi$ for $\psi\in C^\infty(\mathbb{R}^n)$ and $\psi\in C_c^\infty(\mathbb{R}^n)$,

$$
\begin{aligned}
DT_\varepsilon(\varphi) &= T_\varepsilon(-D\varphi)\\
&= \int\int -\frac{\partial\varphi}{\partial x_i}(x)\chi_\varepsilon(x-y)\psi(y)\,dx\,dy\\
&= \int \chi_\varepsilon(u)\left(\int -\frac{\partial\varphi}{\partial x_i}(x)\psi(x-u)\,dx\right)du\\
&= \int \chi_\varepsilon(u)\left(\int \varphi(x)\frac{\partial\psi}{\partial x_i}(x-u)\,dx\right)du\\
&= \int\int \frac{\partial\psi}{\partial y_i}(y)\varphi(x)\chi_\varepsilon(x-y)\,dx\,dy\\
&= (DT)_\varepsilon(\varphi).
\end{aligned}
$$

For a general $T\in\mathfrak{D}(\mathbb{R}^n)$ and $\varphi\in C_c^\infty(\mathbb{R}^n)$,

$$
\begin{aligned}
(DT)_\varepsilon(\varphi) &= (DT)(\varphi_\varepsilon)\\
&= T(-D\varphi_\varepsilon)\\
&= T(-D\varphi)_\varepsilon \qquad \text{(by the previous step)}\\
&= T_\varepsilon(-D\varphi)\\
&= DT_\varepsilon(\varphi),
\end{aligned}
$$

which proves assertion 3.

In particular, we conclude that for any $\psi\in C_c^\infty(\mathbb{R}^n)$

$$D\psi_\varepsilon \to D\psi$$

uniformly, and consequently

$$T_\varepsilon(\psi)\to T(\psi) \qquad \text{as } \varepsilon\to 0.$$

There are, of course, many subtle questions about the convergence of the smoothing process in particular norms, but we need not get into these here.

A current $T \in \mathfrak{D}^q(\mathbb{R}^n)$ may be considered as a differential form

$$T = \sum_{\#I=q} T_I \, dx_I$$

with distribution coefficients T_I defined by

$$T_I(\varphi) = \pm T(\varphi \, dx_{I^0})$$

for $\varphi \in C_c^\infty(\mathbb{R}^n)$. Here I^0 is the index set defined by $*dx_I = \pm dx_{I^0}$. The smoothing

$$T_\varepsilon = \sum_I (T_I)_\varepsilon dx_I$$

satisfies

$$T_\varepsilon(\varphi) \to T(\varphi) \qquad \text{as } \varepsilon \to 0, \qquad \varphi \in A_c^{n-q}(\mathbb{R}^n),$$

and

$$dT_\varepsilon = d(T_\varepsilon).$$

We will now use smoothing to prove some regularity results concerning the Laplace equation on distributions

$$\Delta T = S,$$

where

$$\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}.$$

Lemma. *If* $T \in \mathfrak{D}(\mathbb{R}^n)$ *satisfies* $\Delta T = 0$, *then* $T = T_\varphi$ *for some* $\varphi \in C^\infty(\mathbb{R}^n)$ *with* $\Delta\varphi = 0$.

Proof. Smooth functions φ satisfying $\Delta\varphi = 0$ are said to be *harmonic*. We shall first prove that harmonic functions obey the *mean-value property*

$$\varphi(y) = \int_{\|x-y\|=\varepsilon} \varphi(x)\sigma_y(x),$$

where, if

$$\sigma = C_n \frac{*(r\,dr)}{r^n}$$

is the form encountered in the preceding section, then

$$\sigma_y(x) = \sigma(x-y)$$

is the invariant volume form on the sphere $\|x-y\| = \varepsilon$ having total area 1. Since the Laplacian is invariant under translations and proper rotations, it will suffice to prove the mean value property when $y = 0$.

We shall apply Stokes' theorem twice to spherical shells $B[\delta,\varepsilon] = \{\delta \leqslant \|x\| \leqslant \varepsilon\}$. The first time we take the $(n-1)$-form

$$\eta = \varphi\sigma.$$

Since $d\sigma=0$,

$$d\eta = C_n d\varphi \wedge \frac{*(r\,dr)}{r^n} = \pm C_n *d\varphi \wedge \frac{dr}{r^{n-1}},$$

and Stokes' theorem gives

$$(*) \qquad \pm C_n \int_{B[\delta,\varepsilon]} *d\varphi \wedge \frac{dr}{r^{n-1}} = \int_{\|x\|=\varepsilon} \varphi\sigma - \int_{\|x\|=\delta} \varphi\sigma.$$

We write

$$*d\varphi \wedge \frac{dr}{r^{n-1}} = *d\varphi \wedge d\gamma,$$

where

$$\gamma = \begin{cases} \log r & \text{in case } n=2, \\ \left(-\dfrac{1}{n-2}\right)\dfrac{1}{r^{n-2}} & \text{in case } n \geqslant 3. \end{cases}$$

Now

$$d*d\varphi = \pm\Delta\varphi\, dx = 0,$$

so that

$$*d\varphi \wedge d\gamma = d(\gamma *d\varphi).$$

Applying Stokes' theorem once again, we may express the integral on the left of $(*)$ as a difference of integrals

$$C_n \int_{\|x\|=\rho} \gamma *d\varphi.$$

For fixed ρ, this integral is a constant times

$$\int_{\|x\|=\rho} *d\varphi = \int_{\|x\|\leqslant\rho} d*d\varphi = 0.$$

Thus, for a harmonic function φ,

$$\int_{\|x\|=\delta} \varphi\sigma = \int_{\|x\|=\varepsilon} \varphi\sigma.$$

If we let $\delta \to 0$, the left-hand side tends to $\varphi(0)$, and the mean-value property is established.

Now we assume that $\chi(x)$ is radically symmetric. Then a harmonic function φ satisfies $\varphi_\varepsilon=\varphi$ for $\varepsilon>0$. By the formal properties 1–3 of smoothing, a harmonic distribution T satisfies $T_\delta=T$ for $\delta>0$. More precisely, by property 3,

$$\Delta T_\varepsilon = (\Delta T)_\varepsilon = 0,$$

so that $T_\varepsilon = T_{\psi_\varepsilon}$ for a harmonic function ψ_ε. Then

$$(T_\varepsilon)_\delta = T_{(\psi_\varepsilon)_\delta} = T_{\psi_\varepsilon} = T_\varepsilon,$$

and for $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} T(\varphi) &= \lim_{\varepsilon\to 0} T_\varepsilon(\varphi) \\ &= \lim_{\varepsilon\to 0} (T_\varepsilon)_\delta(\varphi) \\ &= \lim_{\varepsilon\to 0} T_\varepsilon(\varphi_\delta) \\ &= T(\varphi_\delta) \qquad \text{(by property 2)} \\ &= T_\delta(\varphi); \end{aligned}$$

i.e.,

$$T = T_\delta$$

is a smooth distribution as desired. Q.E.D.

We now extend regularity to the inhomogeneous equation.

Lemma. *If* $T \in \mathfrak{D}(\mathbb{R}^n)$ *satisfies*

$$\Delta T = \eta \in C_c^\infty(\mathbb{R}^n),$$

then $T = T_\psi$ *for some* $\psi \in C^\infty(\mathbb{R}^n)$ *such that* $\Delta\psi = \eta$.

Proof. We will write down an explicit solution $\rho \in C^\infty(\mathbb{R}^n)$ to the equation

$$\Delta\rho = \eta,$$

using the classical *Green's function*

$$G(x,y) = \begin{cases} \dfrac{1}{\|x-y\|^{n-2}}, & n \geqslant 3, \\ \log\|x-y\|, & n=2. \end{cases}$$

Then

$$\Delta(T - T_\rho) = 0,$$

and this lemma follows from the preceding one.

We shall assume that $n \geqslant 3$, the case $n=2$ being essentially the same. Define

$$\begin{aligned} \rho(x) &= C_n \int_{y\in\mathbb{R}^n} \frac{\eta(y)\,dy}{\|x-y\|^{n-2}} \\ &= \pm C_n \int_{u\in\mathbb{R}^n} \frac{\eta(x-u)\,du}{\|u\|^{n-2}}, \end{aligned}$$

where the equality follows from the change of variables $y = x - u$. The second expression shows that $\rho(x)$ is smooth and

$$\Delta\rho(x) = \pm C_n \int_{u\in\mathbb{R}^n} \frac{\Delta\eta(x-u)\,du}{\|u\|^{n-2}}.$$

We will prove that this integral is $\eta(x)$. By translating x to the origin, what must be verified is the *Poisson formula*

$$\eta(0) = C_n \int_{\mathbb{R}^n} \frac{\Delta\eta(x)\,dx}{\|x\|^{n-2}}.$$

In polar coordinates $x = r\omega$, where $r = \|x\|$ and $\omega \in S^{n-1}$,

$$\begin{aligned}\frac{\Delta\eta(x)\,dx}{\|x\|^{n-2}} &= \frac{d*d\eta}{r^{n-2}} \\ &= d\left(\frac{*d\eta}{r^{n-2}}\right) \pm \left(\frac{1}{n-2}\right)\frac{dr\wedge *d\eta}{r^{n-1}} \\ &= d\left(\frac{*d\eta}{r^{n-2}}\right) \pm \left(\frac{1}{n-2}\right)\frac{d\eta\wedge *(r\,dr)}{r^{n}} \\ &= d\left(\frac{*d\eta}{r^{n-2}}\right) \pm \left(\frac{1}{n-2}\right)d(\eta\sigma).\end{aligned}$$

We apply Stokes' theorem to the region $\mathbb{R}^n - \{\|x\| \leqslant \varepsilon\}$ and to each of the forms on the right-hand side. Thus

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\Delta\eta(x)\,dx}{\|x\|^{n-2}} &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n - \{\|x\|\leqslant\varepsilon\}} \frac{\Delta\eta(x)\,dx}{\|x\|^{n-2}} \\ &= A_\varepsilon + B_\varepsilon,\end{aligned}$$

where

$$\begin{aligned}A_\varepsilon &= \pm\int_{\|x\|=\varepsilon}\frac{*d\eta}{\varepsilon^{n-2}} = \frac{1}{\varepsilon^{n-2}}\int_{\|x\|\leqslant\varepsilon}\Delta\eta\,dx \\ &\to 0 \qquad \text{as } \varepsilon\to 0,\end{aligned}$$

since $\Delta\eta$ is C^∞, and so

$$\int_{\|x\|\leqslant\varepsilon}\Delta\eta = O(\varepsilon^n);$$

and where

$$\begin{aligned}B_\varepsilon &= \text{constant}\int_{\|x\|=\varepsilon}\eta\sigma \\ &\to \eta(0) \qquad \text{as } \varepsilon\to 0\end{aligned}$$

for a suitable choice of the constant C_n. This proves the Poisson formula, and hence the lemma.

Regularity also works locally:

Lemma. *Given an open set* $U \subset \mathbb{R}^n$ *and* $T \in \mathfrak{D}(U)$ *with* $\Delta T = 0$, *then* $T = T_\psi$ *for a function* ψ *harmonic in* U.

Proof. Given $V \subset U$ a relatively compact open subset, then for $\varphi \in C_c^\infty(V)$ and ε sufficiently small, $\operatorname{supp}\varphi_\varepsilon \subset U$. We can then define T_ε by

$$T_\varepsilon(\varphi) = T(\varphi_\varepsilon),$$

and repeat the previous argument to conclude that $T_\varepsilon = T_{\psi_V}$ for some ψ_V harmonic in V. Since ψ_V is the same for all ε, if $V \subset W \subset U$ we have $\psi_W|_V = \psi_V$. Consequently there is a harmonic function ψ in U such that $T = T_\psi$. Q.E.D.

As an application we have the

Regularity for the $\bar\partial$-operator. *If* $U \subset \mathbb{C}^n$ *is an open set and* $T \in \mathfrak{D}(U)$ *satisfies* $\bar\partial T = 0$, *then* $T = T_f$ *for some* $f \in \mathcal{O}(U)$.

Proof. By one of our Hodge identities from Section 6 of Chapter 0,

$$\Delta = \sqrt{-1}\,\Lambda\partial\bar\partial$$

on $\mathbb{C}^n$. Thus, $\bar\partial T = 0 \Rightarrow \Delta T = 0$, and so $T = T_f$ for some $f \in C^\infty(U)$ by the preceding lemma. But then $0 = \bar\partial T_f = T_{\bar\partial f} \Rightarrow \bar\partial f = 0$ and $f \in \mathcal{O}(U)$. Q.E.D.

Finally, we will tie up the remaining loose end in Section 6 of Chapter 0 on the proof of the Hodge theorem. Namely, referring to Regularity Lemma I in the subsection entitled "Proof of the Hodge Theorem II: Global Theory" we want to prove that if φ lies in the Sobolev space $\mathcal{H}_s^{p,q}(M)$ and $\psi \in \mathcal{H}_0^{p,q}(M)$ is a weak solution of the equation

$$\Delta\psi = \varphi,$$

then $\psi \in \mathcal{H}_{s+2}^{p,q}(M)$. Writing $P = \bar\partial + \bar\partial^*$, $P^2 = \Delta$. Therefore, we consider the weak equation

$$P\theta = \eta \tag{$*$}$$

and show that if $\eta \in \mathcal{H}_s^{p,q}(M)$, then $\theta \in \mathcal{H}_{s+1}^{p,q}(M)$. If $\rho \in C^\infty(M)$, then

$$\begin{aligned} P(\rho\theta) &= P(\rho)\wedge\theta + \rho P(\theta) \\ &= P(\rho)\wedge\theta + \rho\eta. \end{aligned}$$

It will consequently suffice to prove the regularity assertion about weak solutions of the equation $(*)$ for forms compactly supported in a fixed coordinate patch on M. This coordinate patch may be taken to be diffeomorphic to $\mathbb{R}^n$, so that what we must show is the following

Regularity Lemma II. *Let* $Pu = Qu + Ru$, *where*

$$(Qu)_i = \sum_{k,j} a_{ij}^k(x)\frac{\partial u_j(x)}{\partial x_k},$$

$$(Ru)_i = \sum_j b_{ij}(x)u_j(x),$$

be a first-order differential operator with C^∞ *coefficients satisfying the Gårding inequality*

$$\|Pu\|_0 + \|u\|_0 \geqslant \|u\|_1$$

for compactly supported u. *If the distribution equation*

$$Pu = v$$

holds for some compactly supported v *in the Sobolev space* $\mathcal{H}_s$, *then* $u \in \mathcal{H}_{s+1}$.

Proof. We define the smoothing

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} u_\varepsilon(y)\chi_\varepsilon(x-y)\,dy$$

as above. The L^2 norm

$$\|u_\varepsilon - u\|_0^2 \to 0 \qquad \text{as } \varepsilon \to 0,$$

since the convergence $u_\varepsilon \to u$ is uniform. If we can prove that the Sobolev norms

$$\|u_\varepsilon\|_{s+1}$$

are uniformly bounded for $0<\varepsilon\leqslant\varepsilon_0$, then, taking a sequence $\varepsilon_k \downarrow 0$, a subsequence of u_{ε_k} will converge weakly to an element u' of $\mathcal{H}_{s+1}$, and u' must be equal to u.

By the Gårding inequality in our assumption, we can bound the $\mathcal{H}_{s+1}$-norm of u_ε in terms of the $\mathcal{H}_s$-norms of Qu_ε and u_ε. Inductively, we may assume that $u \in \mathcal{H}_s$, and then the s-norm of u_ε is bounded by the s-norm of u.

It remains to bound the s-norm of Qu_ε. We know how to bound the s-norm of $(Qu)_\varepsilon = -(Ru)_\varepsilon + v_\varepsilon$, and so we must bound the s-norm of the difference

$$(**) \qquad (Qu)_\varepsilon - Q(u_\varepsilon).$$

For constant-coefficient operators this is zero, and so in general we may expect a bound in terms of the s-norm of u and 1-norm of the $a_{ij}^k(x)$'s. For simplicity we do the case $s=0$, the general argument being the same. The ith component of (**) is

$$\frac{\partial}{\partial x^k}\Big(\sum_{j,k} a_{ij}^k u_j\Big) - \sum_{j,k} a_{ij}^k \frac{\partial}{\partial x^k}(u_j)_\varepsilon - \left[\sum_{j,k}\frac{\partial a_{ij}^k}{\partial x^k}u_j\right]_\varepsilon.$$

The last term is bounded by a constant times the L^2-norm of u. The other term is

$$\frac{1}{\varepsilon^{n+1}}\sum_{j,k}\int_{\mathbb{R}^n}(D_k\chi)\Big(\frac{y}{\varepsilon}\Big)\big(a_{ij}^k(x-y) - a_{ij}^k(x)\big)u_j(x-y)\,dy,$$

and the Minkowski inequality implies that its L_2-norm is less than

$$\left(\frac{C}{\varepsilon^{n+1}}\int_{\|y\|\leqslant \varepsilon K}\left|(D_k\chi)\left(\frac{y}{\varepsilon}\right)\right||y|\,dy\right)\|u\|_0 \leqslant C'\|u\|_0$$

for suitable constants C, C'. Q.E.D.

Cohomology of Currents

On a manifold M we have defined the *complex of currents* $(\mathfrak{D}^*(M), d)$. The inclusion of smooth forms into the currents gives a natural map

$$H^*_{\mathrm{DR}}(M) \to H^*(\mathfrak{D}^*(M), d)$$

from de Rham cohomology into the cohomology computed from currents. We will prove that this mapping is an isomorphism. By de Rham's theorem the same is true of the mapping

$$H^*(M, \mathrm{sing}) \to H^*(\mathfrak{D}^*(M), d)$$

from the cohomology of piecewise smooth singular chains into the cohomology of currents. If Γ is a piecewise smooth $(n-p)$-cycle, then there will be a smooth, closed p-form ψ such that the equation of currents

$$T_\Gamma = T_\psi + dR$$

will be satisfied. Although we will not prove it, one may think of R as the current defined by a $(p-1)$-form η that is integrable on M, C^∞ on $M-\Gamma$, and where $d\eta = -\psi$ on $M-\Gamma$. Then the equation above becomes

$$dT_\eta - T_{d\eta} = T_\Gamma,$$

which is a residue formula of the sort discussed above.

Before doing this, we note that if M is a complex manifold, then we also have the complex $(\mathfrak{D}^{p,*}(M), \bar\partial)$ of currents of type (p,q). We will also prove that the map

$$H^{p,*}_{\bar\partial}(M) \to H^*(\mathfrak{D}^{p,*}(M), \bar\partial)$$

is an isomorphism. Since both proofs are essentially the same, we will do the complex case.

Let $\mathfrak{D}^{p,q}$ be the sheaf of currents of type (p,q). Then there is a *complex of sheaves*

$$(*) \qquad 0 \to \Omega^p \to \mathfrak{D}^{p,0} \xrightarrow{\bar\partial} \mathfrak{D}^{p,1} \to \cdots \xrightarrow{\bar\partial} \mathfrak{D}^{p,n} \to 0.$$

Since distributions may be multiplied by C^∞ functions, the sheaves $\mathfrak{D}^{p,q}$ admit partitions of unity. Consequently, $H^k(M, \mathfrak{D}^{p,q}) = 0$ for $k > 0$, and the sheaf-theoretic proof of the de Rham and Dolbeault theorems from Section 3 of Chapter 0 will apply verbatim if we can prove that $(*)$ is exact. In other words, we must establish the $\bar\partial$-Poincaré lemma for currents.

The first step is just the regularity theorem for the $\bar{\partial}$-operator. Note that this step is trivial for the full exterior derivative d.

To prove the $\bar{\partial}$-Poincaré lemma for higher-degree currents, we shall give another proof for the C^∞ case that can be adapted to currents. This proof will be based on finding a homotopy operator

$$K\colon A_c^{0,q}(\mathbb{C}^n) \to A^{0,q-1}(\mathbb{C}^n).$$

The construction of K is based on the Bochner-Martinelli formula above, and the explicit expression will turn out to be useful in proving the holomorphic Lefschetz fixed-point formula.

Some notation will be helpful in defining K. Given complex manifolds M and N with local holomorphic coordinates z and w, the forms on the product $M \times N$ decompose into *bitype*, where, e.g.,

$$A^{(p,q)(r,s)}(M \times N)$$

denotes the C^∞ forms having type (p,q) in dz's and (r,s) in dw's, and therefore type $(p+r,q+s)$ on $M \times N$. We set

$$\Phi(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n,$$

$$\Phi_i(\zeta) = (-1)^{i-1}\zeta_i\, d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_i} \wedge \cdots \wedge d\zeta_n,$$

and define the *Bochner-Martinelli kernel* on $\mathbb{C}^n \times \mathbb{C}^n$ by

$$k(z,w) = C_n \frac{\sum \overline{\Phi_i(z-w)} \wedge \Phi(w)}{\|z-w\|^{2n}}.$$

This form has singularities along the diagonal $z=w$ and is integrable on $\mathbb{C}^n \times \mathbb{C}^n$. Its decomposition into bitype is

$$k(z,w) \in \bigoplus_{q=1}^{n} L^{(0,q-1)(n,n-q)}(\mathbb{C}^n \times \mathbb{C}^n, \mathrm{loc}).$$

We then define

$$K\colon A_c^{0,q}(\mathbb{C}^n) \to A^{0,q-1}(\mathbb{C}^n)$$

by

$$(K\varphi)(z) = \int_{w \in \mathbb{C}^n} k(z,w) \wedge \varphi(w).$$

This integral makes sense, since k is integrable and φ has compact support. With the usual change of variables $u = z - w$,

$$(K\varphi)(z) = \int_{u \in \mathbb{C}^n} k(z,z-u) \wedge \varphi(z-u)$$

is C^∞ in z since only $\|u\|^{2n}$ will appear in the denominator of the integrand.

We note that $K\varphi$ does not have compact support. There are analogues of $K\varphi$ for forms $\varphi \in A_c^{p,q}(\mathbb{C}^n)$ $(p \neq 0)$, but we shall leave it to the reader to write these out.

What we need to know about K is the *homotopy formula*

$$\bar{\partial} K + K \bar{\partial} = \text{identity}.$$

Proof. Since

$$\begin{aligned}
\bar{\partial}\left(\frac{\sum \overline{\Phi_i(\zeta)}}{\|\zeta\|^{2n}} \right) &= \frac{n\,\overline{\Phi(\zeta)}}{\|\zeta\|^{2n}} - \frac{n\bar{\partial}(\zeta,\zeta) \wedge \overline{\sum \Phi_i(\zeta)}}{\|\zeta\|^{2n+2}} \\
&= \frac{n}{\|\zeta\|^{2n}} \left[\overline{\Phi(\zeta)} - \frac{\left(\sum \zeta_i\, \overline{d\zeta_i}\right) \wedge \left(\overline{\sum \Phi_i(\zeta)} \right)}{\|\zeta\|^2} \right] \\
&= 0,
\end{aligned}$$

we see that *formally* $\bar{\partial} k(z,w) = 0$. Ignoring for a moment the singularities, for a test form $\psi \in A_c^{n,n-q+1}(\mathbb{C}^n)$ Stokes' theorem gives

$$\begin{aligned}
0 &= \int_{\mathbb{C}^n \times \mathbb{C}^n} d(\psi(z) \wedge k(z,w) \wedge \varphi(w)) \\
&= \int_{\mathbb{C}^n \times \mathbb{C}^n} \bar{\partial}(\psi(z) \wedge k(z,w) \wedge \varphi(w)) \\
&= \int_{\mathbb{C}^n \times \mathbb{C}^n} \bar{\partial}\psi(z) \wedge k(z,w) \wedge \varphi(w) \pm \int_{\mathbb{C}^n \times \mathbb{C}^n} \psi(z) \wedge k(z,w) \wedge \bar{\partial}\varphi(w).
\end{aligned}$$

This equation says that, considering $K_\varphi = K\varphi \in A^{0,q-1}(\mathbb{C}^n)$ as a current operating on $A_c^{n,n-q+1}(\mathbb{C}^n)$,

$$(*) \qquad \bar{\partial} K_\varphi + K_\varphi \bar{\partial} = 0.$$

Of course this formal computation is not correct, because in applying Stokes' theorem the singularities of the kernel along the diagonal come into the picture. Referring to the previously established Bochner-Martinelli formula,

$$\eta(0) = C_n \int_{\mathbb{C}^n} \bar{\partial}\eta(\zeta) \wedge \frac{\sum \overline{\Phi_i(\zeta)} \wedge \Phi(\zeta)}{\|\zeta\|^{2n}}$$

for $\eta \in C_c^\infty(\mathbb{C}^n)$, it seems pretty clear that the correction term that must be added to the right-hand side of $(*)$ is just the identity. This may be proved by writing everything out and using the Bochner-Martinelli formula, but since it is completely straightforward to carry out the computation, we will not do it here.

A first application of the homotopy formula is another proof of the $\bar{\partial}$-Poincaré lemma for smooth forms. Given a $\bar{\partial}$-closed form $\varphi \in A^{0,q}(U)$, where $U \subset \mathbb{C}^n$ is an open set, we may find a relatively compact open subset $V \subset U$ and bump function $\rho \in C_c^\infty(U)$ with $\rho \equiv 1$ on V. Then $\rho\varphi \in A_c^{0,q}(\mathbb{C}^n)$, and

$$(\rho\varphi)(z) = \bar{\partial}(K\rho\varphi)(z) + K\big(\bar{\partial}(\rho\varphi)\big)(z).$$

Restricting to V,

$$\varphi(z) = \bar{\partial}(K\rho\varphi)(z) \qquad (z \in V).$$

Now, suppose we say that a current $T \in \mathfrak{D}^{0,q}(\mathbb{C}^n)$ is *compactly supported* if, for some relatively compact open set $U \subset \mathbb{C}^n$, $T(\varphi) = 0$ whenever $\operatorname{supp}\varphi \subset \mathbb{C}^n - U$. Such a current may then be defined on all of $A^{n,n-q}(\mathbb{C}^n)$, not just on the forms with compact support. Using this device, we may define KT for a compactly supported current T by

$$KT(\varphi) = T(K\varphi), \qquad \varphi \in A_c^{n,n-q+1}(\mathbb{C}^n).$$

Then $KT \in \mathfrak{D}^{0,q-1}(\mathbb{C}^n)$, and for a test form $\psi \in A_c^{n,n-q}(\mathbb{C}^n)$,

$$\begin{aligned}\big(\bar{\partial}(KT)\big)(\psi) + \big(K(\bar{\partial}T)\big)(\psi) &= (KT)\big(\bar{\partial}\psi\big) + (\bar{\partial}T)(K\psi)\\ &= T\big(K\bar{\partial}\psi + \bar{\partial}K\psi\big)\\ &= T_\psi,\end{aligned}$$

so that, with this interpretation, the homotopy formula makes sense for compactly supported currents. In particular, the proof of the $\bar{\partial}$-Poincaré lemma for smooth forms may be extended verbatim to prove the result for currents.

This completes the argument establishing the isomorphisms

$$H_{\bar{\partial}}^{p,*}(M) \longrightarrow H^{p,*}\big(\mathfrak{D}^{p,*}(M), \bar{\partial}\big),$$
$$H_{\mathrm{DR}}^{*}(M) \longrightarrow H^{*}(\mathfrak{D}^{*}(M), d),$$

which we shall refer to as *smoothing of cohomology*.

2. APPLICATIONS OF CURRENTS TO COMPLEX ANALYSIS

Currents Associated to Analytic Varieties

Let M be a complex manifold. The currents $\mathfrak{D}^{p,p}(M)$ of type (p,p) are the continuous linear functionals on the compactly supported forms $A_c^{n-p,n-p}(M)$. A (p,p) current T is *real* in case $T = \overline{T}$ in the sense that

$\overline{T(\varphi)} = T(\bar{\varphi})$ for all $\varphi \in A_c^{n-p,n-p}(M)$, and a real current is *positive* in case

$$(\sqrt{-1})^{p(p-1)/2} T(\eta \wedge \bar{\eta}) \geqslant 0, \qquad \eta \in A_c^{n-p,0}(M).$$

Especially noteworthy are the *closed, positive currents*. Note that for real $T \in \mathfrak{D}^{p,p}(M)$,

$$dT = 0 \Leftrightarrow \partial T = \bar{\partial} T = 0.$$

The positivity of a current implies that it is order zero in the sense of distributions. For example, a current $T \in \mathfrak{D}^{1,1}(M)$ is locally written as

$$T = \frac{\sqrt{-1}}{2} \sum_{i,j} t_{ij}\, dz_i \wedge d\bar{z}_j,$$

a differential form with distribution coefficients defined by

$$t_{ij}(\alpha) = (-1)^{n+i+j}\big(\alpha\, dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n\big).$$

The current is real if $\overline{t_{ij}} = t_{ji}$, and positive if for any $\lambda_1, \ldots, \lambda_n$ the distribution

$$\alpha \to T(\lambda)(\alpha) = \Big(\sum_{i,j} t_{ij} \lambda_i \bar{\lambda}_j\Big)(\alpha)$$

is nonnegative on positive functions. In this case, by taking monotone limits we may extend the domain of definition of $T(\lambda)$ to a suitable class of functions—including all the continuous functions—in $L^1(M, \mathrm{loc})$ that are integrable for the positive measure $T(\lambda)$. A similar discussion applies to positive (p,p) currents.

Examples

1. Let $Z \subset M$ be a codimension-p analytic subvariety with $Z^* = Z - Z_s$ the set of smooth points. In the subsection on calculus on complex manifolds in Section 2 of Chapter 0 we proved what, in the language of currents, amounts to the assertion that the map

$$\varphi \to \int_{Z^*} \varphi, \qquad \varphi \in A_c^{n-p,n-p}(M),$$

defines a closed, positive current T_Z. This example is of fundamental importance.

The cohomology class defined by T_Z together with the isomorphism

$$H^*_{\mathrm{DR}}(M) \cong H^*(\mathfrak{D}^*(M), d)$$

is the fundamental class of Z.

2. A smooth $(1,1)$ form

$$\omega = \frac{\sqrt{-1}}{2}\Big(\sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j\Big)$$

is rėal if $\bar{h}_{ij} = h_{ji}$, strictly positive if the matrix $h_{i\bar{j}}$ is positive definite, and closed exactly when the corresponding hermitian metric

$$ds^2 = \sum_{i,j} h_{ij}\, dz_i\, d\bar{z}_j$$

is Kähler. The powers ω^p of a Kähler form define closed, positive (p,p) currents.

3. A real function $\varphi \in L^1(M, \text{loc})$ is said to be *plurisubharmonic* in case $\sqrt{-1}\ \partial\bar{\partial}\varphi$ is a positive $(1,1)$ current. Here the derivatives are taken in the sense of distributions. Plurisubharmonic functions define potentials especially suitable for complex function theory.

Lemma ($\partial\bar{\partial}$-Poincaré Lemma). *Let* T *be a closed, positive* $(1,1)$ *current. Then locally*

$$T = \sqrt{-1}\ \partial\bar{\partial}\varphi$$

for a real plurisubharmonic function φ, *which is unique up to adding the real part of a holomorphic function.*

Proof. By the $\bar{\partial}$-Poincaré lemma, locally

$$T = -\sqrt{-1}\ \bar{\partial}\eta$$

for some current η of type $(1,0)$. The current $\partial\eta$ is of type $(2,0)$, and $\bar{\partial}(\partial\eta) = -\partial\bar{\partial}\eta = (1/\sqrt{-1}\,)\partial T = 0$. By the regularity theorem for the $\bar{\partial}$-operator, $\partial\eta$ is a closed holomorphic 2-form, and so by the d-Poincaré lemma for holomorphic forms $\partial\eta = d\xi$ for a holomorphic 1-form ξ. Then $T = -\sqrt{-1}\ \bar{\partial}\eta'$, where $\eta' = \eta - \xi$ satisfies $\partial\eta' = 0$. Now, by the ∂-Poincaré lemma, $\eta' = \partial\gamma$ for some distribution γ; $\varphi = \frac{1}{2}(\gamma + \bar{\gamma})$ is then a real distribution satisfying $\sqrt{-1}\ \partial\bar{\partial}\varphi = \text{T}$.

Using the fact that $\sqrt{-1}\ \partial\bar{\partial}\varphi$ is a distribution of order zero, it may be proved that φ is a locally L^1 function, but we will not completely prove this, since we do not need it. Intuitively, the argument is that

$$\Delta\varphi = \sqrt{-1}\ \Lambda\partial\bar{\partial}\varphi$$

is a distribution of order zero, hence it is (more or less) in the Sobolev space H_0. By the regularity theorem, then, φ is (more or less) in the Sobolev space H_2.

The function φ is called a *potential* for T, and it is unique up to adding a real function γ with $\partial\bar{\partial}\gamma = 0$. If γ is any such function then $\partial\gamma$ is a $\bar{\partial}$-closed current of type $(1,0)$, and so again by regularity is a closed holomorphic 1-form. Setting $f(z) = \int_{z_0}^{z} \partial\gamma$, then, we have $\gamma = \text{Re}\, f$. Q.E.D.

If $T = T_\omega$ is the current associated to a Kähler metric, then its potential function φ is smooth. For instance, $\varphi(z) = \|z\|^2$ is a global potential function for the Euclidean metric on $\mathbb{C}^n$.

At the other extreme we have the

Lemma (Poincaré-Lelong Equation). *If the holomorphic function* $f \in \mathcal{O}(M)$ *has divisor the analytic hypersurface* Z, *then the equation of currents*

$$T_Z = \frac{\sqrt{-1}}{\pi}\,\partial\bar\partial \log|f|$$

is valid.

Proof. Around a smooth point we may choose coordinates $(z_1, \ldots, z_n)$ such that $f(z) = z_n$. Then

$$\begin{aligned}\frac{\sqrt{-1}}{\pi}\,\partial\bar\partial\log|f| &= \bar\partial\left(\frac{1}{2\pi\sqrt{-1}}\,\partial\log f\right)\\ &= \bar\partial\left(\frac{1}{2\pi\sqrt{-1}}\,\frac{dz_n}{z_n}\right)\\ &= T_{\{z_n=0\}}\end{aligned}$$

by an obvious extension of the 1-variable Cauchy formula (in distributional form)

$$\bar\partial\left(\frac{1}{2\pi\sqrt{-1}}\,\frac{dz}{z}\right) = \delta_{\{0\}}$$

allowing dependence on parameters.

Next, suppose that $p(w) = w^n + a_1 w^{n-1} + \cdots + a_0$ is a polynomial in one complex variable, possibly with repeated roots. Then we have the distribution equation

$$\bar\partial\left(\frac{1}{2\pi\sqrt{-1}}\,\frac{p'(w)\,dw}{p(w)}\right) = \sum_{p(w_\nu)=0} \delta_{w_\nu}.$$

This means that the 1-form $\partial \log p(w)$ is integrable, and moreover for $\alpha \in C_c^\infty(\mathbb{C})$,

$$\frac{1}{2\pi\sqrt{-1}} \int\!\!\int \frac{\partial\alpha(w)}{\partial\bar w}\,\frac{p'(w)}{p(w)}\,dw\wedge d\bar w = \sum_{p(w_\nu)=0} \alpha(w_\nu).$$

The formula follows by writing $p(w) = \prod_{\nu=1}^n (w - w_\nu)$ and using the Cauchy formula on each factor.

Returning to the general Poincaré-Lelong formula, we must show that $\log|f|$ is integrable and

$$\frac{\sqrt{-1}}{\pi}\int_M \log|f|\,\bar\partial\partial\varphi = \int_{Z^*}\varphi$$

for $\varphi \in A_c^{n-1,n-1}(M)$. The problem is local around a point $p \in Z$ with local coordinates $(z_1,\ldots,z_n)$. We may assume that $f(z_1,\ldots,z_n)$ is a Weierstrass polynomial in z_n and that

$$\varphi = \alpha(z)\, dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1},$$

since these forms generate all forms under coordinate stretchings,

$$z_1' = z_1, \ \ldots, \ z_{n-1}' = z_{n-1}, \qquad z_n' = \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1} + z_n.$$

Effectively, then, we are reduced to 2-variables (z,w), where

$$\varphi = \alpha(z,w)\, dz \wedge \overline{dz}$$

and

$$f(z,w) = w^n + a_1(z) w^{n-1} + \cdots + a_0(z) = \prod_{\nu=1}^{n} (w - w_\nu(z))$$

is a Weierstrass polynomial. By iteration, the integral in question is a constant times

$$\int_{|z|<1} \left(\int_{|w|<1} \log|f(z,w)| \frac{\partial^2 \alpha(z,w)}{\partial w\, \partial \bar{w}}\, dw \wedge d\bar{w} \right) dz \wedge \overline{dz}\,.$$

Applying first Stokes' theorem and then the polynomial result to the inner integral gives

$$\int_{|z|<1} \left(\sum_{\nu} \alpha(z, w_\nu(z)) \right) dz \wedge d\bar{z} = \int_Z \varphi. \qquad \text{Q.E.D.}$$

One may suspect that a general closed, positive current should be somewhere between the smooth currents and those supported by analytic varieties. This turns out to be basically true, and in order to describe what is known, we show how a closed, positive current $T \in \mathfrak{D}^{p,p}(U)$ an open set U in $\mathbb{C}^n$ has associated to each point p a Lelong number

$$\Theta(T,p) \geqslant 0,$$

which is identically zero for smooth currents, and where at the other extreme

$$\Theta(T_Z,p) = \mathrm{mult}_p(Z)$$

gives the multiplicity of an analytic variety Z at a point.

For simplicity we assume $U = \mathbb{C}^n$, p is the origin, and we use the notations

$$B[r] = \{ z \in \mathbb{C}^n : \ \|z\| \leqslant r \},$$

$$\chi(r) = \text{characteristic function of } B[r],$$

$$B[r,R] = \{ z \in \mathbb{C}^n : \ r \leqslant \|z\| \leqslant R \} \qquad (r < R),$$

$$\omega = \frac{\sqrt{-1}}{2} \left(\sum_i dz_i \wedge d\bar{z}_i \right).$$

As mentioned above, by taking monotone limits the current T may be defined on suitable L^1-forms, such as $\chi(r)\omega^{n-p}$. In case $T = T_Z$,

$$T_Z(\chi(r)\omega^{n-p}) = \int_{Z \cap B[r]} \omega^{n-p}$$
$$= \text{volume of } (Z \cap B[r])$$

by the Wirtinger theorem from Section 2 of Chapter 0. In general we set

$$\Theta(T,p,r) = \frac{1}{r^{2n-2p}} T(\chi(r)\omega^{n-p})$$

and shall prove the

Lemma. $\Theta(T,p,r)$ *is an increasing function of* r.

Proof. The smoothing T_ε of a closed, positive current is again a closed, positive current. Using this it will suffice to prove the lemma when $T = T_\psi$ for a smooth, closed (p,p) form ψ. Then

$$\Theta(T,p,r) = \frac{1}{r^{2n-2p}} \int_{B[r]} \psi \wedge \omega^{n-p}$$

$$= \left(\frac{\sqrt{-1}}{2}\right)^{n-p} \frac{1}{r^{2n-2p}} \int_{B[r]} \psi \wedge (\partial\bar{\partial}\|z\|^2)^{n-p}$$

$$= \left(\frac{\sqrt{-1}}{2}\right)^{n-p} \frac{1}{r^{2n-2p}} \int_{B[r]} d(\psi \wedge \bar{\partial}\|z\|^2 \wedge (\partial\bar{\partial}\|z\|^2)^{n-p-1},$$

since ψ is closed,

$$= \left(\frac{\sqrt{-1}}{2}\right)^{n-p} \frac{1}{r^{2n-2p}} \int_{\partial B[r]} \psi \wedge \bar{\partial}\|z\|^2 \wedge (\partial\bar{\partial}\|z\|^2)^{n-p-1}$$

by Stokes' theorem.

Now on the sphere $\|z\| = r$,

$$0 = d(z,z) = (dz,z) + (z,dz)$$

$$\Rightarrow \partial\bar{\partial}\log(z,z) = \partial\left(\frac{(z,dz)}{(z,z)}\right) = \frac{(dz,dz)}{(z,z)},$$

since $(dz,z)\wedge(z,dz) = -(dz,z)\wedge(dz,z) = 0$. The last integral is therefore equal to

$$\left(\frac{\sqrt{-1}}{2}\right)^{n-p} \int_{\|z\|=r} \psi \wedge \bar{\partial}\log\|z\|^2 \wedge (\partial\bar{\partial}\log\|z\|^2)^{n-p-1}.$$

By Stokes' theorem, for $r < R$ (remembering that p is the origin)

$$\Theta(T_\psi,p,R) - \Theta(T_\psi,p,r)$$
$$= \int_{\partial B[r,R]} \psi \wedge \left\{ \left(\frac{\sqrt{-1}}{2} \right)^{n-p} \bar{\partial} \log \|z\|^2 \wedge (\partial\bar{\partial} \log \|z\|^2)^{n-p-1} \right\}$$
$$= \pi^{n-p} \int_{B[r,R]} \psi \wedge \Omega^{n-p},$$

where

$$\Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|z\|^2$$

is the pullback to $\mathbb{C}^n$ of the Fubini-Study metric on $\mathbb{P}^{n-1}$. Since $\psi \wedge \Omega^{n-p} \geqslant 0$, we have proved the lemma. Q.E.D.

DEFINITION. The *Lelong number* is

$$\Theta(T,p) = \frac{1}{\pi^{n-p}} \lim_{r \to 0} \Theta(T,p,r).$$

It is clear that $\Theta(T,p) \geqslant 0$ and is identically equal to zero in case T is a smooth current. As indicated above

$$\Theta(T_Z,p) = \text{mult}_p(Z)$$

for currents defined by analytic varieties.

Sketch of proof. By the proof of the previous lemma,

$$\Theta(T_Z,0) = \lim_{r \to 0} \int_{Z[r]} \Omega^{n-1},$$

where $Z[r] \subset \mathbb{P}^{n-1}$ is the set of lines $\overrightarrow{0q}$ for $q \in Z \cap B[r]$ and Ω is the standard Kähler form on $\mathbb{P}^{n-1}$. The limiting position of $Z[r]$ as $r \downarrow 0$ is the tangent cone $C(Z)$ to Z at the origin, and by the Wirtinger theorem applied this time to the projective space $\mathbb{P}^{n-1}$,

$$\int_{C(Z)} \Omega^{n-p} = \text{degree}\,(C(Z))$$
$$= \text{mult}_{\{0\}}(Z). \qquad \text{Q.E.D.}$$

This proof is not too difficult to make precise, and it is essentially obvious in case the origin is a smooth point of Z, which is all we shall use. Since the Lelong number is semicontinuous, it follows that

$$\Theta(T_Z,p) \geqslant 1$$

for $p \in Z$.

Building upon previous work by several authors, Siu has recently proved that for a general closed, positive current T the set of points where

$$\Theta(T,p) \geqslant \varepsilon > 0$$

is supported in a codimension-p analytic subvariety.* We shall not use this result, but it is worthwhile to keep in mind when we discuss the proper mapping theorem in the section after next, where in fact a special case of Siu's theorem will be proved.

Intersection Numbers of Analytic Varieties

Suppose that M is a compact, oriented manifold of real dimension n. Two closed currents T, S of complementary degrees have an *intersection number* defined by

$$T\cdot S = \int_M T_\varepsilon \wedge S_\delta,$$

where T_ε, S_δ are smooth forms in the cohomology classes defined by T and S using the isomorphism

$$H^*(\mathfrak{D}^*(M),d) \cong H^*_{\mathrm{DR}}(M).$$

This intersection number coincides with the usual topological one on piecewise smooth singular cycles, with the cup product on the smooth forms considered as currents, and with the usual pairing

$$\int_\Gamma \psi$$

of forms on cycles when $T = T_\psi$ for a smooth form ψ and $S = T_\Gamma$ for a piecewise smooth chain Γ.

In case M is a complex manifold of complex dimension n, the pairing

$$H^{p,q}_{\bar\partial}(M) \otimes H^{n-p,n-q}_{\bar\partial}(M) \to \mathbb{C}$$

induces an intersection number on $\bar\partial$-closed currents of complementary type (p,q) and $(n-p,n-q)$. In case $p=q$ and T is a real (p,p) current and S a real $(n-p,n-p)$ current, then

$$dT = 0 \Leftrightarrow \bar\partial T = 0,$$
$$dS = 0 \Leftrightarrow \bar\partial S = 0,$$

and the intersection number of closed currents is the same in either the d or $\bar\partial$ sense.

*Y. T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, *Bull. Amer. Math. Soc.*, Vol. 79 (1973), pp. 1200–1205.

Now suppose that T is a closed, positive (p,p) current. Then by the smoothing of cohomology there is a closed, real smooth (p,p) form T_ε in the same cohomology class at T. With some care we could insure that

$$\lim_{\varepsilon \to 0} T_\varepsilon = T.$$

However, we *cannot* say that the T_ε are positive forms. For example, suppose that $M = \tilde{N}_p$ is the blow-up of a two-dimensional complex manifold N at a point p. The fiber over p in $M \to N$ is a curve $E \cong \mathbb{P}^1$ with normal bundle H^*, where $H \to \mathbb{P}^1$ has Chern class $+1$. Thus the self-intersection number $E \cdot E = -1$. If T_ε is a smoothing of T_E, then

$$E \cdot E = \int_E T_\varepsilon$$

shows that we cannot take T_ε to be a positive $(1,1)$ form.

The intuitive reason for this is that T_ε is a smooth form supported in ε-tubular neighborhood of E, and so T_ε has to do with the shape of the normal bundle of E. To say that T_ε is positive would be something like saying that the normal bundle has positive curvature, which is not the case in this example.

Using the theory of currents, we now will reprove the fundamental result from Section 4 of Chapter 0 about positivity of intersection numbers of analytic varieties meeting in isolated points.

Theorem. *Suppose that* Z *and* W *are analytic subvarieties of complementary dimensions* p *and* n − p *in* M *that meet at a finite number of points of* M. *Then the intersection number*

$$Z \cdot W = \sum_{p \in Z \cap W} m_p(Z, W),$$

where $(Z, W)_p$ *depends only on* Z *and* W *in a neighborhood of* p *and satisfies*

$$m_p(Z, W) \geqslant \operatorname{mult}_p(Z)\operatorname{mult}_p(W).$$

Proof. We first argue that we may assume W to be smooth. For this we consider the product $M \times M$. By the formal properties of the Künneth formula and Poincaré duality,

$$Z \cdot W = (Z \times W) \cdot \Delta,$$

where the right-hand side is the intersection number in $M \times M$ of $Z \times W$ with the diagonal Δ. Set-theoretically,

$$(Z \times W) \cap \Delta = \{(p,p) \colon p \in Z \cap W\}.$$

Also, it has been established in Section 1 of Chapter 0 that

$$\operatorname{mult}_{p \times q}(Z \times W) = \operatorname{mult}_p(Z)\operatorname{mult}_q(W).$$

Since the diagonal is smooth, the general case is therefore reduced to the situation when W is smooth.

Next, we may for simplicity assume that Z and W meet in a single point p_0. We may choose a holomorphic coordinate system $(z,w) = (z_1,\ldots,z_p; w_1,\ldots,w_{n-p})$ around p_0 such that W is given by $z=0$ and the projection $(z,w)\to z$ is a finitely sheeted branched covering mapping on Z. Set $U_\varepsilon = \{(z,w): \|z\| < \varepsilon, \|w\| < \varepsilon\}$ and let $U = U_1$. The picture is shown by Figure 1.

Suppose now that we have a current $S \in \mathfrak{D}^{p,p-1}(U)$ such that

$$T_W|U = \bar{\partial} S.$$

Let ρ be a bump function that is 1 in U_{ε_0} and has compact support in U. Then

$$T'_W = T_W - \bar{\partial}(\rho S)$$

is a globally defined current on M in the same cohomology class as T_W. Moreover, $T'_W = (T_W - \rho\bar{\partial}S) - \bar{\partial}\rho \wedge S$ is smooth near $Z \cap W$, and so the integral

$$\int_Z T'_W$$

is defined and computes the intersection number $Z\cdot W$. If $Z_\varepsilon = Z \cap U_\varepsilon$, then since $T'_W = 0$ near p_0,

$$\begin{aligned}\int_Z T'_W &= \lim_{\varepsilon\to 0}\int_{Z-Z_\varepsilon} T'_W \\ &= \lim_{\varepsilon\to 0} -\int_{Z-Z_\varepsilon}\bar{\partial}(\rho S) \\ &= \lim_{\varepsilon\to 0}\int_{\partial Z_\varepsilon} S\end{aligned}$$

by Stokes' theorem. The formula

$$Z\cdot W = \lim_{\varepsilon\to 0}\int_{\partial Z_\varepsilon} S$$

reduces us to a purely local question.

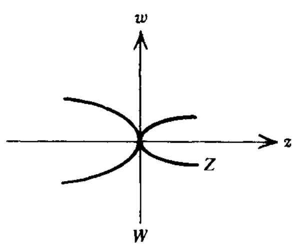

Figure 1

We take for S the current T_β defined by the Bochner-Martinelli form

$$\beta(z) = C_n\left(\partial \log \|z\|^2 \wedge (\partial\bar\partial \log \|z\|^2)^{p-1}\right)$$

discussed in Section 1 of this chapter. The equation

$$\bar\partial T_\beta = T_{\{z=0\}}$$

is just the Bochner-Martinelli formula with trivial dependence on the parameters w. Let $B_\varepsilon \subset \mathbb{C}^p$ be the ball $\{\|z\| < \varepsilon\}$. The projection

$$Z_\varepsilon \xrightarrow{\pi} B_\varepsilon$$

is a $d \geqslant \mathrm{mult}_p(Z)$ sheeted branched covering, and consequently

$$\int_{\partial Z_\varepsilon} S = d\int_{\partial B_\varepsilon} \beta = d. \qquad \text{Q.E.D.}$$

The local intersection numbers $m_p(Z, W)$ will be discussed once again and in greater detail in case Z and W are locally complete intersections in Section 2 of Chapter 5.

In case W is smooth of dimension 1, so that Z is an analytic hypersurface locally defined by a single holomorphic function f, the above proof gives the formula

$$Z \cdot W = \sum_{p \in Z \cap W} \mathrm{ord}_p(f|_W).$$

The Levi Extension and Proper Mapping Theorems

We first recall the statement of Remmert's

Proper Mapping Theorem. *Let* U *and* N *be complex manifolds*, $\mathrm{M} \subset \mathrm{U}$ *an analytic subvariety, and* $\mathrm{f}: \mathrm{U} \to \mathrm{N}$ *a holomorphic mapping whose restriction to* M *is proper. Then the image* f(M) *is an analytic subvariety of* N.

We shall give a proof of this result under one additional technical assumption, which will be trivially satisfied in all of our applications. This is:

For each smooth point $\mathrm{p} \in \mathrm{M}$ *and each* k*-plane* Λ_p *in the tangent space to* M *at* $\mathrm{p}(\mathrm{k} \leqslant \mathrm{n} = \dim \mathrm{M})$, *there is a* k*-dimensional analytic subvariety* Z *of* M *having* Λ_p *as tangent plane at* p.

In practice U will be an open subset of an algebraic variety V, and we may take Z to be a linear section of M.

Our proof of the proper mapping theorem will use the discussion about currents from the preceding sections, together with the following

Levi Extension Theorem (I). *Let f be a meromorphic function defined outside an analytic variety V of codimension* ≥ 2 *on a complex manifold* M. *Then* f *extends to a meromorphic function on* M.

Proof. Let $(f)_\infty$ be the polar divisor of f in $M-V$, and let $\overline{(f)_\infty}$ be its closure in M. If we make the assumption that $\overline{(f)_\infty}$ is an analytic subvariety of M, then we can argue as follows: for any $p \in M$, let $\overline{(f)_\infty}=(g)$ in a neighborhood U of p. Then $g \cdot f = \tilde{h}$ is holomorphic in $U \cap (M-V)$, and hence by Hartogs' theorem extends to a holomorphic function h in U. So h/g gives a meromorphic extension of f to U.

Since the question of whether $\overline{(f)_\infty}$ is an analytic variety is local around a point of M, the theorem is reduced to

Levi Extension Theorem (II). *In the polycylinder* Δ^n *in* $\mathbb{C}^n$ *let* V *be a codimension* ≥ 2 *analytic subvariety, and* D *a subvariety of codimension* 1 *in* $\Delta^n - V$. *Then the closure* $\overline{D}$ *of* D *in* Δ^n *is analytic.*

Proof. This is a geometric variant of Hartogs' theorem. The analogous general result, where

$$\operatorname{codim} D \leq (\operatorname{codim} V) - 1,$$

has been proved by Remmert and Stein.

We begin by making some reductions. If we prove the result when V is nonsingular, then this will imply the general case by the following stratification device: Let V' be the variety of singular points of V, V'' the variety of singular points of V', etc. Applying the nonsingular case to sufficiently small neighborhoods of points $p \in V - V'$, we conclude that D extends to $\Delta^n - V'$. Repeating the argument, D will extend to $\Delta^n - V''$, and so forth.

Next, by localizing around a point of V and choosing coordinates properly, we may assume that V is a linear subspace of $\mathbb{C}^n$. The essential case is thus when $n=2$ and $V=\{z_1=z_2=0\}$ is the origin. We shall prove the result in this situation, from which the general conclusion may be drawn by analogy.

Let $\Delta' = \{|z_1|<1,\ |z_2|<1,\ z_1 \neq 0\} \cong \Delta^* \times \Delta$. Then we have proved in Section 3 of Chapter 0 that

$$H^1(\Delta', \mathcal{O}) = H^2(\Delta', \mathbb{Z}) = 0.$$

From the exact cohomology sequence of the exponential sheaf sequence this implies that $H^1(\Delta', \mathcal{O}^*)=0$. Consequently, if $D^* = D \cap \Delta'$, then the line bundle $[D^*] \to \Delta'$ is trivial and we conclude that the analytic curve D^* is the divisor of some $h \in \mathcal{O}(\Delta')$.

We may assume that D does not contain the line $\{z_1=0\}$, and therefore $D \cap \{z_1=0\}$ consists of a finite number of points in the punctured disc

$0<|z_2|<1$. We may find a circle $|z_2|=\varepsilon$ that does not meet these points. It follows by continuity that, for δ sufficiently small, the locus

$$\{|z_1|\leqslant\delta,\ |z_2|=\varepsilon\}$$

will not meet D (Figure 2). For fixed z with $0<|z_1|\leqslant\delta$ the integral

$$\frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\varepsilon}\frac{dh}{h}$$

is well-defined, continuous, and integer-valued. It follows that D meets each vertical disc $\{z_1=C,|z_2|\leqslant\varepsilon,0<|C|\leqslant\delta\}$ the same number d of times. Thus, projecting $\overline{D}$ on the z_1-axis gives a proper mapping $\pi:\overline{D}\to\Delta$ that restricts to a d-sheeted covering $\pi:D^*\to\Delta^*$ over the punctured disc. If $d=1$, we have the graph of a bounded holomorphic function, and our result follows from the Riemann extension theorem. In general we use the by-now-familiar argument involving the elementary symmetric functions: set

$$\begin{aligned}\varphi_i(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\varepsilon} z_2^i\frac{dh(z_1,z_2)}{h(z_1,z_2)}\\ &= \sum_{\nu=1}^{d} z_{2,\nu}(z_1)^i,\end{aligned}$$

where $\pi^{-1}(z_1)=\{(z_1,z_{2,\nu}(z_1))\}_\nu$. The $\varphi_i(z_1)$ are holomorphic and bounded in $0<|z_1|\leqslant\delta$, and hence they extend to holomorphic functions on the full disc. We may then set

$$F(z_1,z_2) = z_2^d+p_1(\varphi_1(z_1),\dots,\varphi_d(z_1))z_2^{d-1}+\cdots+p_d(\varphi_1(z_1),\dots,\varphi_d(z_1))$$

a polynomial in z_1 whose roots are for fixed $z_1\neq 0$ just the points $(z_1,z_{2,\nu}(z_1))$, and which is holomorphic in the bicylinder. The divisor of F is $\overline{D}$, and we are done. Q.E.D.

Note: The general principle is this: Let $W\subset\Delta^n$ be a closed subset such that (1) the projection $W\xrightarrow{\pi}\Delta^k$ is proper, and (2) outside an analytic subvariety $Z\subset\Delta^k$ this projection $W^*\xrightarrow{\pi}\Delta^k-Z$ ($W^*=W-\pi^{-1}(Z)$) is an

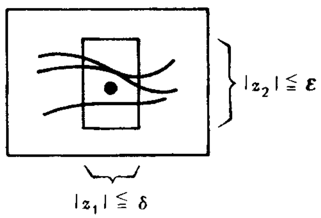

Figure 2

analytic branched covering. Then W is a k-dimensional analytic subvariety of Δ^n.

Now we come to the proof of the proper mapping theorem. We shall give several preliminary reductions before coming to the essential point.

1. Since $f(M)$ is a closed subset of N, the question is local around a point $p \in f(M)$ in N. So we may assume that N is a polycylinder $\Delta^N = \Delta$ in $\mathbb{C}^N$.

Also, we may assume that M is irreducible, since only a finite number of components of M will have images meeting a given compact set in N.

2. The proof is by induction on $n = \dim M$. Let $M^* = M - M_s$ be the complex manifold of smooth points of M, and choose a point $p_0 \in M^*$ where the Jacobian matrix of $f: M^* \to \mathbb{C}^N$ has maximum rank $k \leqslant n$. If $k < n$, by our assumption we may choose a k-dimensional analytic subvariety S in M passing through p_0 and such that $f|S$ has maximum rank k. We call such an S a *horizontal slice* for $f: M \to \mathbb{C}^N$. It is clear from the implicit function theorem that there is a neighborhood W of p_0 such that $f(W) = f(S \cap W)$. It follows that $f(M) = f(S)$, since M is irreducible.

This reduces us to proving the theorem when f has maximum rank $n = \dim M$ at some point $p_0 \in M^*$. We will then prove that $f(M)$ is an n-dimensional analytic subvariety of the polycylinder.

3. At this juncture we may define the current $S \in \mathcal{D}^{p,p}(\Delta)$ $(p = N - n)$, which will turn out to be $T_{f(M)}$ once the theorem is proven. The definition is

$$S(\varphi) = \int_{M^*} f^*(\varphi), \qquad \varphi \in A_c^{n,n}(\Delta).$$

Since f is proper and holomorphic, S is a closed, positive current. What we must prove is that it is the current given by integration over an analytic variety, which must then be $f(M)$.

Note that at each point $q \in f(M)$ the Lelong number

$$\Theta(S,q) \geqslant 1.$$

This is true at points $q = f(p)$, where $p \in M^*$ and f has maximum rank, since then one piece of $f(M)$ passing through q will be a complex manifold, and it is therefore true on all of S by semicontinuity.

4. We next argue that we may assume $N = n+1$, so that $f(M)$ is to be analytic hypersurface. Precisely, we shall show that for a generic choice of coordinate system, the composition g in

$$\begin{array}{ccl} M & \xrightarrow{f} & \Delta^N \\ & {\scriptstyle g}\searrow & \downarrow \pi = \text{projection} \\ & & \Delta^{n+1} \end{array}$$

are proper, at least if we allow ourselves to shrink the polycylinder Δ^N. If this has been established, and if we have proved the result in case $N=n+1$, then a finite number of analytic functions of the form $h\cdot\pi$, where $h\in\mathcal{O}(\Delta^{n+1})$ has divisor $g(M)$, will define $f(M)$.

To prove the existence of these good projections, we let λ be a generic linear form on $\mathbb{C}^N$ and set $\lambda_f=\lambda\circ f$. Then $\lambda_f=0$ defines

$$M_\lambda = f^{-1}\text{ (hyperplane section of } f(M)).$$

By the induction assumption, the image of

$$f\colon\ M_\lambda\to\Delta^{N-1}$$

is an analytic variety of dimension $\leqslant n-1$. For a generic choice of coordinate system, the coordinate projections

$$f(M_\lambda)\to\Delta^n$$

are all proper mappings.

To complete the argument we make an observation: in $\mathbb{C}\times\mathbb{C}^p\times\mathbb{C}^q$ with coordinates $(u,v_1,\ldots,v_p,w_1,\ldots,w_q)=(u,v,w)$ we suppose given a closed subset S of the polycylinder $\Delta\times\Delta^p\times\Delta^q$ defined by $|u|<\varepsilon$, $|v_i|<\varepsilon$, $|u_\alpha|<\varepsilon$. Suppose that we let $S_0=S\cap\{u=0\}$, and assume that the projection $S_0\to\Delta^p$ induced by $(0,v,w)\to v$ is proper. Then, taking a smaller ε if necessary, the projection $S\to\Delta\times\Delta^p$ induced by $(u,v,w)\to(u,v)$ will again be proper.

Following these reductions we come to the essential point.

Completion of the Proof. The idea is this. We are given a proper holomorphic mapping

$$f\colon\ M\to\Delta^{n+1}$$

that has maximal rank n at some point $p_0\in M^*$. We let $W\subset M$ be the union of the singular set of M and subvariety where the Jacobian of f has rank $<n$. By the induction assumption $f(W)$ is an analytic subvariety of codimension $\geqslant 2$ in Δ^{n+1}. The image of a sufficiently small neighborhood of a point $p\in M-W$ is a piece of smooth analytic hypersurface in Δ^{n+1}, and the closure

$$\overline{f(M-W)} = f(M-W)\cup f(W).$$

The problem is therefore to show that the two pieces $f(M-W)$ and $f(W)$ fit together nicely.

What we do know is that

$$\varphi\to\int_{M^*}f^*(\varphi),\qquad \varphi\in A_c^{n,n}(\Delta^{n+1})$$

defines a closed, positive current $S\in\mathfrak{D}^{1,1}(\Delta^{n+1})$. By the $\partial\bar\partial$-Poincaré lemma we may write

$$S = \frac{\pi}{\sqrt{-1}}\,\partial\bar{\partial}\varphi,$$

where φ is a real distribution on Δ^{n+1}. Around a point $q \in f(M-W)$ lying outside the codimension $\geqslant 2$ subvariety $f(W)$, the image $f(M)$ is locally the divisor of a holomorphic function h. By the Poincaré-Lelong formula

$$\partial\bar{\partial}(\varphi - \log|h|) = 0$$

near q, so that $\varphi - \log|h|$ is the real part of a holomorphic function j. This proves that the current

$$\theta = \partial\varphi = d\log h + dj \qquad \text{(locally)}$$

is a closed meromorphic 1-form on $\Delta^{n+1} - f(W)$. By the Levi extension theorem, θ extends to a meromorphic 1-form on all of Δ^{n+1}. The polar divisor of θ contains $f(M-W)$ and therefore is equal to $f(M)$. Equivalently, $f(M)$ is the divisor of the holomorphic function

$$e^{2\pi\sqrt{-1}\int\theta}.$$

This completes the proof of the proper mapping theorem.

3. CHERN CLASSES

Definitions

In this section we will give the definition and some properties of the basic topological invariants of complex vector bundles, the Chern classes. We will not be concerned with holomorphic bundles until later; for the present, all our manifolds and vector bundles will be simply C^∞.

We begin by recalling some of the definitions of Section 5 of Chapter 0. Let M be a manifold, $E \xrightarrow{\pi} M$ a complex vector bundle, and $\mathcal{A}^p(E)$ the sheaf of E-valued p-forms, that is, the sheaf of C^∞ sections of the bundle $\Lambda^p T^*(M) \otimes E$. We define a *connection* D on E to be an operator

$$D\colon\ \mathcal{A}^0(E) \to \mathcal{A}^1(E)$$

satisfying Leibnitz' rule

$$D(f\cdot\xi) = df \otimes \xi + f\cdot D\xi$$

for $f \in C^\infty(U)$, $\xi \in \mathcal{A}^0(E)(U)$. If $\varphi_\alpha : E|U_\alpha \to U_\alpha \times \mathbb{C}^n$ is a trivialization of E over $U_\alpha \subset M$, then we can identify sections ξ of E over U_α with n-vectors $\xi_\alpha = {}^t(\xi_{\alpha,1},\ldots,\xi_{\alpha,n})$ of functions on U_α. If $\{e_{\alpha,i}\}$ is the frame for E over U_α given by the constant vectors $(0,\ldots,1,\ldots,0)$, we can write

$$De_{\alpha,i} = \sum \theta_{\alpha_{ij}} \otimes e_{\alpha,j}.$$

The matrix $\theta_\alpha = (\theta_{\alpha_{ij}})$ of 1-forms is called the *connection matrix* for D; we have for a general section $\xi \in \mathscr{Q}^0(E)(U_\alpha)$

$$D\xi_\alpha = d\xi_\alpha + {}^t\theta_\alpha \cdot \xi_\alpha.$$

If $\varphi_\beta : E|U_\beta \to U_\beta \times \mathbb{C}^n$ is another trivialization of E over $U_\beta \subset M$ with $\varphi_\alpha = g_{\alpha\beta} \cdot \varphi_\beta$ and θ_β is the connection matrix for D in terms of φ_β, then

$$\theta_\alpha = g_{\alpha\beta} \cdot \theta_\beta \cdot g_{\alpha\beta}^{-1} + dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.$$

Note that the dependence of θ on the choice of frame is nonlinear—i.e., θ is not a tensor field of E. Indeed, by solving the equations $g_{\alpha\beta}(x_0) =$ identity and $dg_{\alpha\beta}(x_0) = -\theta_\alpha(x_0)$, we can find a trivialization of E in a neighborhood of any point $x_0 \in M$ in terms of which the connection matrix $\theta_\beta(x_0)$ of D vanishes at x_0.

We extend the connection D to an operator $D : \mathscr{Q}^q(E) \to \mathscr{Q}^{q+1}(E)$ by forcing Leibnitz' rule; that is, by setting, for $\xi \in \mathscr{Q}^0(E)$ and η a q-form,

$$D(\eta \otimes \xi) = d\eta \otimes \xi + (-1)^q \eta \wedge D\xi \in \mathscr{Q}^{q+1}(E).$$

We then define the *curvature operator* Θ by

$$\Theta = D^2 : \mathscr{Q}^q(E) \to \mathscr{Q}^{q+2}(E).$$

In terms of a trivialization φ_α, we have

$$(\Theta\xi)_\alpha = \Theta_\alpha \xi_\alpha,$$

where Θ_α is the matrix of 2-forms

$$\Theta_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha;$$

Θ_α is called the *curvature matrix* of D in terms of φ_α. If φ_β is another trivialization with $\varphi_\alpha = g_{\alpha\beta}\varphi_\beta$,

$$\Theta_\alpha = g_{\alpha\beta} \Theta_\beta \cdot g_{\alpha\beta}^{-1}.$$

This transition rule just expresses the directly verifiable fact that Θ is linear over $C^\infty(M)$, i.e., that $\Theta \in A^2(\mathrm{Hom}(E, E))$.

In the case of E a line bundle the curvature matrix is, according to the transition rule above, a global 2-form, and we have seen that the cohomology class $[(\sqrt{-1}/2\pi)\Theta]$, the Chern class of E, reflects the topological structure of E. In order to define the general Chern classes of a vector bundle, we digress for a moment to consider those functions of a variable matrix which are invariant under conjugation.

Let $\mathfrak{M}_n \cong \mathbb{C}^{n^2}$ denote the vector space of $n \times n$ matrices. A polynomial function $P : \mathfrak{M}_n \to \mathbb{C}$, homogeneous of degree k in the entries, is said to be *invariant* if

$$P(A) = P(gAg^{-1})$$

for all $A \in \mathfrak{M}_n$, $g \in \mathrm{GL}_n$. The basic examples of such polynomials $P(A)$ are the elementary symmetric polynomials of the eigenvalues of A, i.e., the polynomials $P^i(A)$ defined by the relation

$$\det(A+t\cdot I) = \sum_{k=0}^{n} P^{n-k}(A)\cdot t^k.$$

In particular, $P^n(A)=\det(A)$ and $P^1(A)=\operatorname{trace}(A)$; in general, if for any multiindexes I, $J\subset\{1,\ldots,n\}$ we let $A_{I,J}$ denote the (I,J)th minor $(A_{ij})_{i\in I,j\in J}$ of A, we can write

$$\begin{aligned} P^k(A) &= \sum_{{}^\# I=k} \det(A_{I,I}) \\ &= \operatorname{trace}(\textstyle\bigwedge^k A). \end{aligned}$$

The polynomials P^i are called the *elementary invariant polynomials*. In fact, any holomorphic function f on $\mathfrak{M}_n$ invariant under conjugation is expressible as a power series in the P^i: If we set

$$F(\lambda_1,\ldots,\lambda_n) = f\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

then F is a symmetric holomorphic function in $\lambda_1,\ldots,\lambda_n$. Write

$$F(\lambda_1,\ldots,\lambda_n) = G(\sigma_1,\ldots,\sigma_n),$$

where $\sigma_1,\ldots,\sigma_n$ are the elementary symmetric polynomials in the λ_i; the equality

$$f(A) = G(P^1(A),\ldots,P^n(A))$$

then holds throughout the connected and dense open set of semisimple (i.e., diagonalizable) matrices in GL_n, hence in all of $\mathfrak{M}_n$.

Now, a k-linear form

$$\tilde{P}\colon \mathfrak{M}_n\times\cdots\times\mathfrak{M}_n\to\mathbb{C}$$

is called *invariant* if for any $A_1,\ldots,A_k\in\mathfrak{M}_n$, $g\in GL_n$,

$$\tilde{P}(A_1,\ldots,A_k) = \tilde{P}(gA_1g^{-1},\ldots,gA_kg^{-1}).$$

An invariant form $\tilde{P}$ clearly gives an invariant polynomial P by

$$P(A) = \tilde{P}(A,\ldots,A).$$

In fact, the converse is also true: any invariant polynomial P of degree k can be realized as the restriction of a symmetric invariant k-linear form $\tilde{P}$ on $\mathfrak{M}_n\times\cdots\times\mathfrak{M}_n$ to the diagonal. The form $\tilde{P}$, called the *polarization* of P, is uniquely determined by P. For example, for $k=2$ we have

$$\tilde{P}(A,B) = \tfrac{1}{2}(P(A+B)-P(A)-P(B)).$$

In general, to polarize P^k, if for $(A^1,\ldots,A^k)\in(\mathfrak{M}_n)^k$, $\tau\in S_k$ a permutation and $I \subset \{1,\ldots,n\}$ a multiindex of order k, we let A_I^τ be the $k \times k$ matrix whose ith column is the ith column of $A_{I,I}^{\tau(i)}$, then

$$\tilde{P}^k(A_1,\ldots,A_k) = \frac{1}{k!} \sum_{\tau \in S^k} \sum_{\#I=k} \det(A_I^\tau);$$

and the polarizations of a general invariant polynomial—expressed as a polynomial in the elementary invariant polynomials P^i—can be written out in a similarly unenlightening way.

We return now to our complex vector bundle $E \xrightarrow{\pi} M$ of rank n. Let $\{U_\alpha\}$ be an open cover of M with φ_α a trivialization of E over U_α and θ_α and Θ_α the connection and curvature matrices of the connection D on E in terms of φ_α. Then, since the wedge product is commutative on forms of even degree, for any invariant polynomial P of degree k on $\mathfrak{M}_n$ the expression

$$P(\Theta_\alpha)$$

is a well-defined form of degree $2k$ on U_α; since

$$\Theta_\alpha = g_{\alpha\beta} \cdot \Theta_\beta \cdot g_{\alpha\beta}^{-1},$$

we see that

$$P(\Theta_\alpha) = P(\Theta_\beta)$$

in $U_\alpha \cap U_\beta$, so that $P(\Theta) = P(\Theta_\alpha)$ *is a well-defined global* 2k-*form on* M, *independent of the trivializations chosen*. The basic fact is

Lemma. *For* P *any invariant polynomial of degree* k,

1. $dP(\Theta_\alpha) = 0$,
2. *The cohomology class* $[P(\Theta_\alpha)] \in H_{DR}^{2k}(M)$ *is independent of the connection chosen for* E.

Proof. Writing $P(\Theta_\alpha) = \tilde{P}(\Theta_\alpha,\ldots,\Theta_\alpha)$ for $\tilde{P}$ a polarization of P, by linearity

$$dP(\Theta_\alpha) = \sum \tilde{P}(\Theta_\alpha,\ldots,d\Theta_\alpha,\ldots,\Theta_\alpha).$$

Now $\Theta_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha$, so $d\Theta_\alpha = d\theta_\alpha \wedge \theta_\alpha - \theta_\alpha \wedge d\theta_\alpha$. But $P(\Theta_\alpha)$ is invariant under change of frame for E, and as we saw for any $x_0 \in M$, we can find a frame for E in terms of which the connection matrix θ_β vanishes at x_0. Thus

$$\begin{aligned} dP(\Theta_\alpha) &= dP(\Theta_\beta) \\ &= \sum \pm \tilde{P}(\Theta_\beta,\ldots,d\theta_\beta \wedge \theta_\beta - \theta_\beta \wedge d\theta_\beta,\ldots,\Theta_\beta) \\ &\Rightarrow dP(\Theta_\beta)(x_0) = 0 \\ &\Rightarrow dP(\Theta) \equiv 0. \end{aligned}$$

In order to prove part 2, we need to establish an identity for invariant forms. We consider the holomorphic function on GL_n given by

$$f(g)=P(gA_1g^{-1},\ldots,gA_kg^{-1})$$

for any choice of $A_1,\ldots,A_k\in\mathfrak{M}_n$. Using as coordinates on GL_n the entries of $g'=g-I$, we compute the linear term f_1 of the power series expansion for f around I. First,

$$(I+g')^{-1}=I-g'+[2].$$

Thus

$$\begin{aligned}f(g)&=P(gA_1g^{-1},\ldots,gA_kg^{-1})\\&=P((I+g')A_1(I-g'),\ldots,(I+g')A_k(I-g'))+[2]\\&=P(A_1,\ldots,A_k)+\sum_i P(A_1,\ldots,g'A_i-A_ig',\ldots,A_k)+[2].\end{aligned}$$

But if P is invariant,

$$f=P(A_1,\ldots,A_k);$$

thus all higher-order terms in the power series for f vanish, and in particular

$$\sum_i P(A_1,\ldots,g'A_i-A_ig',\ldots,A_k)=0.$$

Now if φ is a 1-form, g a matrix of functions, and A_i a matrix of forms of degree d_i, by multilinearity

$$\begin{aligned}\sum_i(-1)^{d_1+\cdots+d_{i-1}}&P(A_1,\ldots,\varphi\wedge gA_i,\ldots,A_k)\\&=\sum_i\varphi\wedge P(A_1,\ldots,gA_i,\ldots,A_k)\\&=\sum_i\varphi\wedge P(A_1,\ldots,A_ig,\ldots,A_k)\\&=\sum_i(-1)^{d_1+\cdots+d_i}P(A_1,\ldots,A_i\wedge\varphi g,\ldots,A_k).\end{aligned}$$

In general, if θ is any matrix of 1-forms, θ can be written $\sum\theta_\alpha g_\alpha$, where θ_α is a 1-form and g_α is a matrix of functions; by linearity again,

$$(*)\quad\begin{aligned}\sum_i(-1)^{d_1+\cdots+d_{i-1}}&P(A_1,\ldots,\theta\wedge A_i,\ldots,A_k)\\&=\sum_i(-1)^{d_1+\cdots+d_i}P(A_1,\ldots,A_i\wedge\theta,\ldots,A_k).\end{aligned}$$

Now we can prove Part 2. Let $D,\tilde{D}$ be two connections on E, with local connection and curvature matrices θ_α and $\tilde{\theta}_\alpha,\Theta_\alpha$ and $\tilde{\Theta}_\alpha$. In terms of the trivialization φ_α, we have

$$D\xi_\alpha=d\xi_\alpha+{}^t\theta_\alpha\xi_\alpha,\qquad\tilde{D}\xi_\alpha=d\xi_\alpha+{}^t\tilde{\theta}_\alpha\xi_\alpha;$$

consequently the operator $\eta = D - \tilde{D}$ is linear over $C^\infty(M)$, and so it is given in terms of the trivialization φ_α as multiplication by the transpose of the matrix $\eta_\alpha = \theta_\alpha - \tilde{\theta}_\alpha$, which transforms by the rule

$$\eta_\alpha = g_{\alpha\beta}\eta_\beta g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta^{-1}$. Consider the homotopy

$$D_t = \tilde{D} + t\eta, \qquad 0 \leqslant t \leqslant 1,$$

between $D_0 = \tilde{D}$ and $D_1 = D$. D_t has connection matrix $\theta_t = \tilde{\theta} + t\eta$, hence curvature matrix

$$\Theta_t = d(\tilde{\theta} + t\eta) - (\tilde{\theta} + t\eta) \wedge (\tilde{\theta} + t\eta).$$

Let P be an invariant polynomial of degree k. We claim that

$$[P(\Theta)] = [P(\tilde{\Theta})] \in H^{2k}_{\mathrm{DR}}(M).$$

To prove this, we will consider the arc in $A^{2k}(M)$ given by

$$t \mapsto P(\Theta_t)$$

and show that its tangent vector $(\partial/\partial t)P(\Theta_t)$ lies in the subspace $dA^{2k-1}(M) \subset A^{2k}(M)$; this will show that the image curve $t \mapsto [P(\Theta_t)] \in H^{2k}_{\mathrm{DR}}(M)$ is constant. The calculation goes as follows:

$$\frac{\partial}{\partial t}\Theta_t = d\eta - (\tilde{\theta} \wedge \eta + \eta \wedge \tilde{\theta}\,) - 2t\eta \wedge \eta,$$

hence

$$\begin{aligned}\frac{\partial}{\partial t}P(\Theta_t) &= \frac{\partial}{\partial t}\tilde{P}(\Theta_t, \ldots, \Theta_t)\\ &= k \cdot \tilde{P}\left(\frac{\partial}{\partial t}\Theta_t, \Theta_t, \ldots, \Theta_t\right)\\ &= k\tilde{P}(d\eta, \Theta_t, \ldots, \Theta_t) - k\tilde{P}(\tilde{\theta} \wedge \eta + \eta \wedge \tilde{\theta}, \Theta_t, \ldots, \Theta_t)\\ &\quad - 2kt\tilde{P}(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t).\end{aligned}$$

Applying the identity (*) with $\theta = \eta$,

$$\begin{aligned}&\tilde{P}(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t) - (k-1)\tilde{P}(\eta, \eta \wedge \Theta_t, \Theta_t, \ldots, \Theta_t)\\ &\qquad = -\tilde{P}(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t) - (k-1)\tilde{P}(\eta, \Theta_t \wedge \eta, \Theta_t, \ldots, \Theta_t),\end{aligned}$$

so that

$$2kt\tilde{P}(\eta \wedge \eta, \Theta_t, \ldots, \Theta_t) = tk(k-1)\tilde{P}(\eta, \eta \wedge \Theta_t - \Theta_t \wedge \eta, \ldots, \Theta_t, \ldots).$$

Similarly, by (*),

$$\begin{aligned}&\tilde{P}(\tilde{\theta} \wedge \eta, \Theta_t, \ldots, \Theta_t) - (k-1)\tilde{P}(\eta, \tilde{\theta} \wedge \Theta_t, \ldots, \Theta_t, \ldots)\\ &\qquad = -\tilde{P}(\eta \wedge \tilde{\theta}, \Theta_t, \ldots, \Theta_t) - (k-1)\tilde{P}(\eta, \Theta_t \wedge \tilde{\theta}, \ldots, \Theta_t, \ldots),\end{aligned}$$

and so

$$-k\tilde{P}(\tilde{\theta} \wedge \eta + \eta \wedge \tilde{\theta}, \ldots, \Theta_t, \ldots) = k(k-1)\tilde{P}(\eta, \Theta_t \wedge \tilde{\theta} - \tilde{\theta} \wedge \Theta_t, \ldots, \Theta_t, \ldots).$$

Thus we have

$$\frac{\partial}{\partial t}P(\Theta_t) = k\tilde{P}(d\eta,\Theta_t,\ldots,\Theta_t) + k(k-1)\tilde{P}\big(\eta,\Theta_t\wedge(\tilde{\theta}+t\eta)-(\tilde{\theta}+t\eta)\wedge\Theta_t,\ldots,\Theta_t,\ldots,\big).$$

But for any connection θ with curvature Θ, $d\Theta=\theta\wedge\Theta-\Theta\wedge\theta$ and consequently

$$d\Theta_t = (\tilde{\theta}+t\eta)\wedge\Theta_t - \Theta_t\wedge(\tilde{\theta}+t\eta);$$

so we can write, finally,

$$\frac{\partial}{\partial t}P(\Theta_t) = k\tilde{P}(d\eta,\Theta_t,\ldots,\Theta_t) - k(k-1)\tilde{P}(\eta,d\Theta_t,\ldots,\Theta_t)$$

$$= k\cdot d\tilde{P}(\eta,\Theta_t,\ldots,\Theta_t). \qquad \text{Q.E.D.}$$

Note: For those accustomed to the general formalism of differential geometry, we can sketch the above calculation in more intrinsic terms as follows: First, the operator $D: A^q(E)\to A^{q+1}(E)$ may be extended to an operator on all tensor bundles of E. Then, since the connection matrix of D can be made to vanish at any point, we obtain the *Bianchi identity*

$$D\Theta = 0,$$

and applying P to the algebra $A^*(\mathrm{Hom}(E,E))$,

$$d\tilde{P}(A_1,\ldots,A_q) = \sum_i (-1)^{d_1+\cdots+d_{i-1}}\tilde{P}(A_1,\ldots,DA_i,\ldots,A_q),$$

where $A_i\in A^{d_i}(\mathrm{Hom}(E,E))$; thus $dP(\Theta)\equiv 0$. Finally, with D_t, θ_t, Θ_t, and η as above we see that $(\partial/\partial t)\Theta_t = D_t\eta$, and so

$$d(k\tilde{P}(\eta,\Theta_t,\ldots,\Theta_t)) = k\tilde{P}(D_t\eta,\Theta_t,\ldots,\Theta_t)$$

$$= k\tilde{P}\left(\frac{\partial}{\partial t}\Theta_t,\Theta_t,\ldots,\Theta_t\right)$$

$$= \frac{\partial}{\partial t}P(\Theta_t).$$

To restate the lemma: If we let Φ denote the graded algebra of invariant polynomials, then for any vector bundle $E\to M$, we obtain a well-defined homomorphism of algebras

$$\Phi \xrightarrow{w} H^{2*}_{\mathrm{DR}}(M)$$

given by

$$P \overset{w}{\mapsto} [P(\Theta)],$$

where Θ is the curvature matrix of any connection in E; w is called the *Weil homomorphism*.

In particular, let P^i denote again the elementary invariant polynomials. We define the *Chern forms* $c_i(\Theta)$ of the curvature Θ in E by

$$c_i(\Theta) = P^i\left(\frac{\sqrt{-1}}{2\pi}\Theta\right),$$

and we define the *Chern classes* $c_i(E)$ by

$$c_i(E) = \left[P^i\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)\right] \in H^{2i}_{\mathrm{DR}}(M).$$

The *total Chern class* $c(E)$ is the sum of the Chern classes:

$$c(E) = \sum_{i \geqslant 0} c_i(E) \in H^{2*}_{\mathrm{DR}}(M),$$

where we set $c_0(E) = 1 \in H^0_{\mathrm{DR}}(M)$. Also, for M a complex manifold, we take the Chern classes $c_i(M)$ of M to be the Chern classes of its holomorphic tangent bundle $T'(M)$.

Note that the definition of $c_1(E)$ here agrees with our former definition of the Chern class of a holomorphic line bundle. In general—as will be clear by the end of this section—the Chern classes of a vector bundle are likewise purely topological invariants. The basic properties of the Chern classes are these:

1. First, if $f: M \to N$ is any C^∞ map, $E \to N$ a complex vector bundle, then

$$c_r(f^*E) = f^*c_r(E).$$

To see this, note that if D is a connection on E, $\underline{U} = \{U_\alpha\}$ an open cover of N with $e_{1,\alpha}, \ldots, e_{k,\alpha}$ a frame for E over U_α and θ_α the connection matrix for D relative to $\{e_{i,\alpha}\}$, then the matrices

$$f^*(\theta_\alpha) \qquad \text{in } f^{-1}(U_\alpha)$$

define a connection D^* on $f^*E \to M$ with curvature

$$\Theta(D^*) = f^*(\Theta(D)).$$

2. Next, let $E \to M$, $F \to M$ be two vector bundles with connections D, D' and curvature matrices Θ, Θ', respectively. Then the operator

$$D'' = D \oplus D': \mathcal{A}^0(E \oplus F) \to \mathcal{A}^1(E \oplus F)$$

is a connection for the bundle $E \oplus F$, with curvature matrix

$$\Theta'' = \begin{pmatrix} \Theta & 0 \\ 0 & \Theta' \end{pmatrix}.$$

Then we have

$$\det(\Theta'' + tI) = \det(\Theta + tI) \cdot \det(\Theta' + tI)$$

as polynomials in t; i.e.,

$$c(E \oplus F) = c(E) \cdot c(F).$$

This is the *Whitney product formula.*

3. Similarly, if E is a vector bundle of rank r and L is a line bundle, we have seen that for appropriate connections on E, L, and $E \otimes L$,

$$\Theta_{E \otimes L} = \Theta_E \otimes 1 + I_r \otimes \Theta_L,$$

so that

$$c_1(E \otimes L) = \left[\operatorname{trace} \frac{\sqrt{-1}}{2\pi} \Theta_{E \otimes L}\right] = c_1(E) + r \cdot c_1(L).$$

4. Finally for now, if Θ is the curvature matrix of a connection in a complex vector bundle E, then the dual connection in E^* has curvature matrix $-\Theta$; thus

$$c_r(E^*) = (-1)^r c_r(E).$$

We can use the Whitney product formula to evaluate the Chern classes $c_i(\mathbb{P}^n)$ of projective space, as follows. Let

$$\pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$$

be the standard projection map; let $X_0, \ldots, X_n$ be linear coordinates on $\mathbb{C}^{n+1}$ and

$$x_i = X_i / X_0, \qquad i = 1, \ldots, n$$

corresponding affine coordinates on $\mathbb{P}^n$. Then we have

$$\pi^* dx_i = \frac{X_0 \cdot dX_i - X_i \cdot dX_0}{X_0^2},$$

and so, at a point $X \in \mathbb{C}^{n+1}$, the image under π of the tangent vector $\partial / \partial X_i$ is given by

$$\pi_* \frac{\partial}{\partial X_i} = \frac{1}{X_0} \cdot \frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n,$$

$$\pi_* \frac{\partial}{\partial X_0} = -\sum \frac{X_i}{X_0^2} \cdot \frac{\partial}{\partial x_i}.$$

It follows from this calculation that:

1. If $l(X)$ is any linear functional on $\mathbb{C}^{n+1}$, the vector field

$$v(X) = l(X) \frac{\partial}{\partial X_i}$$

descends to $\mathbb{P}^n$—that is, $\pi_* v(X) = \pi_* v(\lambda X)$ for any $X \in \mathbb{C}^{n+1}$, $\lambda \in \mathbb{C}$. We will denote by $\pi_* v$ the induced vector field on $\mathbb{P}^n$.

2. The tangent space $T'_x(\mathbb{P}^n)$ to $\mathbb{P}^n$ at a point $x=\pi(X)$ is spanned by the vectors

$$\left\{\pi_* \frac{\partial}{\partial X_i}\right\}_{i=0,\ldots,n}$$

with the single relation

$$\sum X_i \frac{\partial}{\partial X_i} = 0.$$

Now, recalling that the fiber of the hyperplane line bundle $H \to \mathbb{P}^n$ over a point $x=\pi(X) \in \mathbb{P}^n$ corresponds to linear functionals on the line $\mathbb{C}\{X\} \subset \mathbb{C}^{n+1}$, we can define a bundle map

$$H^{\oplus(n+1)} = \underbrace{H \oplus \cdots \oplus H}_{n+1} \xrightarrow{\mathcal{E}} T'(\mathbb{P}^n)$$

by setting, for $\sigma = (\sigma_0, \ldots, \sigma_n)$ a section of $H^{\oplus(n+1)}$,

$$\mathcal{E}(\sigma) = \pi_*\left(\sum \sigma_i(X) \frac{\partial}{\partial X_i}\right).$$

By the second observation the map $\mathcal{E}$ is surjective, with kernel the trivial line bundle spanned by the section

$$\tau = (X_0, \ldots, X_n).$$

Thus we have an exact sequence of bundles on $\mathbb{P}^n$:

$$0 \longrightarrow \mathbb{C} \longrightarrow H^{\oplus(n+1)} \xrightarrow{\mathcal{E}} T'(\mathbb{P}^n) \longrightarrow 0,$$

called the *Euler sequence*. Now, from the C^∞ decomposition

$$H^{\oplus(n+1)} = T'(\mathbb{P}^n) \oplus \mathbb{C}$$

and the Whitney product formula, we find

$$c(T'(\mathbb{P}^n)) = c(H^{\oplus(n+1)}) = c(H)^{n+1} = (1+\omega)^{n+1},$$

where $\omega = \eta_H \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the class of a hyperplane.

The Gauss-Bonnet Formulas

As we have seen, the first Chern class of a holomorphic line bundle is Poincaré dual to the cycle represented by the zero-locus of a global holomorphic section. A similar geometric description of the general Chern classes—or rather their Poincaré duals in homology—is available, and the remainder of this section will be spent in deriving it. The computation will not be made directly; instead, we will first compute the Chern classes of the universal bundles on the Grassmannians, and then by the functoriality of the Chern classes draw conclusions for general vector bundles.

Recall from Section 6 of Chapter 1 that for any strictly increasing flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n$$

of linear subspaces of $\mathbb{C}^n$, and for any nonincreasing sequence of k integers $a_i : 0 \leqslant a_i \leqslant n-k$, we define the *Schubert cycle* $\sigma_a(V) \subset G(k,n)$ in the Grassmannian of k-planes in $\mathbb{C}^n$ by

$$\sigma_a(V) = \left\{\Lambda:\ \dim\left(\Lambda \cap V_{n-k+i-a_i}\right) \geqslant i\right\}.$$

$\sigma_a(V)$ is an analytic subvariety of $G(k,n)$ of codimension $\sum a_i$, with fundamental class σ_a independent of the flag V; as we saw, the integral homology of the Grassmannian is freely generated by the classes σ_a.

Recall also that the *universal subbundle* $S \to G(k,n)$ is defined to be the subbundle of the trivial bundle $\mathbb{C}^n \times G(k,n)$ whose fiber over a point $\Lambda \in G(k,n)$ is just the k-plane $\Lambda \subset \mathbb{C}^n$. Letting σ_a^* denote the Poincaré dual of the cycle σ_a, our fundamental result is the

Gauss-Bonnet Theorem I

$$c_r(S) = (-1)^r \cdot \sigma^*_{1,\ldots,1}.$$

Proof. By our computation of the intersection pairing in $H_*(G(k,n),\mathbb{Z})$, we must show that for any Schubert cycle σ_a of dimension r,

$$\begin{aligned} c_r(S)(\sigma_a) &= (-1)^r \#(\sigma_{1,\ldots,1} \cdot \sigma_a) \\ &= \begin{cases} (-1)^r, & \text{if } a = n-k,\ldots,n-k,n-k-1,\ldots,n-k-1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We first note that if $\sigma_a(V)$ is any Schubert cycle of dimension r, and $a \neq n-k,\ldots,n-k,n-k-1,\ldots,n-k-1$, then a_{k-r+1} *must necessarily be* $n-k$, i.e., $\sigma_a(V) \subset \{\Lambda \in G(k,n) : \Lambda \supset V_{k-r+1}\}$. Thus if we take $e_1,\ldots,e_{k-r+1}$ any basis for $V_{k-r+1} \subset \mathbb{C}^n$, the corresponding sections e_i of the trivial bundle $\mathbb{C}^n \times G(k,n)$ all lie in S over $\sigma_a(V)$. The sections e_i of $S|_{\sigma_a(V)}$ then extend to give $k-r+1$ everywhere linearly independent sections $\tilde{e}_i$ of S over an open set $U \subset G(k,n)$ containing $\sigma_a(V)$. Let $S' \to U$ be the trivial subbundle of $S|_U$ spanned by $\tilde{e}_1,\ldots,\tilde{e}_{k-r+1}$, and let $S'' \to U$ be the quotient of $S|_U$ by S'. Since S' is trivial, we have $c(S')=1$; by the Whitney formula,

$$c(S|_U) = c(S''),$$

and hence

$$c_r(S)(\sigma_a(V)) = c_r(S'')((\sigma_a(V))=0,$$

since S'' has rank $r-1$.

Now set

$$Z_r = \sigma_{n-k,\ldots,n-k,n-k-1,\ldots,n-k-1}(V) = \left\{\Lambda:\ V_{k-r} \subset \Lambda \subset V_{k+1}\right\}.$$

It remains to check that

$$c_r(S)(Z_r) = (-1)^r.$$

To see this, let

$$\begin{aligned} Z_k &= \sigma_{n-k-1,\dots,n-k-1}(V) \\ &= \{\Lambda : \Lambda \subset V_{k+1}\} \subset G(k,n). \end{aligned}$$

$Z_k \cong \mathbb{P}(V_{k+1})^*$ is just the dual projective space of hyperplanes in V_{k+1}, and $Z_r \subset Z_k$ the linear subspace of hyperplanes containing V_{k-r}. The bundle $S|_{Z_k}$, moreover, is just the subbundle of the trivial bundle $V_{k+1} \times Z_k$ whose fiber over any $\Lambda \in Z_k$ is the hyperplane $\Lambda \subset V_{k+1}$. The quotient Q of $V_{k+1} \times Z_k$ by S is thus the universal quotient bundle on $Z_k \cong \mathbb{P}^k$, that is, the hyperplane line bundle. Letting ω denote the class of a hyperplane in $Z_k \cong \mathbb{P}^k$, we have then

$$c(Q) = 1 + \omega,$$

and since as C^∞ bundles

$$\begin{aligned} V_{k+1} \times Z_k &= S|_{Z_k} \oplus Q, \\ 1 &= c(S|_{Z_k}) \cdot (1+\omega), \end{aligned}$$

hence

$$c(S|_{Z_k}) = 1 - \omega + \omega^2 - \omega^3 + \cdots,$$

i.e.,

$$c_r(S|_{Z_k}) = (-1)^r \omega^r.$$

Thus

$$\begin{aligned} c_r(S)(Z_r) &= c_r(S|_{Z_r}) \cdot (Z_r) \\ &= (-1)^r \omega^r(\mathbb{P}^r) \\ &= (-1)^r, \end{aligned}$$

and the theorem is proved. Q.E.D.

Note that by the relation giving the Chern classes of a dual bundle

$$c_r(S^*) = (-1)^r c_r(S) = \sigma^*_{1,\dots,1}$$

and via the isomorphism $(S^* \to G(n-k,n)) \cong (Q \to G(k,n))$, we see that

$$c_r(Q) = \sigma^*_r,$$

where Q is the universal quotient bundle.

The Gauss-Bonnet formula gives us a relatively concrete interpretation of Chern classes in general as follows. Let M be a compact, oriented manifold, $E \to M$ a complex vector bundle of rank k, and $\sigma = (\sigma_1, \dots, \sigma_k)$ k global C^∞ sections of E. We define the *degeneracy set* $D_i(\sigma)$ to be the set of points $x \in M$, where $\sigma_1, \dots, \sigma_i$ are linearly dependent, i.e.,

$$D_i(\sigma) = \{x : \sigma_1(x) \wedge \cdots \wedge \sigma_i(x) = 0\}.$$

We say that the collection σ of sections is *generic* if, for each i, σ_{i+1} intersects the subspace of E spanned by $\sigma_1,\ldots,\sigma_i$ transversely—so that $D_{i+1}(\sigma)$ is, away from $D_i(\sigma)$, a submanifold of codimension $2(k-i)$—and if, moreover, integration over $D_{i+1}(\sigma)-D_i(\sigma)$ defines a closed current as discussed in Section 1 of this chapter. (By the results of Section 2 this will occur if everything is complex analytic and the dimensions are correct.) In this case, we can give the smooth locus $D_i - D_{i-1}$ an orientation: In a neighborhood of a point $x_0 \in D_i - D_{i-1}$, complete the sections $e_1 = \sigma_1,\ldots,e_{i-1}=\sigma_{i-1}$ to a frame for E, and write

$$\sigma_i(x) = \sum_j f_j(x)\cdot e_j(x).$$

D_i is then given near x_0 as the locus $(f_i = \cdots = f_k = 0)$; let Φ_i be the orientation on D_i near x_0 such that the form

$$\Phi_i \wedge \frac{\sqrt{-1}}{2}\left(df_{i+1}\wedge d\bar{f}_{i+1}\right)\wedge\cdots\wedge\frac{\sqrt{-1}}{2}\left(df_k\wedge d\bar{f}_k\right)$$

is positive for the given orientation on M. By the theorem on smoothing of cohomology given in Section 1 of this chapter, the locus D_i together with the orientation Φ_i on $D_i - D_{i-1}$ represents a cycle in homology, called the *degeneracy cycle* of the sections σ.

Now suppose $\sigma_1,\ldots,\sigma_k$ are generic sections of E. Using a partition of unity on M, we can then construct additional sections $\sigma_{k+1},\ldots,\sigma_n$ of E such that together $\sigma_1(x),\ldots,\sigma_n(x)$ span the fiber E_x of E over each point $x\in M$. By the construction of Section 6 of Chapter 1, then, the sections $\sigma_1,\ldots,\sigma_n$ give us a map

$$\iota:\ M \to G(k,n).$$

In terms of a trivialization of E, we can express $\sigma_1,\ldots,\sigma_n$ as k-vectors $V_1,\ldots,V_n$ of C^∞ functions; the map ι is given by

$$x \xrightarrow{\ \iota\ } \left[\left(V_1(x),\ldots,V_n(x)\right)\right] \in G(k,n).$$

Since the subspace $\iota(x)=\Lambda\subset\mathbb{C}^n$ corresponds to linear functionals on the fiber E_x of E over x, moreover, we see as before that $\iota^*(S)=E^*$, i.e.,

$$\iota^*(S^*) = E.$$

Now for each $r=1,\ldots,k$ let $V_{n-k+r-1}=\{e_{k-r+2},\ldots,e_n\}\subset\mathbb{C}^n$. Then for any $x\in M$, the k-plane $\Lambda=\iota(x)\in G(k,n)$ will intersect $V_{n-k+r-1}$ in a space of dimension r or greater if and only if the sections $\sigma_1,\ldots,\sigma_{k-r+1}$ are linearly dependent at x—i.e., $\iota(M)$ meets the Schubert cycle $\sigma_{1,\ldots,1}(V)\subset G(k,n)$ exactly in the degeneracy set D_{k-r+1} of the sections $\sigma_1,\ldots,\sigma_k$. Moreover the condition that $\sigma_1,\ldots,\sigma_k$ be generic assures that $\iota(M)$ meets $\sigma_{1,\ldots,1}(V)$ transversely. If α is any cycle of real dimension $2r$ on M meeting

D_{k-r+1} transversely at points p_i, then, $\iota_*\alpha$ will meet $\sigma_{1,\ldots,1}(V)$ transversely at the points $\iota(p_i)$, and by our choice of orientation for D_{k-r+1} the intersection number of $\iota_*\alpha$ with $\sigma_{1,\ldots,1}(V)$ at $\iota(p_i)$ will be that of α with D_{k-r+1} at p_i. Thus

$$^{\#}(\iota_*\alpha\cdot\sigma_{1,\ldots,1}) = {}^{\#}(\alpha\cdot D_{k-r+1}),$$

and we see that

$$\begin{aligned} c_r(E)(\alpha) &= \iota^*(c_r(S^*))(\alpha) \\ &= c_r(S^*)(\iota_*\alpha) \\ &= {}^{\#}(\iota_*\alpha\cdot\sigma_{1,\ldots,1}) \\ &= {}^{\#}(\alpha\cdot D_{k-r+1}). \end{aligned}$$

We thus have the

Gauss-Bonnet Formula II. *The* r*th Chern class* $c_r(E)$ *is Poincaré dual to the degeneracy cycle* D_{k-r+1}.

Example. We can now make a second computation for the Chern classes of projective space. Let $X_0,\ldots,X_n$ be linear coordinates on $\mathbb{C}^{n+1}$, and let $\mathcal{E}$ and π_* be as in the Euler sequence above. Let $A=(\alpha_{ij})$ be an $(n+1)\times(n+1)$ matrix all of whose minors are distinct and nonzero, and consider the vector fields

$$\begin{aligned} v_i &= \mathcal{E}(\alpha_{i,0}X_0,\ldots,\alpha_{i,n}X_n) \\ &= \pi_*\sum_j \alpha_{ij}X_i\frac{\partial}{\partial X_j}. \end{aligned}$$

We will leave it as an exercise to verify that, under the assumptions made about A, $v_1,\ldots,v_n$ are generic sections of $T'(\mathbb{P}^n)$ (this is simply a matter of writing v_i out in terms of Euclidean coordinates on $\mathbb{P}^n$), and compute the degeneracy cycles D_i of $v_1,\ldots,v_i$. First, we see that v_1 vanishes at $X\in\mathbb{P}^n$ exactly when

$$[\alpha_{1,0}X_0,\ldots,\alpha_{1,n}X_n] = [X_0,\ldots,X_n],$$

and since by assumption $\alpha_{1i}\neq 0$ for all i and $\alpha_{1i}\neq\alpha_{1j}$ for all $i\neq j$, this is the case only for $X=p_i$, where

$$p_i = [0,\ldots,0,1,0,\ldots,0], \qquad i=0,\ldots,n.$$

Thus $c_n(\mathbb{P}^n)=n+1$. Now v_1 and v_2 will be linearly dependent at $X\in\mathbb{P}^n$ when there exist $(\lambda_1,\lambda_2)\neq 0$ such that

$$[\lambda_1\alpha_{10}X_0+\lambda_2\alpha_{20}X_0,\ldots,\lambda_1\alpha_{1n}X_n+\lambda_2\alpha_{2n}X_n] = [X_0,\ldots,X_n],$$

and by the assumption that all 2×2 minor determinants of A are distinct and nonzero, this will be the case only when all but two of the homogeneous coordinates of X are zero, i.e., when X lies on a line $\overline{p_ip_j}$ for some

$0 \leqslant i \neq j \leqslant r$. D_2 thus consists of the union of the $\binom{n+1}{2}$ lines $\overline{p_i p_j}$; or in other words, if ω is the hyperplane class on $\mathbb{P}^n$,

$$c_{n-1}(\mathbb{P}^n) = \binom{n+1}{2} \cdot \omega^{n-1}.$$

In general, $v_1, \ldots, v_q$ will be linearly dependent at X exactly when all but q of the homogeneous coordinates of X vanish, i.e.,

$$D_q = \bigcup_{\# I = q} \overline{p_{i_1}, p_{i_2}, \ldots, p_{i_q}}$$

consists of the union of the coordinate $(q-1)$-planes spanned by the points p_i. Thus

$$c_q(\mathbb{P}^n) = \binom{n+1}{q} \cdot \omega^q,$$

as we computed before.

As an immediate application, we can add one more identity to those previously mentioned. If $E \to M$ is a complex vector bundle of rank k, then the first Chern class $c_1(E)$ is dual to the cycle $D_k \subset X$ given as the locus where k generic sections $\sigma_1, \ldots, \sigma_k$ of E are linearly dependent. But the k sections σ_i of E together give one section

$$\sigma = \sigma_1 \wedge \cdots \wedge \sigma_k$$

of the line bundle $\Lambda^k E \to M$, and the degeneracy set D_1 of σ is equal to D_k. Checking that the orientations are in fact the same, we have

$$c_1(\wedge^k E) = c_1(E).$$

Finally, note that given generic sections $\sigma_1, \ldots, \sigma_i$ we can define degeneracy cycles

$$D_i^{(j)} = \left\{ x : \dim \overline{\sigma_1(x), \ldots, \sigma_i(x)} \leqslant i - j \right\}.$$

If, as before, we complete the sections $\sigma_1, \ldots, \sigma_i$ to a collection $\sigma_1, \ldots, \sigma_n$ spanning each fiber, then the degeneracy cycle $D_i^{(j)}$ will be the inverse image, under the corresponding map $\iota : M \to G(k,n)$, of the Schubert cycle

$$\sigma_{\underbrace{j, \ldots, j}_{k-i+1}}\left(V_{n-i-j+1}\right) = \left\{ \Lambda : \dim \Lambda \cap V_{n-i-j+1} \geqslant k - i + 1 \right\}.$$

Composing ι *with the isomorphism* $*$: $G(k,n) \to G(n-k,n)$, *we find that*

$$D_i^{(j)} = (*\iota)^{-1}\Big(\sigma_{\underbrace{k-i+1, \ldots, k-i+1}_{j}} \Big);$$

and since $c_r(E) = (*\iota)^* \sigma_r$, we may combine this with Giambelli's formula on p. 205 to obtain

Porteous' Formula. *For $\sigma_1,\ldots,\sigma_i$ suitably generic, the Poincaré dual of the degeneracy cycle* $D_i^{(j)}$ *is*

$$D_i^{(j)*} = \det\begin{vmatrix} c_{k-i+1}(E) & \cdots & c_{k-i+j}(E) \\ \vdots & & \vdots \\ c_{k-i-j+2}(E) & \cdots & c_{k-i+1}(E) \end{vmatrix}$$

Finally, we will specialize our general Gauss-Bonnet formula to obtain a more classical form, and also to explain the terminology. Suppose that M is of real dimension $2n$, $E\to M$ of complex rank n, and σ a global C^∞ section of E having nondegenerate zeros at points $p_\nu\in M$. For each ν, let $e_1,\ldots,e_n$ be a frame for E around p_ν, $x=(x_1,\ldots,x_{2n})$ oriented real coordinates on X centered around p_ν, and write

$$\sigma(x) = \sum\left(a_{ak}^{\nu}+\sqrt{-1}\ b_{ak}^{\nu}\right)\cdot x_a\cdot e_k(x)+[2], \qquad a_{ak}^{\nu}, b_{ak}^{\nu}\in\mathbb{R}.$$

Let A_{p_ν} be the $2n\times 2n$ matrix (A^ν,B^ν), where $A^\nu=(a_{ak}^\nu)$ and $B^\nu=(b_{ak}^\nu)$. Then, if we write

$$\sigma(x) = \sum f_k(x)\cdot e_k(x)$$

as before, we have

$$\left(df_k\wedge d\bar{f}_k\right)(p_\nu) = \sum_a\left(a_{ak}^{\nu}+\sqrt{-1}\ b_{ak}^{\nu}\right)dx_a\wedge\sum_b\left(a_{bk}^{\nu}-\sqrt{-1}\ b_{bk}^{\nu}\right)dx_b$$

$$= -2\sqrt{-1}\sum a_{ak}^{\nu}\cdot b_{bk}^{\nu}\cdot dx_a\wedge dx_b,$$

and so by linear algebra the sign of the point p_ν in the degeneracy cycle D_1 of σ is

$$(-1)^{n(n-1)/2}\cdot\operatorname{sgn}\det\left(A_{p_\nu}\right).$$

Thus by Gauss-Bonnet II we have

$$c_n(E) = \sum(-1)^{n(n-1)/2}\operatorname{sgn}\det\left(A_{p_\nu}\right).$$

Specializing still further, let M be a complex manifold of dimension n, $E=T'(M)\to M$ its holomorphic tangent bundle and σ a C^∞ section of E having nondegenerate zeros at $p_\nu\in M$. Let $z=(z_1,\ldots,z_n)$ be local holomorphic coordinates centered around p_ν, and write

$$z_i = x_{2i-1}+\sqrt{-1}\ x_{2i}$$

so that $x=(x_1,\ldots,x_{2n})$ is an oriented real coordinate system for M near p_ν. Then, if

$$v(z) = \sum\left(a_{jk}^{\nu}+\sqrt{-1}\ b_{jk}^{\nu}\right)z_j\frac{\partial}{\partial z_k}+[2]$$

and if $A_{p_\nu} = (A^\nu, B^\nu)$ as above, we have

$$c_n(M) = \sum (-1)^{n(n-1)/2} \operatorname{sgn}\det(A_{p_\nu}).$$

Now let

$$v'(z) = \tfrac{1}{2}\left(v(z) + \overline{v(z)}\right)$$

be the real vector field obtained from v by the real projection $T'(M) \to T_{\mathbb{R}}(M)$. Then

$$\begin{aligned} v'(z) &= \frac{1}{2} \sum \left(a^\nu_{jk} + \sqrt{-1}\, b^\nu_{jk}\right) z_j \cdot \left(\frac{\partial}{\partial x_{2k-1}} - \sqrt{-1}\, \frac{\partial}{\partial x_{2k}}\right) \\ &\quad + \frac{1}{2} \sum \left(a^\nu_{jk} - \sqrt{-1}\, b^\nu_{jk}\right) \bar{z}_j \cdot \left(\frac{\partial}{\partial x_{2k-1}} + \sqrt{-1}\, \frac{\partial}{\partial x_{2k}}\right) + [2] \\ &= \sum a^\nu_{jk} x_j \frac{\partial}{\partial x_{2k-1}} + \sum b^\nu_{jk} x_j \cdot \frac{\partial}{\partial x_{2k}} + [2], \end{aligned}$$

so the index of v' at p_ν is $(-1)^{n(n-1)/2}$ times the sign of the determinant of A_{p_ν}. Thus by the Hopf index theorem (to be proved in the next section)

$$\chi(M) = \sum (-1)^{n(n-1)/2} \operatorname{sgn}\det(A_{p_\nu}),$$

and so we have

Gauss-Bonnet Formula III. $c_n(M) = \chi(M)$.

We have, in this discussion, inverted the historical order of things. The Chern classes of complex vector bundles and the analogous Steifel-Whitney classes of real vector bundles were originally defined using obstruction theory; in terms of this definition, the classes were visibly the Poincaré duals of degeneracy cycles. Chern then discovered the remarkable fact that these global topological invariants of a vector bundle could in fact be computed from the local hermitian differential geometric structure of the vector bundle; Chern's theorem has since been frequently adopted as a definition.

Some Remarks—Not Indispensable—Concerning Chern Classes of Holomorphic Vector Bundles

Suppose that $E \to M$ is a holomorphic vector bundle with base space a complex manifold M. If we choose a hermitian connection as in Section 5 of Chapter 0, then the hermitian symmetry $\Theta + {}^t\overline{\Theta} = 0$ of the curvature matrix in a unitary frame implies the relations

$$c_p(\Theta) \text{ has type } (p,p), \qquad c_p(\Theta) = \overline{c_p(\Theta)}$$

on the Chern forms. In case M is a compact Kähler manifold, these imply that

$$c_p(E) \in H^{p,p}(M) \cap H^{2p}(M,\mathbb{Z});$$

i.e., *the Chern classes are integral and of Hodge type* (p, p).

In case M is a projective algebraic variety, which by the Kodaira embedding theorem is equivalent to the existence of a positive holomorphic line bundle $L \to M$, we can say more. We assume that L is the hyperplane bundle relative to a projective embedding. First, the Chern classes of $E \cong (E \otimes L^k) \otimes L^{-k}$ can be expressed as polynomials in the Chern classes of $E \otimes L^k$ and L^{-k}. The Chern class of L^{-k} is

$$c(L^{-k}) = 1 - k\eta_D,$$

where D is a hyperplane section of M, and by Theorem B in Section 5 of Chapter 1 we may find a holomorphic embedding

$$M \to G(r,N) \qquad (r = \operatorname{rank} E)$$

inducing $E \otimes L^k$ from the universal bundle over the Grassmannian. According to the preceding discussion, the Chern classes of $E \otimes L^k$ are Poincaré dual to the intersection of M with suitable Schubert cycles. In summary, *the Chern classes of a holomorphic vector bundle over an algebraic variety are represented by fundamental classes of algebraic cycles.*

There is also a notion of *positivity* for the Chern classes of holomorphic vector bundles. We shall not enter into this in detail, as it will not be used in the study of specific varieties, but will offer two observations. If $E \to M$ is generated by its global holomorphic sections, we have seen at the end of Section 5 in Chapter 0 that there is a hermitian connection whose curvature matrix has the local form

$$\Theta^{\alpha}_{\beta} = \sum_{\mu} A^{\alpha}_{\mu} \wedge \bar{A}^{\beta}_{\mu},$$

where $A^{\alpha}_{\mu} = \sum A^{\alpha}_{\mu j} dz_j$ is a matrix of (1,0) forms. The qth Chern polynomial is then

$$c_q(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^q \left\{ \sum_{\alpha_1 < \cdots < \alpha_q} \left(\frac{1}{q!} \sum_{\pi} \operatorname{sgn} \pi \, \Theta^{\alpha_1}_{\alpha_{\pi(1)}} \wedge \cdots \wedge \Theta^{\alpha_q}_{\alpha_{\pi(q)}} \right) \right\}$$

$$= \left(\frac{\sqrt{-1}}{2\pi}\right)^q (-1)^{q(q-1)/2} \sum_{\substack{\alpha_1 < \cdots < \alpha_q \\ \mu_1 < \cdots < \mu_q}} \frac{\operatorname{sgn} \pi}{q!} A^{\alpha_1}_{\mu_1} \wedge \cdots \wedge A^{\alpha_q}_{\mu_q} \wedge \bar{A}^{\alpha_{\pi(1)}}_{\mu_1} \wedge \cdots \wedge \bar{A}^{\alpha_{\pi(q)}}_{\mu_q}$$

$$= \sqrt{-1}^{\,q^2} \sum_{\mu = (\mu_1, \ldots, \mu_q)} \eta_{\mu} \wedge \bar{\eta}_{\mu},$$

where

$$\eta_\mu = \left(\frac{1}{2\pi}\right)^q \frac{1}{q!} \sum_{\substack{\alpha_1 < \cdots < \alpha_q \\ \pi}} \operatorname{sgn}\pi \cdot A_{\mu_1}^{\alpha_{\pi(1)}} \wedge \cdots \wedge A_{\mu_q}^{\alpha_{\pi(q)}}$$

is a form of type $(q,0)$. It follows that

$$\int_Z c_q(\Theta) \geqslant 0$$

for any q-dimensional analytic subvariety Z in M.

Perhaps more interesting are the *Schwarz-type inequalities*. The simplest of these is

$$\int_Z c_1(\Theta)^2 \geqslant 2\int_Z c_2(\Theta),$$

where Z is a two-dimensional analytic subvariety of M. For simplicity of notation we prove this when the rank is 2, omitting the exterior multiplication symbol and summing repeated indices. Then

$$c_1(\Theta)^2 = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left(A_\mu^1 \bar{A}_\mu{}^1 + A_\mu^2 \bar{A}_\mu{}^2\right)\left(A_\lambda^1 \bar{A}_\lambda{}^1 + A_\lambda^2 \bar{A}_\lambda{}^2\right)$$

$$= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ -\left(A_\mu^1 A_\lambda^1 \bar{A}_\mu{}^1 \bar{A}_\lambda{}^1 + A_\mu^2 A_\lambda^2 \bar{A}_\mu{}^2 \bar{A}_\lambda{}^2\right) + 2A_\mu^1 \bar{A}_\mu{}^1 A_\lambda^2 \bar{A}_\lambda{}^2 \right\}$$

$$2c_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ 2A_\mu^1 \bar{A}_\mu{}^1 A_\lambda^2 \bar{A}_\lambda{}^2 - 2A_\mu^1 \bar{A}_\mu{}^2 A_\lambda^2 \bar{A}_\lambda{}^1 \right\}.$$

Then

$$c_1(\Theta)^2 - 2c_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ A_\mu^1 \bar{A}_\mu{}^1 A_\lambda^1 \bar{A}_\lambda{}^1 + A_\mu^2 \bar{A}_\mu{}^2 A_\lambda^2 \bar{A}_\lambda{}^2 - 2A_\mu^1 \bar{A}_\mu{}^2 A_\lambda^2 \bar{A}_\lambda{}^1 \right\},$$

and the inequality

$$\left\langle c_1(\Theta)^2 - 2c_2(\Theta), \left(\frac{2}{\sqrt{-1}}\right)^2 \tau_1 \wedge \bar{\tau}_1 \wedge \tau_2 \wedge \bar{\tau}_2 \right\rangle \geqslant 0$$

for (1,0) vectors τ_1, τ_2 follows from the usual Cauchy-Schwarz inequality. The inequalities

$$\begin{cases} c_q(E) \geqslant 0, \\ c_1^2(E) \geqslant 2c_2(E), \text{ etc.,} \end{cases}$$

are valid for any holomorphic vector bundle that is positive in a suitable sense. We shall not give the proof here, but the reader may consult S.

Bloch and D. Gieseker, The positivity of the Chern classes of an ample vector bundle, *Invent. Math.*, Vol. 12 (1971), 112–117.

4. FIXED-POINT AND RESIDUE FORMULAS

The Lefschetz Fixed-Point Formula

We now derive Lefschetz's formula for the number of fixed points, properly counted, of an endomorphism $f: M \to M$ on a compact oriented manifold M of dimension n in terms of the action of f^* on the cohomology of M. That such a formula should exist is not hard to see: a fixed point of f corresponds to a point of intersection of the graph $\Gamma_f \subset M \times M$ of f with the diagonal $\Delta \subset M \times M$, and as we have seen the intersection number $^{\#}(\Gamma_f \cdot \Delta)_{M \times M}$ depends only on the homology classes of Γ_f and Δ in $M \times M$. Nor is the calculation itself difficult; it will come out readily once we have obtained an expression for the cohomology class $\eta_\Delta \in H^n(M \times M)$ of the diagonal $\Delta \subset M \times M$. We do this as follows: First, for each q, let $\{\psi_{\mu,q}\}$ be a collection of closed q-forms on M representing a basis for $H^q_{\text{DR}}(M)$, and let $\{\psi^*_{\mu,n-q}\}$ be $(n-q)$-forms representing the dual basis for $H^{n-q}_{\text{DR}}(M)$, i.e., such that

$$\int_M \psi_{\mu,q} \wedge \psi^*_{\nu,n-q} = \delta_{\mu,\nu}.$$

Let π_1 and π_2 denote the two projection maps $M \times M \to M$. By the Künneth formula, the forms

$$\{\varphi_{\mu,\nu,p,q} = \pi_1^*\psi_{\mu,p} \wedge \pi_2^*\psi^*_{\nu,q}\}_{p+q=k}$$

represent a basis for $H^k_{\text{DR}}(M \times M)$. The dual basis for $H^{2n-k}_{\text{DR}}(M \times M)$ is then represented by

$$\{\varphi^*_{\mu,\nu,n-p,n-q} = (-1)^{q(p+q)}\pi_1^*\psi^*_{\mu,n-p} \wedge \pi_2^*\psi_{\nu,n-q}\}_{p+q=k}$$

since by a direct computation using iteration of the integral

$$\int_{M \times M} \varphi_{\mu,\nu,p,q} \wedge \varphi^*_{\mu',\nu',n-p',n-q'} = \delta_{\mu,\mu'} \cdot \delta_{\nu,\nu'} \cdot \delta_{p,p'} \cdot \delta_{q,q'}.$$

The Poincaré dual η_Δ of the homology class of the diagonal $\Delta \subset M \times M$ is thus represented by the form

$$\varphi_\Delta = \sum_{p,\mu,\nu} c_{p,\mu,\nu}\varphi_{\mu,\nu,p,n-p},$$

where

$$c_{p,\mu,\nu} = \int_\Delta \varphi^*_{\mu,\nu,n-p,p} = (-1)^{n-p}\delta_{\mu,\nu},$$

i.e., η_Δ is represented by

$$\varphi_\Delta = \sum_{p,\mu} (-1)^{n-p} \varphi_{\mu,\mu,p,n-p}.$$

Now let $f: M \to M$ be a C^∞ map. We say that a fixed point $p \in M$ of f is *nondegenerate* if it is isolated and in terms of local coordinates $x_1, \dots, x_n$ on M centered around p, the Jacobian matrix

$$\mathcal{J}_f(p): \; T_p(M) \to T_p(M)$$

satisfies

$$\det(\mathcal{J}_f(p) - I) \neq 0;$$

under these circumstances, we define the *index* $\iota_f(p)$ of f at p to be

$$\iota_f(p) = \operatorname{sgn}\det(\mathcal{J}_f(p) - I).$$

We can give another interpretation of the nondegeneracy condition and the index as follows: let $\Gamma_f = \{(p, f(p))\} \subset M \times M$ be the graph of f. Γ_f is a submanifold of $M \times M$; we give it the orientation induced by the map

$$\tilde{f}: \; p \mapsto (p, f(p)).$$

Let p be a fixed point of f, $x_1, \dots, x_n$ an oriented coordinate system for M centered around p; take as coordinates around $(p,p) \in M \times M$ the functions

$$y_i = \pi_1^* x_i \quad \text{and} \quad z_i = \pi_2^* x_i.$$

An oriented basis for $T_{(p,p)}(\Delta) \subset T_{(p,p)}(M \times M)$ is then given by

$$\Delta_*\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial y_n} + \frac{\partial}{\partial z_n}\right),$$

where Δ is the diagonal map $x \mapsto (x,x)$, and an oriented basis for $T_{(p,p)}(\Gamma_f) \subset T_{(p,p)}(M \times M)$ is given by

$$\tilde{f}_*\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) = \left(\frac{\partial}{\partial y_1} + \sum \frac{\partial f_i}{\partial x_1} \cdot \frac{\partial}{\partial z_i}, \dots, \frac{\partial}{\partial y_n} + \sum \frac{\partial f_i}{\partial x_n} \cdot \frac{\partial}{\partial z_i}\right).$$

The combined collection

$$\left(\Delta_*\left(\frac{\partial}{\partial x_1}\right), \dots, \Delta_*\left(\frac{\partial}{\partial x_n}\right), \tilde{f}_*\left(\frac{\partial}{\partial x_1}\right), \dots, \tilde{f}_*\left(\frac{\partial}{\partial x_n}\right)\right)$$

is consequently obtained from the standard oriented basis $(\partial/\partial y_1, \dots, \partial/\partial y_n, \partial/\partial z_1, \dots, \partial/\partial z_n)$ for $T_{(p,p)}(M \times M)$ by the matrix

$$\begin{pmatrix} I_n & I_n \\ I_n & \mathcal{J}_f(p) \end{pmatrix};$$

we see accordingly that the cycles Γ_f and Δ intersect transversely at (p,p) exactly when

$$\det\begin{pmatrix} I & I \\ I & \mathcal{J}_f(p) \end{pmatrix} = \det(\mathcal{J}_f(p) - I)$$

is nonzero, i.e., when p is a nondegenerate fixed point of f; and in this case the index of f at p is just the intersection number of Δ with Γ_f at p. Thus if f has only nondegenerate fixed points,

$$\sum_{f(p)=p} \iota_f(p) = {}^{\#}(\Delta \cdot \Gamma_f)_{M \times M},$$

and we can evaluate this intersection number by

$$\begin{aligned} {}^{\#}(\Delta \cdot \Gamma_f) &= \int_{\Gamma_f} \varphi_\Delta \\ &= \sum_p (-1)^{n-p} \int_{\Gamma_f} \sum_\mu \pi_1^* \psi_{\mu,p} \wedge \pi_2^* \psi^*_{\mu,n-p}; \end{aligned}$$

since $\tilde{f}^* \pi_2^* = f^*$, this is

$$\begin{aligned} &= \sum_p (-1)^{n-p} \int_M \sum_\mu \psi_{\mu,p} \wedge f^* \psi^*_{\mu,n-p} \\ &= \sum_p (-1)^{n-p} \cdot \operatorname{trace}(f^*|_{H^{n-p}_{\mathrm{DR}}(M)}) \\ &= \sum_p (-1)^p \operatorname{trace}(f^*|_{H^p_{\mathrm{DR}}(M)}). \end{aligned}$$

The number $\sum (-1)^p \operatorname{trace}(f^*|_{H^p_{\mathrm{DR}}(M)})$ is called the *Lefschetz number* of the map f, and is usually denoted $L(f)$; we have proved the

Lefschetz Fixed-Point Formula.

$$\sum_{f(p)=p} \iota_f(p) = L(f).$$

Note that without computing signs the number of fixed points of f must be at least the absolute value of $L(f)$, i.e.,

$${}^{\#}\{p \in M: f(p) = p\} \geqslant |L(f)|,$$

and in particular,

$$L(f) \neq 0 \Rightarrow f \text{ has a fixed point.}$$

As an immediate corollary to the Lefschetz fixed-point formula we will prove the Hopf index theorem. Let M be as above, and let v be a global C^∞ vector field on M. We say a zero p of v is *nondegenerate* if it is isolated and, in terms of local coordinates $x_1,\ldots,x_n$ centered around p,

$$v(x) = \sum a_{ij}x_i\frac{\partial}{\partial x_j} + [2]$$

with $\Delta=(a_{ij})$ nonsingular; in this case we define the *index* $\iota_v(p)$ of v at p to be the sign of the determinant of Δ. Now, integrating the vector field v to time t gives a flow

$$f_t\colon\ M\to M.$$

For t small, the fixed points of f_t will be exactly the zeros of v, and if v is given as above near a zero p, then in terms of the coordinates x,

$$\mathcal{J}_{f_t}(p) = e^{tA} + \text{higher-order terms.}$$

Thus

$$\mathcal{J}_{f_t}(p) - I = t\left(A+\frac{tA^2}{2}+\frac{t^2A^3}{6}+\cdots\right),$$

and for t positive and sufficiently small,

$$\iota_{f_t}(p) = \operatorname{sgn}\det\left(\mathcal{J}_f(p)-I\right) = \operatorname{sgn}\det A = \iota_v(p).$$

Since f_t is homotopic to the identity, f_t^* acts as the identity on the cohomology of M, so that

$$\operatorname{trace} f_t^*|_{H^p_{\mathrm{DR}}(M)} = \dim H^p_{\mathrm{DR}}(M),$$

i.e.,

$$L(f_t) = \chi(M);$$

and we have the

Hopf Index Theorem.

$$\sum_{v(p)=0} \iota_v(p) = \chi(M).$$

The Holomorphic Lefschetz Fixed-Point Formula

Suppose now that M is a compact complex manifold of dimension n and $f\colon M\to M$ a holomorphic map. Then f acts not only on the de Rham cohomology of M but on the Dolbeault cohomology groups as well, and we may hope, by analogy with the Lefschetz fixed-point formula, that the action of f on $H^{*,*}_{\bar\partial}(M)$ will be reflected in the local behavior of f around its fixed points. This is in fact the case, and we will spend the remainder of this section deriving the corresponding formula.

Our starting point, as before, is a computation of the Dolbeault cohomology class of the diagonal $\Delta\subset M\times M$. To this end, for each p and q

let

$$\{\psi_{p,q,\mu}\}$$

be a collection of $\bar{\partial}$-closed (p,q)-forms representing a basis for $H_{\bar{\partial}}^{p,q}(M)$, and let

$$\{\psi_{n-p,n-q,\mu}^*\}$$

be $\bar{\partial}$-closed forms representing the dual basis for $H_{\bar{\partial}}^{n-p,n-q}(M)$ under the pairing

$$H_{\bar{\partial}}^{p,q}(M)\otimes H_{\bar{\partial}}^{n-p,n-q}(M)\to\mathbb{C}$$

given by

$$\psi\otimes\varphi\mapsto\int_M\psi\wedge\varphi.$$

By the Künneth formula from Section 6 of Chapter 0 a basis for $H_{\bar{\partial}}^{n,n}(M\times M)$ is represented by the forms

$$\{\varphi_{p,q,\mu,\nu}=\pi_1^*\psi_{p,q,\mu}\wedge\pi_2^*\psi_{n-p,n-q,\nu}^*\},$$

and the dual basis for $H_{\bar{\partial}}^{n,n}(M\times M)$ is represented, as in the real case above, by

$$\{\varphi_{n-p,n-q,\mu,\nu}^*=\pi_1^*\psi_{n-p,n-q,\mu}^*\wedge\pi_2^*\psi_{p,q,\nu}\}.$$

The Dolbeault class η_Δ of the diagonal is

$$\varphi_\Delta=\sum_{p,q,\mu}(-1)^{p+q}\varphi_{p,q,\mu,\mu}.$$

Now let $f:M\to M$ be a holomorphic map with isolated nondegenerate zeros; let $\Gamma_f=\{(p,f(p))\}\subset M\times M$ be its graph. If we compute the intersection number of Δ and Γ_f in $M\times M$, we find only that

$$\begin{aligned}L(f)&=\int_{\Gamma_f}\varphi_\Delta\\&=\sum_{p,q}(-1)^{p+q}\int_{\Gamma_f}\sum_{\mu}\pi_1^*\psi_{p,q,\mu}\wedge\pi_2^*\psi_{n-p,n-q,\mu}^*\\&=\sum(-1)^{p+q}\operatorname{trace}f^*|_{H_{\bar{\partial}}^{p,q}(M)}.\end{aligned}$$

This tells us nothing essentially new: in case M is Kähler, this follows from the ordinary Lefschetz fixed-point formula and the Hodge decomposition; in general, it follows from the Lefschetz formula and the Fröhlicher spectral sequence relating Dolbeault and de Rham cohomology given in Section 5 of this chapter. To obtain finer information about the action of f on the Dolbeault groups of M, let $\eta_\Delta^{p,q}$ be the (p,q)th component of the class η_Δ under the decomposition into bitype

$$H_{\bar{\partial}}^{n,n}(M\times M)=\bigoplus_{p,q}(\pi_1^*H_{\bar{\partial}}^{p,q}(M)\otimes\pi_2^*H_{\bar{\partial}}^{n-p,n-q}(M)),$$

and set

$$\eta_\Delta^0 = \sum_q \eta_\Delta^{0,q}.$$

η_Δ^0 is then represented by the form

$$\varphi_\Delta^0 = \sum_{q,\mu} (-1)^q \varphi_{0,q,\mu,\mu},$$

and so the value of η_Δ^0 on the cycle Γ_f is given by

$$\begin{aligned}\eta_\Delta^0(\Gamma_f) &= \int_{\Gamma_f} \varphi_\Delta^0 \\ &= \sum_q (-1)^q \int_M \sum_\mu \psi_{0,q,\mu} \wedge f^* \psi_{n,n-q,\mu}^* \\ &= \sum_q (-1)^q \operatorname{trace} f^*|_{H_{\bar\partial}^{n,n-q}(M)} \\ &= \sum_q (-1)^q \operatorname{trace} f^*|_{H_{\bar\partial}^{0,q}(M)}\end{aligned}$$

by Kodaira-Serre duality. The number $\sum_q (-1)^q \operatorname{trace} f^*|_{H_{\bar\partial}^{0,q}(M)}$ is called the *holomorphic Lefschetz number* of the map f, and is denoted $L(f, \mathcal{O})$.

We ask accordingly whether we can evaluate the number $\eta_\Delta^0(\Gamma_f)$ in terms of the local behavior of f around its fixed points. What makes this possible is the fact that while the full decomposition of forms on $M \times M$ into bitype (cf. Section 2 in this chapter)

$$A^{p,q}(M \times M) = \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} A^{(p_1,q_1),(p_2,q_2)}(M \times M),$$

does *not* commute with the $\bar\partial$-operator, the coarser direct-sum decomposition

$$A^{p,q}(M \times M) = \bigoplus_{p_1} A^{(p_1,*),(p-p_1,q-*)}(M \times M)$$

does. Here $*$ represents an index running from zero to q. It follows that if T_Δ^0 is the component of the current T_Δ of bitype $(0,*),(n,n-*)$—i.e., the current defined by the linear function

$$T_\Delta^0(\varphi) = \int_\Delta \sum_q \varphi^{(n,n-q),(0,q)}$$

on test forms φ, then *T_Δ^0 is $\bar\partial$-closed and represents the Dolbeault cohomology class η_Δ^0*. To compute $\eta_\Delta^0(\Gamma_f)$, then, we need only smooth the current T_Δ^0—that is, solve the equation of currents

$$(*) \qquad T_\Delta^0 = \varphi + \bar\partial k$$

with k any $(n,n-1)$-current on $M \times M$ and φ a smooth form; we will then have

$$\eta_\Delta^0(\Gamma_f) = \int_{\Gamma_f} \varphi.$$

In fact, as we will see, it will suffice just to solve the equation (*) locally around the fixed points. We proceed as follows.

Recall from the subsection "Definitions; Residue Formulas" in Section 1 of this chapter the Bochner-Martinelli kernel on $\mathbb{C}^n \times \mathbb{C}^n$ is given by

$$k(z,\zeta) = C_n \frac{\sum \overline{\Phi_i(z-\zeta)} \wedge \Phi(\zeta)}{\|z-\zeta\|^{2n}},$$

where

$$\begin{cases} \Phi_i(x) = (-1)^{i-1} x_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \\ \Phi(x) = dx_1 \wedge \cdots \wedge dx_n. \end{cases}$$

This form has bitype $(0, *-1), (n, n-*)$ in the variables $(dz, d\bar{z})(d\zeta, d\bar{\zeta})$. Also, from $\bar{\partial}\Phi_i(z-\zeta) = \overline{\Phi_i(z-\zeta)}$ it follows that

$$\bar{\partial} k(z,\zeta) = 0 \qquad \text{on } \mathbb{C}^n \times \mathbb{C}^n - \Delta,$$

so that the current defined by $k(z,\zeta)$ has distributional derivative $\bar{\partial} k$ supported on the diagonal. In fact, the homotopy formula proved in the subsection "Cohomology of Currents" is equivalent to the distributional equation

$$\bar{\partial} k = T_\Delta^0,$$

giving the desired "smoothing" of T_Δ^0 in $\mathbb{C}^n \times \mathbb{C}^n$.

Now we return to our complex manifold M and map $f: M \to M$. Assume that f has isolated, nondegenerate fixed points $\{p_\alpha\}$ and, in terms of local coordinates $z_{\alpha i}$ around p_α, write

$$f(z_\alpha)_i = \sum b_{ij} z_{\alpha j} + [2],$$

i.e.,

$$f(z_\alpha) = B_\alpha z_\alpha + [2],$$

where $B_\alpha = (b_{ij})$; by nondegeneracy, $(I - B_\alpha)$ is nonsingular. Let $B_\varepsilon(p_\alpha; p_\alpha)$ be a ball of radius ε around (p_α, p_α) in $M \times M$, and let ρ_α be a bump function with

$$\begin{aligned} \rho_\alpha &\equiv 1 \qquad \text{in } B_\varepsilon(p_\alpha, p_\alpha) \\ \rho_\alpha &\equiv 0 \qquad \text{in } M \times M - B_{2\varepsilon}(p_\alpha, p_\alpha); \end{aligned}$$

let k be the current on $M \times M$ given by

$$k = \sum_\alpha \rho_\alpha \cdot k(z_\alpha, \xi_\alpha),$$

where $k(z_\alpha, \xi_\alpha)$ is the Bochner-Martinelli kernel. Then in $B_\varepsilon(p_\alpha, p_\alpha)$ we have

$$\bar{\partial} k = \bar{\partial} k(z_\alpha, \xi_\alpha) = T^0_\Delta.$$

Moreover, k is smooth on $M \times M - \Delta$, so that if we set

$$\varphi = T^0_\Delta - \bar{\partial} k,$$

φ will be a $\bar{\partial}$-closed current representing η^0_Δ, *smooth in an open set containing* Γ_f, and equal to $-\bar{\partial} k$ away from Δ. Then

$$\begin{aligned}\eta^0_\Delta(\Gamma_f) &= \int_{\Gamma_f} \varphi \\ &= -\int_{\Gamma_f - \cup B_\varepsilon(p_\alpha, p_\alpha)} \bar{\partial} k \\ &= \sum_\alpha \int_{\partial(\Gamma_f \cap B_\varepsilon(p_\alpha, p_\alpha))} k \\ &= \lim_{\varepsilon \to 0} \sum_\alpha \int_{\|z_\alpha\| = \varepsilon} k(z_\alpha, f(z_\alpha)).\end{aligned}$$

Now if we set $w_\alpha = z_\alpha - f(z_\alpha)$, then

$$dw_{\alpha_1} \wedge \cdots \wedge dw_{\alpha_n} = \det(I - \mathcal{J}(f)) \cdot dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_n},$$

and we have

$$\begin{aligned}&\int_{\|z\| = \varepsilon} k(z_\alpha, f(z_\alpha)) \\ &= C_n \int_{\|z\| = \varepsilon} \frac{\sum (-1)^{i-1} \bar{w}_{\alpha_i} d\bar{w}_{\alpha_1} \wedge \cdots \wedge \widehat{d\bar{w}_{\alpha_i}} \wedge \cdots \wedge d\bar{w}_{\alpha_n} \wedge dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_n}}{\|w_\alpha\|^{2n}} \\ &= C_n \int_{\|z\| = \varepsilon} \frac{\sum (-1)^{i-1} \bar{w}_{\alpha_i} d\bar{w}_{\alpha_1} \wedge \cdots \wedge \widehat{d\bar{w}_{\alpha_i}} \wedge \cdots \wedge d\bar{w}_{\alpha_n} \wedge dw_{\alpha_i} \wedge \cdots \wedge dw_{\alpha_n}}{\|w_\alpha\|^{2n} \det(I - \mathcal{J}(f))} \\ &= \frac{1}{\det(I - \mathcal{J}(f)(0))} = \frac{1}{\det(I - B_\alpha)}\end{aligned}$$

by the Bochner-Martinelli formula proved in Section 1 of this chapter. Putting this all together, we have the *holomorphic Lefschetz fixed-point formula*:

$$L(f, \mathcal{O}) = \sum_{f(p_\alpha) = p_\alpha} \frac{1}{\det(I - B_\alpha)}.$$

The Bott Residue Formula

We ask now whether there exist refinements of the Gauss-Bonnet formula for holomorphic vector bundles on complex manifolds. The answer, in general, is no, for the reason that a zero of a section σ of a holomorphic vector bundle E on a complex manifold carries no nonobvious local structure: since we can choose a frame $e=(e_1,\ldots,e_k)$ for E and a local holomorphic coordinate system $z=(z_1,\ldots,z_n)$ for M independently, the local expansion

$$\sigma(z) = \sum b_{ij}z_i \cdot e_j + \sum b_{ijl}z_iz_j \cdot e_l + \ldots$$

for σ can be given virtually arbitrary form. The exception to this occurs when E is a holomorphic tensor bundle, e.g., when $E=T'(M)$ is the holomorphic tangent bundle of M: in this case a local coordinate system (z_i) determines naturally a frame $\{\partial/\partial z_i\}$ for $T'(M)$. Thus, in the neighborhood of a zero of the holomorphic vector field v, we set $A_p=(a_{ij})$, where

$$v(z) = \sum a_{ij}z_i\frac{\partial}{\partial z_j} + [2].$$

If $w=f(z)$ is any other coordinate system around $z=0$ and we let $A'_p=(a'_{ij})$ be given by

$$v(z) = \sum a'_{ij}\cdot w_i \cdot \frac{\partial}{\partial w_j} + [2],$$

then for $g=(g_{ij})=\mathcal{J}(f)$ the Jacobian of the change of coordinates,

$$\frac{\partial}{\partial z_i} = \sum g_{ji}\frac{\partial}{\partial w_j},$$

and hence

$$v(z) = \sum_{i,j,k,l} a_{ij}\cdot g_{ik}^{-1}\cdot w_k \cdot g_{lj}\cdot\frac{\partial}{\partial w_l} + [2].$$

Thus

$$A'_p = {}^t g^{-1}\cdot A_p \cdot {}^t g,$$

i.e., *A_p is determined up to conjugation*. The value $P(A_p)$ of any invariant polynomial P on A is therefore an invariant of v and p, and we may hope that the numbers $P(A_p)$ carry some global information. This is in fact the case: if Θ is any curvature matrix in the holomorphic tangent bundle $T'(M)$ of the compact complex manifold M, P any invariant polynomial of degree $n=\dim M$, v a global holomorphic vector field and A_p as above, we have the

Bott Residue Formula

$$\sum_{v(p)=0}\frac{P(A_p)}{\det(A_p)} = \int_M P\left(\frac{\sqrt{-1}}{2\pi}\Theta\right),$$

i.e., if we write P *as a polynomial*

$$P = Q(P^1, \ldots, P^n)$$

in the elementary invariant polynomials P^i,

$$\sum_{v(p)=0} \frac{P(A_p)}{\det(A_p)} = Q(c_1(M), \ldots, c_n(M)).$$

Proof.* The outline of the proof is this: we choose a metric in $T'(M)$ that is Euclidean in a neighborhood of the zeros $\{p_\nu\}$ of v, and let Θ be the curvature matrix of the metric connection on $T'(M)$. Then

$$\Theta \equiv 0$$

in a ball $B_\epsilon(p_\nu)$ around each p_ν. We will construct a $C^\infty(n, n-1)$ form Λ on $M^* = M - \{p_\nu\}$ such that

$$d\Lambda = \bar{\partial}\Lambda = P(\Theta);$$

we will then have

$$\int_M P\left(\frac{\sqrt{-1}}{2\pi}\Theta\right) = \int_{M - \cup B_\epsilon(p_\nu)} P\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)$$
$$= -\left(\frac{\sqrt{-1}}{2\pi}\right)^n \sum_i \int_{\partial B_\epsilon(p_\nu)} \Lambda,$$

and since our construction of Λ is essentially a local process, we will be able to evaluate the last integrals in terms of the local behavior of v at p_ν.

So: let $\{p_\nu\}$ denote the zeros of v, and $z_1, \ldots, z_n$ local holomorphic coordinates in $B_{2\epsilon}(p_\nu)$, and let h_ν be the Euclidean metric in $B_{2\epsilon}(p_\nu)$ given by

$$\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \delta_{jk}.$$

Let h_0 be any metric on $M^* = M - \{p_\nu\}$, and $\{\rho_0, \rho_\nu\}$ a partition of unity for the covering of M by $U_0 = M - \cup B_\epsilon(p_\nu)$ and $U_\nu = B_{2\epsilon}(p_\nu)$; we take as our metric on M

$$h = \rho_0 \cdot h_0 + \sum \rho_\nu h_\nu.$$

Let Θ hereafter be the curvature matrix of the associated metric connection D; clearly $\Theta \equiv 0$ in $B_\epsilon(p_\nu)$ for each ν.

*This proof is due to S. S. Chern, "*Meromorphic vector fields and characteristic numbers, Scripta Mathematica*, Vol. XXIX, pp. 243–251.

Now consider the bundle map

$$\wedge^{p-1}T'\otimes\wedge^{q}T'' \xrightarrow{\wedge v} \wedge^{p}T'\otimes\wedge^{q}T''$$

given by wedge product with $v(z)\in T'_z$. We define the *contraction operator*

$$\iota(v):\ A^{p,q}(M)\to A^{p-1,q}(M)$$

to be the adjoint of $\wedge v$; i.e., such that for any $\varphi\in A^{p,q}(M)$ and $\eta\in C^{\infty}(\wedge^{p-1}T'\otimes\wedge^{q}(T''))$,

$$\langle\iota(v)\varphi,\eta\rangle = \langle\varphi, v\wedge\eta\rangle.$$

Thus, if in terms of local coordinates z_i

$$v(z) = \sum_j v^j(z)\cdot\frac{\partial}{\partial z_j},$$

we have

$$\iota(v)(dz_i) = v^i$$

and in general

$$\iota(v)(f(z)\cdot dz_I) = \sum_\alpha(-1)^{\alpha-1}v^{i_\alpha}(z)f(z)dz_{I-\{i_\alpha\}}.$$

In particular, since the coefficient functions v^i are holomorphic it follows from sign considerations that

$$\bar\partial\cdot\iota(v)+\iota(v)\cdot\bar\partial = 0.$$

The essential step in our construction of Λ is to express the tensor

$$\iota(v)(\Theta)\in A^{0,1}(T'\otimes T'^*)$$

as $\bar\partial$ of a global section of $T'\otimes T'^*$. To do this, we recall from Section 5 of Chapter 0 the definition of the *torsion* associated to a metric on $T'(M)$. As we saw then, if $v_1,\ldots,v_n$ is a unitary frame for $T'(M)$, $\varphi_1,\ldots,\varphi_n$ the dual coframe for T'^*, θ the connection matrix of D in terms of $\{v_i\}$ and $\theta^* = -{}^t\theta$ the connection matrix of the dual connection D^* on T'^* in terms of $\{\varphi_i\}$, then

$$d\varphi_i = \sum\theta^*{}_{ij}\wedge\varphi_j+\tau_i$$

with τ_i of type (2,0); the vector $\tau=(\tau_1,\ldots,\tau_n)$ of 2-forms is called the *torsion* of the metric in terms of $\{v_i\}$. Now if

$$\left\{v_i'=\sum g_{ij}\cdot v_j\right\}$$

is another frame, $\{\varphi_i\}$ the dual coframe, and θ' and θ'^* the connection and curvature matrices of D and D^* in terms of $\{v_i'\}$ and $\{\varphi_i'\}$, then in matrix notation and setting $g^*={}^tg^{-1}$

$$\varphi' = g^*\cdot\varphi,$$
$$\theta'^* = g^*\cdot\theta^*\cdot{}^tg + d(g^*)\cdot{}^tg$$

and

$$\begin{aligned}\tau' &= d\varphi' - \theta'^* \wedge \varphi' \\ &= d(g^*\varphi) - \theta'^* \wedge g^*\varphi \\ &= d(g^*)\cdot\varphi + g^*\varphi - g^*\cdot\theta^*\cdot {}^t g\cdot g^*\cdot\varphi - d(g^*)\cdot {}^t g\cdot g^*\cdot\varphi \\ &= g^*\varphi - g^*\cdot\theta^*\wedge\varphi \\ &= g^*\cdot\tau\end{aligned}$$

i.e., *the quantity*

$$\tilde{\tau} = \sum \tau_i \otimes v_i \in A^{2,0}(T')$$

is a tensor invariant of the metric, called the *torsion tensor*. Note that by our calculation, if $v_1',\ldots,v_n'$ is any other frame for T', not necessarily unitary but with φ, θ, and θ^* as before, we still have

$$\tilde{\tau} = \sum_i \left(d\varphi_i' - \sum \theta'^*_{ij} \wedge \varphi_j'\right) \otimes v_i'.$$

Now let $(z_1,\ldots,z_n)$ be local coordinates on M and $\theta = (\theta_{ij})$ the connection matrix of D in terms of the frame $\{\partial/\partial z_i\}$ for $T'(M)$. Write

$$\theta_{ij} = \sum_k \Gamma^j_{ik} dz_k,$$

so that

$$Dv = D\left(\sum_i v^i \frac{\partial}{\partial z_i}\right) = \sum_{j,k}\left(\frac{\partial v^j}{\partial z_k} + \sum_i \Gamma^j_{ik} v^i\right)\frac{\partial}{\partial z_j} \otimes dz_k.$$

The torsion tensor $\tilde{\tau}$ is given by

$$\begin{aligned}\tilde{\tau} &= -\sum_{i,j} \theta^*_{ji} \wedge dz_i \otimes \frac{\partial}{\partial z_j} \\ &= \sum_{i,j} \theta_{ij} \wedge dz_i \otimes \frac{\partial}{\partial z_j} \\ &= \frac{1}{2}\sum_{i,j,k} \left(\Gamma^j_{ik} - \Gamma^j_{ki}\right)\frac{\partial}{\partial z_j} \otimes (dz_i \wedge dz_k),\end{aligned}$$

and so the contraction $\iota(v)\cdot\tau \in C^\infty(T' \otimes T'^*)$ of τ by v is given by

$$\iota(v)\cdot\tau = \sum_{i,j,k} \left(\Gamma^j_{ik} - \Gamma^j_{ki}\right) v^i \cdot \frac{\partial}{\partial z_j} \otimes dz_k.$$

Thus the tensor

$$\begin{aligned}E &= -Dv + \iota(v)\cdot\tau \\ &= -\sum_{j,k}\left\{\frac{\partial v^j}{\partial z_k} + \sum_i \Gamma^j_{ki} v^i\right\}\frac{\partial}{\partial z_j} \otimes dz_k\end{aligned}$$

is a well-defined global section of the holomorphic vector bundle $T'\otimes T'^*$, and

$$\bar{\partial}E = -\sum\left(\frac{\partial}{\partial\bar{z}_l}\left\{\frac{\partial v^j}{\partial z_k}+\sum\Gamma^j_{ki}v^i\right\}\cdot\frac{\partial}{\partial z_j}\otimes dz_k\right)d\bar{z}_l$$
$$= -\sum\left(\frac{\partial\Gamma^j_{ki}}{\partial\bar{z}_l}\cdot v^i\cdot\frac{\partial}{\partial z_j}\otimes dz_k\right)d\bar{z}_l.$$

On the other hand, the curvature tensor $\Theta\in A^2(T\otimes T^*)$ is given by

$$\Theta = \sum\Theta_{ij}\frac{\partial}{\partial z_j}\otimes dz_i,$$

where

$$\Theta_{ij} = \bar{\partial}\theta_{ij} = -\sum\frac{\partial\Gamma^j_{ik}}{\partial\bar{z}_l}\cdot dz_k\wedge d\bar{z}_l,$$

so that from the formula

$$\iota(v)\cdot\Theta = -\sum_{i,j,k}\left(\frac{\partial\Gamma^j_{ik}}{\partial\bar{z}_l}\cdot v^k\cdot\frac{\partial}{\partial z_j}\otimes dz_i\right)d\bar{z}_l$$

we deduce the desired relation

$$\iota(v)\Theta = \bar{\partial}E.$$

Now, consider Θ, E, and $\bar{\partial}E=\iota(v)\Theta$ again as matrix-valued 2-, 0-, and 1-forms, respectively; if P is any invariant polynomial of degree n on $GL(n)$ and $\tilde{P}$ its polarization, set

$$P_r(E,\Theta) = \binom{n}{r}\tilde{P}(\underbrace{E,\ldots,E}_{n-r},\underbrace{\Theta,\ldots,\Theta}_{r})\in A^{r,r}(M).$$

Since $\bar{\partial}\Theta=0$ and $\bar{\partial}E=\iota(v)\Theta$,

$$\bar{\partial}P_r(E,\Theta) = \binom{n}{r}\sum_{i=1}^{n-r}\tilde{P}(E,\ldots,\iota(v)\cdot\Theta,\ldots,E,\Theta,\ldots,\Theta)$$
$$= \iota(v)\cdot P_{r+1}(E,\Theta).$$

Let $\omega\in A^{1,0}(M^*)$ be the form dual to v under the metric on M; set

$$\Phi_r = \omega\wedge(\bar{\partial}\omega)^{n-r-1}\wedge P_r(E,\Theta)\in A^{n,n-1}(M^*).$$

We have, for $0\leqslant r\leqslant n-1$,

$$\bar{\partial}\Phi_r = (\bar{\partial}\omega)^{n-r}\wedge P_r(E,\Theta)-\omega\wedge(\bar{\partial}\omega)^{n-r-1}\wedge\iota(v)P_{r+1}(E,\Theta);$$

since $\iota(v)\omega=1$,

$$0 = \iota(\bar{\partial}v)\omega+\iota(v)\bar{\partial}\omega\Rightarrow\iota(v)\bar{\partial}\omega = 0,$$

and so

$$\iota(v)\bar{\partial}\Phi_r = (\bar{\partial}\omega)^{n-r}\wedge\iota(v)P_r(E,\Theta) - (\bar{\partial}\omega)^{n-r-1}\wedge\iota(v)P_{r+1}(E,\Theta).$$

$\iota(v)P_0(E,\Theta)$ is trivially zero, and so if we set

$$\Phi = \sum_{i=0}^{n-1}\Phi_i,$$

we see that

$$\begin{aligned}\iota(v)\bar{\partial}\Phi &= \sum_{i=1}^{n-1}(\bar{\partial}\omega)^{n-r}\wedge\iota(v)P_i(E,\Theta) - \sum_{i=1}^{n}(\bar{\partial}\omega)^{n-i}\wedge\iota(v)\cdot P_i(E,\Theta)\\ &= -\iota(v)P_n(\Theta)\\ &= -\iota(v)\cdot P(\Theta).\end{aligned}$$

Since $\bar{\partial}\Phi$ and $P(\Theta)$ are both forms of top degree,

$$\iota(v)\big(\bar{\partial}\Phi + P(\Theta)\big) = 0$$

implies that

$$\bar{\partial}\Phi + P(\Theta) = 0$$

and we have constructed our explicit solution to $\bar{\partial}\Lambda = P(\Theta)$.

It remains now to evaluate the integral of Φ over the boundary of $B_\varepsilon(p_\nu)$. First of all, recall that by our choice of metric, Θ—and hence $P_r(E,\Theta)$ for $r>0$—vanishes identically in $B_\varepsilon(p_\nu)$; thus

$$\int_{\partial B_\varepsilon(p_\nu)}\Phi = \int_{\partial B_\varepsilon(p_\nu)}P_0(E,\Theta) = \int_{\partial B_\varepsilon(p_\nu)}\omega\wedge(\bar{\partial}\omega)^{n-1}P(E).$$

Let $z=(z_1,\dots,z_n)$ be as before local coordinates around p_i such that

$$\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right) = \delta_{ij}$$

in $B_\varepsilon(p_\nu)$; write

$$v(z) = \sum v^j(z)\cdot\frac{\partial}{\partial z_j}.$$

Then since our metric is Euclidean in $B_\varepsilon(p_\nu)$, the connection D is zero and

$$E_{jk} = -\left(\frac{\partial v^j}{\partial z_k} + \sum_i \Gamma^j_{ki}v^i\right) = -\frac{\partial v^j}{\partial z_k},$$

i.e.,

$$P(E)(p_\nu) = -P(A_{p\nu}).$$

Moreover,

$$\omega = \frac{\sum \bar{v}^i dz_i}{\sum v^i \bar{v}^i} = \frac{(dz,v)}{(v,v)},$$

so

$$\bar{\partial}\omega = -\frac{(dz,dv)}{(v,v)} + \frac{(dz,v)\wedge(v,dv)}{(v,v)^2}$$

Thus, since $(dz,v)\wedge(dz,v)=0$,

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^n \omega\wedge(\bar{\partial}\omega)^{n-1} = (-1)^{n-1}\left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{(dz,v)\wedge(dz,dv)^{n-1}}{(v,v)^n}$$

$$= -C_n \frac{\sum_i (-1)^{i-1}\bar{v}^i d\bar{v}^1\wedge\cdots\wedge\widehat{d\bar{v}^i}\wedge\cdots\wedge d\bar{v}^n\wedge dz_1\wedge\cdots\wedge dz_n}{(v,v)^n},$$

where C_n is the constant appearing in the Bochner-Martinelli formula from Section 1 of this chapter,

$$= -\frac{1}{\det A_p}\beta(v,\bar{v}),$$

where β is the form appearing in that formula. Putting everything together,

$$\int_M P\left(\frac{\sqrt{-1}}{2\pi}\Theta\right) = \int_{M-\cup B_\epsilon(p_\nu)} P\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)$$

$$= -\sum_\nu \int_{\partial B_\epsilon(p_\nu)} \Phi$$

$$= \sum_\nu \int_{\partial B_\epsilon(p_\nu)} \frac{P(A_{p_\nu})}{\det A_{p_\nu}}\beta(v,v) \qquad \text{where } A = (\partial v_i/\partial z_j),$$

$$= \sum_\nu \frac{P(A_{p_\nu})}{\det A_{p_\nu}}$$

by the Bochner-Martinelli formula. Q.E.D.

As an example of a computation involving the Bott residue theorem, we calculate for the third (and last) time the Chern classes of projective space. Let $X=(X_0,\ldots,X_n)$ be linear coordinates on $\mathbb{C}^{n+1}$, π_* and $\mathfrak{S}$ as in Section 3 of this chapter, and $(\alpha_0,\ldots,\alpha_n)$ any $(n+1)$-vector of distinct nonzero complex numbers. Consider the vector field on $\mathbb{P}^n$,

$$v(X) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i}$$

(since $\pi_* \sum X_i(\partial/\partial X_i) \equiv 0$, we may as well take $\sum \alpha_i = 0$). As we have seen, v vanishes exactly at the points $p_i = [0,\dots,1_i,\dots,0]$; in terms of Euclidean coordinates

$$x_j = \frac{X_j}{X_i}, \qquad j \neq i,$$

on $\mathbb{P}^n$ around p_i, we have

$$\pi_*\left(X_j \frac{\partial}{\partial X_j}\right) = x_j \frac{\partial}{\partial x_j}, \qquad j \neq i,$$

and

$$\pi_*\left(X_i \frac{\partial}{\partial X_i}\right) = -\sum_{j \neq i} x_j \frac{\partial}{\partial x_j}.$$

Thus

$$v(x) = \sum_{j \neq i} (\alpha_j - \alpha_i) x_j \frac{\partial}{\partial x_j},$$

i.e., the matrix A_{p_i} for v near p_i is just the diagonal matrix with entries $(\alpha_j - \alpha_i), j \neq i$. According to the Bott residue formula, then,

$$\begin{aligned} c_1(\mathbb{P}^n)^n &= \sum_{i=0}^{n} \frac{(\operatorname{trace}(A_{p_i}))^n}{\det(A_{p_i})} \\ &= \sum_i \frac{\left(\sum_{j \neq i}(\alpha_j - \alpha_i)\right)^n}{\prod_{j \neq i}(\alpha_j - \alpha_i)} \\ &= \sum_i \frac{(-(n+1)\alpha_i)^n}{\prod_{j \neq i}(\alpha_j - \alpha_i)}, \end{aligned}$$

since $\sum \alpha_k = 0$. To evaluate this expression, consider the meromorphic functions f, g on the Riemann sphere given in terms of a Euclidean coordinate by

$$f(z) = \prod_{k=0}^{n} (\alpha_k - z), \qquad g(z) = z^n;$$

then $(g(z)/f(z))dz = \varphi$ is a meromorphic differential with simple poles at

$z=\alpha_k$ and $z=\infty$, and

$$\operatorname{Res}_{\alpha_i}(\varphi) = \frac{-\alpha_i^n}{\Pi_{j\neq i}(\alpha_j-\alpha_i)},$$
$$\operatorname{Res}_{\infty}(\varphi) = (-1)^n.$$

By the residue theorem

$$\sum_i \left(\frac{\alpha_i^n}{\Pi_{j\neq i}(\alpha_j-\alpha_i)}\right) = (-1)^n$$

and consequently

$$c_1(\mathbb{P}^n)^n = \sum_i \frac{(-1)^n(n+1)^n\alpha_i^n}{\Pi_{j\neq i}(\alpha_j-\alpha_i)} = (n+1)^n;$$

since the nth power of the hyperplane class ω in $\mathbb{P}^n$ is 1, this implies that

$$c_1(\mathbb{P}^n) = (n+1)\omega.$$

Now to compute the rest of the Chern classes of $\mathbb{P}^n$ we need only evaluate the Chern numbers $c_1(\mathbb{P}^n)^{n-r}\cdot c_r(\mathbb{P}^n)$. By Bott residue applied to v,

$$\begin{aligned} c_1(\mathbb{P}^n)^{n-r}\cdot c_r(\mathbb{P}^n) &= \sum_{i=0}^{n} \frac{\left(\sum_{j\neq i}(\alpha_j-\alpha_i)\right)^{n-r}\cdot\left(\sum_{i\notin I}^{\#I=r}\Pi_{j\in I}(\alpha_j-\alpha_i)\right)}{\Pi_{j\neq i}(\alpha_j-\alpha_i)} \\ &= \sum_{i=0}^{n} \frac{(-1)^{n-r}(n+1)^{n-r}\alpha_i^{n-r}\cdot\sum_{i\notin I}^{\#I=r}\Pi_{i\in I}(\alpha_j-\alpha_i)}{\Pi_{j\neq i}(\alpha_j-\alpha_i)}. \end{aligned}$$

Again, for $f(z)$ as above, $g(z)=z^{n-r}\sum_{\#I=r}\Pi_{k\in I}(\alpha_k-z)$, and $\varphi=(g/f)dz$

$$\operatorname{Res}_{\alpha_i}(\varphi) = \frac{-\alpha_i^{n-r}\sum_{\#I=r}\Pi_{j\in I}(\alpha_j-\alpha_i)}{\Pi_{j\neq i}(\alpha_j-\alpha_i)}$$
$$\operatorname{Res}_{\infty}(\varphi) = (-1)^{n-r}\binom{n+1}{r}$$

and the residue theorem together with $c_1(\mathbb{P}^n)=(n+1)\omega$ imply

$$c_r(\mathbb{P}^n) = \binom{n+1}{r}\omega^r.$$

The General Hirzebruch-Riemann-Roch Formula

Consider now how we arrived at the identity

$$c_n(M) = \chi(M)$$

for a compact complex manifold M of dimension n: On the one hand, the general Gauss-Bonnet formula tells us that we can realize $c_n(M)$ as the number of zeros, properly counted, of a generic C^∞ vector field on M; on the other hand, the Lefschetz fixed-point formula tells us that the Euler characteristic of M is equal to the number of fixed points, properly counted, of the map $\varphi_v: M \to M$ obtained by integrating v—that is, again the number of zeros of v. Now we have obtained refinements of both the Gauss-Bonnet and the Lefschetz fixed-point formulas in the holomorphic case, and we may try to apply them in the same way to arrive at a formula for the holomorphic Euler characteristic of a complex manifold.

So, suppose again that M is a compact Kähler manifold of dimension n and let v be a holomorphic vector field on M having isolated nondegenerate zeros. Let

$$f_t = \exp(tv): \ M \to M$$

be the map obtained by integrating the corresponding real vector field to time t; f_t is readily seen to be holomorphic. Moreover, if $z_1, \ldots, z_n$ are local coordinates around a zero p of v and

$$v(z) = \sum a_{ij} z_i \frac{\partial}{\partial z_j} + [2],$$

then the Jacobian of $f_t(z)$ at p is given by

$$B_p = e^{t \cdot A_p},$$

where $A_p = (a_{ij})$. Now for t small, f_t will have a fixed point exactly where v has a zero, and by the holomorphic Lefschetz fixed-point formula,

$$L(f_t, \mathcal{O}) = \sum_{v(p)=0} \frac{1}{\det(I - B_p)}.$$

Since f_t is homotopic to the identity, f_t^* is the identity on $H_{\bar{\partial}}^{p,q}(M)$, and so this formula reads

$$\chi(\mathcal{O}_M) = \sum_{v(p)=0} \frac{1}{\det(I - e^{tA_p})}$$

$$= \sum_{v(p)=0} \frac{1}{\det A_p} \cdot \left(\frac{\det A_p}{\det(I - e^{tA_p})} \right).$$

Now, for each t the holomorphic function

$$F_t(A) = \det(A) \cdot (\det(I - e^{tA}))^{-1}$$

on GL_n is invariant under conjugation, and hence uniquely expressible as a power series in the elementary invariant polynomials P^i on GL_n. Explicitly, for $A \in \mathrm{GL}_n$ semisimple with eigenvalues $\lambda_1,\ldots,\lambda_n$,

$$\begin{aligned}
F_t(A) &= \frac{\det A}{\det(I-e^{tA})} \\
&= \prod_{i=1}^{n}\left(\frac{\lambda_i}{1-e^{t\lambda_i}}\right) \\
&= (-1)^n t^{-n}\left\{1-\left(\frac{\sum\lambda_i}{2}\right)t+\left(\frac{\sum\lambda_i^2}{12}+\frac{\sum\lambda_i\lambda_j}{4}\right)t^2\right. \\
&\quad \left.-\left(\frac{\sum\lambda_i\lambda_j\lambda_k}{8}+\frac{\sum\lambda_i^2\lambda_j+\sum\lambda_i\lambda_j^2}{24}\right)t^3+\cdots\right\} \\
&= (-1)^n t^{-n}\left\{1-P^1(A)t+\left(\frac{P^1(A)^2+P^2(A)}{12}\right)t^2\right. \\
&\quad \left.-\frac{P^1(A)P^2(A)}{24}t^3+\cdots\right\},
\end{aligned}$$

where the summations occur over increasing indices. In general the coefficient of t^i in the bracketed power series may be expressed as a polynomial in the elementary invariant polynomials. The *Todd polynomials* Td_i are then defined by

$$\frac{\det A}{\det(I-e^{-tA})} = (-1)^n t^{-n}\left\{\sum_i Td_i(P^1(A),\ldots,P^i(A))t^i\right\}$$

Now we may express the Lefschetz fixed-point formula as applied to v and f_t by

$$\begin{aligned}
\chi(\mathcal{O}_M) &= \sum_{v(p)=0}\frac{1}{\det A_p}\cdot\frac{\det A_p}{\det(I-e^{tA_p})} \\
&= (-1)^n t^{-n}\sum_i\left[\sum_{v(p)=0}(-1)^i\frac{Td_i(P^1(A_p),\ldots,P^i(A_p))}{\det A_p}\right]\cdot t^i.
\end{aligned}$$

But $\chi(\mathcal{O}_M)$ is obviously independent of t, and so *all terms on the right involving nonzero powers of* t *are necessarily zero*; thus

$$\chi(\mathcal{O}_M) = \sum_{v(p)=0}\frac{Td_n(P^1(A_p),\ldots,P^n(A_p))}{\det A_p},$$

and finally, by the Bott residue formula, we can evaluate this last term to arrive, in this special case, at the famous

Hirzebruch-Riemann-Roch Formula

$$\chi(\mathcal{O}_M) = Td_n(c_1(M),\ldots,c_n(M)).$$

For a curve, the formula reads

$$\chi(\mathcal{O}_M) = \tfrac{1}{2}c_1(M),$$

which is equivalent to *Riemann's relation*

$$g = \tfrac{1}{2}b_1(M).$$

This we proved by harmonic theory. For a surface, we have *Noether's formula*

$$\chi(\mathcal{O}_M) = \frac{c_1(M)^2 + c_2(M)}{12},$$

which we will prove and use extensively in the next chapter.

Unfortunately, our analogy between the Gauss-Bonnet III and Riemann-Roch formulas fails in one crucial aspect: while any differentiable manifold has many C^∞ vector fields to use as props in the proof of Gauss-Bonnet III, relatively few compact manifolds have any global holomorphic vector fields. (Cf. the theorem of Carrell and Liebermann proved in Section 4 of Chapter 5.) Of course, since the Riemann-Roch formula itself has nothing to do with the vector field v used to obtain it, we may suspect that the role of v is only auxiliary. This is in fact the case—the formula holds for any compact complex manifold—but we do not have available here the techniques necessary to prove it. Our derivation of the formula thus remains only a suggestion, and not a proof; we will, however, give a geometric proof of the formula for algebraic surfaces in the next chapter.

5. SPECTRAL SEQUENCES AND APPLICATIONS

Spectral Sequences of Filtered and Bigraded Complexes

Spectral sequences are algebraic tools for working with cohomology; basically they form an array of long exact sequences fit into a systematic pattern and are to be applied in a similar fashion. To someone who works with cohomology, they are essential in the same way that the various integration techniques are essential to a student of calculus. We shall use spectral sequences in rather limited circumstances, but it seems worthwhile to give the general definitions.

A *complex* $(K^*, d) = \{K^0 \xrightarrow{d} K^1 \xrightarrow{d} K^2 \rightarrow \cdots\}$ is a sequence of Abelian groups with differentials

$$d\colon\ K^p \rightarrow K^{p+1}$$

satisfying $d \circ d = 0$. The *cohomology* of the complex is

$$H^*(K^*) = \bigoplus_{p \geq 0} H^p(K^*),$$

where

$$H^p(K^*) = \frac{Z^p}{dK^{p-1}}$$

with $Z^p = \ker\{d : K^p \to K^{p+1}\}$ the group of *cycles* and $dK^{p-1} = B^p \subset Z^p$ the subgroup of *boundaries*. A *subcomplex* (J^*, d) is given by subgroups $J^p \subset K^p$ with $dJ^* \subset J^*$. The *quotient complex* (L^*, d) is defined by $L^* = K^*/J^*$ with the obvious differential. We then have an *exact sequence of complexes*

$$0 \to J^* \to K^* \to L^* \to 0;$$

by an easy and well-known argument, this gives rise to a *long exact cohomology sequence*

$$\cdots \to H^p(J^*) \to H^p(K^*) \to H^p(L^*) \to H^{p+1}(J^*) \to \cdots.$$

Generalizing the notion of a subcomplex is that of a *filtered complex* (F^pK^*, d), defined as a decreasing sequence of subcomplexes

$$K^* = F^0K^* \supset F^1K^* \supset \cdots \supset F^nK^* \supset F^{n+1}K^* = \{0\}.$$

The single subcomplex mentioned above corresponds to the filtration

$$K^* \supset J^* \supset \{0\},$$

and the spectral sequence of a filtered complex will generalize the long exact cohomology sequence. Before coming to this, we need a few more definitions.

The *associated graded complex* to a filtered complex (F^pK^*, d) is the complex

$$\operatorname{Gr} K^* = \bigoplus_{p \geq 0} Gr^p K^*$$

where

$$\operatorname{Gr}^p K^* = \frac{F^pK^*}{F^{p+1}K^*}$$

and the differential is the obvious one. The filtration F^pK^* on K^* also induces a filtration $F^pH^*(K^*)$ on the cohomology by

$$F^pH^q(K^*) = \frac{F^pZ^q}{F^pB^q}.$$

The *associated graded cohomology* is

$$\operatorname{Gr} H^*(K^*) = \bigoplus_{p,q} \operatorname{Gr}^p H^q(K^*),$$

where

$$\operatorname{Gr}^p H^q(K^*) = \frac{F^p H^q(K^*)}{F^{p+1} H^q(K^*)}.$$

DEFINITION. A *spectral sequence* is a sequence $\{E_r, d_r\}$ $(r \geqslant 0)$ of bigraded groups

$$E_r = \bigoplus_{p,q \geqslant 0} E_r^{p,q}$$

together with differentials

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}, \qquad d_r^2 = 0,$$

such that

$$H^*(E_r) = E_{r+1}.$$

When working with spectral sequences it is useful—even essential—to draw the "picture" (Figure 3).

In practice we will always have $E_r = E_{r+1} = \cdots$ for $r \geqslant r_0$; we call this limit group E_∞ and say that *the spectral sequence* $\{E_r\}$ *converges to* E_∞.

Proposition. *Let* K^* *be a filtered complex. Then there exists a spectral sequence* $\{E_r\}$ *with*

$$E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}},$$

$$E_1^{p,q} = H^{p+q}(\operatorname{Gr}^p K^*),$$

$$E_\infty^{p,q} = \operatorname{Gr}^p(H^{p+q}(K^*)).$$

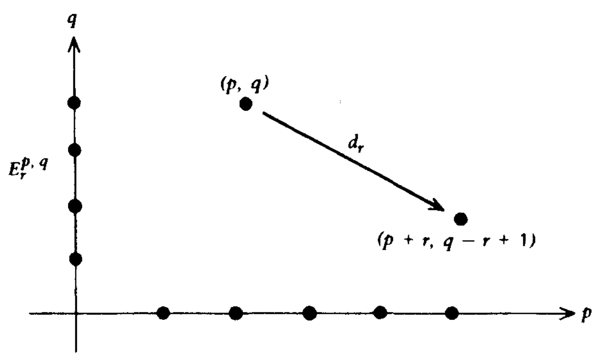

Figure 3

The last statement is usually written

$$E_r \Rightarrow H^*(K^*)$$

and we say that the spectral sequence abuts *to* H*(K*).

Proof. The initial term has been defined, and

$$\begin{array}{ccc} d_0:\ E_0^{p,q} & \longrightarrow & E_0^{p,q+1} \\ \| & & \| \\ F^pK^{p+q}/F^{p+1}K^{p+q} & \longrightarrow & F^pK^{p+q+1}/F^{p+1}K^{p+q+1} \end{array}$$

is obtained from the given differential d by passing to the quotient. The cohomology of $\{E_0, d_0\}$ is

$$\begin{aligned} E_1^{p,q} = \frac{\operatorname{Ker} d_0}{\operatorname{Im} d_0} &= \frac{\{a \in F^pK^{p+q}:\ da \in F^{p+1}K^{p+q+1}\}}{d(F^pK^{p+q-1}) + F^{p+1}K^{p+q}} \\ &= H^{p+q}\left(\frac{F^pK^*}{F^{p+1}K^*}\right) \\ &= H^{p+q}(\operatorname{Gr}^p K^*) \end{aligned}$$

as specified.

If $[a]$ is a class in $E_1^{p,q}$ as just above, then

$$da \in \frac{\{b \in F^{p+1}K^{p+q+1}:\ db \in F^{p+2}K^{p+q+2}\}}{d(F^{p+1}K^{p+q}) + F^{p+2}K^{p+q+1}}$$

defines a class in $E_1^{p+1,q}$, and this gives the differential

$$d_1:\ E_1^{p,q} \to E_1^{p+1,q}.$$

It follows that

$$\operatorname{Ker} d_1 = \frac{\{a \in F^pK^{p+q}:\ da \in F^{p+2}K^{p+q+1}\}}{d(F^pK^{p+q-1}) + F^{p+1}K^{p+q}},$$

$$\operatorname{Im} d_1 = \frac{d(F^{p-1}K^{p+q-1})}{d(F^pK^{p+q-1}) + F^{p+1}K^{p+q}},$$

so that

$$E_2^{p,q} = \frac{\{a \in F^pK^{p+q}:\ da \in F^{p+2}K^{p+q+1}\}}{d(F^{p-1}K^{p+q-1}) + F^{p+1}K^{p+q}}.$$

Here, the denominator is not a subgroup of the numerator; the meaning is that we take {denominator as written} ∩ {numerator}. A similar remark applies during the remainder of this proof.

Continuing in this way, we define in general

$$E_r^{p,q} = \frac{\{a \in F^pK^{p+q}:\ da \in F^{p+r}K^{p+q+1}\}}{d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}},$$

and for $[a]\in E_r^{p,q}$ we define

$$d_r a=[da]\in\frac{\{b\in F^{p+r}K^{p+q+1}:\ db\in F^{p+2r+1}K^{p+q+2}\}}{d(F^pK^{p+q})+F^{p+r+1}K^{p+q+1}}=E_r^{p+r,q-r+1}$$

A computation—straightforward but messy—gives

$$H^*(E_r)\cong E_{r+1}.$$

For r sufficiently large,

$$\begin{aligned}E_r^{p,q}&=\frac{\{a\in F^pK^{p+q}:\ da=0\}}{dK^{p+q-1}+F^{p+1}K^{p+q}}\\&\cong\frac{F^pH^{p+q}(K^*)}{F^{p+1}H^{p+q}(K^*)}\\&=\mathrm{Gr}^p H^{p+q}(K^*).\end{aligned}$$

This completes the proof of the proposition. Q.E.D.

One of our main examples is the spectral sequence associated to a *double complex*. This latter is a bigraded group

$$K^{*,*}=\bigoplus_{p,q\geqslant 0}K^{p,q}$$

together with differentials

$$\begin{aligned}d&:\ K^{p,q}\to K^{p+1,q},\\ \delta&:\ K^{p,q}\to K^{p,q+1},\end{aligned}$$

satisfying

$$d^2=\delta^2=0,\qquad d\delta+\delta d=0.$$

The double complex will be denoted $(K^{*,*};d,\delta)$. The *associated single complex* (K^*,D) is defined by

$$\begin{aligned}K^n&=\bigoplus_{p+q=n}K^{p,q}\\ D&=d+\delta.\end{aligned}$$

There are two filtrations on (K^*,D) given by

$$'F^pK^n=\bigoplus_{\substack{p'+q=n\\ p'\geqslant p}}K^{p',q},$$

$$''F^qK^n=\bigoplus_{\substack{p+q''=n\\ q''\geqslant q}}K^{p,q''}.$$

If, e.g., M is a complex manifold and

$$K^{p,q}=A^{p,q}(M),\qquad d=\partial,\qquad \delta=\bar\partial,$$

then $'F^pA^n(M)$ means "n-forms having at least p-dz's."

There are two spectral sequences, $\{'E_r\}$ and $\{''E_r\}$, both abutting to $H^*(K^*)$. By symmetry we may consider the first one. Then

$$'E_0^{p,q} = \frac{K^{p,q} + K^{p+1,q-1} + \cdots}{K^{p+1,q-1} + \cdots} \cong K^{p,q}.$$

The differential d_0 is induced from $D = d + \delta$ by passing to the quotient. Thus, under the above isomorphism $d_0 = \delta$ and

$$'E_1^{p,q} \cong H_\delta^q(K^{p,*}),$$

where the right-hand side denotes the qth cohomology group of the complex

$$\cdots \to K^{p,q-1} \xrightarrow{\delta} K^{p,q} \xrightarrow{\delta} K^{p,q+1} \to \cdots .$$

The differential d_1 is computed from $D = d + \delta$ on $'E_1$. Since $\delta = 0$ on $'E_1$, we see that $d_1 = d$ and

$$'E_2^{p,q} = H^*('E_1^{p,q}, d_1) \cong H_d^p(H_\delta^q(K^{*,*})).$$

The last expression denotes the cohomology of

$$\cdots \to H_\delta^q(K^{p-1,*}) \xrightarrow{d} H_\delta^q(K^{p,*}) \xrightarrow{d} H_\delta^q(K^{p+1,*}) \to \cdots,$$

which has meaning, since $d\delta + \delta d = 0$. Summarizing:

Associated to a bigraded complex $(\mathrm{K}^{*,*}; \mathrm{d}, \delta)$ *are two spectral sequences both abutting to the cohomology of the total complex and where*

$$\begin{cases} 'E_2^{p,q} \cong H_d^p(H_\delta^q(K^{*,*})) \\ ''E_2^{p,q} \cong H_\delta^q(H_d^p(K^{*,*})). \end{cases}$$

There is one point to be careful of here. A class $[a] \in {}'E_1^{p,q}$ is given by $a \in K^{p,q}$ satisfying $\delta a = 0$ and taken modulo $\delta K^{p,q-1}$. Then a class $[a] \in {}'E_2^{p,q}$ is given by $a \in K^{p,q}$ satisfying

$$\begin{cases} \delta a = 0, \\ da \in \delta K^{p+1,q-1} \end{cases}$$

and taken modulo

$$\delta K^{p,q-1} + dK^{p-1,q} \cap \operatorname{Ker} \delta;$$

we cannot assume $da = 0$, but only that $[da] = 0$ in $H_\delta^q(K^{p+1,*})$.

Examples

If M is a complex manifold and

$$\begin{cases} K^{p,q} = A^{p,q}(M), \\ d = d \quad \text{and} \quad \delta = \bar{\partial}, \end{cases}$$

then the associated single complex is the de Rham complex $(A^*(M),d)$. In general not much seems to be known about the resulting *Fröhlicher spectral sequences* $\{'E_r\}$ and $\{''E_r\}$, both of which abut to $H^*_{\mathrm{DR}}(M)$.

If, however M is compact Kähler, then every class $[a] \in {}'E_1^{p,q} \cong H^{p,q}_{\bar\partial}(M)$ has a harmonic representative for the $\bar\partial$-Laplacian $\Delta_{\bar\partial}$. By the Kähler assumption, $2\Delta_{\bar\partial} = \Delta_d$, and consequently $da = 0$. Thus

$$'E_1 \cong {}'E_2 \cong \cdots \cong {}'E_\infty,$$

and the filtration on $H^*_{\mathrm{DR}}(M)$ is the *Hodge filtration* defined by

$$F^p H^n_{\mathrm{DR}}(M) \cong H^{n,0}(M) \oplus \cdots \oplus H^{p,n-p}(M).$$

If M is compact but not Kähler, it may happen that $'E_1 \neq {}'E_2$, but no example seems to be known where $'E_2 \neq {}'E_\infty$. An example of $'E_1 \neq {}'E_2$ is provided by the *Iwasawa manifold*

$$M = \frac{G}{\Gamma},$$

where G is the Lie group of all complex matrices

$$g = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

and $\Gamma \subset G$ is the discrete subgroup all of whose entries are Gaussian integers $\alpha + i\beta (\alpha, \beta \in \mathbb{Z})$. Under the mapping $g \to (a,c)$ we may check that M is a holomorphic fiber bundle over a complex 2-torus with fiber a complex 1-torus. The entries in the Maurer-Cartan matrix $dg \cdot g^{-1}$ are right-invariant holomorphic forms on G and hence descend to M. These entries are

$$\omega_1 = da, \qquad \omega_2 = dc, \qquad \omega_3 = -c\,da + db.$$

In particular,

$$d\omega_3 = \omega_1 \wedge \omega_2,$$

so that ω_3 is a nonclosed holomorphic form on M. If we consider ω_3 as defining a class in $'E_1^{1,0} \cong H^{1,0}_{\bar\partial}(M) = H^0(\Omega^1_M)$, then

$$d_1[\omega_3] = [d\omega_3] = [\omega_1 \wedge \omega_2]$$

is nonzero in $'E_2^{2,0}$.

Since $\dim E_r \geqslant \dim E_{r+1}$ and the Euler characteristic is invariant under taking cohomology, we have the *Fröhlicher relations*

$$\sum_{p+q=r} h^{p,q} \geqslant b_r,$$

$$\sum_{p,q} (-1)^{p+q} h^{p,q} = \sum_r (-1)^r b_r = \chi(M),$$

where $h^{p,q} = \dim H^{p,q}_{\bar\partial}(M)$ and b_r is the rth Betti number.

At the other extreme, we suppose M is a noncompact complex manifold and that the Dolbeault cohomology

$$(*) \qquad H^{p,q}_{\bar\partial}(M) = 0, \qquad q > 0.$$

This happens if M is what is called a *Stein manifold*—e.g., in Section 3 of Chapter 0 we proved $(*)$ when

$$M = \Delta^{*k} \times \Delta^{n-k}$$

is a *punctured polycylinder* defined by

$$\{z \in \mathbb{C}^n : |z_i| < 1, z_1 \cdots z_k \neq 0\}.$$

If $(*)$ is satisfied, then $'E_1^{p,q} = 0$ for $q > 0$ and the first spectral sequence is trivial from E_2 onward; i.e., $'E_2 \cong 'E_\infty$. What this implies is

$$H^*_{\mathrm{DR}}(M) \cong H^*_{\mathrm{DR}}(M, \mathrm{hol}),$$

where the right-hand side is the de Rham cohomology computed from the complex of holomorphic forms.

Hypercohomology

This is a useful generalization of ordinary sheaf cohomology. On a topological space X, a *complex of sheaves* $(\mathcal{K}^*, d)$ is given by sheaves of Abelian sheaves $\mathcal{K}^p$ together with sheaf maps

$$\mathcal{K}^0 \to \cdots \to \mathcal{K}^p \xrightarrow{d} \mathcal{K}^{p+1} \to \cdots$$

satisfying $d^2 = 0$. In this discussion the notation does *not* mean that the sheaf sequence is exact. We sometimes write

$$(\mathcal{K}^*, d) = \left\{ \mathcal{K}^0 \xrightarrow{d} \mathcal{K}^1 \xrightarrow{d} \mathcal{K}^2 \to \cdots \right\}.$$

Associated to a complex of sheaves $(\mathcal{K}^*, d)$ are the *cohomology sheaves* $\mathcal{H}^q = \mathcal{H}^q(\mathcal{K}^*)$: Setting $\mathcal{K}^q(U) = H^0(U, \mathcal{K}^q)$, the *presheaf*

$$U \mapsto \frac{\operatorname{Ker}\{d : \mathcal{K}^q(U) \to \mathcal{K}^{q+1}(U)\}}{d\mathcal{K}^{q-1}(U)}$$

gives rise to a sheaf $\mathcal{H}^q$ whose stalk is

$$\mathcal{H}^q_x = \lim_{U \ni x} \frac{\operatorname{Ker}\{d : \mathcal{K}^q(U) \to \mathcal{K}^{q+1}(U)\}}{d\mathcal{K}^{q-1}(U)}.$$

A section σ of $\mathcal{H}^q$ over an open set $U \subset X$ is given by a covering $\{U_\alpha\}$ of U and $\sigma_\alpha \in \mathcal{K}^q(U_\alpha)$ such that

$$d\sigma_\alpha = 0,$$

$$\sigma_\alpha - \sigma_\beta = d\eta_{\alpha\beta}, \qquad \eta_{\alpha\beta} \in \mathcal{K}^{q-1}(U_\alpha \cap U_\beta);$$

the section is zero in case

$$\sigma_\alpha = d\eta_\alpha, \qquad \eta_\alpha \in \mathcal{K}^{q-1}(U_\alpha),$$

after perhaps refining the given covering. We note that essentially by definition:

The cohomology sheaves $\mathcal{H}^q = 0$ *for* $q > 0 \Leftrightarrow$ *the Poincaré lemma holds for the complex of sheaves* $(\mathcal{K}^*, d)$.

Now let $\underline{U} = \{U_\alpha\}$ be a covering of X and $C^p(\underline{U}, \mathcal{K}^q)$ the Čech cochains of degree p with values in $\mathcal{K}^q$. The two operators

$$\delta:\ C^p(\underline{U}, \mathcal{K}^q) \to C^{p+1}(\underline{U}, \mathcal{K}^q),$$

$$d:\ C^p(\underline{U}, \mathcal{K}^q) \to C^p(\underline{U}, \mathcal{K}^{q+1}),$$

satisfy $\delta^2 = d^2 = 0, d\delta + \delta d = 0$; and hence gives rise to a double complex

$$\{C^{p,q} = C^p(\underline{U}, \mathcal{K}^q); \delta, d\}.$$

Let $(C^*(\underline{U}), D)$ be the associated single complex. A refinement $\underline{U}' < \underline{U}$ of coverings induces mappings

$$C^p(\underline{U}, \mathcal{K}^q) \to C^p(\underline{U}', \mathcal{K}^q),$$
$$H^*(C^*(\underline{U})) \to H^*(C^*(\underline{U}')),$$

and we define the *hypercohomology*

$$\mathbb{H}^*(X, \mathcal{K}^*) = \lim_{\underline{U}} H^*(C^*(\underline{U}), D).$$

Now the spectral sequences $'E, ''E$ associated to the double complex $(C^p(\underline{U}, \mathcal{K}^q), \delta, d)$ behave well with respect to refinements of the covering, and passing to the limit we obtain two spectral sequences abutting to $\mathbb{H}^*(X, \mathcal{K}^*)$ with

$$'E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{K}^*)),$$
$$''E_2^{p,q} = H_d^q(H^p(X, \mathcal{K}^*)).$$

Explanations. $H^*(X, \mathcal{H}^*(\mathcal{K}^*))$ is the Čech cohomology of the cohomology sheaves $\mathcal{H}^*(\mathcal{K}^*)$, and $H_d^*(H^*(X, \mathcal{K}^*))$ is the cohomology of the complex

$$H^*(X, \mathcal{K}^0) \xrightarrow{d} H^*(X, \mathcal{K}^1) \xrightarrow{d} \cdots.$$

Before giving some examples, we need one lemma. A map

$$j:\ \mathcal{L}^* \to \mathcal{K}^*$$

between complexes of sheaves is a *quasi-isomorphism* if it induces an isomorphism on cohomology sheaves:

$$j_*:\ \mathcal{H}^q(\mathcal{L}^*) \longrightarrow \mathcal{H}^q(\mathcal{K}^*), \qquad q \geqslant 0.$$

Lemma. *If* $j:\mathcal{L}^*\to\mathcal{K}^*$ *is a quasi-isomorphism, then the induced map on hypercohomology*

$$j_*:\ \mathbb{H}^*(X,\mathcal{L}^*)\to\mathbb{H}^*(X,\mathcal{K}^*)$$

is an isomorphism.

Proof. Clearly j induces mappings on the spectral sequences, and

$$j_*:\ H^p(X,\mathcal{H}^q(\mathcal{L}^*))\longrightarrow H^p(X,\mathcal{H}^q(\mathcal{K}^*))$$

is an isomorphism by our assumption. It is a reasonably obvious general fact that a map between filtered complexes that induces an isomorphism on any term $\{E_r\}$ in the spectral sequences necessarily induces an isomorphism on the total cohomology. Q.E.D.

Here are some examples.

1. *De Rham's theorem revisited.* Suppose M is a manifold and $(\mathcal{A}^*,d)$ the de Rham complex of sheaves of smooth forms

$$\mathcal{A}^0\xrightarrow{d}\mathcal{A}^1\xrightarrow{d}\mathcal{A}^2\to\cdots.$$

We denote by $\mathbb{R}^*$ the trivial complex

$$\mathbb{R}\to 0\to 0\to\cdots$$

with $\mathbb{R}$ in degree zero and nothing elsewhere. By the d-Poincaré lemma,

$$\mathcal{H}^q(\mathcal{A}^*)=0\qquad\text{for } q>0,\qquad \mathcal{H}^0(\mathcal{A}^*)\cong\mathbb{R}.$$

Consequently, the inclusion

$$i:\ \mathbb{R}^*\to\mathcal{A}^*$$

is a quasi-isomorphism, and by the lemma

$$\mathbb{H}^*(M,\mathbb{R}^*)\cong\mathbb{H}^*(M,\mathcal{A}^*).$$

Evidently

$$('E_{\mathbb{R}^*})_2^{p,q}=\begin{cases}H^p(M,\mathbb{R}), & q=0,\\ 0, & q>0,\end{cases}$$

so the first spectral sequence for $\mathbb{R}^*$ is trivial and

$$H^*(M,\mathbb{R})\cong\mathbb{H}^*(M,\mathbb{R}^*).$$

On the other hand, by the partition of unity argument $H^q(M,\mathcal{A}^*)=0$ for $q>0$, and so

$$(''E_{\mathcal{A}^*})_2^{p,q}=\begin{cases}H^p_{\rm DR}(M), & q=0,\\ 0, & q>0.\end{cases}$$

Combining the previous remarks yields again the de Rham isomorphism

$$H^*(M,\mathbb{R})\cong H^*_{\rm DR}(M).$$

This is, of course, essentially the previous sheaf-theoretic proof of the theorem. However, it is cast in such a way that the essential aspects are more clearly isolated, thus leading naturally to the generalizations to appear shortly.

2. *Same for Dolbeault.* Suppose M is a complex manifold, and let $(\mathcal{Q}^{p,*},\bar\partial)$ denote the Dolbeault complex of sheaves

$$\mathcal{Q}^{p,0}\xrightarrow{\bar\partial}\mathcal{Q}^{p,1}\xrightarrow{\bar\partial}\mathcal{Q}^{p,2}\rightarrow\cdots,$$

and Ω^{p^*} the trivial complex

$$\Omega^p\rightarrow 0\rightarrow 0.$$

Then, by the $\bar\partial$-Poincaré lemma the inclusion

$$\Omega^{p^*}\rightarrow\mathcal{Q}^{p,*}$$

is a quasi-isomorphism. Repeating the argument just given for de Rham's theorem gives the Dolbeault isomorphism

$$H^q(M,\Omega^p)\cong H^{p,q}_{\bar\partial}(M).$$

3. *The complex of holomorphic forms.* We now show how to compute the ordinary cohomology $H^*(M,\mathbb{C})$ of a complex manifold M purely in terms of the holomorphic differentials. First, note that the Poincaré lemma holds for these forms: If φ is a closed holomorphic p-form ($p>0$), then locally $\varphi=d\eta$ for a holomorphic $(p-1)$-form η. The proof may be done by the same method as the $\bar\partial$-Poincaré lemma—a much more sophisticated lemma will be proved when we discuss the log complex in the next example.

Now the holomorphic de Rham complex

$$\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d}\Omega^2\rightarrow\cdots$$

and trivial complex

$$\mathbb{C}\rightarrow 0\rightarrow 0\rightarrow\cdots$$

are such that the inclusion

$$\mathbb{C}^*\rightarrow(\Omega^*,d)$$

is a quasi-isomorphism, and repeating the previous argument gives

$$(*)\qquad H^*(M,\mathbb{C})\cong\mathbb{H}^*(M,\Omega^*),$$

expressing the complex Čech cohomology in terms of the holomorphic forms.

Concerning the right-hand side of $(*)$, the second spectral sequence has

$$''E_2^{p,q}=H^p_d(H^q(M,\Omega^p)).$$

Two cases are noteworthy: If M is compact Kähler, then $d=0$ on $H^q(M,\Omega^p)\cong H^{p,q}_{\bar\partial}(M)$, since $2\Delta_{\bar\partial}=\Delta_d$; thus $''E_2=E_\infty$ and

$$\mathbb{H}^n(M,\Omega^*)\cong\bigoplus_{p+q=n}H^q(M,\Omega^p),$$

which is the Hodge decomposition. In the Stein case, $H^q(M,\Omega^*)=0$ for $q>0$ and (*) reduces to the previously noted isomorphism

$$H^*(M,\mathbb{C}) \cong H^*_{\mathrm{DR}}(M,\mathrm{hol}).$$

4. *The log complex.* We now come to an interesting situation. Suppose M is a complex manifold and D a divisor on M. We say that D has *normal crossings* in case $D=\Sigma_\nu D_\nu$, where the irreducible components D_ν of D are smooth and meet transversely. At a point p through which k of the D_ν pass, we may choose local holomorphic coordinates $(z_1,\dots,z_n)$ in a neighborhood $U=\{|z_i|<1\}$ of $p=(0,\dots,0)$ such that

$$D \cap U = \{z_1\cdots z_k=0\}$$

is the union of coordinate hyperplanes. The complement

$$U^* = U - U\cap D = (\Delta^*)^k \times \Delta^{n-k}$$

is a punctured polycylinder $P^*(k,n)$ given by

$$\{z:\ |z_i|<1, z_1\cdots z_k\neq 0\}.$$

Topologically, $P^*(k,n)$ is a product $\times^k S^1$ of k circles.

Denote by $\Omega^p(*D)=\bigcup_{k\geq 0}\Omega^p(kD)$ the sheaf on M of meromorphic p-forms that are holomorphic on $M^*=M-D$ and have poles of arbitrary (finite) order on D. Similarly, we define $\mathcal{A}^p(*D)$ to be the sheaf on M coming from the presheaf

$$U \longrightarrow A^p(U-U\cap D).$$

Both of these fit into complexes of sheaves $(\Omega^*(*D),d)$ and $(\mathcal{A}^*(*D),d)$ on M.

Next, we define $\Omega^p(\log D)$ to be the subsheaf of $\Omega^p(*D)$ generated by the holomorphic forms and the logarithmic differentials dz_i/z_i $(i=1,\dots,k)$. Symbolically,

$$\Omega^p(\log D) = \Omega^p\left\{\frac{dz_1}{z_1},\dots,\frac{dz_k}{z_k}\right\}.$$

Clearly

$$d\Omega^p(\log D) \subset \Omega^{p+1}(\log D),$$

and the resulting complex $(\Omega^*(\log D),d)$ is called the *log complex*. An intrinsic characterization is given by the following

Lemma. *If* f *is a local defining equation for* D, *then* $\Omega^p(\log D)$ *is given by those meromorphic forms* φ *such that both*

$$f\varphi \quad \text{and} \quad fd\varphi$$

are holomorphic.

Proof. Obviously we may take $f=z_1\cdots z_k$, and then the necessary condition is clear.

Suppose, conversely, that $f\varphi$ and $fd\varphi$ are holomorphic. Using the notations

$$\begin{aligned} I &= (1,\ldots,k), \qquad J,K,L \subset (1,\ldots,n) \text{ are index sets,} \\ z_J &= z_{j_1}\cdots z_{j_q}, \qquad dz_J = dz_{j_1}\wedge\cdots\wedge dz_{j_q}, \end{aligned}$$

we may write

$$\varphi = \sum_{\substack{J\subset I \\ K\cap I=\varnothing}} \frac{\varphi_{JK}}{z_J}\frac{dz_{I-J}}{z_{I-J}}\wedge dz_K,$$

where φ_{JK} is holomorphic. Computing modulo terms T such that fT is holomorphic,

$$\begin{aligned} d\varphi &\equiv -\sum_{J,K}\sum_{j\in J}\frac{\varphi_{JK}}{z_J}\frac{dz_j}{z_j}\wedge\frac{dz_{I-J}}{z_{I-J}}\wedge dz_K \\ &= \sum_{L,K}\psi_{LK}\frac{dz_L}{z_L}\wedge dz_K, \end{aligned}$$

where

$$z_{I-L}\psi_{LK} = \pm\sum_{i\in L}\frac{\varphi_{(I-L)\cup\{i\},K}}{z_i}$$

is holomorphic. It follows that φ_{JK}/z_J is holomorphic, as was to be proved.

Q.E.D.

Intuitively, if φ contains a term with $1/z_i$ but no dz_i in the numerator, then $d\varphi$ will contain dz_i/z_i^2—what we have verified is that no cancellation occurs.

The main local result, which as we will see plays the role of a Poincaré lemma in the present context, is the following

Lemma. *The two inclusions*

$$\begin{cases} \Omega^*(\log D)\subset\mathcal{A}^*(*D), \\ \Omega^*(*D)\subset\mathcal{A}^*(*D), \end{cases}$$

are both quasi-isomorphisms.

Proof. At a point $p\notin D$, the stalks are

$$\begin{cases} \Omega^*(\log D)_p=\Omega^*(*D)_p=\Omega^*{}_p, \\ \mathcal{A}^*(*D)_p=\mathcal{A}^*{}_p, \end{cases}$$

and the result follows from the usual holomorphic and C^∞ Poincaré lemmas, respectively.

Around $p\in D$ we consider neighborhoods U as above. By the de Rham theorem for the (open) manifold $P^*(k,n)$

$$H^q_{\mathrm{DR}}(U-U\cap D)\cong H^q(\times^k S^1,\mathbb{C})=\wedge^q H^1(\times^k S^1,\mathbb{C}),$$

and so the stalk

$$\mathcal{H}^q(\mathcal{Q}^*(*D))_p\cong H^q(\times^k S^1,\mathbb{C}).$$

Since the cohomology of $U^*=U-U\cap D$ has as basis the forms

$$\frac{dz_J}{z_J}\qquad (J\subset I),$$

the stalks $\mathcal{H}^q(\Omega^*(\log D))_p$ and $\mathcal{H}^q(\Omega^*(*D))_p$ both map *onto* $\mathcal{H}^q(\mathcal{Q}^*(*D))_p$. What must be verified is:

(*) *Let φ be a closed meromorphic* p-*form on the polycylinder such that φ has poles on* D *and $\varphi=0$ in $\mathrm{H}^{\mathrm{p}}_{\mathrm{DR}}(\mathrm{P}^*(\mathrm{k},\mathrm{n}))$. Then $\varphi=\mathrm{d}\eta$, where η is meromorphic with poles on* D. *If φ is in the log complex and $\varphi=0$ in $\mathrm{H}^{\mathrm{p}}_{\mathrm{DR}}(\mathrm{P}^*(\mathrm{k},\mathrm{n}))$, then $\varphi=\mathrm{d}\eta$ for a form η in the log complex.*

Before giving the proof, we remark that on two previous occasions we have proved the isomorphism

$$H^*(M,\mathbb{C})\cong H^*_{\mathrm{DR}}(M,\mathrm{hol})$$

for a complex manifold M satisfying

$$H^q(M,\Omega^p)=0,\qquad q>0.$$

Since this latter is true for $M=P^*(k,n)$, we may write $\varphi=d\eta$ where η is holomorphic in $P^*(k,n)$ but may have an *essential singularity* on the divisor $(z_1\cdots z_k)=0$. By being careful we must show that η may be taken to be meromorphic.

Proof. The argument is not difficult but is a little long. We shall concentrate on writing $\varphi=d\eta$, where η has at most a pole on the divisor $(z_1\cdots z_k)=0$. The argument will also show that η is in the log complex in case this is true of φ.

Write $(z_1,\ldots,z_n)=(u_1,\ldots,u_k,v_1,\ldots,v_{n-k})=(u,v)$, so that $P^*(k,n)$ is given by

$$\{(u,v):\ 0<|u_i|<1,|v_j|<1\}$$

and the divisor D by $u_i\cdots u_k=0$. We first eliminate the v's from the picture. Following the procedure in the proof of the $\bar\partial$-Poincaré lemma in Section 2 of Chapter 0, we suppose that $\varphi\equiv 0(du,dv_1,\ldots,dv_l)$ and write

$$\varphi=\varphi'+\varphi''\wedge dv_l,$$

where $\varphi',\varphi''\equiv 0(du,dv_1,\ldots,dv_{l-1})$. Then $d\varphi=0\Rightarrow(\partial\varphi'/\partial v_j)=(\partial\varphi''/\partial v_j)=0$

for $j>l$, where if $\alpha=\Sigma\alpha_I dx_I$ is a differential form,

$$\frac{\partial\alpha}{\partial x_j}=\sum_I \frac{\partial\alpha_I}{\partial x_j}dx_I.$$

Since φ is holomorphic in v, we may use formal integration of power series to solve

$$\varphi''=\frac{\partial\eta}{\partial v_l},$$

where η has the same order pole in u as φ'' and $\partial\eta/\partial v_j=0$ for all $j>l$. Then $\varphi-d\eta\equiv 0(du,dv_1,\dots,dv_{l-1})$. Continuing in this way, we may assume that $\varphi\equiv 0(du)$. Then $d\varphi=0 \Rightarrow (\partial\varphi/\partial v_j)=0$, and so the v's may effectively be ignored.

Inductively, we assume the theorem for $u'=(u_1,\dots,u_{k-1})$ and write

$$\varphi=\psi'+\psi''\wedge du_k,$$

where $\psi',\psi''\equiv 0(du')$. Consider the Laurent series

$$\psi''=\sum_{\nu=-N}^{\infty}\psi''(u')_\nu u_k^\nu.$$

Then, by formally integrating the series insofar as possible, we may write

$$\psi''-\frac{\psi''(u')_{-1}}{u_k}=\frac{\partial\eta}{\partial u_k},$$

where η has the same order pole in u' and one less order pole in u_k. Clearly

$$\tilde{\varphi}=\varphi-d\eta=\xi'+\xi''\wedge\frac{du_k}{u_k},$$

where $\xi''\equiv 0(u',du')$ and $\xi'\equiv 0(du')$. Since $\tilde{\varphi}$ is closed, we deduce that $\xi'\equiv 0(u',du')$ and

$$d\xi'=0=d\xi''.$$

Now $\tilde{\varphi}=0$ in $H^q_{\mathrm{DR}}((\Delta^*)^k)$, and thus the restriction of $\tilde{\varphi}$ to $H^q_{\mathrm{DR}}((\Delta^*)^{k-1})$ is zero, where $(\Delta^*)^{k-1}\subset(\Delta^*)^k$ is given by $u_k=$constant. This restriction is just ξ', and by induction $\xi'=d\gamma'$, where γ' has at most a pole in u'.

Finally we consider $\tilde{\tilde{\varphi}}=\tilde{\varphi}-d\gamma'=\xi''\wedge du_k/u_k$. Writing $(\Delta^*)^k=(\Delta^*)^{k-1}\times\Delta^*$ and using Künneth, $\tilde{\tilde{\varphi}}=0$ in $H^q_{\mathrm{DR}}((\Delta^*)^k)\Rightarrow\xi''=0$ in $H^{q-1}_{\mathrm{DR}}((\Delta^*)^{k-1})$. Then $\xi''=d\gamma''$ where γ'' has at most a pole in u', and $\tilde{\tilde{\varphi}}=d(\gamma''\wedge du_k/u_k)$.

Q.E.D.

We now draw some conclusions from the lemma. The sheaves $\mathcal{Q}^*(*D)$ admit partitions of unity, and therefore $H^q(M,\mathcal{Q}^*(*D))=0$ for $q>0$ and, by the spectral sequence for hypercohomology,

$$\begin{aligned}\mathbb{H}^*(M,\mathcal{Q}^*(*D)) &\cong H_d^*(H^0(M,\mathcal{Q}^*(*D)))\\ &= H_{\mathrm{DR}}^*(M-D)\\ &\cong H^*(M-D,\mathbb{C}).\end{aligned}$$

Using this together with the lemma on quasi-isomorphisms, we deduce the isomorphisms

$$(*)\qquad \begin{array}{ccc}\mathbb{H}^*(M,\Omega^*(\log D)) & \cong & H^*(M-D,\mathbb{C})\\ \downarrow & & \|\\ \mathbb{H}^*(M,\Omega^*(*D)) & \cong & H^*(M-D,\mathbb{C}).\end{array}$$

This gives a method for computing the cohomology of the complement of a divisor with normal crossing by using meromorphic forms that are holomorphic in $M-D$ and have poles along D.

Using the resolution of singularities theorem*, the second isomorphism

$$\mathbb{H}^*(M,\Omega^*(*D)) \cong H^*(M-D,\mathbb{C})$$

holds with no assumptions on the singularities of D.

Suppose now that the line bundle $[D]\to M$ is positive. By Theorem B,

$$H^q(M,\Omega^p(kD)) = 0 \qquad \text{for } q>0,\ k \geqslant k_0.$$

If we set $U=M-D$ and denote by

$$H_{\mathrm{DR}}^*(U,\mathrm{alg})$$

the cohomology of the complex of meromorphic forms that are holomorphic in U and have poles on D, then by the degeneration of the second spectral sequence of hypercohomology we obtain

Grothendieck's Algebraic de Rham Theorem

$$H_{\mathrm{DR}}^*(U,\mathrm{alg}) \cong H^*(U,\mathbb{C}).$$

The reason for this description of the result is this. An *affine algebraic variety* U is a complex submanifold of $\mathbb{C}^N$ defined by polynomial equations. We denote by $\Omega^*(U,\mathrm{alg})$ the complex of holomorphic forms on U that are the restrictions of rational differential forms in $\mathbb{C}^N$. This notation is consistent, since if we take the projective closure M_0 of $U\subset\mathbb{C}^N\subset\mathbb{P}^N$ and apply Hironaka's theorem to obtain a resolution of singularities

$$M \xrightarrow{\pi} M_0$$

that is an isomorphism on U, then $\Omega^*(U,\mathrm{alg})$ are just the meromorphic forms on M that are holomorphic in U—cf. Section 4 of Chapter 1. The

*H. Hironaka, *On Resolution of Singularities*, Proc. Int. Congress Math., Stockholm (1962), pp. 507–525.

algebraic de Rham theorem then asserts that cohomology $H^*(U,\mathbb{C})$ may be computed from the complex $\Omega^*(U,\text{alg})$.

Differentials of the Second Kind*

Let M be a smooth algebraic variety. A *differential of the first kind* is the classical terminology for a holomorphic p-form on M. By Hodge theory these inject to give the part $H^{p,0}(M)$ of the cohomology $H^p(M,\mathbb{C})$ of M.

DEFINITION. A *differential of the second kind* is given by a closed meromorphic p-form φ on M such that, for some divisor D with complement $U = M - D$, φ is holomorphic in U and is in the image of

$$H^p_{\text{DR}}(M) \to H^p_{\text{DR}}(U).$$

Equivalently, a differential of the second kind is a closed meromorphic p-form on M, holomorphic on $M - D$, which can be extended, up to an exact form on $M - D$, to a C^∞ closed form on M. We let ρ_p be the dimension of the space

$$\frac{(p\text{-forms of the second kind})}{d\,(\text{meromorphic }(p-1)\text{-forms})}.$$

Historically, differentials of the second kind for $p = 1,2$ played a pivotal role in the early development of the theory of algebraic surfaces. They furnished the technique for the first proof that the irregularity q of an algebraic surface was equal to $\frac{1}{2}b_1$—so that in particular b_1 is even—and the original proof that the Neron-Severi group defined below

$$\frac{\{\text{divisors on } S\}}{\{\text{divisors algebraically equivalent to zero}\}}$$

is finitely generated. For $p \geqslant 3$ the differentials of the second kind are only partially understood, and even that is fairly recent. Because of their historical importance and close tie-in with the algebraic de Rham theorem, we shall give a brief discussion of differentials of the second kind with special emphasis on the cases $p = 1,2$.

We begin by amplifying the definition of second kind in two ways. Given a closed meromorphic p-form φ and divisor D such that φ is holomorphic in $U = M - D$, we define a *residue* to be an integral

$$\int_\gamma \varphi,$$

where $\gamma \in H_p(U,\mathbb{Z})$ is a p-cycle that is homologous to zero in M. It is clear

*This treatment is based on M. F. Atiyah and W. V. D. Hodge, Integrals of the second kind on an algebraic variety, *Annals of Math.*, Vol. 62 (1955), pp. 56–91.

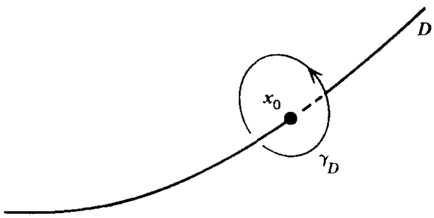

Figure 4

that φ *is of the second kind* $\Leftrightarrow$ *it has no residues in open sets* $\mathrm{U}=\mathrm{M}-\mathrm{D}$ *for sufficiently large divisors* D.

For $p=1$ we may obtain a clear picture of what the residues look like. Let D be an irreducible divisor and $x_0 \in D$ a simple point. The boundary γ_D of a normal disc to D in M at x_0 is then a 1-cycle in $H_1(M-D,\mathbb{Z})$ that bounds in M, and the class of γ_D is independent of x_0, since the smooth points of D form a connected manifold (Figure 4). Now suppose that $D=D_1+\cdots+D_k$ is a divisor with irreducible components D_i. We may choose the γ_{D_i} to lie in $U=M-D$, and we claim that any cycle γ in

$$\operatorname{Ker}\{H_1(U,\mathbb{Z})\longrightarrow H_1(M,\mathbb{Z})\}$$

is homologous to a linear combination of the γ_{D_i}. Indeed, by assumption $\gamma=\partial\Delta$, where Δ is a 2-chain in M. Since the singularities of D are in real codimension 4, we may assume that Δ meets D transversely at simple points. If $x_0\in D_i$ is such an intersection point, then near x_0 we may picture the part Δ_ε of Δ lying within distance ε of D_i as a normal disc at x_0 (Figure 5), and so $\partial\Delta_\varepsilon=\gamma_{D_i}$. Consequently $\gamma-\gamma_{D_i}$ has one less intersection point with D, and repeating the argument gives a homology

$$\gamma\sim\sum m_i\gamma_{D_i}.$$

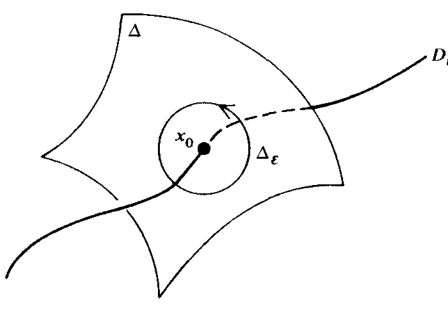

Figure 5

A consequence of this is:

For p = 1, *a closed meromorphic* 1-*form* φ *is of the second kind* $\Leftrightarrow$ φ *has no residues in* any *open set of the form* U = M − D *where it is holomorphic.*

The argument also makes it pretty clear that residues will be complicated when $p \geqslant 3$.

We shall now show that

For a p-*form* φ *of the second kind, given any point* $x_0 \in M$ *there is a meromorphic* (p − 1)-*form* ψ *such that*

$$\varphi - d\psi = \eta$$

is holomorphic near x_0. *The converse is true when* p = 1.

Proof. Given $x_0 \in M$, we may find an ample divisor D not passing through x_0, and then for $U = M - D$ by the algebraic de Rham theorem,

$$H^*_{\mathrm{DR}}(U) \cong H^*_{\mathrm{DR}}(U, \mathrm{alg}).$$

In fact, we may take U to be an affine neighborhood of x_0 as discussed at the end of the preceding section. Then for any divisor $D' \supset D$, $M - D' = U' \subset U$ will also be affine and consequently

$$H^*_{\mathrm{DR}}(U') \cong H^*_{\mathrm{DR}}(U', \mathrm{alg}).$$

We may find a U' such that φ is holomorphic in U' and is the image of a class $\Phi \in H^p_{\mathrm{DR}}(M)$. In the diagram

$$\begin{array}{ccc} H^p_{\mathrm{DR}}(M) & \longrightarrow & H^p_{\mathrm{DR}}(U', \mathrm{alg}) \\ & \searrow \quad \nearrow & \\ & H^p_{\mathrm{DR}}(U, \mathrm{alg}) & \end{array}$$

the restriction of Φ to U will be represented by a closed p-form η that is meromorphic on M and holomorphic in U. Restricting to U', we find the desired presentation

$$\varphi - \eta = d\psi,$$

where ψ is a meromorphic $(p-1)$-form on M that is holomorphic in U'.

When $p = 1$, it is clear from our above description of residue cycles that a closed meromorphic 1-form that has local presentations

$$\varphi = d\psi + \eta$$

will have no residues, and consequently φ is of the second kind. Q.E.D.

To give the interpretations of ρ_1 and ρ_2, we define the *Picard number* ρ to be the rank of the image

$$H^1(M,\mathcal{O}^*) \xrightarrow{c_1} H^2(M,\mathbb{Z}).$$

Equivalently, according to the proof of the Lefschetz (1,1) theorem from Section 2 of Chapter 1, ρ is the rank of $H^{1,1}(M)\cap H^2(M,\mathbb{Z})$, which is the rank of the quotient group

$$\frac{\text{divisors on } M}{\text{homological equivalence}}$$

of all divisors on M modulo those homologous to zero. We shall prove that

$$\begin{cases} \rho_1 = b_1, \\ \rho_2 = b_2 - \rho. \end{cases}$$

Proof. Recall that for a divisor D on M, $\Omega^p(*D)$ denotes the subsheaf of the sheaf $\mathfrak{M}^p$ of all meromorphic p-forms consisting of those having poles only on D. We let

$$\Omega^p(*) = \bigcup_{D\in \operatorname{Div}(M)} \Omega^p(*D)$$

be the subsheaf of $\mathfrak{M}^p$ of meromorphic p-forms whose polar loci are a part of a global divisor on M. Clearly

$$\Omega(*):\ \Omega^0(*) \xrightarrow{d} \Omega^1(*) \to \cdots \xrightarrow{d} \Omega^n(*)$$

gives a complex of sheaves, and as usual $\mathcal{H}^p(\Omega(*))$ denotes the pth cohomology sheaf. Evidently

$$\mathcal{H}^0(\Omega(*)) \cong \mathbb{C},$$

and we shall prove the

Lemma. $\mathcal{H}^1(\Omega(*)) \cong \bigoplus_{D\in \operatorname{Div} M} \mathbb{C}_D$, *where* $\mathbb{C}_D$ *is the constant sheaf concentrated on divisor* D.

Proof. We let φ be a closed meromorphic 1-form given in a sufficiently small polycylindrical neighborhood W of a point $x_0\in M$. The polar divisor of φ is $D=D_1+\cdots+D_k$, where the D_i are irreducible and are divisors of holomorphic functions $f_i\in\mathcal{O}(W)$. By the same argument as above, if $W^*=W-D$, then $H_1(W^*,\mathbb{Z})$ is generated by 1-cycles γ_i consisting of circles turning once around D_i. If

$$\lambda_i = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma_i}\varphi,$$

then

$$\varphi - \sum_i \lambda_i \frac{df_i}{f_i} = \psi$$

will have no periods, and consequently

$$g = \int \psi$$

will be a meromorphic function in W with

$$(*) \qquad \varphi = \sum_i \lambda_i \frac{df_i}{f_i} + dg.$$

We define the *residue map*

$$R: \mathcal{H}^1(\Omega(*)) \to \bigoplus_{D \in \operatorname{Div} M} \mathbb{C}_D$$

by

$$R(\varphi) = \bigoplus_i \lambda_i \cdot 1_{D_i}.$$

The notation means that $R(\varphi)$ is the constant λ_i on the divisor D_i. The local presentation $(*)$ shows that R is an isomorphism. Q.E.D.

Now we write out the two spectral sequences abutting to $\mathbb{H}^*(\Omega(*))$. One of these has

$$''E_2^{p,q} = H_d^p(H^q(M,\Omega(*))).$$

Since the ample divisors are cofinal among all divisors,

$$H^q(M,\Omega(*)) = 0 \qquad \text{for } q > 0.$$

Consequently, $''E_2^{p,q} = 0$ for $q > 0$ and

$$\mathbb{H}^p(\Omega(*)) \cong \frac{\{\text{closed meromorphic } p\text{-forms}\}}{\{\text{exact forms}\}}.$$

We may therefore think of $\mathbb{H}^*(\Omega(*))$ *as the de Rham cohomology of the function field of* M.

For the other spectral sequence

$$(**) \qquad 'E_2^{p,q} \cong H^p(M, \mathcal{H}^q(\Omega(*))).$$

Now any spectral sequence gives an exact sequence in low degrees, which in this case is

$$0 \to E'^{1,0}_2 \to \mathbb{H}^1 \to {'E_2^{0,1}} \xrightarrow{d_2} {'E_2^{2,0}} \to \mathbb{H}^2 \to G \to 0,$$

where $G = \mathbb{H}^2/F^2\mathbb{H}^2$ has the subgroup $G' = \mathbb{H}^2/F^1H^2$ with

$$G' \oplus G/G' \subset {'E_2^{1,1}} \oplus {'E_2^{2,0}}$$

a subgroup of $\ker d_2$. Substituting $(**)$ in this exact sequence, we obtain

$$0 \to H^1(M,\mathbb{C}) \to \mathbb{H}^1(\Omega(*)) \xrightarrow{R} H^0\Big(\bigoplus_{D \in \operatorname{Div} M} \mathbb{C}_D\Big) \xrightarrow{i} H^2(M,\mathbb{C}) \to \mathbb{H}^2(\Omega(*)) \to G.$$

The interpretations of the maps in this sequence are (we omit the proofs that diagrams commute):

1. Using the previously established isomorphism

$$(***) \qquad H^*(M,\mathbb{C}) \cong \mathbb{H}^*(\Omega^*),$$

the first map is the natural one

$$\mathbb{H}^1(\Omega^*) \to \mathbb{H}^1(\Omega(*))$$

induced from the inclusions $\Omega^p \to \Omega^p(*)$.

2. The second map assigns to a closed meromorphic 1-form its residue as in the proof of the lemma above.

3. The map i assigns to 1_D the fundamental class $\eta_D \in H^2(M,\mathbb{Z})$ of the divisor D.

4. The map $H^2(M,\mathbb{C}) \to \mathbb{H}^2(\Omega(*))$ is again induced by the isomorphism $(***)$ and inclusion $\Omega^p \hookrightarrow \Omega^p(*)$.

Now the isomorphism

$$\begin{aligned} H^1(M,\mathbb{C}) &\cong \ker R \\ &= \frac{\{\text{1-forms of the second kind}\}}{\{\text{exact forms}\}} \end{aligned}$$

gives $\rho_1 = b_1$.

Next, we have

$$\begin{aligned} \frac{H^2(M,\mathbb{C})}{\left\{\begin{matrix}\text{Chern classes}\\ \text{of holomorphic}\\ \text{line bundles}\end{matrix}\right\}} &= \frac{H^2(M,\mathbb{C})}{iH^0(\oplus\mathbb{C}_D)} \\ &\cong \text{image}\{H^2(M,\mathbb{C}) \to \mathbb{H}^2(\Omega(*))\} \\ &= \frac{\{\text{2-forms of the second kind}\}}{\{\text{exact forms}\}}, \end{aligned}$$

and so $\rho_2 = b_2 - \rho$. Q.E.D.

It is clear that the identification

$$\text{image}\{'E^{0,p} \to \mathbb{H}^p(\Omega(*))\} \cong \frac{\{p\text{-forms of the second kind}\}}{\{\text{exact forms}\}}$$

allows the above proof to continue, but the subsequent interpretation of the numbers ρ_p has yet to yield much geometric information. So we shall conclude with some further remarks on the cases $p = 1, 2$.

For $p = 1$ perhaps the most interesting case is when M is an algebraic curve of genus g. Our definition of differentials of the second kind agrees with that given in Section 2 of Chapter 2 on Riemann surfaces. We will prove the result:

Let $\mathrm{D}=\mathrm{p}_1+\cdots+\mathrm{p}_{\mathrm{g}}$ *be a nonspecial divisor of degree* g. *Then there is an isomorphism*

$$\left\{\begin{array}{l}\text{1-forms } \varphi \text{ having}\\ \text{no residues and}\\ \text{polar divisor } 2\mathrm{D}\end{array}\right\} \cong \frac{\{\text{1-forms of the second kind}\}}{\{\text{exact forms}\}}.$$

Proof. By the Riemann-Roch theorem

$$h^0(D)=\deg D-g+1+i(D)=1,$$

so the only meromorphic functions with polar divisor D are the constants. Again by Riemann-Roch applied to the line bundle K_C+2D,

$$\begin{aligned} h^0(K_C+2D) &= \deg(K_C+2D)-g+1+i(K_C+2D)\\ &= 2g-2+2g-g+1\\ &= 3g-1, \end{aligned}$$

so that the space of meromorphic differentials having polar divisor $2D$ has dimension $3g-1$. The equations

$$\sum \operatorname{Res}_{p_i}(\varphi)=0$$

impose exactly $g-1$ independent conditions on this space, due to the residue theorem

$$\sum_i \operatorname{Res}_{p_i}(\varphi)=0,$$

and observation that we may find $\varphi\in H^0(\mathcal{O}_C(K+2D))$ with prescribed residues subject only to the residue theorem (cf. Section 2 in Chapter 2). So the space of 1-forms of the second kind with polar divisor $2D$ has dimension

$$3g-1-(g-1)=2g,$$

and none of these can be exact by our remark about meromorphic functions with polar divisor D. Q.E.D.

We turn now to the case $p=2$. To first explain how the relation

$$\rho_2=b_2-\rho$$

was used classically, we refer to the exact sequence

$$\mathbb{H}^1(\Omega(*))\xrightarrow{R} H^0\Big(\bigoplus_{D\in\operatorname{Div}M}\mathbb{C}_D\Big)\xrightarrow{i} H^2(M,\mathbb{C}),$$

which appeared in the proof above. We may interpret it in the following manner:

> *If* D *is a divisor on* M *with fundamental class* $\eta_{\mathrm{D}} \in \mathrm{H}^2(\mathrm{M}, \mathbb{Z})$, *then* η_{D} *is a torsion element if and only if there exists a closed, meromorphic* 1*-form* φ *whose residue* $\mathrm{R}(\varphi) = \mathrm{D}$.

This was proved by Picard, and Severi showed that a multiple $\lambda D\,(\lambda \in \mathbb{Z})$ is algebraically equivalent to zero (to be explained momentarily) if and only if there is a closed meromorphic 1-form whose residue is D. Combining these, it follows that the Neron-Severi group

$$NS(M) = \frac{\{\text{divisors on } M\}}{\left\{\begin{array}{l}\text{divisors algebraically}\\ \text{equivalent to zero}\end{array}\right\}}$$

is finitely generated (*theorem of the base*). The structure of the group of divisors on M may be pictured by the diagram

$$H^{11}(M) \cap H^2(M, \mathbb{Z})$$

$$\underbrace{\underbrace{\mathrm{Div} \supset \mathrm{Div}_h}_{NS} \;\supset\; \underbrace{\mathrm{Div}_a \supset \mathrm{Div}_l}_{\mathrm{Pic}^0}}_{\mathrm{Pic}}$$

where Div_h, Div_a, Div_l are the divisors homologous, algebraically equivalent, and linearly equivalent to zero.

We shall give the precise difinitions and derive the finiteness theorem in a different way. Two effective divisors D_1, D_2 are *algebraically equivalent in the strong sense*, written

$$D_1 \overset{=}{=} D_2,$$

if there is a connected parameter variety T with marked points $t_1, t_2 \in T$ and divisor D on $M \times T$ such that

$$D \cdot M \times \{t_i\} = D_i \qquad (i = 1, 2).$$

Intuitively, there is an algebraic family $D_t (t \in T)$ of divisors connecting D_1 and D_2.

Two divisors D_1, D_2 are *algebraically equivalent*, written $D_1 \equiv D_2$, if there is a divisor D such that both of $D + D_i$ are effective and $D + D_1 \overset{=}{=} D + D_2$. We will see in a minute that this is an equivalence relation compatible with the group structure on $\mathrm{Div}(M)$. The divisors algebraically equivalent to zero then form a subgroup "$\equiv$" of $\mathrm{Div}(M)$, and the quotient

$$\mathrm{Div}(M)/\text{“}\equiv\text{”} = \mathrm{NS}(M)$$

is called the *Neron-Severi group*.

The basic result we need is the

Lemma. *Two divisors* D_1, D_2 *are algebraically equivalent if and only if they are homologous.*

Proof. It is clear that $D_1 \equiv D_2 \Rightarrow \eta_{D_1} = \eta_{D_2}$ in $H^2(M, \mathbb{Z})$. For the converse we assume that $\eta_{D_1} = \eta_{D_2}$, which is equivalent to $c_1([D_1]) = c_1([D_2])$ by the proposition in Section 1 of Chapter 1, and shall show that $D_1 \equiv D_2$. Let D_i^- be the part of D_i appearing with negative coefficients and add $E = D_1^- + D_2^-$ of each of D_1, D_2 to obtain effective divisors, thereby reducing us to proving that $D_1 \equiv D_2$ for effective divisors in the same homology class.

Now we come to the point. Recall that the Picard variety $\text{Pic}^0(M) = H^1(M, \mathcal{O})/H^1(M, \mathbb{Z})$ parametrizes line bundles with first Chern class zero; we denote by $\{P_\xi \to M\}$ $(\xi \in H^1(M, \mathcal{O})/H^1(M, \mathbb{Z}))$ this family. Since $[D_1] \otimes [D_2]^*$ has zero Chern class,

$$[D_1] \otimes [D_2]^* = P_{\xi_0}$$

for some ξ_0. By the last result proved in the subsection "Intrinsic Formulations" in Section 6 of Chapter 2, we may find a line bundle $L \to M$ and sections $\theta_\xi \in H^0(M, \mathcal{O}(L \otimes P_\xi))$ such that $\theta_\xi \neq 0$ for generic ξ. In fact from the proof we may assume that $\theta_e, \theta_{\xi_0} \neq 0$. Setting $D_\xi = (\theta_\xi)$ from $[D_1 - D_2 + D_e] = [D_{\xi_0}]$, we deduce that the *linear* equivalence

$$D_1 + D_e \sim D_2 + D_{\xi_0}$$

holds. In particular

$$D_1 + D_e \equiv D_2 + D_{\xi_0},$$

and it is clear that

$$D_e \equiv D_{\xi_0}$$

via the family of divisors $\{D_\xi\}$ $(\xi \in \text{Pic}^0(M))$. Thus $D_1 \equiv D_2$ and we are done. Q.E.D.

As a corollary we deduce the theorem of the base: $\text{NS}(M)$ *is a finitely generated group of rank* $\rho = b_2 - \rho_2$.

We have not dwelt on rational and algebraic equivalence of divisors or of general algebraic cycles, partly because we do not need these for our study of any specific varieties, and partly because the codimension-one theory is—at least as matters now stand—misleading as regards higher codimensional cycles.

The Leray Spectral Sequence

This is in many ways the most useful general spectral sequence, and so we want at least to say what it is and give an illustration. Suppose we are given topological spaces X, Y with a continuous mapping

$$f\colon X \to Y$$

and sheaf $\mathcal{F}$ over X. The qth *direct image sheaf* is the sheaf $R_f^q(\mathcal{F})$ on Y associated to the presheaf

$$U \to H^q(f^{-1}(U), \mathcal{F}).$$

The *Leray spectral sequence*, which exists under very mild restrictions (cf. the references at the end of this chapter) is a spectral sequence $\{E_r\}$ with

$$\begin{cases} E_\infty \Rightarrow H^*(X, \mathcal{F}), \\ E_2^{p,q} = H^p(Y, R_f^q(\mathcal{F})). \end{cases}$$

Suppose that $E \xrightarrow{\pi} B$ is a differentiable fiber bundle with compact fiber F. Then E, B, and F are manifolds, π is a C^∞ mapping, and

$$\pi^{-1}(U) \cong U \times F$$

for sufficiently small open sets $U \subset B$. For the constant sheaf $\mathbb{Q}$ on E, by the Künneth formula

$$H^q(\pi^{-1}(U), \mathbb{Q}) \cong H^q(F, \mathbb{Q}).$$

This suggests that as a first approximation

$$R_f^q(\mathbb{Q}) \cong H^q(F, \mathbb{Q})$$

is a constant sheaf on B. This is not quite correct, since account must be taken of how the fundamental group $\pi_1(B, x_0)$ acts on the cohomology $H^q(F_{x_0}, \mathbb{Q})(F_x = \pi^{-1}(x))$. More precisely, displacement of homology cycles in the fibers over a path γ from x_0 to x induces an isomorphism

$$H^q(F_x, \mathbb{Q}) \cong H^q(F_{x_0}, \mathbb{Q})$$

that depends only on the homotopy class of γ. This is reasonably intuitive and is proven in standard books on topology. The upshot is that first there is a representation

$$\rho\colon \pi_1(B, x_0) \to \operatorname{Aut}(H^q(F_{x_0}, \mathbb{Q}))$$

that describes how cycles change when they are displaced around closed paths. Second, *any* representation of the fundamental group

$$\rho\colon \pi_1(B, x_0) \to \operatorname{Aut}(V)$$

gives locally constant sheaf $\mathcal{V}_\rho$ on B. To construct $\mathcal{V}_\rho$, we take the vector bundle

$$V_\rho = \tilde{B} \times_{\pi_1} V$$

associated to the universal covering $\tilde{B} \to B$, and then the sections of $\mathcal{V}_\rho$ over an open set $U \subset B$ are just those which lift to constant sections of $\tilde{B} \times V$. Third, the qth direct image sheaf $R_\pi^q(\mathbb{Q})$ is the sheaf constructed in this way from the representation of $\pi_1(B, x_0)$ on $H^q(F_{x_0}, \mathbb{Q})$.

It is instructive to sketch the derivation of the Leray spectral sequence in de Rham cohomology. At any point $p \in E$ we let

$$T_p(F) = \ker\{\pi_*: T_p(E) \to T_{\pi(p)}(B)\}$$

be the tangent space to the fiber $F_{\pi(p)}$ passing through p. Setting

$$F^p(\wedge^n T_p(E)) = (\wedge^p T_p(F)) \wedge (\wedge^{n-p} T_p(E))$$

defines a filtration $\{F^p(\wedge^n T(E))\}$ on the exterior powers of the tangent bundle $T(E)$, and we let $\{F^p(\wedge^n T^*(E))\}$ be the dual filtration of the exterior powers of $T^*(E)$ given by

$$F^p(\wedge^n T^*(E)) = \mathrm{Ann}(F^{n-p+1}(\wedge^n T(E))).$$

This gives a filtration $F^pA^n(E)$ on the space of C^∞ differential forms of degree n on E, and, setting $A^n = A^n(E)$, we have

$$\begin{cases} A^n = F^0A^n \supset F^1A^n \supset \cdots \supset F^nA^n \supset F^{n+1}A^n = 0 \\ d: F^pA^n \to F^pA^{n+1}. \end{cases}$$

To picture this filtration, we choose local product coordinates (x,y) in E with $\pi(x,y) = x$. Then $T_p(F)$ is spanned by the vectors $\partial/\partial y_i$, and

$$F^pA^n = \left\{ \varphi = \sum_{\substack{\#I + \#J = n \\ \#I > p}} \varphi_{IJ}(x,y)\, dx_I \wedge dy_J \right\},$$

from which the two above properties of the filtration are apparent.

According to the general mechanism, once we have such a filtered complex $\{F^pA^*\}$, there is an associated spectral sequence $\{E_r\}$ with

$$E_\infty \Rightarrow H^*(A^*) = H^*_{\mathrm{DR}}(E).$$

We will calculate the terms E_1 and E_2.

Recall that

$$E_0^{p,q} = \frac{F^pA^{p+q}}{F^{p+1}A^{p+q}},$$

and d_0 is obtained from d by passing to the quotient. Taking a local product isomorphism

$$\pi^{-1}(U) \cong U \times F,$$

we may represent $E_0^{p,q}$ by forms

$$\varphi = \sum_{\#I = p} \eta_I(x,y,dy) \wedge dx_I,$$

where the η_I are q-forms on F. Computing modulo $F^{p+1}A^*$,

$$d_0\varphi = \sum_{\#I = p} d_y\eta_I \wedge dx_I,$$

where d_y is the exterior derivative in the F-direction relative to the product decomposition. It follows that elements of $E_1^{p,q}$ are locally represented by

$$\bar{\varphi} = \sum_{\#I=p} \bar{\eta}_I \wedge dx_I,$$

where

$$\bar{\eta}_I(x,y,dy) \in H^q_{\mathrm{DR}}(F_x).$$

Intuitively, we may think of $E_1^{p,q}$ as the p-forms on B with values in the bundle $H^q_{\mathrm{DR}}(F)$ whose fibers are

$$H^q_{\mathrm{DR}}(F)_x = H^q_{\mathrm{DR}}(F_x).$$

We now compute $d_1\bar{\varphi}$. For φ as above with $d_0\varphi = d_y\varphi = 0$,

$$d_1\bar{\varphi} = \overline{d\varphi}\,.$$

Thus

$$d_1\bar{\varphi} = d_x\left(\sum_{\#I=p} \eta_I(x,y,dy)\wedge dx_I\right),$$

and so $E_2^{p,q}$ is given by

$$E_2^{p,q} = H^p_{\mathrm{DR}}(B, H^q_{\mathrm{DR}}(F)),$$

where the right-hand side may be defined by first interpreting $H^q_{\mathrm{DR}}(F) \to B$ as a *flat vector bundle*—i.e., a vector bundle associated to a representation of the fundamental group—whose locally constant sections are just the sheaf $R^q_\pi(\mathbb{C})$, and then taking the de Rham cohomology of forms with values in this bundle. Granted that this interpretation needs some amplification, but once this is done we have derived the spectral sequence of a differentiable fibration.

These spectral sequences are generally nontrivial—i.e., $E_2 \neq E_\infty$—and may be extremely complicated. Even the simplest nontrivial fibration, the *Hopf fibration*,

$$\pi:\ S^{2n+1} \to \mathbb{P}^n,$$

has an interesting spectral sequence: The fiber is the circle S^1, and since $\mathbb{P}^n$ is simply connected,

$$E_2^{p,q} \cong H^q(S^1) \otimes H^p(\mathbb{P}^n).$$

Figure 6 pictures the E_2 term. If $\eta \in E_2^{0,1} \cong H^1(S^1)$ is a generator, then $d_2\eta \neq 0$, since $H^q(S^{2n+1}) = 0$ for $q \neq 0, 2n+1$. Thus

$$d_2\eta = \omega,$$

where $\omega \in E_2^{2,0} \cong H^2(\mathbb{P}^n)$ is a generator. If we represent S^{2n+1} as the unit sphere $\{z : \|z\| = 1\}$ in $\mathbb{C}^{n+1}$, then

$$\omega = dd^c \log\|z\|^2$$

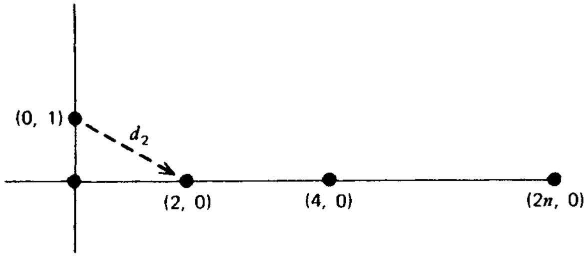

Figure 6

is the standard Kähler form on $\mathbb{P}^n$. Up on S^{2n+1},

$$\omega = d\eta,$$

where it is straightforward to check that

$$\eta = d^c \log\|z\|^2$$

restricts to the generator of $H^1(S^1)$ for each fiber. So, in this case the relation

$$d_2\eta = \omega$$

is quite visible. We also note that

$$d_2(\eta\wedge\omega^q) = \omega^{q+1} \qquad (0 \leqslant q \leqslant n).$$

By way of contrast, we suppose that E, B are compact Kähler manifolds and

$$\pi : E \to B$$

is a surjective, holomorphic mapping of maximal rank. This is a differentiable fiber bundle whose fiber F is a compact Kähler manifold, and we shall prove:*

The Leray spectral sequence degenerates at E_2; *i.e.*,

$$E_2 \cong E_\infty$$

so that

$$H^*(E, \mathbb{Q}) \cong H^*(B, R_\pi^*(\mathbb{Q})).$$

Before giving the proof, we wish to suggest two interpretations of this result. One is as another reflection of the extraordinary topological properties, such as those encountered in Sections 1 and 3 of Chapter 1, possessed

*Cf. P. Deligne, Théorème de Lefschetz et critéres de dégénérescence de suite spectrales, *Publ. Math. I.H.E.S.*, Vol. 35 (1968), pp. 107–126.

by an algebraic variety. The other interpretation is as focusing attention on the extremely important role played by the *monodromy group*, which is by definition the image of $\pi_1(B,x_0)$ in $\mathrm{Aut}(H^*(G,\mathbb{Q})$ under the representation obtained by displacing cycles around closed paths.

Proof. We first remark that a closed k-form φ given on the total space E defines classes in $E_r^{p,k}$ for all r, and moreover that multiplication by φ induces

$$\varphi:\ E_r^{p,q}\to E_r^{p,q+k}$$

commuting with

$$d_r:\ E_r^{p,q}\to E_r^{p+r,q-r+1},$$

again for all r. These assertions are clear from our proof of the spectral sequence using differential forms and also were verified in the little example above.

Now let ω be a Kähler form on E, and denote by L the map induced by multiplication by ω. Then from the definition of the direct image sheaves, $L: R_\pi^q(\mathbb{C})\to R_\pi^{q+2}(\mathbb{C})$ is defined and if $\dim F=n$ the hard Lefschetz theorem

$$L^k:\ R_\pi^{n-k}(\mathbb{C})\to R_\pi^{n+k}(\mathbb{C})$$

is valid, simply because each stalk

$$R_\pi^q(\mathbb{C})_x\cong H^q(F_x,\mathbb{C})$$

and we may apply the usual hard Lefschetz theorem. Continuing this line of thought, if we define the *primitive Leray sheaf* by

$$P^{n-k}=\ker\{L^{k+1}:\ R^{n-k}\to R^{n+k+2}\},\qquad R^q=R_\pi^q(\mathbb{C}),$$

then for the same reasons the Lefschetz decomposition

$$R^q\cong\bigoplus_k L^kP^{q-2k}\qquad (q\leqslant n)$$

is valid.

We shall show that

$$d_2:\ E_2^{p,q}\to E_2^{p+2,q-1}$$

is zero, with the proof for the higher d_r's being the same. Since

$$E_2^{p,q}=H^p(B,R^q)$$

and

$$L^k:\ E_2^{p,n-k}\cong E_2^{p,n+k}$$

is an isomorphism commuting with d_2, it will suffice to show that $d_2=0$ on

$E_2^{p,n-k}$. Passing to the Lefschetz decomposition, we consider the commutative diagram

$$\begin{array}{ccc} H^p(B,P^{n-k}) & \xrightarrow{d_2} & H^{p+2}(B,R^{n-k-1}) \\ \downarrow L^{k+1}=0 & & \downarrow L^{k+1} \\ H^p(B,R^{n+k+2}) & \xrightarrow{d_2} & H^{p+2}(B,R^{n+k+1}). \end{array}$$

The right-hand vertical arrow is an isomorphism by hard Lefschetz, and the left-hand one is zero by definition of primitive. Thus $d_2=0$. Q.E.D.

REFERENCES

This chapter gives a potpourri of general analytic, topological, and homological methods applied to complex manifolds and algebraic varieties. Some specific references were given in the text, and here we mention one or two sources for each topic that may assist the reader in amplifying the discussions in the book and serve as a guide to the literature.

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4
SURFACES

Perhaps the most striking aspect of the theory of algebraic surfaces, when first encountered, is how different it is in character from the theory of Riemann surfaces. Whereas curves, having the genus as their sole discrete invariant, fall into an orderly sequence of families, surfaces possess a variety of numerical invariants and are not so readily classified. Conversely, while curves have a natural continuous invariant—their periods, realized geometrically by the Jacobian—no fully satisfactory continuous invariant has been found for surfaces. As a result, the theory of algebraic surfaces does not possess the natural cohesiveness of the theory of curves; it tends to concentrate more on the study of special classes of surfaces. This is reflected in our treatment: with the exception of the basic tools presented in Sections 1 and 2 and the proof of Noether's formula, virtually all our results either describe or characterize specific families of surfaces.

Sections 1 and 2 contain all the techniques used in our study. For the most part, these results are special cases of general phenomena discussed before; the one new idea introduced here is the notion of a *rational map*. This is an important aspect of the theory of varieties in dimension two or more; in the case of surfaces we are able to give a complete description of birational maps.

In Section 3 we describe the general rational surface, and obtain in consequence the answer to some problems posed in curve theory. Section 4 is complementary to 3: its main result is a characterization of rational surfaces by numerical invariants.

Section 5 discusses the classification theorem for surfaces; this essentially amounts to a description, in varying detail, of all surfaces except those of general type.

It remains in Section 6 to prove Noether's formula. To do this, we introduce another technique of general interest: the blow-up of a complex manifold along any submanifold. Using this construction together with some remarks on singularities of surfaces in $\mathbb{P}^3$ we represent a general surface as a smooth divisor in a blow-up of $\mathbb{P}^3$, and obtain formulas for the numerical characters of a surface in terms of the projective invariants of a birational embedding in 3-space. Noether's formula is an immediate consequence of these.

1. PRELIMINARIES

Intersection Numbers, the Adjunction Formula, and Riemann-Roch

Let M be an algebraic surface, i.e., a compact complex manifold of dimension 2 that may be embedded in projective space. Since M is an oriented real 4-manifold, the intersection pairing

$$H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$$

is symmetric and nondegenerate. For divisors D and D' on M we define the *intersection number* $D \cdot D'$ of D and D' to be simply the intersection number of their fundamental classes (D), $(D') \in H_2(M,\mathbb{Z})$. Similarly, if $L \to M$ and $L' \to M$ are two line bundles, we take the *intersection number* $L \cdot L'$ of L and L' to be given by

$$L \cdot L' = (c_1(L) \cup c_1(L'))[M],$$

and likewise we define the intersection number $L \cdot D$ of a line bundle L with a divisor D to be just the value of the Chern class $c_1(L) \in H^2(M,\mathbb{Z})$ on the fundamental class $(D) \in H_2(M,\mathbb{Z})$ of D. Since intersection of cycles is Poincaré dual to cup product, all these definitions are consistent with the correspondence between divisors and line bundles; i.e., if $L=[D]$ and $L'=[D']$, then $D \cdot D' = L \cdot D' = L' \cdot D = L \cdot L'$.

There are a few points to be made about the intersection of divisors on an algebraic surface:

1. If L is a positive line bundle, then for any effective divisor D

$$L \cdot D = \int_D c_1(L) > 0.$$

2. Any two effective divisors D and D' intersecting in isolated points intersect positively; thus $D \cdot D' \geqslant 0$ unless D and D' have a component in common. In particular, if D is irreducible, then any effective divisor D' not containing D intersects D positively, and if in addition, $D \cdot D \geqslant 0$, then $D \cdot D' \geqslant 0$ for any effective divisor D'.

3. In a somewhat deeper vein, recall that by the Hodge-Riemann bilinear relations the intersection form is negative definite on the primitive cohomology $P^{1,1}(M) \subset H^{1,1}(M)$. By the Lefschetz decomposition, $P^{1,1}$ has codimension 1 in $H^{1,1}$; thus if D is any divisor on M with $D \cdot D > 0$, the

intersection pairing is negative definite on the orthogonal complement of η_D in $H^{1,1}(M)$. In particular, *if* $D \cdot D > 0$, *then for any divisor* D' *on* M *such that* $D' \cdot D = 0$, *either* $D' \cdot D' < 0$ *or* $(D') = 0$; this is commonly called the *index theorem*.

By way of terminology, we define a *curve* C on the surface M to be any effective divisor on M; a curve C is called *smooth* if it is the locus of a submanifold of M taken with multiplicity 1 and *irreducible* if it is not the sum of two nontrivial effective divisors.

Let C be a smooth, irreducible curve on M. By the adjunction formula from Section 2 of Chapter 1

$$K_C = (K_M + C)|_C,$$

where K_C and K_M denote, as usual, the canonical line bundles of C and M. If g is the genus of the curve C, it follows that

$$\begin{aligned} g &= \tfrac{1}{2}\deg K_C + 1 \\ &= \tfrac{1}{2}\deg(K_M + C)|_C + 1 \\ &= \frac{K_M \cdot C + C \cdot C}{2} + 1. \end{aligned}$$

This formula is also referred to as the adjunction formula. In general, we define the *virtual genus* $\pi(C)$ of an arbitrary curve C on M by

$$\pi(C) = \frac{K \cdot C + C \cdot C}{2} + 1.$$

Now let D be any smooth, irreducible curve on the surface M, and let $L = [D]$ be its associated line bundle. From the long exact cohomology sequence associated to the exact sheaf sequence

$$0 \to \mathcal{O}_M \to \mathcal{O}_M(L) \to \mathcal{O}_D(L) \to 0$$

we obtain

$$\chi(L) = \chi(\mathcal{O}_M) + \chi(\mathcal{O}_D(L)).$$

Now, by Riemann-Roch for D,

$$\begin{aligned} \chi(\mathcal{O}_D(L)) &= -\pi(D) + \deg(L|_D) + 1 \\ &= -\pi(D) + L \cdot L + 1. \end{aligned}$$

But by the adjunction formula,

$$\pi(D) = \frac{L \cdot L + L \cdot K}{2} + 1;$$

thus we have

$$\chi(L) = \chi(\mathcal{O}_M) + \frac{L \cdot L - L \cdot K}{2}.$$

This formula holds for an arbitrary line bundle L on M. We just choose a divisor D on M sufficiently positive so that both the linear series $|D|$ and $|L+D|$ contain smooth, irreducible divisors; setting

$$L' = L + D$$

the exact sequence

$$0 \to \mathcal{O}_M(L) \to \mathcal{O}_M(L') \to \mathcal{O}(L') \to 0$$

gives

$$\chi(L) = \chi(L') - \chi(L'|_D).$$

But

$$\begin{aligned} \chi(L'|_D) &= -\pi(D) + \deg L'|_D + 1 \\ &= -\frac{D\cdot D + D\cdot K}{2} + L'\cdot D; \end{aligned}$$

so

$$\begin{aligned} \chi(L) &= \chi(\mathcal{O}_M) + \frac{L'\cdot L' - L'\cdot K}{2} + \frac{D\cdot D + D\cdot K - 2(L'\cdot D)}{2} \\ &= \chi(\mathcal{O}_M) + \frac{(L'\cdot L' - 2L'\cdot D + D\cdot D) - (L'\cdot K - D\cdot K)}{2} \\ &= \chi(\mathcal{O}_M) + \frac{L\cdot L - L\cdot K}{2}; \end{aligned}$$

this is the *Riemann-Roch formula* for line bundles on a surface.

As suggested in the last chapter, the holomorphic Euler characteristic $\chi(\mathcal{O}_M)$ of M is itself expressible as a polynomial in the Chern classes of M: the formula

$$\begin{aligned} \chi(\mathcal{O}_M) &= \tfrac{1}{12}\left(c_1(M)^2 + c_2(M)\right) \\ &= \tfrac{1}{12}(K\cdot K + \chi(M)) \end{aligned}$$

is called *Noether's formula*. We defer the proof until the last section of this chapter.

We observe that the Riemann-Roch theorem for line bundles gives a direct proof of the index theorem for divisors, as follows: let E be a positive divisor on S. We have seen (p.164) that the intersection pairing on the group $H^{1,1}(S) \cap H^2(S,\mathbb{Z})$ of divisors modulo homology is nondegenerate; if it had two positive eigenvalues, it would of course have at least one in the orthogonal complement of the class of E; i.e., we could find a divisor D with

$$D\cdot E = 0 \quad \text{and} \quad D\cdot D = d > 0.$$

We will show that such a divisor D cannot exist. First, since E has strictly positive intersection number with any effective divisor, neither mD nor

$-mD$ can be effective for any $m \neq 0$. Applying Riemann-Roch, we find

$$h^0(mD) - h^1(mD) + h^2(mD) = \frac{1}{2}m^2 d - \frac{m}{2}K \cdot D + \chi(\mathcal{O}_S);$$

i.e.,

$$h^0(K - mD) = h^2(mD) \geq \frac{1}{2}m^2 d - \frac{m}{2}K \cdot D + \chi(\mathcal{O}_S) + h^1(mD)$$

becomes arbitrarily large as m goes to either $-\infty$ or $+\infty$. In particular, $K + mD$ is linearly equivalent to an effective divisor E_m for all $m \gg 0$. But now the map

$$|K - mD| \longrightarrow |2K|$$

given by

$$G \mapsto G + E_m$$

is injective, and so the dimension of $|K - mD|$ is bounded—a contradiction.

Blowing Up and Down

We recall some definitions from Chapter 1, Section 5: let M be a complex manifold of dimension n, $z = (z_1, \ldots, z_n)$ holomorphic coordinates in an open set $U \subset M$ centered around the point $p \in M$. The *blow-up* $\tilde{M}$ of M at p is then taken to be the complex manifold obtained by adjoining to $M - \{p\}$ the manifold

$$\tilde{U} = \{(z, l) : z \in l\} \subset U \times \mathbb{P}^{n-1}$$

via the isomorphism

$$\tilde{U} - (z = 0) \cong U - \{p\}$$

given by $(z, l) \mapsto z$. There is a natural projection map $\pi : \tilde{M} \to M$ extending the identity on $M - \{p\}$. The inverse image $E = \pi^{-1}(p)$ is naturally isomorphic to $\mathbb{P}(T_p(M)) \cong \mathbb{P}^{n-1}$ and is called the *exceptional divisor* of the blow-up $\tilde{M} \to M$.

When blow-ups were introduced in the course of the Kodaira embedding theorem, we were primarily concerned with the local geometry of M and $\tilde{M}$ near p and E, respectively. We would now like to relate the global geometry of $\tilde{M}$ to that of M. We begin by considering the topology of M and $\tilde{M}$: we set $M^* = M - \{p\}$, $\tilde{M}^* = \pi^{-1}M^* = \tilde{M} - E$, $U^* = U - \{p\}$ and $\tilde{U}^* = \pi^{-1}U^* = \tilde{U} - E$, and compare the Mayer-Victoris sequences of $M = M^* \cup U$ and $\tilde{M} = \tilde{M}^* \cup U$:

$$\begin{array}{ccccccc} H_i(\tilde{U}^*) & \to & H_i(\tilde{U}) \oplus H_i(\tilde{M}^*) & \to & H_i(\tilde{M}) & \to & H_{i+1}(\tilde{U}^*) \\ \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* & & \downarrow \pi_* \\ H_i(U^*) & \to & H_i(U) \oplus H_i(M^*) & \to & H_i(M) & \to & H_{i+1}(U^*) \end{array}$$

Now, π_* is an isomorphism between $H_*(\tilde{U}^*)$ and $H_*(U^*)$, and between $H_*(\tilde{M}^*)$ and $H_*(M^*)$. On the other hand, we may choose our open set U a ball around the point p; and the standard contraction $z\mapsto tz$ of U onto p induces, via π, a contraction of $\tilde{U}$ onto E. Thus we have

$$H_i(\tilde{M}) = H_i(M)\oplus H_i(E), \qquad i>0,$$

Since all the cohomology of $E\cong\mathbb{P}^{n-1}$ is represented by analytic cycles,

$$h^{i,i}(\tilde{M}) = h^{i,i}(M)+1, \qquad i>0,$$

with all other Hodge numbers of $\tilde{M}$ equal to those of M.

We make here one new definition. Let p, M, $\tilde{M}$, and π be as above, and let $V\subset M$ be any analytic subvariety of M. Then we define the *proper transform* $\tilde{V}\subset\tilde{M}$ of V to be the closure in $\tilde{M}$ of the inverse image

$$\tilde{V} = \overline{\pi^{-1}(V-\{p\})} = \overline{\pi^{-1}(V)-E}$$

of V away from x. Clearly π maps $\tilde{V}-E=\pi^{-1}(V-\{p\})$ isomorphically onto $V-\{p\}$. To get a picture of $\tilde{V}$ near the exceptional divisor, let $z=(z_1,\ldots,z_n)$ be holomorphic coordinates around $p\in M$, $\tilde{U}_i$ the open set $(l_i\neq 0)$ in $\tilde{U}=\pi^{-1}(U)$, and

$$z(i)_j = \frac{z_j}{z_i} = \frac{l_j}{l_i}, \qquad j\neq i,$$
$$z_i = z_i$$

holomorphic coordinates on $\tilde{U}_i$ as on p. 184. Recall that the divisor E is given in $\tilde{U}_i$ as $(z_i=0)$, and that the coordinates $\{z(i)_j\}_{j\neq i}$ restrict to Euclidean coordinates on $E\cong\mathbb{P}^{n-1}$. Now let f be any holomorphic function near $p\in M$, $V=(f)$ its divisor. Write

$$f(z) = \sum_{m\geq 0} f_m(z),$$

where

$$f_m(z) = \sum_{|a|=m} c_a\cdot z_1^{a_1}\cdots z_n^{a_n}$$

is the mth homogeneous component of f in terms of the coordinates z around p. Setting $\tilde{f}=\pi^*f$, $\tilde{f}_m=\pi^*f_m$, we have

$$\tilde{f}(z) = \sum\tilde{f}_m(z)$$

and

$$\tilde{f}_m(z) = \sum_{|a|=m} c_a\cdot(z_i\cdot z(i)_1)^{a_1}\cdots z_i^{a_i}\cdots(z_i\cdot z(i)_n)^{a_n}$$
$$= z_i^m\cdot\sum c_a\cdot z(i)_1^{a_1}\cdots\widehat{z_i^{a_i}}\cdots z(i)_n^{a_n}.$$

Consequently if f vanishes to order m_0 at x—i.e., if $f_0=f_1=\cdots=f_{m_0-1}=0$

—then $\tilde{f}$ vanishes to order m_0 along E, and

$$\tilde{V} = \pi^* V - \mathrm{mult}_p(V) \cdot E$$
$$= \pi^* V - \mathrm{ord}_E(\pi^* V) \cdot E.$$

Moreover, we see that

$$\tilde{V} \cap E = (z_i^{-m_0} \cdot f_m)$$
$$= \left(\sum_{|a| = m_0} c_a \cdot l_1^{a_1} \cdots l_n^{a_n} \right),$$

i.e., *under the identification* $E \cong \mathbb{P}(T_p'(M))$, $\tilde{V} \cap E$ *is just the projective tangent cone to* V *at* p. Figure 1 illustrates the case of a surface M and a curve V in M with an ordinary double point at p.

Note finally that if p is a smooth point of the subvariety $V \subset M$, then the proper transform $\tilde{V}$ of V under the blow-up of M at p is just the blow-up of V at p. Accordingly, we sometimes refer to the proper transform $\tilde{V} \subset \tilde{M}$ of a subvariety $V \subset M$ as the blow-up of V at p, even when p is a singular point of V.

We now consider the case of a surface M and its blow-up $\tilde{M} \xrightarrow{\pi} M$ at $p \in M$. First, we see that if C is any curve on $\tilde{M}$ not containing the exceptional divisor E, then C is the proper transform of its image $\pi(C)$ in M; thus

$$(*) \qquad \mathrm{Div}(\tilde{M}) = \pi^* \mathrm{Div}(M) \oplus \mathbb{Z}\{E\}.$$

We can now compute intersection numbers readily. We have seen in Chapter 1 that the normal bundle to E in $\tilde{M}$ is just the dual of the hyperplane bundle on $E \cong \mathbb{P}^1$; thus

$$(E \cdot E) = \deg([E]|_E) = \deg(N_E) = -1.$$

Since the map π has degree 1—that is, the image under π_* of the fundamental class $[\tilde{M}] \in H_4(\tilde{M}, \mathbb{Z})$ of $\tilde{M}$ is just the fundamental class of M

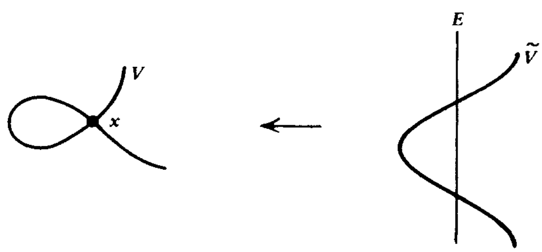


Figure 1

—we deduce that for any divisors D, D' on M,

$$\pi^*D \cdot \pi^*D' = D \cdot D',$$

and since the class (E) of the exceptional divisor of the blow-up is in the kernel of π_*,

$$\pi^*D \cdot E = {}^{\#}\big(D \cdot \big(\pi_*(E)\big)\big) = 0$$

for any divisor D on M. Summarizing, the isomorphism $(*)$ above is an isomorphism of inner product spaces.

Note in particular that if D, D' are two divisors on M intersecting transversely at p, $\tilde{D}$ and $\tilde{D}'$ their proper transforms in $\tilde{M}$, we have

$$\begin{aligned}\tilde{D} \cdot \tilde{D}' &= (\pi^*D - E) \cdot (\pi^*D' - E)\\ &= \pi^*D \cdot \pi^*D' + E \cdot E\\ &= D \cdot D' - 1.\end{aligned}$$

This is as we would expect from our picture of the proper transforms of curves: for every point p' of intersection of D and D' other than p, $\tilde{D}$ and $\tilde{D}'$ will meet at $\pi^{-1}(p')$; since D and D' have distinct tangents at p, however, $\tilde{D}$ and $\tilde{D}'$ will not meet at any point of $E = \pi^{-1}(p)$.

One point that should be brought out here is that if $\{D_\lambda\}$ is a linear system of curves on the surface M, the proper transforms $\{\tilde{D}_\lambda\}$ of the curves D_λ on $\tilde{M}$ *do not necessarily form a linear system on* $\tilde{M}$. Indeed, since

$$\tilde{D}_\lambda = \pi^*D_\lambda - \operatorname{mult}_p(D_\lambda) \cdot E$$

and the curves $\{\pi^*D_\lambda\}$ do form a linear system, $\{\tilde{D}_\lambda\}$ will be a linear system if and only if all the curves D_λ have the same multiplicity at p. Thus when we speak of the *proper transform of a linear system* $\{D_\lambda\}$, we will mean the linear system of curves $\{\pi^*D_\lambda - mE\}$, where $m = \min\{\operatorname{mult}_p(D_\lambda)\}_\lambda$ is the multiplicity of the generic curve D_λ at p.

We see from all the above that the blow-up $\tilde{M}$ of a surface M is very closely related to M. An important question to ask, then, is the converse: *Given a surface* M *and a curve* C *on* M, *when can we realize* M *as the blow-up* $\tilde{\mathrm{N}}_{\mathrm{x}_0}$ *of some surface* N, *with* $\mathrm{C} = \pi^{-1}(\{\mathrm{x}_0\})$? Clearly, necessary conditions are that C be rational and that $C \cdot C = -1$; in fact, the following result says that these are also sufficient.

Castelnuovo-Enriques Criterion. *Let* M *be an algebraic surface,* $\mathrm{C} \subset \mathrm{M}$ *a smooth rational curve on* M *of self-intersection* -1. *Then there exists a smooth algebraic surface* N *and a map* $\pi\colon\colon \mathrm{M} \to \mathrm{N}$ *such that* $\mathrm{M} \xrightarrow{\pi} \mathrm{N}$ *is the blow-up of* N *at* $\mathrm{p}_0 \in \mathrm{N}$, *and* $\mathrm{C} = \pi^{-1}(\mathrm{p}_0)$.

Proof. The proof here is along the lines of the Kodaira embedding theorem, but with a twist: we want to find a map $f: M \to \mathbb{P}^m$ that is one-to-one away from C, maps C to a point, and has smooth image. Accordingly, we look first for a line bundle $L \to M$ that is sufficiently positive away from C to have global sections, but whose restriction to C is trivial.

To find such a bundle, we start with a very ample line bundle L on M; choosing L sufficiently large, we may assume that

$$H^1(M, \mathcal{O}(L)) = 0.$$

Let $m = L \cdot C$ and consider, for each $k = 0, 1, \ldots, m$ the sequences

$$(*_k) \qquad 0 \to \mathcal{O}_M(L+(k+1)C) \to \mathcal{O}_M(L+kC) \to \mathcal{O}_C(L+kC) \to 0.$$

We note first that if $H \to C \cong \mathbb{P}^1$ is the point bundle on $\mathbb{P}^1$, $(L+kC)|_C = (m-k)H$, so that

$$H^1(C, \mathcal{O}(L+kC)) = 0 \qquad \text{for } k < m+1.$$

It follows from the long exact cohomology sequence associated to $(*_k)$ that $H^1(M, \mathcal{O}(L+(k-1)C))$ surjects onto $H^1(M, \mathcal{O}(L+kC))$ for $k \leqslant m+1$; and since by hypothesis $H^1(M, \mathcal{O}(L)) = 0$,

$$H^1(M, \mathcal{O}(L+kC)) = 0 \qquad \text{for } k \leqslant m+1$$

so the restriction map $H^0(M, \mathcal{O}(L+kC)) \to H^0(C, \mathcal{O}((m-k)H))$ is surjective for $k \leqslant m+1$. In particular, this tells us that the linear system $|L+kC|$ has no base points on C for $k \leqslant m$; since $|L|$ itself has no base points, it follows that *$|L+kC|$ has no base points for $k \leqslant m$.*

Consider now the map $\iota_{L'}$ given by the linear system $|L'| = |L+mC|$. Since $|L'|$ contains the subseries $|L| + mC$ and L is very ample, $\iota_{L'}$ embeds $M - C$ and separates points of C from points of $M - C$. On the other hand, since $L'|_C$ is trivial, any section $\sigma \in H^0(M, \mathcal{O}(L'))$ vanishing at a point of C vanishes identically along C; so $\iota_{L'}$ maps C to a point. To conclude the argument, we must show that $\iota_{L'}(C)$ is a smooth point of the image $\iota_{L'}(M)$; to see this, note that by the sequence $(*_{m-1})$ the restriction map

$$H^0(M, \mathcal{O}(L' - C)) \to H^0(C, \mathcal{O}(H))$$

is surjective. Let $p_1 \neq p_2 \in C$, and let ξ_1 be a section of L^1 vanishing on C that restricts, via the map above, to a section of H over C vanishing at p_1; let ξ_2 be a global section of L' restricting to a section of H vanishing at p_2. Let ξ_0 be any section of L' not vanishing identically on C (and hence nonzero on C), and set

$$z_1 = \frac{\xi_1}{\xi_0}, \qquad z_2 = \frac{\xi_2}{\xi_0}.$$

Let $U_1 = C - \{p_2\}$, $U_2 = C - \{p_1\}$. Then in some open set $\tilde{U}_1 \subset M$ containing U_1, z_1/z_2 is holomorphic; in fact, for $p \in U_1$ we have $d(z_1/z_2) \neq 0$ on $T'_p(C) \subset T'_p(M)$ and $dz_2 \neq 0$ on $T'_p(M)/T'_p(C)$, so that if we choose $\tilde{U}_1$ sufficiently small, we may take z_2, z_1/z_2 local coordinates on $\tilde{U}_1$. Similarly,

z_1 and z_2/z_1 furnish local coordinates on an open set $\tilde{U}_2 \subset M$ containing U_2. We see from this that the functions (z_1, z_2) map a neighborhood of C in M onto a neighborhood of the origin in $\mathbb{C}^2$, a mapping that is holomorphic outside C. This proves that $\iota_{L'}(C)$ is a smooth point of $\iota_{L'}(M)$, completing the proof of the Castelnuovo-Enriques criterion. Q.E.D.

A smooth rational curve of self-intersection -1 on a surface is called an *exceptional divisor of the first kind.*

The Quadric Surface

We now consider a smooth surface of degree 2 in $\mathbb{P}^3$. Such a surface is given as the locus of

$$(x \cdot Qx) = \sum q_{ij} x_i x_j = 0$$

for $Q = (q_{ij})$ a symmetric matrix; since

$$\frac{\partial}{\partial x_i}(x \cdot Qx) = 2\sum_j q_{ij} x_j,$$

we see that S is smooth exactly when the matrix Q is nonsingular. All nondegenerate symmetric quadratic forms on $\mathbb{C}^4$ are isomorphic, and it follows that *any two smooth quadric surfaces in* $\mathbb{P}^3$ *are projectively isomorphic.* Consider in particular the Segré map

$$\sigma: \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

given by

$$([s_0, s_1], [t_0, t_1]) \mapsto [s_0 t_0, s_0 t_1, s_1 t_0, s_1 t_1].$$

σ is clearly an embedding, and the image of σ is contained in—hence equal to—the smooth quadric

$$S_0 = (X_1 X_4 - X_2 X_3 = 0).$$

Thus *any quadric surface* $S \subset \mathbb{P}^3$ *is isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1$.

Of particular interest is the set of lines in $\mathbb{P}^3$ lying on a smooth quadric S. We see that under the Segré map σ, the curves $\{s\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{t\}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ are sent into lines in $\mathbb{P}^3$. We will call these two families of lines on S the *A-lines* and the *B-lines*; clearly every A-line meets every B-line, and any two A-lines are disjoint, as are any two B-lines. These are, moreover, all the lines on S: if $L \subset S$ is any line, then obviously L must meet at least one A-line L_1 and one B-line L_2. But L_1 and L_2 meet, and the plane they span in $\mathbb{P}^3$ can meet S in at most two lines, so either $L = L_1$ or $L = L_2$.

We can describe the set of lines on a general smooth quadric S directly as follows: first, note that if Q is any point in the intersection of S with its tangent plane $T_p(S) \subset \mathbb{P}^3$ at P, the line $\overline{PQ}$ meets S three times—once at Q and twice at P—and so must lie in S. The locus $S \cap T_p(S)$ must therefore consist of a union of lines; since $S \cap T_p(S)$ has degree 2, it must consist of two lines. Conversely any line on S through P must lie in the locus $S \cap T_p(S)$, and so we see that *through every point* $P \in S$ *there pass exactly two lines on* S, *comprising the locus* $S \cap T_p(S)$. (Note that these two lines are necessarily distinct: if $T_p(S)$ met S in only one line L, $T_p(S)$ would have to be tangent to S everywhere along L, and so no other line on S could meet L. But the intersection of S with a general tangent plane $T_Q(S)$ not containing L will consist of a union of lines, and must meet L, so this cannot be the case.)

Now, pick one line $L_0 \subset S$ and call any line on S an A-line if it is either equal to or disjoint from L, a B-line if it meets L in one point. If two lines $L, L' \neq L_0$ on S meet in a point, the plane they span in $\mathbb{P}^3$ meets L_0 in a point, which must be a point of either L or L'; so one of the two is an A-line and the other a B-line. Conversely, if $L \neq L_0$ is an A-line and L' a B-line, the plane spanned by L' and L_0 must meet L in a point, which by definition cannot be a point of L_0; so L and L' intersect. Thus two lines on S meet if and only if they are of different type; since there will be a unique B-line passing through every point of L_0 and likewise a unique A-line passing through each point of a fixed B-line, we see that the families of A-lines and B-lines are each parametrized by $\mathbb{P}^1$ In sum, then, *the set of lines on* S *consists of two disjoint families, each parametrized by* $\mathbb{P}^1$, *with two lines meeting if and only if they are from different families*. It follows that $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

We can obtain another description of a quadric surface S by projecting from a point p of S onto a plane H in $\mathbb{P}^3$. Of course, the projection map $\pi_p : S - \{p\} \to H$ is not defined at, and does not extend over, the point p. If we let $\tilde{S}$ be the blow-up of S at p, however, we can extend the map π_p continuously over the exceptional divisor $E \subset \tilde{S}$, obtaining a holomorphic map $\tilde{\pi} : \tilde{S} \to H$: for a point $r \in E$ corresponding via the identification $E \cong \mathbb{P}(T_p'(S))$ to the line $\bar{r} \subset T_p(S)$, take $\tilde{\pi}(r)$ to be the point of intersection of H with the line $L_r \subset \mathbb{P}^3$ through p with tangent line $\bar{r}$.

Now let L_1, L_2 be the two lines on S passing through p, and let q_1 and q_2 be their points of intersection with H. Then for any point $q \in H$ other than q_1 and q_2, the line $\overline{pq}$ will either meet S in one point other than p, or be simply tangent to S at p; in either case q will be the image under $\tilde{\pi}$ of a single point of $\tilde{S}$. The inverse images of q_1 and q_2, on the other hand, will be the proper transforms $\tilde{L}_1$ and $\tilde{L}_2$ of L_1 and L_2 in $\tilde{S}$. (Note that the A-lines of S—i.e., lines meeting L_1—are mapped into the pencil of lines in H containing q_1, the B-lines into the pencil of lines through q_2, and the exceptional divisor E onto the line $\overline{q_1 q_2}$.) We see from this that $\tilde{\pi}$ is one-to-one on $\tilde{S} - \tilde{L}_1 - \tilde{L}_2$, and maps $\tilde{L}_1$ and $\tilde{L}_2$ onto q_1 and q_2—i.e., $\tilde{\pi} : \tilde{S} \to \mathbb{P}^2$ *is just the blow-up of* $\mathbb{P}^2$ *at* q_1 *and* q_2. Thus we may obtain $\mathbb{P}^2$ from a quadric surface S by blowing up one point p of S and blowing down the proper transforms of the two lines on S through p. In reverse: *we may obtain a quadric surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ from $\mathbb{P}^2$ by blowing up two points q_1, q_2 on $\mathbb{P}^2$ and blowing down the proper transform of the line $\overline{q_1 q_2} \subset \mathbb{P}^2$*. We will see this operation more explicitly following our discussion of the cubic surface.

Note that since the only invariant of a symmetric quadratic form on $\mathbb{C}^n$ is its rank, there are all in all only three quadric surfaces without multiple components in $\mathbb{P}^3$: (1) those given as the locus of a nondegenerate form

$$X_0^2 + X_1^2 + X_2^2 + X_3^2$$

on $\mathbb{C}^4$—these are the smooth quadrics; (2) those given as the locus of a form

$$X_0^2 + X_1^2 + X_2^2;$$

such a quadric is the cone over a plane conic curve and singular at the vertex $[0,0,0,1]$ of the cone; and (3) those given as the locus of a form

$$X_0^2 + X_1^2;$$

these consist of the union of two planes.

The Cubic Surface

We now describe a smooth cubic surface in $\mathbb{P}^3$. We will first construct such a surface by blowing up six "general" points in $\mathbb{P}^2$ and embedding the blown-up surface in $\mathbb{P}^3$ as a cubic; we will then show that in fact every nonsingular cubic surface may be obtained in this way.

Choose six points $p_1, \ldots, p_6 \in \mathbb{P}^2$ such that

1. $p_1, \ldots, p_6$ do not all lie on a conic curve; and
2. no three of them lie on a line.

Let $\tilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_6$, E_i the exceptional divisor over p_i, and consider the complete linear system $|\tilde{C}|$ where

$$\tilde{C} = \pi^* 3H - E_1 - \cdots - E_6.$$

If C is any cubic curve in the plane passing through all six points p_i, then the curve $\pi^{-1}(C) - E_1 - \cdots - E_6$ is certainly in the linear system $|\tilde{C}|$; conversely, if D is any curve in $|\tilde{C}|$, then

$$\pi(D) \cdot H = D \cdot \pi^* H = 3$$

so $\pi(D)$ is a cubic curve, and

$$D \cdot E_i = -E_i \cdot E_i = 1$$

so D meets every exceptional divisor E_i, i.e., $\pi(D)$ passes through all six points p_i. Thus the system $|\tilde{C}|$ consists exactly of curves

$$\pi^{-1}(C) - E_1 - \cdots - E_6,$$

where C is a cubic plane curve containing $p_1,\ldots,p_6$.

We claim now that *the linear system* $|\tilde{C}|$ *embeds* $\tilde{\mathbb{P}}^2$ *as a cubic surface in* $\mathbb{P}^3$. This involves quite a bit of checking: we have to show that

1. $\tilde{C}\cdot\tilde{C}=3$,
2. $\dim|\tilde{C}|=3$,
3. $\iota_{\tilde{C}}$ separates points $p \neq q \in \mathbb{P}^2$, for
   a. $p,q \in \tilde{\mathbb{P}}^2 - \cup E_i$,
   b. $p \in E_i$, $q \in \tilde{\mathbb{P}}^2 - \cup E_i$,
   c. $p \in E_i$, $q \in E_j$, and
   d. $p,q \in E_i$; and
4. $\iota_{\tilde{C}}$ has nonzero differential at p, for
   a. $p \in \tilde{\mathbb{P}}^2 - \cup E_i$, and
   b. $p \in E_i$.

Assertion 1 is immediate: we have

$$\tilde{C}\cdot\tilde{C} = \pi^*3H\cdot\pi^*3H + E_1\cdot E_1 + \cdots + E_6\cdot E_6 = 9-6 = 3.$$

The other assertions, however, are of a different character. For example, we know that the complete linear system of cubic curves in the plane has dimension 9, and that the requirement that a cubic pass through any one of the points p_i imposes one linear condition on the system $|3H|$; the statement $\dim|\tilde{C}|=3$ amounts to saying that the conditions imposed by the six points p_i are independent. This, and the last two assertions as well, will follow from the

Lemma. *Eight points* $p_1,\ldots,p_8 \in \mathbb{P}^2$ *fail to impose independent conditions on cubics only if either*

1. *All eight lie on a conic curve*; *or*
2. *Five of the points* p_i *are collinear.*

Proof. The first step in the proof is to show that *seven points* $p_1,\ldots,p_7 \in \mathbb{P}^2$ *fail to impose independent conditions on cubics only if five are collinear.* To see this, we argue as follows: Assume that $p_1,\ldots,p_7$ fail to impose independent conditions on cubics. Then for some point p_i, any cubic containing the other six will contain p_i; reordering, we may take p_i to be p_1. Let L_{ij} denote the line $\overline{p_i,p_j}$. The cubic curve

$$L_{23} + L_{45} + L_{67}$$

contains $p_2,\ldots,p_7$ and hence p_1 as well; thus p_1 is collinear with two other points p_i, which we may take to be p_2 and p_3.

Suppose now that the line $L=\overline{p_1p_2p_3}$ also contains one of the points $p_4,\dots,p_7$, say p_4. Then since the cubic

$$L_{25}+L_{36}+L_{47}$$

contains p_1, we must have either p_5, p_6, or p_7 lying on L as well; thus we have five collinear points. If, on the other hand, none of the points $p_4,\dots,p_7$ lies on the line L, then since the cubics

$$L_{24}+L_{35}+L_{67}, \qquad L_{24}+L_{36}+L_{57}, \quad \text{and} \quad L_{25}+L_{36}+L_{47}$$

all contain p_1 but the lines L_{24}, L_{25}, L_{35}, and L_{36} cannot, p_1 must lie on L_{47}, L_{57}, and L_{67}; thus p_4, p_5, and p_6 all lie on the line L_{17}, and again we have five collinear points.

The lemma follows readily from this first step. Suppose we have eight points $p_1,\dots,p_8$ in the plane imposing only seven or fewer conditions on cubics, and assume that no five are collinear. By our first step, then, any seven of the eight points p_i do impose independent conditions, and it follows that any cubic passing through *any* seven of the points contains them all. Choose three noncollinear points; call them p_1, p_2, and p_3 and let C be a conic containing the remaining points $p_4,\dots,p_8$. By hypothesis, the cubics

$$C+L_{12}, \qquad C+L_{13}, \quad \text{and} \quad C+L_{23}$$

each contain all eight points; since each of the points p_1, p_2, and p_3 lies outside one of the lines L_{12}, L_{13}, and L_{23}, it follows that the conic C contains all eight. Q.E.D.

The statement of the lemma also holds in case p_1 is *infinitely near* p_2, that is, if p_1 is a point on the exceptional divisor E of the blow-up $\tilde{\mathbb{P}}^2 \xrightarrow{\pi} \mathbb{P}^2$ of $\mathbb{P}^2$ at p_2. In this case, we say that a curve C in the plane contains p_1 and p_2 if C passes through p_2 and the curve $\pi^{-1}(C)-E$ contains p_1, i.e., if either C is smooth at p_2 with tangent line corresponding to p_1, or C is singular at p_2. Thus, for example, we say that the points p_1, p_2, and p_3 are collinear if the proper transform in $\tilde{\mathbb{P}}^2$ of the line $\overline{p_2p_3}$ contains p_1. Of course, the linear condition imposed by p_1 on the system of cubics is defined only on the subsystem of cubics passing through p_2, but the independence of the conditions imposed by $p_1,\dots,p_8$ is still well-defined.

The argument for the lemma in case p_1 is infinitely near p_2 runs as follows: as before, we first want to show that any seven points $p_1,\dots,p_7$, with p_1 infinitely near p_2, impose independent conditions on cubics unless five are collinear. Assuming that no five are collinear, we know from the first argument that the points $p_2,\dots,p_7$ impose six conditions; if $p_1,\dots,p_7$ fail to impose seven, every cubic passing through $p_2,\dots,p_7$ will contain p_1. Now, if two of the points $p_3,\dots,p_7$ lie on the line $L=L_{12}$—say p_3 and p_4—we are done: the cubic $L_{25}+L_{36}+L_{47}$ contains p_1, and so either p_5, p_6, or p_7 lies on L, giving us five collinear points. If exactly one of the points $p_3,\ldots,p_7$—say p_3—lies on L, then the cubics

$$L_{24}+L_{35}+L_{67},\qquad L_{24}+L_{36}+L_{57},\quad \text{and}\quad L_{25}+L_{36}+L_{47}$$

all contain p_1, and so must be singular at p_2; thus p_2 lies on L_{47}, L_{57}, and L_{67}, i.e., p_4, p_5, and p_6 all lie on the line L_{27}. If, finally, none of the points $p_3,\ldots,p_7$ lies on L, then since the cubic

$$L_{27}+L_{34}+L_{56}$$

contains p_1, either L_{34} or L_{56}—say L_{34}—must contain p_2. In this case, take L' any line through p_7 missing all the other points p_i; the cubic

$$L_{23}+L_{56}+L'$$

contains p_1; thus p_2 lies on L_{56}. But then since

$$L_{27}+L_{35}+L_{46}$$

contains p_1, either L_{35} or L_{46} must pass through p_2, and in either case it follows that p_2, p_3, p_4, p_5, and p_6 are collinear.

The lemma now follows just as in the original case: given eight points $p_1,\ldots,p_8$, with p_1 infinitely near p_2 and no five collinear, by the first step any conic containing all but any three noncollinear points p_i contains all eight. Q.E.D.

We leave to the reader the proof of the lemma in three additional cases: when p_1 and p_2 are infinitely near p_3, when p_1 is infinitely near p_2 and p_3 is infinitely near p_4, and when p_1 is infinitely near p_2, which is itself infinitely near p_3.

A note: this lemma will reappear as a consequence of the general duality theory discussed in Section 4 of Chapter 5.

Let us return now to the blow-up $\pi:\tilde{\mathbb{P}}^2\to\mathbb{P}^2$ of $\mathbb{P}^2$ at six points $p_1,\ldots,p_6$ as specified earlier, and the linear system $|\tilde{C}|=|\pi^*3H-E_1-\cdots-E_6|$. As an immediate consequence of the lemma, we see that the points $p_1,\ldots,p_6$ impose independent conditions on cubics, so that $\dim|\tilde{C}|=3$. The remaining assertions 3a–d and 4a and b likewise follow from the lemma: respectively, they may be restated as saying that the points $p_1,\ldots,p_6,p$ and q impose independent conditions on cubics in case

3a. $p\neq q\in\mathbb{P}^2-\{p_1,\ldots,p_6\}$,
3b. p infinitely near p_i, $q\in\mathbb{P}^2-\{p_1,\ldots,p_6\}$,
3c. p infinitely near p_i, q infinitely near p_j,
3d. $p\neq q$ infinitely near p_i,
4a. p infinitely near $q\in\mathbb{P}^2-\{p_1,\ldots,p_6\}$,
4b. p infinitely near q infinitely near p_i.

In each of these cases, we see that since no three of the points p_i are collinear, no five of the points $p_1,\ldots,p_6,p,q$ are; and since the points p_i do not all lie on a conic, certainly $p_1,\ldots,p_6$, p and q do not. By the lemma, then, the points $p_1,\ldots,p_6$, q and p impose independent conditions on cubics, and the map $\iota_{\tilde{C}}$ embeds $\tilde{\mathbb{P}}^2$ as a cubic $S\subset\mathbb{P}^3$.

Before proceeding to study the geometry of S, we make one observation. Recall that a smooth quadric surface $S\subset\mathbb{P}^3$ may be obtained by blowing up two points q_1,q_2 on $\mathbb{P}^2$, and blowing down the proper transform in $\tilde{\mathbb{P}}^2_{q_1q_2}$ of the line $\overline{q_1q_2}$. The reader may wish to verify, by the techniques of the preceding argument, that the linear system $|\pi^*2H-E_1-E_2|$ on $\tilde{\mathbb{P}}^2_{q_1q_2}$ —corresponding to conic curves in $\mathbb{P}^2$ passing through q_1 and q_2—does indeed give a map of $\tilde{\mathbb{P}}^2_{q_1q_2}$ onto a quadric surface in $\mathbb{P}^3$, one-to-one except along the proper transform of $\overline{q_1q_2}$.

Now return to our cubic surface $S\cong\tilde{\mathbb{P}}^2_{p_1,\ldots,p_6}$ in $\mathbb{P}^3$. Consider first the image of the exceptional divisors $E_1,\ldots,E_6$ in S. Since $\tilde{C}\cdot E_i=1$, we see that each of the curves E_i has degree 1 in $\mathbb{P}^3$, hence must be a line. Likewise, if F_{ij} $(j>i)$ is the proper transform in $\tilde{\mathbb{P}}^2$ of the line $L_{ij}=\overline{p_ip_j}$ in $\mathbb{P}^2$, then

$$\begin{aligned} F_{ij}\cdot\tilde{C} &= (\pi^*L_{ij}-E_i-E_j)\left(\pi^*3H-\textstyle\sum E_k\right)\\ &= L_{ij}\cdot 3H-2=3-2=1, \end{aligned}$$

so that the image of F_{ij} in $S\subset\mathbb{P}^3$ is again a line; there are 15 such lines. Note that

$$\begin{aligned} F_{ij}\cdot F_{ij} &= (\pi^*L_{ij}-E_i-E_j)(\pi^*L_{ij}-E_i-E_j)\\ &= L_{ij}\cdot L_{ij}-2=-1 \end{aligned}$$

so that the lines F_{ij} are exceptional divisors of the first kind on S. Also, if G_i is the proper transform in $\tilde{\mathbb{P}}^2$ of the conic C_i in $\mathbb{P}^2$ through the five points $p_1,\ldots,\hat{p}_i,\ldots,p_6$,

$$\begin{aligned} G_i\cdot\tilde{C} &= \Bigl(\pi^*C_i-\sum_{j\neq i}E_j\Bigr)\left(\pi^*3H-\textstyle\sum E_i\right)\\ &= C_i\cdot 3H-5=6-5=1. \end{aligned}$$

Thus $G_i\subset S\subset\mathbb{P}^3$ is again a line, and

$$\begin{aligned} G_i\cdot G_i &= \Bigl(\pi^*C_i-\sum_{i\neq j}E_j\Bigr)\Bigl(\pi^*C_i-\sum_{i\neq j}E_j\Bigr)\\ &= C_i\cdot C_i-5=4-5=-1 \end{aligned}$$

so G_i is exceptional of the first kind.

Now if L is any line in S, we consider the locus $\pi(L)\subset\mathbb{P}^2$. Assuming L is not one of the exceptional divisors E_i, L can meet each line E_i at most once, and that transversely. Thus $\pi(L)$ will be a smooth rational curve in $\mathbb{P}^2$, hence by the genus formula either a line or a conic. Now

$$L=\pi^*\pi(L)-\sum_{p_i\in\pi(L)}E_i$$

and so

$$1 = \tilde{C}\cdot L = \left(\pi^*3H - \sum E_i\right)\cdot\left(\pi^*\pi(L) - \sum_{p_i\in\pi(L)} E_i\right)$$
$$= 3H\cdot\pi(L) + \sum_{p_i\in\pi(L)} E_i\cdot E_i.$$

This tells us that $\pi(L)$ must contain exactly two of the points p_i in case $\pi(L)$ is a line, five of the points p_i if $\pi(L)$ is a conic, and hence that L must be one of the lines F_{ij}, G_i.

Thus there are exactly 27 *lines on the cubic surface we have constructed*: *six* E_i*'s*, 15 F_{ij}*'s*, *and six* G_i*'s*. The incidence relations among the lines are clearly seen from their description as curves in $\mathbb{P}^2$: the line E_i will meet all lines on S coming from plane curves passing through p_i, that is, F_{ij} for any j and G_j for any $j\neq i$. The line F_{ij} will meet, apart from E_i and E_j, any line coming from a curve in $\mathbb{P}^2$ having a point of intersection with $\overline{p_ip_j}$ other than p_i or p_j, that is, F_{kl} for $k,l\neq i,j$, G_i, and G_j. The line G_i will meet E_j for $j\neq i$, and F_{ij} for all j. Note in particular that *every line of* S *meets exactly ten other lines*; some other interesting aspects of this configuration of 27 lines are:

1. *There are exactly* 72 *sets of six disjoint lines on* S: these are

$$\{E_i\} \qquad (1),$$
$$\{E_i, E_j, E_k, F_{lm}, F_{mn}, F_{ln}\} \qquad (20),$$
$$\{E_i, G_i, F_{jl}, F_{jk}, F_{jm}, F_{jn}\} \qquad (30),$$
$$\{G_i, G_j, G_k, F_{lm}, F_{mn}, F_{ln}\} \qquad (20),$$

and

$$\{G_i\} \qquad (1).$$

As the reader may verify, there is a unique automorphism of the configuration of 27 lines on S (not an automorphism of S) that carries any of these 72 into any other, in any assignment; thus *there are* $72\cdot 6! = 51{,}840$ *symmetries of the configuration of lines on* S.

2. *If two of the lines* L, L′ *on* S *intersect*, *then there is a unique other line on* S *that intersects them both*: the hyperplane in $\mathbb{P}^3$ containing L and L' must intersect S in a cubic curve including L and L', hence in a third line. In fact, the planes in $\mathbb{P}^3$ that meet S in a union of three lines are

$$H_{ij} = \overline{E_iG_jF_{ij}} \quad\text{and}\quad H_{ijklmn} = \overline{F_{ij}F_{kl}F_{mn}}\,.$$

3. Recall from our discussion of Grassmannians in Section 6 of Chapter 1 that if L_1, L_2, L_3, L_4 are four disjoint lines in $\mathbb{P}^3$, then there are exactly two lines $L, L'\subset\mathbb{P}^3$ meeting all four. Now if the lines L_i all lie on S,

then the lines L and L' meet S in four points, hence must also lie on S. Thus *for any four disjoint lines on a cubic surface* S *there will be exactly two other lines on* S *meeting all four.*

We now want to show that every smooth cubic surface S in $\mathbb{P}^3$ can be obtained by blowing up $\mathbb{P}^2$ in six points. We first locate six exceptional divisors on S to blow down; to find these we look for the cohomology classes they represent. Let $S_0=\tilde{\mathbb{P}}^2_{p_1,\ldots,p_6}$ be the cubic surface constructed above, and let S be an arbitrary smooth cubic in $\mathbb{P}^3$. Let $W=|3H|\cong\mathbb{P}^{19}$ be the linear system of all cubic surfaces in $\mathbb{P}^3$, and consider the incidence correspondence

$$X = \{(S,p):\ p\in S\} \subset W\times\mathbb{P}^3.$$

The subset $V\subset W$ of singular cubics is a proper analytic subvariety of W, and so $W-V$ is connected; take $\gamma:I\to W-V$ a C^∞ embedding of the unit interval $I\subset\mathbb{R}$ in $W-V$ with $\gamma(0)=S_0$, $\gamma(1)=S$. Let $\pi:X\to W$ be the projection map on the first factor. The inverse image $X'=\pi^{-1}(\gamma(I))\subset X$ is a smooth manifold, and the map $\gamma^{-1}\circ\pi:X'\to I$ is smooth. By standard manifold theory, then, X' is diffeomorphic to the product $I\times S_0$, and consequently S is diffeomorphic to S_0: since X' is compact and the map $\gamma^{-1}\circ\pi$ is smooth, we can by a partition of unity lift the vector field $-\partial/\partial t$ on I to a vector field v on X'; the flow $\varphi_t=\varphi_t(v)$ on X' will then map $\pi^{-1}(\gamma(t))$ diffeomorphically onto $\pi^{-1}(\gamma(0))=S_0$. Note also that if $H\subset\mathbb{P}^3$ is any hyperplane meeting S and S_0 transversely, the set V' of cubic surfaces tangent to H is again an analytic subvariety of W. We may therefore choose our path γ to lie in $W-V-V'$, so that $Y'=X'\cap(W\times H)$ will be a submanifold of X' mapping smoothly onto I. Take v' a vector field on Y' lifting $-\partial/\partial t$ and choose v to extend v'; the diffeomorphism $\varphi=\varphi_1:S\to S_0$ will then carry the hyperplane section $H\cap S$ to $H\cap S_0$. This argument shows in general that *any two smooth hypersurfaces of degree* d *in* $\mathbb{P}^n$ *are diffeomorphic via a map carrying a hyperplane section of one to a hyperplane section of the other.*

Now by the adjunction formula applied to $S\subset\mathbb{P}^3$,

$$\begin{aligned} K_S &= (K_{\mathbb{P}^3}+S)|_S \\ &= -H|_S \end{aligned}$$

and similarly $K_{S_0}=-H|_{S_0}$. Since our diffeomorphism $\varphi:S\to S_0$ carries $S\cap H$ to $S_0\cap H$, we deduce that

$$c_1(K_S) = \varphi^* c_1(K_{S_0}).$$

Let $\eta_{E_i}\in H^2(S_0,\mathbb{Z})$ be the cohomology class of the exceptional divisor $E_i\subset S_0=\tilde{\mathbb{P}}^2_{p_1,\ldots,p_6}$, and set $\mu_i=\varphi^*\eta_{E_i}$. Since K_S is negative, it clearly cannot have any global sections, so $h^{2,0}(S)=h^{0,2}(S)=0$ and the classes $\mu_i\in H^2(S,\mathbb{Z})$ are necessarily of type $(1,1)$. By the Lefschetz $(1,1)$ theorem, there exists a holomorphic line bundle $L_i \to S$ with $c_1(L_i) = \mu_i$. Since intersection numbers are topologically invariant,

$$L_i \cdot L_i = -1, \qquad L_i \cdot L_j = 0 \qquad (i \neq j)$$
$$L_i \cdot K_S = E_i \cdot K_{S_0} = -1.$$

Applying the Riemann-Roch

$$\chi(L_i) = \frac{L_i \cdot L_i - L_i \cdot K_S}{2} + \chi(\mathcal{O}_S) = \chi(\mathcal{O}_S).$$

Now, as remarked, $h^{2,0}(S) = 0$; by the Lefschetz hyperplane theorem

$$H^1(S, \mathbb{Z}) \cong H^1(\mathbb{P}^2, \mathbb{Z}) = 0,$$

so $h^{1,0}(S) = 0$ and consequently $\chi(\mathcal{O}_S) = 1$. Moreover, by Kodaira-Serre duality $h^2(L_i) = h^0(K_S - L_i)$; but

$$\deg(K_S - L_i)|_{S \cap H} = K_S \cdot H_S - L_i \cdot H_S = K_{S_0} \cdot H_{S_0} - E_i \cdot H_{S_0} = -3 - 1 = -4,$$

so $K_S - L_i$ cannot have any global sections. Thus

$$h^0(L_i) \geqslant 1,$$

so L_i has a nonzero global section, and μ_i is the cohomology class of an effective divisor D_i.

Since $D_i \cdot H_S = 1$, D_i is a line on S in $\mathbb{P}^3$. Thus D_i is a smooth rational curve on S with self-intersection -1 and so can be blown down. Moreover, since $D_i \cdot D_j = 0$, the lines D_i on S are disjoint, so that the image $\pi_i(D_j)$ of D_j under the blowing-down π_i of D_i is again a smooth rational curve of self-intersection -1. Thus we can blow down all six divisors D_i in turn; let $\tilde{S}$ be the surface obtained by blowing them down. We observe first that the Betti numbers of $\tilde{S}$ are

$$b^0(\tilde{S}) = b^4(\tilde{S}) = 1,$$
$$b^1(\tilde{S}) = b^3(\tilde{S}) = 0,$$
$$b^2(\tilde{S}) = b^2(S) - 6 = 7 - 6 = 1.$$

Note also that if $\pi : S \to \tilde{S}$ is the blowing-down map, then

$$K_S = \pi^* K_{\tilde{S}} + D_1 + \cdots + D_6,$$

and since K_S is negative, we deduce that for any curve $D \subset \tilde{S}$,

$$D \cdot K_{\tilde{S}} = \pi^* D \cdot (K_S - D_1 - \cdots - D_6)$$
$$\leqslant \pi^* D \cdot K_S < 0;$$

so the bundle $K_{\tilde{S}}$ is certainly not positive. Consequently our argument that S is $\mathbb{P}^2$ blown up six times will be complete once we prove the

Lemma. *If* M *is an algebraic surface with the same Betti numbers as* $\mathbb{P}^2$ *and* K_M *is not positive, then* $M \cong \mathbb{P}^2$.

Proof. Since $b_1(M)=0$, we have $\text{Pic}(M)\cong H^2(M,\mathbb{Z})\cong\mathbb{Z}$. Since M is algebraic, there exists a positive line bundle L' on M; let L be the generator of $\text{Pic}(M)$ such that $L'=L^n$ for some $n>0$. (Note that L' positive implies that L is positive). By hypothesis, $H^2(M,\mathbb{C})=\mathbb{C}$ and since $H^{1,1}(M,\mathbb{C})=\mathbb{C}$, we have $h^{2,0}(M)=0$; $b^1(M)=0$ implies $h^{1,0}(M)=0$ and hence $\chi(\mathcal{O}_M)=1$. The topological Euler characteristic $\chi(M)=3$, and so by Noether's formula

$$1=\chi(\mathcal{O})=\frac{K_M\cdot K_M+\chi(M)}{12}\Rightarrow K_M\cdot K_M=9.$$

Since $c_1(L)$ generates $H^2(M,\mathbb{Z})$, by Poincaré duality

$$L\cdot L=(c_1(L)\cup c_1(L))[M]=\pm 1,$$

and since L^k is effective for $k\gg 0$ and L is positive,

$$L\cdot L^k=k(L\cdot L)>0\Rightarrow L\cdot L=1.$$

Thus, if we write $K_M=L^m$, m must be negative since K_M is not positive, and so

$$9=K_M\cdot K_M=m^2(L\cdot L)\Rightarrow m=-3,$$

i.e., $K_M=L^{-3}$. Apply Riemann-Roch for L:

$$h^0(L)-h^1(L)+h^2(L)=1+\frac{L\cdot L-K\cdot L}{2}=1+\frac{1-(-3)}{2}=3.$$

But $h^2(L)=h^0(K-L)=h^0(L^{-4})=0$ since L^{-4} is negative. Also, by the Kodaira vanishing theorem,

$$h^1(L)=h^1(K+4L)=0,$$

since L^4 is positive. Consequently

$$h^0(L)=3.$$

Now if $D\in|L|$ is any divisor in the linear system $|L|$, D must be irreducible: if $D=D_1+D_2$ where $D_1,D_2>0$, we would obtain

$$1=L\cdot L=L\cdot D_1+L\cdot D_2$$

Moreover, D must be a smooth curve: if $p\in D$ is a singular point, since $\dim|L|=2$ we can find $D'\neq D\in|L|$ such that $p\in D'$; we would then have

$$1=L\cdot L=D\cdot D'>1.$$

The genus of the curve D is given by the adjunction formula:

$$\pi(D)=\frac{D\cdot D+D\cdot K}{2}+1=\frac{1-3}{2}+1=0,$$

i.e., $D\cong\mathbb{P}^1$. The restriction $L|_D$ is then the hyperplane (i.e., point) bundle $H_{\mathbb{P}^1}$; and from the cohomology of the exact sequence

$$0\to\mathcal{O}_M\to\mathcal{O}_M(L)\to\mathcal{O}_{\mathbb{P}^1}(H_{\mathbb{P}^1})\to 0$$

and the fact that $H^1(M,\mathcal{O}_M)=0$, it follows that

$$H^0(M,\mathcal{O}_M(L)) \to H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(H_{\mathbb{P}^1})) \to 0.$$

$H_{\mathbb{P}^1}$ is very ample on $\mathbb{P}^1$ so the linear system $|L|$ separates points on each curve $D\in|L|$. But since $\dim|L|=2$, we see that for every two points $p,q\in M$ we can find a curve $D\in|L|$ passing through p and q. Thus the linear system $|L|$ has no base points, and the map

$$\iota_L: M\to\mathbb{P}^2$$

separates points; it follows that ι_L is surjective, and hence is an isomorphism. Q.E.D.

We have now shown that every smooth cubic surface $S\subset\mathbb{P}^3$ is of the form $\tilde{\mathbb{P}}^2_{p_1,\ldots,p_6}$. Suppose that three of the points p_i lay on a line $L\subset\mathbb{P}^2$. Then the proper transform $\tilde{L}$ of L in S would be a smooth rational curve of self-intersection $1\cdot1-3=-2$, and by the adjunction formula,

$$0=\pi(\tilde{L})=\frac{\tilde{L}\cdot\tilde{L}+K_S\cdot\tilde{L}}{2}+1,$$

i.e., $K_S\cdot\tilde{L}=0$. But the canonical bundle of S is negative, a contradiction. Similarly, suppose that the six points p_i lay on a conic curve $C\subset\mathbb{P}^2$. By the above C would have to be smooth, and so its proper transform $\tilde{C}$ in S would again be a smooth rational curve with self-intersection $2\cdot2-6=-2$; the same argument shows this cannot happen. Thus if $S\cong\tilde{\mathbb{P}}^2_{p_1,\ldots,p_6}$, the points p_i necessarily satisfy the conditions 1 and 2 of p. 480; thus we see that

Every smooth cubic surface $S\subset\mathbb{P}^3$ may be obtained by blowing up $\mathbb{P}^2$ at six points $p_1,\ldots,p_6$, no three collinear and not all six on a conic, and embedding the blow-up in $\mathbb{P}^3$ by the proper transform of the linear system of cubics passing through the points p_i.

In particular, we see that our discussion of the lines on the surface constructed before applies to all smooth cubics.

As we will see in the following sections, the quadric and cubic surfaces are the only smooth hypersurfaces in $\mathbb{P}^3$ that may be obtained from $\mathbb{P}^2$ by a series of blow-ups and blow downs.

2. RATIONAL MAPS

Rational and Birational Maps

One of the basic geometric operations on algebraic varieties $V\subset\mathbb{P}^n$ is the projection

$$\pi_p: V \longrightarrow \mathbb{P}^{n-1}$$

of a variety V from a point $p \in \mathbb{P}^n$ not lying on V to a hyperplane. In Chapter 2 we saw that, if V is a curve, the map π_p is well-defined even in case p lies on V: π_p, defined a priori only on $V-\{p\}$, may be extended by mapping p to the image in $\mathbb{P}^{n-1}$ of the tangent line to V at p. In general, however, if V has dimension greater than one and $p \in V$, the map π_p is not well-defined at, nor can it be extended over, the point p. This is not hard to see: for any point $q \in \mathbb{P}^{n-1} \cap T_p(V)$ in the image of the tangent plane to V at p there is a sequence $\{q_i\}$ of points on $V-\{p\}$ tending to p, such that $\pi_p(q_i)$ tends to q. Despite the fact that it is not everywhere defined, however, π_p is a natural geometric operation—as we have already had occasion to see in the previous section—and it is recognized as such in algebraic geometry. π_p is an example of a large class of transformations called *rational maps*, which we will now discuss.

We begin with a definition.

DEFINITION. A *rational* (or *meromorphic*) map of a complex manifold M to projective space $\mathbb{P}^n$ is a map

$$f: z \longrightarrow [1, f_1(z), \dots, f_n(z)]$$

given by n global meromorphic functions on M. A rational map $f: M \to N$ to the algebraic variety $N \subset \mathbb{P}^n$ is a rational map $f: M \to \mathbb{P}^n$ whose image lies in N.

One difficulty in understanding rational maps $f: M \to \mathbb{P}^n$ is the fact that they are not, strictly speaking, maps: they need not be defined on all of M. Let us first see how this occurs.

As we saw in several contexts in the chapter on curves, any collection $f_1, \dots, f_n$ of meromorphic functions on a Riemann surface S serves to define a holomorphic map

$$f: z \mapsto [1, f_1(z), \dots, f_n(z)]$$

from S to $\mathbb{P}^n$: while f is defined a priori only away from the poles of the functions f_i, at any point $p = (z=0) \in S$ we may set

$$m = \max\{-\operatorname{ord}_p(f_i)\}_i,$$

and the map

$$\tilde{f}: z \mapsto [z^m, z^m f_1(z), \dots, z^m f_n(z)]$$

extends f over p. The fact that we are using here to extend f is simply that the point p is a divisor on the Riemann surface S; i.e., it is defined by a single function z, and any function vanishing at p must be divisible by z. This, of course, fails in higher codimension, and so we may expect that a general rational map will not be everywhere defined. The simplest case is the rational map

$$f: \mathbb{C}^2 \longrightarrow \mathbb{P}^1$$

given by the single meromorphic function $f(x,y)=y/x$, i.e., by

$$f(x,y)=\left[1,\frac{y}{x}\right]=[x,y]:$$

f is well-defined and holomorphic away from the origin $(0,0)\in\mathbb{C}^2$, but cannot be extended to a map on all of $\mathbb{C}^2$.

Another way to represent a rational map $f: M\to\mathbb{P}^n$ is by an $(n+1)$-tuple of holomorphic functions: if f is given by meromorphic functions $f_1,\ldots,f_n$, write each of the functions locally as

$$f_i=\frac{g_i}{h_i}$$

with h_i, g_i holomorphic and relatively prime; let h_0 be the least common multiple of the functions h_i. Then f may be given locally by

$$f: z\mapsto[1,f_1(z),\ldots,f_n(z)]=[h_0(z),f_1(z)h_0(z),\ldots,f_n(z)h_0(z)];$$

of course the functions $\tilde{f}_0=h_0$ and $\tilde{f}_i=h_0f_i$ are holomorphic, and f will be well-defined away from their common zero locus $\bigcap(\tilde{f}_i)$.

Note that the functions $\tilde{f}_i$ have no common factors: if k is any irreducible function dividing h_0 exactly m times, then k^m divides h_i for some i. Since k cannot then divide g_i, it follows that k cannot divide $f_i\cdot h_0=g_i\cdot h_0/h_i$. Thus no function vanishing at p can divide all the functions h_0, h_0f_i. It follows that the locus $\bigcap(\tilde{f}_i)$ contains no divisors, i.e., that *a rational map* f *is defined away from a subvariety of codimension* 2 *or more.* Conversely, if $V\subset M$ is any analytic subvariety of codimension at least 2, $f: M-V\to\mathbb{P}^n$ a holomorphic map, then by the Levi theorem from Section 2 of Chapter 3 the pullback to $M-V$ of the Euclidean coordinate functions $x_i=X_i/X_0$, $i=1,\ldots,n$, on $\mathbb{P}^n$ extend to meromorphic functions f_i on M; the map $f=[1,f_1(z),\ldots,f_n(z)]$ is thus rational. This affords a second point of view on rational maps, namely

A rational map

$$f: M\longrightarrow N$$

from the complex manifold M *to the algebraic variety* N *is given by a holomorphic map*

$$f: M-V\longrightarrow N$$

defined on the complement of a subvariety V *of codimension* 2 *or more in* M.

Next, we would like to relate rational maps to $\mathbb{P}^n$ to linear systems of divisors and sections of line bundles, as we have done with holomorphic maps. Let $L\to M$ be a line bundle and $\sigma_0,\ldots,\sigma_n\in H^0(M,\mathcal{O}(L))$ a collection of linearly independent global holomorphic sections of L. Then the meromorphic functions

$$f_i = \frac{\sigma_i}{\sigma_0}$$

determine a rational map

$$f: M \longrightarrow \mathbb{P}^{n^*}.$$

In terms of divisors, suppose $|D_\lambda|_{\lambda \in \mathbb{P}^n}$ is a linear system on M. Let E be the fixed component of $\{D_\lambda\}$—that is, the largest effective divisor such that $D_\lambda - E > 0$ for every λ—so that the divisors $\{D'_\lambda = D_\lambda - E\}$ form a linear system with base locus of codimension at least 2. Then we may define a rational map

$$f: M \longrightarrow \mathbb{P}^{n^*}$$

by setting

$$f(p) = \{\lambda: D'_\lambda \ni p\} \in \mathbb{P}^{n^*};$$

this is well-defined away from the base locus of $\{D'_\lambda\}$. Of course, if $\{D_\lambda\}$ is the linear system

$$D_\lambda = (\lambda_0\sigma_0 + \cdots + \lambda_n\sigma_n)$$

associated to the vector space $\{\sigma_0, \ldots, \sigma_n\}$ of sections of the line bundle L above, then the maps f given by the system $\{D_\lambda\}$ and the meromorphic functions σ_i/σ_0 are the same.

Note that while any linear system gives in this way a rational map, we have an exact correspondence

$$\left\{\begin{array}{l}\text{linear systems of divisors}\\ \text{on } M \text{ with base locus}\\ \text{of codimension} \geqslant 2\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{rational maps}\\ f\colon M \to \mathbb{P}^n, \text{ up to}\\ \text{automorphisms of } \mathbb{P}^n\end{array}\right\}$$

From yet another viewpoint, we may consider a rational map $f: M \to \mathbb{P}^n$ as a subvariety of $M \times \mathbb{P}^n$. Explicitly, we define the *graph* $\Gamma_f \subset M \times \mathbb{P}^n$ of f to be the closure in $\mathbb{P}^n$ of the graph

$$\{(p, X): f(p) = X\}$$

of f where defined. Note that this is an analytic subvariety: if f is given locally by

$$f: p \mapsto [g_0(p), \ldots, g_n(p)],$$

where $g_0, \ldots, g_n$ are holomorphic functions with no common factor, then Γ_f will be contained in the variety

$$\Gamma_0 = (g_i(p) \cdot X_j - g_j(p) X_i = 0)$$

and will agree with Γ_0 over the domain of definition M_0 of f in M. Γ_f is thus the irreducible component of Γ_0 containing $\Gamma_0 \cap M_0 \times \mathbb{P}^n$. Conversely, suppose $\Gamma \subset M \times \mathbb{P}^n$ is any k-dimensional analytic subvariety having intersection number

$$^{\#}(\Gamma, \{p\} \times \mathbb{P}^n) = 1$$

with the fibers of $M \times \mathbb{P}^n$ over M. For each $p \in M$, Γ will either meet the fiber $\{p\} \times \mathbb{P}^n$ transversely in a single point $(p, f(p))$—in which case by the implicit function theorem Γ is the graph of a holomorphic map near p —or have at least a curve in common with it. The former is clearly generically the case. Indeed, the locus V of points $p \in M$ where the latter case occurs must have codimension at least 2: if V were of dimension $k-1$, the inverse image of V in Γ would have dimension k, and so would form a component of the irreducible variety Γ. Γ thus defines a rational map $f: M \to \mathbb{P}^n$. We have then:

A rational map $f: M \to \mathbb{P}^n$ *is given by an irreducible* k*-dimensional subvariety of* $M \times \mathbb{P}^n$ *having intersection number* 1 *with the fibers* $\{p\} \times \mathbb{P}^n$ *of* $M \times \mathbb{P}^n$ *over* M.

One point that emerges readily from this description is that for M compact, the image of M under a rational map $f: M \to \mathbb{P}^n$—that is, the closure of the image of f where defined—is an algebraic subvariety of $\mathbb{P}^n$. This follows from the proper mapping theorem, once we observe that the image of the closure of the graph Γ_f of f in $M \times \mathbb{P}^n$ is indeed just the closure of the image of f.

Birational Maps. We say that a rational map $f: M \to N$ is birational if there exists a rational map $g: N \to M$ such that $f \circ g$ is the identity as a rational map; two algebraic varieties are said to be *birationally isomorphic*, or simply *birational*, if there exists a birational map between them. In particular, a variety is called *rational* if it is birational to $\mathbb{P}^n$, i.e., if there exist n meromorphic functions on it providing local coordinates almost everywhere. Note that a rational map $f: M \to N$ is birational if and only if it is generically one-to-one: if, for generic $p \in N$, $f^{-1}(p)$ is a single point, then the graph $\Gamma_f \subset M \times N$ of f has intersection number 1 with the fibers $M \times \{p\}$, and so defines an inverse rational map.

Birational isomorphism represents an important intermediate notion of equivalence among varieties. Birational varieties are alike in more ways than they differ; to the classical geometers they were different manifestations of the same variety. This point of view is immediately clear to an algebraist, in whose terms the local rings of functions around points $p \in M$, $q \in N$ on two varieties M and N are isomorphic as local rings if and only if there is a birational map $f: M \to N$ taking p to q and biregular around p. It will take us somewhat longer to appreciate the close relationship between birational manifolds. To start, note the following:

Let $f: M \to N$ be a rational map, defined and holomorphic on the complement $M - V$ of a subvariety V of codimension $\geqslant 2$. If φ is any global holomorphic p-form on N, then by Hartogs' theorem the pullback $f^*\varphi$ on $M - V$ extends uniquely to a p-form on all of M; thus we have a map

$$f^*: H^0(N, \Omega_N^p) \longrightarrow H^0(M, \Omega_M^p)$$

for each p. More generally, if $E_M \to M$ is any contravariant tensor bundle, the natural map f^* from sections of E_N over N to sections of E_M over $M - V$ gives a map

$$f^*: H^0(N, \mathcal{O}(E_N)) \longrightarrow H^0(M, \mathcal{O}(E_M)).$$

If f is a birational map, of course, then all the functions f^* are isomorphisms; *thus the space of sections of any contravariant holomorphic tensor bundle is a birational invariant*; in particular, the Hodge numbers $h^{p,0}(M)$ are. Several of these invariants have been given names:

1. The number $h^{1,0}(S)$ of holomorphic 1-forms on a Riemann surface is its genus $g(S)$. In general, the number $h^{n,0}(M)$ of holomorphic forms of top degree on a compact complex n-manifold M is called the *geometric genus* of M and denoted $p_g(M)$.

2. An alternative generalization of the notion of genus is the number

$$p_a(M) = h^{n,0}(M) - h^{n-1,0}(M) + \cdots + (-1)^{n-1} h^{1,0}(M),$$

called the *arithmetic genus* of M. Using $h^{q,0}(M) = h^{0,q}(M)$ we can also write

$$p_a(M) = (-1)^n (\chi(\mathcal{O}_M) - 1).$$

3. The number $h^{1,0}(M)$ of holomorphic 1-forms on a compact complex manifold M is often denoted $q(M)$ and called the *irregularity* of M. If M is Kähler, of course, the irregularity is simply half the first Betti number.

4. Of interest also are the dimensions

$$P_n(M) = h^0(M, \mathcal{O}(K_M^n)),$$

of the spaces of sections of the nth powers of the canonical bundle, called collectively the *plurigenera* of M.

5. The fundamental group $\pi_1(M)$ of an algebraic variety is also a birational invariant: suppose

$$f: M \longrightarrow N$$

is a birational map, defined away from the subvariety $U \subset M$ and one-to-one away from the subvariety $V \subset M$. If γ is any loop on M, then we can find a loop γ' in M homotopic to γ and disjoint from U; and the class of $f(\gamma')$ on N will be independent of the choice of γ': since U has real codimension at least 4, if $\gamma' - \gamma''$ is the boundary of a disc in M, then it is the boundary of a disc in $M - U$. We thus obtain maps

$$f_*: \pi_1(M) \longrightarrow \pi_1(M)$$

and

$$f_*^{-1}: \pi_1(N) \longrightarrow \pi_1(M)$$

inverse to one another; and so $\pi_1(M) \cong \pi_1(N)$.

Another way in which a birational map carries structure is this: if $f: M \to N$ is a birational map, we may define two maps

$$f_*: \operatorname{Div}(M) \longrightarrow \operatorname{Div}(N),$$

called the *proper transform* and the *total transform*. The proper transform of a divisor D in M is defined to be the closure in N of the image of D under f where defined, while the total transform is defined to be the image in N of the inverse image of D in the graph $\Gamma \subset M \times N$ of f. The reader may verify that the total transform map preserves linear equivalence while the proper transform map does not.

Examples of Rational and Birational Maps

1. Any holomorphic map $M \to \mathbb{P}^n$ is trivially rational.

2. If $\tilde{M} \xrightarrow{\pi} M$ is the blow-up of an algebraic variety M at a collection of points $\{p_i\}$, then the inverse map

$$M - \{p_i\} \xrightarrow{\pi^{-1}} \tilde{M}$$

is clearly rational, so π is a birational isomorphism. A holomorphic map $f: \tilde{M} \to \mathbb{P}^n$ thus gives a rational map from M to $\mathbb{P}^n$; in fact—as we shall prove later in this section in case M is a surface—the converse is true: any rational map $f: M \to \mathbb{P}^n$ is induced by a holomorphic map on a (possibly multiple) blow-up $\tilde{M}$ of M.

3. As mentioned at the beginning of this section, the projection map

$$\pi_p: C - \{p\} \longrightarrow \mathbb{P}^{n-1}$$

of a curve $C \subset \mathbb{P}^n$ from a point p on C into a hyperplane in $\mathbb{P}^n$ extends to a holomorphic map on all of C. In general, if $V \subset \mathbb{P}^n$ is any variety, $p \in V$ any point, the projection map π_p of V from p to a hyperplane is a rational map. Indeed, π_p may always be extended to a holomorphic map on the blow-up $\tilde{V}$ of V at p by sending a point $r \in E$ in the exceptional divisor to the point of intersection of $\mathbb{P}^{n-1}$ with the tangent line to V at p corresponding to r.

Note that in case V is a quadric hypersurface the map π_p is a birational isomorphism. We have already seen this in the case of Q a quadric surface in $\mathbb{P}^3$, where the map π_p consists of the blow-up of the point p, followed by the blowing down of the two lines of Q through p.

4. If $\varphi: M \to \mathbb{P}^n$ is any holomorphic map of a k-dimensional manifold to $\mathbb{P}^n$, the associated *Gauss map*

$$\mathcal{G}: M^* \longrightarrow G(k+1, n+1),$$

sending any smooth point of the image to its tangent plane in $\mathbb{P}^n$, is a rational map: explicitly, if φ is given locally by

$$\varphi(z) = [\varphi_0(z), \ldots, \varphi_n(z)],$$

then the map $\mathcal{G}$ is given in terms of the Plücker embedding

$$G(k+1, n+1) \longrightarrow \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$$

by the minors of the Jacobian matrix $\partial\varphi_i/\partial z_\alpha$ of φ.

5. If $V \subset \mathbb{P}^n$ is any variety, we may define a rational map

$$V^k \longrightarrow G(k, n+1),$$

from the k-fold product of V with itself to the Grassmannian of $(k-1)$-planes in $\mathbb{P}^n$, by

$$(v_1, \ldots, v_k) \longrightarrow \overline{v_1 \cdots v_k}\,.$$

The graph of this map is just the main irreducible component of incidence correspondence $I \subset V^k \times G(k, n+1)$ given by

$$I = \{(p_1, \ldots, p_k; \Lambda): \ p_i \in \Lambda \quad \text{for all } i\}.$$

6. We previously encountered the two birational maps

$$\mu^{(g)}: \ S^{(g)} \longrightarrow J(S)$$

and

$$\mu^{(g-1)}: \ S^{(g-1)} \longrightarrow \Theta$$

from the g-fold symmetric product of a Riemann surface S to its Jacobian $J(S)$, and from the $(g-1)$st symmetric product of S to the theta-divisor $\Theta \subset J(S)$. The latter map involves both of the two previous examples of rational maps: if $\mathcal{G}: \Theta \to \mathbb{P}^{g-1}$ is the Gauss map, as defined in Section 6 of Chapter 2, then the composition

$$S^{g-1} \xrightarrow{\ \pi\ } S^{(g-1)} \xrightarrow{\ \mu^{(g-1)}\ } \Theta \xrightarrow{\ \mathcal{G}\ } \mathbb{P}^{g-1*}$$

(π the standard quotient map) is just the map defined in example 4 above, applied to the canonical curve $S \subset \mathbb{P}^{g-1}$.

7. A birational map of the projective plane $\mathbb{P}^2$ to itself is called a *Cremona transformation*. One basic example of a Cremona transformation may be given as follows: let a, b, c be noncollinear points of $\mathbb{P}^2$, and $\tilde{\mathbb{P}}^2$ the blow-up of $\mathbb{P}^2$ at these three points. The proper transforms $\tilde{L}_{ab}, \tilde{L}_{bc}$, and $\tilde{L}_{ac}$

of the lines $\overline{ab}$, $\overline{bc}$, and $\overline{ac}$ are then disjoint rational curves of self-intersection -1 in $\tilde{\mathbb{P}}^2$ and may all be blown down. (See Figure 2.) The resulting surface S, by the last lemma of the previous section, is isomorphic to $\mathbb{P}^2$; so we have given, up to an automorphism of $\mathbb{P}^2$, a birational map $\varphi_{a,b,c}$ of $\mathbb{P}^2$ to itself. In terms of linear series, the map $\varphi_{a,b,c}$ is given by the linear system $|2H|_{a+b+c}$ of conics in $\mathbb{P}^2$ passing through the three points a, b, and c; in homogeneous coordinates, if

$$a=[1,0,0], \qquad b=[0,1,0], \qquad c=[0,0,1],$$

the map φ_{abc} is

$$\varphi_{abc}: \ [X_0,X_1,X_2] \longrightarrow [X_1X_2, X_0X_2, X_0X_1].$$

Defined as it is by a linear system of conics, $\varphi_{a,b,c}$ is called a *quadratic transformation* of the plane. Note that a general line L through the point a is carried over into a line through the image point of L_{bc}, while a line L not containing any of the points a,b,c is carried over into a conic passing through all three image points $d=\varphi(L_{ab})$, $e=\varphi(L_{bc})$ and $f=\varphi(L_{ac})$. This reflects the fact that $\varphi_{edf}\circ\varphi_{abc}$ is holomorphic.

Another Cremona transformation has been implicitly mentioned in the last section. Let $a_1,\ldots,a_6$ be six points in $\mathbb{P}^2$ in general position with

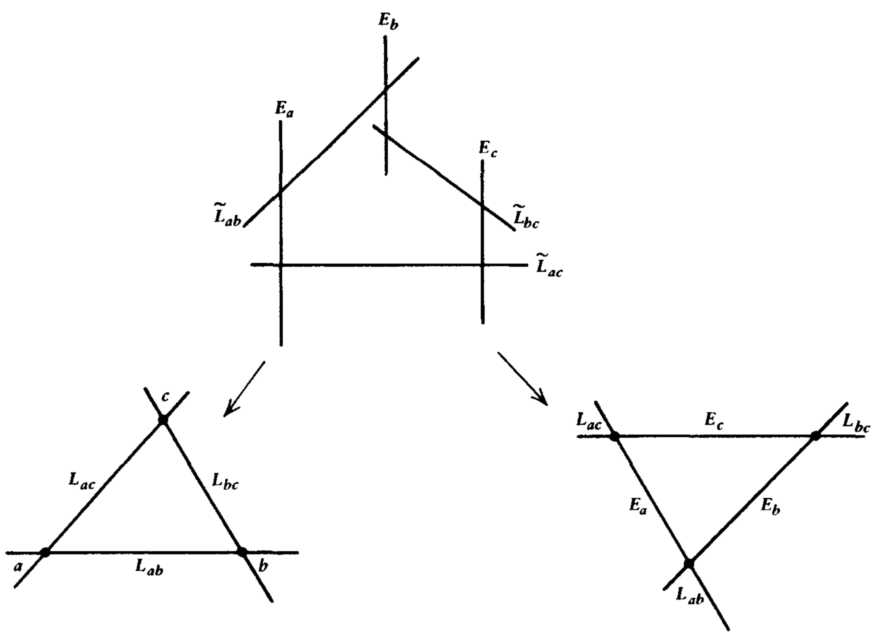


Figure 2

respect to lines and conics. Then in the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at the points a_i the proper transforms of the conics $G_i\{a_j\}_{j\neq i}$ in $\mathbb{P}^2$ passing through five of the six points a_i are disjoint rational curves of self-intersection -1, and so in turn may be blown down. The resulting surface is $\mathbb{P}^2$; thus we have another Cremona transformation ψ. The reader may verify that the birational map ψ is given by the linear system of quintic curves in $\mathbb{P}^2$ having double points at each of the points a_i. Of course, blowing down any of the 72 sets of six disjoint lines on the cubic $\tilde{\mathbb{P}}^2$ yields a Cremona transformation.

It is a classical result that the group of Cremona transformations is generated by the set of quadratic transformations φ_{abc}. An interesting exercise is to check this in the case of the map ψ above by expressing ψ as a composition of quadratic transformations; three will be needed.

We will return later in this section to prove a structure theorem for birational maps on surfaces; before we do that, however, we need to know some more about curves on surfaces.

Curves on an Algebraic Surface

We begin our discussion of curves on surfaces by proving a fact mentioned in Chapter 2: that *if* $C\subset S$ *is any irreducible curve on an algebraic surface, then there exists a compact Riemann surface* $\tilde{C}$ *and a holomorphic map*

$$\psi:\ \tilde{C}\longrightarrow C\subset S$$

that is one-to-one over smooth points of C. The Riemann surface $\tilde{C}$ together with the map ψ is called a *desingularization* of C.

To start, observe that the problem is a local one: we want to complete the (possibly) open Riemann surface $C^*=C-C_s$ to a compact one, and we may do this over one singular point at a time. Indeed, since the local irreducible components of C^* around a singular point $p\in C$ are all disjoint, we may proceed by completing one local component of C^* at a time. Explicitly, suppose $p_1,\dots,p_m$ are the singular points of C, and $C_1^i,\dots,C_{a_i}^i$ the irreducible components of C at p_i. If we can find maps

$$\psi_{i,\alpha}:\ \Delta_\varepsilon\longrightarrow C_\alpha^i$$

one-to-one away from $0\in\Delta_\varepsilon$ and $p_i\in C_\alpha^i$, then we take our desingularization $\tilde{C}$ to be the union

$$C^*\cup_{\psi_{1,1}}\Delta\cup_{\psi_{1,2}}\Delta\cup\cdots\cup_{\psi_{1,a_1}}\Delta\cup_{\psi_{2,1}}\Delta\cup\cdots\cup_{\psi_{m,a_m}}\Delta$$

of the smooth locus C^* of C with the discs Δ via the maps $\{\psi_{i\alpha}\}$.

Now, let p be a singular point of C, and suppose C is irreducible in a small neighborhood Δ of p and smooth in $\Delta-\{p\}$. Let z,w be local holomorphic coordinates on S around $p=(0,0)$, and let C be given in Δ by the holomorphic function $f(z,w)$. After a holomorphic change of coordinates we may take f to be a Weierstrass polynomial in w and write

$$f(z,w) = w^k + p_1(z)w^{k-1} + \cdots + p_k(z)$$

with $p_i(0)=0$ for every i. Now for ε small and $0<|z|<\varepsilon$, the polynomial $f(z,w)$ will have k distinct roots $a_r(z)$; the functions $a_r(z)$ will be locally single-valued holomorphic functions of $z\neq 0$, and

$$f(z,w) = \prod_{v=1}^{k} (w - a_r(z)).$$

Geometrically, this means that the projection map $\pi(z,w)\to z$ on the z-plane expresses the inverse image $\pi^{-1}(\Delta_\varepsilon^*)\subset C^*$ of the punctured disc $\Delta_\varepsilon^* = \Delta_\varepsilon - \{0\}$ as a topological covering space of Δ_ε^*.

Analytic continuation of the function $a_r(z)$ around the origin in the z-plane gives a new function element $a_{\sigma(r)}(z)$, where σ is a permutation of $(1,\ldots,k)$—that is, if we lift the path $t\mapsto z\cdot e^{2\pi i t}$ from the z-plane to C, starting at the point $(z,a_r(z))\in C$, we end up at $(z,a_{\sigma(r)}(z))$. Since C is irreducible at p, the covering space $\pi^{-1}(\Delta_\varepsilon^*)\to\Delta_\varepsilon^*$ is connected and the permutations $\{\sigma^r\}$ act transitively; thus σ has order exactly k.

We now construct our local desingularization map

$$\psi\colon \Delta_{\varepsilon'} \longrightarrow C \qquad (\varepsilon'^k = \varepsilon)$$

as follows: Consider the function

$$b(\zeta) = a_1(\zeta^k),$$

defined a priori in a neighborhood of $\zeta = r$. Writing $\zeta = re^{i\theta}$, we see that as θ increases from 0 to $2\pi/k$, ζ^k turns once around the origin and, continuing b,

$$b(e^{2\pi i/k}\zeta) = a_{\sigma(1)}(\zeta^k).$$

Continuing further, we have

$$b(e^{2\pi i\mu/k}\zeta) = a_{\sigma^\mu(1)}(\zeta^k),$$

and so, since σ has order k, the analytic continuation of b around the circle $|\zeta|=r$ agrees with the original function b. Thus $b(\zeta)$ is well-defined in the punctured disc $\Delta_{\varepsilon'}^*$, and, being bounded, it extends to a holomorphic function on $\Delta_{\varepsilon'}$. Now let

$$\psi(\zeta) = (\zeta^k, b(\zeta)).$$

Clearly $f(\zeta^k, a_1(\zeta^k))\equiv 0$, so ψ maps the disc $\Delta_{\varepsilon'}$ into C. Moreover, ψ must be one-to-one: if for some ζ,ζ', we had $\psi(\zeta)=\psi(\zeta')$, it would follow first that $\zeta^k = \zeta'^k$, hence

$$\zeta' = e^{2\pi i\mu/k}\zeta$$

for some $\mu = 1,\ldots,k$; then, since

$$b(\zeta') = a_{\sigma^{\mu}(1)}(\zeta^k) = a_1(\zeta^k) = b(\zeta),$$

we would have

$$\sigma^{\mu}(1) = 1.$$

But since σ acts transitively on $\{1,\ldots,k\}$, this implies that $\mu = k$, i.e., $\zeta = \zeta'$. Consequently the map

$$\psi: \Delta_{\epsilon'} \longrightarrow C$$

restricts to an isomorphism

$$\psi^*: \Delta^*_{\epsilon'} \longrightarrow C - \{p\},$$

and we have our desingularization. (Note that C, being the image of a disc Δ, has a uniquely determined tangent line at p.)

We see that the desingularization of C is unique: if $\pi: \tilde{C} \to C$, $\pi': \tilde{C}' \to C$ are two desingularizations, the isomorphism

$$\tilde{C} - \pi^{-1}(C_s) \longrightarrow C^* \longrightarrow C' - \pi'^{-1}(C_s)$$

extends continuously, hence holomorphically, to an isomorphism of $\tilde{C}$ and $\tilde{C}'$.

The desingularization of algebraic curves gives a second notion of the genus of a singular curve. Recall that for $C \subset S$ an irreducible curve on an algebraic surface S, the *virtual genus* $\pi(C)$ is defined by

$$\pi(C) = \frac{C \cdot C + K_S \cdot C}{2} + 1;$$

by the adjunction formula, $\pi(C)$ is the genus of any smooth curve homologous to C. On the other hand, we may define the *real genus* $g(C)$ of C to be the genus of its desingularization $\tilde{C}$. The focal point of this discussion is a comparison of the two notions of genus; to make this comparison we will use the explicit form of the adjunction formula: the Poincaré residue map.

Let S be a smooth surface, $C \subset S$ a smooth curve given locally in terms of holomorphic coordinates z, w on S by $f(z,w) = 0$. The Poincaré residue map

$$R: \Omega^2_S(C) \longrightarrow \Omega^1_C$$

is given locally by

$$g(z,w)\frac{dz \wedge dw}{f(z,w)} \longmapsto g(z,w)\frac{dz}{(\partial f/\partial w)(z,w)}\bigg|_{f(z,w)=0};$$

as observed in Section 2 of Chapter 1, this is independent of the choice of coordinates z, w. We can extend R to a map from the space of meromorphic 2-forms in S having a pole of order at most one along C to the space of meromorphic 1-forms on C simply by letting $g(z,w)$ be meromorphic in the formula above. For a general meromorphic 2-form ω written as above, the Poincaré residue $R(\omega)$ on C will have the following zeros and poles:

1. At a point p of intersection of the divisor $D=(g)=(\omega)+(f)=K_S+C$ with C, $R(\omega)$ will have order exactly $m_p(D,C)$.
2. At a point where the restriction to C of the 1-form dz vanishes, $R(\omega)$ will have a zero.
3. At a point where the derivative $\partial f/\partial w$ vanishes, $R(\omega)$ will have a pole.

Now, as observed in the case of a curve in $\mathbb{P}^2$, the second and third factors exactly cancel each other out: at a point p with $(\partial f/\partial w)(p)=0$, $(\partial f/\partial z)(p)$ must be nonzero since C is smooth and we have

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial w}dw \equiv 0 \qquad \text{on } C,$$

hence

$$\operatorname{ord}_p\left(\frac{\partial f}{\partial w}\right) = \operatorname{ord}_p(dz)$$

on C. We see then that the canonical divisor $K=(R(\omega))$ on C is just the intersection of C with K_S+C, and consequently

$$2g-2 = \deg K_C = C\cdot(K_S+C),$$

i.e.,

$$g = \frac{C\cdot C + K_S\cdot C}{2} + 1.$$

We would like to see how this goes over when C is singular. In this case, let $\tilde{C}\xrightarrow{\pi} C$ be the desingularization of C. As before, we let $\omega\in\Omega^2_S(C)$ be a meromorphic 2-form with

$$\operatorname{ord}_C(\omega) = -1,$$

written locally as

$$\omega = g(z,w)\frac{dz\wedge dw}{f(z,w)}.$$

Then the pullback

$$\tilde{\omega} = \pi^*R(\omega) = \pi^*\left(g(z,w)\frac{dz}{(\partial f/\partial w)(z,w)}\right)$$

is a well-defined meromorphic 1-form on the inverse image $\pi^{-1}(C^*)$ of the smooth locus of C in $\tilde{C}$; we ask how $\tilde{\omega}$ behaves over the singular points of C. To facilitate the calculation, we first check that we can choose ω so that its divisor (ω) contains no components other than C passing through the singular points of C—that is, in the above expression for ω, g will bé nonzero at all singular points of C. This is easy: if L is any positive line bundle on S, then for k sufficiently large, we can find $\sigma \in H^0(S, \Omega^2(L^k + C))$ and $\tau \in H^0(S, \mathcal{O}(L^k))$ nonzero on C_s; the quotient $\omega = \sigma/\tau$ will then be a meromorphic 2-form of the desired type.

First, we will consider the case of p a point of multiplicity $k > 1$ on C with C locally irreducible at p. Since C is irreducible at p the tangent cone to C at p consists of one line taken k times; let z, w be local coordinates on S centered around p such that the tangent line to C at p is given by $(w = 0)$. Let $f(z, w)$ be a defining function for C around p; we may take f to be a Weierstrass polynomial in w. Write

$$\begin{aligned} f(z,w) &= \sum_i a_i \cdot z^i \cdot w^{k-i} + [k+1] \\ &= \prod_i (\gamma_i z - \delta_i w) + [k+1] \end{aligned}$$

with $(\gamma_i z - \delta_i w)$ the equations of the lines of the tangent cone to C at p; under the assumption that $(w = 0)$ is the only tangent line to C at p,

$$f(z,w) = w^k + [k+1],$$

i.e., the power series expansion of f in z and w contains no terms of degree $\leqslant k$ except w^k. Writing out f as a Weierstrass polynomial,

$$f(z,w) = w^k + p_1(z)w^{k-1} + \cdots + p_k(z),$$

then, *the function* $p_i(z)$ *must vanish to order at least* $i+1$ *at* $z=0$ *for each* i.

Let $\{a_r(z)\}$ be as before the function elements in a punctured disc in the z-plane such that

$$f(z,w) = \sum w^{k-i} p_i(z) = \prod_r (w - a_r(z));$$

set

$$b(\zeta) = a_1(\zeta^k)$$

and let $\pi : \Delta_\varepsilon \to C$ be the desingularization map

$$\pi(\zeta) = (\zeta^k, b(\zeta))$$

constructed above. Then

$$\pi^* dz = d(\zeta^k) = k \cdot \zeta^{k-1} d\zeta$$

and

$$\frac{\partial f}{\partial w} = \sum_r \Big(\prod_{r' \neq r} (w - a_{r'}(z)) \Big)$$

so

$$\begin{aligned}\pi^* \frac{\partial f}{\partial w}(\zeta) &= \sum_r \Big(\prod_{r' \neq r} (b(\zeta) - a_{r'}(\zeta^k)) \Big) \\ &= \sum_r \Big(\prod_{r' \neq r} (b(\zeta) - b(e^{2\pi i r'/k}\zeta)) \Big) \\ &= \prod_{r' \neq 1} (b(\zeta) - b(e^{2\pi i r'/k}\zeta)),\end{aligned}$$

since $\{b(e^{2\pi i r/k}\zeta)\}_r = \{a_r(\zeta^k)\}_r$. We can write

$$\begin{aligned}\prod_r \Big(\pi^* \frac{\partial f}{\partial w}(e^{2\pi i r/k}\zeta) \Big) &= \prod_r \prod_{r' \neq 1} (b(e^{2\pi i r}\zeta) - b(e^{2\pi i (r+r')/k}\zeta)) \\ &= \prod_{r \neq r'} (b(e^{2\pi i r/k}\zeta) - b(e^{2\pi i r'/k}\zeta)).\end{aligned}$$

This last expression is a symmetric polynomial homogeneous of degree $k(k-1)$ in the functions $\{b(e^{2\pi i r/k}\zeta)\}_r = \{a_r(\zeta^k)\}_r$, and so is expressible as a polynomial in the elementary symmetric polynomials $p_i(\zeta^k)$ of $\{a_r(\zeta^k)\}_r$; we have

$$\prod_{r \neq r'} (b(e^{2\pi i r/k}\zeta) - b(e^{2\pi i r'/k}\zeta)) = \sum c_e p_1(\zeta^k)^{e_1} \cdot p_2(\zeta^k)^{e_2} \cdots p_k(\zeta^k)^{e_k}$$

with

$$\sum i \cdot e_i = k(k-1)$$

for each e such that $c_e \neq 0$. But the function p_i vanishes to order at least $i+1$ at 0, and therefore the function $p_i(\zeta^k)$ vanishes there to order $k(i+1)$. Thus the function $\prod_r(\pi^*(\partial f/\partial w)(e^{2\pi i r/k}\zeta))$ vanishes to order at least $k(k-1)(k+1)$ at $\zeta = 0$—and, since the functions $\{\pi^*(\partial f/\partial w)(e^{2\pi i r/k}\zeta)\}_r$ all vanish to the same order at $\zeta = 0$, it follows that $\pi^*(\partial f/\partial w)(\zeta)$ *vanishes to order at least* $(k-1)(k+1)$ *at* $\zeta = 0$.

Summarizing, with ω the 2-form above,

$$\begin{aligned}\tilde{\omega} = \pi^* R(\omega) &= \pi^* g(z,w) \frac{\pi^* dz}{\pi^*(\partial f/\partial z)} \\ &= \pi^* g(z,w) \frac{k \cdot \zeta^{k-1} d\zeta}{\pi^*(\partial f/\partial z)(\zeta)}\end{aligned}$$

extends over $\pi^{-1}(p)$ to a meromorphic 1-form having a pole of order at least $(k-1)(k+1) - (k-1) = k(k-1)$ at $\pi^{-1}(p)$.

The computation in case C may be locally reducible around p is no more difficult. Let z, w, and f be as above, and write

$$f = \prod f_i$$

with f_i irreducible and zero at p; denote by

$$C_i = (f_i = 0)$$

the irreducible components of C around p, and let

$$\pi_i:\ \Delta \to C_i$$

be the corresponding desingularization maps, $\tilde{p}_i = \pi_i^{-1}(p_i)$. Observe that if

$$k_i = \operatorname{mult}_p(C_i),$$

then

$$k = \operatorname{mult}_p(C) = \sum k_i.$$

Consider again the Poincaré residue of ω on $\tilde{C}$ near p_i,

$$\tilde{\omega} = \pi_i^* g(z,w)\cdot \frac{\pi_i^* \, dz}{\pi_i^*(\partial f/\partial w)(z,w)}.$$

We have

$$\frac{\partial f}{\partial w} = \sum_i \left(\frac{\partial f_i}{\partial w} \cdot \prod_{j \neq i} f_j \right)$$

and, since f_i vanishes identically on C_i,

$$\pi_i^* \frac{\partial f}{\partial w} = \pi_i^* \frac{\partial f_i}{\partial w} \cdot \prod_{j \neq i} \pi_i^* f_j.$$

Thus

$$\tilde{\omega} = \pi_i^* g \cdot \frac{1}{\prod\limits_{j \neq i} \pi_i^* f_j} \cdot \frac{\pi_i^* \, dz}{\pi_i^*(\partial f_i/\partial w)}.$$

By our previous computation, the form

$$\frac{\pi_i^* \, dz}{\pi_i^*(\partial f_i/\partial w)}$$

extends over $\tilde{p}_i$ to a meromorphic form having a pole of order at least $k_i(k_i - 1)$ at $\tilde{p}_i$. On the other hand, the function $\prod_{j \neq i} \pi_i^* f_j$ vanishes to order

$$\begin{aligned}
\operatorname{ord}_{\tilde{p}_i}\left(\prod_{j \neq i} \pi_i^* f_j \right) &= m_p\left(C_i, \sum_{j \neq i} C_j \right) \\
&= \sum_{j \neq i} m_p(C_i, C_j) \\
&\geqslant \sum_{j \neq i} k_i k_j.
\end{aligned}$$

We see from this that $\tilde{\omega}$ extends over $\tilde{p}_i$ to a meromorphic 1-form having a pole of order at least

$$\begin{aligned}
k_i(k_i - 1) + \sum_{j \neq i} k_i k_j &= k_i\left(\sum k_j - 1 \right) \\
&= k_i(k-1)
\end{aligned}$$

at $\tilde{p}_i$. The form $\tilde{\omega}$ thus has a total of at least

$$\sum_i k_i(k-1) = k(k-1)$$

poles at the points $\{\tilde{p}_i\}$ lying over p. Summarizing, we have proved that

The form $\tilde{\omega}$ on $\tilde{C}^*$ extends to a meromorphic 1-form on all of $\tilde{C}$, having a total of at least $k(k-1)$ poles at the points of $\tilde{C}$ lying over a point of multiplicity k on C.

Now we count the degree of the meromorphic form $\tilde{\omega}$ on $\tilde{C}$. Away from the inverse images in $\tilde{C}$ of the singular points of C, as in the smooth case $\tilde{\omega}$ will have poles and zeros exactly where $g(z,w)$ does; as before,

$$\deg(\tilde{\omega}|_{\tilde{C}^*}) = C(K_S + C).$$

Letting

$$\delta_p = \sum_{\pi(\tilde{p}_i)=p} -\operatorname{ord}_{\tilde{p}_i}(\tilde{\omega})$$

be the total order of $\tilde{\omega}$ at the points of $\tilde{C}$ lying over p, we have $\delta_p \geqslant k(k-1)$ for p a point of multiplicity k on C, and

$$\begin{aligned} 2g(C) - 2 &= \deg(\tilde{\omega}) \\ &= C\cdot C + C\cdot K_S - \sum_{p\in C_s} \delta_p, \\ g(C) &= \frac{C\cdot C + C\cdot K_S}{2} + 1 - \frac{1}{2}\sum \delta_p. \end{aligned}$$

This gives the basic

Lemma. *If the curve $C \subset S$ has singular points p_i with multiplicities k_i,*

$$g(C) \leqslant \frac{C\cdot C + C\cdot K_S}{2} + 1 - \sum \frac{k_i(k_i-1)}{2}.$$

In particular,

$$g(C) \leqslant \pi(C)$$

with equality holding if and only if C is smooth.

Note, as an important corollary, that

$\pi(C) \geqslant 0$ for any irreducible curve C on a surface S; and if $\pi(C) = 0$, then C is smooth.

This in turn yields a stronger statement of the Castelnuovo-Enriques criterion for blowing down:

An irreducible curve C on an algebraic surface S may be blown down if and only if $C\cdot C$ and $K\cdot C$ are both negative.

Proof.

$$\pi(C) = \frac{C\cdot C + K\cdot C}{2} + 1 \geqslant 0,$$

and if $C\cdot C < 0$ and $K\cdot C < 0$, this implies that

$$C\cdot C = K\cdot C = -1.$$

Then

$$\pi(C) = 0;$$

hence C is smooth and the first version of the blowing-down criterion applies. Q.E.D.

Another important feature of the lemma is that it gives us a means of constructing the desingularization of a curve explicitly, as follows: Suppose C is an irreducible curve lying on the algebraic surface S, $p \in C$ a singular point of multiplicity k. Let $\tilde{S} \xrightarrow{\pi} S$ be the blow-up of S at p, $E = \pi^{-1}(p) \subset \tilde{S}$ the exceptional divisor of the blow-up, and $\tilde{C}$ the proper transform of C in $\tilde{S}$. Then

$$K_{\tilde{S}} \sim \pi^* K_S + E,$$
$$\tilde{C} \sim \pi^* C - kE,$$

so

$$\begin{aligned} \tilde{C}\cdot\tilde{C} &= (\pi^* C\cdot\pi^* C) + k^2(E\cdot E) \\ &= C\cdot C - k^2 \end{aligned}$$

and

$$\begin{aligned} K_{\tilde{S}}\cdot\tilde{C} &= (\pi^* K_S\cdot\pi^* C) - k(E\cdot E) \\ &= K_S\cdot C + k. \end{aligned}$$

Combining, we have

$$\begin{aligned} \pi(\tilde{C}) &= \frac{\tilde{C}\cdot\tilde{C} + K_{\tilde{S}}\cdot\tilde{C}}{2} + 1 \\ &= \frac{C\cdot C + K_S\cdot C}{2} + 1 - \frac{k(k-1)}{2} \\ &= \pi(C) - \frac{k(k-1)}{2}, \end{aligned}$$

i.e., the virtual genus of $\tilde{C}$ will be strictly less than the virtual genus of C. This gives a recipe for the desingularization C. We define a sequence of curves and surfaces $C_i \subset S_i$ by letting C_1 be the proper transform of C in the blow-up $S_1 \xrightarrow{\pi_1} S$ of S at the singular points of C, C_2 the proper transform of C_1 in the blow-up $S_2 \xrightarrow{\pi_2} S_1$ of S_1 at the singular points of C_1, and so forth. If C_i were singular for all i, we would have

$$\pi(C) > \pi(C_1) > \pi(C_2) > \cdots.$$

The lemma tells us, however, that $\pi(C_i) \geqslant 0$ for every i, so this is impossible. Therefore for some i, the proper transform C_i must be smooth. By our construction, the map

$$\pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_i: \; C_i \longrightarrow C$$

is one-to-one away from the singular locus of C, and so $C_i \to C$ is the desingularization of C.

Using this process we can, as promised in Section 5 of Chapter 1, evaluate the effect of any singular point on the genus of a curve. For example, suppose $C \subset S$ has a *tacnode*, that is, a double point whose branches are simply tangent at a point p. (See Figure 3.) If we let $\tilde{S} \to S$ be the blow-up of S at p, $\tilde{C}$ the the proper transform of C in $\tilde{S}$, then the proper transforms of the two branches of C will meet transversely at the point $r \in E$ in the exceptional divisor corresponding to their common tangent line at p. $\tilde{C}$ thus has an ordinary double point at r. If we let $\tilde{S}^{(2)} \to \tilde{S}$ be the blow-up of $\tilde{S}$ at r, then the proper transform $\tilde{C}^{(2)}$ of $\tilde{C}$ in $\tilde{S}^{(2)}$ will be smooth. Now

$$\tilde{C} \cdot \tilde{C} = C \cdot C - 4, \qquad K_{\tilde{S}} \cdot \tilde{C} = K_S \cdot C + 2,$$
$$\tilde{C}^{(2)} \cdot \tilde{C}^{(2)} = \tilde{C} \cdot \tilde{C} - 4, \quad K_{\tilde{S}^{(2)}} \cdot \tilde{C}^{(2)} = K_{\tilde{S}} \cdot \tilde{C} + 2,$$

$\tilde{\tilde{C}}$

$\tilde{C}$

C

Figure 3

so that

$$\pi(\tilde{C}^{(2)}) = \frac{\tilde{C}^{(2)}\cdot\tilde{C}^{(2)} + K_{\tilde{S}}^{(2)}\cdot\tilde{C}^{(2)}}{2} + 1$$
$$= \frac{C\cdot C + K_S\cdot C}{2} + 1 - 2$$
$$= \pi(C) - 2,$$

i.e., *a tacnode drops the genus of a curve by* 2.

Similarly, if p is an ordinary triple point of C—that is, around p C consists of three arcs meeting transversely at p—then the proper transform $\tilde{C}$ of C in the blow-up $\tilde{S}$ of S at p will be smooth over p, and we have

$$\pi(\tilde{C}) = \pi(C) - 3,$$

i.e., *an ordinary triple point drops the genus of a curve by* 3.

One final note, which we will have occasion to use in what follows: if $C \subset S$ is a curve having singular points p_i with multiplicity k_i, C_0 a smooth curve homologous to C, and f: $\tilde{C} \to C$ the desingularization of C, then it has been proved that

$$g(C) \leqslant \pi(C) - \sum \frac{k_i(k_i-1)}{2} = \pi(C_0) - \sum \frac{k_i(k_i-1)}{2},$$

so

$$\chi(\tilde{C}) \geqslant \chi(C_0) + \sum k_i(k_i-1).$$

On the other hand, taking a triangulation of $\tilde{C}$ having all the points $f^{-1}(p_i)$ as vertices, we deduce that

$$\chi(C) = \chi(\tilde{C}) - \sum_i {}^{\#}\{f^{-1}(p_i)\} - 1;$$

since the points of $f^{-1}(p_i)$ correspond exactly to the irreducible components of C around p_i, and there are less than k_i of these, this implies

$$\chi(C) \geqslant \chi(\tilde{C}) - \sum (k_i - 1).$$

Combining, we see that

$$\chi(C) \geqslant \chi(C_0) + \sum (k_i - 1)^2,$$

i.e., *the Euler characteristic of a singular curve on* S *is strictly greater than the Euler characteristic of a smooth curve homologous to it.*

In particular, if C has δ ordinary double points and no other singularities,

$$\chi(C) = \chi(C_0) + \delta.$$

We introduce here a classical formula relating the Euler characteristic of a surface to the structure of a pencil of curves on it. Suppose that M is any algebraic surface, $\{C_\lambda\}$ a pencil of generically irreducible curves on M. Assume in addition that all the curves C_λ are smooth at the base points of the pencil $\{C_\lambda\}$, so that if we blow M up $C_\lambda \cdot C_\lambda = n$ times at the base points of $\{C_\lambda\}$, the proper transforms $\tilde{C}_\lambda$ of the curves C_λ on $\tilde{M}$ form a pencil of disjoint, generically irreducible curves. Consider the map

$$\iota: \tilde{M} \to \mathbb{P}^1$$

given by the pencil $\{\tilde{C}_\lambda\}$. By Bertini's theorem, the generic curve $\tilde{C}_\lambda \cong C_\lambda$ is smooth; let $\tilde{C}_{\lambda_1}, \ldots, \tilde{C}_{\lambda_\mu}$ be the singular elements of the pencil. Then the restricted map

$$\iota: \tilde{M} - \bigcup_i C_{\lambda_i} \to \mathbb{P}^1 - \{\lambda_1, \ldots, \lambda_\mu\}$$

is proper and everywhere nonsingular, so that $\mathrm{M} - \cup \mathrm{C}_{\lambda_i}$ *is a* C^∞ *fiber bundle over* $\mathbb{P}^1 - \{\lambda_1, \ldots, \lambda_\mu\}$. Thus

$$\begin{aligned}\chi\Big(\tilde{M} - \bigcup_i C_{\lambda_i}\Big) &= \chi(\mathbb{P}^1 - \{\lambda_1, \ldots, \lambda_\mu\}) \cdot \chi(C) \\ &= (2-\mu) \cdot \chi(C),\end{aligned}$$

where $\chi(C)$ denotes the Euler characteristic of a generic curve C_λ. Taking a triangulation of $\tilde{M}$ in which the union of the singular fibers appears as a subcomplex, then, we see that

$$\chi(\tilde{M}) = (2-\mu)\chi(C_2) + \sum_{i=1}^{\mu} \chi(C_{\lambda_i})$$

and so

$$\begin{aligned}\chi(M) &= \chi(\tilde{M}) - n \\ &= (2-\mu)\chi(C_\lambda) + \sum \chi(C_{\lambda_i}) - n. \\ &= 2\chi(C) + \sum_\lambda (\chi(C_\lambda) - \chi(C)) - n\end{aligned}$$

If we make the additional assumption that each of the singular curves C_{λ_i} has one double point and no other singularities (a pencil satisfying these conditions is called a *Lefschetz pencil*), then we have

$$\chi(C_{\lambda_i}) = \chi(C) + 1,$$

i.e.,

Proposition. *If* $\{\mathrm{C}_\lambda\}$ *is a Lefschetz pencil of curves on* M, *with self-intersection* n *and containing* μ *singular curves, then*

$$\chi(M) = 2\chi(C) + \mu - n,$$

where $\chi(\mathrm{C})$ *denotes the Euler characteristic of the generic element of the pencil.*

We remark that if $f: M \to B$ is any holomorphic map of a smooth surface S onto a curve B with singular fibers $C_{p_i} = f^{-1}(p_i)$, then the same argument gives

$$\chi(M) = \chi(B - \{p_i\}) \cdot \chi(C) + \sum_i \chi(C_{p_i})$$
$$= \chi(B) \cdot \chi(C) + \sum_i (\chi(C_{p_i}) - \chi(C)),$$

where C is the generic fiber of f; combining this with the inequality $\chi(C_{p_i}) > \chi(C)$ noted above, we see that

If $f: M \to B$ *is any holomorphic map of* M *to a curve* B, C *the generic fiber of* f, *then*

$$\chi(M) \geqslant \chi(B) \cdot \chi(C).$$

The Structure of Birational Maps Between Surfaces

As we mentioned in the introduction to this chapter, we can give a comprehensive picture of birational maps on surfaces. This is the

Theorem. *Any birational map between surfaces may be obtained by a sequence of blow-ups followed by a sequence of blowing-downs; i.e., if* M *and* N *are algebraic surfaces and*

$$f: M \to N$$

a birational map, then there exists a surface $\tilde{M}$ *and maps* π_1, π_2

$$\begin{array}{ccc} & \tilde{M} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ M & \xrightarrow{f} & N \end{array}$$

such that $f = \pi_2 \circ \pi_1^{-1}$, *and* π_1, π_2 *are blowing-up maps.*

Proof. The proof of this theorem consists of two parts: we will show that

1. If $f: M \to N$ is any rational map on the surface M, then there exists a blow-up $\tilde{M} \xrightarrow{\pi_1} M$ such that $\pi_1 \circ f$ is holomorphic; and
2. Any holomorphic birational map $\tilde{M} \xrightarrow{\pi_2} N$ is a sequence of blow-ups.

To prove part 1, we must prove that if $\{D_\lambda\}_{\lambda\mathbb{P}^n}$ is any linear system of divisors on M having only isolated base points, then there exists a blow-up $\tilde{M} \xrightarrow{\pi_1} M$ such that the proper transform in $\tilde{M}$ of the linear system $\{D_\lambda\}$ has no base points. This is fairly easy. Suppose that the point $p \in M$ is a base point of multiplicity k for the linear system $\{D_\lambda\}$ (i.e., p has multiplicity k in the generic D_λ). Let $\tilde{M} \xrightarrow{\pi} M$ be the blow-up of M at p, $E = \pi^{-1}(p)$ the exceptional divisor of the blow-up. Then

$$\tilde{D}_\lambda = \pi^* D_\lambda - kE,$$

and therefore $\tilde{D}_\lambda$ has self-intersection

$$\begin{aligned}\tilde{D}_\lambda \cdot \tilde{D}_\lambda &= (\pi^* D_\lambda \cdot \pi^* D_\lambda) + k^2(E \cdot E) \\ &= D_\lambda \cdot D_\lambda - k^2 < D_\lambda \cdot D_\lambda.\end{aligned}$$

Now define a sequence of blow-ups $M_i \xrightarrow{\pi_i} M_{i-1}$ and linear systems $\{D_\lambda^i\}$ as follows: let $M_1 \xrightarrow{\pi_1} M$ be the blow-up of M in the base points of $\{D_\lambda\}$ and $\{D_\lambda^1\}$ the proper transform in M_1 of the system $\{D_\lambda\}$, $M_2 \xrightarrow{\pi_2} M_1$ the blow-up of M_1 in the base points of $\{D_\lambda^1\}$ and $\{D_\lambda^2\}$ the proper transform of $\{D_\lambda^1\}$ in M^2, etc. If every series $\{D_\lambda^i\}$ had base points, we would have

$$D_\lambda \cdot D_\lambda > D_\lambda^1 \cdot D_\lambda^1 > D_\lambda^2 \cdot D_\lambda^2 > \cdots .$$

But for generic λ, λ', and any i, the divisors D_λ^i and D_λ^i have no common components, so

$$D_\lambda^i \cdot D_\lambda^i \geqslant 0$$

for all i. Thus the linear system $\{D_\lambda^i\}$ is base-point-free for some i, and we have proved part 1 of our theorem.

Part 2 is somewhat deeper. Suppose that $\pi: M \to N$ is a holomorphic birational map, that is, a holomorphic map one-to-one away from a finite collection of points in N. Note that for any point $p \in N$ the inverse image $f^{-1}(p) \subset M$ is connected: if it were not, we could find disjoint relatively compact open sets $U_1, U_2 \subset M$ each containing connected components of $f^{-1}(p)$; being open, they could not map to p nor to any curve through p, so by the proper mapping theorem the image of each would contain a neighborhood of p—contradicting the hypothesis that π is generically one-to-one. The inverse image of any point $p \in N$ is thus either a single point or a connected divisor. We claim now that

If the inverse image $\pi^{-1}(\mathrm{p})$ *of a point* $\mathrm{p} \in \mathrm{N}$ *is a curve* C, *then* C *contains an exceptional curve of the first kind.*

To prove this, we will use the index theorem. Let $C_1, \ldots, C_m$ be the irreducible components of C. We can certainly find a positive divisor E on N that does not pass through p; we have

$$(\pi^* E \cdot \pi^* E) = (E \cdot E) > 0$$

but

$$(\pi^* E \cdot C_i) = 0 \qquad \text{for every } i.$$

It follows from the index theorem that *the intersection pairing is negative definite on the subspace of* $\mathrm{H}^2(\mathrm{M}, \mathbb{Q})$ *spanned by the classes* $\{\mathrm{C}_i\}$.

Let ω be any meromorphic 2-form on N, regular at p. The pullback $\pi^*\omega$ is a meromorphic 2-form on M, vanishing everywhere in C; we can thus write

$$K_M = (\pi^*\omega) = D + \sum a_i C_i$$

with D disjoint from C and $a_i > 0$ for every i. Now from the index theorem

$$\left(\sum a_i C_i \cdot \sum a_i C_i\right) < 0.$$

But since D is disjoint from C,

$$\left(\sum a_i C_i \cdot \sum a_i C_i\right) = \left(\sum a_i C_i \cdot K_M\right),$$

which implies that

$$C_i \cdot K < 0$$

for some i. But by the index theorem again,

$$C_i \cdot C_i < 0,$$

and so by the stronger version of the Castelnuovo-Enriques criterion, C_i is an exceptional curve of the first kind; we have thus proved the claim.

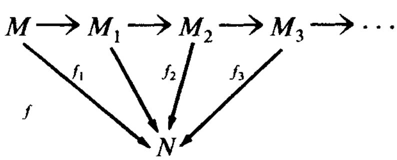
Assertion 2, and hence the main theorem, follow readily. If $f: M \to N$ is holomorphic and birational, but not biholomorphic, then for some $p \in N$, $f^{-1}(p)$ will be a curve and so contain an exceptional curve C_1 of the first kind. Let $M \xrightarrow{\pi_1} M_1$ be the blow-down of C_1; the map

$$f: M_1 - \pi_1(C) \longrightarrow M - C \to N$$

extends continuously, hence holomorphically, to a map

$$f_1: M_1 \to N.$$

Again, if f_1 is not biholomorphic, we can find an exceptional curve of the first kind C_2 in M_1 lying over a point of N; let M_2 be the blow-down of C_2, and so define inductively a sequence of blow-downs $M_i \to M_{i+1}$ and holomorphic birational maps f_i



The Betti numbers satisfy

$$b_2(M) > b_2(M_1) > b_2(M_2) > \cdots$$

and, since $b_2(M)$ is finite, it follows that for some i, $M_i \xrightarrow{f_i} N$ is biholomorphic; the theorem is proved.

3. RATIONAL SURFACES I

Noether's Lemma

The next two sections will be devoted to a discussion of rational surfaces, that is, algebraic surfaces birationally isomorphic to $\mathbb{P}^2$. In this section our goal will be a description of all rational surfaces; as corollaries of the main theorem we will answer two questions left open in Chapter 2.

We start with

Noether's Lemma. *An algebraic surface* S *is rational if and only if it contains an irreducible rational curve* C *with* $\dim|C| \geqslant 1$.

Proof. One direction is clear: if $\pi: S \to \mathbb{P}^2$ is a birational map, then for the generic line $L \subset \mathbb{P}^2$, the pullback $C = \pi^* L$ on S will be such a curve. Conversely, suppose $C \subset S$ is an irreducible rational curve varying in a nontrivial linear system. Choose a pencil $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ containing C in the complete linear system $|C|$. We have seen that if we blow up S sufficiently many times at the base points of the pencil $\{C_\lambda\}$, we obtain a surface $\tilde{S}$ on which the proper transforms $\{\tilde{C}_\lambda\}$ of the curves C_λ form a pencil without base points; clearly the curves $\tilde{C}_\lambda$ will again be rational. Thus we may as well assume from the start that S contains a pencil $\{C_\lambda\}$ of rational curves, not all reducible, having no base points.

Since any point of intersection of two distinct elements $C_\lambda, C_{\lambda'}$ of our pencil is a base point for the whole pencil,

$$C_\lambda \cdot C_\lambda = C_\lambda \cdot C_{\lambda'} = 0.$$

Now suppose a particular curve C_0 in the pencil is reducible; write

$$C_0 = \sum a_\nu C_\nu$$

with C_ν irreducible, $a_\nu > 0$. Since each C_ν is disjoint from $C_\lambda \sim C_0$ for $\lambda \neq 0$,

$$0 = C_0 \cdot C_\nu = \sum_{\nu'} a_{\nu'}(C_\nu \cdot C_{\nu'}).$$

But $C_\nu \cdot C_{\nu'} \geqslant 0$ for $\nu \neq \nu'$, and C_0, being the limiting position of irreducible curves C_λ, must be connected; we deduce that $C_\nu \cdot C_{\nu'} > 0$ for some $\nu' \neq \nu$. It follows that $C_\nu \cdot C_\nu < 0$ for all ν. By the adjunction formula,

$$\begin{aligned} \pi(C_0) &= \frac{C_0 \cdot C_0 + C_0 \cdot K}{2} + 1 = 0 \\ &\Rightarrow C_0 \cdot K = \sum a_\nu C_\nu \cdot K = -2 \\ &\Rightarrow C_{\nu_0} \cdot K < 0 \qquad \text{for some } \nu_0. \end{aligned}$$

Thus, by the strong version of the Castelnuovo-Enriques criterion, C_{ν_0} can be blown down. Let $S \xrightarrow{\pi} \tilde{S}'$ be the blowing-down of C_{ν_0}. Since every C_λ other than C_0 is disjoint from C_{ν_0}, the curves $\pi(C_\lambda)$ form a pencil of rational curves without base points on $\tilde{S}'$; by the same argument, if any curve $\pi(C_\lambda)$ is reducible, $\tilde{S}'$ can again be blown down. Since we can only blow down a surface a finite number of times, we see that after a finite number of steps we obtain a surface $\tilde{S}$ with map $\pi: S \to \tilde{S}$ such that the curves $\pi(C_\lambda)$ form a pencil of irreducible—hence smooth—disjoint rational curves. Such a surface is called a *rational ruled surface*; the proof of Noether's lemma will be completed in the following discussion.

Rational Ruled Surfaces

Let S be a rational ruled surface, $\{C_\lambda\}$ a pencil of disjoint smooth rational curves on S, and consider the map $\iota: S \to \mathbb{P}^1$ given by the pencil $\{C_\lambda\}$. We claim first that $\iota: S \to \mathbb{P}^1$ *is a holomorphic fiber bundle over* $\mathbb{P}^1$ *with fiber* $\mathbb{P}^1$, *i.e., for every* $\lambda_0 \in \mathbb{P}^1$, *there exists a neighborhood* $\mathrm{U} \ni \lambda_0$ *in* $\mathbb{P}^1$ *and an isomorphism* $\varphi: \iota^{-1}(\mathrm{U}) \cong \mathrm{U} \times \mathbb{P}^1$ *fibering over* U. To see this, let $L \to S$ be a positive line bundle, sufficiently positive so that $H^1(S, \mathcal{O}(L - C)) = 0$. Then from the exact cohomology sequence associated to the sequence

$$0 \to \mathcal{O}S(L - C) \to \mathcal{O}S(L) \to \mathcal{O}C_\lambda(L) \to 0$$

we find

$$H^0(S, \mathcal{O}(L)) \to H^0(C_\lambda, \mathcal{O}(L)) \to 0$$

for each λ. Let $L \cdot C_\lambda = n$—so that $L|_{C_\lambda} \cong H^n$, where H is the hyperplane bundle on $C_\lambda \cong \mathbb{P}^1$—and let $\sigma_0, \ldots, \sigma_n$ be global sections of L whose restrictions to some fiber $C_0 = C_{\lambda_0}$ span $H^0(C_0, \mathcal{O}(L)$. Then for λ in some neighborhood U of λ_0 in $\mathbb{P}^1$, the restrictions of $\sigma_0, \ldots, \sigma_n$ to C_λ span $H^0(C_\lambda, \mathcal{O}(L)$, i.e., the map $\iota_\sigma: \iota^{-1}(U) \to \mathbb{P}^n$ given by $[\sigma_0, \ldots, \sigma_n]$ is well-defined and embeds each curve C_λ as a rational normal curve in $\mathbb{P}^n$.

Choose $n-1$ distinct points $p_1, \ldots, p_{n-1}$ on C_0. Since the fibers of the map $\iota: S \to \mathbb{P}^1$ are smooth, we can find holomorphic arcs $\gamma_1, \ldots, \gamma_{n-1}: \Delta \to S$ with γ_i meeting C_0 transversely at p_i; for λ in some open set U' around λ_0, then, the curve C_λ will likewise meet the arc γ_i transversely in a point $p_i(\lambda)$. For each $\lambda \in U'$, let $V(\lambda) \subset \mathbb{P}^n$ be the $(n-2)$-plane spanned by the points $\iota_\sigma(p_i(\lambda))$, $i = 1, \ldots, n-1$. Choose a line $L \subset \mathbb{P}^n$ disjoint from $V(\lambda)$ for all $\lambda \in U'$—restricting U' again if necessary—and let π_λ denote the projection map from $\mathbb{P}^n - V(\lambda)$ onto L. Note that since $p_i(\lambda)$—and hence $\iota_\sigma(p_i(\lambda))$—varies holomorphically with λ, the map π_λ likewise varies holomorphically with λ; in particular the map

$$\pi': \iota^{-1}(U') \to L \cong \mathbb{P}^1$$

given by

$$\pi'|_{C_\lambda} = \pi_\lambda \longrightarrow C_\lambda \xrightarrow{\sim} L$$

is holomorphic. The map

$$\varphi = (\iota, \pi') : \iota^{-1}(U') \xrightarrow{\approx} U' \times \mathbb{P}^1$$

then gives the bundle structure.

In general, if $E \to M$ is a holomorphic vector bundle on a complex manifold M, we define the *associated projective bundle* $\mathbb{P}(E) \to M$ to be the fiber bundle over M whose fiber over any point $x \in M$ is the projective space $\mathbb{P}(E_x)$. If $\{U_\alpha\}$ is an open cover of M, $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^r$ a trivialization of E over U_α for each α, the maps φ_α induce maps $\tilde{\varphi}_\alpha : \mathbb{P}(E)|_{U_\alpha} \to U_\alpha \times \mathbb{P}^{r-1}$, giving $\mathbb{P}(E)$ the structure of a holomorphic $\mathbb{P}^{r-1}$-bundle over M. Note that if $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_r\}$ are the transition functions for the trivializations φ_α of E, then transition functions for $\mathbb{P}(E)$ relative to $\tilde{\varphi}_\alpha$ are given by the composition $\tilde{g}_{\alpha\beta}$ of $g_{\alpha\beta}$ with the standard projection map $\mathrm{GL}_r \to \mathrm{PGL}_r$. In particular, if L is any line bundle over M with transition functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, then $E \otimes L$ is given by transition functions $g'_{\alpha\beta} = h_{\alpha\beta} \cdot g_{\alpha\beta}$; since $\tilde{g}_{\alpha\beta} = \tilde{g}'_{\alpha\beta}$, we see that $\mathbb{P}(E) = \mathbb{P}(E \otimes L)$. Conversely, if E, E' are any two vector bundles over M with $\mathbb{P}(E) \cong \mathbb{P}(E')$, it follows that $E' = E \otimes L$ for some line bundle $L \to M$.

We claim now that any holomorphic $\mathbb{P}^{r-1}$ bundle P over $\mathbb{P}^1$ is of the form $\mathbb{P}(E)$ for some vector bundle $E \to \mathbb{P}^1$ of rank r. To see this, let $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}_r$ be transition functions for P relative to some open cover $\{U_\alpha\}$ of $\mathbb{P}^1$. Assuming $\{U_\alpha\}$ is sufficiently fine, we can find liftings $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_r$ of $\tilde{g}_{\alpha\beta}$ (the group $\mathrm{SL}_r \subset \mathrm{GL}_r$ of matrices with determinant 1 forms an unbranched r-sheeted cover of PGL_r); on $U_\alpha \cap U_\beta \cap U_\gamma$, set

$$h_{\alpha\beta\gamma} = g_{\alpha\beta} \times g_{\beta\gamma} \times g_{\gamma\alpha}.$$

Since $\tilde{h}_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta} \times \tilde{g}_{\beta\gamma} \times \tilde{g}_{\gamma\alpha} = I$, we see that $h_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{C}^*$; i.e., $\{h_{\alpha\beta\gamma}\} \in Z^2(\underline{U}, \mathcal{O}^*)$. But from the exact cohomology sequence of $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 1$ and $H^2(\mathbb{P}^1, \mathcal{O}) = H^3(\mathbb{P}^1, \mathbb{Z}) = 0$, we deduce that $H^2(\mathbb{P}^1, \mathcal{O}^*) = 0$, so we can write

$$h_{\alpha\beta\gamma} = j_{\alpha\beta} \times j_{\beta\gamma} \times j_{\gamma\alpha}$$

for some Čech cochain $\{j_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*\}$. The functions $g_{\alpha\beta} \times j_{\alpha\beta}^{-1}$ then are the transition functions for a vector bundle $E \to \mathbb{P}^1$ with $P = \mathbb{P}(E)$. (Note that this argument works as well for a projective bundle over any Riemann surface.)

Summarizing, we have shown that *any rational ruled surface is of the form* $\mathbb{P}(E)$ *for some holomorphic vector bundle* E *of rank two over* $\mathbb{P}^1$. The following lemma gives a complete description of such vector bundles:

Lemma. *Any holomorphic vector bundle on* $\mathbb{P}^1$ *is decomposable—that is, a direct sum of line bundles.*

Proof. First note that a vector bundle E is decomposable if and only if $E\otimes H^k$ is decomposable for any k. From the exact sequence

$$0\to\mathcal{O}(E\otimes H^{k-1})\to\mathcal{O}(E\otimes H^k)E_x\otimes H_x^k\to 0,$$

we find that $H^1(\mathbb{P}^1,\mathcal{O}(E\otimes H^{k-1}))=0 \Rightarrow H^0(\mathbb{P}^1,\mathcal{O}(E\otimes H^k))\neq 0$; i.e., for $k\gg 0$, $E'=E\otimes H^k$ has a nontrivial global holomorphic section σ. Now suppose σ vanishes at n points on $\mathbb{P}^1$; then, multiplying σ by a meromorphic function on $\mathbb{P}^1$ with poles exactly at the zeros of σ, we obtain another section σ' of E' with σ and σ' everywhere linearly dependent and nowhere both zero. Together, they span a subline bundle L in E' of degree n. Note that by Riemann-Roch for $\mathbb{P}^1, h^0(L)=n+1$, and, since $H^0(\mathbb{P}^1,\mathcal{O}(\mathrm{L}))$ injects into $\mathrm{H}^0(\mathbb{P}^L,\mathcal{O}(E)$, we have $n\leqslant h^0(E)$; thus *no global section of* E *can have more than* $\mathrm{h}^0(\mathrm{E})-1$ *zeros.*

Assume now that rank $E=2$. Let n be the greatest number of zeros of a global section of E, and let σ_0 be a global section of E with n zeros. Let L_1 be the corresponding subline bundle of E and $L_2=E/L_1$ the quotient bundle, we have an exact sequence of bundles

$$0\to L_1\to E\to L_2\to 0.$$

We claim now that $m=\deg L_2\leqslant \deg L_1=n$. Otherwise, let $\tilde{\tau}$ be a section of L_2 vanishing at $m>n$ points $p_1,\dots,p_m\in\mathbb{P}^1$. Since $H^1(\mathbb{P}^1,\mathcal{O}(L_1))=0$,

$$H^0(\mathbb{P}^1,\mathcal{O}(E))\to H^0(\mathbb{P}^1,\mathcal{O}(L_2))\to 0;$$

i.e., $\tilde{\tau}$ is the projection onto L_2 of a section τ of E; since $\tilde{\tau}(p_i)=0, \tau(p_i)\in (L_1)_{p_i}$ for all i. For any collection $q_0,\dots,q_n$ of $n+1$ points in $\mathbb{P}^1$, $\deg(L_1-q_0-\cdots-q_n)=-1$, so

$$H^1(\mathbb{P}^1,\mathcal{O}(L_1-(q_0+\cdots+q_n)))=0$$
$$\Rightarrow H^0(\mathbb{P}^1,\mathcal{O}(L_1-(q_0+\cdots+q_{n-1})))\to\mathbb{C}_{q_n}\to 0.$$

Thus there exist sections of L_1 vanishing at $q_0,\dots,q_{n-1}$ and nonzero at q_n. Let τ_i be the section of L_1 vanishing at $p_1,\dots,\widehat{p_i},\dots,p_{n+1}$ and taking the value $\tau(p_i)$ at p_i; then

$$\tau-\sum\tau_i$$

is a nonzero section of E vanishing at $n+1$ points, contradicting our assumption that no section of E vanishes at more than n points.

Next consider the sequence of bundles

$$0 \to \mathrm{Hom}(L_2, L_1) \to \mathrm{Hom}(L_2, E) \to \mathrm{Hom}(L_2, L_2) \to 0.$$

Since $\deg L_1 \geqslant \deg L_2$, $\deg(\mathrm{Hom}(L_2, L_1)) = \deg L_1 - \deg L_2 \geqslant 0$; so $H^1(\mathbb{P}^1, \mathcal{O}(\mathrm{Hom}(L_2, L_1))) = 0$ and

$$H^0(\mathbb{P}^1, \mathcal{O}(\mathrm{Hom}(L_2, E))) \to H^0(\mathbb{P}^1, \mathcal{O}(\mathrm{Hom}(L_2, L_2))) \to 0.$$

Let $\iota : L_2 \to E$ be a section of $\mathrm{Hom}(L_2, E)$ that maps onto the identity section of $\mathrm{Hom}(L_2, L_2)$. Since ι composed with the projection map $E \to L_2$ is the identity, ι gives an inclusion of bundles $L_2 \to E$, with $\iota(L_2)_x$ disjoint from $(L_1)_x$ for all x; thus

$$E \cong L_1 \oplus L_2.$$

To prove the lemma for bundles of general rank, we use induction on the rank: if rank $E = r$ and the lemma is proved for all bundles of rank $< r$, take again a section σ and corresponding subline bundle L_1 of maximal degree; then

$$E' = \frac{E}{L_1} \cong \bigoplus_{i=2}^{r} L_i.$$

The same argument shows that $\deg L_i \leqslant \deg L_1$ for all i, hence

$$H^1(\mathbb{P}^1, \mathrm{Hom}(E', L_1))) = \bigoplus_{i=2}^{n} H^1(\mathbb{P}^1, \mathrm{Hom}(L_i, L_1))) = 0,$$

so that the exact sequence

$$0 \to L_1 \to E \to E' \to 0$$

again splits. Q.E.D.

By the lemma, any rational ruled surface is of the form

$$\mathbb{P}(E) = \mathbb{P}(L_1 \oplus L_2) = \mathbb{P}((L_1 \otimes L_2^*) \oplus \mathbb{C}_{\mathbb{P}^1}) = \mathbb{P}(H^n \oplus \mathbb{C}_{\mathbb{P}^1})$$

for some $n \geqslant 0$ (here $\mathbb{C}_{\mathbb{P}^1}$ stands for the trivial line bundle over $\mathbb{P}^1$); the bundle $\mathbb{P}(H^n \oplus \mathbb{C})$ is denoted S_n.

Let $E_0 \subset S_n$ be the image of the section $(0,1)$ of $H^n \oplus \mathbb{C}_{\mathbb{P}^1}$; E_0 is called the *zero-section* of S_n. More generally, if σ is any holomorphic section of H^n, let E_σ be the image in S_n of the section $(\sigma, 1)$ of $H^n \oplus \mathbb{C}_{\mathbb{P}^1}$. Clearly E_σ is homologous to E_0, and since for σ a nontrivial section E_σ will meet E_0 exactly n times, the intersection number $E_0 \cdot E_0$ is n. (See Figure 4.)

Consider the section $(\sigma, 0)$ of $H^n \otimes \mathbb{C}_{\mathbb{P}^1}$ where σ is any section of H^n. Away from the zeros of σ, $(\sigma, 0)$ gives a curve in S_n; let E_∞ denote the closure of this curve. (Clearly E_∞ is independent of the choice of section σ, since $(\sigma, 0)$ and $(\sigma', 0)$ have the same image in S_n away from a finite number of points.) More generally, if σ is any meromorphic section of H^n,

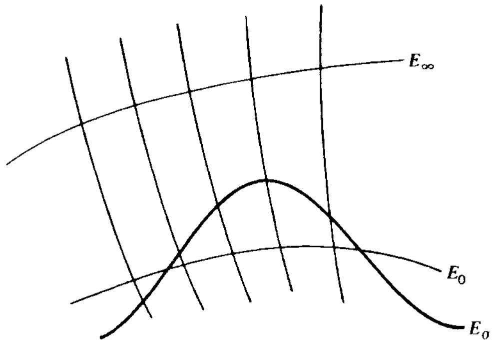


Figure 4

let E_σ denote the closure of the curve given by $(\sigma, 1)$ away from the poles of σ. Letting C be any fiber of the bundle map $S_n \to \mathbb{P}^1$, we have

$$E_0 \cdot E_0 = n$$
$$E_\sigma \cdot E_0 = \text{number of zeros of } \sigma,$$
$$E_\sigma \cdot E_\infty = \text{number of poles of } \sigma,$$
$$E_0 \cdot E_\infty = 0,$$
$$E_0 \cdot C = E_\sigma \cdot C = E_\infty \cdot C = 1.$$

$S_n - C_\lambda - E_0$ forms a $\mathbb{C}$-bundle over $\mathbb{P}^1 - \{\lambda\} \cong \mathbb{C}$, and therefore is contractible; thus

$$H_2(S_n, \mathbb{Z}) \cong H_2(C_\lambda \cup E_0, \mathbb{Z}) \cong \mathbb{Z}\{(C_\lambda), (E_0)\}$$

and

$$H_1(S_n, \mathbb{Z}) = H_1(C_\lambda \cup E_0, \mathbb{Z}) = 0.$$

Since $H^1(S_n, \mathbb{Z}) = 0$ and $H^2(S_n, \mathbb{Z})$ is spanned by $(1,1)$-classes, we deduce that

$$H^1(S_n, \mathcal{O}) = H^2(S_n, \mathcal{O}) = 0.$$

In particular, the Chern class map

$$H^1(S_n, \mathcal{O}^*) \xrightarrow{c_1} H^2(S_n, \mathbb{Z})$$

is an isomorphism; i.e., two curves on S_n are linearly equivalent if and only if they are homologous. Consequently we can write

$$E_\infty \sim m_1 \cdot E_0 + m_2 \cdot C$$

for $m_1, m_2 \in \mathbb{Z}$. But then $E_\infty \cdot C = 1 \Rightarrow m_1 = 1$, and $E_\infty \cdot E_0 = 0 \Rightarrow m_2 = -n$; i.e.,

$$E_\infty \sim E_0 - n \cdot C,$$

and so

$$E_\infty \cdot E_\infty = E_0 \cdot E_0 - 2nC \cdot E_0 = -n.$$

Similarly, for σ a meromorphic section of H^n,

$$E_\sigma \sim E_0 + m \cdot C,$$

where m is the number of poles of the section σ.

Now suppose D is any irreducible curve on S_n. If $D \neq E_\infty$, then since D and E_∞ are irreducible, we have $D \cdot E_\infty \geqslant 0$; since D cannot contain every curve C_λ, and since C_λ is irreducible, $D \cdot C \geqslant 0$. If

$$D = m_1 \cdot E_0 + m_2 \cdot C,$$

we see that $D \cdot E_\infty \geqslant 0 \Rightarrow m_2 \geqslant 0$, and $D \cdot C \geqslant 0 \Rightarrow m_1 \geqslant 0$; consequently

$$D \cdot D = n \cdot m_1^2 + 2m_1 m_2 \geqslant 0.$$

From this it follows that E_∞ is the only irreducible curve on S_n with negative self-intersection: for $n \neq 0$, then, *S_n is the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ having an irreducible curve of self-intersection* $-$n. In particular, we see that the spaces $\{S_n\}_{n \geqslant 0}$ are all distinct as abstract compact complex manifolds.

Note that the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at a point $p \in \mathbb{P}^2$ is an S_n: the proper transforms $\tilde{L}_\lambda$ of the pencil of lines $L_\lambda \subset \mathbb{P}^2$ through p form a pencil of disjoint irreducible rational curves on $\tilde{\mathbb{P}}^2$. Since the exceptional divisor E of the blow-up has self-intersection -1,

$$\tilde{\mathbb{P}}^2 \cong S_1.$$

To determine the class of the canonical bundle K of S_n, note that by the adjunction formula,

$$0 = \pi(E_0) = \frac{E_0 \cdot E_0 + K \cdot E_0}{2} + 1$$

$$\Rightarrow K \cdot E_0 = -n - 2$$

and

$$0 = \pi(C) = \frac{C \cdot C + C \cdot K}{2} + 1$$

$$\Rightarrow K \cdot C = -2.$$

Thus if $K = m_1 E_0 + m_2 C$, $K \cdot C = -2$ implies that $m_1 = -2$, and $E_0 \cdot K = -n - 2$ implies that $m_2 = n - 2$; i.e.,

$$K = -2E_0 + (n-2)C.$$

Finally, we would like to relate the surfaces S_n to one another geometrically. To do this, let $x \in S_n$ be any point not on E_∞, say $x \in C_\lambda$. Blow up x to obtain a surface $\tilde{S}_n \xrightarrow{\pi_1} S_n$; the proper transform $\tilde{C}_\lambda$ of C_λ will then have self-intersection -1 and can be blown down. If $\tilde{S}_n \xrightarrow{\pi_2} S$ is the blowing-down map, we notice that the curves $\{\pi_2(\pi_1^*(C_\lambda))\}_\lambda$ form a pencil of irreducible rational curves on S with self-intersection 0, and hence S is again a ruled surface. Moreover, since $\pi_1^* E_\infty \cdot \tilde{C}_{\lambda_0} = 1$,

$$\pi_2 \pi_1^* E_\infty \cdot \pi_2 \pi_1^* E_\infty = E_\infty \cdot E_\infty + 1 = -n + 1,$$

i.e., S contains an irreducible curve of self-intersection $-n+1$, and hence S is biholomorphic to S_{n-1}. As Figure 5 attempts to show, the image $\pi_2(E_x) \subset S$ of the exceptional curve E_x of π_1 becomes an element of the pencil $\pi_2 \pi_1^* C_\lambda$, while $\tilde{E}_0^{(2)} = \pi_2 \pi_1^* E_0$ is the curve corresponding to a section τ of H^{n-1} with a single pole; the role of E_0 is taken over by $\pi_2 \pi_1^* E_\sigma$ for some E_σ passing through x. To obtain S_{n+1} from S_n, conversely, we blow up a point x on E_∞ and blow down the proper transform of the curve C_λ through x. We have seen this process once before, when we showed that $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ could be obtained by blowing up a point $q \notin E$ on the blow-up $\tilde{\mathbb{P}}^2 = S_1$ of $\mathbb{P}^2$ at a point p, and blowing down the proper transform of the line $\overline{pq}$.

The proof of Noether's lemma is at last complete: since the surfaces S_n are all obtained from one another by blowing up and down, and since $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and S_1 are rational, it follows that *all the surfaces* $\mathrm{S_n}$ *are rational.*

The General Rational Surface

Having given a fairly thorough account of the rational ruled surfaces S_n, we may now complete our picture of rational surfaces in general with the

Theorem. *Every rational surface is the blow-up of* $\mathbb{P}^2$ *or* $\mathrm{S_n}$.

Proof. We begin by making two observations. First, we claim that *any surface* S *with a pencil* |C| *of irreducible rational curves is either* $\mathbb{P}^2$ *or* $\mathrm{S_n}$. This is not hard: as we have seen, if we blow up the base points of the pencil $|C|$, we obtain a surface $\tilde{S} \to S$ with a pencil of disjoint irreducible

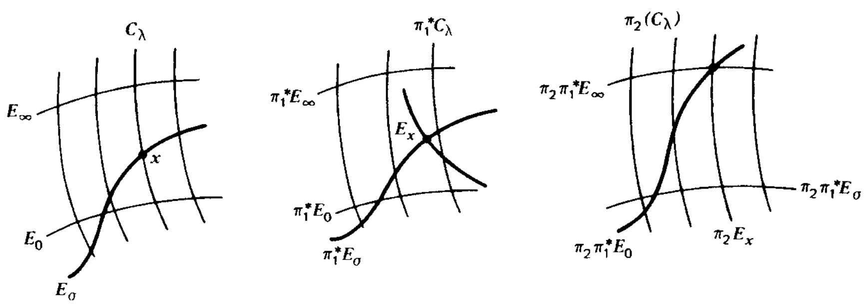


Figure 5

rational curves; $\tilde{S}$ must then be a rational ruled surface. But $b_2(S_n)=2$, and since the second Betti number increases by 1 every time we blow up, it follows that either S is rational ruled, or $b_2(S)=1$. In the latter case, since S is rational, we have

$$b_1(S) = b_3(S) = 0;$$

moreover, $P_n(S)=H^0(S,\mathcal{O}(K^n))=0$ for all n implies that K_S is not positive. By the lemma at the end of the discussion on cubic surfaces, then, we find that $S\cong\mathbb{P}^2$.

Our second point is more obvious. Suppose C is a curve on an algebraic surface S, L a positive line bundle, and $C\cdot L=n$. Then C *cannot be linearly equivalent to a sum of more than* n *effective divisors* D_i: if $C=\sum_{i=1}^{n+1}D_i$, we would have $L\cdot C=\sum L\cdot D_i\geqslant n+1$.

Now let S be any rational surface, $\{C_\lambda\}$ a pencil of rational curves on S, not all reducible. We want to show that either

1. S can be blown down; or
2. S is rational ruled or $\mathbb{P}^2$.

Since any surface can be blown down only finitely many times, this will suffice to prove the theroem.

If all the curves C_λ on S are irreducible, then by the above argument we are done. Suppose then that C_0 is reducible and write

$$C_0=\sum_{\nu=1}^{k} a_\nu C_\nu, \qquad a_\nu>0.$$

We note first that *all the curves* C_ν *must be rational*. This follows from writing

$$\begin{aligned} 0=\pi(C_\lambda) &= \frac{\left(\sum a_\nu C_\nu\right)\cdot\left(\sum a_\nu C_\nu\right)+K_S\cdot\left(\sum a_\nu C_\nu\right)}{2}+1\\ &=\frac{1}{2}\Big[\left(\sum (a_\nu-1)C_\nu\right)\cdot\left(\sum a_\nu C_\nu\right)\\ &\qquad+\sum_{\nu\neq\nu'} a_{\nu'}C_\nu\cdot C_{\nu'}+\sum_\nu a_\nu C_\nu\cdot C_\nu+\sum a_\nu\cdot K_S\cdot C_\nu+2\Big]\\ &=\frac{1}{2}\left(\sum (a_\nu-1)C_\nu\right)\cdot\left(\sum a_\nu C_\nu\right)+\sum a_\nu\pi(C_\nu)+\sum_{\nu\neq\nu'} a_{\nu'}C_\nu\cdot C_{\nu'}-(k-1). \end{aligned}$$

$C_\nu\cdot(\sum a_\nu C_\nu)=C_\nu\cdot C_\lambda\geqslant 0$, so the first term is nonnegative; since C_0 is connected, $\sum_{\nu\neq\nu'}C_\nu\cdot C_{\nu'}\geqslant k-1$ and so the third term is also nonnegative. It follows that $\sum a_\nu\pi(C_\nu)=0$, and hence $\pi(C_\nu)=0$. Since $C_\lambda\cdot C_\lambda\geqslant 0$, by the adjunction formula

$$0=\pi(C_\lambda)=\frac{C_\lambda\cdot C_\lambda+K_S\cdot C_\lambda}{2}+1,$$

we see that $K_S \cdot C_\lambda < 0$, hence $K_S \cdot C_{\nu_0} < 0$ for some ν_0. Now if $C_{\nu_0} \cdot C_{\nu_0} < 0$, it follows that $C_{\nu_0} \cdot C_{\nu_0} = C_{\nu_0} \cdot K = -1$ and hence C_{ν_0} can be blown down; in this case we are done. Suppose on the other hand $C_{\nu_0} \cdot C_{\nu_0} \geqslant 0$; then, since S is rational, $\chi(\mathcal{O}_S) = 1$, and by Riemann Roch,

$$h^0(C_{\nu_0}) + h^0(K_S - C_{\nu_0}) \geqslant 1 + \frac{C_{\nu_0} \cdot C_{\nu_0} - C_{\nu_0} \cdot K_S}{2} > 1.$$

But $h^0(K_S) = 0$ implies $h^0(K_S - C_{\nu_0}) = 0$ and therefore $h^0(C_{\nu_0}) > 1$, i.e., C_{ν_0} *is itself an element of a pencil of rational curves*, which we will denote $\{C_\lambda^1\}$.

If all the curves C_λ^1 are irreducible, we are done; if not, let C_0^1 be a reducible element of $\{C_\lambda^1\}$ and write

$$C_0^1 = \sum a_\nu^1 \cdot C_\nu^1.$$

Again, $\pi(C_\lambda^1) = 0$ implies $K \cdot C_\lambda^1 < 0$, so $K \cdot C_{\nu_0}^1 < 0$ for some $C_{\nu_0}^1$, and either $C_{\nu_0}^1$ can be blown down or $C_{\nu_0}^1$ is itself an element of a pencil of rational curves $\{C_\lambda^2\}$. If S has no exceptional curves and is not $\mathbb{P}^2$ or S_n, we can continue to generate new pencils in this way. But after n steps we can write

$$C_\lambda = \sum_{\nu \neq \nu_0} C_\nu + \sum_{\nu \neq \nu_0} C_\nu^1 + \cdots + \sum C_\nu^n,$$

and so eventually either every element of the pencil we obtain will be irreducible or S will contain an exceptional curve. Q.E.D.

Surfaces of Minimal Degree

In Section 3 of Chapter 1 we showed that the smallest possible degree of an irreducible, nondegenerate variety $M \subset \mathbb{P}^n$ of dimension m is $n - m + 1$. We can now describe exactly the surfaces that achieve this minimal degree. We will start by constructing some such surfaces, and then show that all surfaces of minimal degree may be obtained in this way.

Consider the linear system $|E_0 + kC|$ on the rational ruled surface $S_n = \mathbb{P}(H^n \oplus \mathbb{C})$. If σ is any meromorphic section of H^n having exactly k poles, then, as we have seen, the corresponding curve

$$E_\sigma = \sigma(\mathbb{P}^1) \subset S_n$$

is homologous, hence linearly equivalent, to $E_0 + kC$. Conversely, suppose D is any irreducible curve linearly equivalent to $E_0 + kC$. Then D meets each fiber C of the projection map $S_n \xrightarrow{\pi} \mathbb{P}^1$ exactly once, and away from the k points $\mu_1, \ldots, \mu_k \in \mathbb{P}^1$ over which D meets E_∞ we can define a section σ of H^n by

$$\overline{(\sigma(\mu), 1)} = D \cap C_\mu.$$

σ then extends to a global meromorphic section of H^n, having poles at $\mu_1, \ldots, \mu_k$, such that $D = E_\sigma$.

Note in particular that the linear system $|E_0+kC|$ has no base points; we denote the corresponding map ι_{E_0+kC} by $\varphi_{k,n}$.

To specify a global meromorphic section of σ having k poles, we have to specify first its polar divisor $(\sigma)_\infty=\mu_1+\cdots+\mu_k$—involving k degrees of freedom—and then specify σ as an element of the vector space $H^0(\mathbb{P}^1,\mathcal{O}(H^n+\mu_1+\cdots+\mu_k))$—involving $h^0(H^n+\mu_1+\cdots+\mu_k)=n+k+1$ degrees of freedom. Thus we may expect that the linear system $|E_0+kC|$ is at least $k+(n+k+1)=(n+2k+1)$-dimensional. The Riemann-Roch theorem confirms our guess: since $h^0(K_{S_n})=0$, $h^2(E_0+kC)=h^0(K_{S_n}-E_0-kC)=0$, and we have

$$\begin{aligned} h^0(E_0+kC) &\geqslant 1+\frac{(E_0+kC)\cdot(E_0+kC)-(E_0+kC)\cdot(-2E_0+(n-2)C)}{2} \\ &= 1+\frac{n+2k+2n+2k-n+2}{2} \\ &= n+2k+2. \end{aligned}$$

But now $\varphi_{k,n}$ maps S_n into projective space of dimension $h^0(E_0+kC)-1$ as a nondegenerate surface of degree

$$(E_0+kC)\cdot(E_0+kC) = n+2k,$$

and so $h^0(E_0+kC)-1\leqslant n+2k+1$. Thus equality must hold, and we see that *the image* $S_{k,n}$ *of* S_n *under* $\varphi_{k,n}$ *is a surface of minimal degree* $n+2k$ *in* $\mathbb{P}^{n+2k+1}$.

We can give a nice description of the surfaces $S_{k,n}$ as follows: recall first that

$$\begin{aligned} E_0\cdot(E_0+kC) &= n+k, \\ E_\infty\cdot(E_0+kC) &= k, \\ C\cdot(E_0+kC) &= 1. \end{aligned}$$

In fact, we see from the correspondence between irreducible curves $E_\sigma\in|E_0+kC|$ and meromorphic sections σ of H^n having k poles that the restrictions of $|E_0+kC|$ to the curves E_0, E_∞, and $C\cong\mathbb{P}^1$ are the complete linear systems $|H_{\mathbb{P}^1}^{n+k}|$, $|H_{\mathbb{P}^1}^{k}|$, and $|H_{\mathbb{P}^1}|$, respectively: given any collection of points $\mu_1,\ldots,\mu_{n+k},\nu_1,\ldots,\nu_k\in\mathbb{P}^1$ we can always find a meromorphic section σ of H^n with

$$(\sigma)_0=\sum\mu_i,\qquad (\sigma)_\infty=\sum\nu_i;$$

likewise, given any point $\overline{(\xi,1)}\in C_\lambda$, we can find σ with $\sigma(\lambda)=\xi$. Thus:

1. The image D_0 of E_0 under $\varphi_{k,n}$ is a rational normal curve in some linear subspace $V_{n+k}\subset\mathbb{P}^{n+2k+1}$.
2. The image D_∞ of E_∞ under $\varphi_{k,n}$ is a rational normal curve in some linear subspace $V_k\subset\mathbb{P}^{n+2k+1}$.
3. The image L_λ of the curves C_λ under $\varphi_{k,n}$ are straight lines meeting D_0 and D_∞. Note that $S_{k,n}$ thus lies in the linear span of V_k and V_{n+k}, so

that these subspaces are necessarily complementary (i.e., disjoint) in $\mathbb{P}^{n+2k+1}$

The surface $S_{k,n}$ *consists of the union of the straight lines joining points* p *on the rational normal curve* $D_0 \subset V_{n+k}$ *with corresponding points* $\psi(p)$ *on the rational normal curve* $D_\infty \subset V_k$.

Conversely, suppose that V_k, V_{n+k} are complementary linear subspaces in $\mathbb{P}^{n+2k+1}$, D_∞ and D_0 rational normal curves in V_k and V_{n+k}, respectively, and $\psi: D_0 \to D_\infty$ an isomorphism between the two curves. For each point $\mu \in D_0$, let L_μ be the line $\overline{\mu, \psi(\mu)}$ in $\mathbb{P}^{n+2k+1}$, and let

$$S = \bigcup_{\mu \in D_0} L_\mu$$

be the corresponding surface. To compute the degree of S, note that the generic hyperplane $H \subset \mathbb{P}^{n+2k+1}$ containing D_∞ meets D_0 transversely in $\deg D_0 = n+k$ points $\mu_1, \ldots, \mu_{n+k}$; we then have

$$H \cap S = D_\infty + L_{\mu_1} + \cdots + L_{\mu_{n+k}},$$

all components occurring with multiplicity 1. Therefore

$$\deg S = \deg H \cap S = n + 2k,$$

and S is a surface of minimal degree.

Now the lines $\{L_\mu\}$ are disjoint: If L_μ met $L_{\mu'}$, the 2-plane spanned by L_μ and $L_{\mu'}$ in $\mathbb{P}^{n+2k+1}$ would have to meet V_{n+k} in the line $\overline{\mu\mu'}$ and V_k in the line $\overline{\psi(\mu), \psi(\mu')}$, so that V_k and V_{n-k} would intersect. Thus *the map*

$$S \xrightarrow{\pi} D_0 \cong \mathbb{P}^1$$

sending L_μ *to* μ *expresses* S *as a rational ruled surface*. To determine which rational ruled surface S is, consider again a hyperplane section $D = H \cdot S = D_\infty + L_{\mu_1} + \cdots + L_{\mu_{n+k}}$ above. We have

$$\begin{aligned} n + 2k &= D \cdot D \\ &= D_\infty \cdot D_\infty + 2D_\infty \cdot \left(\sum L_{\mu_i}\right). \end{aligned}$$

But each line L_{μ_i} meets D_∞ transversely (otherwise it would lie in V_k), and so

$$D_\infty \cdot \left(\sum L_{\mu_i}\right) = n + k,$$

and hence

$$D_\infty \cdot D_\infty = -n,$$

so

$$S \cong S_n$$

with D_∞ corresponding to E_∞. Moreover, the hyperplane section of S is

$$\begin{aligned} D &= D_\infty + L_{\mu_1} + \cdots + L_{\mu_{n+k}} \\ &\sim E_\infty + (n+k)C \\ &\sim E_0 + kC, \end{aligned}$$

so that indeed $S = S_{k,n}$.

The surfaces $S_{k,n}$ are called *rational normal scrolls*. Note that in case $n>0$ the curve $D_\infty \subset S_{k,n}$ is unique; it is called the *directrix* of $S_{k,n}$.

Now we can apply our theorem on rational surfaces to prove the

Proposition. *Every nondegenerate irreducible surface of degree* $m-1$ *in* $\mathbb{P}^m$ *is either a rational normal scroll or the Veronese surface* $\iota_{2H}(\mathbb{P}^2) \subset \mathbb{P}^5$.

Proof. Note first that if S is an irreducible surface of degree $m-1$ in $\mathbb{P}^m$, then any line L meeting S in three or more points must lie in S. To see this, suppose that L meets S in three points p_1, p_2, p_3 but does not lie in S. The points of intersection of S with a generic $(m-2)$-plane V_{m-2} containing L span V_{m-2}, and so $V_{m-2} \cap S$ must contain at least $m-3$ points $q_1, \ldots, q_{m-3}$ lying outside L—but $^{\#}(V_{m-2} \cdot S) = m-1$ and so it follows that V_{m-2} has a curve in common with S. The image $\pi_L(S)$ of S under projection from L into an $(m-2)$-plane W_{m-2} thus meets every $(m-4)$-plane in W_{m-2} in a curve, and so has dimension 3, an absurdity.

In particular, we see that if S has a singular point p, then for any point $q \in S$ the line $\overline{pq}$ must lie in S. S must therefore be the cone $\cup_{q \in C} \overline{pq}$ through p over any hyperplane section $C = S \cap H$ of S not containing p. Now C is nondegenerate and irreducible, since S is, and has degree $m-1$ in $H \cong \mathbb{P}^{m-1}$, hence is a rational normal curve. Thus $S = S_{0,m-1}$ *is the cone over a rational normal curve.*

The argument for S smooth is by induction. The result clearly holds for $m=3$: as we have seen, the smooth quadric surface in $\mathbb{P}^3$ is just the image $S_{1,0}$ of $S_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Suppose the result is proved for all $m \leq m_0$, $m_0 \geq 4$, and let S be a smooth irreducible nondegenerate surface of degree $m-1$ in $\mathbb{P}^m$. Assume first that S contains only finitely many lines. A generic point p of S will then lie on no lines on S, and since any line meeting S three times lies on S, it follows that no two points of S are collinear with p. This means that the projection map

$$\pi_p : S \to \mathbb{P}^{m-1}$$

gives an embedding of the blow-up $\tilde{S}$ of S at p as a surface of degree $m-2$ in $\mathbb{P}^{m-1}$. By induction hypothesis, then, we see that

1. S is rational; and
2. $\chi(S) = \chi(\tilde{S}) - 1 \leq 3$.

But we have seen that the only rational surface of Euler characteristic 3 is $\mathbb{P}^2$; so $S \cong \mathbb{P}^2$. Now, any base-point-free linear system on $\mathbb{P}^2$ has degree k^2 and dimension at most $((k+1)(k+2)/2)-1$ for some k, and so the only embedding of $\mathbb{P}^2$ as a surface of minimal degree is that given by the complete linear system of conics; thus S must be the Veronese surface.

Suppose now that S does contain in irreducible one-parameter family $\{L_\mu\}_{\mu \in C}$ of lines. Note first that two generic lines in the family must be disjoint: if every two lines met, then every three lines would either lie in a plane—in which case every line in that plane would meet S three times and so lie in S—or meet in a point, with independent directions—impossible since S is assumed smooth. Set $b=[m/2]$ and choose b lines $L_1,\ldots,L_b$ of the family. $L_1,\ldots,L_b$ together span at most an $(m-1)$-plane; take H a hyperplane containing $L_1,\ldots,L_b$ and consider the intersection $H\cdot S$. Since H has intersection number 1 with a line, there must be a unique irreducible component of the divisor $H\cdot S$ having intersection number 1 with L_μ; call this curve D_∞. The remaining components of $H\cdot S$, having intersection number 0 with L_μ, must be themselves lines of the family; thus we can write

$$H\cdot S = D_\infty + L_1 + \cdots + L_b + L_{b+1} + \cdots + L_c.$$

Consider the curve D_∞. By the above, D_∞ has degree $k=m-c-1$. On the other hand, the linear span of D_∞ must be a least a k-plane: otherwise, for any $m-k$ points $p_1,\ldots,p_{m-k}$ of $S-D_\infty$ we could find a hyperplane H' containing D_∞ and the points $p_1,\ldots,p_{m-k}$, and hence also the lines of the family passing through $p_1,\ldots,p_{m-k}$; the degree of $H'\cdot S$ would then be at least m, which is impossible. Thus D_∞ spans a k-plane, i.e., *D_∞ is a rational normal curve.*

Now let $L_1,\ldots,L_k$ be any lines of the family. $L_1,\ldots,L_k$ span at most a $(2k-1)$-plane in $\mathbb{P}^m$ (note that since $c \geqslant b=[m/2]$, $k=m-c-1$ must be strictly less than $m/2$, so $L_1,\ldots,L_k$ all lie in a proper subspace of $\mathbb{P}^m$) which intersects the linear span $\overline{D_\infty}$ in at least the $(k-1)$-plane spanned by the points of intersection $L_1\cdot D_\infty,\ldots,L_k\cdot D_\infty$. In fact, the span of the lines L_i cannot contain D_∞: if it did, for any $m-2k$ points $p_1,\ldots,p_{m-2k} \in S-D_\infty - \cup L_i$, the lines L_i and the points p_i would all lie in a hyperplane, which would then contain the curve D_∞, the k lines L_i, and the $m-2k$ lines of the family passing through the points p_i—altogether a curve of degree m. Thus we can find a hyperplane H in $\mathbb{P}^m$ containing the lines $L_1,\ldots,L_k$ but not D_∞.

Again, the hyperplane section $H\cdot S$ will contain one component having intersection number 1 with L_μ; call this curve D_0. Note that

$$D_0\cdot D_\infty = (H\cdot S - L_1 - \cdots - L_k)\cdot D_\infty = 0$$

so D_0 and D_∞ will be disjoint. Since every line L_μ meets both D_0 and D_∞,

S lies in the linear span of D_0 and D_∞, so D_0 must span at least an $(m-k-1)$-plane; on the other hand,

$$\deg D_0 \leqslant \deg(H\cdot S - L_1 - \cdots - L_k) = m-k-1$$

and it follows that D_0 *is a rational normal curve of degree* k, *in a* k-*plane complementary to the span of* D_∞. Thus S is the rational normal scroll $S_{k,m-2k-1}$, and the result is proved. Q.E.D.

The reader may find it amusing to verify directly what was in effect proved on p. 520: that the image of $S_{k,n}$ under projection from a point lying off the directrix $D_\infty \subset S_{k,n}$ is $S_{k,n-1}$, while the image of $S_{k,n}$ under projection from a point $q \in D_\infty$ is $S_{k-1,n+1}$.

Curves of Maximal Genus

We gave, in the section on linear systems on curves, Castelnuovo's upper bound on the genus of an irreducible nondegenerate curve B of degree d in $\mathbb{P}^n$. Briefly, we showed that if D was the hyperplane divisor of $B \subset \mathbb{P}^n$, then

$$h^0(kD) - h^0((k-1)D)\begin{cases} \geqslant k(n-1)+1, & k \leqslant m = \left[\dfrac{d-1}{n-1}\right], \\ = d, & k \geqslant m \end{cases}$$

leading directly to

$$(*)\qquad \begin{cases} h^0(D) \geqslant n+1, \\ h^0(2D) \geqslant 3(n-1)+3, \\ \quad\vdots \\ h^0((m+j)D) \leqslant \dfrac{m(m+1)}{2}(n-1)+m+1+jd, \end{cases}$$

and the last equality, for $j \gg 0$, gives by Riemann-Roch

$$\begin{aligned} g(B) &\leqslant m\left(d - \frac{m+1}{2}(n-2) - 1\right). \\ &= \frac{m(m-1)}{2}(n-1) + m\varepsilon \qquad \text{where} \qquad d-1 = m(n-1)+\varepsilon. \end{aligned}$$

We can now give a fairly complete description of those curves of degree $d > 2n$ that achieve this bound—called *Castelnuovo curves*—and, in so doing, verify that the bound is indeed sharp for all d and n.

To begin with, we note that if $C \subset \mathbb{P}^n$ is a Castelnuovo curve, equality must hold in all the inequalities above. In particular, we see that

$$h^0(C, \mathcal{O}(2H)) = 3n;$$

and since $h^0(\mathbb{P}^n, \mathcal{O}(2H)) = (n+1)(n+2)/2$, this implies that

The linear system W *of quadrics in* $\mathbb{P}^n$ *containing* C *has dimension at least* $(n-1)(n-2)/2-1$.

Since, moreover, no quadric containing C can contain a hyperplane, the restriction of W to a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ is injective; thus the linear system of quadrics in $n-1$ containing the points $\Gamma = C \cap \mathbb{P}^{n-1}$ likewise has dimension at least $(n-1)(n-2)/2-1$. Inasmuch as the linear system of all quadrics in $\mathbb{P}^{n-1}$ has dimension only $n(n+1)/2-1$, this means that

The points of a generic hyperplane section $\Gamma = C \cap \mathbb{P}^{n-1}$ *of* C *impose only* $2n-1$ *conditions on quadrics.*

Now this is certainly a very strong statement. As we saw in the original discussion of Castelnuovo's bound, *any* $2n-1$ points in general position in $\mathbb{P}^{n-1}$ must impose independent conditions on quadrics; here we have an arbitrary number $d = \deg C$ of points imposing only this smallest possible number of conditions. Indeed, from our previous encounter with the phenomena of superabundance in the discussion of cubic surfaces, we may expect that the extreme failure of the points of Γ to impose independent conditions on quadrics should have strong geometric consequences. The problem is, it simply is not obvious how one should proceed from this hypothesis. To Castelnuovo, however, it must have been clear; after spending the first 30 pages of his original article arriving at this point, he draws the correct conclusion in a paragraph. The essential point seems to have been a familiarity with certain projective-geometric constructions called *Steiner constructions*, which we now describe.

Steiner Constructions

Let p_1 and p_2 be points in the plane. Parameterize the two pencils $\{L_1(\lambda)\}$ and $\{L_2(\lambda)\}$ of lines through p_1 and p_2 respectively by $\lambda \in \mathbb{P}^1$, choosing the parameterizations so that the one line $\overline{p_1 p_2}$ common to the two pencils corresponds to different values of λ—that is, so that $L_1(\lambda) \neq L_2(\lambda)$ for all λ. Then the curve

$$C = \bigcup_{\lambda} L_1(\lambda) \cap L_2(\lambda)$$

is clearly irreducible and nondegenerate, containing the points p_1 and p_2 but not lying in the line $\overline{p_1 p_2}$. Its intersection with a general line $L \subset \mathbb{P}^2$ consists of the fixed points of the automorphism of L sending the point

$L \cap L_1(\lambda)$ to $L \cap L_2(\lambda)$; since there can be at most two such fixed points it follows that C *is a conic curve.*

Note that given three additional points p_3, p_4, and p_5 in the plane, no two collinear with either p_1 or p_2, we may choose our parameterizations of the pencils L_1 and L_2 so that $L_1(0)$ and $L_2(0)$ both contain p_3, $L_1(1)$ and $L_2(1)$ contain p_4, and $L_1(\infty)$ and $L_2(\infty)$ contain p_5. If in addition we assume that p_3, p_4, and p_5 lie off the line $\overline{p_1p_2}$ and are not all three collinear, then these parameterizations satisfy our requirement that the line $\overline{p_1p_2}$ correspond to different λ in the two pencils. If indeed $\overline{p_1p_2} = L_1(\lambda_0) = L_2(\lambda_0)$, then the automorphism of the line $L = \overline{p_3p_4}$ taking $L \cap L_1(\lambda)$ to $L \cap L_2(\lambda)$ would fix the three points p_3, p_4, and $L \cap \overline{p_1p_2}$, and so would be the identity; we would then have

$$p_5 = L_1(\infty) \cap L_2(\infty) = L_1(\infty) \cap L \in L.$$

We see, accordingly, that we may construct a smooth conic through any five points in the plane, no three collinear.

Classically, the common parameterization of the two pencils L_1 and L_2 was given geometrically by choosing two auxiliary lines M_1 and M_2 and an auxiliary point $q \notin M_1, M_2$, and for each line $M(\lambda)$ through q letting $L_1(\lambda)$ be the line joining p_1 and the point $M_1 \cap M(\lambda)$, $L_2(\lambda)$ the line joining p_2 and $M_2 \cap M(\lambda)$. Thus, for example, to construct the conic through $p_1, \ldots, p_5$ one could take

$$M_1 = \overline{p_3p_4}, \qquad M_2 = \overline{p_3p_5}, \qquad q = \overline{p_1p_5} \cap \overline{p_2p_4}.$$

This construction may be generalized to higher dimensional space in many ways, two of which are the following:

1. If V_1, V_2 are two $(n-2)$-planes in $\mathbb{P}^n$, we may choose any parameterization of the two pencils of hyperplanes $\{H_1(\lambda)\}$ and $\{H_2(\lambda)\}$ through V_1 and V_2 respectively such that $H_1(\lambda) \neq H_2(\lambda)$ for all λ, and consider the locus

$$Q = \bigcup_{\lambda} H_1(\lambda) \cap H_2(\lambda).$$

As in the previous construction, Q is readily seen to be an irreducible, nondegenerate hypersurface, and hence a quadric, intersecting a general line $L \subset \mathbb{P}^n$ in the fixed points of the automorphism of L sending $L \cap H_1(\lambda)$ to $L \cap H_2(\lambda)$. In the terminology of Section 1 of Chapter 6, Q is a quadric of rank either 3 or 4, with vertex $V_1 \cap V_2$.

2. Let $p_1, \ldots, p_n$ be linearly independent points in $\mathbb{P}^n$, and let $\{H_i(\lambda)\}$ be the pencil of hyperplanes containing the $(n-2)$-plane V_i spanned by $p_1, \ldots, \hat{p}_i, \ldots, p_n$; choose the parameterizations so the one hyperplane $V = \overline{p_1, \ldots, p_n}$ common to all the pencils corresponds to n different values of λ. Then for each λ, the planes $H_1(\lambda), \ldots, H_n(\lambda)$ meet only in a point: if none

of the planes $H_i(\lambda)$ are equal to V then the intersection $H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ cannot meet V and so must be a point; while if $H_i(\lambda) = V$ then $H_j(\lambda) \cap H_i(\lambda)$ is just V_j, and the intersection $H_1(\lambda) \cap \ldots \cap H_n(\lambda) = p_i$. Thus the curve

$$C = \bigcup_{\lambda} H_1(\lambda) \cap \ldots \cap H_n(\lambda)$$

is irreducible; and as before it is nondegenerate, containing the points $p_1, \ldots, p_n$ but not lying in the hyperplane V they span. Its degree must therefore be at least n; and since its intersection with a general hyperplane $H \subset \mathbb{P}^n$ consists of the fixed points of the automorphism of H sending the point $H \cap H_1(\lambda) \cap \ldots \cap H_{n-1}(\lambda)$ to $H \cap H_2(\lambda) \cap \ldots \cap H_n(\lambda)$—that is, the eigenspaces of the corresponding linear transformation of H—we see that the degree of C must be exactly n, that is, *C is a rational normal curve.*

Note that if we set

$$Q_{ij} = \bigcup_{\lambda} H_i(\lambda) \cap H_j(\lambda)$$

then C will be the intersection of the quadrics Q_{ij}; thus we see that

A rational normal curve is cut out by quadrics.

Now, choose three additional points p_{n+1}, p_{n+2}, and p_{n+3} such that the points $p_1, \ldots, p_{n+3}$ are in general position. As in the construction of the plane conic, then, we can choose our parameterizations of the pencils H_i so that

$$p_{n+1} \in H_i(0), \qquad p_{n+2} \in H_i(1), \qquad p_{n+3} \in H_i(\infty)$$

for all i. This choice satisfies our requirement. If for some $\lambda \in \mathbb{P}^1$ we had $H_i(\lambda) = H_j(\lambda) = V$, the automorphism of the line $L = \overline{p_{n+1}p_{n+2}}$ sending $L \cap H_i(\lambda)$ to $L \cap H_j(\lambda)$ would fix the points p_{n+1}, p_{n+2} and $L \cap V$, and so would be the identity; $H_i(\infty) \cap H_j(\infty)$ would then meet L and so the $n+1$ points $p_1, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n, p_{n+1}, p_{n+2}$, and p_{n+3} would all lie in a hyperplane. Having chosen our parameterizations in this way, we then see that all the points $p_1, \ldots, p_{n+3}$ will lie on C; thus we can find a rational normal curve in $\mathbb{P}^n$ containing any $n+3$ points in general position. Indeed, such a curve is unique: if D is another rational normal curve containing $p_1, \ldots, p_{n+3}$, then each hyperplane $H_i(\lambda)$ will meet D in $p_1, \ldots, \hat{p}_i, \ldots, p_n$ and one more point, which we may denote $q_i(\lambda)$. But now the automorphism ϕ_{ij} of D sending $q_i(\lambda)$ to $q_j(\lambda)$ for each λ fixes p_{n+1}, p_{n+2}, and p_{n+3}, and so is the identity; thus $q_i(\lambda) = H_1(\lambda) \cap \ldots \cap H_n(\lambda)$ and correspondingly $D = C$. In sum, then,

Through any $n+3$ *points in general position in* $\mathbb{P}^n$ *there passes a unique rational normal curve.*

We note in passing some of the other variations on the theme of Steiner constructions. For example, we may take three nets of planes in $\mathbb{P}^3$ and parameterize each by $\lambda \in \mathbb{P}^2$; the union of the intersection of corresponding planes will then be a cubic surface. One can also take two pencils of planes in $\mathbb{P}^3$ parameterized by $\lambda \in \mathbb{P}^1$ and a correspondence $T: \mathbb{P}^1 \to \mathbb{P}^1$, and take the union of corresponding pairs of planes. If T has bidegree $(1,2)$ the resulting surface will be a cubic surface with a double line, while if T has bidegree $(2,2)$ the resulting surface is a quartic with two double lines; both of these surfaces will be discussed in Section 6 of this chapter.

Now we can without difficulty prove

Castelnuovo's Lemma. *A collection* $p_1, \ldots, p_d$ *of* $d \geq 2n+3$ *points in general position in* $\mathbb{P}^n$ *which impose only* $2n+1$ *conditions on quadrics lies on a rational normal curve.*

Proof. First note that, since any $2n+1$ points in general position in $\mathbb{P}^n$ impose independent conditions on quadrics, any quadric containing $2n+1$ of the points $p_1, \ldots, p_d$ will contain them all. Now let $\{H_i(\lambda)\}$ and $\{H(\lambda)\}$ be the pencils of hyperplanes in $\mathbb{P}^n$ through $p_1, \ldots, \hat{p}_i, \ldots, p_n$ and $p_{n+1}, \ldots, p_{2n-1}$, respectively, parameterized so that

$$p_{2n} \in H_i(0), H(0); \quad p_{2n+1} \in H_i(1), H(1); \qquad p_{2n+2} \in H_i(\infty), H(\infty)$$

for all i. Then the quadrics

$$Q_i = \bigcup_{\lambda} H_i(\lambda) \cap H(\lambda),$$

containing the points $p_1, \ldots, \hat{p}_i, \ldots, p_{2n+2}$ must contain the points $p_{2n+3}, \ldots, p_d$ as well; that is, the remaining points $p_{2n+3}, \ldots, p_d$ also lie on corresponding hyperplanes of the pencils H_i, and hence lie on the rational normal curve

$$C = \bigcup_{\lambda} H_1(\lambda) \cap \ldots \cap H_n(\lambda).$$

We have shown then that $p_1, \ldots, p_n, p_{2n}, \ldots, p_d$ all lie on a rational normal curve, and hence after rearranging that any $d-n+1 > n+3$ of the points $p_1, \ldots, p_n$ do also; since a rational normal curve is determined by any $n+3$ points this implies that all the points $p_1, \ldots, p_n$ lie on a rational normal curve. Q.E.D.

Returning to our Castelnuovo curve $C \subset \mathbb{P}^n$, it is now straightforward to describe C explicitly. As we have seen, the linear system W of quadrics in $\mathbb{P}^n$ through C cuts out on a general hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ a linear system of quadrics through the hyperplane section $\Gamma = C \cap \mathbb{P}^{n-1}$ having codimension $2n-1$ in the system of all quadrics in $\mathbb{P}^{n-1}$; clearly these are all the quadrics containing Γ. Now, by Castelnuovo's lemma, Γ lies on a rational normal curve $D \subset \mathbb{P}^{n-1}$; since Γ consists of $d \geq 2n-1$ points, any quadric will contain Γ if and only if it contains D; and since a rational normal curve is cut out by quadrics it follows that the base locus of the linear system W intersects the hyperplane $\mathbb{P}^{n-1}$ in the rational normal curve D. But then the base locus of W must be a surface of degree $n-1$ in $\mathbb{P}^n$; so by our previous result,

A Castelnuovo curve lies on either a rational normal scroll or the Veronese surface.

It is easily checked that any smooth plane curve, mapped to $\mathbb{P}^5$ via the Veronese map, is a Castelnuovo curve; in what follows we shall assume that C lies on a rational normal scroll $S = S_{k,l}$. We ask first for the homology class of the curve C on S; a priori we may write

$$C \sim aH + bL$$

where H and L are the classes of a hyperplane section of and line on S respectively. We have

$$d = \deg C = H \cdot C = a(n-1) + b$$

so $b = d - a(n-1)$; and applying the adjunction formula,

$$\begin{aligned} g(C) \leqslant (C) &= \frac{C \cdot C + K_S \cdot C}{2} + 1 \\ &= \frac{(aH + (d - a(n-1))L) \cdot ((a-2)H + (d - (a-1)(n-1) - 2)L)}{2} + 1 \\ &= \tfrac{1}{2}(a(a-2)(n-1) + (a-2)(d - a(n-1)) + a(d - a(n-1) + n - 3)) \\ &= \frac{(a-1)(a-2)}{2}(n-1) + (a-1)(d - (a-1)(n-1) - 1) \end{aligned}$$

This in fact achieves our bound exactly when $a = m+1$, and, in case $\varepsilon = 0$, when $a = m$ as well. We see, then, that the curve C must be smooth, and have either class $(m+1)H - (n-2-\varepsilon)L$ in general or class $mH + L$ when $C = 0$. Another way to express this, since the linear series cut out on a rational normal scroll by hypersurfaces of degree m is complete for all m, is to say that C plus any $n-2-\varepsilon$ lines of S form the complete intersection of S with a hypersurface of degree $m+1$ in $\mathbb{P}^n$, or, in the exceptional case, that C together with the directrix E_∞ of S and any $n-k-2$ lines of S form the complete intersection of I with a hypersurface of degree $m+1$.

Finally, to see that smooth irreducible curves with this homology class exist on the surface $S = S_{k,1}$ (at least for some k), we simply write

$$(m+1)H - (n-2-\varepsilon)L = (m+1)E_0 + (m(n-k-1) - k + 1 + \varepsilon)L.$$

Since the coefficient of L can, by a suitable choice of k, be made positive, we see that the linear system $|(m+1)H + (n-2-\varepsilon)L|$ on S has no base points and so the generic element is smooth; since any two components of a reducible curve homologous to $(m+1)H+(n-2-\varepsilon)L$ would meet, it follows that the generic element is irreducible as well.

Summing up, then, we can say that

The greatest possible genus of an irreducible nondegenerate curve C *of degree* d *in* $\mathbb{P}^n$ *is* $m(m-1)/2+m\varepsilon$, *where* $m=[(d-1)/(n-1)]$ *and* $d-1=m(n-1)+\varepsilon$. *Moreover, any curve achieving this bound is either*

1. *Residual to either* $n-2-\varepsilon$ *lines of* $n-k-2$ *lines plus the directrix in a complete intersection of a rational normal scroll* $S_{k,l}\subset\mathbb{P}^n$ *with a hypersurface of degree* $m+1$; *or*
2. *A smooth curve on the Veronese surface in* $\mathbb{P}^5$.

The Enriques-Petri Theorem

Recall from our initial discussion of curves of maximal genus that the curves of degree $d=2n$ in $\mathbb{P}^n$ having maximal genus are just the canonical curves of genus $g=n+1$. Much of the preceding analysis of extremal curves of degree greater than $2n$ applies as well in this case: for $C\subset\mathbb{P}^n$ a canonical curve, we have

$$h^0(L) = n+1, \qquad h^0(2L) = 3n,$$

and so the linear system $W\subset|2H|$ of quadrics in $\mathbb{P}^n$ containing C again has dimension

$$\dim W \geqslant \tfrac{1}{2}(n-1)(n-2)-1.$$

In addition, we see just as before that the restriction of W to a hyperplane is injective, and hence that the hyperplane section $\Gamma=C\cap\mathbb{P}^{n-1}$ imposes only $2n-1$ conditions on quadrics. At this point, our previous analysis breaks down: lacking $2n+1$ points, we cannot apply Castelnuovo's argument to prove that the hyperplane section of C lies on a rational normal curve. *If we hypothesize the existence of just one point of* S *not lying on C,* however, Castelnuovo's argument is again in force. To see this, we need only prove a slight strengthening of the basic general position lemma of Section 3, Chapter 2:

Basic Lemma II. *Let* $C\subset\mathbb{P}^n$ *be any nondegenerate curve,* $p\in\mathbb{P}^n$ *any point not lying on infinitely many chords of* C. *Then for* H *a generic hyperplane passing through* p, *the points*

$$\{p\}\cup(H\cap C)$$

are in general position.

Proof. We first show that for a generic hyperplane H containing p, the point p is linearly independent from any $n-1$ points of $H\cap C$. This is clear: by hypothesis, for generic H the projection map π_p of C from p onto a hyperplane $\mathbb{P}^{n-1}$ is one-to-one on $H \cap C$, and for any $n-1$ points $p_1,\ldots,p_{n-1} \in H \cap C$, p will lie in the linear span of $p_1,\ldots,p_{n-1}$ if and only if the points $\{\pi_p(p_i)\}$ are linearly dependent in $\mathbb{P}^{n-1}$. But by our original basic lemma, the generic hyperplane in $\mathbb{P}^{n-1}$ contains no such collection of points in $\pi_p(C)$.

To see that the generic H containing p will not contain n linearly dependent points of C, consider the incidence correspondence

$$I \subset C^n \times \mathbb{P}^{n*}$$

given by

$$I = \{(p_1,\ldots,p_n,H): \ p_i \in H\}$$

and let $J \subset I$ be given by

$$J = \left\{(p_1,\ldots,p_n,H): \ \dim \overline{p_1,\ldots,p_n} < n-1\right\}.$$

The projection map

$$\pi_1: \ J \to C^n$$

has fiber dimension at least 1. From the first half of our proof, moreover, we see that if every hyperplane through p contained n linearly dependent points $p_1,\ldots,p_n$ of C, then for a generic such hyperplane H the points $p_1,\ldots,p_n$ would uniquely determine H, so that the image $\pi_1(J) \subset C^n$ would have dimension at least $n-1$. But then J would have dimension at least n, and since the projection

$$\pi_2: \ J \to \mathbb{P}^{n*}$$

is finite-to-one, this would imply that $\pi_2(J) = \mathbb{P}^{n*}$, i.e., that every hyperplane section of C contained n linearly dependent points, contradicting our first basic lemma. Q.E.D.

Now let $C \subset \mathbb{P}^n$ again be a canonical curve, $W \subset |2H|$ the linear system of quadrics through C, and suppose that the base locus S of W is not equal to C. If any point $p \in S - C$ lay on a chord $L = \overline{qr}$ of C the line L, meeting each quadric $Q \in W$ in the three points q, r and p, would lie in Q and hence in S; it follows that we may choose a point $p \in S - C$ not lying on infinitely many chords of C. By our basic lemma II, if H is a generic hyperplane through p the $2n+1$ points

$$\{p\} \cup (H \cap C)$$

are in general position. But now the restriction $W|_H$ to H of W is a linear system of quadrics of dimension at least $\frac{1}{2}(n-1)(n-2)-1$ with base locus containing $2n+1$ points in general position, and so by Castelnuovo's argument, *the base locus* $S \cap H$ *of* $W|_H$ *must be a rational normal curve*, hence S *is a surface of minimal degree.*

If S is the Veronese surface $\iota_{2H}(\mathbb{P}^2)\subset\mathbb{P}^5$, then clearly C is just a quintic plane curve. On the other hand, if S is one of the ruled surfaces $S_{k,l}$, then by the computation of p. 532, the curve C—having maximal genus—must be linearly equivalent to

$$(m+1)H-(n-2-\varepsilon)C=3H-(n-3)C$$

since

$$m=\left[\frac{d-1}{n-1}\right]=\left[\frac{2n-1}{n-1}\right]=2.$$

In particular, we see that C has intersection number 3 with each of the lines of the surface $S_{k,l}$. C is thus expressible as a 3-sheeted cover of $\mathbb{P}^1$; such a curve is called *trigonal*.

Conversely, suppose the canonical curve $C\subset\mathbb{P}^n$ is trigonal, $\pi:C\to\mathbb{P}^1$ a threefold cover. Then the divisors $\{\pi^{-1}(\lambda)=p_1^\lambda+p_2^\lambda+p_3^\lambda\}_{\lambda\in\mathbb{P}^1}$ form a linear system of degree 3 and dimension 1 on C; by the geometric version of Riemann-Roch (p. 248), it follows that the points p_1^λ, p_2^λ, and p_3^λ are collinear for each λ. The line $L_\lambda=\overline{p_1^\lambda p_2^\lambda p_3^\lambda}$ then meets every quadric Q containing C in three points, and so lies on Q; the surface

$$S'=\bigcup_{\lambda\in\mathbb{P}^1}L_\lambda$$

is contained in—hence equal to—the intersection of all quadrics containing C.

Similarly, if C is a plane quintic curve, then by the adjunction formula

$$K_C=[2H_{\mathbb{P}^2}|_C],$$

so that the canonical map on C is just the restriction to C of the Veronese map $\iota_{2H}:\mathbb{P}^2\to\mathbb{P}^5$. In particular, if L is any line in $\mathbb{P}^2$, $\iota_{2H}(L)$ is a conic curve in $\mathbb{P}^5$ meeting C in five points; as before, any quadric containing C will have to contain $\iota_{2H}(L)$. The intersection of the quadrics containing $C\subset\mathbb{P}^5$ thus contains—hence equals—the Veronese surface.

Summarizing, we have proved*

Theorem (Enriques; Petri). *For* $C\subset\mathbb{P}^n$ *any canonical curve, either*

1. C *is entirely cut out by quadric hypersurfaces; or*
2. C *is trigonal, in which case the intersection of all quadrics containing* C *is the rational normal scroll swept out by the trichords of* C; *or*
3. C *is a plane quintic, in which case the intersection of the quadrics containing* C *is the Veronese surface* $\iota_{2H}(\mathbb{P}^2)\subset\mathbb{P}^5$, *swept out by the conic curves through five coplanar points of* C.

*Cf. B. Saint-Donat, On Petri's Analysis of the linear system of quadrics through a canonical curve, *Math. Annalen*, Vol. 206 (1973), pp. 157–175.

Note that since the rational normal scrolls $S_{k,l}$ (other than $S_{0,1} \subset \mathbb{P}^3$) contain only one family of lines, a trigonal curve of genus $g \geqslant 5$ can contain only one linear system of degree 3 and dimension 1.

4. RATIONAL SURFACES II

The Castelnuovo-Enriques Theorem

Now that we have a fairly complete picture of rational surfaces, a natural question is whether we can characterize them by numerical birational invariants. Clearly, if S is rational, $q(S)=p_g(S)=P_n(S)=0$; we now prove a converse.

Theorem of Castelnuovo-Enriques. *If* S *is an algebraic surface with* q(S)= P_2(S)=0, *then* S *is rational.*

Proof. First of all, we can blow down S to obtain a surface birational to S that does not contain any exceptional curves of the first kind; thus we may assume from the start that S contains no such curves.

To apply Noether's lemma we must show that S contains an irreducible curve C with $\pi(C)=0$ and $\dim|C|>0$. To begin with, we transpose the problem slightly: since $P_2(S)=0$, we have $p_g(S)=0$—a nontrivial section σ of K_S yields a nontrivial section $\sigma \otimes \sigma$ of K_S^2—so

$$\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S) = 1.$$

Moreover, for any curve C on S,

$$h^2(C) = h^0(K_S - C) = 0,$$

and the Riemann-Roch formula tells us that

$$h^0(C) \geqslant \frac{C\cdot C - K\cdot C}{2} + 1.$$

Now if C is a rational curve and $C\cdot C \geqslant 0$, then by the adjunction formula $K\cdot C \leqslant -2$; thus $h^0(C) \geqslant 2$ and $\dim|C| \geqslant 1$. It will suffice, then, to find an irreducible curve C on S such that

$$(*) \qquad \begin{cases} \pi(C)=0, \\ C\cdot C \geqslant 0. \end{cases}$$

Before we proceed with the proof, we want to make explicit a special corollary to Bertini's theorem.

Lemma. *If* $\{D_\lambda\}$ *is a pencil of curves on a surface and the generic element of* $\{D_\lambda\}$ *is reducible—i.e.,*

$$D_\lambda = E + \sum C_{\nu_\lambda},$$

where E *is the fixed component of* $\{D_\lambda\}$, *then* $C_\nu \cdot C_\nu \geqslant 0$ *for each* ν.

Proof. Let $\{D'_\lambda\} = \{D_\lambda - E\}$ be the reduced pencil; $\{D_\lambda\}$ will have only isolated base points. Let $\tilde{S} \xrightarrow{\pi} S$ be the blow-up of S at the base points of $\{D'_\lambda\}$, so that the proper transforms

$$\tilde{D}'_\lambda = \sum \tilde{C}_{\nu_\lambda}$$

form a linear system $\{\tilde{D}'_\lambda\}$ without base points on $\tilde{S}$. Then, since a point of intersection of $\tilde{C}_{\nu_\lambda}$ with $\tilde{C}_{\nu'_\lambda}$ would be a singular point of $\tilde{D}'_\lambda$, we see that for generic λ, the curves $\tilde{C}_{\nu_\lambda}$ are disjoint. Thus,

$$\tilde{C}_\nu \cdot \tilde{C}_\nu = \tilde{C}_\nu \cdot \tilde{D}' = 0$$

and consequently $C_\nu \cdot C_\nu \geqslant 0$. Q.E.D.

The proof of the Castelnuovo-Enriques theorem is in three cases, $K \cdot K < 0$, $K \cdot K = 0$, and $K \cdot K > 0$, the last of which is the most difficult. We start with

Case 1. $K \cdot K = 0$

First, by Riemann-Roch applied to the divisor $-K$,

$$h^0(-K) + h^2(-K) = h^0(-K) + h^0(2K) \geqslant 1;$$

but $h^0(2K) = P_2(S) = 0$, and so $h^2(-K) \geqslant 1$; i.e., there exists an effective divisor D linearly equivalent to $-K$. Note that $D \neq 0$, since the bundle K is nontrivial.

Let $L = [E]$ be a very ample line bundle on S; we may assume that $h^0(L - D) \neq 0$. Since E is positive,

$$E \cdot K = -E \cdot D < 0;$$

hence

$$E \cdot (E + mK) < 0 \qquad \text{for } m \gg 0.$$

This implies that $h^0(E + mK) = 0$ for $m \gg 0$, since if $E + mK$ were linearly equivalent to an effective divisor, we would have $E \cdot (E + mK) > 0$. Choose n such that

$$h^0(E + nK) > 0,$$
$$h^0(E + (n+1)K) = 0.$$

Now let $D' \in |E + nK|$, and write $D' = \sum a_\nu C_\nu$. Then

$$K \cdot D' = K \cdot (E + nK) = K \cdot E < 0;$$

thus $K \cdot C_{\nu_0} < 0$ for some ν_0. By Riemann-Roch applied to the divisor $-C_{\nu_0}$,

$$h^0(-C_{\nu_0}) + h^0(K + C_{\nu_0}) \geqslant \frac{C_{\nu_0} \cdot C_{\nu_0} + C_{\nu_0} \cdot K}{2} + 1$$
$$= \pi(C_{\nu_0}).$$

But $h^0(-C_{\nu_0})=0$ clearly, and since $K+C_{\nu_0}<K+D'$,

$$h^0(K+C_{\nu_0}) \leqslant h^0(K+D') = h^0(E+(n+1)K) = 0;$$

thus we have

$$\pi(C_{\nu_0}) = 0.$$

By the adjunction formula $C_{\nu_0}\cdot K<0$ implies $C_{\nu_0}\cdot C_{\nu_0}\geqslant -1$; but if $C_{\nu_0}\cdot C_{\nu_0}=-1$, then C_{ν_0} is an exceptional curve of the first kind, and we assumed that S contains no such curve. Consequently C_{ν_0} satisfies the numerical conditions (*), and we are done.

Case 2: $K\cdot K<0$

We claim first that, in this case, if E is any divisor on S,

$$h^0(E+nK) = 0 \qquad \text{for } n \gg 0.$$

To see this, first choose n_0 large enough that

$$K\cdot(E+n_0K) = K\cdot E + n_0K\cdot K < 0.$$

Now suppose $h^0(E+mK)\neq 0$ for some $m\geqslant n_0$; let $D\in|E+mK|$ and write $D=\sum_\nu a_\nu C_\nu$. $K\cdot D\leqslant K\cdot(E+n_0K)<0$, so $K\cdot C_{\nu_0}<0$ for some ν_0; then if $C_{\nu_0}\cdot C_{\nu_0}$ were negative, we would have $K\cdot C_{\nu_0}=C_{\nu_0}\cdot C_{\nu_0}=-1$, i.e., C_{ν_0} would be an exceptional curve of the first kind, contrary to assumption. Thus $C_{\nu_0}\cdot C_{\nu_0}\geqslant 0$ and hence, by the remark of p. 470, $C_{\nu_0}\cdot D'\geqslant 0$ for any effective divisor D'. Then, since $K\cdot C_{\nu_0}<0$, we have for $m'\gg 0$,

$$(E+m'K)\cdot C_{\nu_0} < 0 \Rightarrow h^0(E+m'K) = 0,$$

and our assertion is proved.

Let E be a very ample divisor with $h^0(E+K)\geqslant 2$; choose n such that

$$h^0(E+nK) \geqslant 2, \qquad h^0(E+(n+1)K) \leqslant 1.$$

Let D be a generic element of the system $|E+nK|$; by our corollary to Bertini's theorem, if we write

$$D = E + \sum C_\nu,$$

where E is the fixed component of D, then $C_\nu\cdot C_\nu\geqslant 0$ for all ν. Since $h^0(-C_{\nu_0})$ is clearly 0, we have by Riemann-Roch for $-C_\nu$,

$$h^0(K+C_\nu) \geqslant \frac{C_\nu\cdot C_\nu + C_\nu\cdot K}{2} + 1 = \pi(C_\nu).$$

But

$$h^0(K+C_\nu) \leqslant h^0(K+D) \leqslant 1,$$

and so $\pi(C_\nu)=0$ or $\pi(C_\nu)=1$ for all ν. If $\pi(C_\nu)=0$, we are done, since $C_\nu\cdot C_\nu\geqslant 0$; assume that $\pi(C_\nu)=1$. In this case $h^0(K+C_\nu)=h^0(K+D)=1$;

let $D' \in |K+C_\nu|$ and write

$$D' = \sum a_\mu E_\mu.$$

Note that $D' \neq 0$, since $D'=0 \Rightarrow K \sim -C_\nu \Rightarrow K\cdot K = C_\nu \cdot C_\nu < 0$. By the adjunction formula,

$$\pi(C_\nu) = 1 \Rightarrow K\cdot C_\nu = -C_\nu \cdot C_\nu \Rightarrow D' \cdot C_\nu = (K+C_\nu)\cdot C_\nu = 0;$$

and since $C_\nu \cdot C_\nu \geqslant 0 \Rightarrow C_\nu \cdot E_\mu \geqslant 0$ for all μ, it follows that $E_\mu \cdot C_\nu = 0$ for all μ.

Since $K\cdot C_\nu < 0$ and $K\cdot K<0$, we have $D'\cdot K<0$; thus $E_{\mu_0}\cdot K<0$ for some μ_0. But then

$$\begin{aligned} 0 > E_{\mu_0}\cdot K &= E_{\mu_0}\cdot (K+C_\nu) \\ &= a_{\mu_0} E_{\mu_0}\cdot E_{\mu_0} + \sum_{\mu\neq\mu_0} E_{\mu_0}\cdot E_\mu \\ &\geqslant a_{\mu_0} E_{\mu_0}\cdot E_{\mu_0}. \end{aligned}$$

Thus $E_{\mu_0}\cdot K = E_{\mu_0}\cdot E_{\mu_0} = -1$, i.e., E_{μ_0} is an exceptional curve of the first kind, a contradiction.

Case 3: $K\cdot K>0$

To begin with, since $h^0(2K)=0$, we have by Riemann-Roch for $-K$,

$$h^0(-K) \geqslant 1 + \frac{K\cdot K-(K\cdot -K)}{2} > 1,$$

i.e., $|-K|$ contains a pencil of curves. Let D be a generic element of $|-K|$; by our lemma,

$$D = E + \sum a_\nu C_\nu$$

with E the fixed component of $|D|$ and $C_\nu \cdot C_\nu \geqslant 0$ for all ν. Now if D is reducible—i.e., $D\neq C_1$—we have

$$h^2(-C_1) = h^0(K+C_1) = h^0\Big(-(a_1-1)C_1 - E - \sum_{\nu\neq 1} a_\nu C_\nu\Big) = 0,$$

and of course

$$h^0(-C_1) = 0,$$

so that by Riemann-Roch for $-C_1$,

$$0 \geqslant \frac{C_1\cdot C_1 + K\cdot C_1}{2} + 1 = \pi(C_1);$$

but then $\pi(C_1)=0$ and $C_1\cdot C_1 \geqslant 0$, so we are done. Thus we may assume $D=C_1$ is an irreducible curve on S; since $D\sim -K$, we have $D\cdot K = -D\cdot D$, and hence $\pi(D)=1$.

Now if every very ample line bundle on S were a multiple of K, it would follow then that every bundle on S is a multiple of K; i.e.,

$$H^2(S,\mathbb{Z}) = H^{1,1}(S,\mathbb{Z}) \cong \mathbb{Z}$$

with $c_1(K)$ as a generator. But then by Poincaré duality, we would have $K\cdot K=1$, and by Noether's formula

$$1 = \chi(\mathcal{O}_S) = \frac{K\cdot K+\chi(S)}{12} = \frac{1+3}{12},$$

a contradiction. Thus we can find a very ample line bundle $[E]$ on S that is not a multiple of K and such that $h^0(E+K)\geqslant 1$. Since $E\cdot K=-E\cdot D<0$, we see that $E\cdot(E+nK)<0$, and hence $h^0(E+nK)=0$, for $n\gg 0$; let n_0 be the integer such that

$$h^0(E+n_0K) \geqslant 1,$$
$$h^0(E+(n_0+1)K) = 0.$$

Let D' be a generic element of $|E+n_0K|$ and write $D'=\Sigma a_\nu C_\nu$; we know that $D'\neq 0$ because $E\neq -n_0K$. Now D is irreducible, and so $D\cdot D\geqslant 0\Rightarrow K\cdot C_\nu=-D\cdot C_\nu\leqslant 0$ for all ν. Again, $h^0(K+C_\nu)\leqslant h^0(K+D')=0$, and $h^0(-C_\nu)=0$, so by Riemann-Roch

$$0 \geqslant \frac{C_\nu\cdot C_\nu+K\cdot C_\nu}{2}+1 = \pi(C),$$

i.e., $\pi(C)=0$. Now we know $K\cdot C_\nu\leqslant 0$; if $K\cdot C_\nu<-1$, then $C_\nu\cdot C_\nu\geqslant 0$ and we're done. On the other hand, if $K\cdot C_\nu=-1$, then C_ν is exceptional of the first kind. Thus we may assume

$$C_\nu\cdot C_\nu = -2, \qquad K\cdot C_\nu = 0.$$

Apply Riemann-Roch to the divisor $D-C_\nu=-C_\nu-K$; since

$$h^0(2K+C_\nu) \leqslant h^0(2K+D') = 0,$$
$$h^0(D-C_\nu) \geqslant \frac{(D-C_\nu)\cdot(D-C_\nu)-K\cdot(D-C_\nu)}{2}+1$$
$$= \frac{D\cdot D+C_\nu\cdot C_\nu+D\cdot D}{2}+1$$
$$= K\cdot K > 0;$$

Let $\Gamma\in|D-C_\nu|$ and write $\Gamma=\Sigma b_\nu\Gamma_\nu$; $\Gamma\neq 0$, since $\Gamma=0\Rightarrow C_\nu=D=-K\Rightarrow K\cdot K=C_\nu\cdot C_\nu=-2$. Applying Riemann-Roch to $-\Gamma_\nu$, we have

$$h^0(K+\Gamma_\nu) \leqslant h^0(K+\Gamma) = h^0(-C_\nu) = 0$$
$$\Rightarrow 0 = h^0(-\Gamma_\nu) \geqslant \frac{\Gamma_\nu\cdot\Gamma_\nu+\Gamma_\nu\cdot K}{2}+1 = \pi(\Gamma_\nu),$$

i.e., $\pi(\Gamma_\nu)=0$ for all ν. But now

$$\Gamma\cdot K = (-K-C_\nu)\cdot K < 0 \Rightarrow \Gamma_{\nu_0}\cdot K < 0 \qquad \text{for some } \nu_0.$$

Consequently, either $\Gamma_{\nu_0}\cdot\Gamma_{\nu_0}=-1$—in which case Γ_{ν_0} is exceptional of the first kind—or $\Gamma_{\nu_0}\cdot\Gamma_{\nu_0}\geqslant 0$, and we are done. Q.E.D.

As an immediate corollary, we have

Luroth's Theorem. *If* M *is a rational surface,* f : M→N *a surjective holomorphic map, then* N *is rational.*

Proof. The proof is clear: if $P_2(N)$ were nonzero, then the vanishing of the pullback to M of a nonzero section of K_N^2 would imply that the Jacobian of f was everywhere zero, and hence that the image of f could not be all of N. Similarly, if N had a nonzero holomorphic 1-form, its pullback to M would have to vanish, so that again the Jacobian of f would be identically zero. Q.E.D.

The Enriques Surface

We will now show that the hypotheses $q=P_2=0$ of the Castelnuovo-Enriques theorem cannot be weakened. The condition $q=0$ clearly cannot be eliminated: if $S=\mathbb{P}^1\times E$ is the product of $\mathbb{P}^1$ with a Riemann surface of genus 1, then $K_S=-2(\{p\}\times E)$ and hence $P_n(S)=0$ for all n; but S cannot be rational, since $q(S)=1$. To show that the requirement $P_2=0$ cannot be replaced by the weaker condition $p_g=0$ is somewhat more difficult. Enriques did it by constructing a class of surfaces satisfying $q=p_g=0$ and $P_2\neq 0$, one of which we now describe.

Let $[X_0,X_1,X_2,X_3]$ be homogeneous coordinates on $\mathbb{P}^3$, and S the quartic Fermat surface, given as the locus of

$$F(X)=X_0^4+X_1^4-X_2^4-X_3^4=0.$$

S is a smooth surface, and by the adjunction formula, the canonical bundle

$$K_S=(K_{\mathbb{P}^3}+S)|_S$$

is trivial. Let T be the automorphism of $\mathbb{P}^3$ given by

$$T:\ [X_0,X_1,X_2,X_3]\mapsto\left[X_0,\sqrt{-1}\,X_1,-X_2,-\sqrt{-1}\,X_3\right].$$

T preserves $S\subset\mathbb{P}^3$, and so generates a group $\{T^n\}$ of automorphisms of S, of order 4. T has four fixed points in $\mathbb{P}^3$, $[0,0,0,1]$, $[0,0,1,0]$, $[0,1,0,0]$, and $[1,0,0,0]$, none of which lies on S. On the other hand, T^2 has two fixed lines

$$l_1=(X_0=X_2=0)\quad\text{and}\quad l_2=(X_1=X_3=0),$$

each of which intersects S transversely in four points; thus T^2 has eight isolated fixed points $p_1,\ldots,p_8$ on S.

The quotient of S by the group of automorphisms $\{T^n\}$ cannot be given the structure of a complex manifold—for one thing, a punctured neighborhood of the image of a fixed point p_i in $S/\{T^n\}$ has fundamental group $\mathbb{Z}/2$. If we let $\tilde{S}=\tilde{S}_{p_1,\ldots,p_8}\xrightarrow{\pi} S$ be the blow-up of S at the points $p_1,\ldots,p_8$, however, the automorphisms T^n on $\tilde{S}-E_1-\cdots-E_8\cong S-\{p_i\}$ extend to automorphisms $\{\tilde{T}^n\}$ of $\tilde{S}$, and the quotient $\tilde{S}/\{\tilde{T}^n\}$ *is* a complex manifold. To see this, let

$$x=\frac{X_1}{X_0},\qquad y=\frac{X_2}{X_0},\qquad z=\frac{X_3}{X_0}$$

be Euclidean coordinates on $(X_0\neq 0)$ in $\mathbb{P}^3$, so that S is given by

$$f(x,y,z)=1+x^4-y^4-z^4=0$$

and consider the fixed point $p=[1,0,1,0]=(0,1,0)$ of T^2 on S. In a neighborhood U of p in S (which we may take to be preserved by T^2) the functions x and z furnish local coordinates; if we let $\tilde{U}_1$ and $\tilde{U}_2$ be the complements in $\tilde{U}=\pi^{-1}U$ of the proper transforms of $(x=0)$ and $(z=0)$, respectively, then we may correspondingly take as local coordinates in U_1 the functions

$$x'=x,\qquad z'=\frac{z}{x},$$

and in U_2 the functions

$$x''=\frac{x}{z},\qquad z''=z.$$

Now T^2 is given in U by

$$T^2\colon\ (x,z)\to(-x,-z),$$

and so $\tilde{T}^2$ is given in U_1 and U_2 by

$$\tilde{T}^2(x',z')=(-x',z'),\qquad \tilde{T}^2(x'',z'')=(x'',-z'');$$

we see then that $\tilde{T}^2$ extends by the identity map on $E=\pi^{-1}(p)$ to an automorphism of $\tilde{U}$. On the image of U_1 in the quotient $\tilde{U}/\{\tilde{T}^2\}$, moreover, the functions $v=x'^2$ and z' provide local coordinates; on the image of U_2 in $\tilde{U}/\{\tilde{T}^2\}$, similarly, x'' and $u=z''^2$ provide local coordinates, giving the quotient the structure of a complex manifold. Since all the fixed points behave similarly, we see that *the quotient* $S''=\tilde{S}/\{\tilde{T}^{2n}\}$ *has naturally the structure of a complex manifold*; *and the quotient map* $\iota\colon\tilde{S}\to S''$ *is a double cover simply branched at the divisors* $E_1,\ldots,E_8$.

Now the automorphism T likewise induces an automorphism $\tilde{T}$ on $\tilde{S}$, which then descends to S''. $\tilde{T}$ is fixed-point-free on S'', moreover, so that the quotient $S'=S''/\{\tilde{T}^n\}=\tilde{S}/\{\tilde{T}^n\}$ naturally inherits a complex structure; the quotient map $\iota'\colon S''\to S'$ is, of course, an unbranched double cover.

Note that S'' (and hence S') is an algebraic variety: the reader may verify directly that the space of sections of H^4 on S invariant under T^2 (i.e., homogeneous polynomials of degree 4 involving only monomials $X_0^{\alpha_0}X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}$ with $\alpha_1+\alpha_3$ even) and vanishing to order 2 at the points p_i (or in other words, sections of $[\pi^*4H-2E_1-\cdots-2E_8]$ on $\tilde{S}$ invariant under $\tilde{T}^2$) embeds S'' as a surface of degree 16 in $\mathbb{P}^9$. In fact, S'' has two linearly independent positive line bundles: the space of sections of H^6 on S invariant under T^2 and vanishing to order 4 at each p_i embeds S'' as a surface of degree 8 in $\mathbb{P}^5$.

First observe that $q(S')=0$: if η were any holomorphic 1-form on S', $\iota'^*\eta$ would, of course, be a holomorphic 1-form on $\tilde{S}$, hence zero, since $q(\tilde{S})=q(S)=0$. Similarly, if ω were a holomorphic 2-form on S', $\iota'^*\omega$ would be a holomorphic 2-form on $\tilde{S}$, and so would give a holomorphic 2-form on S *invariant under* T. But we know that $H^0(S,\Omega^2)\cong\mathbb{C}$, and we see from the Poincaré residue map that a generator of $H^0(S,\Omega^2)$ is

$$\varphi=\frac{dx\wedge dz}{\partial f/\partial y}=\frac{dx\wedge dz}{4y^3}.$$

Since

$$T^*\varphi=-\varphi,$$

it follows that S has no holomorphic 2-form invariant under T, and hence that $p_g(S')=0$.

On the other hand, we have

$$T^{2*}\varphi=\varphi,$$

so φ descends to give a holomorphic 2-form ψ on S'' away from the branch locus $\iota(E_1+\cdots+E_8)$ of the map ι. We claim that in fact ψ extends over $\iota(E_i)$; to see this, let x',z' and x'',z'' be the coordinates introduced above on the neighborhood $\tilde{U}$ of the exceptional divisor $E\subset\tilde{S}$. We can write

$$\begin{aligned} x&=x', & z&=z'x',\\ dx&=dx', & dz&=z'dx'+x'dz', \end{aligned}$$

so

$$dx\wedge dz=x'dx'\wedge dz'.$$

Thus

$$\begin{aligned}\pi^*(\varphi)&=\frac{x'}{(1+x'^4-z'^4x'^4)^{3/4}}(dx'\wedge dz')\\ &=\iota^*\frac{(dv\wedge dz')}{2(1+v^2-z'^4v^2)^{3/4}},\end{aligned}$$

so that indeed ψ extends over all of S''. Finally, since $\varphi\otimes\varphi$ is invariant under T, $\psi\otimes\psi$ must likewise be invariant under the induced involution $\tilde{T}$

of S''. Thus $\psi\otimes\psi$ descends to give a nonzero holomorphic section of $K_{S'}\otimes K_{S'}$, and we see that $P_2(S')\neq 0$; so S' cannot be rational.

The surface S' is called an *Enriques surface*. We can compute some of its invariants as follows: first, since $K_S=0$,

$$K_{\tilde{S}} = E_1 + \cdots + E_8.$$

Now if ω is any meromorphic 2-form on S'', we see that

$$K_{\tilde{S}} = (\iota^*\omega) = \iota^* K_{S''} + E_1 + \cdots + E_8,$$

so the canonical bundle of S'' has Chern class 0. The holomorphic 2-form ψ on S'' is therefore nowhere zero, and $K_{S''}$ is trivial. Since

$$K_{S''} = \pi^* K_{S'},$$

moreover, we see that $K_{S'}$ has Chern class zero modulo torsion.

Since we know that $c_1(S)^2=0$, we can apply Noether's formula to obtain

$$\chi(S) = c_2(S) = 12\chi(\mathcal{O}_S) = 24,$$

and similarly, since $c_1(S'')^2=c_1(S')^2=0$,

$$\chi(S'') = 24, \qquad \chi(S') = 12.$$

So $b_1(S')=b_3(S')=0$, and $b_2(S')=h^{1,1}(S')=10$. S' cannot be simply connected—otherwise it would not have a connected two-sheeted unbranched cover. In fact, by the Lefschetz theorem, S is simply connected, and hence so is $\tilde{S}$. Now any loop γ in S'' lifts to $\tilde{S}$—just take the base point of γ to be in the branch locus $\iota(E_1+\cdots+E_8)$ of ι—so it follows that S'' is again simply connected, and hence

$$H_1(S',\mathbb{Z}) = \mathbb{Z}/2.$$

Another way to see that S'' is simply connected is as follows: we know that the fundamental group $\pi_1(S'')$ is torsion, since $q(S'')=0$; if $\pi_1(S'')$ had a subgroup of index d, then there would exist a d-sheeted covering space $M\xrightarrow{j} S''$ of S''. M would be a compact complex manifold with

$$K_M = j^* K_{S''}$$

trivial, and

$$\chi(M) = d\cdot K_{S''} = 24\cdot d.$$

Thus by Noether's formula

$$\chi(\mathcal{O}_M) = \frac{c_1^2+c_2}{24} = 2d.$$

But since K_M is trivial, $p_g(M)=1$, and $q(S'')=0$ implies $q(M)=0$, so we must have

$$\chi(\mathcal{O}_M) = 2, \qquad d = 1.$$

Cubic Surfaces Revisited

As an application of the techniques developed in the last two sections, we will now go back and give a shorter, if less ingenuous, analysis of a smooth cubic surface. We will then consider the correspondence between the lines on a cubic surface and the bitangents to a quartic curve in $\mathbb{P}^2$.

Let $S \subset \mathbb{P}^3$ be any smooth cubic surface. By the Lefschetz theorem, S is simply connected and so $q(S)=0$; by the adjunction formula

$$K_S = (K_{\mathbb{P}^3} S)|_S = -H|_S,$$

the canonical bundle of S is negative, and so $P_n(S)=0$ for all n. By the Castelnuovo-Enriques theorem, then, we see immediately that S is rational.

Now by Noether's formula,

$$1 = \chi(\mathcal{O}_S) = \frac{c_1(S)^2 + c_2(S)}{12},$$

and since $c_1(S) = -H$, $c_1(S)^2 = 3$ and it follows that

$$\chi(S) = c_2(S) = 9.$$

By our theorem on rational surfaces, S must be a ruled surface S_n blown up five times. But for any irreducible curve C on S we have

$$K \cdot C = -H \cdot C < 0,$$

and so by the adjunction formula $C \cdot C \geqslant -1$, i.e., S cannot contain an irreducible curve of self-intersection -2 or less. Thus S is S_0 or S_1 blown up five times, or, what is the same thing, $\mathbb{P}^2$ blown up six times. In fact, S must be $\mathbb{P}^2$ blown up in six distinct points $p_1, \ldots, p_6$: if at any stage in the sequence of blow-ups of $\mathbb{P}^2$ we blow up a point on the exceptional divisor of a previous blow-up, the proper transform of that exceptional divisor in S will have self-intersection < -1. Likewise, the points $p_1, \ldots, p_6$ must be "general" in the sense of p. 480: if three of the points p_i lay on a line $L \subset \mathbb{P}^2$, the proper transform $\tilde{L}$ of L in S would have self-intersection $\leqslant -2$, and similarly if all six lay on a conic $C \subset \mathbb{P}^2$, the proper transform $\tilde{C}$ of C would have self-intersection -2. Finally, the embedding line bundle of $S \cong \tilde{\mathbb{P}}^2_{p_1, \ldots, p_6} \xrightarrow{\pi} \mathbb{P}^2$ is given by

$$\begin{aligned} H &= -K_S = -(\pi^* K_{\mathbb{P}^2} + E_1 + \cdots + E_6) \\ &= \pi^*(3H) - E_1 - \cdots - E_6. \end{aligned}$$

Thus we can show as on p. 485 that S contains exactly 27 lines.

Now let S be, as above, a smooth cubic surface, $P \in S$ any point not lying on any of the 27 lines of S, and consider the projection map

$$\pi_P : S - \{P\} \to \mathbb{P}^2$$

of S from P onto a hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$. As we have seen, π_P extends to a holomorphic map

$$\tilde{\pi}_P : \tilde{S} \to \mathbb{P}^2$$

on the blow-up $\tilde{S}$ of S at P; $\tilde{\pi}_P$ expresses $\tilde{S}$ as a 2-sheeted branched cover of $\mathbb{P}^2$.

Let $B \subset \mathbb{P}^2$ be the branch locus of $\tilde{\pi}_P$. If $l \subset \mathbb{P}^2$ is a generic line, the plane $H_l = \overline{l, P} \subset \mathbb{P}^3$ spanned by l and P meets S in a smooth curve C_l: the generic hyperplane in $\mathbb{P}^3$ through P will not be tangent to S at P, and by Bertini's theorem will not be tangent to S anywhere else. The map $\tilde{\pi}_P$ then maps the proper transform $\tilde{C}_l \cong C_l$ onto $l \cong \mathbb{P}^1$ as a 2-sheeted cover; since C_l has genus 1, $\tilde{\pi}_P|_{\tilde{C}_l}$ must have four branch points, and since no branch point of a 2-sheeted cover can have multiplicity greater than 1, these points are all distinct. Thus the generic line l meets B in four points, and consequently B has degree 4. Note, moreover, that if $q \in \mathbb{P}^2$ is any point, then for the generic line $l \subset \mathbb{P}^2$ containing q the plane H_l meets S transversely: the generic plane in $\mathbb{P}^3$ containing $\overline{Pq}$ will not be tangent to S at any of the finite number of points of $\overline{Pq} \cap S$, and by Bertini's theorem will not be tangent to S elsewhere. Thus by the same argument the generic line through q in $\mathbb{P}^2$ meets B in four distinct points, so q cannot be a singular point of B; it follows that *B is a smooth quartic curve.*

We have argued that if the plane H_l meets S in a smooth curve, then l meets B in four distinct points, i.e., l is nowhere tangent to B. Conversely, let $Q \in S$ be any point of the branch locus $\tilde{B}$ of π_P, $q = \pi_P(Q) \in B$, and $l_0 = T_q(B)$ the tangent line to B at q. The line $\overline{PQ} \subset \mathbb{P}^3$ is clearly in the tangent plane $T_Q(S)$ to S at Q, and so is the tangent line $T_Q(\tilde{B})$ to $\tilde{B}$ at Q, hence so is the line $\pi_P(T_Q(B)) = l_0$; thus

$$T_Q(S) = H_{l_0}.$$

Summarizing, we see that for any line $l \subset \mathbb{P}^2$ through $q = \pi_P(Q) \in B$, l is tangent to B at Q if and only if H_l is tangent to S at Q, if and only if $C_l = H_l \cap S$ is singular at Q.

Now let $L_1, \ldots, L_{27}$ be the lines on S, and $l_1, \ldots, l_{27}$ their images in $\mathbb{P}^2$ under $\tilde{\pi}_P$. l_i is a line in $\mathbb{P}^2$ by our assumption that $P \notin L_i$. If $l_i = l_j$ for some $i \neq j$, moreover, the plane $H_{l_i} = H_{l_j}$ in $\mathbb{P}^3$ would contain both L_i and L_j and so meet S in the sum of three lines—but $P \in H_{l_i} \cap S$ does not lie on any line in S, so this cannot happen; thus the lines l_i are distinct. Note also that under the assumption that $P \notin L_i$, no line on S lies in the tangent plane T to S at P: if $L_i \subset T \cap S$, then we must have

$$T \cap S = L_i + C,$$

where C is a conic curve in T. But then C would be singular at P, hence would consist of two lines containing P.

Now consider the intersections of S with the planes H_{l_i}, $i=1,\ldots,27$. H_{l_i} meets S in a curve of degree 3 containing L_i and no other line; thus

$$H_{l_i} \cap S = L_i + C_i,$$

where C_i is a smooth conic curve in H_{l_i}. C_i then meets L_i in two points Q_i and R_i (not necessarily distinct), which are singular points of $H_{l_i} \cap S$. By what we said above, then, the points $q_i = \pi_P(Q_i)$ and $r_i = \pi_P(R_i)$ are points of tangency of l_i with B. Thus either

1. $q_i \neq r_i$—i.e., C_i meets L_i transversely—in which case l_i is a bitangent line to B, or
2. $q_i = r_i$—i.e., C_i is tangent to L_i at $Q_i = R_i$—in which case every line L through P in H_{l_i} other than $\overline{PQ_i}$ will meet C_i and L_i in two distinct points, so that l_i will meet B *only* at q_i—i.e., l_i will have contact or order 4 with B at q_i. Such a line is called a *hyperflex* of B.

Finally, let $l_{28} \subset \mathbb{P}^2$ be the image under π_P of the tangent plane T to S at p—or, in other words, the image under $\tilde{\pi}_P$ of the exceptional divisor $E \subset \tilde{S}$. T intersects S in a cubic curve $C \subset T$ with a singularity at P, either a node or a cusp. The tangent lines to C at P in T map via π_P to points of tangency of l_{28} with B, and no other line through P in T maps to a point of B, so that either

1. P is a node of C, i.e., l_{28} is bitangent to B, or
2. P is a cusp of C, i.e., l_{28} is a hyperflex of B.

Conversely, let $l \subset \mathbb{P}^2$ be any bitangent to B. If $H_l \neq T$, then C_l must have two singular points. Since C_l is a cubic curve in H_l, it must then contain the line joining these two points, i.e., $l = l_i$ for some i. Similarly, if $l \subset \mathbb{P}^2$ is a hyperflex of B and $l \neq l_{28}$, then C_l maps down to l via π_P as a double cover branched only over $l \cap B = \{q\}$. Since $l - \{q\}$ is simply connected, $C_l - \pi_P^{-1}(q)$ is disconnected. Thus C_l is reducible; since C_l is a cubic curve, one of its irreducible components must be a line, and again we have $l = l_i$, for some i.

Note that if we realize S as $\mathbb{P}^2$ blown up in six points $x_1,\ldots,x_6$ with $x_7 \in \mathbb{P}^2$ corresponding to $P \in S$, then the linear system giving the map $\tilde{\pi}_P : \tilde{S} \to \mathbb{P}^2$ is just the proper transform of the system of cubic curves $C \subset \mathbb{P}^2$ passing through $x_1,\ldots,x_7$. In particular if

1. $L_i = E_i$; then $C_{l_i} = H_i \cap S$ corresponds to the cubic curve $C \subset \mathbb{P}^2$ through $x_1,\ldots,x_7$ singular at x_i. l_i will be a bitangent if C has a node at x_i, a hyperflex if C has a cusp at x_i.
2. $L_i = G_{ij}$; then C_{l_i} corresponds to the line L through x_i and x_j in $\mathbb{P}^2$ plus the conic C through $\{x_k : k \neq i,j\}$. l_i will be a bitangent if C meets L transversely, a hyperflex is C is tangent to L.

3. $L_i = F_i$; then C_{l_i} is the line L through x_i and x_7 in $\mathbb{P}^2$ plus the conic $C \subset \mathbb{P}^2$ through $\{x_k : k \neq i, 7\}$; again, l_i will be a hyperflex or a bitangent according to whether these curves are tangent or not.

We see from the above discussion that every quartic curve in $\mathbb{P}^2$ obtained as the branch locus of the projection of a cubic surface from a point on the surface has exactly 28 bitangents and hyperflexes, and that the generic quartic curve of this form has no hyperflexes and 28 bitangents.

Now we show that in fact every nonsingular quartic curve $B \subset \mathbb{P}^2$ can be realized as the branch locus of the projection of a cubic surface $S \subset \mathbb{P}^2$ from a point $p \in S$. We first show that we can construct a surface $\tilde{S}$ that is a double cover of $\mathbb{P}^2$ branched at B. To do this, fix an isomorphism of line bundles on $\mathbb{P}^2$

$$H^2 \otimes H^2 \longrightarrow H^4$$

and let $\sigma \in H^0(\mathbb{P}^2, \mathcal{O}(H^4))$ be a section defining B, i.e., such that $(\sigma) = B$. Then in the total space of the bundle $H^2 \xrightarrow{\pi} \mathbb{P}^2$, consider the locus

$$\tilde{S} = \{(p, \xi) : \xi \otimes \xi = \sigma(p)\}.$$

$\tilde{S}$ is readily seen to be a submanifold of H^2, and the projection map $\pi : H^2 \to \mathbb{P}^2$ expresses $\tilde{S}$ as a double cover of $\mathbb{P}^2$ branched exactly along B. (In general, a similar construction can be made of a double cover of $\mathbb{P}^2$ branched along any given curve of even degree; the singularities of the double cover will occur exactly over the singular points of the curve.)

Let $\tilde{B} \subset \tilde{S}$ be the branch locus of $\pi : \tilde{S} \to \mathbb{P}^2$ in $\tilde{S}$. If ω is any meromorphic 2-form on $\mathbb{P}^2$, we see that

$$K_{\tilde{S}} = (\pi^*\omega) = \pi^*(\omega) + \tilde{B} = \pi^*(-3H) + \tilde{B}.$$

But $2\tilde{B} = \pi^*(4H)$, and so

$$2K_{\tilde{S}} = \pi^*(-6H) + \pi^*(4H) = \pi^*(-2H).$$

Thus

$$\begin{aligned} 4K_{\tilde{S}} \cdot K_{\tilde{S}} &= \pi^*(-2H) \cdot \pi^*(-2H) \\ &= 2 \cdot (-2H) \cdot (-2H) \\ &= 8, \end{aligned}$$

i.e., $K_{\tilde{S}} \cdot K_{\tilde{S}} = 2$.

Now, taking a triangulation of $\mathbb{P}^2$ extending a triangulation of B and lifting it to $\tilde{S}$, we see that

$$\begin{aligned} \chi(\tilde{S}) &= 2 \cdot \chi(\mathbb{P}^2) - \chi(B) \\ &= 6 - (-4) \\ &= 10. \end{aligned}$$

Thus by Noether's formula

$$\chi(\mathcal{O}_{\tilde{S}}) = \frac{10+2}{12} = 1.$$

But $2K_{\tilde{S}}$ is the inverse of an effective divisor, so

$$P_2(\tilde{S}) = p_g(\tilde{S}) = 0.$$

Hence $q(\tilde{S})=0$ as well, and by Castelnuovo-Enriques, $\tilde{S}$ is rational. Since $\chi(\tilde{S})=10$, $\tilde{S}$ must then be the blow-up six times of some ruled surface S_n. But now for any irreducible curve D on $\tilde{S}$,

$$\begin{aligned} K_{\tilde{S}}\cdot D &= \tfrac{1}{2}\pi^*(-2H)\cdot D \\ &= -H\cdot\pi(D) < 0, \end{aligned}$$

since $\pi(D)$ must again be effective and nonzero. It follows from the adjunction formula that

$$D\cdot D \geqslant -1,$$

and hence, as on p. 545, $\tilde{S}$ must be either S_1 or S_0 blown up in six distinct points, or equivalently $\mathbb{P}^2$ blown up in seven distinct points $p_1,\ldots,p_7$. We see, moreover, that no three of the points p_i can lie on a line L: if they did, the proper transform $\tilde{L}$ of L in $\tilde{S}$ would have self-intersection $\leqslant -2$; similarly, no six of the points can lie on a conic curve $C\subset\mathbb{P}^2$: $\tilde{C}$ would have self-intersection $\leqslant -2$. Thus if we blow down any of the exceptional divisors E_i of $\tilde{S}=\tilde{\mathbb{P}}^2_{p_1,\ldots,p_7}$, the resulting surface $S=\tilde{\mathbb{P}}^2_{p_1,\ldots,\hat{p}_i,\ldots,p_7}$ may be embedded in $\mathbb{P}^3$ as a smooth cubic surface S.

To complete our argument, then, we claim that the map

$$\pi_p:\ \tilde{S}\to\mathbb{P}^2,$$

obtained by projecting S from the image point P of the exceptional divisor $E_i\subset\tilde{S}$, is the same as our original map $\pi:\tilde{S}\to\mathbb{P}^2$. But this is clear: on the one hand, the hyperplane section of $S\subset\mathbb{P}^3$ is just the dual $-K_S$ of the canonical bundle of S, and so the proper transforms in $\tilde{S}$ of hyperplane sections of S through P are elements of $|-K_{\tilde{S}}|$, i.e.,

$$\pi_P = \iota_{-K_{\tilde{S}}}.$$

On the other hand, we have seen that

$$2K_{\tilde{S}} = \pi^*(-2H);$$

since $\tilde{S}$ is rational and so has no torsion in $\mathrm{Pic}(S)=H^2(S,\mathbb{Z})$, it follows that $K_{\tilde{S}}=\pi^*(-H)$, i.e.,

$$\pi = \iota_{-K_{\tilde{S}}} = \pi_P.$$

Thus the branch locus of the projection of $S\subset\mathbb{P}^3$ from P is the quartic curve B we started with.

In conclusion, then, *every smooth quartic curve in* $\mathbb{P}^2$ *has exactly 28 bitangents and/or hyperflexes.*

The Intersection of Two Quadrics in $\mathbb{P}^4$

Recall that in the section on Grassmannians we saw that the set of lines lying on a quadric hypersurface $Q \subset \mathbb{P}^n$ represented the Schubert cycle $4 \cdot \sigma_{21}$ in the Grassmannian $G(2, n+1)$ of lines in $\mathbb{P}^n$. In particular, this suggested that the generic intersection of two quadrics $Q, Q' \subset \mathbb{P}^4$ contained

$$(4\sigma_{21} \cdot 4\sigma_{21})_{G(2,5)} = 16$$

lines. We can now show that this is indeed the case for any smooth intersection, $Q \cap Q'$.

Let Q and Q' be any two quadric hypersurfaces in $\mathbb{P}^4$ intersecting transversely in a surface S. First, by the adjunction formula,

$$K_Q = (K_{\mathbb{P}^4} + Q)|_Q = -3H|_Q$$

and

$$K_S = (K_Q + Q')|_S = -H|_S.$$

In particular,

$$c_1^2 = H \cdot H = \deg S = 4$$

and, since K_S is negative,

$$p_g(S) = P_2(S) = 0.$$

Now Q is a positive divisor on $\mathbb{P}^4$ and S a positive divisor on Q, so by the Lefschetz theorem,

$$H^1(S, \mathbb{Z}) = H^1(Q, \mathbb{Z}) = H^1(\mathbb{P}^4, \mathbb{Z}) = 0.$$

Hence

$$q(S) = 0,$$

and, by Castelnuovo-Enriques, S is rational. By Noether's formula,

$$1 = \chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12} = \frac{4 + c_2}{12},$$

therefore $c_2(S) = \chi(S) = 8$. Then, since K_S is negative, for any irreducible curve D on S,

$$K_S \cdot D < 0,$$

which implies

$$D \cdot D \geqslant -1.$$

By our classification of rational surfaces, S *must be* $\mathbb{P}^2$ *blown up in five distinct points*. No three of these points may be collinear, moreover, since the proper transform in S of a line in $\mathbb{P}^2$ containing three of them would have self-intersection < -1.

Now let $E_1,\ldots,E_5$ be the five exceptional divisors of the blow-up, $\tilde{L}_{ij}$ the proper transform in S of the line $L_{ij}=\overline{p_ip_j}\subset\mathbb{P}^2$, and $\tilde{C}$ the proper transform in S of the conic $C\subset\mathbb{P}^2$ through all five points. Since the hyperplane section H of $S\subset\mathbb{P}^4$ is given by

$$H = -K_S = \pi^*(3H_{\mathbb{P}^2}) - E_1 - \cdots - E_5,$$

we see that

$$\begin{aligned} E_i\cdot H &= -E_i\cdot E_i = 1,\\ \tilde{L}_{ij}\cdot H &= 3(L_{ij}\cdot H_{\mathbb{P}^2}) - 2 = 1,\\ \tilde{C}\cdot H &= 3(C\cdot H_{\mathbb{P}^2}) - 5 = 1, \end{aligned}$$

i.e., E_i, $\tilde{L}_{ij}$, *and* $\tilde{C}$ *are all lines on* S. Conversely, if $D\neq E_i\subset S$ is any line, then since D can meet each line E_i in at most one point, its image $\pi(D)\subset\mathbb{P}^2$ is a smooth rational curve, hence either a line or a conic. If $\pi(D)$ is a line, then

$$\begin{aligned} 1 = H\cdot D &= \pi(D)\cdot H_{\mathbb{P}^2} - D\cdot\textstyle\sum E_i\\ &= 3 - D\cdot\textstyle\sum E_i, \end{aligned}$$

so D meets two of the exceptional divisors E_i; thus $\pi(D)$ contains two of the points p_i and so $D=\tilde{L}_{ij}$ for some i,j. Similarly, if $\pi(D)$ is a conic,

$$\begin{aligned} 1 = H\cdot D &= \pi(D)\cdot H_{\mathbb{P}^2} - D\cdot\textstyle\sum E_i\\ &= 6 - D\cdot\textstyle\sum E_i \end{aligned}$$

tells us that $\pi(D)$ contains all five points $p_1,\ldots,p_5$, i.e., $D=\tilde{C}$. Thus E_i, $\tilde{L}_{ij}$, and $\tilde{C}$ are all the lines on S, and consequently S contains exactly $5+10+1=16$ lines, as expected.

Note that any line on S will meet exactly five other lines on S: the line C will meet the five lines $\{E_i\}_i$, the line E_i will meet the line C and the four lines $\{L_{ij}\}_j$, and the line L_{ij} will meet the two lines E_i and E_j and the three lines $\{L_{kl}\}_{k\neq i,j;\, l\neq i,j}$.

We claim that conversely if $p_1,\ldots,p_5$ are any five points no three of which are collinear, and $S\xrightarrow{\pi}\mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at these points, then the linear system $|-K_S|=|\pi^*(3H_{\mathbb{P}^2})-E_1-\cdots-E_5|$ embeds S in $\mathbb{P}^4$ as the intersection of two quadrics. First, note that if $p_6\in S$ is any point lying off the inverse image $\pi^{-1}C$ of the conic $C\subset\mathbb{P}^2$ containing $p_1,\ldots,p_5$, our argument of p. 481 shows that the proper transform in the blow-up $\tilde{S}$ of S at p_6 of the linear system of curves $D\in|-K_{\tilde{S}}|$ passing through p_6 embeds $\tilde{S}$ as a cubic in $\mathbb{P}^3$; thus a fortiori the complete linear system $|-K_S|$ embeds S as a surface of degree 4 in $\mathbb{P}^4$. Now, since $-2K_S$ is positive, by the Kodaira vanishing theorem

$$H^1(S,\mathcal{O}(-2K_S)) = H^1(S,\Omega^2(-3K_S)) = 0,$$

and likewise

$$H^2(S,\mathcal{O}(-2K_S)) = H^2(S,\Omega^2(-3K_S)) = 0.$$

Thus by Riemann-Roch,

$$\begin{aligned} h^0(-2K_S) &= 1 + \frac{(-2K_S\cdot -2K_S)-(-2K_S\cdot K_S)}{2} \\ &= 1+3K_S\cdot K_S \\ &= 13. \end{aligned}$$

But the linear system $|2H|$ on $\mathbb{P}^4$ has dimension $\binom{6}{2}-1=14$; since its restriction to S has dimension $13-1=12$, it follows that S must lie in—hence equal—the intersection of two quadric hypersurfaces in $\mathbb{P}^4$.

Note that we can also find the 16 lines on the intersection S of two quadrics Q,Q' by our knowledge of cubic surfaces in P^3. To see this, let $p\in S$ be any point not lying on a line of S; projection from p onto a hyperplane $\mathbb{P}^3$ then defines a holomorphic map

$$\pi_p:\ \tilde{S}\to\mathbb{P}^3$$

on the blow-up $\tilde{S}$ of S at p. This map is in fact an embedding: if any line L in $\mathbb{P}^4$ through p meet S twice away from p, it would have three points in common with each of the quadrics Q and Q', and so would lie on S—but we assumed to begin with that p lay on no line of S. The image $\pi_p(S)$ is a smooth cubic and so has 27 lines L_i on it, including the image of the exceptional divisor E of S. For each $L\neq\pi_p(E)\subset\pi_p(S)$, the inverse image $\tilde{L}$ of L in S will be either

1. a line on S, if L is disjoint from $\pi_p(E)$; or
2. a conic curve on S, if L meets $\pi_p(E)$.

Since, as we have seen, exactly 10 lines of $\pi_p(S)$ will meet $\pi_p(E)$, it follows that S contains $27-1-10=16$ lines.

5. SOME IRRATIONAL SURFACES

The Albanese Map

In this section we will discuss the overall classification of surfaces and briefly describe some basic types of surfaces other than the rational ones. We will classify surfaces by means of birational invariants, and will often assume that our surfaces are *minimal*; i.e., that they contain no exceptional curves of the first kind.

A basic new technique to be employed is the Albanese variety $\mathrm{Alb}(S)$ and Albanese mapping

$$\mu:\ S\to\mathrm{Alb}(S)$$

for a surface S. We recall (pp. 331–332) that $\mathrm{Alb}(S) = V/\Lambda$ where $V = H^0(S,\Omega^1)^*$ and Λ are the linear functions obtained by integrating over cycles in $H_1(S,\mathbb{Z})$. Explicitly, if $\eta_1,\dots,\eta_q$ are a basis for the holomorphic one-forms on S, then $V \cong \mathbb{C}^q$ and Λ is the lattice of vectors

$$\left(\int_\gamma \eta_1,\dots,\int_\gamma \eta_q\right), \qquad \gamma \in H_1(S,\mathbb{Z}).$$

The mapping μ is given by choosing a base point p_0, and then for $p \in S$ setting

$$\mu(p) = \left(\int_{p_0}^{p} \eta_1,\dots,\int_{p_0}^{p} \eta_q\right).$$

As in the curve case, the map μ induces isomorphisms

$$\mu_*:\ H_1(M,\mathbb{Z})/\text{torsion} \longrightarrow H_1(A,\mathbb{Z})$$

and

$$\mu^*:\ H^0(A,\Omega_A^1) \longrightarrow H^0(M,\Omega_M^1),$$

hence an isomorphism

$$\begin{aligned}\mathrm{Pic}^0(A) &= H^1(A,\mathcal{O}_A)/H^1(A,\mathbb{Z}) \underset{\mu^*}{\rightarrow} H^1(M,\mathcal{O}_M)/H^1(M,\mathbb{Z})\\ &= \mathrm{Pic}^0(M).\end{aligned}$$

Irrational Ruled Surfaces

In Section 3 of this chapter we defined a *rational ruled surface* to be a holomorphic $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Similarly, we define an *irrational ruled surface* to be a holomorphic $\mathbb{P}^1$-bundle $S \xrightarrow{\Psi} E$ over an irrational curve E.

The first thing to notice about such a surface is that the pullback map Ψ^* on holomorphic 1-forms is injective. Conversely, since the fibers C of Ψ are rational, any holomorphic 1-form η on S restricts to zero on the fibers. It follows that η is the pullback of a 1-form on E: in a neighborhood of any fiber of Ψ we may choose a point p_0 and set

$$f(p) = \int_{p_0}^{p} \eta;$$

f, being constant along the (connected) fibers of Ψ, is the pullback of a function g on an open set in E, and we can write

$$\eta = \Psi^*\xi, \qquad \xi = dg.$$

Inasmuch as η determines ξ, ξ is globally defined on E. The pullback map

$$\Psi^*:\ H^0(E,\Omega_E^1) \rightarrow H^0(S,\Omega_S^1)$$

is thus an isomorphism; in particular, the irregularity $q(S)$ of S is the genus of E.

Second, the fibers C of Ψ have self-intersection 0 and so by adjunction

$$K \cdot C = -2.$$

But since $C \cdot C = 0$, any effective curve on S has nonnegative intersection number with C. Thus no multiple of K can be effective, i.e.,

$$P_m(S) = 0 \qquad \text{for all } m.$$

Also, since S is a $\mathbb{P}^1$-bundle over a curve of genus $q(S)$,

$$\chi(S) = 2 \cdot \chi(E) = 4 - 4q,$$

and it follows from Noether's formula

$$1 - q = \chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12}$$

that

$$K \cdot K = 8 - 8q.$$

Finally, by either the Leray spectral sequence or the exact homotopy sequence of a fiber bundle, we see that the map

$$\Psi_*: H_1(S, \mathbb{Z}) \to H_1(E, \mathbb{Z})$$

is an isomorphism; it follows that the Albanese variety of S is just the Jacobian of E, and the Albanese map $S \to \text{Alb}(S)$ the composition of Ψ with the natural map $\mu: E \to \mathcal{J}(E)$.

This is as far as we will go into the geometry of ruled surfaces. Our previous analysis of rational ruled surfaces applies to irrational ruled surfaces in one respect: any ruled surface $S \to E$ is, by the same argument as given earlier, the projective bundle associated to a vector bundle of rank 2 on E. Here the analogy ends: it is *not* the case that any vector bundle over an irrational curve is the direct sum of line bundles. To study ruled surfaces in any greater detail thus requires more knowledge of vector bundles on curves than we have. One point to bear in mind, however, is this: in large measure the geometry of a ruled surface is a reflection of the geometry of its base curve.

In the remainder of this discussion we will give two numerical criteria for a surface S to be ruled. The first is

The Castelnuovo-de Franchis Theorem. *If* S *is minimal, i.e., contains no exceptional curves of the first kind, and if* $\chi(S) < 0$, *then* S *is an irrational ruled surface.*

Proof. Before we begin, we prove the

Lemma. *If* S *is minimal,* E *a curve, and* $\pi: S \to E$ *any holomorphic map whose generic fiber is irreducible and rational, then* S *is a* $\mathbb{P}^1$*-bundle over* E.

Proof. We will show that $S \xrightarrow{\pi} E$ can contain no reducible fibers; since all fibers of π have the same virtual genus 0, this will imply that all fibers are smooth. From the argument used for rational ruled surfaces (p. 514) it will follow that S is a $\mathbb{P}^1$-bundle.

Suppose that π has a reducible fiber

$$C = \sum n_i C_i, \qquad n_i > 0, \quad C_i \text{ irreducible.}$$

We may then write

$$0 = C \cdot C_i = n_i C_i \cdot C_i + \sum_{j \neq i} n_j C_i \cdot C_j$$

and, since all fibers are connected, the latter term is strictly positive; thus

$$C_i \cdot C_i < 0$$

for all i. On the other hand,

$$\pi(C) = \frac{C \cdot C + K \cdot C}{2} + 1 = 0,$$

so

$$K \cdot C = -2$$

and hence $K \cdot C_{i_0} < 0$ for some i_0. C_{i_0} then has negative intersection both with itself and with the canonical bundle, and so is an exceptional curve of the first kind. Q.E.D.

Thus, to prove the Castelnuovo-de Franchis theorem, we need only exhibit a map $\pi : S \to E$ whose generic fiber is irreducible and rational. We do this in two steps: we first find π under the assumption that S has two independent holomorphic 1-forms with wedge product identically zero; we then go back and show that every surface S with $\chi(S) < 0$ has two such forms. In following the proof of the former assertion, it is helpful to bear in mind the actual picture: for $S \xrightarrow{\pi} E$ ruled, any two 1-forms are pullbacks of forms η_1, η_2 on E. The quotient $f = \eta_1 / \eta_2$ then gives a map of E to $\mathbb{P}^1$ with the fibers of the composed map $\pi' : S \to E \to \mathbb{P}^1$ consisting of combinations of fibers of π, and we may reconstruct E as the set of connected components of fibers of π'.

Suppose that $\omega_1, \omega_2 \in H^0(S, \Omega^1)$ are linearly independent and that $\omega_1 \wedge \omega_2 \equiv 0$. Then the vectors

$$\omega_1(p), \omega_2(p) \in T_p^*(S)$$

are linearly dependent at every point $p \in S$, and therefore

$$\omega_1 = f \omega_2$$

for f some global meromorphic function on S; since ω_1 and ω_2 are linearly independent, f is nonconstant.

In a small polydisc Δ around any point $p_0 \in S$ we may set

$$\Psi(p) = \left(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2 \right).$$

Then since $\omega_1 \wedge \omega_2 \equiv 0$ the Jacobian of Ψ has rank one at a generic p, and so the image is an analytic arc C in $\mathbb{C}^2$. Now, if the arc C is given in a neighborhood of the origin as the locus of the holomorphic function $g(z_1, z_2)$, then we have

$$f = \Psi^*\left(\frac{\partial g / \partial z_1}{\partial g / \partial z_2} \right).$$

Since f is locally the pullback of a meromorphic function *on the curve* C, then, it follows that *the zero and polar divisors of* f *are disjoint*, and hence that f gives a holomorphic—rather than rational—map

$$\pi' : S \to \mathbb{P}^1$$

with fibers $C_\lambda = \pi'^{-1}(\lambda) = (f - \lambda)_0$. The fibers of π' may be (indeed will be) reducible; if so, by Bertini the irreducible components of the generic C_λ will be disjoint, and

$$C_\lambda = C_{\lambda,1} + \cdots + C_{\lambda,n}$$

with $C_{\lambda,i}$ connected and generically irreducible, and

$$C_{\lambda,i} \cdot C_{\lambda,j} = 0 \qquad \text{for all } i \neq j;$$

since, of course, $C_\lambda \cdot C_{\lambda,i} = 0$, it follows that

$$C_{\lambda,i} \cdot C_{\lambda,i} = 0$$

as well.

Consider the set of connected components $E = \{ C_{\lambda,i} \}_{\lambda,i}$ of curves in the pencil $\{ C_\lambda \}$. E forms a branched cover of $\mathbb{P}^1$ via the map λ and so inherits the structure of an algebraic curve. This is intuitively clear; a formal proof of the fact is based on two observations: all the curves C_{λ_i} (taken with proper multiplicity) are homologous, and only finitely many pairs $C_{\lambda,i}$ and $C_{\mu,j}$ are linearly equivalent. The first follows from the fact that the curves $\{ C_{\lambda_i} \}_{\lambda,i}$ must form a connected family: if instead the pairs (λ, i) broke up into two families A and B,

$$S = \left(\bigcup_{(\lambda,i) \in A} C_{\lambda,i} \right) \cup \left(\bigcup_{(\lambda,i) \in B} C_{\lambda,i} \right)$$

would itself be reducible. To see the second point, note that if two generic components $C_{\lambda,i}$ and $C_{\mu,j}$ are linearly equivalent, they span a pencil $\{ D_\alpha \}$ of curves. Every curve $C_{\lambda,i}$ will then meet at least one D_α, and so lie in D_α; thus the pencil $\{ D_\alpha \}$ is the pencil $\{ C_\lambda \}$, and $C_{\lambda,i}$ and $C_{\mu,j}$ constitute the entire fibers C_λ, C_μ—but since the generic C_λ has two or more connected components, there can be only finitely many such fibers. We may thus choose λ_0, i_0 and define

$$E' \subset S \times \mathrm{Pic}^0(S)$$

by

$$E' = \left\{(p,\delta):\ p \in C_{\lambda,i} \quad \text{and} \quad \left[C_{\lambda,i} - C_{\lambda_0,i_0}\right] = \delta\right\};$$

the desingularization E of E' is the curve we seek.

Now, the map

$$\pi' : S \to \mathbb{P}^1$$

factors through E via the map

$$S \xrightarrow{\pi} E,$$

sending a point p to the pair (λ, i) such that $p \in C_{\lambda,i}$. The fibers $C_{\lambda,i}$ of π are irreducible, and both the forms ω_1 and ω_2 vanish identically on all fibers of π. It follows as on p. 553 that the forms ω_i are the pullbacks of 1-forms η_1, η_2 on the base E. Since E has at least the two one-forms η_1 and η_2, then,

$$g(E) \geqslant 2,$$

so

$$\chi(E) < 0.$$

Recalling from p. 510 the formula

$$\chi(S) \geqslant \chi(E) \cdot \chi(F)$$

for the Euler characteristic of a surface S mapped to a curve E with generic fiber F, it follows from $\chi(S) < 0$ that the fibers of π have positive Euler characteristic, and so must be rational; thus S is ruled.

To complete the proof of the Castelnuovo-de Franchis theorem, we now argue that any surface S with $\chi(S) < 0$ contains two independent 1-forms with wedge product identically zero. To see this, note that from the Hodge decomposition,

$$\begin{aligned} 4q &= 2 + 2p_g + h^{1,1} - \chi(S) \\ &\geqslant 2p_g + 3 - \chi(S). \end{aligned}$$

It follows that

$$2q \geqslant p_g + \left(\frac{3 - \chi(S)}{2}\right).$$

In particular, $q \geqslant 2$—if q were 1, the Albanese map would take S to an elliptic curve, and by the formula of p. 510 quoted above the Euler characteristic of S would be nonnegative—so S contains at least two 1-forms ω_1, ω_2.

Now, since $\pi_1(S)$ contains at least a $\mathbb{Z}$-factor, we may for any m construct an m-sheeted covering space $\tilde{S} \xrightarrow{\pi} S$; $\tilde{S}$ inherits from S the structure of an algebraic surface. We then have

$$\chi(\tilde{S}) = m \cdot \chi(S),$$

so by taking m large we may assume $\chi(\tilde{S}) \leqslant -5$. Now consider the map

$$\wedge^2\left(H^0(\tilde{S},\Omega^1_{\tilde{S}})\right) \xrightarrow{\rho} H^0(\tilde{S},\Omega^2_{\tilde{S}})$$

given by wedge product. The kernel of ρ has codimension at most

$$p_g(\tilde{S}) \leqslant 2q(\tilde{S}) - \left(\frac{3-\chi(\tilde{S})}{2}\right) \leqslant 2q(\tilde{S}) - 4$$

in $\wedge^2 H^0(\tilde{S},\Omega^1_{\tilde{S}})$. On the other hand, the cone of decomposable vectors

$$\{\eta_1 \wedge \eta_2\} \subset \wedge^2 H^0(\tilde{S},\Omega^1_{\tilde{S}})$$

has dimension $2q(\tilde{S})-3$, and so must meet the kernel of ρ. $\tilde{S}$ thus contains two independent 1-forms with wedge product 0, and so by the first part of the argument is birational to a ruled surface. But as we have seen, on a ruled surface the wedge product of *any* two 1-forms is identically zero—in particular, the pullbacks $\pi^*\omega_1, \pi^*\omega_2$ of the two 1-forms ω_1, ω_2 on S have wedge product zero, and hence so do ω_1 and ω_2. Q.E.D.

Note also that we can take any surface S with $\chi(S) < 0$ and, blowing down, arrive at a minimal surface S_0 with $\chi(S_0) \leqslant \chi(S) < 0$; thus

Any surface with $\chi(S) < 0$ *is the blow-up of a ruled surface.*

There is a second similar criterion for a surface to be ruled:

Theorem. *If* S *is minimal and* $c_1^2(S) < 0$, *then* S *is irrational ruled.*

Proof. We start by proving the

Lemma. *If* S *is minimal and* $P_m(S) \neq 0$ *for any* $m > 0$*—that is, if any multiple* mK *of the canonical divisor on* S *is linearly equivalent to an effective divisor* D*—then* $c_1^2(S) \geqslant 0$.

Proof. Write

$$mK \sim D = \sum n_i D_i,$$

with $n_i > 0$ and D_i irreducible. Suppose that $K \cdot K$ were negative; we would then have

$$\begin{aligned} K \cdot mK = K \cdot \sum n_i D_i &< 0, \\ \Rightarrow K \cdot D_i &< 0 \qquad \text{for some } i. \end{aligned}$$

But then

$$0 > K \cdot n_i D_i = n_i D_i \cdot D_i + \sum_{j \neq i} n_j D_i \cdot D_j$$
$$\geqslant n_i^2 D_i \cdot D_i,$$

so $D_i \cdot D_i < 0$ and D_i is an exceptional curve of the first kind, contrary to the hypothesis that S is minimal. Q.E.D.

From this we see that if S is minimal and $c_1^2(S) < 0$, then $P_m(S) = 0$ for all m; in particular, $P_2(S) = p_g(S) = 0$. If the irregularity q of S were zero, it would follow from Section 4 that S was rational, and from Section 3 that S was either rational ruled or $\mathbb{P}^2$; in either case $c_1^2(S)$ would be positive. Thus $q(S) > 0$; and since $p_g(S) = 0$, the Albanese map $\overline{\Psi}: S \to \mathrm{Alb}(S)$ maps S onto a curve $\overline{E} \subset \mathrm{Alb}(S)$. Now let

$$\pi: E \to \overline{E}$$

be the desingularization of $\overline{E}$. The map

$$\Psi = \pi^{-1} \circ \overline{\Psi}: S \to E$$

is defined outside the divisor $\overline{\Psi}^{-1}(\overline{E}_s)$ on S, and is given, in terms of a local coordinate z around a point $p \in \pi^{-1}(\overline{E}_s)$, by a bounded holomorphic function; by Riemann's extension theorem the map Ψ extends to all of S. Note that since all holomorphic 1-forms on S are induced via $\overline{\Psi}$ from $\mathrm{Alb}(S)$ (and hence via Ψ from E), E must have genus at least q; on the other hand, since the pullback map

$$\psi^*: H^0(E, \Omega_E^1) \to H^0(S, \Omega_S^1)$$

is injective, it follows that E must have genus exactly q.

The final point in setting up the proof is that the fibers of the map $\Psi: S \to E$ are generically irreducible. To see this, we first note that they are generically smooth (apply Bertini's theorem to the pullback $\{\Psi^*(D_\lambda)\}$ of any pencil $\{D_\lambda\}$ on E) and so, if the generic fiber were reducible, it would have more than one connected component. We could then make the construction used in the last argument (p. 557) to factor the map Ψ

$$\begin{array}{ccc} S & \xrightarrow{\Psi} & E \\ & {\scriptstyle \Psi'} \searrow \quad \nearrow {\scriptstyle \alpha} & \\ & F & \end{array}$$

through the curve F consisting of connected components of fibers of Ψ. If E were of genus $q \geqslant 2$ or if the map α were branched, then by the

Riemann-Hurwitz formula

$$g(F) = m(q-1) + 1 + b > q,$$

where b is the number of branch points of α and m its sheet number; since the pullback Ψ'^* on 1-forms is injective, this is a contradiction. On the other hand, if q were 1 and α unbranched, then the image

$$\alpha_*(H_1(F,\mathbb{Z})) \subset H_1(E,\mathbb{Z})$$

would have positive index—contradicting the fact that the composed map

$$\begin{array}{ccccccc} \overline{\Psi}_*: \mathrm{H}_1(\mathrm{S},\mathbb{Z})/\text{torsion} & \xrightarrow{\Psi'_*} & \mathrm{H}_1(\mathrm{F},\mathbb{Z}) & \xrightarrow{\alpha_*} & \mathrm{H}_1(\mathrm{E},\mathbb{Z}) & \xrightarrow{\pi_*} & \mathrm{H}_1(\mathrm{Alb(S)},\mathbb{Z}) \\ & & \wr\Vert & & \wr\Vert & & \wr\Vert \\ & & \mathbb{Z}^2 & & \mathbb{Z}^2 & & \mathbb{Z}^2 \end{array}$$

is an isomorphism. Thus α cannot have degree 2 or more, and the fibers of Ψ are generically irreducible.

This completes the setting up. Our object now is to show that the fibers of Ψ are rational; by the lemma used in the proof of the Castelnuovo-de Franchis theorem, this will establish that S is ruled. The first step is to prove the

Lemma. *If* $c_1^2(S)<0$, *then* S *contains an irreducible curve* D *having non-negative self-intersection, and negative intersection number with* K.

Proof. First, since $K\cdot K<0$, by the index theorem the intersection pairing has one positive eigenvalue on the orthogonal complement $K^\perp$ of the canonical class in $H^{1,1}(S,\mathbb{Z})$; i.e., we can find a divisor class D_1 on S with

$$D_1\cdot K = 0, \qquad D_1\cdot D_1 > 0.$$

By Riemann-Roch, for any m,

$$h^0(mD_1) + h^0(K-mD_1) \geqslant 1 - q + \frac{m^2 D_1\cdot D_1}{2},$$

so for large m, either mD_1 or $K-mD_1$ will be effective. $K\cdot(K-mD_1)=K\cdot K$ is negative, so in the latter case we may take $D_2=K-mD_1$. On the other hand, if mD_1 is effective, we apply Riemann-Roch to the divisor mD_1+K to obtain

$$h^0(mD_1+K) + h^0(-mD_1) \geqslant 1 - q + \frac{m^2 D_1\cdot D_1}{2}$$

—but since mD_1 is effective, $h^0(-mD_1)=0$, and consequently mD_1+K is effective. $(mD_1+K)\cdot K<0$, so we may set $D_2=mD_1+K$. In either event we obtain an effective curve D_2 with $D_2\cdot K<0$, and of course some irreducible component D of D_2 will have negative intersection with K. Note that since $D\cdot K<0$ and S is minimal, $D\cdot D\geqslant 0$. Q.E.D.

Now, assume the fibers of Ψ are irrational of genus g. Let D be an irreducible curve on S with $D\cdot K<0$, and consider the linear systems

$$|D+nK|, \qquad n=0,1,2,\ldots.$$

These are eventually empty: for sufficiently large n

$$D\cdot(D+nK)<0;$$

but D, being irreducible with nonnegative self-intersection, has nonnegative intersection number with every effective divisor on S. We can thus choose an integer n such that

$$h^0(D+nK)\geqslant 1,$$
$$h^0(D+(n+1)K)=0.$$

Let G be a generic curve in the system $|D+nK|$; since

$$G\cdot K=D\cdot K+nK\cdot K<0,$$

some irreducible component G_0 of G will have negative intersection number with K. Let us first see that G_0 cannot be contained in a fiber C of Ψ: if we write

$$C=\sum n_iC_i,$$

then

$$0=C_i\cdot C=n_iC_i\cdot C_i+\sum_{j\neq i} n_jC_i\cdot C_j;$$

C being connected, the latter term is strictly positive, and so

$$C_i\cdot C_i<0 \qquad \text{for all } i.$$

Since S contains no exceptional curves of the first kind, then,

$$C_i\cdot K\geqslant 0 \qquad \text{for all } i.$$

Thus, *no component of a fiber of* Ψ *has negative intersection number with* K; accordingly, our curve G_0 cannot lie in a fiber, and so it must have positive intersection number $k=G_0\cdot C$ with C.

The argument now splits up into two cases:

Case 1: $q\geqslant 2$

Consider the representation

$$\Psi:\ G_0\to E$$

of G_0 as a k-sheeted cover of E; if $\Psi:G_0\to E$ has b branch points, by Riemann-Hurwitz

$$\pi(G_0)=k(q-1)+1+\frac{b}{2}$$

and in particular, if $k\geqslant 2$, then

$$\pi(G_0)>q.$$

In this case, from the Poincaré residue sequence

$$0 \to \Omega_S^2 \to \Omega_S^2(G_0) \to \Omega_{G_0}^1 \to 0$$

and

$$h^0(\Omega_{G_0}^1) = \pi(G_0) > h^1(\Omega_S^2)$$

we deduce that

$$h^0(\Omega_S^2(G_0)) \neq 0.$$

But

$$\begin{aligned} h^0(\Omega_S^2(G_0)) &= h^0(K+G_0) \leqslant h^0(K+G) \\ &= h^0(D+(n+1)K) \\ &= 0, \end{aligned}$$

so this is impossible. G_0 must thus have intersection number 1 with C and therefore is mapped biholomorphically onto E by Ψ. Note that since

$$\pi(G_0) = q \quad \text{and} \quad G_0 \cdot K < 0,$$

we have

$$G_0 \cdot G_0 > 2q - 2.$$

Now let C_λ and $C_{\lambda'}$ be two distinct generic fibers of $\Psi: S \to E$, and

$$G_1 = G_0 + C_{\lambda'} - C_\lambda.$$

G_1 being homologous to G_0, by Riemann-Roch

$$\begin{aligned} h^0(G_1) &\geqslant \chi(\mathcal{O}_S) + \frac{G_1 \cdot G_1 - G_1 \cdot K}{2} \\ &= 1 - q + \frac{G_0 \cdot G_0 - G_0 \cdot K}{2} \\ &\geqslant 1, \end{aligned}$$

so G_1 is effective. On the other hand, since G_0 and G_1 are homologous and not linearly equivalent, G_1 cannot contain G_0; thus G_1 and G_0 each meet the generic fiber C in a single point, and these points are distinct. But inasmuch as the line bundles $[C_\lambda]$ and $[C_{\lambda'}]$ on S both restrict to the trivial bundle on C,

$$[G_1]|_C = [G_0]|_C.$$

The single points $G_0 \cdot C$ and $G_1 \cdot C$ are thus linearly equivalent on C, contrary to the hypothesis that C is irrational.

Case 2: $q = 1$

Here our previous argument fails—a priori, G_0 could very well be a multisheeted unbranched cover of E— and so we must employ a somewhat subtler approach. Certainly, from the above we may conclude that

$$\pi(G_0) = q = 1.$$

Write

$$G_0 \cdot K = -d, \qquad d > 0;$$

then, by adjunction,

$$G_0 \cdot G_0 = d.$$

Now choose a fixed point $\lambda_0 \in E$, and for each $\lambda \in E$ set

$$G_\lambda = G_0 + C_\lambda - C_{\lambda_0}.$$

G_λ is once more homologous to G_0, and by Riemann-Roch

$$\begin{aligned} h^0(G_\lambda) &\geqslant 1 - q + \frac{G_0 \cdot G_0 - G_0 \cdot K}{2} \\ &\geqslant d; \end{aligned}$$

in particular G_λ is effective. Note that in fact we must have $h^0(G_\lambda) = d$: since no curve in the system $|G_\lambda|$ can contain G_0, $|G_\lambda|$ cuts out on G_0 a linear system of degree d and dimension $h^0(G_\lambda) - 1$; if $h^0(G_\lambda)$ were $d+1$ or greater, this would imply that G_λ was rational.

Now choose $d-1$ generic points $p_1, \ldots, p_{d-1}$ on S. For generic $\lambda \in E$, there will be a unique curve in the system $|G_\lambda|$ passing through the points P_i; we will denote this curve by H_λ. For two generic points $\lambda, \lambda' \in E$, H_λ and $H_{\lambda'}$ will intersect in d points, consisting of $p_1, \ldots, p_{d-1}$ and an additional point, which we will call $Q(\lambda, \lambda')$. Note that the points $Q(\lambda, \lambda')$ fill up the surface S, since the curves $\{H_\lambda\}_\lambda$ do and on any H_λ the divisors

$$\begin{aligned} Q(\lambda, \lambda') - Q(\lambda, \lambda'') &= (H_{\lambda'} - H_{\lambda''})|_{H_\lambda} \\ &\cong (C_{\lambda'} - C_{\lambda''})|_{H_\lambda} \end{aligned}$$

fill up $\mathrm{Pic}^0(H_\lambda)$. Now, considering the elliptic curve E as a group with origin λ_0, for any $\mu \in E$ set

$$F_\mu = \bigcup_{\lambda \in E} Q(\lambda, \mu - \lambda).$$

The points of F_μ are parametrized by the quotient of E by the involution $\lambda \mapsto \mu - \lambda$, and so F_μ is a priori either a point or a rational curve; since the points $\{Q(\lambda, \mu - \lambda)\}_{\mu, \lambda}$ fill up S, it follows that for generic $\mu \in E$, F_μ is a rational curve. S thus contains a one-parameter family of rational curves —but since any rational curve on S lies in a fiber of Ψ and the generic fiber is irreducible, this is impossible. Q.E.D.

Note that by the Riemann-Roch formula

$$\chi(\mathcal{O}_S) = 1 - q + p_g = \frac{c_1^2 + c_2}{12},$$

if the holomorphic Euler characteristic $\chi(\mathcal{O}_S)$ of S is negative, then either c_1^2 or c_2 is. We have, accordingly, a third

Theorem. *If* S *is minimal and* $\chi(\mathcal{O}_S)<0$, *then* S *is ruled.*

A Brief Introduction to Elliptic Surfaces

An *elliptic surface* with base E is a surface S and a map $\Psi: S\to E$ to a curve E such that the generic fiber of Ψ is an irreducible elliptic curve. Elliptic surfaces form a far more varied class than ruled surfaces: for one thing, whereas all fibers of a ruled surface are irreducible and smooth, an elliptic surface may have singular, reducible, and/or multiple fibers; for another, while all fibers of a ruled surface are of necessity isomorphic to one another, the complex structure of the fibers of an elliptic surface will, in general, vary from fiber to fiber. The various questions arising from these considerations—what configurations of curves may occur as reducible fibers of elliptic surfaces and how they affect the global geometry of the surface; what variations of the complex structure of the fiber are possible, especially around singular fibers—are both fascinating and, to a large degree, tractable. We are not able, in the present context, to go into these questions fully; the interested reader is referred to the papers of Kodaira, "On compact complex analytic surfaces," I, II, III, and IV, listed at the end of this chapter. One phenomenon associated to elliptic surfaces, however, unlike anything we have dealt with previously and warranting some discussion is that of *multiple fibers*, which we now describe.

Let us first say what a multiple fiber is. If $\Psi: S\to E$ is any holomorphic map of a surface S onto a curve E, then for generic $p\in E$, the pullback Ψ^*z of a local coordinate z on E centered around p will vanish simply to order 1 along the fiber $\Psi^{-1}(p)$ of Ψ over p. A fiber $C=\Psi^{-1}(p)$ of Ψ along which the pullback of a local defining function z for $p\in E$ vanishes to order $m\geq 2$ is called a *multiple fiber* of *multiplicity* m. (More properly, if $C=\Sigma C_i$ is reducible, and Ψ^*z vanishes to order n_i along C_i, we say that C is multiple if the greatest common divisor of the n_i is $m\geq 2$.) We note first a few points about a multiple fiber C_p:

1. Since the divisors $C_\lambda=(\Psi^*(z-\lambda))$ on S are all homologous, including $(\Psi^*z)=mC_p$, we see that $\mathrm{m}\cdot\mathrm{C_p}$ *is homologous to a generic fiber* C *of* Ψ. In particular,

$$C_p\cdot C_p=0 \quad\text{and}\quad C_p\cdot K=\frac{1}{m}C_\lambda\cdot K.$$

2. Similarly, if $\gamma:\Delta\to S$ is any holomorphic arc meeting the multiple fiber C_p transversely at $\gamma(0)$, the pullback $\gamma^*\Psi^*z$ vanishes to order m at 0; thus the map $\Psi\circ\gamma$ expresses Δ locally as an m-sheeted cover of its image in

E, with a branch point of multiplicity m at 0; in particular, $\gamma(\Delta)$ will meet every fiber near C_p not once but a total of m times. Likewise, if $\gamma(\Delta)$ meets C_p with multiplicity k, then $\Psi \circ \gamma$ expresses Δ as an mk-sheeted cover of its image, totally branched at 0, and $\gamma(\Delta)$ will meet fibers near C_p a total of mk times.

Now, it is easy to see that if the generic fiber of a map $\Psi: S \to E$ is rational, then no fiber C_p may be multiple: we would have

$$C_p \cdot C_p = 0 \quad \text{and} \quad C_p \cdot K = -\frac{2}{m},$$

contradicting at the very least the integrality of the virtual genus $\pi(C_p)$. If the generic fiber of Ψ is of genus $g \geqslant 2$, then we see in the same way that a multiple fiber C_p must have multiplicity m dividing $g-1$, and the genus of C_p will be $g' = (g-1)/m + 1 < g$. On the other hand, if $\Psi: S \to E$ is elliptic, Ψ may have multiple fibers of any multiplicity, all of the same genus 1.

We can construct a map $\Psi: S \to \Delta$ of an open surface S to the unit disc $\Delta \subset \mathbb{C}$ with a multiple fiber at 0 as follows. Let F be any elliptic curve, given as the complex plane with Euclidean coordinate w modulo the lattice $\Lambda = \{1, \tau\}$. Let z be the coordinate in the disc Δ, and consider the automorphism

$$\varphi: \Delta \times F \to \Delta \times F$$

given by

$$\varphi(z, w) = \left(e^{2\pi i/m} \cdot z,\ w + \frac{\tau}{m}\right).$$

φ has order m and all powers φ^i of φ are fixed-point-free; let S be the quotient of $\Delta \times F$ by the group $\{\varphi^i\}$ and $\Delta \times F \xrightarrow{\pi} S$ the quotient map. The map

$$\begin{aligned} \psi': \Delta \times F &\longrightarrow \Delta \\ (z, w) &\mapsto z^m \end{aligned}$$

factors through S to give a map

$$\Psi: S \to \Delta$$

whose fibers $C_\lambda = \Psi^{-1}(\lambda)$ are all elliptic curves. Now for $\lambda \neq 0 \in \Delta$ the divisor $\pi^* C_\lambda$ consists of the m elliptic curves $\{\{\varepsilon\} \times F : \varepsilon^m = \lambda\}$ each taken singly, while the divisor $\pi^* C_0$ consists of the single curve $\{0\} \times F$, *taken with multiplicity 1 since the map π is unbranched.* C_0 is thus a multiple fiber of Ψ, of multiplicity m.

More generally, we can alter any elliptic surface $S_0 \xrightarrow{\Psi_0} E$ to create multiple fibers, as follows. Start with any point $p \in E$ with C_p smooth and nonmultiple. By way of preparation, let U be a small disc around p so that Ψ_0 has no singular fibers over U, let z be a local coordinate in U centered around p such that $U=\{|z|<1\}$, and let $\Sigma_0=\Psi_0^{-1}(U)$. Take a section α of Ψ_0—that is, a map $\alpha: U\to S_0$ such that $\Psi_0\circ\alpha=id$, and for each $z\in U$ consider the fiber C_z as a group with $\alpha(z)\in C_z$ as the origin. The points of order exactly m in the fibers C_z form an unbranched cover of U and so break up into disjoint arcs; choose one such arc and call it β. The elliptic curve C_z may be realized as $\mathbb{C}$ modulo the lattice $\{1,\tau(z)\}$, with $\alpha(z)$ corresponding to the origin and $\beta(z)$ the point $1/m$. For any complex number t, then, let $t\cdot\beta(z)$ denote the point of C_z corresponding to $(t/m)\in\mathbb{C}$; this is well-defined.

Now define

$$\Sigma\subset\Delta\times\Sigma_0$$

by

$$\Sigma = \{(w,r):\ z(r)=w^m\}.$$

Note that the projection $w:\Sigma\to\Delta$ expresses Σ as elliptic over Δ, and that the fibers over points w and $e^{2\pi i/m}w$ are naturally identified with the fiber C_{w^m} of Ψ_0; accordingly, we can define an automorphism φ on Σ by

$$\varphi(w,r) = (e^{2\pi i/m}w,\ r+\beta(w^m)).$$

The quotient Σ_1 of Σ by the finite group $\{\varphi^i\}$ is, as in the first example, an elliptic surface over Δ via the map $\Psi_1(w,r)=w^m=z$ with a fiber of multiplicity m over $w=0$. Moreover, the fiber $\tilde{C}_z$ of Σ_1 over z is isomorphic to the fiber of the original surface S over z; indeed, *the inverse image* $\Psi_1^{-1}(\Delta-\{0\})$ *of the punctured disc in* Σ_1 *and the inverse image* $\Psi_0^{-1}(\mathrm{U}-\{0\})$ *are isomorphic as elliptic surfaces* via the map $\alpha:\Sigma\to S_0$ induced by

$$\bar{\alpha}:\ \Sigma\to S_0$$
$$:\ (w,r)\mapsto r+\left(\frac{1}{2\pi\sqrt{-1}}\log w\right)\cdot\beta(w^m)$$

away from the fibers over 0. We can thus glue the new surface Σ_1 into S_0 in place of the original Σ_0; i.e., we can set

$$S_1 = S_0-\Psi_0^{-1}(0)\cup_\alpha\Sigma_1$$

to obtain an elliptic surface $S_1\to E$, isomorphic to S_0 away from $C_p=\Psi_0^{-1}(0)$ and having a fiber of multiplicity m over p.

This operation—replacing an ordinary fiber of an elliptic surface with a multiple one—is called a *logarithmic transformation*; it was invented by Kodaira. One warning: if S_1 is obtained from S_0 by a logarithmic transformation on the fiber C_p as above, then it is clear from the formulas that under the isomorphism

$$S_0-C_p\cong S_1-\tilde{C}_p,\qquad \tilde{C}_p=\Psi_1^{-1}(0),$$

a curve in S_0 transverse to the fiber C_p may be mapped to a curve in $S_1 - \tilde{C}_p$ having an essential singularity along $\tilde{C}_p$. There may be, accordingly, very little correlation between closed curves in S_0 and in S_1; indeed, S_0 may be algebraic and S_1 not, or vice versa.

One basic fact about multiple fibers is the

Lemma. *If* B *is a smooth multiple fiber of multiplicity* m *in the elliptic surface* $\Psi : S \to E$, *then the normal bundle* $N_{B/S} = [B]|_B$ *of* B *is torsion of order exactly* m *in* $\mathrm{Pic}^0(B)$.

Proof. It is easy to see that the normal bundle to B is m-torsion: the bundles $\{[\Psi^*(p)]|_B\}_{p \in E}$ form a continuous family, trivial for $p \neq q \in \Psi(B)$ and hence for all p, including $\Psi^*(q) = [mB]$. To see that $N_{B/S}$ is indeed of order m, choose a local coordinate w on E centered around $q = \Psi(B)$, and a covering $\underline{U} = \{U_\alpha\}$ of a neighborhood U of B in S by small polydiscs. Since the function $\Psi^* w$ vanishes to order m along B, we can in each U_α choose an mth root of $\Psi^* w$, i.e., a holomorphic function z_α on U_α with

$$z_\alpha^m = \Psi^* w;$$

write

$$z_\alpha = e^{2\pi i k_{\alpha\beta}/m} \cdot z_\beta \qquad \text{in } U_\alpha \cap U_\beta$$

for $k_{\alpha\beta} \in \{0, 1, \ldots, m-1\}$.

Since z_α vanishes to order 1 along B in U_α, the 1-form dz_α restricted to B gives a nonzero section of the conormal bundle $N^*_{B/S}$ in $B \cap U_\alpha$; transition functions for $N^*_{B/S}$ are thus given by the constant functions

$$\frac{dz_\alpha}{dz_\beta} = e^{2\pi i k_{\alpha\beta}/m}.$$

Suppose that the dth power $N^d_{B/S}$ (and hence the dth power $(N^*_{B/S})^d$ is trivial. Then the cocycle

$$\left\{ g_{\alpha\beta} = e^{2\pi i d k_{\alpha\beta}/m} \right\} \in Z^1(B, \mathbb{C}^*)$$

is a coboundary, i.e., we can find constants $l_\alpha \in \mathbb{C}$ such that for all α, β,

$$e^{2\pi i d k_{\alpha\beta}/m} = \frac{l_\alpha}{l_\beta}.$$

We may normalize the l_α's by taking $l_1 = 1$; it then follows from the last equation that every l_α is a (d/m)th root of unity. Now let

$$z'_\alpha = l_\alpha \cdot z_\alpha^d.$$

The functions $z'_\alpha \in \mathcal{O}(U_\alpha)$ agree on the overlaps $U_\alpha \cap U_\beta$ and so define a single function $z' \in \mathcal{O}(U)$, with

$$z'^m = \Psi^* w^d.$$

Moreover, z', being constant along the fibers of Ψ, is in fact induced from a function w' on a neighborhood of q in E satisfying $w'^m = w^d$; but since w^d vanishes to order exactly d at q, $d = m \cdot \operatorname{ord}_q(w')$ is a multiple of m. We see, then, that $N^d_{B/S}$ is trivial if and only if m divides d. Q.E.D.

The proof of this lemma suggests a way to invert the logarithmic transformation; we will sketch this construction, leaving the details as an exercise for the reader. Let B be a smooth fiber of multiplicity m on $S_1 \xrightarrow{\Psi_1} E$, $U = \{|w| < 1\} \subset E$ a neighborhood of $\Psi_1(B)$, and $\Sigma_1 = \Psi_1^{-1}(U)$. Consider the set Σ of pairs (p, z_α), where p is a point of $\Sigma_1 \subset S_1$ and z_α a function element around p satisfying $z_\alpha^m = \Psi_1^* w$. By the proof of the lemma, Σ forms a connected unbranched, m-sheeted cover of Σ_1, and the map

$$\Sigma \xrightarrow{z = z_\alpha} \Delta$$

expresses Σ as an elliptic surface over the disc, with *no multiple fiber*: for $\lambda \neq 0$, the fiber $z^{-1}(\lambda)$ maps one-to-one onto the fiber $\Psi_1^{-1}(\lambda^m)$ of S_1, while $z^{-1}(0)$ forms an m-sheeted cover of the multiple fiber B of S_1. Now take an arc γ in S_1 transverse to B; γ then forms an m-sheeted cover of its image in E, branched totally over 0. The inverse image of γ in Σ then consists of m disjoint arcs, each transverse to the fibers of z_α. Choose one of these components and call it $\tilde{\gamma}$, and let φ_λ be the isomorphism

$$\varphi_\lambda = z^{-1}(\lambda) \longrightarrow z^{-1}(e^{2\pi i/m} \cdot \lambda)$$

consisting of the natural identification

$$z^{-1}(\lambda) \cong \Psi_1^{-1}(\lambda^m) \cong z^{-1}(e^{2\pi i/m} \cdot \lambda)$$

composed with a translation, and carrying $\tilde{\gamma} \cdot z^{-1}(\lambda)$ to $\tilde{\gamma} \cdot z^{-1}(e^{2\pi i/m} \cdot \lambda)$. The automorphism φ of Σ given by

$$\varphi(p, z_\alpha) = \left(\varphi_\lambda(p), e^{2\pi i/m} z_\alpha\right) \qquad (\lambda = z_\alpha(p))$$

then has order m and is fixed-point-free away from $z^{-1}(0)$, where it is the identity. The quotient Σ_0 of Σ by the group $\{\varphi^i\}$ is again elliptic via the map $w = z^m$ and is without multiple fibers; moreover, the complement of $w^{-1}(0)$ in Σ_0 is isomorphic to the complement of $\Psi_1^{-1}(0)$ in Σ_1, and so we may plug Σ_0 into $S_1 - \Psi_1^{-1}(0)$ to obtain a surface S_0, isomorphic to S_1 outside $\Psi_1^{-1}(0)$ and having a nonmultiple fiber over $q \in E$. The reader may verify that the operation is indeed inverse to the logarithmic transformation—i.e., that if we perform a logarithmic transformation to S_0 (making suitable choices of arcs α, β), we get S_1 back again—and so show that *every elliptic surface with smooth multiple fibers is obtained from an elliptic surface without multiple fibers by means of logarithmic transformations.*

We come now to the main point of our analysis of elliptic surfaces: the formula for the canonical divisor. We will consider here an elliptic surface $S \xrightarrow{\Psi} E$, all of whose multiple fibers are smooth—all of our formulas apply as well to the general case, but the proof is substantially harder.

To start, let $C_{\lambda_1}, \ldots, C_{\lambda_n}$ be n generic fibers of Ψ. Consider the Poincaré residue map

$$0 \to \Omega_S^2 \to \Omega_S^2(C_{\lambda_1} + \cdots + C_{\lambda_n}) \to \oplus \Omega^1_{C_{\lambda_i}} \to 0.$$

The image in $H^0(\oplus\Omega^1_{C_{\lambda_i}}) = \oplus H^0(C_\lambda, \Omega^1_{C_{\lambda_i}}) \cong \mathbb{C}^n$ of $H^0(S, \Omega_S^2(\Sigma C_{\lambda_i}))$ has codimension at most $h^1(S, \Omega_S^2) = q(S)$; thus

$$h^0\left(K + \sum_{i=1}^n C_{\lambda_i}\right) \geqslant n + p_g - q.$$

In particular, for n large, we see that $K + \Sigma_{i=1}^n C_{\lambda_i}$ is linearly equivalent to an effective divisor D. Now any fiber C of S has self-intersection 0 and hence, by adjunction, intersection number 0 with K. D, accordingly, has intersection number

$$D \cdot C_\lambda = \left(K + \sum_{i=1}^n C_{\lambda_i}\right) \cdot C_\lambda = 0$$

with the fiber C_λ and so must consist of a linear combination of fibers and components of fibers. We claim that D cannot contain a component of a reducible fiber C unless it contains the entire fiber. To see this, decompose C into irreducible components

$$C = \sum n_i C_i$$

and write

$$D = D' + \sum m_i C_i.$$

with D' disjoint from C. By the standard argument (p. 555), every component C_i of C has strictly negative self-intersection, and so by the hypothesis that S contains no exceptional curves of the first kind, $K \cdot C_i \geqslant 0$ for all i. Writing

$$K = D - \sum C_{\lambda_i},$$

this implies that for any i

$$\sum_{j \neq i} m_j C_i \cdot C_j \geqslant -m_i C_i \cdot C_i.$$

Now,

$$0 = C \cdot C_i = n_i C_i \cdot C_i + \sum_{j \neq i} n_j C_j C_i,$$

and so we obtain the inequality

$$\frac{\sum_{j\neq i} m_j C_i\cdot C_j}{\sum_{j\neq i} n_j C_i\cdot C_j} \geqslant \frac{-m_i C_i\cdot C_i}{-n_i\cdot C_i\cdot C_i} = \frac{m_i}{n_i} \qquad \text{for any } i;$$

it follows that *all the ratios* m_i/n_i *are equal.* Thus, if C is nonmultiple (i.e., the coefficients n_i have no common divisor), D contains the entire fiber $C=\sum n_i C_i$ with some multiplicity.

From the above, we see that we can write the canonical divisor

$$K = \Psi^* D + \sum \rho_i B_i$$

as the pullback of a divisor D on E, plus a linear combination of the multiple fibers B_i of S. If B_i has multiplicity m_i, then we can incorporate any integral multiple of $m_i B_i$ into D; thus we may take

$$0 \leqslant \rho_i \leqslant m_i - 1.$$

This determines ρ_i: by the adjunction formula,

$$K_{B_i} = [K+B_i]|_{B_i} = [(\rho_i+1)B_i]|_{B_i} \equiv 0;$$

since the bundle $[B_i]|_{B_i}$ is torsion of order exactly m_i, this implies

$$\rho_i = m_i - 1,$$

i.e.,

$$K = \Psi^* D + \sum (m_i - 1) B_i.$$

Finally, we ask for the degree d of D. We will find d by computing $h^0(K+\sum C_{\lambda_i})$ for n generic fibers $C_{\lambda_1},\ldots,C_{\lambda_n}$ of S in two ways: by Riemann-Roch on E, and on S. First, note that since

$$\left[K+\sum C_{\lambda_i}\right]\Big|_{B_i} = K|_{B_i} = [(m_i-1)B_i]|_{B_i},$$

the divisor $\sum(m_i-1)B_i$ *is a fixed component of the series* $|K+\sum C_{\lambda_i}|$. We have thus for n large

$$\begin{aligned} h^0\left(S,\mathcal{O}_S\left(K+\sum C_{\lambda_i}\right)\right) &= h^0\left(E,\mathcal{O}\left(D+\sum \lambda_i\right)\right)\\ &= \deg D + n - g(E) + 1 \end{aligned}$$

by Riemann-Roch on E. On the other hand, by Riemann-Roch on S,

$$h^0\left(K+\sum C_{\lambda_i}\right) - h^1\left(K+\sum C_{\lambda_i}\right) = \chi(\mathcal{O}_S),$$

inasmuch as $h^2(K+\sum C_{\lambda_i}) = h^0(-\sum C_{\lambda_i}) = 0$, and $K\cdot C_{\lambda_i} = C_{\lambda_i}\cdot C_{\lambda_i} = 0$. The problem hinges on determining $h^1(K+\sum C_{\lambda_i})$; to this end, consider the sequence

$$0 \to \mathcal{O}_S\left(-\sum C_{\lambda_i}\right) \to \mathcal{O}_S \to \oplus \mathcal{O}_{C_{\lambda_i}} \to 0.$$

From the long exact cohomology sequence

$$0 \to H^0\left(S, \mathcal{O}_S\left(-\sum C_{\lambda_i}\right)\right) \to H^0(S, \mathcal{O}_S) \to \oplus H^0\left(C_{\lambda_i}, \mathcal{O}_{C_{\lambda_i}}\right)$$
$$\to H^1\left(S, \mathcal{O}_S\left(-\sum C_{\lambda_i}\right)\right) \to H^1(S, \mathcal{O}_S) \to \oplus H^1\left(C_{\lambda_i}, \mathcal{O}_{C_{\lambda_i}}\right)$$

associated to this sequence, we see that

$$h^1\left(S, \mathcal{O}\left(K + \sum C_{\lambda_i}\right)\right) = h^1\left(S, \mathcal{O}\left(-\sum C_{\lambda_i}\right)\right)$$
$$= 0 - 1 + n + k,$$

where k is the dimension of the kernel of the map $H^1(S, \mathcal{O}_S) \to \oplus H^1(C_{\lambda_i}, \mathcal{O}_{C_{\lambda_i}})$ induced by restriction. To compute k, note that by the functoriality of the Dolbeault isomorphism—i.e., the commutativity of the diagram

$$\begin{array}{ccc} H^1(S, \mathcal{O}_S) & \longrightarrow & \oplus H^1(C_{\lambda_i}, \mathcal{O}_{C_{\lambda_i}}) \\ \| & & \| \\ H^{0,1}_{\bar\partial}(S) & \longrightarrow & \oplus H^{0,1}_{\bar\partial}(C_{\lambda_i}) \\ \| & & \| \\ \overline{H^{1,0}_{\bar\partial}(S)} & \longrightarrow & \oplus \overline{H^{1,0}_{\bar\partial}(C_{\lambda_i})} \end{array}$$

—this is just the number of holomorphic 1-forms on S whose restriction to each of the C_{λ_i} is identically zero.

If η is any 1-form on S with $\eta|_{C_{\lambda_i}} \equiv 0$, then the restriction of η to any smooth fiber C_λ of S is likewise zero in the de Rham cohomology of C_λ, and hence identically zero. Thus any 1-form vanishing on a single fiber of S vanishes on all fibers, and so by our previous argument (p. 553), is induced from a 1-form on the base E; conversely, of course, the pullback $\Psi^*\omega$ of any 1-form ω on E vanishes on every fiber of Ψ. The number of 1-forms on S vanishing on the curves C_{λ_i} is thus just the genus g of E; we have, accordingly,

$$h^1(K + \Sigma C_{\lambda_i}) = h^1(-\Sigma C_{\lambda_i}) = g + n - 1$$

and

$$h^0(K + \Sigma C_{\lambda_i}) = \chi(\mathcal{O}_S) + g + n - 1.$$

Combining this with the formula for $h^0(K + \Sigma C_{\lambda_i})$ obtained from

Riemann-Roch on E, we see that

$$\chi(\mathcal{O}_S) + g + n - 1 = h^0(K + \Sigma C_{\lambda_i}) = \deg D + n - g + 1,$$

and hence

$$\deg D = 2g - 2 + \chi(\mathcal{O}_S).$$

Summarizing, we have

The canonical bundle K *of an elliptic surface* $S \xrightarrow{\Psi} E$ *with multiple fibers* B_i *of multiplicity* m_i *is given by*

$$K = \psi^* D + \sum (m_i - 1) B_i,$$

where

$$\deg D = 2g(E) - 2 + \chi(\mathcal{O}_S).$$

Note that a suitable multiple mK of K will always be the pullback of a bundle on the base E, of degree

$$m\left(2g - 2 + \chi(\mathcal{O}_S) + \sum \frac{m_i - 1}{m_i}\right).$$

Kodaira Number and the Classification Theorem I

In terms of their gross characteristics, algebraic curves may be said to fall into three classes—genus 0, genus 1, and genus $g \geqslant 2$—according to whether the canonical bundle of the curve has negative, zero, or positive degree. Analogously, in trying to classify surfaces, we consider the behavior of their canonical bundles. Of course, we have for surfaces no notion completely analogous to the degree of a line bundle on a curve; nor does it suffice to consider only the dimension of the linear system $|K|$: as we have seen in our discussion of elliptic surfaces, there may exist surfaces on which a multiple of K is effective and nonzero, while $p_g = 0$. Indeed, taking the quotient of the Fermat quintic

$$S' = (X_0^5 + X_1^5 + X_2^5 + X_3^5 = 0) \subset \mathbb{P}^3$$

by the fixed-point-free group of automorphisms generated by

$$\varphi[X_0, X_1, X_2, X_3] = [X_0, e^{2\pi i/5} X_1, e^{4\pi i/5} X_2, e^{6\pi i/5} X_3],$$

we obtain a surface S (called a *Godeaux surface*) on which the canonical bundle is actually positive but has no sections. We therefore consider not just the dimension of the linear system $|K|$ but all the plurigenera $P_m(S) = H^0(S, \mathcal{O}(mK))$. There are, in broad terms, four possible types of behavior for the sequence $P_m(S)$:

1. It may be—as, for example, on a rational surface—that $P_m(S)=0$ for all m. Such a surface is said to have *Kodaira number* -1.

2. Assuming that $P_m(S)\neq 0$ for some m, we may ask whether the integers $P_m(S)$ are bounded. If in fact they are, then $P_m(S)$ must be either 0 or 1 for all m: if for some m the bundle mK had two linearly independent global holomorphic sections σ and τ, then the bundle mnK would possess at least the $n+1$ independent sections

$$\sigma^n, \sigma^{n-1}\otimes\tau, \ldots, \sigma\otimes\tau^{n-1}, \tau^n.$$

A surface whose plurigenera are bounded but not all 0 is said to have *Kodaira number* 0.

3. If the sequence $P_m(S)$ is unbounded, but

$$P_m(S) \leqslant c\cdot m$$

for some constant c, then S is said to have *Kodaira number* 1.

4. Finally, if the sequence $P_m(S)/m$ is unbounded, S is said to have *Kodaira number* 2, and to be of *general type*.

The Kodaira number of a surface S, usually written $\kappa(S)$, can also be thought of as either one less than the transcendence degree over $\mathbb{C}$ of the quotient field of the *pluricanonical* (graded) *ring*

$$\bigoplus_{m=0}^{\infty} H^0(S,\mathcal{O}(mK))$$

or alternatively as the dimension of the image of S under the rational map given by the linear system $|mK|$ for m large (-1 if the map is not defined —i.e., if $|mK|$ is empty—for all m).

For example, any ruled surface $S \xrightarrow{\Psi} E$ has Kodaira number -1, as we have seen. An elliptic surface $S \xrightarrow{\Psi} E$, on the other hand, may have Kodaira number -1, 0, or 1. As we saw in the last discussion, if $\{B_i\}$ are the multiple fibers of Ψ, m_i the multiplicity of B_i, and m a common multiple of the m_i, then

$$mK_S = \Psi^* L$$

for some line bundle $L\to E$ of degree

$$\deg(L) = m\left(2g-2+\chi(\mathcal{O}_S)+\sum \frac{m_i-1}{m_i}\right),$$

where g is the genus of E. The Kodaira number of S is clearly -1 if

$$\sum \frac{m_i-1}{m_i} < 2-2g-\chi(\mathcal{O}_S)$$

and 1 if

$$\sum \frac{m_i - 1}{m_i} > 2 - 2g - \chi(\mathcal{O}_S),$$

while in case $\deg(L) = 0$, S will have Kodaira number either -1 or 0 depending on whether some power of L is trivial or not (we shall see later that in fact $\deg(L) = 0$ implies $L^n \equiv 0$ for some n). Note one point that emerges from our description of the canonical bundle of an elliptic surface S: if $K \cdot D > 0$ for any effective curve D on S (in particular if some multiple of K is effective and nonzero), then $\kappa(S) = 1$.

The version of the classification theorem for surfaces we shall prove here consists of describing in turn surfaces of Kodaira number -1, 0, and $+1$. The description of surfaces of general type is not yet in as complete a form. We begin with the relatively easy case $\kappa = +1$; we will prove that

Any surface S *with Kodaira number* 1 *is elliptic.*

The proof is fairly straightforward. We note first that if S is minimal with $\kappa(S) = 0$ or 1, then $c_1^2(S) = 0$: if $c_1^2(S)$ were negative, then S would be ruled, while if $c_1^2(S)$ were positive, we would have by Riemann-Roch

$$h^0(mK) + h^0(-(m-1)K) \geqslant \frac{mK \cdot mK - mK \cdot K}{2} + \chi(\mathcal{O}) > \frac{c_1^2}{4} m^2$$

for m large, so that either $h^0(-(m-1)K) \gg 0$—in which case $\kappa(S) = -1$—or $h^0(mK) \geqslant c_1^2 m^2/4$, i.e., $\kappa(S) = 2$.

Lemma. *If any multiple* $D = mK$ *of the canonical bundle on a minimal surface is effective, and* $D \cdot D = 0$, *then all irreducible components* D_i *of* D *satisfy*

$$K \cdot D_i = 0, \qquad D_i \cdot D_i = 0 \text{ or } -2.$$

Proof. Write

$$D = \sum n_i D_i, \qquad n_i > 0, \quad D_i \text{ irreducible};$$

for each i we have

$$K \cdot D_i = \frac{1}{m}\Big(n_i D_i \cdot D_i + \sum_{j \neq i} n_j D_j \cdot D_i\Big) \geqslant \frac{n_i}{m} D_i \cdot D_i,$$

and it follows that $K \cdot D_i \geqslant 0$ for all i since otherwise D_i would be exceptional of the first kind. Now from the equation

$$0 = D \cdot D = m \sum n_i K \cdot D_i$$

we see that in fact $K \cdot D_i = 0$ for all i, and hence that $D_i \cdot D_i \leqslant (n_i/m) K \cdot D_i = 0$

must be either 0 or -2 for each i. Each component D_i is thus either rational or elliptic. Q.E.D.

Note that the lemma applies as well to curves D homologous to a multiple of K.

We may use this lemma to describe surfaces S with $\kappa(S)=1$. For some m, the linear system $|mK|$ contains at least a pencil $\{D_\lambda\}$, and by the lemma all components of each curve D_λ are either elliptic or rational. Since the rational components all have negative self-intersection, moreover, they cannot vary; thus if F is the fixed component of the pencil $\{D_\lambda\}$, the generic element of the pencil

$$\{D'_\lambda = D_\lambda - F\}$$

will contain only elliptic compoments $D'_{\lambda_1},\ldots,D'_{\lambda_n}$. Inasmuch as all components of D_λ have intersection number 0 with K, and hence with D_λ, we have

$$\begin{aligned} D'_\lambda\cdot D'_\lambda &= D_\lambda\cdot D_\lambda - 2D_\lambda\cdot F + F\cdot F \\ &\leqslant D_\lambda\cdot D_\lambda + F\cdot F \\ &\leqslant D_\lambda\cdot D_\lambda = 0. \end{aligned}$$

But since D'_λ moves in a pencil without fixed component, $D'_\lambda\cdot D'_\lambda \geqslant 0$; thus $D'_\lambda\cdot D'_\lambda=0$. Finally, by the lemma $D'_{\lambda_i}\cdot D'_{\lambda_i}=0$ and obviously $D'_{\lambda_i}\cdot D'_{\lambda_j}\geqslant 0$ for $i\neq j$, so it follows that $D'_{\lambda_i}\cdot D'_{\lambda_j}=0$ for all i and j, i.e.,

The pencil $\{D'_\lambda\}$ *has no base points, and its generic element* D'_λ *consists of a disjoint collection of elliptic curves.*

By the construction introduced in the proof of the Castelnuovo-de Franchis theorem (p. 557), the map

$$\pi:\ S\to\mathbb{P}^1$$

given by the pencil $\{D'_\lambda\}$ factors

$$\pi:\ S\xrightarrow{\pi'} E\to\mathbb{P}^1$$

through the curve $E=\{D'_{\lambda_i}\}_{\lambda,i}$ consisting of connected components of curves in the pencil $\{D'_\lambda\}$, and the fibers of π' are generically irreducible elliptic curves; thus S is elliptic. Q.E.D.

Note, incidentally, that by this argument a minimal surface S with $K\cdot K=0$ cannot have Kodaira number 2: if $P_m(S)>1$ for any m, S will be elliptic and so $\kappa(S)\leqslant 1$.

We next consider surfaces of Kodaira number -1. We will prove that

A minimal surface S *with* $\kappa(S)=-1$ *is either* $\mathbb{P}^2$ *or ruled.*

A large part of the work involved in proving this has already been done: if $q(S)=0$, then by the Castelnuovo-Enriques theorem of Section 4 the vanishing of $P_2(S)$ implies that S is rational, and our discussion in Section 3 of minimal rational surfaces shows that S is either $\mathbb{P}^2$ or rational ruled. On the other hand, if either $c_1^2(S)$ or $c_2(S)$ is negative, then, as we have seen, S is ruled, so we may assume both are nonnegative; from this it follows that

$$0 \leqslant \frac{c_1^2+c_2}{12} = \chi(\mathcal{O}_S) = 1-q,$$

i.e.,

$$c_1^2 = c_2 = \chi(\mathcal{O}_S) = 0 \quad \text{and} \quad q(S) = 1.$$

Let S be a surface with these numerical characters, and assume that S is not ruled. We will describe the geometry of S; our principal object for the present will be to show that $P_m(S)\neq 0$ for some m.

We start with some generalities. First, the Albanese map $\Psi: S \to E$ maps S to an elliptic curve E; assuming S is not ruled,the fibers will have genus $g \geqslant 1$. Since, moreover, $\chi(S)=0$, by the formula of p. 510, all fibers of Ψ must be smooth irreducible curves of genus g. Note in particular that S *contains no rational curves*, since any such curve would necessarily lie in a fiber of the Albanese map. Also, since S is not ruled, by the argument of p. 561, S contains no effective curve having negative intersection number with the canonical bundle K; from this follows the basic

Lemma. *If* D *is any curve on* S *with*

$$K\cdot D = D\cdot D = 0,$$

then D *consists of a disjoint collection of smooth elliptic curves* D_i, *each satisfying* $D_i\cdot D_i = K\cdot D_i = 0$.

Proof. The proof is not difficult: writing

$$D = \sum n_i D_i,$$

we see first that, since no D_i can have negative intersection number with K, and

$$K\cdot D = \sum n_i K\cdot D_i = 0,$$

we must have

$$K\cdot D_i = 0 \qquad \text{for all } i.$$

Then, since S contains no rational curves,

$$D_i\cdot D_i \geqslant 0 \qquad \text{for all } i,$$

and the inequality

$$0 = D \cdot D = \sum n_i^2 D_i \cdot D_i + \sum_{i \neq j} n_i n_j D_i \cdot D_j$$
$$\geqslant \sum n_i^2 D_i \cdot D_i \geqslant 0$$

implies that

$$D_i \cdot D_j = 0 \qquad \text{for all } i,j,$$

so the curves D_i are disjoint. Finally, since S contains no rational curves, any irreducible curve of virtual genus $\pi = 1$ on S is smooth. Q.E.D.

The main point of our study of S will be to show that S is elliptic *with rational base*. The argument for this proceeds in three steps: we show first that S must contain an elliptic curve transverse to the fibers C of Ψ, second that it must contain two disjoint such elliptic curves, and finally that it must contain an elliptic pencil transverse to the fibers of Ψ.

Step 1. S *contains an irreducible curve* F *with* $K \cdot F = F \cdot F = 0$, $F \cdot C > 0$, *transverse to the fibers of* Ψ.

This is the hardest of the three parts of the argument. We shall have to use two different approaches, depending on whether the genus g of the fibers of the Albanese map Ψ is 1, or more than 1.

Case 1: $g \geqslant 2$

In this case, we will show that S contains an effective curve homologous to $2K$; by the last lemma, every component of such a curve will be elliptic, and as we shall see, one must be transverse to the fibers.

Note first that since the fibers C_λ of Ψ have self-intersection 0, by the adjunction formula

$$K \cdot C_\lambda = 2g - 2.$$

For each $\lambda \in E$, consider the linear system

$$|2K + C_\lambda|.$$

Since

$$h^2(2K + C_\lambda) = h^0(-K - C_\lambda) \leqslant h^0(-K) = 0,$$

by Riemann-Roch we have

$$h^0(2K + C_\lambda) \geqslant \frac{(2K + C_\lambda)(K + C_\lambda)}{2} = 3g - 3.$$

On the other hand, the line bundle $[2K + C_\lambda]$ on S restricts to the bicanonical bundle $2K_{C_0}$ on any fiber C_0, and by Riemann-Roch for curves,

$$h^0\left(C_0, \mathcal{O}_{C_0}(2K_{C_0})\right) = 3g - 3.$$

Now, if for any λ the restriction map

$$r_\lambda : H^0(S, \mathcal{O}_S(2K + C_\lambda)) \to H^0\left(C_0, \mathcal{O}_{C_0}(2K_{C_0})\right)$$

failed to be injective, $2K + C_\lambda - C_0$ would be effective, and we would be done. Assume, on the other hand, that r_λ is injective (and hence an isomorphism) for all λ. In this case, if we choose any divisor $D = P_1 + \cdots + P_{4g-4}$ in the bicanonical series $|2K_{C_0}|$, then for every λ there will be a unique curve D_λ in the series $|2K + C_\lambda|$ cutting out D on C_0. Consider then the incidence correspondence

$$I \subset E \times S$$

defined by

$$I = \{(\lambda, p) : p \in D_\lambda\}$$

Since the curves D_λ are distinct, the image of I under the projection π_2 onto S cannot be a curve; since I is compact, it follows that $\pi_2 : I \to S$ is surjective. Thus, for any $Q \in C_0$, $Q \neq P_i$, there will be some curve D_λ containing Q—but then D_λ, containing the $4g-3$ points $P_1, \ldots, P_{4g-4}$ and Q on C_0, will contain C_0. $F = D_\lambda - C_0 \in |2K + C_\lambda - C_0|$ is thus effective, and we are done.

Case 2: The fibers of the Albanese map have genus $g = 1$

Note first that if the Albanese map $\Psi : S \to E$ had multiple fibers, then by the formula for the canonical class of an elliptic surface, S would have Kodaira number 1; assume therefore that Ψ has no multiple fibers. In particular, this means the canonical class of S is zero in homology. Let H be any curve on S having positive intersection number m with the fibers C_λ of Ψ. In each fiber C_λ of Ψ, consider the set of points

$$\left\{ p_i^\lambda \in C_\lambda : \left[m p_i^\lambda \right] = [H]|_{C_\lambda} \in \operatorname{Pic}(C_\lambda) \right\}.$$

Inasmuch as the map

$$E \to \operatorname{Pic}^0(E) \cong E$$

given by

$$p \mapsto [mp - mp_0]$$

is simply multiplication by m on the group E, there are exactly m^2 points $\{p_i^\lambda\}$ in each fiber C_λ, differing from one another by $1/m$ lattice points. The curve

$$F = \bigcup_{\lambda, i} \{p_i^\lambda\}$$

is thus an unbranched m^2-sheeted cover of E, and so every component F_i of F is elliptic. Since K is homologous to 0, $K\cdot F_i=0$ and hence $F_i\cdot F_i=0$, so we are done.

The remaining two steps apply equally in cases $g=1$, $g\geqslant 2$.

Step 2. *S contains two disjoint irreducible elliptic curves* F *and* F′ *satisfying* $K\cdot F=K\cdot F'=0$, $F'\cdot C>0$, $F\cdot C>0$, *transverse to the fibers of* Ψ.

We have already located one such elliptic curve F in step 1. For any n,

$$[nK+nF]|_F = nK_F \equiv 0;$$

consider the sequences

$$0\to \mathcal{O}_S(nK+(n-1)F)\to \mathcal{O}_S(nK+nF)\to \mathcal{O}_F\to 0.$$

We have, for $n\geqslant 2$

$$h^2(nK+(n-1)F) = h^0(-(n-1)K-(n-1)F) = 0$$

and likewise

$$h^2(nK+nF) = h^0(-(n-2)K-nF) = 0.$$

But

$$h^1(\mathcal{O}_F) = 1,$$

and so it follows that $h^1(nK+nF)\geqslant 1$. By Riemann-Roch, then,

$$\begin{aligned} h^0(nK+nF) &= \frac{(nK+nF)((n-1)K+nF)}{2}+\chi(\mathcal{O}_S)+h^1(nK+nF)\\ &= h^1(nK+nF)\geqslant 1, \end{aligned}$$

so $nK+nF$ is equivalent to an effective curve G_n. We note that G_n cannot be simply a multiple of F for all n: if, for example, we had

$$G_n = nK+nF = mF$$

and

$$G_{n+1} = (n+1)K+(n+1)F = m'F,$$

we would have

$$K\sim (m'-m-1)F,$$

and since $G_{n+1}\cdot C>G_n\cdot C, m'>m$, so this would imply $p_g\neq 0$. Thus, some G_n contains at least one component F' distinct from F, and by our lemma, F' is elliptic with $K\cdot F'=0$. Finally, since $G_n\cdot F=F\cdot F=0$, F' is disjoint from F, and since $F\cdot C>0$ this in turn implies that F' is transverse to the fibers C of Ψ.

Step 3. S *is elliptic with rational base.*

Let F and F' be the two disjoint elliptic curves found above. We have

$$[2K+2F+2F']|_F=[2K+2F]|_F = 2K_F\equiv 0$$

and likewise for F'; consider the sequence

$$0 \to \mathcal{O}_S(2K+F+F') \to \mathcal{O}_S(2K+2F+2F') \to \mathcal{O}_F \oplus \mathcal{O}_{F'} \to 0.$$

We have

$$h^2(2K+F+F') = h^0(-K-F-F') = 0$$

and likewise for $h^2(2K+2F+2F')$; since

$$h^1(\mathcal{O}_F \oplus \mathcal{O}_{F'}) = h^1(\mathcal{O}_F) \oplus h^1(\mathcal{O}_{F'}) = 2,$$

it follows that

$$h^1(2K+2F+2F') \geqslant 2.$$

By Riemann-Roch, then,

$$\begin{aligned} h^0(2K+2F+2F') &= \frac{(2K+2F+2F')(K+2F+2F')}{2} \\ &\quad + \chi(\mathcal{O}_S) + h^1(2K+2F+2F') \\ &= h^1(2K+2F+2F') \geqslant 2, \end{aligned}$$

i.e., the system $|2K+2F+2F'|$ contains at least a pencil $\{G_\lambda\}$. By the lemma of p. 576, every G_λ consists of a disjoint collection of elliptic curves; and we may apply once more the construction made in the proof of Castelnuovo's theorem to obtain a map

$$S \xrightarrow{\pi} B$$

of S onto a curve B, with fibers consisting of the variable components of the curves $\{G_\lambda\}$: thus S is elliptic. Finally, since $G_\lambda \cdot F = F \cdot F = 0$, every component of G_λ is either equal to or disjoint from F and so, as before, transverse to the fibers C of Ψ. The base B of an elliptic pencil on S must therefore be rational: if not, we could lift a holomorphic 1-form from B to S to obtain a 1-form on S not vanishing along the fibers of Ψ.

The assertion we set out to prove is easily seen, now that we have expressed S as an elliptic surface $\pi: S \to \mathbb{P}^1$ with rational base. If the fibers C of the Albanese map Ψ on S have genus $g \geqslant 2$, then the canonical bundle K_S has positive intersection with C, and we need only know that S is elliptic to conclude that $\kappa(S) = 1$. On the other hand, if the fibers of Ψ have genus 1, then we have seen that the canonical bundle of S is homologous to 0—but some multiple of K_S is the pullback $\pi^* L$ of a line bundle L on $\mathbb{P}^1$ and L, having degree 0, is trivial; thus $\kappa(S) = 0$. This completes the proof that a minimal surface S with Kodaira number -1 is ruled.

In fact, however, we can prove a bit more: namely,

Enriques' Theorem. *A minimal surface* S *with* $P_4(S) = P_6(S) = 0$ *is ruled or* $\mathbb{P}^2$.

Proof. This is clear in case $q(S) = 0$ or $q(S) \geqslant 2$. If $q(S) = 1$, on the other hand, and S is not ruled, then we have seen that either

1. The Albanese map Ψ on S has elliptic fibers, with some multiple fibers; or
2. S is elliptic with rational base.

In case 1, applying the formula for the canonical bundle of an elliptic surface to $\Psi: S \to E$, we have

$$K_S = \Psi^* D + \sum (m_i - 1) B_i,$$

where $\deg D = 0$; so

$$\begin{aligned} 2K_S &= 2\Psi^* D + \sum (2m_i - 2) B_i \\ &= \left(2\Psi^* D + \sum m_i B_i\right) + \sum (m_i - 2) B_i. \end{aligned}$$

The first term in this expression is the pullback of a divisor of positive degree on E, and so effective; thus $2K_S$ is effective and $P_2(S) \neq 0$.

Assume then that case 2 holds, and let

$$\pi: S \to \mathbb{P}^1$$

be the map expressing S as an elliptic surface with rational base. Let B be the generic fiber of π, $B_1, \ldots, B_k$ the multiple fibers of π, and $m_1, \ldots, m_k$ their multiplicities. We have then

$$K_S = -2B + \sum_{i=1}^{k} (m_i - 1) B_i$$

and, since $\kappa(S) \geqslant 0$,

$$(*) \qquad \sum_{i=1}^{k} \frac{m_i - 1}{m_i} \geqslant 2$$

with equality holding when $\kappa(S) = 0$ (note that by $(*)$, $k \geqslant 3$). Order the fibers B_i so that $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_k$; we separate the possible values of $\{m_i\}$ into four cases:

1. $k \geqslant 4$,
2. $k = 3$, $m_1 = 2$, $m_2 = 3$; in this case by $(*)$ we must have $m_3 \geqslant 6$,
3. $k = 3$, $m_1 = 2$, $m_3 \geqslant m_2 \geqslant 4$, and
4. $k = 3$, $m_3 \geqslant m_2 \geqslant m_1 \geqslant 3$.

In case 1,

$$\begin{aligned} 2K_S &= -4B + \sum_{i=1}^{k} (2m_i - 2) B_i \\ &= -4B + \sum_{i=1}^{k} m_i B_i + \sum (m_i - 2) B_i \\ &\geqslant \sum (m_i - 2) B_i \geqslant 0 \end{aligned}$$

is effective, so $P_2(S)\neq 0$. In case 4 we have

$$\begin{aligned} 3K_S &= -6B+\sum(3m_i-3)B_i \\ &= -6B+\sum 2m_iB_i+\sum(m_i-3)B_i \\ &\geqslant \sum(m_i-3)B_i \geqslant 0, \end{aligned}$$

so $P_3(S)\neq 0$. In case 3 we have

$$\begin{aligned} 4K_S &= -8B+4B_1+(4m_2-4)B_2+(4m_3-4)B_3 \\ &= -6B+3m_2B_2+3m_3B_3+(m_2-4)B_2+(m_3-4)B_3 \\ &= (m_2-4)B_2+(m_3-4)B_3 \geqslant 0, \end{aligned}$$

so $P_4(S)\neq 0$. Finally, in case 2 we see that

$$\begin{aligned} 6K_S &= -12B+6B_1+12B_2+(6m_3-6)B_3 \\ &= -5B+5m_3B_3+(m_3-6)B_3 \\ &= (m_3-6)B_3 \geqslant 0, \end{aligned}$$

so $P_6(S)\neq 0$.

Thus in every case either $P_4(S)\neq 0$ or $P_6(S)\neq 0$, and so Enriques' theorem is proved. Q.E.D.

Note that there are exactly four collections of integers $m_i\geqslant 2$ that satisfy equality in (*): they are

$$(2,2,2,2),\quad (2,3,6),\quad (2,4,4),\quad (3,3,3).$$

We will see in the following discussion that each of these in fact represents the multiplicities of the multiple fibers of a nonruled elliptic surface with rational base, and so Enriques' theorem is sharp.

The Classification Theorem II

To complete the classification theorem, we discuss surfaces of Kodaira number 0; we will find four distinct types of such surfaces. We make two observations before starting: by the remarks of p. 574, a minimal surface S with $\kappa(S)=0$ must have $c_1^2(S)=0$; and inasmuch as $\chi(\mathcal{O}_S)\geqslant 0$ and $p_g(S)\leqslant 1$, the irregularity $q(S)$ must be 0, 1, or 2. We proceed now by cases:

Case 1: $q=0$

There are two possibilities in this case: either $p_g=0$ or $p_g=1$. We consider these separately:

Case 1a: $p_g=1$

Here we have $\chi(\mathcal{O}_S)=2$, and by Riemann-Roch for $2K$

$$h^0(2K)+h^0(-K)\geqslant\frac{2K\cdot 2K-2K\cdot K}{2}+\chi(\mathcal{O}_S)$$
$$=2.$$

But $h^0(2K)=1$, and consequently $-K$ must be effective; since K and $-K$ are both effective, it follows that *the canonical bundle* K *of* S *is trivial.* A surface with these invariants—$q=0$ and $K\equiv 0$—is called a *K*-3 *surface*; we will give a brief description of these surfaces later in this section.

Case 1b: $p_g=0$

By the Castelnuovo-Enriques theorem, since $q(S)=0$ and S has Kodaira number 0, we must have $P_2(S)=1$. Now by Riemann-Roch applied to $3K$ we have

$$h^0(3K)+h^0(-2K)\geqslant\chi(\mathcal{O}_S)=1.$$

But $h^0(3K)$ must be zero: We know that there exists a global section σ of $2K$ not identically zero; if we had as well a nontrivial section τ of $3K$, then since $P_6(S)\leqslant 1$ we would have

$$\sigma^3=\lambda\cdot\tau^2$$

for some $\lambda\in\mathbb{C}$. But then if σ vanished to order k along any curve C in S, τ would vanish to order $3k/2$, and so τ/σ would be a global holomorphic section of K. Thus $h^0(3K)=0$, and so from Riemann-Roch $h^0(-2K)\neq 0$. As before, then, $h^0(2K)$, $h^0(-2K)\neq 0$ implies that $2K_S$ *is trivial.* A surface with these numerical characters—$q=p_g=0$ and $2K\equiv 0$—is called an *Enriques surface*; we have already seen one example of such a surface in Section 4 of this chapter, and we will discuss them in general at the end of this section.

Case 2: $q=2$

We will prove that

Any algebraic surface S *with* q = 2 *and Kodaira number* 0 *is an Abelian variety.*

Note first that since $\chi(\mathcal{O}_S)$ is nonnegative, we must have $p_g=1$. Let η_1,η_2 be generators for the space of holomorphic 1-forms on S. If the wedge product $\eta_1\wedge\eta_2$ were identically zero, the Albanese map Ψ on S would map S to a curve of genus 2; since $\chi(S)=0$ and S is not ruled, the fibers of Ψ would have genus 1. But we have seen that an elliptic surface over a base of genus $g\geqslant 2$ has Kodaira number 1; thus, under the assumption that $\kappa(S)=0$, the wedge product $\omega=\eta_1\wedge\eta_2$ is a generator for $H^0(S,\Omega_S^2)$.

Consider now the Albanese map Ψ of S onto the two-dimensional Abelian variety $A = \text{Alb}(S)$. The Jacobian determinant of Ψ is, of course, zero exactly where the forms η_1 and η_2 are dependent, i.e., on the canonical divisor $D = (\omega = \eta_1 \wedge \eta_2)$; we consider what this locus may be.

We first dispense with the possibility that the image of D in A has dimension 0. If this were the case, then the map

$$S - D \xrightarrow{\Psi} A - \Psi(D)$$

would be an unbranched covering. But then, since

$$\pi_1(A - \Psi(D)) \cong \pi_1(A) \cong \mathbb{Z}^4$$

and $\pi_1(S - D)$ surjects onto $\pi_1(S)$, it would follow from the isomorphism

$$H_1(S, \mathbb{Z})/\text{torsion} \longrightarrow H_1(A, \mathbb{Z})$$

that Ψ was 1-sheeted—i.e., that Ψ was birational. By the structure theorem for birational maps (Section 2 of this chapter), then, Ψ would be a blowing-down map, contrary to the hypothesis that S is minimal.

We see then that if $D \neq 0$, its image in A must contain a curve. Suppose that this is the case. By the lemma of p. 574, every component of the divisor D is either elliptic or rational, and since the Abelian variety A contains no rational curves, it follows that D has an elliptic component D_i, with $E = \Psi(D_i) \subset A$ again elliptic. We may take the origin $0 \in A$ to lie on E, and consider the map

$$\mu: A \to \text{Pic}^0(A)$$

given by

$$\mu: \lambda \mapsto [t_\lambda(E) - E],$$

where t_λ is translation by λ in the group A. Since any map between Abelian varieties is, up to translation, a homomorphism, $E = \Psi(D_i)$ is a subgroup of A, so translation by any point $\lambda \in E$ fixes E; on the other hand the reader may check, either directly or via the set-up of pp. 315–317, that $[E]$ cannot be fixed by all of A. The fibers of μ are thus one-dimensional, and the image $B = \mu(A)$ a curve; indeed, since the fiber of μ over 0 is a subgroup of A and hence smooth, E constitutes one connected component of the fiber $\mu^{-1}(0)$. Making the construction used in the proof of Castelnuovo's theorem (p. 557), we obtain a map

$$\tilde{\mu}: A \longrightarrow \tilde{B}$$

of A onto a (a priori possibly) branched cover $\tilde{B}$ of B, with E the fiber over a point; composing $\tilde{\mu}$ with Ψ, we obtain a map of S onto $\tilde{B}$ with D_i a fiber. S is thus an elliptic surface—but we have seen that any elliptic surface with an effective, nonzero canonical divisor has Kodaira number 1. We conclude that the divisor D must be zero, i.e., the canonical bundle of S is trivial. The map Ψ is then an unbranched covering, and S is an Abelian variety.

Case 3: $q=1$

A surface S with these characters—$\kappa(S)=0$ and $q(S)=1$—is called *hyperelliptic*, and we can give a fairly complete account of such surfaces. We start by showing that the geometric genus $p_g(S)$ of S must be zero: to see this, note that since $\pi_1(S)$ contains a $\mathbb{Z}$-factor, we can construct for any m an unbranched m-sheeted cover $\tilde{S}\xrightarrow{\pi}S$. If $p_g(S)$ were 1, then, we would have $\chi(\mathcal{O}_S)=1$,

$$\chi(\mathcal{O}_{\tilde{S}}) = m\cdot\chi(\mathcal{O}_S) = m,$$

and so

$$p_g(\tilde{S}\,) \geqslant m.$$

But now a section $\sigma\in H^0(\tilde{S},\Omega^2_{\tilde{S}})$ induces a section $\pi_*\sigma\in H^0(S,\mathcal{O}_S(K_S^m))$: since for any $p\in S$ and any $q\in\pi^{-1}(p)$ the fibers of K_S at p and $K_{\tilde{S}}$ at q are naturally identified via π, we may set

$$\pi_*\sigma(p) = \sigma(q_1)\otimes\cdots\otimes\sigma(q_m)\in K_{S,p}^{\otimes m},$$

where $\pi^{-1}(p)=\{q_1,\dots,q_m\}$. Clearly $\pi_*\sigma$ is not identically zero if σ is not. For $m\geqslant 2$, then, we could find a section σ of $K_{\tilde{S}}$ vanishing at some point $q\in\tilde{S}$ and another section τ of $K_{\tilde{S}}$ nonzero over $\pi(q)$; the images $\pi_*\tau$ and $\pi_*\sigma$ would then be two independent sections of $K_S^{\otimes m}$—so S would have Kodaira number $\geqslant 1$.

We conclude that a surface S with $\kappa(S)=0$ and $q=1$ must have $p_g=0$, and hence $c_1^2=c_2=0$. We have already discussed surfaces with these numerical invariants. Recall from our previous discussion that for such a surface S the fibers $\{C_\lambda\}$ of the Albanese map $\Psi:S\to E$ are elliptic, and none are multiple. S also contains a pencil $\{F_\lambda\}$ of curves, transverse to C_λ and having rational base. Let F be a nonmultiple element of the second pencil $\{F_\lambda\}$, and in the product $S\times F$ consider the surface

$$\tilde{S} = \{(p,q):\ \Psi(p)=\Psi(q)\}.$$

The projection $\pi_1:\tilde{S}\to S$ on the first factor expresses $\tilde{S}$ as an unbranched cover of S; the projection $\pi_2:\tilde{S}\to F$ expresses S as an elliptic surface with base F, again with no multiple or singular fibers. By our formula for the canonical bundle of an elliptic surface, we can write

$$K_{\tilde{S}} = \tilde{C}_\lambda - \tilde{C}_{\lambda'},$$

where $\tilde{C}_\lambda, \tilde{C}_{\lambda'}$ are fibers of $\pi_2 : \tilde{S} \to F$. Now let

$$\tilde{F} = \{(p,p) : p \in F\} \subset \tilde{S}.$$

$\tilde{F}$ maps via π_2 one-to-one onto F, and likewise one-to-one via π_1 onto $F \subset S$. But now we have

$$(C_\lambda - C_{\lambda'})|_{\tilde{F}} = K_{\tilde{S}}|_{\tilde{F}} = \pi_1^* K_S|_{\tilde{F}} = K_S|_F \equiv 0,$$

and hence

$$K_{\tilde{S}} \equiv 0.$$

Thus $p_g(\tilde{S}) = 1$, and since

$$\chi(\mathcal{O}_{\tilde{S}}) = m\chi(\mathcal{O}_S) = 0,$$
$$q(\tilde{S}) = 2.$$

Finally, $\tilde{S}$, like S, must have Kodaira number 0: if $K_{\tilde{S}}^{\otimes n}$ contained more than one linearly independent section, we could construct as above two independent sections of $K_S^{\otimes mn}$. Thus we see from our last argument that $\tilde{S}$ is an Abelian variety.

Indeed, we can be even more specific. Let C_0 be a fiber of the map π_2, and choose as the origin in $\tilde{S}$ the point of intersection of C_0 with $\tilde{F}$. Then we can define maps

$$\mu_1 : \tilde{S} \longrightarrow F \quad \text{and} \quad \mu_2 : \tilde{S} \longrightarrow C_0$$

by $\mu_1 = \pi_2$ and

$$\mu_2(\lambda) = t_\lambda(\tilde{F}) \cap C_0.$$

These maps give an isomorphism

$$\mu : \tilde{S} \longrightarrow C_0 \times F.$$

Thus, we see that *a surface* S *of Kodaira number* 0 *and irregularity* 1 *is the quotient, by a finite fixed-point-free group, of the product of two elliptic curves.*

We can construct these surfaces explicitly, as follows: let F and C be two arbitrary elliptic curves, with Euclidean coordinates z and w, and suppose C is given as $\mathbb{C}$ modulo the lattice $\Lambda = \{1, \tau\}$. Let $\zeta : C \to C$ be any automorphism on C of finite order m having fixed points (note that under this hypothesis the quotient of C by the group $\{\zeta^i\}$ is rational, because the quotient map $C \to C/\{\zeta^i\}$ is branched). Let φ be the automorphism of $F \times C$ defined by

$$\varphi(z,w) = \left(z + \frac{\tau}{m}, \zeta(w)\right).$$

φ is then fixed-point-free of order m, and the quotient S of $F \times C$ by the group $\{\varphi^i\}$ is a smooth algebraic surface. Since the 1-form dz on $F \times C$ is invariant under φ, it descends to give a 1-form on S. On the other hand,

$$\zeta^*(dw) = e^{2\pi ik/m}\cdot dw$$

for some $k\in\mathbb{Z}$, and since the quotient of C by $\{\zeta^i\}$—or any nonzero subgroup of $\{\zeta^i\}$—is rational, we see that k must be relatively prime to m. Thus neither of the forms dw nor $dz\wedge dw$ on S is invariant under φ. Since any holomorphic form on S lifts to a holomorphic form on $F\times C$ invariant under φ, it follows that

$$q(S)=1, \qquad p_g(S)=0;$$

more generally, since the generator $(dz\wedge dw)^{\otimes n}$ of $H^0(F\times C, \mathcal{O}(K^n))$ is invariant under φ if and only if m divides n,

$$\begin{aligned} nK_S &\equiv 0, \quad \text{if } m|n,\\ nK_S &\not\equiv 0, \quad \text{otherwise.}\end{aligned}$$

Note that the Albanese map Ψ sends S to the curve $E=\mathbb{C}/\{1, \tau/m\}$, with fibers isomorphic to C, while the second pencil $\{F_p\}$ of elliptic curves on S consists of the images in S of the fibers $F\times\{p\}$ of $F\times C$. In particular, if $p\in C$ is not a fixed point of any multiple of ζ other than the identity, the curve F_p forms via Ψ an m-sheeted cover of E, meeting a fiber C of Ψ in the points $\{\zeta^i(p)\}_i$. On the other hand, if q is fixed under a subgroup of order k in $\{\zeta^i\}$, then $F\times\{q\}$ maps k-to-1 onto its image F_q; F_q will then be a multiple fiber of multiplicity k in the pencil $\{F_p\}$, meeting a fiber C of Ψ in the m/k points of the orbit $\{\zeta^i(q)\}$.

We have four examples of this construction:

1. If C is any elliptic curve, we may take $\zeta(w)=-w$ to obtain a surface S with $2K_S\equiv 0$, $K_S\not\equiv 0$. Note that the second elliptic pencil $\{F_p\}$ has four double fibers, corresponding to the four fixed points p_i of ζ shown in Figure 6. S in this case is said to be of type I_a.

2. If C is the elliptic curve given as $\mathbb{C}$ modulo the lattice $\Lambda=\{1,i\}$, we may take $\zeta(w)=iw$ to obtain a surface with $4K$ trivial, but $p_g=P_2=0$. The

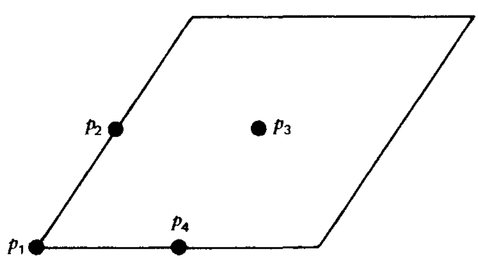


Figure 6

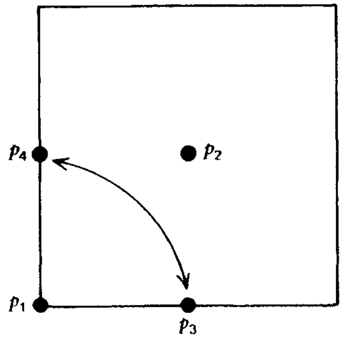


Figure 7

pencil $\{F_p\}$ on S has two quadruple fibers and one double, corresponding to the two fixed points p_1, p_2 of ζ and the fixed pair p_3, p_4 of ζ^2, as shown in Figure 7. S here is called type II_a.

3. If C is the elliptic curve $\mathbb{C}/\{1,e^{\pi i/3}\}$, we may take $\zeta(w)=e^{2\pi i/3}w$; S then has canonical bundle of order 3. The pencil $\{F_p\}$ then has three triple fibers, corresponding to the three fixed points p_i of ζ in Figure 8. S is said to be of type III_a.

4. With $C=\mathbb{C}/\{1,e^{\pi i/3}\}$ as above, we may set $\zeta(w)=e^{\pi i/3}w$; the canonical bundle of S then has order exactly 6. The pencil $\{F_p\}$ has one sextuple fiber, one triple fiber, and one double corresponding to the orbits $\{p_1\}$, $\{p_2,p_3\}$, and $\{p_4,p_5,p_6\}$ of ζ, respectively; see Figure 9. Note that S is the quotient of a surface of type III_a by an involution; S itself is said to be of type $\mathbf{III}_b$.

Note that the multiplicities of the multiple fibers of the surfaces S expressed as elliptic surfaces with rational base—that is, (2,2,2,2), (2,4,4), (2,3,6), and (3,3,3) in cases 1 through 4, respectively—correspond to all four solutions of the equation (*) of p. 581.

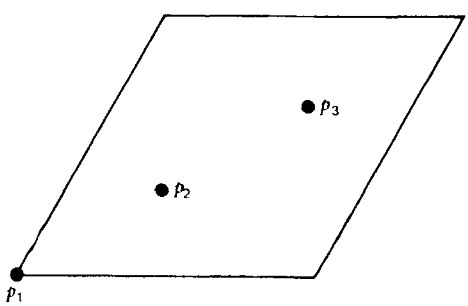


Figure 8

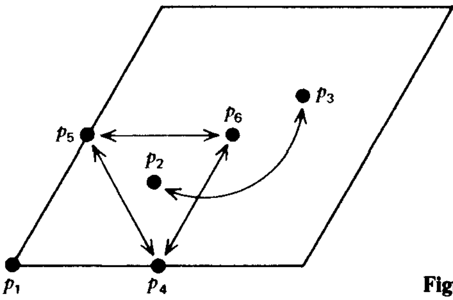


Figure 9

Finally, we can introduce one additional twist into our construction. Suppose that with C, F, ζ, and φ as above, we let

$$\zeta' : C \to C$$

be a translation of order n on C that commutes with the automorphism ζ (i.e., translation by a fixed point of ζ). We may then define a second automorphism of $F \times C$ by

$$\varphi'(z,w) = \left(z + \frac{1}{n}, \zeta'(w)\right).$$

φ and φ' then generate a finite, fixed-point-free group of automorphisms of $F \times C$, and the quotient $\tilde{S} = F \times C / \{\varphi^i \varphi'^j\}$ is again hyperelliptic: since the automorphism $\bar{\varphi}'$ induced by φ' on the partial quotient $S = F \times C / \{\varphi^i\}$ described above preserves all the forms and multiforms on $\tilde{S}$, the numerical invariants of $\tilde{S}$ are the same as those of S. The Albanese map Ψ sends $\tilde{S}$ to the curve $E = \ /\{1/n, \tau/m\}$ with fiber C, and the elements of the second pencil $\{F_p\}$ on S all form n-sheeted unbranched covers of their images $\tilde{F}_p$ in $\tilde{S}$, giving a second elliptic pencil on $\tilde{S}$ with multiple fibers corresponding exactly to those of $\{F_p\}$ on S. Explicitly, in each of the four cases above:

1. If $\zeta(w) = -w$ as in case 1 above, we may take ζ' to be translation by any of the points $\{p_i\}$ of order 2 on C, e.g.,

$$\varphi'(z,w) = \left(z + \tfrac{1}{2}, w + \tfrac{1}{2}\right).$$

The resulting surface $\tilde{S}$ is said to be of type I_b.

2. In case 2 above we must take ζ' to be translation by p_2, i.e.,

$$\varphi'(z,w) = \left(z + \frac{1}{2}, w + \frac{1+i}{2}\right);$$

$\tilde{S}$ is said to be of type II_b.

3. In case 3 we may take ζ to be translation by either of the points

p_2, p_3, e.g.,

$$\varphi'(z,w) = \left(z + \frac{1}{3}, w + \frac{3+\sqrt{3}\,i}{6}\right).$$

$\tilde{S}$ is then called type III_c.

4. In the last case above no nontrivial translation commutes with ζ, and we cannot make this construction.

We have described now seven classes of hyperelliptic surfaces: namely, I_a, I_b, II_a, II_b, III_a, III_b, and III_c. The reader may, by examining in general finite commutative groups of automorphisms on elliptic curves C, see that we have constructed all hyperelliptic surfaces.

In sum we have the following version of the

Classification Theorem (Enriques; Kodaira)

1. *A minimal surface* S *with* $\kappa(\mathrm{S}) = -1$ *is either* $\mathbb{P}^2$ *or ruled.*
2. *A minimal surface* S *with* $\kappa(\mathrm{S}) = 0$ *is either*
   (a) *a K-3 surface, if* $\mathrm{q}=0$ *and* $\mathrm{p_g}=1$,
   (b) *an Enriques surface, if* $\mathrm{q}=0$ *and* $\mathrm{p_g}=0$,
   (c) *a hyperelliptic surface as constructed above, if* $\mathrm{q}=1$; *or*
   (d) *an Abelian variety, if* $\mathrm{q}=2$.
3. *A surface* S *with* $\kappa(\mathrm{S}) = 1$ *is elliptic.*

K-3 Surfaces

To conclude this section, we wish to study in some more detail two types of surfaces encountered in the course of the classification theorem: K-3 surfaces and Enriques surfaces.

Let us first dispose of the numerical characters of a K-3 surface S. By definition,

$$q(S) = 0 \quad \text{and} \quad K_S \equiv 0,$$

so $p_g(S) = 1$ and $c_1^2(S) = 0$. By Riemann-Roch,

$$2 = \chi(\mathcal{O}_S) = \frac{c_2}{12},$$

so the topological Euler characteristic

$$\chi(S) = 24.$$

The Hodge diamond of S is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Suppose S is a K-3 surface embedded in $\mathbb{P}^n$, and suppose that S is normal—that is, the embedding is given by a complete linear system. Let $C = H \cdot S$ be a generic hyperplane section of S, and consider the standard sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Since $K_S \equiv 0$, we see by the adjunction formula that

$$\mathcal{O}_C(C) = \mathcal{O}_C(K_S + C) = \Omega^1_C,$$

i.e., the linear system cut out on C by hyperplane sections of $S \subset \mathbb{P}^n$ is a subsystem of the canonical series on C. Since, moreover, the linear system of hyperplane sections of S is the complete series $|H^0(S, \mathcal{O}_S(C))|$, and

$$h^1(S, \mathcal{O}_S) = q(S) = 0,$$

$H^0(\mathbb{P}^n, \mathcal{O}(H))$ surjects onto $H^0(S, \mathcal{O}(C))$ which surjects onto $H^0(C, \Omega^1_C)$. The hyperplanes in $\mathbb{P}^n$ thus cut out the complete canonical series on C, i.e., $C \subset \mathbb{P}^{n-1}$ *is a canonical curve*. Accordingly C has genus n and degree $2n-2$; in particular,

A normal K-3 surface $S \subset \mathbb{P}^n$ *has degree* $2n-2$.

We may also see this directly from Riemann-Roch: if $S \subset \mathbb{P}^n$ is a normal K-3 surface, C a hyperplane section of S, then since C is positive,

$$h^1(S, \mathcal{O}(C)) = h^1(S, \Omega^2_S(C)) = 0$$

and likewise

$$h^2(S, \mathcal{O}(C)) = 0$$

by the Kodaira vanishing theorem. Riemann-Roch then tells us

$$n + 1 = h^0(S, \mathcal{O}(C)) = \frac{C \cdot C}{2} + \chi(\mathcal{O}_S)$$
$$= \frac{\deg(S)}{2} + 2,$$

so $\deg(S) = 2n-2$.

We will see, in the four cases $n = 2$, 3, 4, and 5, how we may realize such a surface. The easiest is $n = 3$, i.e., S a quartic surface in $\mathbb{P}^3$. By the Lefschetz hyperplane theorem, a smooth quartic in $\mathbb{P}^3$ has irregularity

$$q(S) = q(\mathbb{P}^3) = 0,$$

and by adjunction,

$$K_S = (K_{\mathbb{P}^3} + S)|_S = (-4H + 4H)|_S \equiv 0,$$

so S is a K-3 surface. Note that since the linear system of quartics in $\mathbb{P}^3$ has dimension 34 and PGL_4 dimension 15, *the family of quartic K-3's has* *dimension*

$$34 - 15 = 19.$$

The second case is that of a sextic K-3 surface S in $\mathbb{P}^4$. Observe that if C is the hyperplane section of S, then the system of quadrics in $\mathbb{P}^4$ cuts out on S a system of dimension at most

$$h^0(S, \mathcal{O}(2C)) - 1 = \frac{2C \cdot 2C}{2} + 2 - 1 = \frac{24}{2} + 1 = 13$$

—but the linear system of quadrics in $\mathbb{P}^4$ is 14-dimensional, and so S must lie on a quadric hypersurface $Q \subset \mathbb{P}^4$. Similarly, since

$$h^0(S, \mathcal{O}(3C)) = \frac{54}{2} + 2 = 29$$

and

$$h^0(\mathbb{P}^4, \mathcal{O}(3H)) = \frac{5 \cdot 6 \cdot 7}{6} = 35,$$

S must lie on a five-dimensional family of cubics in $\mathbb{P}^4$—but the system of cubics containing the quadric Q is only $h^0(\mathbb{P}^4, \mathcal{O}(H)) - 1 = 4$-dimensional, and so S must lie on a cubic Q' not containing Q. Because Q is irreducible, Q' must meet Q in a surface of degree 6 or less, and hence exactly in S. Thus, *a sextic K-3 in $\mathbb{P}^4$ is the complete intersection of a quadric and a cubic*. Conversely, if $S = Q \cap Q'$ is such a smooth complete intersection, then by the Lefschetz theorem on hyperplane sections applied twice, $q(S) = 0$, and by adjunction

$$\begin{aligned} K_S &= (K_{Q'} + Q)|_S \\ &= (K_{\mathbb{P}^4} + Q' + Q)|_S \\ &= (-5H + 3H + 2H)|_S \equiv 0, \end{aligned}$$

so S is K-3. Note, finally, that such a K-3 is determined by choosing a quadric Q out of the 14-dimensional family of quadrics in $\mathbb{P}^4$ and then a cubic Q' in the $35 - 5 - 1 = 29$-dimensional family of cubics in $\mathbb{P}^4$ modulo those containing Q. Since PGL_5 has dimension 24, we see that again *the family of sextic K-3's in $\mathbb{P}^4$ has dimension*

$$14 + 29 - 24 = 19.$$

Next, consider an octic K-3 surface $S \subset \mathbb{P}^5$. By Riemann-Roch

$$h^0(S, \mathcal{O}(2C)) = \frac{2C \cdot 2C}{2} + 2 = 18,$$

while $h^0(\mathbb{P}^5, \mathcal{O}(2H)) = 21$. S will thus always lie on three independent quadrics in $\mathbb{P}^5$; generically, S will be their complete intersection. Conversely, as in the last case, by the Lefschetz theorem and the adjunction formula any smooth complete intersection of three quadrics in $\mathbb{P}^5$ is a K-3 surface. Counting parameters, a generic octic K-3 is specified by a net of quadrics in $\mathbb{P}^5$—in other words, a point of the Grassmannian $G(3,H^0(\mathbb{P}^5,\mathcal{O}(2H)))$—and so the family of octic K-3's has dimension once again

$$\dim G(3,21)-\dim \mathrm{PGL}_6 = 54-35 = 19.$$

The fourth case we shall look at is that of $n=2$—that is, K-3 surfaces S expressed as double covers $S \xrightarrow{\pi} \mathbb{P}^2$ of the plane. The "hyperplane section" of S—i.e., the inverse image $\pi^{-1}(l)$ of a line $l \subset \mathbb{P}^2$—is a curve of genus 2, expressed by π as a double cover of $l \cong \mathbb{P}^1$. π is thus branched over six points in l; and the branch locus of π a sextic curve in $\mathbb{P}^2$. Conversely if $B \subset \mathbb{P}^2$ is any smooth sextic curve, we can construct a double cover $S \xrightarrow{\pi} \mathbb{P}^2$ of $\mathbb{P}^2$ branched along B by the construction of p. 548; and the surface S is a K-3 surface: as in Section 4, if $\tilde{B}=\pi^{-1}(B)$ is the branch locus of π in S, we have

$$K_S = \pi^* K_{\mathbb{P}^2} + \tilde{B},$$

and so

$$\begin{aligned} 2K_S &= 2\pi^* K_{\mathbb{P}^2} + 2\tilde{B} \\ &= \pi^*(-6H) + \pi^* B \equiv 0. \end{aligned}$$

(Note that this implies S is minimal.) Also, since B has genus 10,

$$\chi(B) = 2-2g(B) = -18,$$

and

$$\chi(S) = 2\chi(\mathbb{P}^2) - \chi(B) = 24,$$

so, by the classification theorem, S must be a K-3. We count parameters once again: the system of sextic curves in $\mathbb{P}^2$ has dimension 27, and is acted on by PGL_3, so the family of K-3 surfaces expressed as double covers of $\mathbb{P}^2$ is $27-8=19$-dimensional.

This is as much as we shall prove about K-3 surfaces. One comment, however, should be made: while the general statement which extrapolates our computations—that *for any* n, *there is a* 19-*dimensional irreducible family* Γ_n *of K-*3 *surfaces of degree* $2n-2$ *in* $\mathbb{P}^n$—is true, it may give a false impression. In fact, if we drop the requirement of projectivity and simply define a K-3 surface to be a compact complex 2-manifold, simply connected and having trivial canonical bundle, then all K-3 surfaces will form an irreducible 20-dimensional family, the generic member of which is not algebraic; the families Γ_n form a countable union of subvarieties of this one moduli space. The picture is not unlike that for complex tori/Abelian varieties: in a family $\{S_\lambda\}$ of complex K-3 surfaces, parametrized by a polydisc, we may consider the cohomology group $H^2(S_\lambda,\mathbb{C})$ as a fixed vector space V and the subgroup $H^2(S_\lambda, \mathbb{Z})$ of integral classes likewise as a fixed lattice inside V. The subspace $H^{1,1}(S_\lambda) \subset V$, however, varies as the complex structure of S_λ varies; S_λ will belong to the family Γ_n only when $H^{1,1}(S_\lambda)$ meets a lattice point corresponding to an integral cohomology class of self-intersection $2n-2$. (See Figure 10.) Note that more generally the group of divisors modulo homology on an algebraic S_λ is exactly the intersection of $H^{1,1}(S_\lambda)$ with $H^2(S_\lambda, \mathbb{Z})$, and indeed it turns out to be the case that

The family of K-3 surfaces having k *or more divisors independent in homology forms a dense countable union of subvarieties of dimension* $20-\mathrm{k}$ *in the family of all K-3's; in particular, on the generic algebraic K-3 surface all divisors are homologous to multiples of the hyperplane class.*

The reader may verify this in one particular case, by showing that a K-3 expressed as a double cover of $\mathbb{P}^2$ branched along a sextic curve B contains two or more independent curves if and only if there is in $\mathbb{P}^2$ a rational curve of degree d tangent to B at $3d$ points (i.e., a tritangent line, etc.).

Enriques Surfaces

We turn our attention now to Enriques surfaces. First, for the numerical invariants: by definition,

$$p_g = q = 0, \qquad \chi(\mathcal{O}_S) = 1,$$

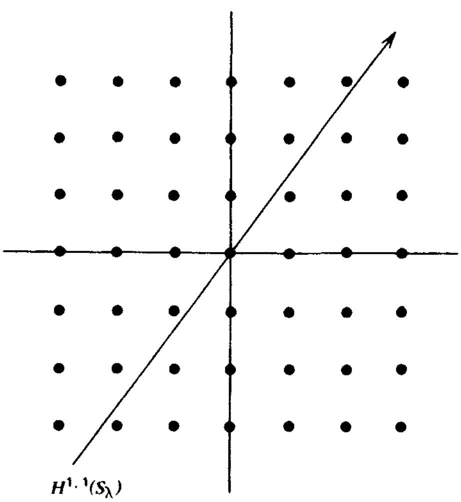


Figure 10

and, since $2K \equiv 0$,

$$c_1^2 = 0;$$

it follows from Riemann-Roch that

$$\chi(S) = 12$$

and the Hodge diamond of S is

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Note in particular that since $H^{2,0}(S)=0$, all the second cohomology of S is represented by algebraic curves. The group of divisors modulo rational homology on any Enriques surface is thus $\mathbb{Z}^{10}$.

We have already constructed one Enriques surface in Section 4 of this chapter by letting S be the surface $(X_0^4+X_1^4-X_2^4-X_3^4=0)$ and T the automorphism sending $[X_0,X_1,X_2,X_3]$ into $[X_0,iX_1,-X_2,-iX_3]$; we blew up the fixed points of T^2 on S and then took the quotient S'' of the blow-up by T^2; the Enriques surface S' was then the quotient of S'' by the involution T. S' was thus the quotient of the surface S''—which we can now identify as K-3—by a fixed-point-free involution. Indeed, it is not hard to see that any Enriques surface S arises as a quotient of a K-3 surface: simply let $\sigma \in H^0(S,\mathcal{O}(2K))$ be a nonzero section of the bundle $K \otimes K$, and consider, in the total space of the bundle K, the locus

$$X = \{(p,\zeta): \zeta \in K_p, \zeta \otimes \zeta = \sigma(p)\}.$$

Since σ is nowhere zero, X projects to S as an unbranched 2-sheeted covering space. We have, then,

$$q(X) = q(S) = 0,$$
$$\chi(X) = 2\chi(S) = 24,$$

and

$$\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_S) = 2.$$

In particular, we see from the first and third equality that $p_g(X)=1$, and indeed the section is visible: since $X \to S$ is unbranched, for any point $(p,\zeta) \in X$

$$K_X((p,\zeta)) = K_S(p),$$

and so we can define a section $\tilde{\sigma} \in H^0(X,\mathcal{O}(K_X))$ simply by

$$\tilde{\sigma}((p,\zeta)) = \zeta.$$

Clearly $\tilde{\sigma}$ never vanishes, and so K_X is trivial; consequently X is a K-3 surface:

Every Enriques surface is the quotient of a K-3 surface by a fixed-point-free involution.

A second way to represent an Enriques surfaces is as an elliptic surface S with rational base. To show that S has an elliptic pencil, we begin with the observation

If E *is any smooth elliptic curve on* S, *then* $h^0(2E) \geq 2$.

Since $h^0(2E) \geq h^0(E)$, if E moves in a pencil we are done, so we may suppose that E itself does not move, i.e., that $h^0(E)=1$. Now, since

$$K \cdot E = 0,$$

it follows from adjunction that

$$E \cdot E = 0,$$

and hence, by Riemann-Roch, that

$$\begin{aligned} h^1(E) &= h^0(E) - \frac{E \cdot E - K \cdot E}{2} + \chi(\mathcal{O}_S) \\ &= 1 - 1 = 0, \end{aligned}$$

since $h^2(E) = h^0(K-E) = 0$. Now we have

$$[2E]|_E = [2K+2E]|_E = 2K_E \equiv 0,$$

and so from the long exact sequence associated to the sequence

$$0 \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_S(2E) \longrightarrow \mathcal{O}_E \longrightarrow 0$$

we deduce that

$$\begin{aligned} h^0(S, \mathcal{O}(2E)) &= h^0(S, \mathcal{O}(E)) + h^0(E, \mathcal{O}_E) \\ &= 2, \end{aligned}$$

and the assertion is proved.

Now $2E$, like E, has virtual genus 1; thus, to show that S is elliptic, we need only locate a smooth elliptic curve on S.

We start by locating an effective divisor of self-intersection 0 (and hence virtual genus 1) on S. By the index theorem the intersection pairing on $H^2(S,\mathbb{Z}) \cong \mathbb{Z}^{10}$ is unimodular with one positive and nine negative eigenvalues; so we can find a class $\alpha \neq 0 \in H^2(S,\mathbb{Z})$ with $\alpha \cdot \alpha = 0$.* Since $p_g(S)=0$, α is of type (1, 1), and consequently by the Lefschetz (1, 1) theorem is the

*Cf. J.-P. Serre, *Cours d'Arithmétique*, Presses Universitaires de France, Paris, 1970, Chap. 5.

class of a divisor D' on S. By Riemann-Roch,

$$h^0(D') + h^0(K - D') \geqslant \frac{D'\cdot D' - K\cdot D'}{2} + \chi(\mathcal{O}_S) = 1,$$

so either D' or $K - D'$ is effective; call the effective one D. In either case, $D\cdot D = 0$.

Let $X \xrightarrow{\pi} S$ be the twofold covering of S by a K-3 surface X, and $\tilde{D} = \pi^* D$ the inverse image of D. We have

$$\tilde{D}\cdot\tilde{D} = 2D\cdot D = 0,$$

and so by Riemann-Roch on X,

$$h^0(\tilde{D}) \geqslant \frac{\tilde{D}\cdot\tilde{D} - K_X\cdot\tilde{D}}{2} + \chi(\mathcal{O}_X) = 2,$$

i.e., $\tilde{D}$ moves in a linear system on X. To separate out the fixed components of $|\tilde{D}|$ we write

$$|\tilde{D}| = |C| + \sum n_i E_i,$$

where the E_i's are irreducible, $n_i > 0$, and the linear system $|C|$ has no fixed components. E_i being fixed, we find

$$1 = h^0(E_i) \geqslant \frac{E_i\cdot E_i}{2} + 2,$$

while

$$0 \leqslant \pi(E_i) = \frac{E_i\cdot E_i}{2} + 1;$$

it follows that *the curves* $\mathrm{E_i}$ *are all rational of self-intersection* -2. If for each i we set

$$k_i = (\tilde{D} - n_i E_i)\cdot E_i = \Big(C + \sum_{j\neq i} n_j E_j\Big)\cdot E_i,$$

then,

$$\begin{aligned} 0 = \tilde{D}\cdot\tilde{D} &= \tilde{D}\cdot\Big(C + \sum n_i E_i\Big) \\ &= \tilde{D}\cdot C + \sum n_i(\tilde{D} - n_i E_i)\cdot E_i + \sum n_i^2 E_i\cdot E_i \\ &= C\cdot C + \sum n_i C\cdot E_i + \sum n_i k_i - 2\sum n_i^2. \end{aligned}$$

Now $C\cdot C \geqslant 0$, since C moves in a linear system without fixed components. If $C\cdot C = 0$, set $\tilde{D}_1 = C$; if $C\cdot C > 0$, then it follows that for some i_0,

$$k_{i_0} \leqslant 2n_{i_0}.$$

In this case, set

$$\tilde{D}_1 = \tilde{D} - (2n_{i_0} - k_{i_0})E_{i_0}$$

We have then

$$\begin{aligned}\tilde{D}_1\cdot\tilde{D}_1 &= \tilde{D}\cdot\tilde{D}-2(2n_{i_0}-k_{i_0})(\tilde{D}\cdot E_{i_0})+(2n_{i_0}-k_{i_0})^2(E_{i_0}\cdot E_{i_0})\\ &= -2(2n_{i_0}-k_{i_0})(k_{i_0}-2n_{i_0})-2(2n_{i_0}-k_{i_0})^2\\ &= 0.\end{aligned}$$

Now once more by Riemann-Roch

$$h^0(\tilde{D}_1)+h^0(K-\tilde{D}_1)\geqslant\frac{\tilde{D}_1\cdot\tilde{D}_1}{2}+2=2,$$

and since

$$K-\tilde{D}_1=K+(2n_{i_0}-k_{i_0})E_{i_0}-\tilde{D}$$

cannot be effective, $\tilde{D}_1$ must move in a linear system. Since $2n_{i_0}-k_{i_0}>0$, we deduce that if the linear system $|\tilde{D}|$ has a rational fixed component, we may subtract an effective curve from $\tilde{D}$ to obtain a divisor $\tilde{D}_1$ moving in a linear series, again with self-intersection 0. If the system $|\tilde{D}_1|$ has rational fixed components, then, we may deduct an effective curve again to obtain a divisor $\tilde{D}_2>0$ with $\tilde{D}_2\cdot\tilde{D}_2\geqslant 0$, $h^0(\tilde{D}_2)\geqslant 2$, and so on. But, as was pointed out on p. 521, the divisor $\tilde{D}$ cannot be written as the sum of arbitrarily many effective curves, and so we arrive at a divisor $\tilde{D}_\alpha$ of self-intersection 0, moving in a linear system without fixed components. By the lemma of p. 576, every component of $\tilde{D}_\alpha$ is elliptic, with self-intersection 0.

Consider now the image $D_{\alpha,i}$ in S of a component $\tilde{D}_{\alpha,i}$ of $\tilde{D}_\alpha$. If $\tilde{D}_{\alpha,i}$ maps two-to-one to $D_{\alpha,i}$, then $D_{\alpha,i}$ is a smooth elliptic curve on S and by the lemma we are done. On the other hand, if $\pi:\tilde{D}_{\alpha,i}\to D_{\alpha,i}$ is generically one-to-one, then $D_{\alpha,i}$ will be singular exactly at those points $p\in S$ such that both points of $\pi^{-1}(p)\subset X$ lie in $\tilde{D}_{\alpha,i}$. In this case, let

$$\tilde{D}'_{\alpha,i}=\pi^*\pi_*\tilde{D}_{\alpha,i}-\tilde{D}_{\alpha,i}$$

be the remaining component of $\pi^*D_{\alpha,i}$. Since $\pi:\tilde{D}'_{\alpha,i}\to D_{\alpha,i}$ is again generically one-to-one,

$$\pi_*(\tilde{D}'_{\alpha,i})=\pi_*(\tilde{D}_{\alpha,i})$$

and hence

$$\tilde{D}'_{\alpha,i}\cdot\tilde{D}'_{\alpha,i}=0.$$

Since the self-intersection of $\pi^*\pi_*\tilde{D}_{\alpha,i}=\tilde{D}_{\alpha,i}+\tilde{D}'_{\alpha,i}$ is again 0, it follows that $\tilde{D}_{\alpha,i}\cdot\tilde{D}'_{\alpha,i}=0$—i.e., $\tilde{D}_{\alpha,i}$ and $\tilde{D}'_{\alpha,i}$ are disjoint. But if $D_{\alpha,i}$ is singular at a point $p\in S$, then $\tilde{D}_{\alpha,i}$ must pass through both points of $\pi^{-1}(p)$ and $\tilde{D}'_{\alpha,i}$ likewise, so $\tilde{D}_{\alpha,i}$ and $\tilde{D}'_{\alpha,i}$ will meet over p. Thus $D_{\alpha,i}$ is a smooth elliptic curve, and we are done.

Consider now the Enriques surface S as an elliptic surface $\Psi:S\to\mathbb{P}^1$ with rational base. If S has multiple fibers B_i with multiplicity m_i, then since $\chi(\mathcal{O}_S)=1$, we have by our formula for the canonical class

$$K_S = \Psi^*(-p) + \sum (m_i - 1)B_i,$$

and since

$$0 \equiv 2K_S = \Psi^*(-2p) + \sum 2(m_i - 1)B_i,$$

it follows that S has exactly two double fibers, B_1 and B_2. Finally, since $2B_1 = 2B_2 = \Psi^*(p)$, we may write

$$K_S = \Psi^*(-p) + B_1 + B_2 = B_1 - B_2 = B_2 - B_1,$$

i.e., *the canonical divisor of an Enriques surface* S *is just the difference of the two double fibers appearing in an elliptic pencil on* S.

Performing the inverse logarithmic transformation on the two double fibers of an Enriques surface $S \xrightarrow{\Psi} \mathbb{P}^1$, we obtain an elliptic surface $S' \xrightarrow{\Psi'} \mathbb{P}^1$ without multiple fibers. We see immediately that

$$c_1^2(S') = c_1^2(S) = 0$$

and

$$c_2(S') = c_2(S) = 12,$$

so by Riemann-Roch

$$\chi(\mathcal{O}_{S'}) = \chi(\mathcal{O}_S) = 1.$$

By our formula, then,

$$K_{S'} = \Psi^*(-p)$$

and so

$$\kappa(S') = -1;$$

in particular $p_g(S')=0$, and from $\chi(\mathcal{O}_{S'})=1$ we deduce that $q(S')=0$. By Castelnuovo's theorem, S' is rational; and by our classification of rational surfaces S' is some rational ruled surface blown up eight times, or $\mathbb{P}^2$ blown up nine times. In fact, since $-K_{S'}$ is effective and irreducible and has self-intersection 0, every curve on S' has nonpositive intersection number with $K_{S'}$ and hence self-intersection $\geqslant -2$; by the standard argument, S' must be $\mathbb{P}^2$ blown up nine times. Finally, the images in $\mathbb{P}^2$ of the fibers C_λ of $S \xrightarrow{\Psi} \mathbb{P}^1$ under the blowing-down map $\pi: S' \to \mathbb{P}^2$ must represent sections of $|-K_{\mathbb{P}^2}|$, that is, cubic curves; since the C_λ are all disjoint, the nine points blown up are the base locus of the pencil $|\pi(C_\lambda)|$ of cubics. In sum

An Enriques surface may be obtained by blowing up $\mathbb{P}^2$ *at the nine base points of pencil* D_λ *of cubic curves, and applying two logarithmic transformations of order* 2 *to the resulting elliptic surface* $\tilde{\mathbb{P}}^2 \to \mathbb{P}^1$.

Conversely, performing logarithmic transforms in this case will always yield an algebraic surface, which the reader may easily verify is an Enriques surface. Note that this construction enables us to count parameters for Enriques surfaces: to construct an Enriques surface, we specify nine points in $\mathbb{P}^2$ forming the base locus of a pencil of cubics, blow them up, and then specify two fibers of the resulting elliptic surface (i.e., two cubics passing through the nine blown-up points) on which to perform logarithmic transformations. In short, then, the entire process is determined by the choice of two cubic curves in $\mathbb{P}^2$ meeting transversely; since there is a nine-dimensional family of cubics in $\mathbb{P}^2$, and dim $\mathrm{PGL}_3 = 8$, the family of Enriques surfaces is irreducible of dimension

$$9+9-8=10.$$

6. NOETHER'S FORMULA

Noether's Formula for Smooth Hypersurfaces

A *Riemann-Roch formula* in general is a formula giving the holomorphic Euler characteristic of a vector bundle $E \to M$ on a compact complex manifold M in terms of the Chern classes of E and M. In practice this problem breaks up into two halves: first, expressing the holomorphic Euler characteristic $\chi(\mathcal{O}_M)$ of M in terms of the Chern classes of M—e.g., for curves and surfaces,

$$\chi(\mathcal{O}_M) = \tfrac{1}{2}c_1(M)$$

and

$$\chi(\mathcal{O}_M) = \tfrac{1}{12}\left(c_1^2(M) + c_2(M)\right)$$

—and second, expressing the holomorphic Euler characteristic $\chi(\mathcal{O}(E))$ in terms of the Chern classes of E and M and the holomorphic Euler characteristic $\chi(\mathcal{O}_M)$ of M, e.g., for line bundles L on curves and surfaces

$$\chi(\mathcal{O}(L)) = \chi(\mathcal{O}_M) + c_1(L)$$

and

$$\chi(\mathcal{O}(L)) = \chi(\mathcal{O}_M) + \tfrac{1}{2}\left(c_1^2(L) + c_1(L)c_1(M)\right).$$

Of the two halves of a Riemann-Roch formula, the second is usually much easier: once we know the formula for $(n-1)$-dimensional varieties, we can compute it for the line bundle $L \to M$ associated to a smooth divisor D on the n-dimensional manifold M simply by adding up the Euler characteristics of the sheaves in the sequence

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}_M(L) \longrightarrow \mathcal{O}_D(L) \longrightarrow 0;$$

the resulting formula holds for all line bundles $L \to M$. The first half —expressing $\chi(\mathcal{O}_M)$ in terms of the Chern classes of M—is in general harder. For curves, of course, the formula $\chi(\mathcal{O}_M) = \frac{1}{2}c_1(M)$ is an easy consequence of Hodge theory; it was originally proved, in form $h^0(\Omega^1_M) = b_1(M)/2$ or "the number of independent differentials of the first kind is equal to the number of handles," by Riemann. Our principal object in this section is to prove the analogous formula $\chi(\mathcal{O}_M) = \frac{1}{12}(c_1^2(M) + c_2(M))$ for surfaces, called *Noether's formula*.

To get a sense of the problem, we will first verify the formula for a smooth surface S in $\mathbb{P}^3$ of degree n. To begin with, we establish two general formulas: for $S \subset X$ a smooth surface on a threefold X, from the C^∞ decomposition

$$\begin{aligned} T_X|_S &= T_S \oplus N_{S/X} \\ &= T_S \oplus [S]|_S \end{aligned}$$

and the Whitney product formula, we obtain the *adjunction formulas*

$$(*) \qquad c_1(X)|_S = c_1(S) + (S)|_S$$

and

$$(**) \qquad c_2(X)|_S = c_2(S) + c_1(S)\cdot S|_S.$$

The first of these is, of course, the standard adjunction formula applied to $c_1(S) = -K_S$.

Applying these formulas to our smooth surface $S \subset \mathbb{P}^3$ of degree n, and using the values

$$c_1(\mathbb{P}^3) = 4H, \qquad c_2(\mathbb{P}^3) = 6H^2,$$

we have

$$\begin{aligned} c_1(S)^2 &= (n-4)^2 H^2 \\ &= n(n-4)^2 \\ &= n^3 - 8n^2 + 16n, \end{aligned}$$

since $H\cdot H = n$ on S; and by $(**)$,

$$\begin{aligned} 6H^2 &= c_2(S) + c_1(S)\cdot c_1(N_S) \\ &= c_2(S) + (4-n)\cdot n\cdot H^2, \end{aligned}$$

i.e.,

$$\begin{aligned} \chi(S) = c_2(S) &= (n(n-4)+6)H^2 \\ &= n(n(n-4)+6) \\ &= n^3 - 4n^2 + 6n. \end{aligned}$$

$\chi(\mathcal{O}_S)$ is likewise readily expressed in terms of n. By the Lefschetz theorem, $q(S) = 0$; to compute $p_g(S)$, consider the Poincaré residue map

$$0 \to \Omega^3_{\mathbb{P}^3} \to \Omega^3_{\mathbb{P}^3}(S) \to \Omega^2_S \to 0.$$

We have $h^0(\Omega^3_{\mathbb{P}^3})=0$ and $h^1(\Omega^3_{\mathbb{P}^3})=h^{3,1}(\mathbb{P}^3)=0$, so

$$\begin{aligned} p_g(S) &= h^0(\Omega^2_S) = h^0(\Omega^3_{\mathbb{P}^3}(S)) \\ &= h^0(\mathbb{P}^3, \mathcal{O}((n-4)H)) \\ &= \binom{n-1}{3} = \frac{(n-1)(n-2)(n-3)}{6}. \end{aligned}$$

Thus

$$\begin{aligned} \chi(\mathcal{O}_S) &= \frac{(n-1)(n-2)(n-3)}{6} + 1 \\ &= \frac{n^3-6n^2+11n-6}{6} + 1 \\ &= \frac{2n^3-12n^2+22n}{12} \\ &= \frac{c_1(S)^2+c_2(S)}{12}, \end{aligned}$$

and the Riemann-Roch formula is proved for S.

This computation illustrates the general principle that if we know the cohomology ring and Chern classes of a variety M, we can compute most of these invariants—and hence verify Riemann-Roch—for a smooth divisor on M. Of course, a general surface S cannot be realized as a smooth divisor in $\mathbb{P}^3$: while we can embed S in some large projective space and project to $\mathbb{P}^3$ to obtain a generically one-to-one map $\phi: S \to \mathbb{P}^3$, the image $S_0=\phi(S)$ will in general be singular. To prove Noether's formula, then, we will extend the formulas obtained above for the numerical characters of a smooth surface S in $\mathbb{P}^3$ to the case of surfaces S_0 with standard singularities; we will do this by re-embedding S_0 as a smooth surface in a threefold X obtained by blowing up $\mathbb{P}^3$. This requires two preliminary steps: we have to describe those types of singularities which arise under a generic projection of a surface to $\mathbb{P}^3$; and, given a surface $S_0 \subset \mathbb{P}^3$ with these singularities, construct a blow-up $X \xrightarrow{\pi} \mathbb{P}^3$ in which the proper transform of S_0 is smooth. The first of these steps is simply a matter of dimension-theoretic case checking, and will be deferred for the moment. The second, on the other hand, involves an important extension of our notion of blowing up; we take this up in the following discussion.

Blowing Up Submanifolds

As previously, we start by constructing the blow-up of a disc along a coordinate plane. Let Δ be an n-dimensional disc with holomorphic coordinates $z_1,\ldots,z_n$, and let $V \subset \Delta$ be the locus $z_{k+1}=\cdots=z_n=0$. Let $[l_{k+1},\ldots,l_n]$ be homogeneous coordinates on $\mathbb{P}^{n-k-1}$, and let

$$\tilde{\Delta} \subset \Delta \times \mathbb{P}^{n-k-1}$$

be the smooth variety defined by the relations

$$\tilde{\Delta} = \{(z,l):\ z_i l_j = z_j l_i,\ k+1 \leqslant i,j \leqslant n\}.$$

The projection $\pi : \tilde{\Delta} \to \Delta$ on the first factor is clearly an isomorphism away from V, while the inverse image of a point $z \in V$ is a projective space $\mathbb{P}^{n-k-1}$. The manifold $\tilde{\Delta}$, together with the map $\pi : \tilde{\Delta} \to \Delta$, is called the *blow-up of* Δ *along* V; the inverse image $E = \pi^{-1}(V)$ is called the *exceptional divisor* of the blow-up.

$\tilde{\Delta}$ may be covered by coordinate patches

$$U_j = (l_j \neq 0), \qquad j = k+1, \ldots, n$$

with holomorphic coordinates

$$\begin{aligned} z_i &= z_i, & i &= 1, \ldots, k, \\ z(j)_i &= \frac{l_i}{l_j} = \frac{z_i}{z_j}, & i &= k+1, \ldots, \hat{j}, \ldots, n, \\ z_j &= z_j \end{aligned}$$

on U_j; the coordinates $\{z(j)_i\}$ are Euclidean coordinates on each fiber $\pi^{-1}(p) \cong \mathbb{P}^{n-k-1}$ of the exceptional divisor.

Note that the blow-up $\tilde{\Delta} \xrightarrow{\pi} \Delta$ is independent of the coordinates chosen in Δ: if $\{z_i' = f_i(z)\}$ is another coordinate system in Δ with V again given as $(z'_{k+1} = \cdots = z'_n = 0)$,

$$\tilde{\Delta}' = \{(z',l'):\ z_i' l_j' = z_j' l_i'\} \subset \Delta \times \mathbb{P}^{n-k-1}$$

the blow-up of Δ in this coordinate system, then the isomorphism

$$\tilde{f}:\ \tilde{\Delta} - E \longrightarrow \tilde{\Delta}' - E'$$

given by $z \mapsto f(z)$ may be extended over E by sending a point (z,l) with $z_{k+1} = \cdots = z_n = 0$ to the point $(f(z), l')$, where

$$l_j' = \sum_{i=k+1}^{n} \frac{\partial f_j}{\partial z_i}(z) \cdot l_i.$$

Indeed, we see from this that the identification of the fiber of $E \xrightarrow{\pi} V$ over a point $z = (z_1, \ldots, z_k, 0, \ldots, 0)$ with the projective normal space $\mathbb{P}(N_{V/\Delta}(z)) = \mathbb{P}(T_z'(\Delta)/T_z'(V))$ made via

$$(z,l) \mapsto \sum_{i=k+1}^{n} l_i \cdot \frac{\partial}{\partial z_i}$$

is natural, i.e., does not depend on the coordinate system used.

This last observation allows us to globalize our construction. Let M be a complex manifold of dimension n and $X \subset M$ a submanifold of dimension k. Let $\{U_\alpha\}$ be a collection of discs in M covering V such that in each disc

Δ_α the subvariety $X \cap \Delta_\alpha$ may be given as the locus $(z_{k+1} = \cdots = z_n = 0)$, and let $\tilde{\Delta}_\alpha \xrightarrow{\pi_\alpha} \Delta_\alpha$ be the blow-up of Δ_α along $X \cap \Delta_\alpha$. We have then isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}(U_\alpha \cap U_\beta) \longrightarrow \pi_\beta^{-1}(U_\alpha \cap U_\beta)$$

and using them, we can patch together the local blow-ups $\tilde{\Delta}_\alpha$ to form a manifold

$$\tilde{\Delta} = \bigcup_{\pi_{\alpha\beta}} \tilde{\Delta}_\alpha$$

with a map

$$\tilde{\Delta} \xrightarrow{\pi} \bigcup \Delta_\alpha.$$

Finally, since π is an isomorphism away from $X \cap (\bigcup \Delta_\alpha)$, we can take

$$\tilde{M} = \tilde{\Delta} \cup_\pi M - X;$$

$\tilde{M}$, together with the map $\pi : \tilde{M} \rightarrow M$ extending π on $\tilde{\Delta}$ and the identity on $M - X$, is called the *blow-up of* M *along* X. By the construction, the blow-up has the following properties

1. π is an isomorphism away from $X \subset M$ and $E = \pi^{-1}(X) \subset \tilde{M}$.
2. The *exceptional divisor* E is a fiber bundle over X with fiber $\mathbb{P}^{n-k-1}$; indeed, $E \xrightarrow{\pi} X$ is naturally identified with the projectivization $\mathbb{P}(N_{X/M})$ of the normal bundle $N_{X/M}$ of X in M.
3. Locally the blow-up is isomorphic to the blow-up of a disc as given above.
4. As the reader may check by the same method as used in the case of blow-ups of a point, blow-ups of submanifolds are unique, in the sense that if

$$N \xrightarrow{\pi} M$$

is any map of complex manifolds that is an isomorphism away from a smooth subvariety X of dimension k in M, and such that the fiber of π over any point $z \in X$ is isomorphic to projective space $\mathbb{P}^{n-k-1}$, then $N \xrightarrow{\pi} M$ is the blow-up of M along X.

5. For any subvariety $Y \subset M$, we may define the *proper transform* $\tilde{Y} \subset \tilde{M}_X$ of Y in the blow-up $\tilde{M}_X$ to be the closure in $\tilde{M}_X$ of the inverse image

$$\pi^{-1}(Y - X) = \pi^{-1}(Y) - E$$

of Y away from the exceptional divisor E. As in the case of blowing up a point, we see that the intersection

$$\tilde{Y} \cap E \subset \mathbb{P}(N_{X/M})$$

corresponds to the image in $N_{X/M}$ of the tangent cones $T_p(Y) \subset T_p(M)$ to Y at points of $Y \cap X$. In particular, for $Y \subset M$ a divisor,

$$\tilde{Y} = \pi^{-1}Y - m \cdot E,$$

where

$$m = \mathrm{mult}_X(Y)$$

is the multiplicity of Y at a generic point of X. Note also that blow-ups are functorial, in the sense that if Y meets X transversely everywhere, the proper transform $\tilde{Y}$ of Y in $\tilde{M}_X$ is just the blow-up of Y along $Y \cap X$.

The Cohomology of a Blow-up. We would like now to describe the relation between the cohomology ring of M and that of its blow-up $\tilde{M} = \tilde{M}_X$ along a submanifold. Cohomology is with $\mathbb{Z}$ coefficients throughout. As in our discussion of point blow-ups, we may take $U \subset M$ a tubular neighborhood of $X \subset M$, $\tilde{U} = \pi^{-1}U$, $U^* = U - X$, $\tilde{U}^* = \tilde{U} - E$, $M^* = M - X$, and $\tilde{M}^* = \tilde{M} - E$, and compare the Mayer-Vietoris sequences for $M = U \cup M^*$ and $\tilde{M} = \tilde{M}^* \cup \tilde{U}$. Since contractions yield, as before, isomorphisms

$$H^*(U) \longrightarrow H^*(X) \quad \text{and} \quad H^*(\tilde{U}) \longrightarrow H^*(E),$$

and clearly π^* gives isomorphisms

$$H^*(U^*) \longrightarrow H^*(\tilde{U}^*) \quad \text{and} \quad H^*(M^*) \longrightarrow H^*(\tilde{M}^*),$$

we obtain

$$\begin{array}{ccccccccc} H^{i-1}(U^*) & \to & H^i(\tilde{M}) & \to & H^i(M^*) & \oplus & H^i(E) & \to & H^i(U^*) \\ \| & & \uparrow & & \| & & & & \| \\ H^{i-1}(U^*) & \to & H^i(M) & \to & H^i(M^*) & \oplus & H^i(X) & \to & H^i(U^*). \end{array}$$

Since the pullback map $\pi^* : H^*(M) \to H^*(\tilde{M})$ is injective (equivalently, and more visibly, the map $\pi_* H_*(\tilde{M}) \to H(_*(M)$ is surjective), we see from this that (additively)

$$H^*(\tilde{M}) = \pi^* H^*(M) \oplus H^*(E)/\pi^* H^*(X)$$

To describe the cohomology of $E \cong \mathbb{P}(N_{X/M})$, as well as the multiplicative structure of $H^*(\tilde{M})$, we need a general result on the cohomology of projective bundles. First a definition: for $E \to X$ a complex vector bundle of rank r and $\mathbb{P}(E) \xrightarrow{\pi} X$ its associated projective bundle, we define the *tautological line bundle* $T \to \mathbb{P}(E)$ to be the subbundle of the pullback bundle $\pi^* E \to \mathbb{P}(E)$ whose fiber at any point $(p, v) \in \mathbb{P}(E)$ is the line in E_p represented by v. Note that the bundle T is *not* determined by the abstract projective bundle $\mathbb{P}(E) \to X$ alone: if $L \to X$ is any line bundle, we have seen that

$$\mathbb{P}(E) \cong \mathbb{P}(E \otimes L);$$

but the tautological line bundles on $\mathbb{P}(E)$ and $\mathbb{P}(E\otimes L)$ will differ. One thing is always true: the restriction of T to each fiber $\mathbb{P}(E)_p \cong \mathbb{P}^{r-1}$ is the universal bundle.

Now, the cohomology ring $H^*(\mathbb{P}(E))$ is, via the pullback map,

$$H^*(X) \xrightarrow{\pi^*} H^*(\mathbb{P}(E)),$$

an algebra over the ring $H^*(X)$. A complete description of $H^*(\mathbb{P}(E))$ is given in these terms by the

Proposition. *For* X *any compact oriented* C^∞ *manifold,* $E \to X$ *any complex vector bundle of rank* r, *the cohomology ring* $H^*(\mathbb{P}(E))$ *is generated, as an* $H^*(X)$-*algebra, by the Chern class*

$$\zeta = c_1(T)$$

of the tautological bundle, with the single relation

$$\zeta^r - c_1(E)\zeta^{r-1} + c_2(E)\zeta^{r-2} + \cdots + (-1)^{r-1}c_{r-1}(E)\zeta + (-1)^r c_r(E) = 0. \tag{$*$}$$

Proof. We first establish the basic relation $(*)$. To do this, let S be the quotient of the pullback π^*E by the tautological subbundle, and set $\eta_i = c_i(E)$. By Whitney; then,

$$(1+\zeta)(1+\eta_1 + \cdots + \eta_{r-1}) = \pi^* c(E)$$

and solving successively, we have

$$\begin{aligned}\eta_1 &= c_1(E) - \zeta\\ \eta_2 &= c_2(E) - \zeta\cdot c_1(E) + \zeta^2\\ &\vdots\\ \eta_{r-1} &= c_{r-1}(E) - \zeta\cdot c_{r-2}(E) + \cdots + (-1)^{r-1}\zeta^{r-1}.\end{aligned}$$

The final equation

$$c_r(E) = \zeta\cdot\eta_{r-1}$$

is then our basic relation.

Now let $\{\psi_{i,\alpha}\}_\alpha$ be a basis for $H^i(X)$, with $\{\psi_{i,\alpha}\}$ and $\{\psi_{n-i,\alpha}\}$ orthogonal—i.e., such that

$$\psi_{i,\alpha} \cup \psi_{n-i,\beta} = \pm\delta_{\alpha,\beta}.$$

We claim that the classes

$$\{\pi^*\psi_{i,\alpha} \cup \zeta^j\}_{1 \leqslant i \leqslant n,\, 1 \leqslant j \leqslant r-1,\,\alpha}$$

are linearly independent in $H^*(\mathbb{P}(E))$. First, for any pair of classes $\psi_{i,\alpha}$ and $\psi_{n-i,\alpha}$, the cup product $\pi^*\psi_{i,\alpha} \cup \pi^*\psi_{n-i,\alpha}$ will be Poincaré dual to plus or minus the class of a fiber $\mathbb{P}(E)_p$ of $\mathbb{P}(E)$. But the restriction of ζ to $\mathbb{P}(E)_p$

is minus the class of a hyperplane in $\mathbb{P}(E)_p$, and consequently

$$\pi^*\psi_{i,\alpha} \cup \pi^*\psi_{n-i,\alpha} \cup \zeta^{r-1} = \pm 1,$$

or, in other words, for any j,

$$\left(\pi^*\psi_{i,\alpha} \cup \zeta^j\right) \cup \left(\pi^*\psi_{n-i,\alpha} \cup \zeta^{r-j-1}\right) = \pm 1.$$

On the other hand, for $\alpha \neq \beta$,

$$\pi^*\psi_{i,\alpha} \cup \pi^*\psi_{n-i,\beta} = 0;$$

likewise, for $i < k$ and any α, β, j,

$$\left(\pi^*\psi_{i,\alpha} \cup \zeta^j\right) \cup \left(\pi^*\psi_{n-k,\beta} \cup \zeta^{r-i-j+k-1}\right) = 0.$$

Therefore the intersection matrix for the classes $\{\psi_{i,\alpha} \cup \zeta^j\}_{i,j,\alpha}$ may be made upper triangular with ± 1's along the diagonal; in particular, we see that it is nonsingular, and so these elements are all linearly independent in $H^*(\mathbb{P}(E))$.

Finally, consider the Leray spectral sequence $(E_r^{p,q}, d_r)$ of the bundle $\mathbb{P}(E) \to X$. Since the cohomology of the fiber has rank at most 1 in each dimension, $\pi_1(X)$ acts trivially on $H^*(\mathbb{P}^{r-1})$, and so

$$E_2 \cong H^*(X) \otimes H^*(\mathbb{P}^{r-1}).$$

But since the classes $\{\psi_{i,\alpha} \cup \zeta^j\}$ are all independent in $H^*(\mathbb{P}(E))$,

$$\begin{aligned} r \cdot \dim H^*(X) &\leqslant \dim H^*(\mathbb{P}(E)) \\ &= \dim E_\infty \\ &\leqslant \dim E_2 \\ &= r \cdot \dim H^*(X). \end{aligned}$$

Equality must therefore hold everywhere, i.e., the classes $\{\psi_{i,\alpha} \cup \zeta^j\}$ span $H^*(\mathbb{P}(E))$ so that ζ generates $H^*(\mathbb{P}(E))$ as an $H^*(X)$-algebra, and there can be no relations on ζ other than (*) above. Q.E.D.

One observation makes this result particularly applicable to blow-ups: if $\tilde{M} \to M$ is the blow-up of the manifold M along the submanifold X, $E = \mathbb{P}(N_{X/M})$ the exceptional divisor, then *the normal bundle to* E *in* $\tilde{M}$ *is just the tautological bundle on* $E \cong \mathbb{P}(N_{X/M})$. Indeed, for any point $(p,v) \in E$, we easily see that

$$\pi_*: T'_{(p,v)}(\tilde{M}) \to T'_p(M)$$

induces a map

$$\tilde{\pi}_*: N_{E/\tilde{M}}(p,v) \to N_{X/M}(p).$$

To see that the image of $\tilde{\pi}_*$ is just the line v in $N_{X/M}(p)$, it is sufficient to check it for the blow-up $\tilde{\mathbb{C}}^n_V \xrightarrow{\pi} \mathbb{C}^n$ of $\mathbb{C}^n$ along the subspace $V \cong \mathbb{C}^k$, and there it is clear. As a consequence, we see that the restriction to E of the cohomology class $e=c_1([E])$ is

$$e|_E = c_1(N_{E/\tilde{M}}) = c_1(T) = \zeta,$$

and correspondingly, with a knowledge of $H^*(E)$ and the restriction map $H^*(M) \to H^*(X)$, we may compute effectively in the cohomology ring of the blow-up $\tilde{M}_X$.

Chern Classes of Blow-ups. We have seen that if $\tilde{M} \xrightarrow{\pi} M$ is the blow-up of an n-dimensional complex manifold at a point, E the exceptional divisor, then

$$c_1(\tilde{M}) = -K_{\tilde{M}} = \pi^* c_1(M) - (n-1)E.$$

In a similar fashion, it is not hard to verify that for $\tilde{M} \xrightarrow{\pi} M$ the blow-up of M along a k-dimensional submanifold $X \subset M$, E again the exceptional divisor,

$$c_1(\tilde{M}) = \pi^* c_1(M) - (n-k-1)E.$$

This formula may be checked in general as it was in case $k=0$—that is, by writing out transition functions for the canonical bundle $K_{\tilde{M}}$. The computation is substantially easier, however, if we consider only algebraic varieties M. In this case we can find a meromorphic n-form ω on M, with X not contained in the zero or polar divisor of ω. The divisor of the pullback form $\pi^*\omega$ on $\tilde{M}$ is then, away from E, just the pullback of the divisor (ω). To see how $\pi^*\omega$ behaves around E, let p be a generic point of X and $z_1, \ldots, z_n$ local coordinates in a neighborhood U of p with

$$X \cap U = (z_{k+1}, \ldots, z_n = 0);$$

we may write

$$\omega = g(z)\, dz_1 \wedge \cdots \wedge dz_n$$

with g nonzero and holomorphic around p. In terms of coordinates

$$z_i = z_i,\ i = 1, \ldots, k, j,$$

and

$$z(j)_i = \frac{z_i}{z_j},\ i = k+1, \ldots, \hat{j}, \ldots, n,$$

on the open set $U_j \subset \pi^{-1}(U)$ as described above, we have

$$\begin{aligned} \pi^* dz_i &= dz_i, & i &= 1, \ldots, k, j, \\ \pi^* dz_i &= d(z_j z(j)_i) \\ &= z_j\, dz(j)_i + \tilde{z}(j)_i \cdot dz_j, & i &= k+1, \ldots, \hat{j}, \ldots, n. \end{aligned}$$

Thus,

$$\pi^*\omega = \pi^* g(z) \cdot z_j^{n-k-1} \cdot dz_1 \wedge \cdots \wedge dz_k \wedge dz(j)_{k+1} \wedge \cdots \wedge dz(j)_n$$

vanishes to order $n-k-1$ along $E=(\tilde{z}_j)$, and the formula is verified.

Computing the higher Chern classes of a general blow-up is substantially harder; in particular, one has to find the Chern classes of the exceptional divisor, and we do not have at present the requisite formalism of the *Chern character*. We may, however, determine the Chern classes of the blow-up of a threefold by essentially ad hoc methods; we will do this in the following two lemmas.

Lemma. *If* $\tilde{M} \xrightarrow{\pi} M$ *is the blow-up of the algebraic threefold* M *at a point,*

$$c_2(\tilde{M}) = \pi^* c_2(M).$$

Proof. We will prove this by applying the adjunction formulas (*) and (**) of p. 601 to surfaces in $\tilde{M}$ whose Chern classes we know. First, let E be the exceptional divisor of the blow-up, $l \in H^2(E)$ the class of a line in $E \cong \mathbb{P}^2$. We have seen that

$$E|_E = -l$$

while

$$c_1(E) = 3l \quad \text{and} \quad c_2(E) = 3.$$

By the formula (**), then

$$\begin{aligned} c_2(M)|_E &= c_2(E) + c_1(E)\cdot E|_E \\ &= 3 + 3l\cdot(-l) \\ &= 0. \end{aligned}$$

Next, we do the same thing for a surface $S \subset M$ not containing p and its inverse image $S = \pi^{-1}(S) \subset \tilde{M}$. Inasmuch as $\tilde{S} \cong S$ and the fundamental class $\eta_{\tilde{S}} = \pi^* \eta_S$, we have

$$\begin{aligned} c_2(\tilde{M})|_{\tilde{S}} &= c_2(\tilde{S}) + c_1(\tilde{S})\cdot \tilde{S}|_{\tilde{S}} \\ &= c_2(S) + c_1(S)\cdot S|_S \\ &= c_2(M)|_S \\ &= \pi^* c_2(M)|_{\tilde{S}}. \end{aligned}$$

We see from these two computations that the class $c_2(\tilde{M}) - \pi^* c_2(M)$ restricts to zero on—i.e., has intersection number 0 with—the exceptional divisor E and the inverse image of any surface $S \subset M$ not containing p. But any divisor on $\tilde{M}$ is homologous to a linear combination of such surfaces; and since the intersection form

$$H^{1,1}(\tilde{M},\mathbb{Z}) \otimes H^{2,2}(\tilde{M},\mathbb{Z}) \to \mathbb{Z}$$

is nondegenerate, this implies the lemma. Q.E.D.

Using the same approach, we can prove the slightly harder

Lemma. *If* $\pi: \tilde{M} \to M$ *is the blow-up of the algebraic threefold* M *along a smooth curve* $X \subset M$, E *the exceptional divisor of the blow-up, and* $\eta_X \in$

$H^4(M)$ the class of X, then

$$c_2(\tilde{M}) = \pi^*(c_2(M)+\eta_X) - \pi^* c_1(M)\cdot E.$$

Proof. Let $l \in H^2(E)$ denote the class of a fiber in $E \cong \mathbb{P}(N_{X/M}) \xrightarrow{\pi_E} X$; since the class $e = E|_E$ is the class of the tautological bundle on E, we have first

$$l\cdot e = -1.$$

From our basic relation

$$e\cdot e - e\cdot \pi_E^* c_1(N_{X/M}) = 0,$$

then, we have

$$e\cdot e = -c_1(N_{X/M}).$$

Now, by (*),

$$\begin{aligned} c_1(E) &= c_1(\tilde{M}) - E|_E \\ &= \pi^* c_1(M) - 2E|_E \\ &= (c_1(M)\cdot X) - 2e \end{aligned}$$

and hence by (**),

$$\begin{aligned} c_2(\tilde{M})|_E &= c_2(E) + c_1(E)\cdot E \\ &= c_2(E) - c_1(M)\cdot X + 2c_1(N_{X/M}). \end{aligned}$$

But

$$\begin{aligned} c_2(E) &= \chi(E) \\ &= 2\chi(X) \\ &= 2c_1(X) \\ &= 2c_1(M)\cdot X - 2c_1(N_{X/M}) \end{aligned}$$

and so we have

$$\begin{aligned} c_2(\tilde{M})|_E &= c_1(M)\cdot X \\ &= -\pi^* c_1(M)\cdot E|_E \\ &= \big(\pi^*(c_2(M)+\eta_X) - \pi^* c_1(M)\cdot E\big)|_E. \end{aligned}$$

Next, let $S \subset M$ be a smooth surface meeting X transversely, and let $\tilde{S}$ be its inverse image in $\tilde{M}$. $\tilde{S}$ is just the blow-up of S at the points of $S \cap X$, and so

$$c_2(\tilde{S}) = c_2(S) + S\cdot X.$$

By (*),

$$\begin{aligned} c_1(\tilde{S}) &= \big(c_1(\tilde{M}) - \tilde{S}\big)|_{\tilde{S}} \\ &= \pi^* c_1(S) - E|_S \end{aligned}$$

and so from (**) we see that

$$\begin{aligned} c_2(\tilde{M})|_S &= c_2(\tilde{S}) + c_1(\tilde{S})\cdot\tilde{S}|_{\tilde{S}} \\ &= c_2(S) + c_1(S)\cdot S|_S + S\cdot X - E\cdot\tilde{S}|_{\tilde{S}} \\ &= \pi^*c_2(M)|_{\tilde{S}} + \pi^*\eta_X|_{\tilde{S}} - E\cdot\tilde{S}|_{\tilde{S}} \\ &= \pi^*(c_2(M)+\eta_X)|_{\tilde{S}} \end{aligned}$$

since $E\cdot\tilde{S}\cdot\tilde{S} = E\cdot\pi^*(S\cdot S) = 0$. Thus we see that the class $c_2(\tilde{M}) - \pi^*(c_2(M)+\eta_X) + \pi^*c_1(M)\cdot E$ has intersection number 0 with both E and $\tilde{S}$, and hence is zero. Q.E.D.

Ordinary Singularities of Surfaces

Our task in this section is to describe the singularities of a *generic* projection of a surface $S\subset\mathbb{P}^N$ into $\mathbb{P}^3$. To begin with, we recall that for $N>5$, the generic projection of $\tilde{S}\subset\mathbb{P}^N$ from a point is an embedding, so we may take $\tilde{S}$ smooth in $\mathbb{P}^5$ to start and consider the projection map

$$\pi_L:\ \tilde{S}\to\mathbb{P}^3$$

from a generic line $L\subset\mathbb{P}^5$. As we have seen, the map π_L will be 1-1 and smooth at a point $p\in\tilde{S}$ exactly when the 2-plane $\overline{pL}$ meets $\tilde{S}$ transversely at p and nowhere else; accordingly, we will try to determine, at least dimension-theoretically, the number of times a generic line $L\subset\mathbb{P}^5$ meets a chord of $\tilde{S}$, or a tangent plane to $\tilde{S}$, or lies in a 2-plane spanned by points of $\tilde{S}$, etc. One point before we proceed: while we shall argue that the generic projection of *any* surface $\tilde{S}\subset\mathbb{P}^5$ has only ordinary singularities as defined on p. 616, all we need to know for the purpose of proving Noether's formula is that any surface may be embedded in $\mathbb{P}^5$ in such a way that the generic projection has only ordinary singularities. Indeed, some of the subtler questions of "general position" that arise in the latter part of our argument may be decided immediately by taking the embedding $\tilde{S}\to\mathbb{P}^5$ to be of sufficiently high degree; accordingly we will leave the verification of these conditions to the reader, and merely show that they are satisfied for an appropriate embedding of $\tilde{S}$.

Now let $G(2,6)$ be the Grassmannian of lines in $\mathbb{P}^5$ and consider the incidence correspondence

$$I\subset\tilde{S}\times\tilde{S}\times G(2,6)$$

given by

$$I = \left\{(p,q,L):\ p\neq q,\ \dim\overline{pqL}\leqslant 2\right\}.$$

Clearly for any two points $p\neq q\in\tilde{S}$, $\pi_L(p)=\pi_L(q)$ exactly when $(p,q,L)\in I$. But now if we let π_1, π_2, and π_3 be the projections of I onto the three factors of $\tilde{S}\times\tilde{S}\times G(2,6)$, we see that the fiber of the map

$$\pi_1\times\pi_2:\ I\to\tilde{S}\times\tilde{S}$$

over a point (p,q) corresponds to the set of lines $L\in G(2,6)$ meeting the line $\overline{pq}\subset\mathbb{P}^5$, i.e., to a Schubert cycle σ_3. Since σ_3 has codimension 3 in the eight-dimensional $G(2,6)$, the fibers of $\pi_1\times\pi_2$ have dimension 5, and so

$$\dim I = 9.$$

Thus the fiber of the map $\pi_3: I\to G(2,6)$ over a generic point $L\in G(2,6)$ has dimension at most 1. Indeed, as we shall see, $\pi_3^{-1}(L)$ cannot be empty or finite, so in fact $\pi_3^{-1}(L)$ must be one-dimensional for a generic L, and hence *the projection map* $\pi_L:\tilde{S}\to\mathbb{P}^3$ *is one-to-one outside the closure* $\tilde{C}$ *of the curve*

$$\pi_1(\pi_3^{-1}(L))\subset\tilde{S}.$$

The curve $\tilde{C}$ is called the *double curve* of the map π_L; its image $C=\pi_L(\tilde{C})$ in the image surface $S=\pi_L(\tilde{S})\subset\mathbb{P}^3$ is likewise called the *double curve* of S.

We claim now that π_L will be 3-1 only at a finite number of points, and nowhere 4-1 or more. To see this let

$$I'\subset\tilde{S}\times\tilde{S}\times\tilde{S}\times G(2,6)$$

be given by

$$I'=\{(p,q,r,L):\ p\neq q\neq r\neq p, \dim\overline{pqrL}\leqslant 2\}$$

and let π_1, π_2, π_3, and π_4 be the projections of I' on the factors of $\tilde{S}\times\tilde{S}\times\tilde{S}\times G(2,6)$. Clearly the generic triple (p,q,r) of distinct points on $\tilde{S}$ span a 2-plane; thus the fiber of

$$\pi_1\times\pi_2\times\pi_3:\ I'\to\tilde{S}\times\tilde{S}\times\tilde{S}$$

over any point *not* in the locus

$$J=\{(p,q,r):\ p\neq q\neq r\neq p \text{ collinear}\}\subset\tilde{S}\times\tilde{S}\times\tilde{S}$$

corresponds to lines L lying in $\overline{pqr}$ and so is two-dimensional. Thus

$$\dim(\pi_1\times\pi_2\times\pi_3)^{-1}(\tilde{S}\times\tilde{S}\times\tilde{S}-J)=8.$$

On the other hand, since a generic 3-plane $V_3\subset\mathbb{P}^5$ meets $\tilde{S}$ in a collection of points in general position, the generic chord $\overline{pq}$ to $\tilde{S}$ will meet $\tilde{S}$ only at p and q, so J is at most three-dimensional. The fiber of $\pi_1\times\pi_2\times\pi_3$ over a point of J being, as we have seen, five-dimensional, we have

$$\dim(\pi_1\times\pi_2\times\pi_3)^{-1}J\leqslant 8.$$

Thus I' is eight-dimensional, and so the generic fiber of the projection $\pi_4: I'\to G(2,8)$ is finite; this proves the first part of our present claim. In fact, we can say a bit more: from the proof of our basic lemma (p. 249), we see that among those 3-planes V_3 meeting $\tilde{S}$ in a collection of points not in general position, the generic one contains four points of $\tilde{S}$ spanning a 2-plane, and *not three collinear points*. Thus if we define

$$K \subset \tilde{S} \times \tilde{S} \times \tilde{S} \times G(4,6)$$

by

$$K = \{(p,q,r,\Lambda): p \neq q \neq r \in \Lambda \text{ and } r \in \overline{pq}\},$$

we see that the image of K under projection on the factor $G(4,6)$ has codimension at least 2, i.e., dimension at most 6. Assuming that $\tilde{S}$ contains only finitely many lines, then, it follows that $\dim K \leqslant 6$, and since the fibers of the projection map $\pi_1 \times \pi_2 \times \pi_3 : K \to \tilde{S} \times \tilde{S} \times \tilde{S}$ are all four-dimensional, that the dimension of $J = \pi_1 \times \pi_2 \times \pi_3(K)$ is at most 2. If $\tilde{S}$ does contain a family of lines, of course, J will be three-dimensional; but as we will see, this causes no trouble.

Now to see that π_L is never 4-1 or worse, let

$$I'' \subset \tilde{S} \times \tilde{S} \times \tilde{S} \times \tilde{S} \times G(2,6)$$

be given by

$$I'' = \{(p,q,r,t,L): p,q,r,t \text{ distinct and } \dim \overline{pqrtL} \leqslant 2\},$$

and let $\pi_1, \ldots, \pi_5$ be the corresponding projection maps. First off, since the generic triple (p,q,r) of distinct points on $\tilde{S}$ are linearly independent and the 2-plane they span contains no other points of $\tilde{S}$, the map $\pi_1 \times \pi_2 \times \pi_3$ maps I'' into a proper subvariety of $\tilde{S} \times \tilde{S} \times \tilde{S}$ with fiber dimension 2 away from J; i.e.,

$$\dim(\pi_1 \times \pi_2 \times \pi_3)^{-1}(\tilde{S} \times \tilde{S} \times \tilde{S} - J) \leqslant 7.$$

On the other hand, $\pi_1 \times \pi_2 \times \pi_3$ has fiber dimension 5 over J, so if $\tilde{S}$ does not contain a family of lines,

$$\dim(\pi_1 \times \pi_2 \times \pi_3)^{-1} J \leqslant 7.$$

Thus $\dim I'' \leqslant 7$, and the projection $\pi_5 : I'' \to G(2,6)$ cannot be surjective. Finally, if S does contain a family of lines, then I'' may be 10-dimensional —but the generic point of I'' lies in a fiber of π_5 of dimension 4, and so again π_5 cannot be surjective.

In sum, then, we have seen that for generic L, $\pi_L : \tilde{S} \to S \subset \mathbb{P}^3$ is 1-1 outside of the double curve $\tilde{C} \subset \tilde{S}$, generically 2-1 on $\tilde{C}$, and 3-1 over a finite collection of points. We consider now the possible singularities of π_L. First, let

$$I_1 \subset \tilde{S} \times G(2,6)$$

be given by

$$I_1 = \left\{(q,L):\ \dim \overline{T_q(\tilde{S}),L} \leqslant 3\right\}$$

and let π_1, π_2 be the projection maps. For each $q \in \tilde{S}$, the fiber $\pi_1^{-1}(q)$ of π_1 over q is the Schubert cycle σ_2 of lines $L \subset \mathbb{P}^5$ meeting $T_q(\tilde{S})$, and so has dimension 6; thus $\dim I_1 = 8$. The generic fiber of π_2 on I_1 is thus finite, so that for generic L the set

$$\tilde{B} = \pi_1(\pi_2^{-1}(L)) \subset \tilde{S}$$

of points where π_L fails to be smooth is finite. (Note that if $(q,L) \in I_1$, then (q,q,L) lies in the closure of I above; thus we see that $\tilde{B}$ lies in $\tilde{C}$.) Moreover, since for generic $q \in \tilde{S}$ not every line tangent to $\tilde{S}$ at q meets $\tilde{S}$ in another point besides q, the locus

$$I_1' = \{(q,r,L):\ (q,L) \in I_1, (q,r,L) \in I\}$$

has dimension at most 7, so that π_L will be 1-1 at the points of $\tilde{B}$; and since the generic tangent line to S is simply tangent, the 2-plane $\overline{qL}$ will be simply tangent to $\tilde{S}$ at each point of $\tilde{B}$.

All that remains is to check that if $\pi_L(q) = \pi_L(r)$ for two points $q \neq r \in S$, then the images in $\mathbb{P}^3$ of neighborhoods of q and r meet transversely, and likewise for triple points. In the first case, neighborhoods of q and r will fail to meet transversely in $\mathbb{P}^3$ exactly when $\overline{T_q(\tilde{S}),L} = \overline{T_r(\tilde{S}),L}$; thus we consider the variety

$$I_2 \subset I \subset \tilde{S} \times \tilde{S} \times G(2,6)$$

given by

$$I_2 = \left\{(q,r,L):\ \overline{T_q(\tilde{S}),L} = \overline{T_r(\tilde{S}),L}\right\}.$$

Letting π_1, π_2, and π_3 denote the projection maps on I_2, we see that for $(q,r) \in \tilde{S} \times \tilde{S}$, $(\pi_1 \times \pi_2)^{-1}(\tilde{q},\tilde{r})$ will be empty if $T_q(\tilde{S})$ is disjoint from $T_r(\tilde{S})$, the Schubert cycle $\sigma_{3,1} \subset G(2,6)$ of lines lying in $\overline{T_q(\tilde{S}),T_r(\tilde{S})}$ and meeting $\overline{qr}$ if $T_q(\tilde{S})$ and $T_r(\tilde{S})$ meet in a point, and the Schubert cycle σ_3 of lines meeting $\overline{qr}$ if $T_q(\tilde{S})$ and $T_r(\tilde{S})$ have a line or more in common. The reader may check that in fact the only nondegenerate surface $\tilde{S}$ in $\mathbb{P}^5$ every two of whose tangent planes meet is the Veronese surface, and any surface $\tilde{S} \subset \mathbb{P}^5$ such that $T_p(\tilde{S})$ meets $T_q(\tilde{S})$ in a line for all (p,q) in a three-dimensional subvariety of $\tilde{S} \times \tilde{S}$ must contain a family of lines. Except for these exceptional cases, then, the variety I_2 has dimension at most 7; if $\tilde{S}$ is the Veronese surface or contains a family of lines, on the other hand, we see that I_2 is at most eight-dimensional and the fiber of the map π_3 through the generic point of I_2 is positive dimensional. Thus in any case $\pi_3: I_2 \to G(2,6)$ cannot be surjective.

Alternatively, we can guarantee the condition $\dim I_2 \leqslant 7$ by rechoosing our embedding of $\tilde{S}$ in $\mathbb{P}^5$: if $L \to \tilde{S}$ is a positive line bundle, then for $k \gg 0$ we see by the proof of the Kodaira embedding theorem that L^k is very ample and for any two points $p, q \in \tilde{S}$,

$$H^1(\tilde{S}, \mathcal{I}_{p,q}(L^k)) = H^1(\tilde{S}, \mathcal{I}^2_{p,q}(L^k)) = 0.$$

It follows that

$$h^0(\tilde{S}, \mathcal{I}_{p,q}(L^k)) = h^0(\tilde{S}, \mathcal{O}(L^k)) - 2$$

and

$$h^0(\tilde{S}, \mathcal{I}^2_{p,q}(L^k)) = h^0(\tilde{S}, \mathcal{O}(L^k)) - 6,$$

i.e., the linear subspace of divisors $D \in |L^k|$ singular at two fixed points p_0 and q_0 is codimension 6. Thus the generic sublinear system of dimension 5 in $|L^k|$ contains no such divisor, or in other words no hyperplane section of the image of $\iota_{L^k}(\tilde{S})$ under a generic projection to $\mathbb{P}^5$ is singular at both p_0 and q_0. This implies that the tangent spaces to $\tilde{S} \subset \mathbb{P}^5$ at p_0 and q_0 do not both lie in any 4-plane, and so are disjoint. Likewise, we see that for fixed p_0 the locus of divisors $D \in |L^k|$ singular at p_0 and at some other point $q \in \tilde{S}$ as well has codimension 4. The generic five-dimensional sublinear system of $|L^k|$ will then contain at most a finite number of lines from this locus, that is, under a generic projection of $\iota_{L^k}(\tilde{S})$ to $\mathbb{P}^5$, there will be only finitely many $q \in \tilde{S}$ such that a pencil of hyperplane sections of $\tilde{S} \subset \mathbb{P}^5$ is singular at both p_0 and q. The locus of pairs $(p,q) \in \tilde{S} \times \tilde{S}$ with $T_p(\tilde{S})$ meeting $T_q(\tilde{S})$ in a line is thus at most two-dimensional, and so I_2 has dimension $\leqslant 7$.

Lastly, if $\pi_L(p) = \pi_L(q) = \pi_L(r)$ for distinct points $p, q, r \in S$, the images of neighborhoods of the three points will fail to meet transversely at $\pi_L(p) \in \mathbb{P}^3$ exactly when the hyperplanes $\overline{T_p(\tilde{S}), L}$, $\overline{T_q(\tilde{S}), L}$ and $\overline{T_r(\tilde{S}), L}$ intersect in a 3-plane. Since we have seen that the generic $L \subset \mathbb{P}^5$ will not meet any line containing three points of $\tilde{S}$, we consider

$$I_2' \subset \tilde{S} \times \tilde{S} \times \tilde{S} \times G(2,6)$$

given by

$$I_2' = \Big\{ (p,q,r,L) : L \subset \overline{pqr}, \\ \dim\big(\overline{T_p(\tilde{S}), q, r} \cap \overline{T_q(\tilde{S}), p, r} \cap \overline{T_r(\tilde{S}), p, q} \big) \geqslant 3 \Big\}.$$

Again, we leave to the reader the verification that the projection map $\pi_4 : I_2' \to G(2,6)$ cannot be surjective, and argue that for an appropriate embedding of $\tilde{S}$, I_2' will have dimension at most 7. This is not hard: as before, we take L a positive line bundle and choose k such that

$$H^1\big(\tilde{S},\mathcal{I}_{p,q,r}(L^k)\big) = H^1\big(\tilde{S},\mathcal{I}^2_{p,q,r}(L^k)\big) = 0.$$

It then follows that the space of divisors $D \in |L^k|$ singular at p, q, and r has codimension 9, and hence that the spaces E_p, E_q, and $E_r \subset |L^k|$ of divisors $D \in |L^k|$ containing all three points and singular at p, q, or r, respectively, have codimension 5 and are in general position in $|L^k|$. The generic five-dimensional sublinear system of $|L^k|$ will therefore intersect E_p, E_q, and E_r in three linearly independent points, or in other words under the generic projection of $\iota_{L^k}(\tilde{S})$ into $\mathbb{P}^T$, *no pencil of hyperplane sections of* S *through all three points can contain elements singular at* p, q, *and* r. But if $\overline{T_p(\tilde{S}),q,r}$, $\overline{T_q(\tilde{S}),p,r}$, and $\overline{T_r(\tilde{S}),q,p}$ met in a 3-plane, the set of hyperplane sections of $\tilde{S}$ containing that 3-plane would be just such a pencil. Thus the projection

$$(\pi_1\times\pi_2\times\pi_3)\colon\ I_2' \to \tilde{S}\times\tilde{S}\times\tilde{S}$$

maps onto a proper subvariety of $\tilde{S}\times\tilde{S}\times\tilde{S}$; since the fiber dimension of the map is clearly $\leqslant 2$, we see that $\dim I_2' \leqslant 7$.

Putting this all together, then, we see that any singular point of the image $S \subset \mathbb{P}^3$ of $\tilde{S}$ under a generic projection π is one of the following three types:

1. A transverse *double point* of S, i.e., the image of two distinct points of $\tilde{S}$. A neighborhood of p in S then consists of two smooth polydiscs intersecting transversely in the double curve C of S (see Figure 11); in terms of an appropriate local holomorphic coordinate system (u,v,w) for $\mathbb{P}^3$ near p,

$$(*)\qquad\qquad S = (uv=0)$$

and

$$C = (u=0, v=0).$$

Around the two points p', p'' in the inverse image of p, the functions

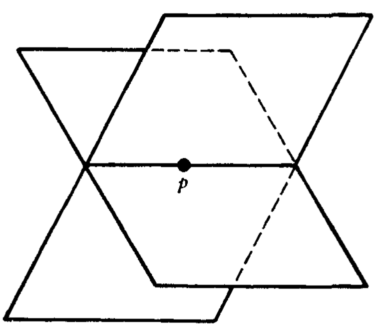


Figure 11

(π^*v, π^*w) and (π^*u, π^*w) furnish local holomorphic coordinates; $\tilde{C}$ is given near p' and p'' by $\pi^*v=0$ and $\pi^*u=0$, respectively. Note in particular that C and $\tilde{C}$ are smooth at p and $\pi^{-1}(p)$, as is the map $\pi:\tilde{C}\to C$.

2. A *triple point* of S, that is, the image of three distinct points of $\tilde{S}$. A neighborhood of p in S then consists of three smooth polydiscs intersecting transversely (see Figure 12); we can choose holomorphic coordinates (u,v,w) for $\mathbb{P}^3$ near p so that

$$(*)\qquad S=(uvw=0),$$
$$C=(u=v=0)\cup(u=w=0)\cup(v=w=0).$$

Around the three points p', p'', and p''' of $\tilde{S}$ lying over p we may take as local coordinates (π^*v, π^*w), (π^*u, π^*w) and (π^*u, π^*v); $\tilde{C}$ is then given in these coordinate systems by $(\pi^*v\cdot\pi^*w=0)$, $(\pi^*u\cdot\pi^*w=0)$, and $(\pi^*u\cdot\pi^*v=0)$, respectively.

3. A *cuspidal* (or *flat*, or *pinch*) *point* of S, i.e., the image in S of a simple tangent line to $\tilde{S}$. Here the local character of the map π near p is not so plain. Choose Euclidean coordinates $u_1,\ldots,u_5$ in $\mathbb{P}^5$ so that the family of 2-planes containing the line of projection L is given by $(u_1,u_2,u_3)=(c_1,c_2,c_3)$, and so that the line $u_1=u_2=u_3=u_4=0$ is tangent to $\tilde{S}$ at p, and choose local coordinates (s,t) on $\tilde{S}$ such that under the inclusion $T_{p'}(\tilde{S})\subset T_{p'}(\mathbb{P}^5)$,

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial u_5}\quad\text{and}\quad\frac{\partial}{\partial s}=\frac{\partial}{\partial u_3}.$$

Then the inclusion $\tilde{S}\subset\mathbb{P}^5$ will have the form

$$(s,t)\mapsto([2],[2],s+[2],[2],t+[2])$$

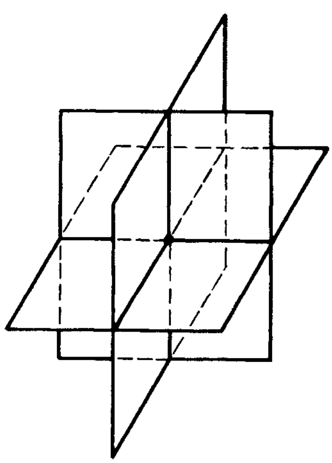


Figure 12

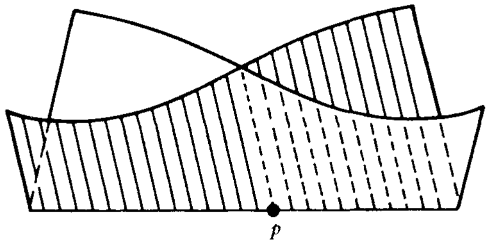


Figure 13

and in terms of Euclidean coordinates u_1, u_2, u_3 on $\mathbb{P}^3$,

$$\pi: \ (s,t) \mapsto ([2], [2], s+[2]).$$

We can then find Euclidean coordinates u, v, w on $\mathbb{P}^3$ such that (after a change in the coordinate s)

$$\pi: \ (s,t) \mapsto (st+[3], t^2+[3], s+[3]).$$

A local defining equation for S will then have the form

$$(*) \qquad f(u,v,w) = u^2 - vw^2 + [4].$$

C is thus given near p by

$$C = (u+[2] = w+[2] = 0);$$

we see that both C and $\tilde{C}$ are smooth near p and p', and that p' is a branch point of the map $\pi: \tilde{C} \to C$. The picture is this (Figure 13): at any point $q \neq p \in C$ near p, the two branches of S—corresponding to the two points of $\tilde{C}$ lying over q—meet transversely; at p, these two branches come together. (Although we shall not need the fact, we can find holomorphic coordinates (s,t) on $\tilde{S}$ and (u,v,w) on $\mathbb{P}^3$ near p such that

$$\pi(s,t) = (st, t^2, s)$$

and correspondingly

$$S = (u^2 - vw^2).)$$

The local defining equations $(*)$ for the surface S given above are called the *normal forms* of the various singular points. A surface in general is said to have *ordinary singularities* if every singular point of S is one of the above types, i.e., if S can be given in a neighborhood of any point by one of the normal forms above.

Noether's Formula for General Surfaces

We have now at our disposal all the tools necessary to prove Noether's formula along the lines suggested earlier. Given any algebraic surface S, embed S in $\mathbb{P}^5$ and choose a generic projection

$$\varphi: S \longrightarrow S_0 \subset \mathbb{P}^3$$

mapping S onto a surface $S_0 \subset \mathbb{P}^3$ of degree n having ordinary singularities. Let $p_1,\ldots,p_t$ be the triple points of S_0, $C_1,\ldots,C_u$ the irreducible components of the double curve C of S_0, and d_i and g_i the degree and genus of C_i, respectively; call $d=\sum d_i$ and $g=\sum g_i$ the degree and genus of C.

Our object is to compute both sides of Noether's formula for S in terms of the numbers n, d, g, u, and t. The first step is to describe a blow-up $X \xrightarrow{\pi} \mathbb{P}^3$ such that the proper transform of S_0 in X is smooth; because of the relatively simple nature of ordinary singularities, this is not difficult. To begin with, let

$$\pi_1: Y \longrightarrow \mathbb{P}^3$$

be the blow-up of $\mathbb{P}^3$ at the triple points $p_1,\ldots,p_t$ of S_0, E_i the exceptional divisor over p_i. In a neighborhood of each exceptional divisor E_i, the proper transform S_1 of S will consist of three smooth sheets, intersecting pairwise in smooth arcs and intersecting $E_i \cong \mathbb{P}^2$ in three lines; the double curve of S_1 is the proper transform of C, that is, the three arcs comprising the pairwise intersections of the three components of S_1, as shown in Figure 14. Explicitly, suppose z_1, z_2, z_3 are local coordinates in an open set U around the triple point p such that S_0 is given in U as the locus $(z_1z_2z_3=0)$. $\pi^{-1}(U)$ is then covered by three open sets U_1, U_2, U_3, where U_i is the complement in $\pi_1^{-1}(U)$ of the proper transform of the coordinate hyperplane $(z_i=0)$; and in terms of coordinates

$$z_i = z_i, \qquad z(i)_j = \frac{z_j}{z_i}, \qquad z(i)_k = \frac{z_k}{z_i}$$

on U_i, we see that

$$\begin{aligned}\pi_1^{-1}(S_0) &= (z_i\cdot z_i z(i)\cdot z_i z(i)_k=0)\\ &= 3E_i + (z(i)_j=0) + (\tilde{z}(i)_k=0)\end{aligned}$$

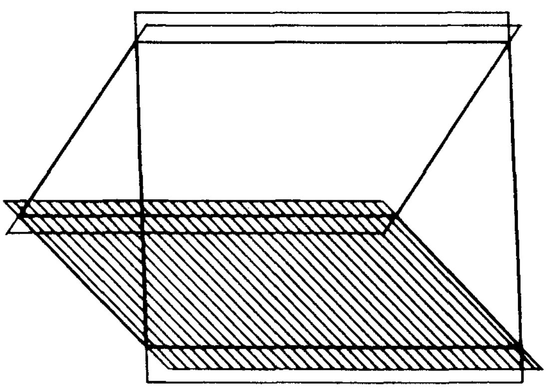


Figure 14

i.e., $S_1 \cap U_i$ consists of the proper transforms of the two coordinate hyperplanes $(z_j = 0)$ and $z_k = 0)$. The double curve $\tilde{C}$ of S_1 is the union of the arcs $(z(i)_j = z(i)_k = 0) \subset U_i$; in particular, we note that $\tilde{C}$ is smooth, and hence that all the irreducible components $\tilde{C}_i$ of $\tilde{C}$ are disjoint.

Next, we let

$$\pi_2 : X \longrightarrow Y$$

be the blow-up of Y along the double curve $\tilde{C} = \bigcup \tilde{C}_i$ of S_1, $\pi = \pi_2 \circ \pi_2$ the combined blow-up. Let F_i denote the exceptional divisor over $\tilde{C}_i$, $\tilde{S}$ the proper transform of S_1 in X, and E_i the inverse image of the exceptional divisor $E_i \subset Y$. Our first observation is that *$\tilde{S}$ is smooth in X*. Clearly this has to be checked at the inverse image of a point $p \in \tilde{C}$. If p is not a pinch point of S_1, then we may take coordinates z_1, z_2, z_3 on a neighborhood U of p in Y such that in U,

$$S_1 = (z_2 z_3 = 0), \qquad C = (z_2 = z_3 = 0).$$

The inverse image of U in X is covered by the complements U_2 and U_3 of the proper transforms of the coordinate hyperplanes $(z_2 = 0)$ and $(z_3 = 0)$. In U_2 we have coordinates

$$z_1 = z_1, \qquad z_2 = z_2, \qquad z(2)_3 = \frac{z_3}{z_2}$$

in terms of which $F = (z_2 = 0)$ and

$$\begin{aligned} \pi_2^{-1}(S_1) &= (z_2 \cdot z_2 z(2)_3 = 0) \\ &= 2F + (z(2)_3 = 0) \end{aligned}$$

so $\tilde{S}$ is smooth in U_2; similarly in terms of coordinates

$$z_1 = z_1, \qquad z(3)_2 = \frac{z_2}{z_3}, \qquad z_3 = z_3$$

on U_3, we have

$$\begin{aligned} \pi_2^{-1}(S_1) &= (z_3 \cdot z_3 \cdot z(3)_2 = 0) \\ &= 2F + (z(3)_2 = 0) \end{aligned}$$

so $\tilde{S}$ is smooth in U_3. Thus $\tilde{S}$ is smooth around $\pi_2^{-1}(p)$; indeed, we see that near p the intersection $\tilde{S} \cap F$ is just the two sections of the bundle $F \xrightarrow{\pi_2} \tilde{C}$ corresponding to the normal directions to $\tilde{C}$ in the two branches of S_1 around p.

If p is a pinch point of S_1, then we may choose coordinates z_1, z_2, z_3 in a neighborhood U of p such that

$$\begin{aligned} S_1 &= (z_2^2 - z_1 z_3^2 = 0), \\ C &= (z_2 = z_3 = 0). \end{aligned}$$

With U_i, $\tilde{z}_i$, and $\tilde{z}(j)_i$ as above, then, we see that in U_2,

$$\begin{aligned} \pi_2^{-1}(S_1) &= (z_2 - z_1 \cdot z_2^2 z(2)_3^2 = 0) \\ &= 2F + (1 - z_1 z(2)_3^2 = 0) \end{aligned}$$

i.e., $\tilde{S}$ intersects each fiber $z_1 = c$ of F (except for $z = 0$) in the two points $\tilde{z}(2)_3 = \pm\sqrt{1/c}$, and in particular $\tilde{S}$ is smooth. In U_3,

$$\begin{aligned} \pi_2^{-1}(S_1) &= \left(z_3^2 z^1(3)_2^2 - z_1 z_3^2 = 0\right) \\ &= 2F + \left(z(3)_2^2 - z_1\right) \end{aligned}$$

so $\tilde{S}$ intersects the fiber $z_1 = c$ of F in the two points $z(3)_2 = \pm\sqrt{c}$, and the fiber $z_1 = 0$ tangentially at the point $z(3)_2 = 0$, i.e., $\pi_2^{-1}(p)$ is just a branch point of the twofold cover $\pi_2 : \tilde{S} \cap F \to \tilde{C}$. Again, we see immediately that $\tilde{S}$ is smooth at $\pi^{-1}(p)$, and thus $\tilde{S}$ is everywhere smooth.

We obtain in this fashion a desingularization of any surface $S_0 \subset \mathbb{P}^3$ having only ordinary singularities. Note that in case S_0 is the image under a generic projection of a smooth surface $S \subset \mathbb{P}^5$, the surface $\tilde{S}$ is not the surface S: the identity map

$$\tilde{S} - \pi^{-1}(C) \longrightarrow S - \varphi^{-1}(C)$$

extends to an isomorphism

$$\tilde{S} - \pi^{-1}(\{p_1, \ldots, p_t\}) \longrightarrow S - \varphi^{-1}(\{p_1, \ldots, p_t\})$$

and then to a holomorphic map $\tilde{S} \to S$ blowing down the curves $\tilde{S} \cap E$ lying over $p_1, \ldots, p_t$; i.e., *$\tilde{S}$ is the blow-up of S at the $3t$ points $\varphi^{-1}(\{p_1, \ldots, p_t\})$*. We will see later that if $S_0 \subset \mathbb{P}^3$ is any surface with ordinary singularities and $\tilde{S}$ is proper transform in the blow-up X of $\mathbb{P}^3$ as constructed above, then the curves of $\tilde{S} \cap E$ are all exceptional of the first kind. If we blow down these curves, we obtain a *minimal desingularization* of S_0, that is, a surface S mapping holomorphically to $S_0 \subset \mathbb{P}^3$ which cannot be blown down to another smooth surface mapping holomorphically to S_0.

One point to be made here is that while the particularly simple nature of ordinary singularities allows us to remove them by only two blow-ups, any singularity may be eventually resolved by this process, i.e.,

Theorem (Resolution of Singularities). *Given any variety* $V \subset \mathbb{P}^n$, *there exists a blow-up* $\tilde{\mathbb{P}}^n \to \mathbb{P}^n$ *such that the proper transform of* V *in* X *is smooth.*

The proof of this theorem, due to Hironaka (cf. p. 453 for the reference), is far beyond the scope of this book.

The next step is to compute the cohomology ring of X, at least in even dimensions. To start, the cohomology ring $H^*(Y)$ is easily expressed: if $E_1, \ldots, E_t$ are the exceptional divisors of $Y \xrightarrow{\pi_1} \mathbb{P}^3$, L_i a line in $E_i \cong \mathbb{P}^2$, then we have

$$\begin{aligned} H^2(Y) &= \pi^* H^2(\mathbb{P}^3) \oplus \mathbb{C}\{E_1, \ldots, E_t\} \\ &= \mathbb{C}\{H, E_1, \ldots, E_t\} \end{aligned}$$

and likewise

$$H^4(Y) = \mathbb{C}\left\{H^2, L_1, \ldots, L_t\right\}.$$

Clearly $H \cdot E_i = E_j \cdot E_i = 0$ for $i \neq j$; and since the restriction $[E_i]|_{E_i} \cong [-L_i]$, $E_i \cdot E_i = -L_i$. In complementary dimensions, then,

$$H \cdot H^2 = 1, \qquad H \cdot L_i = 0,$$
$$E_i \cdot H^2 = 0, \qquad E_i \cdot L_j = -\delta_{ij}.$$

By our formulas,

$$\begin{aligned} c_1(Y) &= \pi_1^* c_1(\mathbb{P}^3) - 2\sum E_i \\ &= 4H - 2\sum E_i \end{aligned}$$

and $c_2(Y) = \pi^* c_2(\mathbb{P}^3) = 6H^2$.

To find the class of $\tilde{C}_i$ in $H^4(Y)$, let τ_{ij} be the number of branches of C at p_j belonging to $\tilde{C}_i$. (Note that $\Sigma_i \tau_{ij} = 3$ for all j, and so $\Sigma_{i,j} \tau_{ij} = 3t$.) Then

$$\tilde{C}_i \cdot H = d_i, \qquad \tilde{C}_i \cdot E_j = \tau_{ij},$$

and accordingly

$$\tilde{C}_i \sim d_i \cdot H^2 - \sum \tau_{ij} L_j.$$

In particular, we have

$$\begin{aligned} c_1(N_{\tilde{C}_i/Y}) &= c_1(T_Y)|_{\tilde{C}_i} - c_1(T_{\tilde{C}_i}) \\ &= 4d_i - 2\sum_j \tau_{ij} + 2g_i - 2. \end{aligned}$$

We proceed now to X. Letting F_i denote as before the exceptional divisor of the blow-up $\pi_2 : X \to Y$ over $\tilde{C}_i$, M_i a fiber of the bundle $\pi_2 : F_i \to \tilde{C}_i$ and E_j and L_j the inverse images of E_i and L_j in Y, we see that

$$H^2(X) = \mathbb{C}\{H, E_1, \ldots, E_t, F_1, \ldots, F_u\}$$

and

$$H^4(X) = \mathbb{C}\{H^2, L_1, \ldots, L_t, M_1, \ldots, M_u\}.$$

The intersection pairing in complementary dimensions is readily determined; we note first that as before

$$H \cdot H^2 = 1, \quad H \cdot L_i = 0, \quad E_i \cdot L_j = -\delta_{ij}, \quad E_i \cdot H^2 = 0,$$

and similarly

$$H \cdot M_i = E_j \cdot M_i = H^2 \cdot F_i = F_i \cdot L_j = 0,$$

since in each case the cycles can clearly be made disjoint. Lastly, since $[F_i]|_{F_i}$ is just the tautological bundle on $F_i = \mathbb{P}(N_{\tilde{C}_i/Y})$

$$F_i \cdot M_j = -\delta_{ij}.$$

In the pairing $H^2(X) \times H^2(X) \to H^4(X)$, the relations

$$H \cdot E_i = 0, \qquad E_i \cdot E_j = -\delta_{ij} L_j$$

are immediate. Since a hyperplane in $\mathbb{P}^3$ will meet C_i in d_i points, its inverse image in X will intersect the exceptional divisor F_i in d_i fibers, i.e.,

$$H \cdot F_i = d_i M_i;$$

and likewise E_j will intersect F_i in the fibers of F_i over the points of intersection $\tilde{C}_i \cap E_j$, so

$$E_j \cdot F_i = \tau_{ij} M_i.$$

Finally, since the class $[F_i]|_{F_i}$ is the class ξ of the tautological bundle on $F_i \cong \mathbb{P}(N_{\tilde{C}_i/Y}) \xrightarrow{\pi_2} \tilde{C}_i$ and by our relation

$$\begin{aligned} 0 &= \xi^2 - \xi \cdot \pi_2^* c_1(N_{\tilde{C}_i/Y}) \\ &= F_i^2|_{F_i} - F_i|_{F_i} \cdot c_1(N_{\tilde{C}_i/Y}) \cdot M_i \\ &= F_i^2|_{F_i} + 4d_i + 2g_i - 2 - 2\sum_j \tau_{ij} \end{aligned}$$

we have

$$F_i^3 = -\left(4d_i + 2g_i - 2 - 2\sum_j \tau_{ij}\right).$$

This, together with the products

$$F_i^2 \cdot H = F_i \cdot (F_i \cdot H) = F_i \cdot d_i M_i = -d_i$$

and

$$F_i^2 \cdot E_j = F_i(F_i \cdot E_j) = F_i \cdot \tau_{ij} M_i = -\tau_{ij}$$

yield the formula

$$F_i^2 = -d_i H^2 + \left(4d_i + 2g_i - 2 - 2\sum_j \tau_{ij}\right) M_i + \sum_j \tau_{ij} L_j.$$

In sum, then, the intersection form on $H^{2*}(X)$ is given by the multiplication tables

	H	E_i	F_i
H^2	1	0	0
L_j	0	$-\delta_{ij}$	0
M_j	0	0	$-\delta_{ij}$

	H	E_j	F_i
H	H^2	0	$d_i M_i$
E_k		$-\delta_{jk} L_j$	$\tau_{ij} M_i$
F_k			$\delta_{ij}(-d_i H^2 + \Sigma_j \tau_{ij} L_j + (4d_i + 2g_i - 2 - 2\Sigma_j \tau_{ij}) M_i)$

Note that the class of $\tilde{S}$ is $nH - 3\Sigma E_j - 2\Sigma F_i$, so that

$$\tilde{S} \cdot H^2 = n, \qquad \tilde{S} \cdot L_j = 3, \quad \text{and} \quad \tilde{S} \cdot M_i = 2.$$

Now, to compute $c_1^2(S)$ and $c_2(S)$ we have only to apply the adjunction formulas (*) and (**) of p. 601. First,

$$\begin{aligned} c_1(X) &= \pi_2^* c_1(Y) - \sum F_i \\ &= 4H - 2\sum E_j - \sum F_i \end{aligned}$$

so

$$\begin{aligned} c_1(\tilde{S}) &= \left(c_1(T_X) - (\tilde{S})\right)|_{\tilde{S}} \\ &= -(n-4)H + \sum E_j + \sum F_i \end{aligned}$$

and

$$\begin{aligned} c_1^2(\tilde{S}) = \tilde{S}\Bigg[&(n-4)^2H^2 - 2(n-4)\sum d_iM_i - \sum L_i \\ &+ 2\sum_{i,j} \tau_{ij}M_i - \sum d_iH^2 \\ &+ \sum_i \Big(4d_i + 2g_i - 2 - 2\sum_j \tau_{ij}\Big)M_i + \sum_{i,j} \tau_{ij}L_j \Bigg] \\ = {}& n(n-4)^2 - 4(n-4)d - 3t + 12t - nd + 8d \\ &+ 4g - 4u - 12t + 9t \\ = {}& n(n-4)^2 - 5nd + 24d + 4g - 4u + 6t. \end{aligned}$$

Since $\tilde{S}$ is the blow-up of S at $3t$ points, then,

$$c_1^2(S) = c_1(\tilde{S}) + 3t = n(n-4)^2 - 5nd + 24d + 4g - 4u + 9t.$$

Next, we have

$$\begin{aligned} c_2(X) &= \pi_2^*\Big(c_2(Y) + \sum \tilde{C}_i\Big) - \pi_2^* c_1(Y) \cdot \sum F_i \\ &= (d+6)H^2 - \sum_i \Big(4d_i - 2\sum_j \tau_{ij}\Big)M_i - 2\sum_{i,j} \tau_{ij}L_j, \end{aligned}$$

hence

$$\begin{aligned} c_2(\tilde{S}) &= \left(c_2(T_X) - c_1(\tilde{S})\cdot\tilde{S}\right)|_{\tilde{S}} \\ &= \tilde{S}\cdot\Bigg[(d+6)H^2 - \sum_i \Big(4d_i - 2\sum_j \tau_{ij}\Big)M_i - 2\sum_{i,j}\tau_{ij}L_j \\ &\quad + n(n-4)H^2 - n\sum d_iM_i - 3\sum_j L_j - 2(n-4)\sum d_iM_i \\ &\quad + 5\sum_{i,j}\tau_{ij}M_i - 2\sum d_iH^2 + 2\sum_{i,j}\tau_{ij}L_j \\ &\quad + 2\sum_i \Big(4d_i + 2g_i - 2 - 2\sum_j \tau_{ij}\Big)M_i\Bigg] \\ &= n(d+6) - 8d + 12t - 9t - 4(n-4)d + n^2(n-4) \\ &\quad - 2dn - 9t + 30t - 2nd + 18t + 16d + 8g - 8u - 24t \end{aligned}$$

and

$$c_2(S) = c_2(\tilde{S}) - 3t$$
$$= n^2(n-4) + 6n + 24d - 7nd + 8g - 8u + 15t.$$

Thus, in sum,

$$\frac{c_1^2(S)+c_2(S)}{12} = \frac{(n-1)(n-2)(n-3)}{6} - (n-4)d + g - u + 2t + 1.$$

To prove Noether's formula we have now to express the holomorphic Euler characteristic $\chi(\mathcal{O}_S)=\chi(\mathcal{O}_{\tilde{S}})$ also in terms of n, d, g, u, and t. To begin with, by the Poincaré residue sequence for $\tilde{S}\subset X$

$$0 \longrightarrow \Omega_X^3 \longrightarrow \Omega_X^3(\tilde{S}) \longrightarrow \Omega_{\tilde{S}}^2 \longrightarrow 0$$

we see that

$$\chi(\mathcal{O}_{\tilde{S}}) = \chi(\Omega_{\tilde{S}}^2) = \chi(\Omega^3(\tilde{S})) - \chi(\Omega_X^3) = \chi(\Omega_X^3(\tilde{S})) + 1$$

since $\chi(\Omega_X^3)=\chi(\Omega_{\mathbb{P}^3}^3)=-1$. To evaluate the holomorphic Euler characteristic

$$\chi(\Omega_X^3(\tilde{S})) = \chi(\mathcal{O}_X(K_X+\tilde{S}))$$
$$= \chi\Big(\mathcal{O}_X\Big((n-4)H-\sum E_j-\sum F_i\Big)\Big)$$

we use a succession of restriction maps. First, we consider the sequence

$$0 \to \mathcal{O}_X\Big((n-4)H-\sum E_j-\sum F_i\Big) \to \mathcal{O}_X\Big((n-4)H-\sum E_j\Big)$$
$$\to \bigoplus_i \mathcal{O}_{F_i}\Big(\Big((n-4)H-\sum E_j\Big)\Big) \to 0.$$

To find the Euler characteristic of the last term, we use Riemann-Roch for line bundles on F_i: the divisor $((n-4)H-\sum E_j)|_{F_i}$ is just the sum of $d_i(n-4)-\sum_j\tau_{ij}$ fibers of the ruled surface $F_i \xrightarrow{\pi_2} \tilde{C}_i$; since the fibers have self-intersection 0 and intersection number -2 with K_{F_i},

$$\chi\Big(\mathcal{O}_{F_i}\Big((n-4)H-\sum E_j\Big)\Big) = \chi(\mathcal{O}_{F_i}) + d_i(n-4) - \sum_j \tau_{ij}$$
$$= 2-2g_i + d_i(n-4) - \sum_j \tau_{ij}.$$

Thus

$$\chi\Big(\bigoplus_i \mathcal{O}_{F_i}\Big((n-4)H-\sum E_j\Big)\Big) = u - g + d(n-4) + 3t$$

and

$$\chi(\mathcal{O}_S) = \chi\Big(\mathcal{O}_X\Big((n-4)H-\sum E_j\Big)\Big) - d(n-4) + g - u + 3t + 1.$$

Next, from the sequence

$$0 \to \mathcal{O}_X((n-4)H - \sum E_j(\to \mathcal{O}_X(n-4)H) \to \bigoplus_j \mathcal{O}_{E_j}((n-4)H) \to 0$$
$$\bigoplus_j \mathcal{O}_{E_j}((n-4)H) = \bigoplus_j \mathcal{O}_{E_j}$$

we see that

$$\chi\Big(\mathcal{O}_X\big((n-4)H - \sum E_j\big)\Big) = \chi(\mathcal{O}_X((n-4)H)) - \chi(\oplus \mathcal{O}_{E_j})$$
$$= \chi(\mathcal{O}_X((n-4)H)) - t$$

and hence

$$\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_X((n-4)H)) - d(n-4) + g - u + 2t + 1.$$

Finally, to evaluate $\chi(\mathcal{O}_X((n-4)H))$, in case $n \geqslant 4$ we let $T \subset \mathbb{P}^3$ be a smooth surface of degree $n-4$ missing $p_1, \ldots, p_t$ and meeting each curve C_i transversely; we let $\tilde{T} \subset X$ be its inverse image and consider the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X((n-4)H) \longrightarrow \mathcal{O}_{\tilde{T}}((n-4)H) \longrightarrow 0.$$

On $\tilde{T}$, we have $((n-4)H)^2 = (n-4)H \cdot K_{\tilde{T}}$, and by Riemann-Roch,

$$\chi(\mathcal{O}_{\tilde{T}}((n-4)H)) = \chi(\mathcal{O}_{\tilde{T}}) = \chi(\mathcal{O}_T) = \frac{(n-1)(n-2)(n-3)}{6} + 1$$

so

$$\chi(\mathcal{O}_X((n-4)H)) = \frac{(n-1)(n-2)(n-3)}{6}.$$

In case $n = 1$, 2, or 3, we let T instead be a surface of degree $4-n$ in $\mathbb{P}^3$, and from the sequence

$$0 \longrightarrow \mathcal{O}_X((n-4)H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\tilde{T}} \longrightarrow 0$$

we deduce again that

$$\chi(\mathcal{O}_X((n-4)H)) = 1 - \chi(\mathcal{O}_{\tilde{T}})$$
$$= 0$$
$$= \frac{(n-1)(n-2)(n-3)}{6}.$$

Thus, in either case, we have

$$\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_{\tilde{S}})$$
$$= \frac{(n-1)(n-2)(n-3)}{6} - d(n-4) + g - u + 2t + 1$$

and Noether's formula is proved.

It should be noted that neither the geometric genus or the irregularity of the surface S appears by itself in the above formulas. The fact is that while, as we shall see, we can usually determine $p_g(S)$ and $q(S)$ for any given surface $S \subset \mathbb{P}^3$, these invariants are not determined by the numbers n, d_i, g_i, u, and t.

To find the geometric genus of a surface $S \to S_0 \subset \mathbb{P}^3$ as given above, we return to the Poincaré residue sequence on the desingularization $\tilde{S} \subset X$:

$$0 \longrightarrow \Omega_X^3 \longrightarrow \Omega_X^3(\tilde{S}) \longrightarrow \Omega_{\tilde{S}}^2 \longrightarrow 0.$$

Since X is rational,

$$h^0(X, \Omega_X^3) = 0 \quad \text{and} \quad h^1(X, \Omega_X^3) = h^0(X, \Omega_X^2) = 0$$

so

$$\begin{aligned} p_g(S) &= h^0(X, \Omega_X^3(\tilde{S})) \\ &= h^0\Big(X, \mathcal{O}_X\big((n-4)H - \sum E_j - \sum F_i\big)\Big). \end{aligned}$$

Now, any section of the line bundle $(n-4)H$ on $\mathbb{P}^3$ vanishing along the curve C gives a section of $(n-4)H - \Sigma E_j - \Sigma F_i$ on X, and conversely by Hartogs' theorem any section $\sigma \in H^0(X, \mathcal{O}_X((n-4)H - \Sigma E_i - \Sigma F_i))$ is the pullback of a section of $(n-4)H$ on $\mathbb{P}^3$ vanishing on C. Thus,

The canonical series $|K_S|$ *of* S *is cut out by surfaces of degree* $n-4$ *in* $\mathbb{P}^3$ *containing the curve* C;

and we may express $p_g(S)$ accordingly: setting $|E| = \pi^*|\mathcal{O}_{\mathbb{P}^3}((n-4)H)| \subset |\mathcal{O}_{\tilde{C}}(\pi^*(n-4)H)|$,

The geometric genus $p_g(S)$ *is the number* $\binom{n-1}{3}$ *of surfaces of degree* $n-4$ *in* $\mathbb{P}^3$*, less the vector space dimension of the linear system* $|E|$ *they cut out on* $\tilde{C}$.

Comparing this with our previous formula for the Euler characteristic $\chi(\mathcal{O}_S) = p_g(S) - q(S) + 1$, we obtain a particularly simple expression for $q(S)$: we have

$$\begin{aligned} q(S) &= p_g(S) - \chi(\mathcal{O}_{\tilde{S}}) + 1 \\ &= \binom{n-1}{3} - \dim|E| - \binom{n-1}{3} \\ &\quad + d(n-4) - g + u - 2t \\ &= \sum_{i=1}^{u} (d_i(n-4) - g_i + 1) - \dim|E| \\ &= h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-4)H)) - \dim|E| \\ &\quad - h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}((n-4)H)) - 2t. \end{aligned}$$

The difference of the first two terms represents the failure of the linear system $|E|$ to be complete; it is called the *deficiency* of the system $|E|$. In these terms we have

The irregularity q(S) *is the deficiency minus the index of speciality of the linear system* $|E|$ *cut out on* $\tilde{C}$ *by surfaces of degree* $n-4$ *in* $\mathbb{P}^3$, *less twice the number of triple points.*

A final point to be made in this context is that the number b of pinch points of a surface $S_0 \subset \mathbb{P}^3$ as given above is determined by the data n, d, g, u, and t: if $D_i \subset \tilde{S}$ is the inverse image of the curve $\tilde{C}_i \subset S_1$, then the number b_i of pinch points of S_0 along C_i is just the number of branch points of the two-sheeted covering map $\pi_2 : D_i \to \tilde{C}_i$, and by Riemann-Hurwitz and adjunction this number is

$$\begin{aligned} b_i &= (2g(D_i)-2)-(2g_i-2) \\ &= D_i \cdot K_{\tilde{S}} + D_i \cdot D_i - 4g_i + 4. \end{aligned}$$

But we have seen that

$$\begin{aligned} K_{\tilde{S}} &= \Big((n-4)H - \sum E_j - \sum F_i\Big)\Big|_{\tilde{S}} \\ &= \Big((n-4)H - \sum E_j\Big)\Big|_{\tilde{S}} - \sum D_i \end{aligned}$$

so

$$D_i \cdot (K_{\tilde{S}} + D_i) = 2(n-4)d_i - 2\sum_j \tau_{ij}$$

and

$$b_i = 2(n-4)d_i - 2\sum_j \tau_{ij} - 4g_i + 4.$$

The total number of pinch points on S is thus

$$b = 2d(n-4) - 6t - 4g + 4u.$$

Some Examples

We consider now some examples of irreducible surfaces $S_0 \subset \mathbb{P}^3$ with ordinary singularities. In each case we will let $S \xrightarrow{\pi} S_0$ be the minimal desingularization of S_0, $C \subset S_0$ the double curve of S_0, and $D \subset S$ its inverse image in S, and take n, d, g, u, and t as above.

To begin with, suppose S_0 is a cubic surface. Inasmuch as the generic plane section $H \cdot S_0$ of S_0 is then an irreducible plane cubic curve singular at the d points of $H \cdot C$, we see that $d \leqslant 1$, i.e., C can be at most a line. Supposing that C is a line, it follows immediately that S is a rational ruled surface: the pencil $\{H_\lambda\}$ of planes in $\mathbb{P}^3$ containing C cuts out on S a pencil $\{L_\lambda\}$ of lines. In terms of the basis $\{E_0, L_\lambda\}$ for $S \cong S_n$, we can write the class of a hyperplane on S as

$$H = aE_0 + bL_\lambda;$$

since $H \cdot L_\lambda = 1$, $a = 1$ and then from

$$3 = H \cdot H = (E_0 + bL_\lambda)^2 = n + 2b$$

it follows that $n = 1$, $H = E_0 + L_\lambda$, and $D = H - L_\lambda = E_0$. Now, we have seen that the complete linear system $|E_0 + L_\lambda|$ embeds the ruled surface S_1 as the Steiner surface $S_{1,1} \subset \mathbb{P}^4$; thus

A cubic surface $S_0 \subset \mathbb{P}^3$ *with a double line is the projection of the Steiner surface* $S_{1,1} \subset \mathbb{P}^4$.

Conversely, any Steiner surface $S \cong S_{1,1} \subset \mathbb{P}^4$ may be realized as the union of lines joining points on a line $D_\infty \subset \mathbb{P}^4$ to corresponding points on a conic D_0 in a complementary 2-plane $W \subset \mathbb{P}^4$; and it is not hard to see that the projection π_p of $S_{1,1}$ from a point $p \in W - D_0$ is one-to-one away from D_0 and maps D_0 two-to-one onto a line. Since any irreducible curve in the two-dimensional system $|E_0|$ may be chosen as D_0 in this construction, any point $p \in \mathbb{P}^4 - S_{1,1}$ lies in the 2-plane spanned by such a curve, and hence

The image of a Steiner surface $S_{1,1} \subset \mathbb{P}^4$ *under projection from any point* $p \in \mathbb{P}^4 - S_{1,1}$ *is a cubic surface with a double line.*

We turn now to quartic surfaces S_0. Note first that since the generic plane section of S_0 is an irreducible quartic with $d = \deg(C)$ singularities, we must have $d \leqslant 3$. Also, any line meeting C three times meets S_0 six times and so lies in S_0; and we accordingly eliminate the possibility of C being the union of three disjoint lines—as the reader may easily verify, the locus of lines in $\mathbb{P}^3$ meeting each of three skew lines is a quadric surface. The remaining possibilities are:

1. C a line,
2. C a smooth plane conic,
3. C the union of two skew lines,
4. C a rational normal curve,
5. C the union of three lines meeting at a point.

Note that in all these cases

$$K_S = -D < 0$$

so that $P_m(S)=0$ for all m; in cases 1, 2, 4, and 5 we see as well that $q(S)=0$, and hence S is rational. By cases, then,

2. C *is a smooth plane conic.* By our formula,

$$\begin{aligned} c_2(S) &= n^2(n-4) + 24d - 7nd + 6n + 8g - 8u + 15t \\ &= 0 + 48 - 56 + 24 + 0 - 8 + 0 \\ &= 8 \end{aligned}$$

so S must be a ruled surface blown up four times, or $\mathbb{P}^2$ blown up five times; since

$$K_S = -D = -H$$

is strictly negative, no curve can have self-intersection less than -1; it follows as usual that S *is* $\mathbb{P}^2$ *blown up in five general points*, no three collinear. We see in addition that each of the five exceptional divisors $E_1,\dots,E_5$ must meet $D=-K_S$ once; thus if $\iota: S\to\mathbb{P}^2$ is the blowing-up map, the image $\iota(D)$ must be a curve of self-intersection

$$D\cdot D + 5 = 9,$$

containing all five blown-up points, i.e.,

$$H = \iota^*3H - E_1 - \cdots - E_5.$$

But we have seen that the linear system $|\iota^*3H - E_1 - \cdots - E_5|$ embeds S in $\mathbb{P}^4$ as the intersection of two quadrics; thus

a quartic S *with a double conic is the projection into* $\mathbb{P}^3$ *of the intersection of two quadrics in* $\mathbb{P}^4$.

3. C *is two disjoint lines* C_1 *and* C_2. Since $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}((n-4)H)) = h^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}) = 1$ while $h^0(C,\mathcal{O}) = h^0(C_1,\mathcal{O}) + h^0(C_2,\mathcal{O}) = 2$, and neither $\mathcal{O}_{C_1}$ nor $\mathcal{O}_{C_2}$ is special, we see that $q(S)=1$; thus S is birationally ruled over an elliptic curve. Indeed, by our formula $c_2(S)=0$, so S is ruled. We can locate the ruling: the pencil $\{H_\lambda\}$ of hyperplanes through C_1 cuts out on S a pencil of conics $\{C_\lambda\}$ which, being singular at the points $H_\lambda \cap C_2$, must all consist of two lines L_λ, L'_λ. Since the pencil $\{C_\lambda\}$ has no base points on S, moreover, the lines L_λ, L'_λ are disjoint there—one passes through each of the points of S lying over the point $H_\lambda \cap C_2$. The curves D_1 and D_2 are thus sections of the ruled surface S.

The reader may find it an amusing exercise to show that the surface S_0 may be realized in two other ways: either as the union of the lines corresponding to the intersection of the Grassmannian $G(2,4)\subset\mathbb{P}^5$ of lines in $\mathbb{P}^3$ with a generic quadric surface $Q\subset\mathbb{P}^3\subset\mathbb{P}^5$; or as the union of the lines joining corresponding points on two skew lines C_1 and C_2 in $\mathbb{P}^3$, under a correspondence of bidegree (2,2) between C_1 and C_2.

4. *C is a rational normal curve.* S is now regular, hence rational; and since by the formula (*)

$$c_2(S) = 4,$$

we see that S is a rational ruled surface. To see the ruling, let Q be a generic element of the net N of quadrics through C; Q intersects S in a curve of type $(4,4)$—that is, homologous to a sum of four lines from each family—and since a rational normal curve has type $(1,2)$ on Q, the residual intersection of Q with S is of type $(4,4)-2(1,2)=(2,0)$—that is, the sum of two lines, L, L', each meeting C twice. Since any quadric $Q \in N$ containing a third point of L' contains L', we have a pencil of quadrics $\{Q_\lambda\}$ containing L'; the residual intersection of $\{Q_\lambda\}$ with S will then be a pencil $\{L_\lambda\}$ of lines.

Now, consider the divisor $D-H$ on S. By our formulas

$$D \cdot D = K \cdot K = 8$$

and since

$$H \cdot D = 6$$

we see that

$$\begin{aligned} h^0(D-H) &\geqslant \frac{(D-H)(D-H-K)}{2}+1 \\ &= \frac{(D-h)(2D-h)}{2}+1 \\ &= \frac{16-18+4}{2}+1 \\ &= 2 \end{aligned}$$

i.e., $D-H$ moves in at least a pencil. Since

$$(D-H)^2 = 8-12+4=0$$

and

$$(D \cdot H) \cdot L_\lambda = \tfrac{1}{2}(D-H)(2H-D) = 1,$$

we see that the curves in $|D-H|$ form a second ruling of S transverse to the first; thus $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ with L_λ and $E = D-H$ the fibers. Finally, we have

$$\begin{aligned} H &= (D-H)+(2H-D) \\ &= E + 2L_\lambda, \end{aligned}$$

so we see that the complete linear system $|H|$ embeds $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ as the rational normal scroll $S_{2,0} \subset \mathbb{P}^5$; so

a quartic surface S_0 *double along a rational normal curve is a projection of the surface* $S_{2,0} \subset \mathbb{P}^5$.

5. C *consists of three lines* C_1, C_2, C_3 *meeting in a point* p. This is the easiest case: we have by our formulas

$$\chi(\mathcal{O}_S) = 1,$$

hence $q(S)=0$ and S is rational. But now

$$c_2(S) = 3$$

and so $S \cong \mathbb{P}^2$. Since the degree of S_0 is four, S_0 is a projection of the Veronese surface $S \subset \mathbb{P}^5$.

Conversely, we can see that the projection of the Veronese surface $S \subset \mathbb{P}^5$ from a generic line $L \subset \mathbb{P}^5$ is such a quartic: we have seen that the chordal variety of the Veronese is equal to the union of the 2-planes spanned by the conic curves (i.e., the images in S of lines in $\mathbb{P}^2$) on S, and that this is a cubic hypersurface in $\mathbb{P}^5$. L then meets this cubic hypersurface in three points; i.e., there are exactly three conics in S whose 2-planes intersect L and under the projection these three conics are mapped two-to-one onto double lines of the image S_0.

We leave it to the reader to show that a quartic surface with a double line is the image of $\mathbb{P}^2$ blown up at nine points under the map given by the system of quartics double at one point and passing through the other eight.

Our last example is perhaps the most interesting. We found, in the previous section, two ways of representing an Enriques surface: as the quotient of a K-3 surface by a fixed-point-free involution, or as an elliptic surface with rational base, having two double fibers. We can now give, in addition, a projective realization of an Enriques surface, as follows. (See Figure 15.) Let T be a tetrahedron in $\mathbb{P}^3$ with vertices p_1, p_2, p_3, p_4, edges $l_{ij} = \overline{p_i p_j}$, and faces $H_{ijk} = \overline{p p_j p_k}$, and let $S \subset \mathbb{P}^3$ be a surface of degree 6 with ordinary singularities whose double curve is exactly the sum of the six lines l_{ij}.

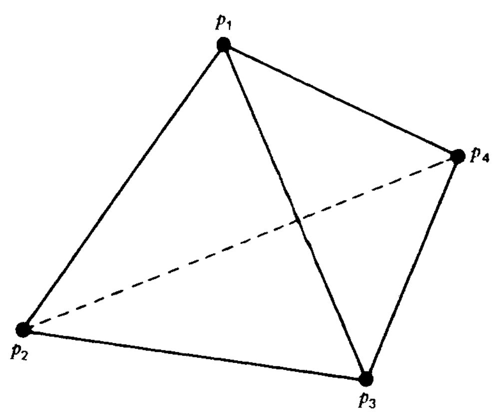


Figure 15

Now if $\tilde{S} \xrightarrow{\pi} S \subset \mathbb{P}^3$ is the desingularization of $\tilde{S}$, $\tilde{l}_{ij} = \pi^{-1} l_{ij}$, then we have

$$K_{\tilde{S}} = \pi^*(2H) - \sum \tilde{l}_{ij},$$

i.e., the canonical series on $\tilde{S}$ is cut out by surfaces of degree 2 in $\mathbb{P}^3$ passing simply through the edges of T. But any quadric containing the lines l_{ij}, l_{jk}, and l_{ik} contains the plane H_{ijk}, so there are no such quadrics; thus

$$p_g(\tilde{S}) = 0.$$

On the other hand, twice the canonical series

$$2K_{\tilde{S}} = \pi^*(4H) - 2\sum \tilde{l}_{ij}$$

is cut out by quartic surfaces in $\mathbb{P}^3$ passing doubly through the edges of T. There is one such quartic: namely the sum

$$Q = \sum H_{ijk}$$

of the four faces of T. Moreover, each plane H_{ijk} meets S in the sextic curve $2l_{ij} + 2l_{jk} + 2l_{ik}$ and nowhere else; thus Q meets S only along the lines l_{ij}, and so the bundle $2K_{\tilde{S}}$ is trivial.

Now, note that the desingularization $\tilde{C}$ of the double curve $C = \Sigma l_{ij}$ of S consists of six disjoint rational curves; so that the system $|E|$ of p. 627 is nonspecial and has deficiency

$$h^0(\tilde{C}, \mathcal{O}(2)) - 10 = 3 \cdot 6 - 10 = 8.$$

The irregularity of $\tilde{S}$ is thus

$$q(\tilde{S}) = 8 - 2t = 0$$

and hence $\tilde{S}$ is an Enriques surface.

We can realize a canonical divisor of $\tilde{S}$ explicitly as follows: let Q be the quadric

$$H_{ijk} + H_{ijl}$$

consisting of two faces of T. Then Q passes doubly through the common line l_{ij} of the two faces, simply through the edges l_{ik}, l_{il}, l_{jk}, and l_{jl} and not at all through the edge l_{kl}. Since the canonical curve of $\tilde{S}$ is $\pi^*2H - \Sigma \tilde{l}_{ij}$, then, we see that

$$K_{\tilde{S}} = \tilde{l}_{ij} - \tilde{l}_{kl},$$

i.e., *the canonical divisor on* $\tilde{S}$ *is the difference of the inverse images of any two opposite edges of* T.

Now, in a previous discussion we showed that an Enriques surface $\tilde{S}$ can be represented as an elliptic surface with rational base, having two double fibers. Indeed, if $\tilde{S}$ is given as a sextic $S \to S$ in $\mathbb{P}^3$ double along the edges of a tetrahedron, we can find three elliptic pencils on S directly: take two disjoint edges l_{ij}, l_{kl} of T and consider the linear system of quadrics in $\mathbb{P}^3$ passing through the remaining four edges $l_{ik}, l_{il}, l_{jk}, l_{jl}$ of T. To see that there is at least a pencil $\{Q_\lambda\}$ of such quadrics, choose arbitrary points $q_1 \in l_{ik}$, $q_2 \in l_{il}$, $q_3 \in l_{jk}$, and $q_4 \in l_{jl}$ distinct from the vertices of T; any quadratic containing the four vertices $\{p_i\}$ of T and the four points $\{q_i\}$ will then have three points in common with each of the edges $l_{ik}, l_{il}, l_{jk}, l_{jl}$ and so will contain them. Since the linear system of quadrics in $\mathbb{P}^3$ is nine-dimensional, there will be at least a $9-8=1$-dimensional family $\{Q_\lambda\}$ of such quadrics. (Conversely, since the four lines comprise the complete intersection of any two quadrics containing them, there can be no more than a pencil of such quadrics.)

Note that the pencil $\{Q_\lambda\}$ will contain exactly two reducible quadrics: $Q_0 = H_{ilk} + H_{jlk}$ and $Q_1 = H_{ijk} + H_{ijl}$. Every other quadric in the pencil will be smooth, inasmuch as it contains, for example, the two disjoint lines l_{ik} and l_{jl}, whereas we have seen that every two lines on an irreducible singular quadric in $\mathbb{P}^3$ meet.

Now write

$$Q_\lambda \cdot S = 2l_{ik} + 2l_{il} + 2l_{jk} + 2l_{jl} + C_\lambda.$$

Consider the pencil $\{C_\lambda\}$ of curves on $\tilde{S}$ and their inverse images $\tilde{C}_\lambda = \pi^* C_\lambda$ in $\tilde{S}$. We note that

$$\tilde{C}_\lambda \cdot \tilde{C}_\lambda = 0,$$

since in particular

$$C_0 = 4l_{kl} \quad \text{and} \quad C_1 = 4l_{ij}$$

and their inverse images

$$\tilde{C}_0 = 2\tilde{l}_{kl} \quad \text{and} \quad \tilde{C}_1 = 2\tilde{l}_{ij}$$

are disjoint. Since $\tilde{S}$ is an Enriques surface, $K \cdot \tilde{C}_\lambda = 0$; so

$$\pi(\tilde{C}_\lambda) = \frac{\tilde{C}_\lambda \cdot \tilde{C}_\lambda}{2} + 1 = 1,$$

and by Bertini the generic $\tilde{C}_\lambda$ is smooth. *The surface* $\tilde{S}$ *is thus an elliptic surface via the map*

$$\Psi : \tilde{S} \longrightarrow \mathbb{P}^1$$

given by the pencil $\{C_\lambda\}$; the two multiple fibers are $2\tilde{l}_{ij}$ and $2\tilde{l}_{kl}$. Note that if $L = [p]$ is the line bundle associated to a point $p \in \mathbb{P}^1$, by our formula

$$K_{\tilde{S}} = -\Psi^* L + \tilde{l}_{ij} + \tilde{l}_{kl} = \tilde{l}_{ij} - \tilde{l}_{kl},$$

since $2\tilde{l}_{k,l} = \Psi^* L$; this agrees with our previous computation.

Finally, while we will not prove that every Enriques surface can be realized as a sextic in $\mathbb{P}^3$, we can suggest this fact by counting parameters for sextics in $\mathbb{P}^3$ double along the edges of a tetrahedron T. To begin with, we may take the vertices of T to be the coordinate points $p_1=[1,0,0,0]$, $p_2=[0,1,0,0]$, $p_3=[0,0,1,0]$, and $p_4=[0,0,0,1]$. The requirement that a sextic $S\subset\mathbb{P}^3$, given as the locus of a polynomial $f(X_0,\ldots,X_3)$, be double along the edges of T is simply that its intersection with each face $H_{jkl}=(X_i=0)$ of T be the double triangle

$$2l_{jk}+2l_{jl}+2l_{kl};$$

e.g., that

$$f(0,X_1,X_2,X_3)=\lambda_0\cdot X_1^2X_2^2X_3^2,$$

$$f(X_0,0,X_2,X_3)=\lambda_1\cdot X_0^2X_2^2X_3^2,$$

and so on. Every term in f other than the four terms $X_j^2X_k^2X_l^2$ must thus contain the factor $X_0X_1X_2X_3$, and so we can write

$$f(X_0,X_1,X_2,X_3)$$
$$=\lambda_0X_1^2X_2^2X_3^2+\lambda_1X_0^2X_2^2X_3^2+\lambda_2X_0^2X_1^2X_3^2+\lambda_3X_0^2X_1^2X_2^2+Q(X)\cdot X_0X_1X_2X_3,$$

where $\lambda_0,\ldots,\lambda_3\in\mathbb{C}$ and $Q(X)$ is some homogeneous quadratic polynomial in $X_0,\ldots,X_3$. Conversely, since the sextics in $\mathbb{P}^3$ given by polynomials f of this type form a linear system without base locus except along the edges of T, for generic Q the locus of f is an Enriques surface.

Now the group of automorphisms of $\mathbb{P}^3$ fixing the tetrahedron T is generated by the permutations

$$[X_0,\ldots,X_3]\mapsto[X_{\sigma(0)},\ldots,X_{\sigma(3)}]$$

of the coordinates, plus the diagonal maps

$$[X_0,\ldots,X_3]\mapsto[\mu_0X_0,\ldots,\mu_3X_3].$$

There is, thus, up to a permutation $\sigma\in\Sigma_4$, a unique automorphism of $\mathbb{P}^3$ carrying any sextic double along the edges of a tetrahedron into the locus of

$$g(X_0,X_1,X_2,X_3)$$
$$=X_1^2X_2^2X_3^2+X_0^2X_2^2X_3^2+X_0^2X_1^2X_3^2+X_0^2X_1^2X_2^2+Q(X)\cdot X_0X_1X_2X_3.$$

We see from this that the family of such sextics, up to projective isomorphism, has dimension $H^0(\mathbb{P}^3,\mathcal{O}(2H))=10$. This is, of course, the dimension of the family of Enriques surfaces as we computed in the previous section; since the family of Enriques surfaces is irreducible and any Enriques surface—having irregularity 0—has only countably many divisor classes, hence at most countably many representations as a sextic in $\mathbb{P}^3$, this tells us that the generic Enriques surface may be realized as a sextic in $\mathbb{P}^3$, double along the edges of a tetrahedron.

Isolated Singularities of Surfaces

Thus far we have dealt only with surfaces in $\mathbb{P}^3$ having positive-dimensional singular locus, for the simple reason that these are the only singularities that necessarily arise from a projection of a smooth surface. Isolated singularities of surfaces are ubiquitous in other contexts, however, and we would be remiss if we did not mention them. Since the general theory is far too complex for our present purposes, we will give here a few examples of surfaces with ordinary isolated double points.

To begin with, if S is a surface lying on a smooth threefold X and $p \in S$ is an isolated point of multiplicity m on S, then the tangent cone to S at p will be a curve of degree m (counting multiplicity) in $\mathbb{P}(T_p(X)) \cong \mathbb{P}^2$; we say that p is an *ordinary* m-fold point of S if the tangent cone is smooth with multiplicity 1. We consider first the simplest possible case, that of an ordinary double point $p \in S$; to avoid complication, assume for the time being that S is smooth away from p. We construct the desingularization of S as follows: choose local coordinates x, y, z on a neighborhood U of p in X so that the defining function $f(x,y,z)$ of S has the form

$$f(x,y,z) = x^2 + y^2 + z^2 + [3].$$

Let $\tilde{X} \xrightarrow{\pi} X$ be the blow-up of X at p. As per Section 5 of Chapter 1, we may take as local coordinates on the complement U_1 in $\pi^{-1}(U)$ of the closure of $\pi^*(x=0, (y,z) \neq 0)$ the functions

$$x_1 = x, \qquad y_1 = \frac{y}{x}, \qquad z_1 = \frac{z}{x};$$

on $U_2 = \pi^{-1}(U) - \overline{\pi^{-1}(y=0, (x,z) \neq 0)}$ the functions

$$x_2 = \frac{x}{y}, \qquad y_2 = y, \qquad z_2 = \frac{z}{y}$$

and on $U_3 = \pi^{-1}(U) - \overline{\pi^{-1}(z=0, (x,y) \neq 0)}$,

$$x_3 = \frac{x}{z}, \qquad y_3 = \frac{y}{z}, \qquad z_3 = z.$$

We have, then,

$$\begin{aligned} x &= x_1 = x_2 y_2 = x_3 z_3, \\ y &= x_1 y_1 = y_2 = y_3 z_3, \\ z &= x_1 z_1 = y_2 z_2 = z_3. \end{aligned}$$

In U_1 we have

$$\begin{aligned} \pi^{-1}(S) &= (\pi^* f) \\ &= x_1^2 + x_1^2 y_1^2 + x_1^2 z_1^2 + [x_1^3] \\ &= (x_1^2)(1 + y_1^2 + z_1^2 + [x_1]), \end{aligned}$$

and so the proper transform $\tilde{S}$ of S is given by

$$\tilde{S} = \pi^{-1}(S) - 2E = \left(1 + y_1^2 + z_1^2 + [x_1]\right),$$

where $E = (x_1)$ is the exceptional divisor of the blow-up; clearly $\tilde{S}$ is smooth over p. Likewise, in U_2, $E = (y_2)$ and we have

$$\begin{aligned}\pi^{-1}(S) &= (\pi^* f)\\ &= x_2^2 y_2^2 + y_2^2 + y_2^2 z_2^2 + [y_2^3]\\ &= (y_2^2)\left(x_2^2 + 1 + z_2^2 + [y_2]\right),\end{aligned}$$

so $\tilde{S} = (x_2^2 + 1 + z_2^2 + [y_2])$, which is smooth in the locus $y_2 = 0$; and similarly we check that $\tilde{S}$ is smooth in $U_3 \cap E$. Since by hypothesis S is smooth away from p, $\tilde{S}$ is everywhere smooth, and the map $\pi : \tilde{S} \to S$ is the desingularization of S. Note that the inverse image of p in $\tilde{S}$ is the smooth conic curve C given, in terms of Euclidean coordinates y_1, z_1 on $E \cong \mathbb{P}^2$, by

$$1 + y_1^2 + z_1^2 = 0.$$

Now we can compute the canonical bundle of $\tilde{S}$ readily: recalling from Section 5 of Chapter 1 that

$$K_{\tilde{X}} = \pi^* K_X + 2E$$

and

$$\tilde{S} \sim \pi^* S - 2E,$$

we see that

$$\begin{aligned}K_{\tilde{S}} &= \left(K_{\tilde{X}} + \tilde{S}\right)|_{\tilde{S}}\\ &= \pi^*(K_X + S),\end{aligned}$$

i.e., the canonical divisor on $\tilde{S}$ is cut out by the linear system $|K_X + S|$ on X, just as for a smooth surface S' of the same class as S on X; in particular,

$$c_1^2(\tilde{S}) = c_1^2(S').$$

Indeed, writing out the Poincaré residue sequence

$$0 \longrightarrow \Omega_X^3 \longrightarrow \Omega_X^3(S) \longrightarrow \Omega_{\tilde{S}}^2 \longrightarrow 0,$$

we see that the codimension in $|K_S|$ of the series cut out by $|K_X + S|$ is the dimension of the kernel of the map $H^1(X, \Omega_X^3) \to H^1(X, \Omega_X^3(S))$, just as it is for S'; thus we have also

$$p_g(\tilde{S}) = p_g(S').$$

Now, since $K_{\tilde{S}} = \pi^*(K_X + S)$,

$$K_{\tilde{S}} \cdot C = 0.$$

By the adjunction formula, then,

$$C \cdot C = -2.$$

Alternatively, another way to find the self-intersection of C is to write

$$\begin{aligned}(C \cdot C)_{\tilde{S}} &= (E \cdot E \cdot \tilde{S})_X \\ &= (E \cdot E \cdot (\pi^* S - 2E))_{\tilde{X}} \\ &= -2(E \cdot E \cdot E)_{\tilde{X}};\end{aligned}$$

since $[E]|_E$ is the dual of the hyperplane bundle H on $E \cong \mathbb{P}^2$, we have

$$(E \cdot E \cdot E)_{\tilde{X}} = (-H \cdot -H)_E = 1,$$

and so $C \cdot C = -2$.

To find $\chi(\mathcal{O}_{\tilde{S}})$ we consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X(-S) \longrightarrow \mathcal{I}_{p,X} \xrightarrow{\pi^*} \mathcal{O}_{\tilde{S}}(-E) \longrightarrow 0;$$

we have

$$\chi(\mathcal{O}_{\tilde{S}}(-E)) = \chi(\mathcal{O}_{\tilde{S}}) + \frac{E \cdot E + K \cdot E}{2} = \chi(\mathcal{O}_{\tilde{S}}) - 1,$$

and from the exact sequence

$$0 \longrightarrow \mathcal{I}_{p,X} \longrightarrow \mathcal{O}_X \longrightarrow \mathbb{C}_p \longrightarrow 0$$

we see that

$$\chi(\mathcal{I}_{p,X}) = \chi(\mathcal{O}_X) - 1.$$

It follows that

$$\chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-S)),$$

which, we see from the analogous exact sequence

$$0 \longrightarrow \mathcal{O}_X(-S') \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

is just the holomorphic Euler characteristic of a smooth surface $S' \subset X$ of the same class as S. Since $p_g(\tilde{S}) = p_g(S)$, it follows that $q(\tilde{S}) = q(S)$; from $c_1^2(\tilde{S}) = c_1^2(S)$ and Riemann-Roch, it follows that $c_2(\tilde{S}) = c_2(S)$. In sum, then,

The desingularization $\tilde{S}$ *of a surface* $S \subset X$ *with an ordinary double point* p *is the proper transform of* S *in the blow-up of* $\mathbb{P}^3$ *at* p; *the inverse image of* p *in* $\tilde{S}$ *is a smooth rational curve of self-intersection* -2, *and all the invariants* q, χ, *and* $|K|$ *of* $\tilde{S}$ *are the same as those of a smooth surface of the same class on* X.

One generalization of this case is straightforward: if p is an ordinary singular point of multiplicity m on $S \subset X$, then the proper transform $\tilde{S}$ of S

in the blow-up of X at p will always be smooth. The invariants of $\tilde{S}$—while not in general equal to those of a smooth surface, equivalent to S as above—are then relatively easy to find. For example, since

$$\tilde{S} \sim \pi^* S - mE,$$

we see that

$$K_{\tilde{S}} = (\pi^* K_X - (m-2)E)|_{\tilde{S}},$$

i.e., the canonical divisor on $\tilde{S}$ is cut out by surfaces in the series $|K_X + S|$ on X containing p with multiplicity $m-2$. Likewise, we see that the inverse image $C = \tilde{S} \cap E \subset \tilde{S}$ of p is a smooth plane curve of degree m in $E \cong \mathbb{P}^2$, having self-intersection $-m$ in $\tilde{S}$, and so on.

Of course, in general the proper transform $\tilde{S}$ of a surface $S \subset X$ in the blow-up of X at a singular point p of S may still be singular at a point over p, necessitating further blow-ups. Consider, for example, a couple of nonordinary double points, both having as their tangent cone two distinct lines:

$$S_1 = \left(x^2 + y^2 + z^3 + [4] = 0\right)$$

and

$$S_2 = \left(x^2 + y^2 + z^4 + [5] = 0\right).$$

Taking $\tilde{X} \xrightarrow{\pi} X$ the blow-up of X at $p = (0,0,0)$ with open sets U_i and coordinate systems (x_i, y_i, z_i) as above, the proper transform $\tilde{S}_1$ of S_1 in $\tilde{X}$ is given as

$$\left(1 + y_1^2 + [x_1]\right) \quad \text{in } U_1,$$
$$\left(1 + x_2^2 + [y_2]\right) \quad \text{in } U_2,$$

and

$$\left(x_3^2 + y_3^2 + [z_3]\right) \quad \text{in } U_3.$$

We see than that $\tilde{S}_1$ is smooth, with $C = \pi^{-1}(p)$ the pair of rational curves C_1 and C_2, given in U_1, U_2, and U_3 by

$$C_1 = (y_1 = i, x_1 = 0) = (x_2 = -i, y_2 = 0) = (y_3 = ix_3, z_3 = 0)$$

and

$$C_2 = (y_1 = -i, x_1 = 0) = (x_2 = i, y_2 = 0) = (y_3 = -ix_3, z_3 = 0),$$

respectively, each having self-intersection -2 and meeting in the point $y_3 = x_3 = z_3 = 0$, as in Figure 16. The proper transform $\tilde{S}_2$ of S_2, however, is given by

$$\left(1 + y_1^2 + x_1^2 z_1^4 + [x_1^3]\right) \quad \text{in } U_1,$$
$$\left(x_2^2 + 1 + y_2^2 z_2^4 + [y_2^3]\right) \quad \text{in } U_2,$$

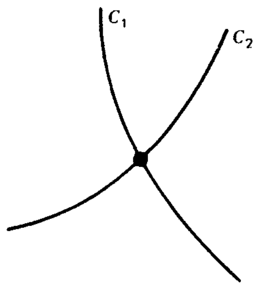


Figure 16

and

$$\left(x_3^2+y_3^2+z_3^2+\left[z_3^3\right]\right) \qquad \text{in } U_3.$$

Once again, the inverse image $\pi^{-1}(p)$ in $\tilde{S}_2$ is the two curves C_1 and C_2, but this time their point $p'=(x_3=y_3=z_3=0)$ of intersection is an ordinary double point of $\tilde{S}_2$. If we let $\tilde{\tilde{X}} \xrightarrow{\pi'} \tilde{X}$ be the blow-up of $\tilde{X}$ at p', then the proper transform $\tilde{\tilde{S}}_2$ of $\tilde{S}_2$ will be smooth, and $\tilde{\tilde{S}}_2 \to S_2$ the desingularizationof S_2. The inverse image $(\pi \circ \pi')^{-1}(p)$ of p in $\tilde{\tilde{S}}_2$ will thus consist of three curves: the proper transforms $\tilde{C}_1$ and $\tilde{C}_2$ of C_1 and C_2 and the inverse image $C_3 = \pi'^{-1}(p')$, forming the configuration shown in Figure 17.

To see how isolated singularities affect the geometry of surfaces in projective space, we consider a cubic surface S in $\mathbb{P}^3$ having δ isolated double points. By what we have said, the arguments of Section 4 of this chapter apply here to show that the desingularization $\tilde{S}$ of S is $\mathbb{P}^2$ blown up at six points $\{p_i\}$; since $\tilde{S}$ will contain δ rational curves of self-intersection -2, however, it is no longer true that the points p_i are necessarily distinct (i.e., some p_i may lie on the exceptional divisor of the blow-up of $\mathbb{P}^2$ at p_j), or that they are in general position.

Suppose first that S has exactly one ordinary double point p; let $\tilde{D}$ be its inverse image in $\tilde{S}$, and $D=\pi(\tilde{D})$ the image of $\tilde{D}$ under the blowing-down map $\pi: \tilde{S} \to \mathbb{P}^2$. There are then three possibilities:

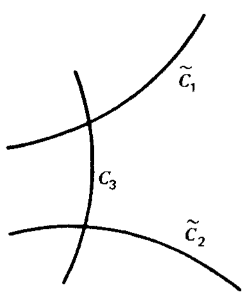


Figure 17

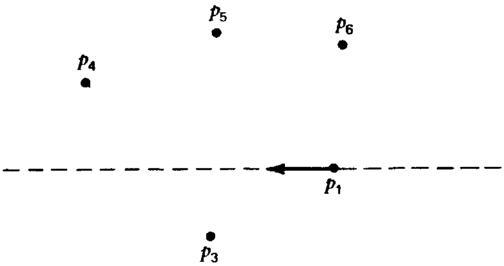


Figure 18

1. $\deg D = 0$, i.e., $\tilde{D}$ is an exceptional divisor. Since $\tilde{D}$ is the only curve of self-intersection < -1 on $\tilde{S}$, this can occur in only one way (see Figure 18): if the points $p_1, p_3, \ldots, p_6$ are all distinct, p_2 is a point on the exceptional divisor E_1 of the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at p_1, and $\tilde{D}$ is the proper transform of E_1 under the blow-up of $\mathbb{P}^2$ at p_2. Also no three of the points $p_1, p_3, \ldots, p_6$ may be collinear, as this would give rise to a second curve of self-intersection -2 on $\tilde{S}$; likewise, if v is the tangent direction to p_1 specified by p_2, then the line through p_1 in the direction v contains none of the points $p_3, \ldots, p_6$, and no conic containing $p_3, \ldots, p_6$ passes through p_1 with tangent v. We can count the number of lines on the surface $S \subset \mathbb{P}^3$: since the map $\tilde{S} \to \mathbb{P}^3$ is given by the inverse canonical series $|-K_S|$, the lines on S are, as before, the rational curves of self-intersection -1 on $\tilde{S}$: namely, the five exceptional divisors $E_2, E_3, \ldots, E_6$, the ten lines $\{L_{ij}\}_{i,j \neq 2}$ plus the line L_{12} through p_1 in the direction v specified by p_2, the four conics $\{C_i\}_{i \neq 1,2}$ passing through the points $p_1, p_3, \ldots, \hat{p}_i, \ldots, p_6$ and having tangent line v at p_1, and the conic C_2 passing through $p_1, p_3, \ldots, p_6$. We have thus a total of 21 lines.

2. $\deg D = 1$. In this case (Figure 19) exactly three of the points p_i—say p_1, p_2, and p_3—lie on the line D_1; apart from that, the points $\{p_i\}$ are in general position. Again, we have 21 lines on S: the six exceptional divisors $E_1, \ldots, E_6$, the 12 lines $\{L_{ij}\}$ for i,j not both in $\{1,2,3\}$, and the three conics $\{C_i\}_{i=1,2,3}$.

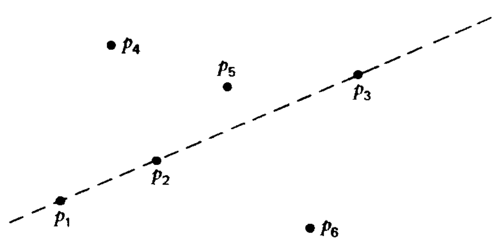


Figure 19

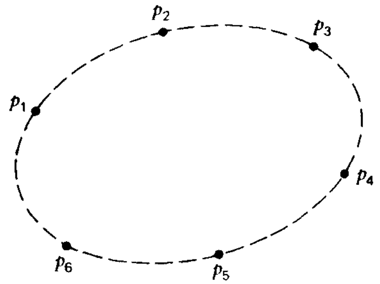


Figure 20

3. $\deg D = 2$. In this case (Figure 20) all the points p_i lie on the conic D and are otherwise in general position. Once more we have 21 lines on S: the six exceptional divisors E_i and the 15 lines L_{ij}.

The same argument as given for the smooth cubic surface shows that $\deg D \leqslant 2$, so these are the only possibilities.

Note, finally, that any S may be realized as any one of these types: in case 1, the six lines $\{L_{1j}\}_{j=2,\ldots,6}$, C_2, and E_2 are disjoint and may be blown down to obtain $\mathbb{P}^2$; the image of the exceptional divisor E_1 is then a conic curve containing all six image points (this blowing-down amounts to projection of S_q from the double point). Likewise, the exceptional divisors E_2, C_2, L_{13}, and $\{L_{3j}\}_{j=4,5,6}$ are all disjoint and may be blown down; under this map to $\mathbb{P}^2$ E_1 maps a line containing the image points of E_2, C_2, and L_{13}.

A cubic surface S with two double points may be obtained by blowing up a configuration as shown in Figure 21; the lines $\overline{p_2p_3p_4}$ and $\overline{p_2p_5p_6}$ become the double points. Such a surface will have 16 lines: the six exceptional divisors, the lines $\{L_{1j}\}_{j=2,\ldots,6}$ and $\{L_{35}, L_{36}, L_{45}, L_{46}\}$, and the conic C_2. Note that apart from the one line E_2 joining the two double points, there will be four lines on S through each of the double points: for example, through the image of $\overline{p_2p_3p_4}$ pass E_3, E_4, L_{15}, and L_{16}.

Of course, the desingularization $\tilde{S}$ of a cubic surface S with two double points may also be realized as the blow-up of $\mathbb{P}^2$ at other configurations of points (for example, projection of S from either of its two double points expresses $\tilde{S}$ as the blow-up of $\mathbb{P}^2$ at five points $p_1,\ldots,p_5$, blown up again at the point of the exceptional divisor E_5 corresponding to the tangent line at p_5 to the conic through $p_1,\ldots,p_5$). The reader may, as an exercise, find all such configurations and then show that they are all equivalent, i.e., that by blowing down a suitable collection of six disjoint lines every cubic with two double points may be realized as $\mathbb{P}^2$ blown up in any of these configurations. (For example, if the surface $\tilde{S}$ is a priori the blow-up described above, we may blow down the exceptional divisors E_1, E_2, $\tilde{L}_{34}$,

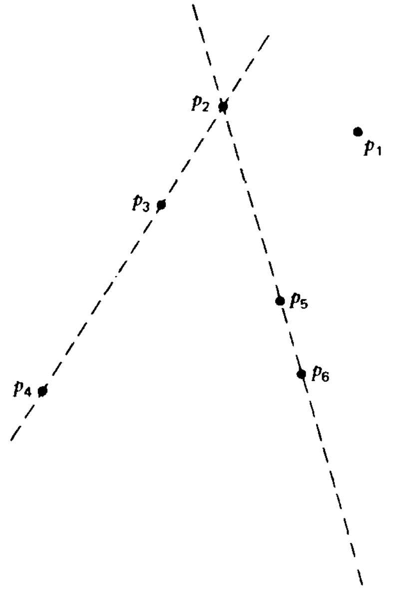


Figure 21

$\tilde{L}_{45}$, $\tilde{L}_{35}$, and E_6 to express $\tilde{S}$ as $\mathbb{P}^2$ blown up in the configuration of Figure 21, with p_1, p_2, $\overline{p_2p_3p_4}$, and $\overline{p_2p_5p_6}$ the images of $\tilde{L}_{34}$, E_6, C_6, and E_5, respectively.) Note one configuration that does not work: in the blow-up $\tilde{S}$ of $\mathbb{P}^2$ at the points $p_1,\ldots,p_6$ shown in Figure 22 the proper transforms of the lines $\overline{p_1p_2p_3}$ and $\overline{p_4p_5p_6}$ meet, and under the map of $\tilde{S}$ to $\mathbb{P}^3$ given by $|-K_{\tilde{S}}|$ they will blow down to form a single nonordinary double point, of the same type as S_1 in the example of p. 639.

A cubic with three double points may be obtained by blowing up the points $p_1,\ldots,p_6$ as shown in Figure 23; the proper transforms of the lines $\overline{L_{123}}$, $\overline{L_{345}}$, and $\overline{L_{156}}$ will map down to the double points. We have in this case 12 lines: $E_1,\ldots,E_6$, L_{24}, L_{25}, L_{26}, L_{41}, L_{46}, and L_{56}.

Note that, apart from the three lines E_1, E_3, E_5 forming the edges of the triangle with vertices at the double points of S, there will be just two lines through each of these double points (e.g., E_2 and L_{46} pass through the image of L_{123}).

To obtain a cubic with four double points, we specialize still further to the configuration of Figure 24. Here we have nine lines: again, the six E_i, plus the three lines L_{24}, L_{15}, and L_{36}. Note that the lines E_i form the edges of the tetrahedron whose vertices are the double points of $S \subset \mathbb{P}^3$, while the lines L_{24}, L_{15}, and L_{36} form a triangle of lines joining opposite edges of the tetrahedron, disjoint from the double points. (See Figure 25.)

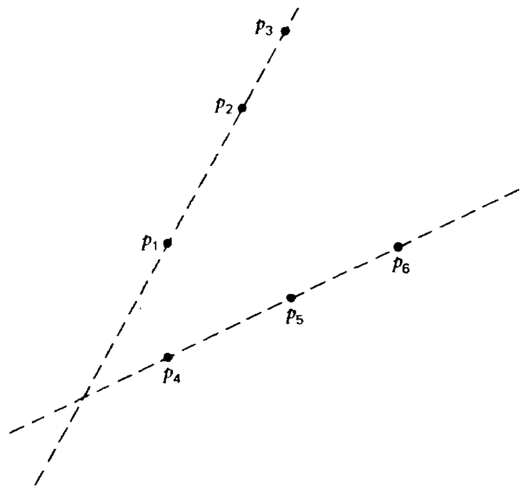


Figure 22

Finally, we note that a cubic surface S cannot contain more than four isolated singularities. To see this, suppose S had five double points $P_1, \ldots, P_5$. (See Figure 26.) Then all the lines $L_{ij} = \overline{P_i P_j}$ lie on S, from which it follows that no four of the points P_i are coplanar: if they were, the plane containing them would have six lines in common with the cubic S and so would necessarily lie in S. Now the points $P_1, \ldots, P_4$ form the vertices of a

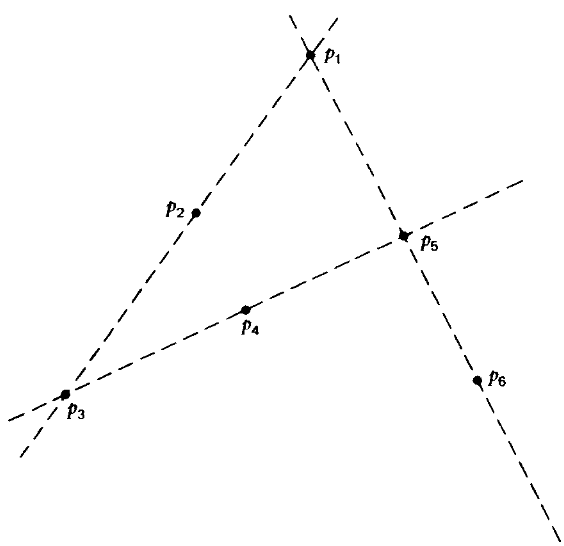


Figure 23

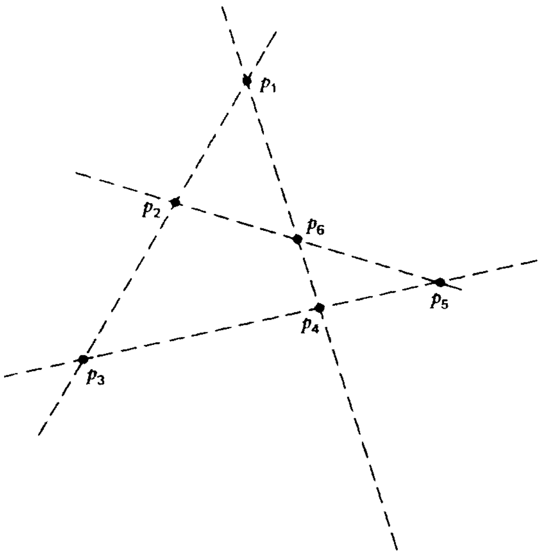


Figure 24

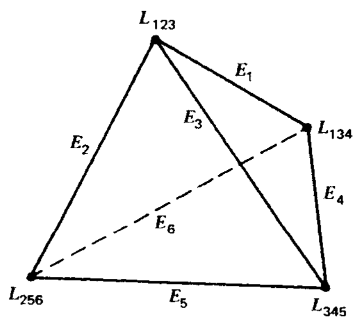


Figure 25

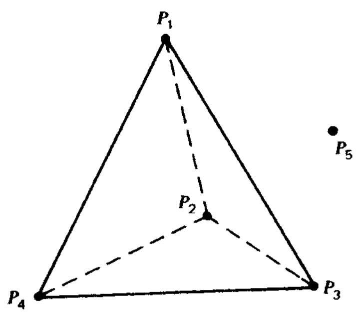


Figure 26

tetrahedron T all of whose edges lie on S, and each face of the tetrahedron will meet S in exactly the three edges lying on it. The line $\overline{P_4P_5}$ must therefore meet the face $\overline{P_1P_2P_3}$ at a point on one of the lines $\overline{P_1P_2}$, $\overline{P_2P_3}$, or $\overline{P_1P_3}$—this implies that P_5 lies on one of the faces of T containing P_4, a contradiction.

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General References

As in the case of curves, the literature is vast. Two sources are especially helpful:

O. Zariski, *Algebraic Surfaces*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York. 1971. Contains a summary of the classical theory and an extensive bibliography that has recently been updated.

E. Bombieri and D. Husemuller, Classification and embeddings of surfaces, *Proc. Symposia in Pure Math.*, Vol. 29, American Math. Society, Providence, 1975, pp. 329–420. This paper provides a survey of, and bibliographical guide to, the recent literature.

Further general references are:

F. Enriques, *Le superficie algebriche*, Zanichelli, Bologna, 1949.

F. Conforto, *Le superficie razionali*, Zanichelli, Bologna, 1939.

And, as mentioned in the introduction,

S. Lefschetz, *L'analysis situs et la géométrie algébrique*, selected papers, Chelsea, N.Y., 1971.

5

RESIDUES

Thus far in this book most of the methods we have developed for studying algebraic varieties have centered around the divisors—especially the linear systems—that lie on the variety. Not only does this technique generally suffice for obtaining a deep understanding of curves and surfaces, but it also entails a minimal amount of analytic and algebraic machinery. On the other hand, many of the outstanding questions in algebraic geometry are concerned with higher-dimensional varieties. Because of the Lefschetz theorems from Chapters 0 and 1, the "new"—i.e., not coming from a lower-dimensional subvariety—cohomology of a smooth n-dimensional algebraic variety M lies in $H^{[n/2]}(M)$, so that going to dimension $n \geqslant 3$ or studying subvarieties of codimension $k \geqslant 2$ are closely related, while divisors pertain to cohomology in degrees 1 and 2. So in this chapter we shall present a modest introduction to some of the methods for dealing with general higher-codimensional problems, both local and global, and in the last chapter we shall investigate a three-dimensional variety.

As in the divisorial case we will develop the theory around the concept of residue. The local residue, given by a variant of the n-variable Cauchy formula, has been present since the early days of several complex variables. It has recently come into focus in an algebraic context in connection with Grothendieck's general duality theorem, which in fact isolated the functorial aspects of the local analytic residue. The subsequent global residue theorem expresses the duality characteristic of a closed variety and should yield many specific applications.

In Section 1 we give an analytic definition of the residue as an integral. It may be alternatively interpreted as a cohomology class, and many of the various integral formulas in several complex variables are manifestations of this same class in different cohomology theories. We then proceed to derive its two most important local properties, the behavior of the residue under a change of variables and local duality theorem. Once the local residue has been properly understood, the global residue theorem turns out, not surprisingly, to be Stokes' theorem.

Next in Section 2 we give some applications of residues. The first two are to intersection numbers and finite holomorphic mappings. These are topics in local analytic geometry, and the use of residues affords an elegant method for studying them. Then we turn to applications of the global residue theorem in projective space. Here it is a kind of Lagrange interpolation formula in several variables, and it provides an amusing technique for studying configurations of points in $\mathbb{P}^2$ leading to several classical results in the theory of plane algebraic curves, including a discussion of the converse to the Bezout theorem.

In Section 3 some of the recent algebraic techniques are introduced. The discussion here is minimal and develops only those methods that will be applied to concrete geometric problems. Following a discussion of Ext, Tor, and Koszul complexes, a synthesis occurs when our analytically defined local residue reappears in a final intrinsic form, one that opens the way for globalization. Other standard applications of computations based on the Koszul complex include Hilbert's syzygy and Noether's "$AF+BG$" theorems.

Next, in this same section, coherent sheaves are introduced. In essentially the only violation of our principle of always proving the "hard" theorems used in the book, we discuss but do not prove the two main facts —Oka's lemma and the finite dimensionality of cohomology. In fact, these are not used in our study of any specific questions, but we felt it would be misleading in a book on algebraic geometry to leave such an important topic unmentioned.

As hinted above, in Section 4 we reap one dividend of the intrinsic understanding of residues when we arrive at a global duality theorem in functorial form. We only prove a special case of the most general duality statement—one that is at the opposite extreme from the Kodaira-Serre duality previously encountered and that suffices for our applications. The methods used will adapt to a more general context.

Our first application is a recent theorem of Carrell and Liebermann concerning vector fields with isolated zeros on compact Kähler manifolds. Following this we derive two "reciprocity formulas" which give methods for calculating the superabundance—or equivalently the measure of the failure to impose independent conditions on a linear system—of a configuration of points on an algebraic surface. Indeed, the second reciprocity formula deals with 0-dimensional schemes and not just points, and uses in an essential way the local and global duality theorems.

Finally we turn to a question, initiated by Schwarzenberger, of understanding the relation between points on a surface and rank-two vector bundles. This illustrates both the global duality theorem and original definition of "Ext" in terms of extensions. The end result is a generalization of the residue theorem to sections of vector bundles and subsequent interpretation of this as imposing necessary and sufficient Abel-type conditions on a configuration of points on a surface to be the zeros of a section of a rank-two vector bundle, perhaps helping to clarify those aspects of the fundamental correspondence between divisors and line bundles that will and will not generalize.

We would like to specifically thank Maurizio Cornalba and David Mumford for extremely valuable help in preparing this chapter.

1. ELEMENTARY PROPERTIES OF RESIDUES

Definition and Cohomological Interpretation

Let U be the ball $\{z \in \mathbb{C}^n : \|z\| < \varepsilon\}$ and $f_1, \ldots, f_n \in \mathcal{O}(\bar{U})$ functions holomorphic in a neighborhood of the closure $\bar{U}$ of U. Since we are interested in the local theory around the origin, we shall allow ourselves to decrease the radius ε as necessary. We assume that the $f_i(z)$ have the origin as isolated common zero, or equivalently that set-theoretically $f^{-1}(0) = \{0\}$, where $f = (f_1, \ldots, f_n)$. We set

$$\begin{aligned} D_i &= (f_i) = \text{divisor of } f_i, \\ D &= D_1 + \cdots + D_n, \\ U_i &= U - D_i, \\ U^* &= U - \{0\} = \bigcup_{i=1}^{n} U_i. \end{aligned}$$

Note that $\underline{U} = \{U_i\}$ gives an open cover of the punctured ball U^*.

We shall be interested in residues associated to a meromorphic n-form

$$\omega = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \qquad (g \in \mathcal{O}(\bar{U}))$$

having polar divisor D. The residue is a variant of the Cauchy integral in several variables, and is defined as follows: Let Γ be the real n-cycle defined by

$$\Gamma = \{z : |f_i(z)| = \varepsilon_i\}$$

and oriented by

$$d(\arg f_1) \wedge \cdots \wedge d(\arg f_n) \geq 0.$$

Then the *residue* is given by

$$\operatorname{Res}_{\{0\}} \omega = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \omega.$$

Here are some elementary properties of the residue.

First, since $\omega \in H^0(U-D, \Omega^n)$ is holomorphic in $U-D$, the exterior derivative $d\omega = 0$. Consequently, *the residue depends only on the homology class of* $\Gamma \in \mathrm{H}_n(\mathrm{U}-\mathrm{D}, \mathbb{Z})$ *and the cohomology class* $[\omega] \in \mathrm{H}^n_{\mathrm{DR}}(\mathrm{U}-\mathrm{D})$ *of* ω.

Second, *the residue is linear in* g *and alternating in the* f_i, the latter being due to the manner in which the cycle Γ has been oriented.

Third, we shall say that $f=(f_1,\ldots,f_n)$ is *nondegenerate* in case the Jacobian determinant

$$\mathcal{J}_f(0) = \left| \frac{\partial(f_1,\ldots,f_n)}{\partial(z_1,\ldots,z_n)}(0) \right| \neq 0$$

is nonzero at the origin. Later on we shall see that the Jacobian is not identically zero. In the nondegenerate case we find that

$$\mathrm{Res}_{\{0\}} = \frac{g(0)}{\mathcal{J}_f(0)}.$$

To prove this, consider the mapping $w = f(z)$, which by the inverse function theorem is biholomorphic in a neighborhood of the origin. Set

$$G(w) = g(f^{-1}(w)),$$

$$K(w) = \frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_n}{w_n} \qquad \text{(Cauchy kernel)},$$

and

$$\mathcal{J}_f(w) = \mathcal{J}_f(f^{-1}(w)).$$

Then

$$\omega = f^*\left(\frac{GK}{\mathcal{J}}\right),$$

and by change of variables in the integral and the usual Cauchy integral formula from Section 1 of Chapter 0,

$$\int_\Gamma \omega = \int_{|w_i|=\epsilon} \frac{G(w)K(w)}{\mathcal{J}_f(w)} = (2\pi\sqrt{-1}\,)^n \frac{G(0)}{\mathcal{J}_f(0)}.$$

Finally, we denote by $I(f) = f_1,\ldots,f_n$ the ideal generated by the f_i's in the ring of germs of holomorphic functions around 0. Then:

$$\mathrm{Res}_{\{0\}}\,\omega = 0 \qquad \text{in case } g \in I(f).$$

To prove this, it suffices by linearity to consider the case $g = hf_1$. But then

$$\omega = \frac{h(z)\,dz_1 \wedge \cdots \wedge dz_n}{f_2(z)\cdots f_n(z)}$$

is holomorphic in the larger open set $U_{\{1\}^0} = U - (D_2 + \cdots + \hat{D}_i + \cdots + D_n)$.

If Γ_i is the chain

$$\Gamma_i = \{z: |f_j(z)| = \varepsilon_j \text{ for } j \neq i, |f_i(z)| \leqslant \varepsilon_i\},$$

then $\Gamma_1 \subset U_{\{1\}^0}$ and $\partial\Gamma_1 = \pm\Gamma$. Hence $\int_\Gamma \omega = \pm\int_{\Gamma_1} d\omega = 0$ by Stokes' theorem.

We now give a sheaf-cohomological interpretation of the residue. To motivate this we note that, even though the meromorphic form ω has polar divisor $D_1 + \cdots + D_n$, it is the origin $\{0\} = D_1 \cap \cdots \cap D_n$ with which we are most concerned. To express this, we consider $\omega \in H^0(U_1 \cap \cdots \cap U_n, \Omega^n)$ as a Čech $(n-1)$-cochain for the sheaf Ω^n and covering $\underline{U} = \{U_i\}$ of U^*. Thus $\omega \in C^{n-1}(\underline{U}, \Omega^n)$, and since trivially $\delta\omega = 0$, we obtain a class in $H^{n-1}(U^*, \Omega^n)$. Denote by η_ω the image of $(1/2\pi\sqrt{-1})^n\omega$ under the Dolbeault isomorphism

$$H^{n-1}(U^*, \Omega^n) \cong H_{\bar\partial}^{n,n-1}(U^*).$$

Now, since $d = \bar\partial$ on forms of type (n,q), there is a natural mapping

$$H_{\bar\partial}^{n,n-1}(U^*) \to H_{\mathrm{DR}}^{2n-1}(U^*).$$

The punctured ball U^* is homotopically just the $2n-1$ sphere, and so the right side is $\mathbb{C}$ with the isomorphism given by

$$\eta \to \int_{S^{2n-1}} \eta,$$

the orientation on the sphere being induced from that in $\mathbb{C}^n$. We shall prove that

Lemma. $\mathrm{Res}_{\{0\}}\,\omega = \eta_\omega$, *or equivalently*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \omega = \int_{S^{2n-1}} \eta_\omega.$$

Proof. Recall that the $\bar\partial$-Poincaré lemma gives exact sheaf sequences

$$0 \to \mathcal{Z}_{\bar\partial}^{n,n-p-1} \to \mathcal{Q}^{n,n-p-1} \xrightarrow{\bar\partial} \mathcal{Z}_{\bar\partial}^{n,n-p} \to 0,$$

where $\mathcal{Q}^{p,q}$ is the sheaf of $C^\infty(p,q)$ forms and $\mathcal{Z}_{\bar\partial}^{p,q} \subset \mathcal{Q}^{p,q}$ is the subsheaf of $\bar\partial$-closed forms. Since the sheaves $\mathcal{Q}^{p,q}$ have no higher cohomology, the Dolbeault isomorphism is a composition of isomorphisms

$$i_p:\ H^p\left(U^*, \mathcal{Z}_{\bar\partial}^{n,n-p-1}\right) \xrightarrow{\sim} H^{p-1}\left(U^*, \mathcal{Z}_{\bar\partial}^{n,n-p}\right)$$

obtained from coboundary maps in the exact cohomology sequence of the above sheaf sequence. For $p = 1$, the right-hand side is to be replaced by

$$H^0\left(U^*, \mathcal{Z}_{\bar\partial}^{n,n-1}\right)/\bar\partial H^0(U^*, \mathcal{Q}^{n,n-1}) = H_{\bar\partial}^{n,n-1}(U^*).$$

Now we may follow our cocycle ω through this sequence of isomorphisms. Beginning with

$$\omega_{n-1} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \omega \in H^{n-1}(U^*, \mathcal{Z}_{\bar\partial}^{n,0}),$$

we let $U_I = \cup_{i\in I} U_i$ and

$$\omega_p = \left\{\omega_{p,I} \in \mathcal{Z}_{\bar\partial}^{n,n-p-1}(U_I)\right\}_{\#I=p}$$

denote a representative of

$$i_{p+1} \circ \cdots \circ i_{n-1}(\omega_{n-1});$$

and then let

$$\xi_p = \left\{\xi_{p,I} \in \mathcal{Q}^{n,n-p-1}(U_I)\right\}_{\#I=p}$$

be cochains such that

$$\delta\xi_p = \omega_p, \qquad \bar\partial\xi_p = \omega_{p-1}.$$

Next, let Γ_I be the chain defined by

$$\Gamma_I = \left\{z : |f_i(z)| = \varepsilon \text{ for } i \in I, |f_j(z)| \leqslant \varepsilon \text{ for } j \notin I\right\}$$

and with orientation

$$d(\arg f_{i_1}) \wedge \cdots \wedge d(\arg f_{i_p}) \wedge \left(\bigwedge_{j\notin I} \frac{\sqrt{-1}}{2} df_j \wedge \overline{df_j}\right) \geqslant 0,$$

where $I = \{i_1 < \cdots < i_p\}$. Then the boundary

$$\partial\Gamma_I = \sum_{j\notin I} (-1)^{(j,I-\{j\})}\Gamma_{I\cup\{j\}},$$

where $(j, I-\{j\})$ is the position from the rear of the index j when $I \cup \{j\}$ is ordered in the usual manner. Now, since $d\xi_I = \bar\partial\xi_I$ we may apply Stokes' theorem to obtain

$$\begin{aligned}
\sum_{\#I=p} \int_{\Gamma_I} \omega_{p-1,I} &= \sum_{\#I=p} \int_{\Gamma_I} d\xi_{p,I} \\
&= \sum_{\#I=p} \int_{\partial\Gamma_I} \xi_{p,I} \\
&= \sum_{\#I=p} \left(\sum_{j\notin I} \int_{\Gamma_{I\cup\{j\}}} (-1)^{(j,I\cup\{j\})}\xi_{p,I}\right) \\
&= \sum_{\#J=p+1} \int_{\Gamma_J} (-1)^{(j,J)}\xi_{p,J-\{j\}} \\
&= \sum_{\#J=p+1} \int_{\Gamma_J} (\delta\xi_p)_J \\
&= \sum_{\#J=p+1} \int_{\Gamma_J} \omega_{p,J}
\end{aligned}$$

by the combinatorial definition of δ. Consequently, *the total sum*

$$\sum_{\#I=p+1}\int_{\Gamma_I}\omega_{p,I}$$

is the same for all p. At the two extremes $p=n-1$ and $p=0$ we obtain

$$\begin{aligned}\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_\Gamma\omega &= \sum_{i\in I}\int_{\Gamma_i}\omega_{0,i}\\ &= \sum_{i\in I}\int_{\Gamma_i}\eta_\omega, \qquad \text{since } \eta_\omega=\omega_{0,i} \text{ in } U_i,\\ &= \int_{\partial\Gamma_0}\eta_\omega, \qquad \text{where } \Gamma_0=\{z:\ |f_1(z)|\leqslant\varepsilon,\ldots,|f_n(z)|\leqslant\varepsilon\},\\ &= \int_{S^{2n-1}}\eta_\omega. \qquad \text{Q.E.D.}\end{aligned}$$

We observe that this lemma does not use the assumption that ω is meromorphic in U with polar divisor D. Only $\omega\in H^0(U-D,\Omega^n)$ is required, so that ω could have a higher order pole or even an essential singularity along D. In case ω is meromorphic with polar divisor D, we may find a distinguished representative for the Dolbeault class η_ω as follows. Set

$$\rho_i=\frac{|f_i|^2}{|f_1|^2+\cdots+|f_n|^2}$$

and observe that

$$\rho_i \text{ is } C^\infty \text{ in } U^*,$$

$$\sum_i\rho_i\equiv 1,$$

and

$$\operatorname{supp}(\rho_i)\subset U_i.$$

Thus $\{\rho_i\}$ looks something like a partition of unity for the covering $\{U_i\}$ of U^* and may be used as such for any ω having first-order poles on the D_i. Indeed, given

$$\omega=\frac{g(z)dz_1\wedge\cdots\wedge dz_n}{f_1(z)\cdots f_n(z)}, \qquad g(z)\in\mathcal{O}(U),$$

we see that

$$\rho_i\omega=\frac{\bar{f}_ig}{\|f\|^2}\cdot\frac{dz_1\wedge\cdots\wedge dz_n}{f_1\cdots\widehat{f_i}\cdots f_n}\in A^{n,0}(U_{\{i\}^0}),$$

so we can set

$$\xi_{\{i\}^0}=\pm\rho_i\omega, \qquad \omega_{\{i\}^0}=\pm\bar{\partial}\rho_i\omega.$$

Proceeding to the next step, for $j \neq i$,

$$\rho_j \omega_{\{i\}^0} \in A^{n,1}(U_{\{i,j\}^0}),$$

so we can set

$$\xi_{\{i,j\}^0} = \pm(\rho_i \omega_{\{j\}^0} - \rho_j \omega_{\{i\}^0}),$$
$$\omega_{\{i,j\}^0} = \pm 2 \bar\partial \rho_i \wedge \bar\partial \rho_j \wedge \omega.$$

Continuing, we finally arrive at

$$\eta_\omega = \omega_{\{i\}} = n!(-1)^{i-1} \bar\partial\rho_1 \wedge \cdots \wedge \widehat{\bar\partial \rho_i} \wedge \cdots \wedge \bar\partial \rho_n \wedge \omega$$

$$= \frac{n!(-1)^{i-1} g \bar\partial\rho_1 \wedge \cdots \wedge \widehat{\bar\partial \rho_i} \wedge \cdots \wedge \bar\partial \rho_n \wedge dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}.$$

But, setting $f = (f_1, \ldots, f_n)$ and $\|f\|^2 = \Sigma_i |f_i|^2$,

$$\bar\partial \rho_i = \frac{f_i d\bar f_i}{\|f\|^2} - \frac{|f_i|^2 \sum f_j d\bar f_j}{\|f\|^4},$$

and so the wedge product

$$\bigwedge_{j \neq i} \bar\partial \rho_j = \frac{\left(\bigwedge_{j\neq i} f_j d\bar f_j\right)}{\|f\|^{2n-2}} - \frac{\sum_{k \neq i} (-1)^{(k,\{i\}^0)}\left(\bigwedge_{j \neq i,k} f_j d\bar f_j\right) \sum f_l d\bar f_l |f_k|^2}{\|f\|^{2n}}$$
$$= \frac{1}{\|f\|^{2n}}\left(\|f\|^2 \bigwedge_{j\neq i} f_j d\bar f_j - \sum_{k \neq i} |f_k|^2 \left(\bigwedge_{j \neq i} f_j d\bar f_j\right)\right.$$
$$\left. - |f_k|^2 \sum_{k \neq i} (-1)^{k-i-1} \bigwedge_{j \neq k} f_j d\bar f_j\right)$$
$$= \frac{1}{\|f\|^{2n}}\left(|f_i|^2 \bigwedge_{j \neq i} f_j d\bar f_j + \sum_{k \neq i} |f_k|^2 (-1)^{k-i} \bigwedge_{j \neq k} f_j d\bar f_j\right)$$
$$= \frac{f_1 \cdots f_n (-1)^i \left(\sum (-1)^k \bar f_k \bigwedge_{j \neq k} \overline{df_j}\right)}{\|f\|^{2n}}.$$

Putting this all together, what we might call the *distinguished Dolbeault representative* of $(1/2\pi\sqrt{-1})^n \omega$ is

$$\eta_\omega = g(z)\left[\frac{C_n \sum (-1)^{i-1} \bar f_i\, \overline{df_1} \wedge \cdots \wedge \widehat{\overline{df_i}} \wedge \cdots \wedge \overline{df_n} \wedge dz_1 \wedge \cdots \wedge dz_n}{\|f\|^{2n}}\right],$$

where C_n is a constant depending only on n.

At this juncture, recall from Section 1 of Chapter 3 the *Bochner-Martinelli kernel*

$$k(z,\xi) = C_n \frac{\sum(-1)^{i-1}(\overline{z_i-\xi_i}) \bigwedge_{j\neq i}(\overline{dz}_j-\overline{d\xi}_j)\wedge d\xi_1\wedge\cdots\wedge d\xi_n}{\|z-\xi\|^{2n}}$$

on $\mathbb{C}^n\times\mathbb{C}^n$. If $F: U\to\mathbb{C}^n\times\mathbb{C}^n$ is defined by

$$F(z) = (z+f(z),z),$$

then

$$\eta_\omega = gF^*k.$$

Taking $f_i(z)=z_i$ and applying our lemma, we obtain another proof of the Bochner-Martinelli formula,

$$\int_{\|z\|=\varepsilon} g(z)\beta(z,\bar{z}) = g(0).$$

Recall also from Section 2 of Chapter 3 on the holomorphic Lefschetz fixed-point formula that we proved that if the origin is an isolated nondegenerate fixed point of a map $f: U\to\mathbb{C}^n$, then

$$\int_{\|z\|=\varepsilon} F^*k = \mathcal{I}_f(0).$$

We now know that for any type of isolated fixed point,

$$\int_{\|z\|=\varepsilon} F^*k = \operatorname{Res}_{\{0\}}\left(\frac{dz_1\wedge\cdots\wedge dz_n}{f_1(z)\cdots f_n(z)}\right).$$

This leads to a corresponding extension of the holomorphic Lefschetz theorem.

The Global Residue Theorem

Suppose that $M\subset M'$ are complex n-manifolds, where M is relatively compact with smooth boundary $\partial M=\overline{M}-M$. The case that $M=M'$ is itself compact will be in many respects the most interesting situation. Suppose that $D_1,\ldots,D_n$ are effective divisors defined in some neighborhood U of $\overline{M}$ in M' and whose intersection $D_1\cap\cdots\cap D_n$ is a discrete—hence finite—set of points in M. By analogy with the previous notation, we set

$$\begin{aligned} D &= D_1+\cdots+D_n,\\ U^* &= U-(D_1\cap\cdots\cap D_n),\\ U_i &= U-D_i, \end{aligned}$$

so that $\underline{U}=\{U_i\}$ is an open covering of U^*. Suppose that

$$\omega\in H^0(U,\Omega^n(D))$$

is a meromorphic n-form on U with polar divisor D. For each point $P \in U_1 \cap \cdots \cap U_n$ we may restrict ω to a neighborhood U_P of P and define the residue

$$\operatorname{Res}_P \omega$$

as in the previous section. On the other hand,

$$\omega \in C^{n-1}(\underline{U}, \Omega^n)$$

defines a class $[\omega] \in H^{n-1}(U^*, \Omega^n)$ that has a Dolbeault representative

$$\eta_\omega \in H_{\bar\partial}^{n,n-1}(U^*) \cong H^{n-1}(U^*, \Omega^n),$$

and we have the

Residue Theorem

$$\sum_P \operatorname{Res}_P \omega = \int_{\partial M} \eta_\omega.$$

In particular, if M *is compact, then*

$$\sum_P \operatorname{Res}_P \omega = 0.$$

Proof. As in the Riemann surface case, we let $U_P(\varepsilon)$ be an ε-ball around P and use $d\eta_\omega = 0$ and Stokes' theorem to write

$$\begin{aligned} \int_{\partial M} \eta_\omega &= \sum_P \int_{\partial U_P(\varepsilon)} \eta_\omega \\ &= \sum_P \operatorname{Res}_P \omega \qquad \text{by the above lemma,} \end{aligned}$$

since $\eta_\omega | U_P^*$ is a Dolbeault representative of $[\omega|_{U_P}] \in H^{n-1}(U_P^*, \Omega^n)$. Q.E.D.

Of course, the essential step here is to convert the original n-dimensional path of integration into one of dimension $2n-1$ so that Stokes' theorem may be utilized.

The Transformation Law and Local Duality

We now explore what might be called the functorial aspects of the residue symbol. To begin with we shall use the residue theorem to derive one of our main techniques, the method of continuity. Suppose that $f_t = (f_{t,1}, \dots, f_{t,n})$ are n-functions of (z,t), holomorphic for z in a neighborhood of $\overline{U}$ where U is a small ball around the origin in $\mathbb{C}^n$, and continuous in a parameter variable $0 \leq t \leq \delta$. We set $f = f_0$, and for a form

$$\omega = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \qquad (g(z) \in \mathcal{O}(\overline{U}))$$

we let

$$\omega_t = \frac{g(z)\,dz_1\wedge\cdots\wedge dz_n}{f_{t,1}(z)\cdots f_{t,n}(z)}$$

If we assume that $f^{-1}(0)$ is a finite set of points interior to U, then $\|f(z)\| \geqslant \varepsilon > 0$ on the boundary $\partial U = \bar{U} - U$, and so $\|f_t(z)\| \geqslant \varepsilon/2 > 0$ for t sufficiently close to 0. Consequently $f_t^{-1}(0)$ will again be a finite set of points interior to U. On the other hand, by the explicit formula

$$\eta_{\omega_t} = C_n g(z) \frac{\sum (-1)^{i-1} \bar{f}_{i,t}\, d\bar{f}_{t,1}\wedge\cdots\wedge \widehat{d\bar{f}_{t,i}}\wedge\cdots\wedge d\bar{f}_{t,n}\wedge dz_1\wedge\cdots\wedge dz_n}{\|f_t(z)\|^2}$$

for the Dolbeault representative of $[\omega_t] \in H^{n-1}(U^*, \Omega^n)$, we see that the boundary integral

$$\int_{\partial U} \eta_{\omega_t}$$

is continuous in t. Going to the residue theorem, we find the *principle of continuity*:

$$(*) \qquad \lim_{t\to 0} \sum_{P_t \in f_t^{-1}(0)} \operatorname{Res}_{P_t} \omega_t = \sum_{P \in f^{-1}(0)} \operatorname{Res}_P \omega.$$

To apply this, we need to discuss perturbations of a given map $f: U \to \mathbb{C}^n$ having $f^{-1}(0) = \{0\}$. A family of maps $f_t: U \to \mathbb{C}^n$ defined and holomorphic in a neighborhood of $\bar{U}$, varying continuously with t and such that $f_0 = f$, is said to be a *good perturbation* of f in case f_t has only nondegenerate zeros for $t \neq 0$. We will be able to easily see the existence of good perturbations when we discuss finite holomorphic mappings below. For the moment they may be deduced from Sard's theorem as follows: Since the critical values of $f: U \to \mathbb{C}^n$ have measure zero in $\mathbb{C}^n$, we can find an arc $\gamma(t)$, $0 \leqslant t \leqslant \varepsilon$, with $\gamma(0) = \{0\}$ and $\gamma(t)$ not a critical value for $t \neq 0$. Then

$$f_t(z) = f(z) - \gamma(t)$$

is a good perturbation of f.

Now we use the existence of good perturbations and continuity method to prove the

Transformation Law. *Suppose* $f = (f_1, \ldots, f_n)$ *and* $g = (g_1, \ldots, g_n)$ *give holomorphic maps* $f, g: \bar{U} \to \mathbb{C}^n$ *with* $f^{-1}(0) = \{0\} = g^{-1}(0)$. *Suppose moreover that*

$$g_i(z) = \sum_j a_{ij}(z) f_j(z)$$

for some holomorphic matrix $A(z) = (a_{ij}(z))$. *Equivalently, the ideals should satisfy*

$$\{g_1, \ldots, g_n\} \subset \{f_1, \ldots, f_n\}.$$

Then, for h(z) $\in \mathcal{O}(\overline{U})$

$$\operatorname{Res}_{\{0\}}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right)=\operatorname{Res}_{\{0\}}\left(\frac{h\det A\,dz_1\wedge\cdots\wedge dz_n}{g_1\cdots g_n}\right).$$

Proof. We prove this in cases of increasing difficulty.

Case 1: $\mathcal{J}_f(0)\neq 0$ and $\det A(0)\neq 0$

Then $\mathcal{J}_g(0)=\mathcal{J}_f(0)\det A(0)$, and by the evaluation of the residue integral in the nondegenerate case

$$\begin{aligned}\operatorname{Res}_{\{0\}}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right)&=(2\pi\sqrt{-1}\,)^n\frac{h(0)}{\mathcal{J}_f(0)}\\&=(2\pi\sqrt{-1}\,)^n\frac{h(0)\det A(0)}{\mathcal{J}_g(0)}\\&=\operatorname{Res}_{\{0\}}\left(\frac{h\det A\,dz_1\wedge\cdots\wedge dz_n}{g_1\cdots g_n}\right).\end{aligned}$$

Case 2: $\det A(0)\neq 0$ but f possibly degenerate

Since the result is local around the origin, we may shrink U and assume that $\det A(z)\neq 0$ in $\overline{U}$. If f_t is a good perturbation of $f=f_0$, then $g_t=A\cdot f_t$ is a good perturbation of $g=g_0$, and by continuity and case 1

$$\begin{aligned}\operatorname{Res}_{\{0\}}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right)&=\lim_{t\to 0}\sum_{P_t\in f_t^{-1}(0)\cap U}\operatorname{Res}_{P_t}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_{t,1}\cdots f_{t,n}}\right)\\&=\lim_{t\to 0}\sum_{P_t\in g_t^{-1}(0)\cap U}\operatorname{Res}_{P_t}\left(\frac{h\det A\,dz_1\wedge\cdots\wedge dz_n}{g_{t,1}\cdots g_{t,n}}\right)\\&=\operatorname{Res}_{\{0\}}\left(\frac{h\det A\,dz_1\wedge\cdots\wedge dz_n}{g_1\cdots g_n}\right).\end{aligned}$$

Case 3: f, g, and A arbitrary

Now we let $A_t(z)$ be a continuous family of holomorphic matrices with $A_0(z)=A(z)$ and $\det A_t(0)\neq 0$ for $t\neq 0$. Set $g_t=A_t\cdot f$, and observe that since $g^{-1}(0)=\{0\}$, $g_t^{-1}(0)=\{P_t\}$ is an isolated set of points interior to U. For $P_t\neq 0, f(P_t)\neq 0$. Suppose $f_1(P_t)\neq 0$ and denote by $A_{i,j}$ the i,jth minor of A. Then by Laplace's expansion of the determinant, for z near P_t

$$\begin{aligned}\det A_t(z)&=\sum_j(-1)^jA_{tj,1}(z)a_{tj,1}(z)\\&=\frac{1}{f_1(z)}\left(\sum_{i,j}(-1)^jA_{tj,1}a_{tj,i}f_i(z)\right),\quad\text{since }\sum_j A_{tj,1}a_{tj,i}=0\text{ for }i\neq 1,\\&=\frac{1}{f_1(z)}\left(\sum_j(-1)^jA_{tj,1}g_{t,j}(z)\right).\end{aligned}$$

Thus $\det A_t(z)$ is in the ideal $\{g_{t,1},\dots,g_{t,n}\}_{P_t}$, and by the second elementary property of the residue integral

$$\operatorname{Res}_{P_t}\left(\frac{h\det A_t\,dz_1\wedge\cdots\wedge dz_n}{g_{t,1}\cdots g_{t,n}}\right)=0 \qquad \text{for } P_t\neq 0.$$

Now then we use this together with case 2 to have:

$$\begin{aligned}\operatorname{Res}_{\{0\}}\left(\frac{h\det A\,dz_1\wedge\cdots\wedge dz_n}{g_1\cdots g_n}\right)&=\lim_{t\to 0}\left(\sum\operatorname{Res}_{P_t}\left(\frac{h\det A_t\,dz_1\wedge\cdots\wedge dz_n}{g_{t,1}\cdots g_{t,n}}\right)\right)\\&=\lim_{t\to 0}\left(\operatorname{Res}_{\{0\}}\left(\frac{h\det A_t\,dz_1\wedge\cdots\wedge dz_n}{g_{t,1}\cdots g_{t,n}}\right)\right)\\&=\lim_{t\to 0}\operatorname{Res}_{\{0\}}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right)\\&=\operatorname{Res}_{\{0\}}\left(\frac{h\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right).\end{aligned}$$

Q.E.D.

Local Duality. We now come to the local duality theorem. Given U a sufficiently small neighborhood of the origin and $f: U\to\mathbb{C}^n$ with $f^{-1}(0)=\{0\}$, or equivalently given an ideal $I=I(f)=\{f_1,\dots,f_n\}$ in the local ring $\mathcal{O}=\mathcal{O}_{\{0\}}$ at the origin and having $\{0\}$ as isolated common zero of the f_i's, we may use the property

$$\operatorname{Res}_{\{0\}}\left(\frac{g\,dz_1\wedge\cdots\wedge dz_n}{f_1\cdots f_n}\right)=0 \qquad \text{for } g\in I,$$

to define a symmetric pairing

$$\operatorname{res}_f:\ \mathcal{O}/I\otimes\mathcal{O}/I\to\mathbb{C}$$

by setting

$$\operatorname{res}_f(g,h)=\operatorname{Res}_{\{0\}}\left(\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\frac{g(z)h(z)\,dz_1\wedge\cdots\wedge dz_n}{f_1(z)\cdots f_n(z)}\right).$$

The basic result is the

Local Duality Theorem I. *The pairing "res_f" is nondegenerate; i.e., if*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n\int_{|f_i(z)|=\epsilon}\frac{g(z)h(z)\,dz_1\wedge\cdots\wedge dz_n}{f_1(z)\cdots f_n(z)}=0$$

for all $h(z)\in\mathcal{O}$, *then* $g(z)$ *lies in the ideal* $\{f_1,\dots,f_n\}$.

Proof. The proof is based on the transformation law and two further general results in local analytic geometry. The first of these is that $f_1,\dots,f_n\in\mathcal{O}$ form a *regular sequence*, which by definition means that

f_i *is not a zero divisor in* $\mathcal{O}/\{f_1,\dots,f_{i-1}\}$ $(1\leqslant i\leqslant n)$.

Intuitively, this amounts to saying that the analytic varieties $V_i = \{f_1(z) = \cdots = f_i(z) = 0\}$ have codimension exactly equal to i, which seems quite reasonable and may be rigorously proved in the case $n=2$ as follows: We need to show that if $gf_2 = 0$ in $\mathcal{O}/\{f_1\}$, then $g = hf_1$ is a multiple of f_1. The problem is unchanged if we multiply the f_i's or g by units, and so we may choose coordinates (z,w) such that $f_1, f_2, g \in \mathcal{O}_z[w]$ are all Weierstrass polynomials. By the division theorem,

$$g = hf_1 + r,$$

where $r \in \mathcal{O}_z[w]$ is a polynomial of degree less than $d = \deg f_1$. For $|z| < \varepsilon$, we denote by $w_1(z), \ldots, w_d(z)$ the roots of

$$f_1(z,w) = 0,$$

where some roots may be repeated. Then, since the equations $f_1(z,w) = f_2(z,w) = 0$ have only $(0,0)$ as common solution, for z^* close to zero all $f_2(z^*, w_\nu(z^*)) \neq 0$. But then, since by assumption $g(z, w_\nu(z)) f_2(z, w_\nu(z)) = 0$, the equation $r(z^*, w) = 0$ will have $d > \deg r$ roots. Hence $r \equiv 0$ and $g = hf_1$.

Q.E.D.

The general statement is:

For U *a sufficiently small neighborhood of the origin and* $\mathrm{f} = (\mathrm{f}_1, \ldots, \mathrm{f}_n) : \mathrm{U} \to \mathbb{C}^n$ *a holomorphic mapping, the conditions*

1. $\mathrm{f}^{-1}(0) = \{0\}$,
2. $\operatorname{codim}\{\mathrm{f}_{i_1}(z) = \cdots = \mathrm{f}_{i_k}(z) = 0\} = \mathrm{k}$,
3. $\mathrm{f}_1, \ldots, \mathrm{f}_n$ *is a regular sequence,*

are all equivalent.

Since we shall be discussing regular sequences in detail in the section on Koszul complexes and shall give another proof of local duality there, we shall let our discussion in the case $n=2$ suffice for the moment.

The second result in local analytic geometry is the *nullstellensatz* for the ideal $\{f_1, \ldots, f_n\}$:

There exists $\mathrm{k}_i > 0$ *such that*

$$z_i^{k_i} \in \{f_1, \ldots, f_n\}.$$

We will prove this in the next section on finite holomorphic mappings.

Now to the proof of local duality. The idea is to directly verify the statement for ideals $\{z_1^{k_1}, \ldots, z_n^{k_n}\}$, and then to use the transformation law to deduce the result for the ideal $\{f_1, \ldots, f_n\} \supseteq \{z_1^{k_1}, \ldots, z_n^{k_n}\}$.

Step One. In case $f_i(z) = z_i^{k_i+1}$, we take $h(z) = z_1^{l_1} \cdots z_n^{l_n}$ and write the power series

$$g(z) = \sum g_{i_1\cdots i_n} z_1^{i_1}\cdots z_n^{i_n}.$$

Then, by iterating the usual Cauchy integral formula,

$$\begin{aligned}\operatorname{res}_f(g,h) &= \sum_{i_1\cdots i_n} g_{i_1\cdots i_n}\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|z_i|=\varepsilon} \frac{dz_1\wedge\cdots\wedge dz_n}{z_1^{k_1+1-l_1-i_1}\cdots z_n^{k_n+1-l_n-i_n}}\\ &= g_{k_1-l_1,\ldots,k_n-l_n}.\end{aligned}$$

Thus, $\operatorname{res}_f(g,h)=0$ for all h is equivalent to $g_{i_1\cdots i_n}=0$ for $i_1\leqslant k_1,\ldots,i_n\leqslant k_n$, in which case $g\in\{z_1^{k_1+1},\ldots,z_n^{k_n+1}\}$. This proves the local duality theorem in this case.

Step Two. We will use the transformation law to prove the

Lemma. *Let* $f_1', f_1,\ldots,f_n\in\mathcal{O}$ *and set* $f'=(f_1',f_2,\ldots,f_n)$, $f=(f_1,\ldots,f_n)$. *Assume that* $f^{-1}(0)=\{0\}=f'^{-1}\{0\}$ *and that* $f_1'\in\{f_1,f_2,\ldots,f_n\}$; *i.e.,* $I(f')\subseteq I(f)$. *Then, if the residue pairing is nondegenerate for* f', *it is nondegenerate for* f.

Proof. Let $\pi:\mathcal{O}/I(f')\to\mathcal{O}/I(f)$ be the natural projection, and write

$$f_1' = \sum_i b_i f_i,$$

so that $f'=Af$, where

$$A=\begin{bmatrix} b_1 & b_2 & \cdots & b_n\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1\end{bmatrix}.$$

If $g=\sum_i c_i f_i$ is in the ideal $I(f)$, then

$$b_1 g = c_1\left(\sum_i b_i f_i\right)+\sum_{i\geqslant 2}(b_1c_i-c_1b_i)f_i$$

is in the ideal $I(f')$. Thus multiplication by b_1 gives a map

$$\alpha:\ \mathcal{O}/I(f)\to\mathcal{O}/I(f')$$

going in the opposite direction to π. Since $\det A=b_1$, the transformation law states exactly that the diagram

$$\begin{array}{ccc} \mathcal{O}/I(f)\otimes\mathcal{O}/I(f) & \xrightarrow{\operatorname{res}_f} & \mathbb{C}\\ \alpha\downarrow \quad\quad \uparrow\pi & & \|\\ \mathcal{O}/I(f')\otimes\mathcal{O}/I(f') & \xrightarrow{\operatorname{res}_{f'}} & \mathbb{C}\end{array}$$

is commutative; i.e.,

$$\operatorname{res}_f(g,h) = \operatorname{res}_{f'}(b_1 g,h)$$

for all $g,h\in\mathcal{O}$. If $\operatorname{res}_f(g,h)=0$ for all h, then by the assumed nondegeneracy of $\operatorname{res}_{f'}$ it follows that $b_1 g\in I(f')$. Thus

$$\begin{aligned} b_1 g &= c_1 f_1' + \sum_{i\geq 2} c_i f_i \\ &= c_1 b_1 f_1 + \sum_{i\geq 2} (b_1 c_i - b_i c_1) f_i, \end{aligned}$$

so that

$$b_1 g = c_1 b_1 f_1 \qquad \text{in } \mathcal{O}/\{f_2,\dots,f_n\}.$$

This implies that either

$$g = c_1 f_1 \qquad \text{in } \mathcal{O}/\{f_2,\dots,f_n\}$$

or

$$b_1 \text{ is a zero-divisor} \qquad \text{in } \mathcal{O}/\{f_2,\dots,f_n\}.$$

In the first case, $g\in\{f_1,f_2,\dots,f_n\}$ as desired. In the second case, $b_1 f_1$ is a zero-divisor in $\mathcal{O}/\{f_2,\dots,f_n\}$, and hence so is $f_1'=b_1 f_1+(b_2 f_2+\cdots+b_n f_n)$. But this contradicts the regular sequence property of $\{f_1',f_2,\dots,f_n\}$. Q.E.D.

Step Three. The theorem now follows easily. Given $f=(f_1,\dots,f_n)$, we inductively choose a coordinate system so that

$$F_i = (z_1,\dots,z_i,f_{i+1},\dots,f_n)$$

has an isolated zero at the origin. Appealing to the nullstellensatz, we may take k_i sufficiently large so that $z_i^{k_i}\in I(F_{i-1})$. Then res_{F_n} is nondegenerate by step one, and by the lemma

$$\begin{aligned} \operatorname{res}_{F_n} \text{ nondegenerate} &\Rightarrow \operatorname{res}_{F_{n-1}} \text{ nondegenerate} \\ &\ \ \vdots \\ &\Rightarrow \operatorname{res}_{F_1} \text{ nondegenerate} \\ &\Rightarrow \operatorname{res}_{F_0} = \operatorname{res}_f \text{ nondegenerate.} \end{aligned}$$

Q.E.D.

2. APPLICATIONS OF RESIDUES

Intersection Numbers

Recall that our discussion of the local structure of those analytic varieties defined by a single function—i.e., analytic hypersurfaces—was based on the one-variable Cauchy formula and subsequent residue theorem. It is similarly possible to use the n-variable residue theorem to derive the local properties of analytic varieties of codimension n defined by n holomorphic functions in $U \subset \mathbb{C}^N$. We shall now carry this out in case the variety is zero-dimensional—i.e., $N = n$. By allowing dependence on parameters, it is possible to adapt the method to the more general situation just mentioned.

We begin by discussing intersection numbers, thereby complementing our previous definitions, which were either topological or used the theory of currents—cf. Section 4 of Chapter 0 and Section 2 of Chapter 3.

Consider an ideal $I(f) = \{f_1, \ldots, f_n\}$ of holomorphic functions $f_i \in \mathcal{O}(\overline{U})$ whose divisors D_i have the origin as set-theoretic intersection—i.e., $f^{-1}(0) = \{0\}$, where $f = (f_1, \ldots, f_n)$. As usual, we allow ourselves to shrink U when necessary. Doing this, we may assume that $f^{-1}(w)$ is a discrete set of points in U for $\|w\| < \varepsilon$, since we will have $|f(z)| \geqslant C > 0$ for $z \in \partial U$.

We write $w = f(z)$, denote by $K = dw_1/w_1 \wedge \cdots \wedge dw_n/w_n$ the *Cauchy kernel*, and set

$$\omega(f_1, \ldots, f_n) = f^*K = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}.$$

The *local intersection number* is defined by

$$(D_1, \ldots, D_n)_{\{0\}} = \mathrm{Res}_{\{0\}} \omega(f_1, \ldots, f_n).$$

We shall give a list of its properties:

(a) $(D_1, \ldots, D_n)_{\{0\}}$ *is an integer that depends only on the ideal* $I(f)$ *and not the choice of generators* f_i. *In particular, it depends only on the divisors* D_i *and not on their defining functions.*

Proof. $(1/2\pi\sqrt{-1}\,)^n \omega(f_1, \ldots, f_n)$ represents an integral cohomology class in $H^n_{\mathrm{DR}}(U - D)$, and so the intersection number is an integer. If

$$f_i' = \sum_j a_{ij} f_j$$

where $\Delta = \det(a_{ij}) \neq 0$, then

$$\frac{df_1' \wedge \cdots \wedge df_n'}{f_1' \cdots f_n'} = \Delta \frac{df_1 \wedge \cdots \wedge df_n}{f_1' \cdots f_n'} + g \frac{dz_1 \wedge \cdots \wedge dz_n}{f_1' \cdots f_n'},$$

where g is in the ideal. By the transformation law

$$\mathrm{Res}_{\{0\}}\left(\frac{\Delta\, df_1 \wedge \cdots \wedge df_n}{f_1' \cdots f_n'} \right) = \mathrm{Res}_{\{0\}}\left(\frac{df_1 \wedge \cdots \wedge df_n}{f_1 \cdots f_n} \right),$$

while

$$\mathrm{Res}_{\{0\}}\left(g \frac{dz_1 \wedge \cdots \wedge dz_n}{f_1' \cdots f_n'} \right) = 0,$$

since g is in the ideal.

(b) *The intersection number is linear in each divisor* D_i.

Proof. If $D_1 = D_1' + D_1''$ corresponds to the factorization $f_1 = f_1' f_1''$, then clearly

$$(*) \qquad \omega(f_1, f_2, \ldots, f_n) = \omega(f_1', f_2, \ldots, f_n) + \omega(f_1'', f_2, \ldots, f_n).$$

This is not yet enough to prove linearity, owing to the complicated nature of the path of integration $\Gamma = \{|f_i| = \varepsilon_i\}$ in the definition of the residue. What is suggested is that we use the Dolbeault isomorphism to convert Γ into the sphere $\|z\| = \varepsilon$.

Thus, we consider $\omega(f_1', f_2, \ldots, f_n)$ as defining a class in $H^{n-1}(\underline{U}', \Omega^n)$, where $\underline{U}' = \{U_1', U_2, \ldots, U_n\}$ is the corresponding covering of $U^* = \overline{U} - \{0\}$. Since $\overline{U}_1 \subset U_1'$, there is a restriction mapping ρ' leading to a commutative diagram

$$\begin{array}{ccc} H^{n-1}(\underline{U}', \Omega^n) & \xrightarrow{\rho'} & H^{n-1}(\underline{U}, \Omega^n) \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\eta'} & \\ & H_{\bar\partial}^{n,n-1}(U^*) & \end{array}$$

where η and η' are Dolbeault maps. Setting $\eta(f_1, \ldots, f_n) = \eta(\omega(f_1, \ldots, f_n))$ and so forth, it follows from $(*)$ that

$$\eta(f_1, f_2, \ldots, f_n) = \eta(f_1', f_2, \ldots, f_n) + \eta(f_1'', f_2, \ldots, f_n)$$

in $H_{\bar\partial}^{n,n-1}(U^*)$. (It is *not* the case that $\eta = \eta' + \eta''$ as differential forms, since the Bochner-Martinelli kernel is nonlinear. What the commutativity of the diagram proves is that $\eta = \eta' + \eta'' + \bar\partial\xi$.) By the lemma on p. 651 above,

$$(D_1, \ldots, D_n)_{\{0\}} = \int_{\|z\| = \varepsilon} \eta(f_1, f_2, \ldots, f_n),$$

from which the linearity of the intersection number follows.

(c) Suppose now that the divisors $D_i = (f_i)$ meet at a finite number of points P_ν interior to U. The total number of intersections of the D_i in U is defined by

$$(D_1, \ldots, D_n)_U = \sum_\nu (D_1, \ldots, D_n)_{P_\nu}.$$

We shall prove:

The total intersection number is invariant under continuous deformation of the D_i.

Proof. We assume that $f_{t,i}(z) \in \mathcal{O}(\overline{U})$ is continuous in t and has divisor $D_i(t)$ with $f_{0,i} = f_i$ and $D_i(0) = D_i$. Since $\Sigma_i |f_i(z,t)|^2 \geq C > 0$ for $z \in \partial U$ and $|t| < \varepsilon$, the divisors $D_i(t)$ will meet at isolated points interior to U. The total intersection number

$$(D_1(t), \ldots, D_n(t))_U$$

is on the one hand an integer and on the other hand, by the continuity principle, continuous in t. Consequently, it is constant. Q.E.D.

Given divisors D_i having the origin as isolated point of intersection, we may perturb them slightly to smooth divisors D_i' having a finite number of transverse intersections near the origin (Figure 1). Each of these transverse intersections has local intersection number $+1$, and $(D_1,\dots,D_n)_{\{0\}}$ is the total number of such intersections.

(d) We now assume that the D_i meet at the origin and that D_1 is nonsingular. Set $D_i' = D_1 \cap D_i$ for $i \geq 2$. Then we claim that

$$(D_1,\dots,D_n)_{\{0\}} = (D_2',\dots,D_n')_{\{0\}}.$$

Proof. We may choose coordinates so that $f_1(z) = z_1$. Set $z = (z_1, z')$ and $f_i'(z') = f_i(0,z') = f_i|_{D_1}$. Then if $\Gamma = \{|f_1(z)| = \cdots = |f_n(z)| = \varepsilon\}$ and $\Gamma' = \{|f_2'(z')| = \cdots = |f_n'(z')| = \varepsilon\}$, we may iterate the Cauchy integral formula to obtain

$$\begin{aligned}(D_1,\dots,D_n)_{\{0\}} &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \frac{dz_1}{z_1}\wedge\frac{df_2}{f_2}\wedge\cdots\wedge\frac{df_n}{f_n}\\ &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^{n-1} \int_{\Gamma'} \frac{df_2'}{f_2'}\wedge\cdots\wedge\frac{df_n'}{f_n'}\\ &= (D_2',\dots,D_n')_{\{0\}}.\end{aligned}$$

Using this, we shall prove:

The Jacobian $\partial(f_1,\dots,f_n)/\partial(z_1,\dots,z_n) \not\equiv 0$ *and the local intersection number* $(D_1,\dots,D_n)_{\{0\}} > 0$.

Proof. The proof is by induction on n, with the case $n = 1$ being clear.

Choose a point z_0 that is a smooth point on D_1 and is very close to the origin. If we assume that $df_1 \wedge df_2 \wedge \cdots \wedge df_n \equiv 0$ and set $f_i' = f_i|D_1$, then $df_2' \wedge \cdots \wedge df_n' \equiv 0$ near z_0. Indeed, we may choose local coordinates

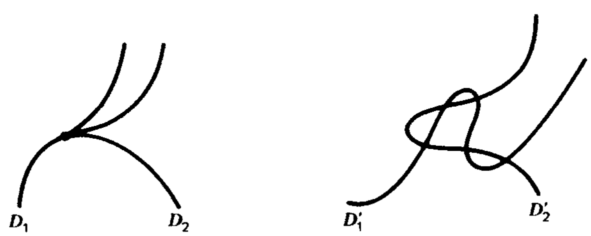


Figure 1

$(u_1, u_2, \ldots, u_n)$ around z_0 such that $f_1 = u_1^m$, where $m > 0$. Then $df_1 \wedge df_2 \wedge \cdots \wedge df_n \equiv 0 \Rightarrow u_1^{m-1} du_1 \wedge df_2 \wedge \cdots \wedge df_n \equiv 0 \Rightarrow df_2 \wedge \cdots \wedge df_n \equiv 0$ modulo $du_1 \Rightarrow df_2' \wedge \cdots \wedge df_n' \equiv 0$. On the other hand, if we let $f_i'(z_0) = w_i'$, then the equations $f_i'(z') = w_i'$ have z_0 as an isolated solution on D_1 near z_0. Setting $D_i' = (f_i' - w_i')$, the local intersection number $(D_2', \ldots, D_n')_{\{z_0\}} > 0$ by induction hypothesis. This is in contradiction to $df_2' \wedge \cdots \wedge df_n' \equiv 0$.

Now, assuming that $df_1 \wedge \cdots \wedge df_n \not\equiv 0$, we shall prove that $(D_1, \ldots, D_n)_{\{0\}} > 0$. The Dolbeault representative is

$$\begin{aligned}\eta(f_1, \ldots, f_n) &= C_n \frac{\sum (-1)^{i-1} \bar{f}_i \, d\bar{f}_1 \wedge \cdots \wedge \widehat{d\bar{f}_i} \wedge \cdots \wedge d\bar{f}_n \wedge df_1 \wedge \cdots \wedge df_n}{\|f\|^{2n}} \\ &= f^*(\beta),\end{aligned}$$

where, according to the Bochner-Martinelli formula in Section 1 of Chapter 3,

$$\beta = C_n \frac{\sum (-1)^{i-1} \bar{w}_i \, d\bar{w}_1 \wedge \cdots \wedge \widehat{d\bar{w}_i} \wedge \cdots \wedge d\bar{w}_n \wedge dw_1 \wedge \cdots \wedge dw_n}{\|w\|^{2n}}$$

is a closed $(n, n-1)$-form in $\mathbb{C}^n - \{0\}$ whose restriction to every sphere $\|w\| = \varepsilon$ is a $(2n-1)$-form with total integral one. On every sphere $\|z\| = \varepsilon$ the form $f^*(\beta)$ is $\geqslant 0$, and it is strictly positive at a point z_0 where $(df_1 \wedge \cdots \wedge df_n)(z_0) \neq 0$. For a sphere passing through such a point,

$$\int_{\|z\| = \|z_0\|} f^*\beta > 0.$$

This proves the positivity of the local intersection number.

(e) In fact it proves more. If we consider $f = (f_1, \ldots, f_n)$ as a mapping

$$f\colon\ U^* \to \mathbb{C}^n - \{0\},$$

then we have essentially shown that

The local intersection number is the topological degree deg(f) *of* f.

Proof. The form β gives an integral generator of $H_{\mathrm{DR}}^{2n-1}(S^{2n-1}(\varepsilon))$ for any sphere $\|w\| = \varepsilon$. By definition of the degree,

$$\begin{aligned}\deg(f) &= \int_{\|z\| = \varepsilon} f^*\beta \\ &= \int_{\|z\| = \varepsilon} \eta(f_1, \ldots, f_n) \\ &= (D_1, \ldots, D_n)_{\{0\}}\end{aligned}$$

by our basic integral formula.

Finite Holomorphic Mappings

We now want to tie in the local intersection number with the properties of f viewed as a holomorphic mapping $f: U \to \mathbb{C}^n$. For this, the following standard terminology will be useful: On a complex manifold M a *zero cycle* is a formal finite sum

$$\Gamma = \sum_\nu m_\nu P_\nu$$

of points $P_\nu \in M$ with multiplicities $m_\nu \in \mathbb{Z}$. We set

$$\underbrace{P + \cdots + P}_{k} = k \cdot P.$$

The zero cycle is *effective* in case all $m_\nu \geqslant 0$. The *degree* of a zero cycle is given by

$$\deg(\Gamma) = \sum_\nu m_\nu.$$

Suppose now that $f: U \to \mathbb{C}^n$ is a holomorphic mapping with $f^{-1}(0) = \{0\}$. We define the *multiplicity of* f *at the origin* to be the topological degree d of $f: U^* \to \mathbb{C}^n - \{0\}$. We then say that *the equation*

$$f(z) = 0$$

has the origin as a solution of multiplicity d.

Now, according to the continuity property (c) of the local intersection numbers, for $\|w\| < \varepsilon$ the equation

$$f(z) = w$$

will have exactly d solutions $z_\nu(w)$ close to the origin. Of course, some of the $z_\nu(w)$ may be repeated. Using the zero-cycle notation, we write

$$f^{-1}(w) = \sum_\nu z_\nu(w).$$

Let $W = \{\|w\| < \varepsilon\}$ and redefine $U = f^{-1}(W)$. Then we claim that the holomorphic mapping

$$f: U \to W$$

has the following properties: It is *surjective*, *open*, and *finite*. The first of these is by definition. The second means that open sets map onto open sets, which is clear, as is the third property. These finite mappings behave quite differently from, for example, blowing-down mappings such as

$$(u, v) \to (u, uv),$$

which are not open. In general, they share most of the properties of maps in one variable, as we shall now prove. For this we need the following:

Lemma. *For* h(z)∈$\mathcal{O}$(U), *the* trace

$$\sigma_h(w) = \sum_{\nu=1}^{d} h(z_\nu(w))$$

is a holomorphic function of w∈W.

Proof. We consider σ_h as a distribution operating on the compactly supported (n,n) forms $A_c^{n,n}(W)$ by the rule

$$\begin{aligned}\sigma_h(\varphi) &= \int_W \sigma_h(w)\varphi(w)\\ &= \int_U h(z)(f^*\varphi)(z),\end{aligned}$$

where $\varphi \in A_c^{n,n}(W)$. By the regularity theorem from Section 1 of Chapter 3 it will suffice to show that $\bar\partial\sigma_h = 0$ in the sense of currents. Now, for $\psi \in A_c^{n,n-1}(W)$, $f^*\psi$ is compactly supported and

$$\begin{aligned}\bar\partial\sigma_h(\psi) &= \int_W \sigma_h(w)\bar\partial\psi(w)\\ &= \int_U h(z)\bar\partial(f^*\psi)(z)\\ &= -\int_U \bar\partial h(z) f^*\psi(z)\\ &= 0,\end{aligned}$$

since $h \in \mathcal{O}(U)$. Q.E.D.

If we apply the lemma to the power sums $\Sigma_\nu h(z_\nu(w))^k$, then we deduce that any symmetric function—such as

$$h(z_1(w))\cdot \cdots \cdot h(z_d(w))$$

—is holomorphic in w.

One application is a proof of the *proper mapping theorem* for $f: U \dot{\rightarrow} W$, and hence for general finite surjective mappings: If $V \subset U$ is an analytic variety defined by equations $\{h_\alpha(z)=0\}$, then $f(V) \subset W$ is defined by $\{H_\alpha(w)=0\}$, where $H_\alpha(w) = h_\alpha(z_1(w)) \cdots h_\alpha(z_d(w))$.

Note in particular that the *discriminant*, or *branch locus*, $D \subset W$, defined as the image $f(R)$ of the *ramification divisor*

$$\left\{ \frac{\partial(f_1,\ldots,f_n)}{\partial(z_1,\ldots,z_n)}(z) = 0 \right\},$$

is an analytic hypersurface. For $w \in W - D$, $h^{-1}(w) = \Sigma_\nu z_\nu(w)$, where the $z_\nu(w)$ are distinct. Choosing a path $w(t)$ with $w(0)=0$ and $w(t) \in W - D$ for $t \neq 0$, we find the explicit good perturbation $f_t(z) = f(z) - w(t)$ of $f(z)$.

As another application of the lemma, we suppose that $h(z)\in\mathcal{O}(U)$ and consider the expression

$$H(z) = \prod_{\nu=1}^{d} \left(h(z)-h(z_\nu(f(z)))\right).$$

On the one hand, $H(z)\equiv 0$, since $z=z_\nu(f(z))$ for some ν. On the other hand, H is a polynomial of the form

$$h(z)^d + a_1(w)h(z)^{d-1} + \cdots + a_d(w) \qquad (w=f(z)),$$

whose coefficients are holomorphic functions of $w=f(z)$. Now, via the mapping $f: U\to W$ the local ring $\mathcal{O}_w=\{$germs of holomorphic functions $h(w)$ defined in some neighborhood of $w=0\}$ injects into $\mathcal{O}_z$, and we have proved

The degree of the extension $[\mathcal{O}_z:\mathcal{O}_w]=d$; *i.e., every* $h\in\mathcal{O}_z$ *satisfies a polynomial equation of degree* $\leqslant d$ *with coefficients in* $f^*\mathcal{O}_w\subset\mathcal{O}_z$, *and moreover* d *is the least such integer.*

As a corollary we have the following special case of the nullstellensatz:

If $h(z)\in\mathcal{O}_z$ *vanishes at* $z=0$, *then*

$$h(z)^d \in \{f_1(z),\ldots,f_n(z)\}.$$

Proof. As $z\to 0$, both $f(z)$ and $z_\nu(f(z))\to 0$. Consequently, the coefficients $a_\nu(w)$ in the polynomial H vanish at $w=0$. This implies that $h^d\equiv 0$ modulo $\{f_1,\ldots,f_n\}$.

(f) Finally, we can give one more interpretation of the local intersection number $(D_1,\ldots,D_n)_{\{0\}}$, where $D_i=(f_i)$. Let $\mathcal{O}=\mathcal{O}_z$ and $I\subset\mathcal{O}$ be the ideal $\{f_1,\ldots,f_m\}$ defined by $f^*(m_w)$ where $m_w\subset\mathcal{O}_w$ is the maximal ideal of functions $h(w)$ with $h(0)=0$. Then we have:

$\mathcal{O}/I$ *is a finite-dimensional complex vector space, and*

$$\dim_{\mathbb{C}}(\mathcal{O}/I) = (D_1,\ldots,D_n)_{\{0\}}.$$

Summarizing: Let $D_i=(f_i)$ be n divisors given in some small neighborhood U of the origin in $\mathbb{C}^n$ with $\cap_i D_i=\{0\}$. Then the local intersection number has the following interpretations:

(1) *Analytic*: The formula

$$(D_1,\ldots,D_n)_{\{0\}} = \operatorname{Res}_{\{0\}}\left(\frac{df_1}{f_1}\wedge\cdots\wedge\frac{df_n}{f_n}\right)$$

was taken as our definition.

(2) *Topological*: Setting $f=(f_1,\ldots,f_n): U^* \to \mathbb{C}^n - \{0\}$,

$$(D_1,\ldots,D_n)_{\{0\}} = \text{degree}(f).$$

Equivalently, $f: U \to W$ is a finite open surjective holomorphic mapping for some neighborhood W of the origin in $\mathbb{C}^n$. Since f is orientation preserving, $(D_1,\ldots,D_n)_{\{0\}}$ is the sheet number of f.

(3) *Algebraic*: If $\mathcal{O}$ is the local ring at the origin and $I \subset \mathcal{O}$ the ideal generated by the f_i, then

$$(D_1,\ldots,D_n)_{\{0\}} = \dim_{\mathbb{C}}(\mathcal{O}/I).$$

In general, if D_i are divisors on a complex manifold M meeting at a finite number of isolated points P_ν, we define the effective zero cycle

$$D_1 \cdot \cdots \cdot D_n = \sum_\nu m_\nu P_\nu,$$

where

$$m_\nu = (D_1,\ldots,D_n)_{P_\nu}.$$

The degree

$$\deg(D_1 \cdot \cdots \cdot D_n) = \sum_\nu m_\nu$$

is the total intersection number of the D_i.

Applications to Plane Projective Geometry

We shall apply the residue theorem to the simplest global case $M = \mathbb{P}^n$, with special attention to the case $n=2$. Suppose then that $D_1,\ldots,D_n$ are hypersurfaces of respective degrees $d_1,\ldots,d_n$ and meeting in isolated points P_ν. According to the discussion in the preceding section, we may write the intersection as a zero cycle

$$D_1 \cdot \cdots \cdot D_n = \sum_\nu m_\nu P_\nu,$$

where the local intersection numbers $m_\nu = (D_1,\ldots,D_n)_{P_\nu}$ are given by a residue, and the global *Bezout theorem*

$$\deg(D_1 \cdot \cdots \cdot D_n) = \sum_\nu m_\nu = d_1 \cdots d_n$$

is valid. In a suitable Euclidean coordinate system $(x_1,\ldots,x_n)$, we may assume that all P_ν lie in $\mathbb{C}^n \subset \mathbb{P}^n$ and that D_i is the divisor of a polynomial $f_i(x_1,\ldots,x_n)$ of degree d_i. The most general meromorphic n-form on $\mathbb{P}^n$ with polar divisor $D = D_1 + \cdots + D_n$ has in $\mathbb{C}^n$ an expression

$$\omega = \frac{g(x)\,dx_1 \wedge \cdots \wedge dx_n}{f_1(x) \cdots f_n(x)}$$

where $g(x)$ is a polynomial. Under a typical change of coordinates in $\mathbb{P}^n$

$$x_1 = \frac{1}{x_1'}, \quad x_2 = \frac{x_2'}{x_1'}, \quad \ldots, \quad x_n = \frac{x_n'}{x_1'},$$

we have

$$dx_1 \wedge \cdots \wedge dx_n = -\frac{1}{(x_1')^{n+1}} dx_1' \wedge dx_2' \wedge \cdots \wedge dx_n',$$

$$f_i(x_1,\ldots,x_n) = \frac{1}{(x_1')^{d_i}} f_i'(x_1',x_2',\ldots,x_n').$$

It follows that ω does not have the hyperplane at infinity as a component of its polar divisor exactly when the degree restriction

$$\deg(g) \leqslant (d_1 + \cdots + d_n) - (n+1)$$

is satisfied. The global residue theorem then gives

$$(*) \qquad \sum_\nu \operatorname{Res}_{P_\nu} \left\{ \frac{g(x)dx_1 \wedge \cdots \wedge dx_n}{f_1(x) \cdots f_n(x)} \right\} = 0.$$

In the case where the D_i meet transversely at $d_1 \cdots d_n$ distinct points, $(*)$ reduces to the *Jacobi relation*

$$\sum_\nu \frac{g(P_\nu)}{(\partial(f_1,\ldots,f_n)/\partial(x_1,\ldots,x_n))(P_\nu)} = 0, \qquad \deg(g) \leqslant \sum d_i - (n+1)$$

proved by him in 1834. For $n=1$ we obtain the *Lagrange interpolation formula*

$$\sum_\nu \frac{g(P_\nu)}{f'(P_\nu)} = 0, \qquad \deg(g) \leqslant \deg(f) - 2;$$

it was in this context that Jacobi was led to his formula.

In the case $n=2$ the Jacobi relation immediately implies the

Cayley-Bacharach Theorem. *If* C *and* D *are curves in* $\mathbb{P}^2$ *of respective degrees* m *and* n *and meeting at* mn *distinct points, then any curve* E *of degree* $\mathrm{m}+\mathrm{n}-3$ *that passes through all but one point of* $\mathrm{C} \cap \mathrm{D}$ *necessarily passes through that remaining point also.*

It is clear that the stronger relation $(*)$ gives a more general statement than the Cayley-Bacharach theorem when C and D may not have transverse intersections. Rather than attempt to formalize this, we shall usually go ahead and use the Cayley-Bacharach theorem in degenerate cases where the proof will be an immediate consequence of $(*)$.

To illustrate, we give an example of a degenerate case:

Suppose that the curves C *and* D *above have intersection*

$$C\cdot D=\sum_{\nu} m_{\nu}P_{\nu},$$

where all the points P_ν *are smooth points of* C. *If* E *is a curve of degree* $m+n-3$ *such that for some* ν_0,

$$\begin{aligned}(C\cdot E)_{P_\nu} &\geqslant m_\nu, \qquad \nu\neq\nu_0,\\ (C\cdot E)_{P_{\nu_0}} &\geqslant m_{\nu_0}-1,\end{aligned}$$

then

$$(C\cdot E)_{P_{\nu_0}}\geqslant m_{\nu_0}.$$

Proof. By hypothesis, we may choose local holomorphic coordinates (z,w) around P_{ν_0} and defining functions $f(z,w)$, $g(z,w)$ for C,D, respectively, such that

$$\begin{cases} f(z,w)=z,\\ g(0,w)=w^{m_{\nu_0}}+\cdots.\end{cases}$$

The defining function $h(z,w)$ for E will then satisfy

$$h(0,w)=\alpha w^{m_{\nu_0}-1}+\cdots$$

and

$$(E,C)_{P_{\nu_0}}\geqslant m_{\nu_0}\Leftrightarrow\alpha=0.$$

Consequently, if we can show that

$$\operatorname{Res}_{\{0\}}\left(\frac{b(z,w)\,dz\wedge dw}{f(z,w)g(z,w)}\right)=0\Leftrightarrow\alpha=0,$$

our assertion will follow from (*). By iterating the integral in the definition of the residue,

$$\begin{aligned}\operatorname{Res}_{\{0\}}\left(\frac{h(z,w)\,dz\wedge dw}{f(z,w)g(z,w)}\right)&=\left(\frac{1}{2\pi\sqrt{-1}}\right)^2\int_{|g(z,w)|=\varepsilon}\left(\int_{|z|=\varepsilon}\frac{h(z,w)\,dz}{g(z,w)z}\right)dw\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{|w|=\varepsilon}\frac{h(0,w)\,dw}{g(0,w)}\\ &=\frac{1}{2\pi\sqrt{-1}}\int_{|w|=\varepsilon}(\alpha+\cdots)\frac{dw}{w}\\ &=\alpha.\end{aligned}$$

Q.E.D.

The first nontrivial case is when $m=n=3$; then we obtain the classical statement:

Suppose that C *and* D *are cubic curves meeting in nine points that are not necessarily distinct but that are simple points of* C. *Then any cubic* E *passing through eight of these points must contain the remaining one also.*

This fact was known in 1748 to Euler, who remarked that as a consequence polynomial functions in two or more variables would necessarily be much more complicated than in one variable, since then it is not generally the case that a set of mn points in plane is the common zero locus of a pair of polynomials.

As another application, we can prove

Pascal's Theorem. *The pairs of opposite sides of a hexagon inscribed in a smooth conic* Q *meet in three collinear points.*

Proof. Suppose that $L_1L_2L_3L_4L_5L_6$ is the inscribed hexagon. Take $C=L_1+L_3+L_5$, $D=L_2+L_4+L_6$, and $E=Q+\overline{P_{12}P_{34}}$, where $P_{ij}=L_i\cap L_j$. Then E passes through the remaining point P_{56} of $C\cap D$. Q.E.D.

There is also a

Converse to Pascal's Theorem. *If* $\mathrm{H}=\mathrm{L_1L_2L_3L_4L_5L_6}$ *is a hexagon such that the opposite sides meet in three collinear points, then the vertices of* H *are on a conic.*

Proof. Set $P_{ij}=L_i\cap L_j$ and let L be the line through P_{14}, P_{25}, and P_{36}. Then if Q is a conic passing through the five vertices $P_{12},P_{23},P_{34},P_{45},P_{56}$ of H, we may take $C=L_1+L_3+L_5$, $D=L_2+L_4+L_6$, and $E=Q+L$ to conclude that Q passes through P_{61}. Q.E.D.

Along similar lines but at a deeper level we shall prove a converse to the Cayley-Bacharach theorem. Suppose that

$$\Gamma = P_1+\cdots+P_{n^2}$$

is a zero-cycle consisting of n^2 distinct points. *We say that* Γ *satisfies the Cayley-Bacharach property if every curve* E *of degree* $2\mathrm{n}-3$ *that passes through all but one point of* Γ *necessarily contains* Γ. Since the dimension of the linear system of curves of degree $2n-3$ is

$$n(2n-3) = 2n^2-3n,$$

there are plenty of such "test curves" E. The result is the following.

Proposition. *Suppose that* $\Gamma=\mathrm{P}_1+\cdots+\mathrm{P}_{\mathrm{n}^2}$ *satisfies the Cayley-Bacharach property. Then* Γ *lies on a pencil of curves of degree* n.

Proof. We consider the *Veronese embedding*

$$i_n: \mathbb{P}^2 \hookrightarrow \mathbb{P}^N, \qquad N = \frac{n(n+3)}{2},$$

given by the complete linear system of curves of degree n—i.e., by $H^0(\mathbb{P}^2, \mathcal{O}(n\mathrm{H}))$. Since the hyperplane sections of $i_n(\mathbb{P}^2)$ are just the curves of degree n, we must show that:

(∗∗) *The points* $\mathrm{i_n(P_\nu)}$ *lie on a* $\mathbb{P}^{\mathrm{N}-2}$ *in* $\mathbb{P}^{\mathrm{N}}$.

This in turn will be the case if any N of the points $i_n(P_\nu)$ are linearly dependent in $\mathbb{P}^N$.

Suppose we select N of the points P_ν, say $P_1,\ldots,P_{n(n+3)/2}$ for simplicity of notation, and let $A \subset \mathbb{P}^N$ be any hyperplane containing $N-1$ of them, say $P_2,\ldots,P_{n(n+3)/2}$. Since

$$n^2 = \frac{n(n+3)}{2} + \frac{n(n-3)}{2}$$

and $\dim H^0(\mathbb{P}^2, \mathcal{O}((n-3)H)) = n(n-3)/2+1$, we may find a curve B of degree $n-3$ passing through $P_{(n(n+3)/2)+1},\ldots,P_{n^2}$. Then $A+B$ is a curve of degree $2n-3$ passing through $P_2,\ldots,P_{n^2}$, and consequently $A+B$ contains P_1. Now in general no $n(n-3)/2+1$ of the points P_ν will lie on a curve of degree $n-3$. In this case P_1 lies on A, and we have proved:

Given N *of the points* $\mathrm{P_\nu}$, *any hyperplane containing* $\mathrm{N}-1$ *of* $\mathrm{i_n(P_\nu)}$ *contains the* N*th point also.*

This clearly implies (∗∗).

In the exceptional case we label our points $P_1,\ldots,P_{n(n+3)/2}$ so that the curve B of degree $n-3$ passes through exactly

$$P_k,\ldots,P_{n^2}, \qquad k \leqslant \frac{n(n+3)}{2}.$$

Given any curve A_i of degree n passing through $P_1,\ldots,\hat{P}_i,\ldots,P_k$, A_i+B will contain Γ and so P_i lies on A_i. *Consequently, the points* $\mathrm{i_n(P_1),\ldots,i_n(P_k)}$ *are linearly dependent in* $\mathbb{P}^{\mathrm{N}}$.

This again implies (∗∗). Q.E.D.

We have now illustrated the application of the residue theorem to points arising as intersections of plane algebraic curves having no multiple components. Multiple components arise naturally when we wish to know not only about the position in $\mathbb{P}^2$ of the points of intersection, but also about higher-order infinitesimal behavior.

For example, let $L \subset \mathbb{P}^2$ be a line. If we mark points $P_1,\ldots,P_n$ on L, then it is trivially possible to find an algebraic curve C of degree n passing through the P_ν—just take C to be a union of lines. It is equally easy to prescribe the tangent lines T_ν that C is to have at P_ν. However, if we assign second-order elements of arc C_ν passing through P_ν and with tangent T_ν, then it is not always possible to find an algebraic curve C having the prescribed second-order behavior C_ν around P_ν. There is one condition here, the *Reiss relation*, which we proceed to derive.

Suppose that C has affine equation $f(x,y)=0$, that L is the line $\{x=0\}$, and that the n points of intersection of C with L are distinct finite points on the y-axis. We shall prove the

Reiss Relation. *With the notations* $f_y=(\partial f/\partial y)(x,y)$, *etc.*

$$\sum \frac{\left(f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2\right)}{f_y^3}=0,$$

the terms in the sum being evaluated at the points $L\cdot C$.

Proof. In a general vein, the mth-order behavior of C near the points of intersection $C\cap L$ will be reflected in the residues of

$$\omega=\frac{p(x,y)\,dx\wedge dy}{xf(x,y)^m}.$$

If $f(x,y)$ has degree n, then ω will not have the line at infinity as a component of its polar divisor provided that $\deg(p)\leqslant mn-2$. In case $m=2$, the restriction $\deg(p)\leqslant 2n-2$ suggests taking p to be of the form $p(x,y)=\alpha f_{xx}f+\beta f_{xy}f+\gamma f_{yy}f+\delta f_x^2+\varepsilon f_xf_y+\kappa f_y^2$. To see what p to choose, we assume that the origin is one of the points of intersection and will prove the

Lemma.

$$\operatorname{Res}_{\{0\}}\left(\frac{p(x,y)\,dx\wedge dy}{xf(x,y)^2}\right)=\frac{p_y}{f_y^2}-\frac{pf_{yy}}{f_y^3}.$$

Proof of Lemma. This is an application of the transformation formula from Section 2. We may assume that $f(x,y)$ has a Taylor series

$$f(x,y)=ax+by+\frac{cx^2}{2}+dxy+\frac{ey^2}{2}+\cdots,\qquad b\neq 0.$$

Consider the ideals

$$I=\{x,y^2\},\qquad I'=\{x,f(x,y)^2\}.$$

For a suitable function $g(x,y)$ holomorphic in a neighborhood of the origin,

$$f(x,y)^2=g(x,y)x+(b^2+bey+\cdots)y^2.$$

Consequently, $I' \subset I$ with transformation matrix

$$A = \begin{pmatrix} 1 & g(x,y) \\ 0 & b^2 + bey + \cdots \end{pmatrix}.$$

Note that the determinant $\Delta = b^2 + bey + \cdots$ is nonzero at the origin. For $h(x,y)$ holomorphic, the transformation law gives

$$\mathrm{Res}_{\{0\}}\left(\frac{h(x,y)\,dx\wedge dy}{xy^2}\right) = \mathrm{Res}_{\{0\}}\left(\frac{\Delta(x,y)h(x,y)\,dx\wedge dy}{xf(x,y)^2}\right)$$

By the Cauchy formula the left-hand side is $h_y(0,0)$. Taking $h = p/\Delta$, we obtain

$$\mathrm{Res}_{\{0\}}\left(\frac{p(x,y)\,dx\wedge dy}{xf(x,y)^2}\right) = \frac{p_y(0)}{f_y(0)^2} - \frac{p(0)f_{yy}(0)}{f_y(0)^3},$$

since $f_y(0) = b$ and $f_{yy}(0) = e$. Q.E.D. for Lemma.

On the basis of the lemma and elementary fiddling around, if we take

$$p = f_x^2 - ff_{xx},$$

then

$$\mathrm{Res}_{\{0\}}\left(\frac{p(x,y)\,dx\wedge dy}{xf(x,y)^2}\right) = -\frac{\left(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2\right)}{f_y^3}.$$

Applying the residue theorem in the form (*) gives the Reiss relation. Q.E.D.

The expression in the Reiss relation has an interpretation encountered in the differential geometry of plane curves—in the calculus sense. Namely, suppose that $(x,y(x))$ is a parametric representation of C near the origin. Differentiating $f(x,y(x)) \equiv 0$ at the origin gives the equations

$$\begin{cases} f_x + f_y y' = 0, \\ f_{xx} + 2f_{xy}y' + f_{yy}y'^2 + f_y y'' = 0, \end{cases}$$

and eliminating y' yields

$$y'' = -\frac{\left(f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2\right)}{f_y^3}.$$

On the other hand, it is elementary calculus that

$$y''(0) = \frac{\kappa}{\sin^3\theta},$$

where κ is the curvature of C at the origin and θ is the angle that the tangent to C makes with the y-axis. Consequently, the Reiss relation may be expressed in the very pretty metric form

$$\sum_{\nu} \frac{\kappa_\nu}{\sin^3\theta_\nu} = 0,$$

where κ_ν is the curvature of C at P_ν and θ_ν is the angle that the tangent T_ν makes with the line L.

We shall now show that the Reiss relation is sufficient. The polynomials $f(x,y)$ of degree n form a vector space of dimmension $(n+1)(n+2)/2$. Those of the form $g(x,y)x^3$ ($\deg(g)=n-3$) form a vector space of dimension $(n-2)(n-1)/2$. The quotient space V has dimension

$$\frac{n^2+3n+2}{2} - \frac{n^2-3n+2}{2} = 3n.$$

Finding a curve C of degree n and with prescribed second-order behavior at points P_ν on the line $\{x=0\}$ is equivalent to finding a suitable point in the projective space $\mathbb{P}(V) \cong \mathbb{P}^{3n-1}$. Each second-order arc element imposes three linear conditions, and so there are $3n$ conditions in all. It follows that the Reiss relation is both necessary and sufficient.

Finally, we wish to point out that the residue theorem from Section 1 applies to configurations of points on general algebraic surfaces, not just $\mathbb{P}^2$. More precisely, suppose that L, L' are holomorphic line bundles over a surface S and $C \in |L|, C' \in |L'|$ are curves meeting transversely at $d = L \cdot L'$ points. Then we have the

Proposition. *Any curve* $D \in |K+L+L'|$ *that passes through all but one point of* $C \cdot C'$ *necessarily contains that remaining point.*

Proof. If $\sigma \in H^0(S, \mathcal{O}(L))$ and $\sigma' \in H^0(S, \mathcal{O}(L'))$ define C and C', and if $\psi \in H^0(S, \mathcal{O}(K+L+L')) = H^0(S, \Omega^2(L+L'))$, then

$$\omega = \frac{\psi}{\sigma \cdot \sigma'}$$

is a meromorphic 2-form on S with polar curve $C + C'$ to which the general residue theorem

$$\sum_{P_\nu \in C \cap C'} \operatorname{Res}_{P_\nu}(\omega) = 0$$

clearly implies the result. Q.E.D.

An extension to general vector bundles will be given in Section 4 at the end of this chapter.

3. RUDIMENTS OF COMMUTATIVE AND HOMOLOGICAL ALGEBRA WITH APPLICATIONS

Commutative Algebra

As the reader is no doubt aware, laying the proper algebraic foundations for the subject of algebraic geometry is an all-consuming task. On the other hand, just as sheaf cohomology greatly facilitates the study of divisors on a variety—a case where the local theory is relatively simple—the introduction of some algebraic machinery will clarify some of the preceding discussion concerning the local properties of a set of analytic equations $f_1(z_1,\dots,z_n)=\cdots=f_n(z_1,\dots,z_n)=0$ having the origin as isolated common zero. This will be especially true of the transformation law and local duality theorem associated to our analytically defined residues; these two results will eventually achieve a very symmetric form.

We use the notation

$$\mathcal{O} = \lim_{\{0\}\in U} \mathcal{O}(U)$$

for the germs of analytic functions defined in some neighborhood U of the origin in $\mathbb{C}^n$. Clearly, $\mathcal{O}=\mathbb{C}\{z_1,\dots,z_n\}$ is the ring of convergent power series. When involved in inductive arguments we shall write $\mathcal{O}_n$ for $\mathcal{O}$. Recall that a *local ring* is a ring having a unique maximal ideal. $\mathcal{O}$ is such a local ring with maximal ideal $m=\{z_1,\dots,z_n\}$ the ideal of functions $f\in\mathcal{O}$ with $f(0)=0$. The units are just $\mathcal{O}^*=\mathcal{O}-m$.

In Section 1 of Chapter 0 we proved that, given $f\neq 0$ in $\mathcal{O}_n$, there is a linear coordinate system $(z_1,z_2,\dots,z_n)=(z',z_n)$ and unique Weierstrass polynomial

$$w(z) = z_n^d + a_1(z')z_n^{d-1} + \cdots + a_d(z') \in \mathcal{O}_{n-1}[z_n],$$

where $a_i(z')\in\mathcal{O}_{n-1}$ are nonunits such that

$$f(z) = u(z)w(z)$$

with $u\in\mathcal{O}^*$. In addition to the Weierstrass preparation theorem, we also proved the division theorem: For $g\in\mathcal{O}_n$,

$$g = hf + r,$$

where $r\in\mathcal{O}_{n-1}[z_n]$ has degree less than that of w. These two results provide the basic tools for studying the local ring $\mathcal{O}$—especially the ideals in $\mathcal{O}$.

The method is frequently by induction on n. For example, the inductive hypothesis and Gauss lemma imply that $\mathcal{O}_{n-1}[z_n]$ is a unique factorization domain, and using the preparation theorem we deduced that

$\mathcal{O}_n$ *is a unique factorization domain.*

Similarly, we shall prove:

$\mathcal{O}_n$ *is a Noetherian ring,*

Proof. We must show that any ideal $I\subset\mathcal{O}$ has a finite number of generators. Let $0\neq f\in I$. We may assume that $f\in\mathcal{O}_{n-1}[z_n]$ is a Weierstrass polynomial. Set $I'=I\cap\mathcal{O}_{n-1}[z_n]$. By induction hypothesis $\mathcal{O}_{n-1}$ is Noetherian, and then the Hilbert basis theorem implies that I' has a finite set $f_1,\dots,f_k\in\mathcal{O}_{n-1}[z_n]$ of $\mathcal{O}_{n-1}$ generators. We claim that

$$I = \{f,f_1,\dots,f_k\}.$$

To see this, let $g\in I$ and apply the division theorem to obtain

$$g = hf + r.$$

Then $r\in I\cap\mathcal{O}_{n-1}[z_n]=I'$ and may be expressed in terms of $f_1,\dots,f_k$.
Q.E.D.

Our discussion of commutative algebra will center around $\mathcal{O}$*-modules*, usually denoted by $M,N,R,\dots$ *and which we always assume to be finitely generated*. Choosing generators $m_1,\dots,m_k$ for an $\mathcal{O}$-module M, there is an exact sequence

$$0\to R\to\mathcal{O}^{(k)}\xrightarrow{\pi}M\to 0$$

of $\mathcal{O}$-modules, where

$$\mathcal{O}^{(k)} = \underbrace{\mathcal{O}\oplus\cdots\oplus\mathcal{O}}_{k}$$

is the *free* $\mathcal{O}$*-module of rank* k,

$$\pi(g_1,\dots,g_k) = g_1m_1+\cdots+g_km_k,$$

and

$$R = \{(g_1,\dots,g_k):\ g_1m_1+\cdots+g_km_k=0\}$$

is the *module of relations* among the m_i's. We claim that R is again finitely generated. The proof is by induction on k, with the case $k=1$ being that of an ideal in $\mathcal{O}$ just discussed. Setting $R'=R\cap\mathcal{O}^{(k-1)}$, in the exact sequence

$$0\to R'\to R\to R/R'\to 0$$

both R' and $R/R'\subset\mathcal{O}$ are finitely generated, and hence so is R.

As examples of $\mathcal{O}$-modules, in addition to the free $\mathcal{O}$-modules mentioned above, the most important ones are

$$\begin{cases} I=\{f_1,\dots,f_k\} & \text{an } \textit{ideal} \text{ in } \mathcal{O},\\ M=\mathcal{O}/\{f_1,\dots,f_k\}. \end{cases}$$

Very roughly speaking, the second of these is the local ring at the origin of the variety $f_1(z)=\cdots=f_k(z)=0$. We shall say more about this later. Given an ideal $\{f_1,\ldots,f_k\}$, where the $f_i\in\mathcal{O}_{n-1}[z_n]$ are Weierstrass polynomials, it is natural to consider $I'=I\cap\mathcal{O}_{n-1}[z_n]$ as a quotient-module of $\mathcal{O}_{n-1}^{(k)}$ over $\mathcal{O}_{n-1}$. Consequently, even though ideals in $\mathcal{O}_n$ may be our primary interest, more general modules arise naturally in inductive arguments.

$\mathcal{O}$-modules admit the operations of linear algebra, such as

$$M\oplus N,\qquad M\otimes_{\mathcal{O}}N,\qquad \operatorname{Hom}_{\mathcal{O}}(M,N).$$

Given an exact sequence of $\mathcal{O}$-modules

$$0\to P\to Q\to R\to 0,$$

the resulting sequences

$$\begin{cases} P\otimes_{\mathcal{O}}M\to Q\otimes_{\mathcal{O}}M\to R\otimes_{\mathcal{O}}M\to 0,\\ 0\to \operatorname{Hom}_{\mathcal{O}}(M,P)\to\operatorname{Hom}_{\mathcal{O}}(M,Q)\to\operatorname{Hom}_{\mathcal{O}}(M,R),\end{cases}$$

are exact. We express this by saying that $\otimes$ *is right-exact* and Hom *is left-exact*. Much of our discussion will be centered around the kernel of $P\otimes_{\mathcal{O}}M\to Q\otimes_{\mathcal{O}}M$ and cokernel of $\operatorname{Hom}_{\mathcal{O}}(M,Q)\to\operatorname{Hom}_{\mathcal{O}}(M,R)$.

Associated to an $\mathcal{O}$-module M is its *fiber* $M_0=M/mM$—the motivation for this terminology will emerge when we discuss coherent sheaves. This is a module over $\mathcal{O}/m=\mathbb{C}$ and is therefore a finite-dimensional vector space. Our main technical tool is the

Nakayama Lemma. *If* $\mathrm{M}=m\mathrm{M}$, *then* $\mathrm{M}=(0)$.

Proof. We define the ideal $I=\{f\in\mathcal{O}: f\cdot M=0\}$ and shall prove that $I=\mathcal{O}$. Suppose $m_1,\ldots,m_k$ generate M and write

$$m_i=\sum_j a_{ij}m_j$$

or equivalently

$$\sum_j(\delta_{ij}-a_{ij})m_j=0,$$

where $a_{ij}\in m$. By Cramer's rule this implies

$$\Delta\cdot m_j=0,$$

where

$$\Delta=\det(\delta_{ij}-a_{ij})\in 1+m.$$

Thus Δ is a unit, and so $I=\mathcal{O}$. Q.E.D.

The Nakayama lemma is most useful in the following form:

$\mathrm{m}_1,\ldots,\mathrm{m}_k$ *generate* $\mathrm{M}\Leftrightarrow$ *they generate* M_0.

Proof. The implication $\Rightarrow$ is obvious. Conversely, assume that $m_1,\ldots,m_k$ generate M_0 and let $S \subset M$ be the submodule of M that they generate. To show that $S = M$ we set $Q = M/S$ and consider the exact sequence

$$0 \to S \to M \xrightarrow{\pi} Q \to 0.$$

If $m \in M$, then $m - s \in mM$ for some $s \in S$. Consequently, $\pi(m) = \pi(m-s) \in mQ$ and $Q = mQ$. Then $Q = (0)$ as desired. Q.E.D.

We note one final version:

If $\varphi: \mathrm{M} \to \mathrm{N}$ *is a homomorphism of* $\mathcal{O}$*-modules such that* $\varphi_0: \mathrm{M}_0 \to \mathrm{N}_0$ *is surjective, then* φ *is surjective.*

Now we come to a main standard definition. An $\mathcal{O}$-module M is *projective* if the following diagram holds:

$$\begin{array}{ccccc} M & & & & \\ {\scriptstyle\gamma}\,\vdots & \searrow^{\beta} & & & \\ K & \xrightarrow[\alpha]{} & L & \to & 0 \end{array}$$

This diagram is to be interpreted as follows: K and L are given $\mathcal{O}$-modules, and α and β are given $\mathcal{O}$-module homomorphisms with α being surjective. Then there exists γ such that the diagram is commutative. Briefly, *the solid arrows are given and the dotted arrows can be filled in*. This notation will be consistently used.

Lemma. M *is projective* $\Leftrightarrow$ *it is free.*

Proof. Assume M is free—we may as well take $M = \mathcal{O}$—and let $l_i \in L$ be generators and $k_i \in K$ with $\alpha(k_i) = l_i$. If $\beta(1) = \Sigma_i f_i l_i$, then we may set $\gamma(1) = \Sigma_i f_i k_i$ to fill in the dotted arrow.

Conversely, assume M is projective. Taking $M = L$ and K to be free, we have

$$\mathcal{O}^{(k)} \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} M \to 0$$

We may assume that k is the minimal number of generators of M, or equivalently that the map $\mathbb{C}^k \xrightarrow{\alpha_0} M_0$ on fibers is an isomorphism. Then $\gamma \circ \alpha$ is surjective and γ is surjective on the fibers. By Nakayama's lemma (third form) γ is surjective and α is an isomorphism. Q.E.D.

Note that the definition of projective may be rephrased as follows:

$$K \to L \to 0$$
$$\Downarrow$$
$$\mathrm{Hom}_{\mathcal{O}}(M,K) \to \mathrm{Hom}_{\mathcal{O}}(M,L) \to 0.$$

On the other hand, since projective modules are free, for M projective we have

$$0 \to P \to Q$$
$$\Downarrow$$
$$0 \to M \otimes_{\mathcal{O}} P \to M \otimes_{\mathcal{O}} Q.$$

Consequently, *both the functors* $\mathrm{Hom}_{\mathcal{O}}(\mathrm{M}, \cdot)$ *and* $\mathrm{M} \otimes_{\mathcal{O}} \cdot$ *are exact for* M *projective*. We shall explore this systematically in the next discussion.

Homological Algebra

We begin by remembering a series of definitions, most of which are probably familiar from algebraic topology.

(a) A *complex* is given by either

$$(K_.) \qquad \to K_n \overset{\partial}{\to} K_{n-1} \overset{\alpha}{\to} \cdots, \qquad \partial^2 = 0,$$

or

$$(K^\cdot) \qquad \to K^n \overset{\delta}{\to} K^{n+1} \overset{\delta}{\to} \cdots, \qquad \delta^2 = 0.$$

Here the K's will always be finitely generated $\mathcal{O}$-modules and maps $\mathcal{O}$-module homomorphisms, although most of the present discussion goes over to modules over general rings. Taking cycles/boundaries gives respectively $H_*(K_.) = \oplus H_n(K_.)$ (*homology*) and $H^*(K^\cdot) = \oplus H^n(K^\cdot)$ (*cohomology*). We shall give the remaining discussion in homology, leaving the dual considerations to the reader.

(b) A *mapping* or *homomorphism of complexes* $\varphi : K_. \to L_.$ is given by a commutative diagram

$$\begin{array}{ccccc} \to K_n & \overset{\partial}{\to} & K_{n-1} & \to & \cdots \\ \downarrow \varphi & & \downarrow \varphi & & \\ \to L_n & \overset{\partial}{\to} & L_{n-1} & \to & \cdots \end{array}$$

It induces a map $\varphi_* : H_*(K_.) \to H_*(L_.)$ on homology. When necessary we shall write $\varphi_n : K_n \to L_n$ and ∂_K, ∂_L for the boundary maps. The set $\mathrm{Hom}(K_., L_.)$ of homomorphisms of complexes is a group with $(\varphi+\psi)_* = \varphi_* + \psi_*$.

(c) A homomorphism of complexes $\varphi : K_. \to L_.$ is *homotopic to zero*, denoted $\varphi \sim 0$, if there is a chain homotopy

$$\varphi = \partial_L \alpha_n + \alpha_{n-1} \partial_K$$

as indicated by the diagram

$$\begin{array}{ccccc} K_{n+1} & \longrightarrow & K_n & \xrightarrow{\partial_K} & K_{n-1} \\ & \swarrow_{\alpha_n} & \downarrow{\scriptstyle\varphi} & \swarrow_{\alpha_{n-1}} & \\ L_{n+1} & \xrightarrow{\partial_L} & L_n & \longrightarrow & L_{n-1} \end{array}$$

In this case $\varphi_* = 0$. Two maps φ and ψ are homotopic if $\varphi - \psi \sim 0$; then $\varphi_* = \psi_*$.

(d) Most importantly, an exact sequence of complexes

$$0 \to K_. \to L_. \to M_. \to 0$$

is defined in the obvious way. It gives rise to a long exact homology sequence

$$\cdots \to H_n(K_.) \to H_n(L_.) \to H_n(M_.) \xrightarrow{\partial_*} H_{n-1}(K_.) \to \cdots .$$

The following definition and proposition are our primary technical tools:

DEFINITION. A *projective resolution* $E_.(M)$ of an $\mathcal{O}$-module M is given by an exact sequence

$$E_.(M): \ \cdots \to E_m \xrightarrow{\partial} E_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_0 \to M \to 0,$$

where the E_m are projective (=free) $\mathcal{O}$-modules.

Note that $H_n(E_.(M)) = 0$ for $n > 0$ and $H_0(E_.(M)) \cong M$. Given a projective resolution $E_.(M)$ and an exact sequence

$$\to F_m \to F_{m-1} \to \cdots \to F_0 \to 0,$$

where the F_m are free, we obtain a new projective resolution $E'_.(M)$ by setting $E'_m = E_m \oplus F_m$. We shall prove later that any projective resolution may be so modified as to have $E'_m = 0$ for $m > n$ (Syzygy theorem).

Proposition. 1. *Projective resolutions exist*;

2. *Given* $\varphi : \mathrm{M} \to \mathrm{N}$ *and projective resolutions* $\mathrm{E_.(M)}$ *and* $\mathrm{E_.(N)}$, *we may find a mapping of complexes* $\Phi : \mathrm{E_.(M)} \to \mathrm{E_.(N)}$ *inducing* φ *in the sense of the*

commutative diagram

$$\begin{array}{ccc} H_0(E.(M)) & \xrightarrow{\Phi_*} & H_0(E.(N)) \\ \wr\| & & \wr\| \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

3. Φ *is unique up to homotopy; and*
4. *If* $0 \to M' \to M \to M'' \to 0$ *is exact, then we may choose projective resolutions and mappings of complexes so that* $0 \to E.(M') \to E.(M) \to E.(M'') \to 0$ *is an exact sequence of complexes.*

Proofs. Since we have proved that the kernel of any surjective map $\mathcal{O}^{(k)} \to M \to 0$ is a finitely generated $\mathcal{O}$-module, assertion 1 follows.

The proofs of assertions 2–4 are all similar, but with increasing complexity of notation. We shall therefore only prove assertion 2, leaving 3 and 4 for the reader to carry out or look up in the references.

Given the solid arrow in the diagram

$$\begin{array}{ccccc} E_0 & \to & M & \to & 0 \\ \Big\downarrow{\scriptstyle \Phi_0} & & \Big\downarrow{\scriptstyle \varphi} & & \\ F_0 & \to & N & \to & 0 \end{array}$$

the dotted arrow Φ_0 exists by the definition of projective. Proceeding to the next step, if R_0 and S_0 are defined by

$$\begin{array}{ccccccccc} 0 & \to & R_0 & \to & E_0 & \to & M & \to & 0 \\ & & \Big\downarrow{\scriptstyle \Phi_0} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow & & \\ 0 & \to & S_0 & \to & F_0 & \to & N & \to & 0 \end{array}$$

then what we have is the solid arrows in the diagram

$$\begin{array}{ccccc} E_1 & \to & R_0 & \to & 0 \\ \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \Phi_0} & & \\ F_1 & \to & S_0 & \to & 0 \end{array}$$

and the dotted arrow fills in by projectivity. Continuing in this manner gives assertion 2. Q.E.D.

DEFINITION. Given finitely generated $\mathcal{O}$-modules M and N,

$$\begin{cases} \operatorname{Ext}^n_{\mathcal{O}}(M,N) = H^n(\operatorname{Hom}_{\mathcal{O}}(E.(M),N)), \\ \operatorname{Tor}^{\mathcal{O}}_n(M,N) = H_n(E.(M) \otimes_{\mathcal{O}} N). \end{cases}$$

We shall derive the main properties of Ext, for the most part leaving the analogous properties of Tor to the reader. We first note that by 2 and 3 Ext is well-defined independently of the projective resolution $E_.(M)$. More generally, maps

$$\varphi: M \to M', \qquad \psi: N \to N'$$

induce

$$\Phi^*: \operatorname{Ext}_{\mathcal{O}}(M',N) \to \operatorname{Ext}_{\mathcal{O}}(M,N),$$
$$\Psi_*: \operatorname{Ext}^*_{\mathcal{O}}(M,N) \to \operatorname{Ext}_{\mathcal{O}}(M,N'),$$

with functoriality properties such as

$$M'' \xrightarrow{\gamma} M' \xrightarrow{\varphi} M, \quad \lambda: M'' \to M$$
$$\Downarrow$$
$$\Lambda^* = \Phi^* \circ \Gamma^*.$$

Thus, $\operatorname{Ext}^*_{\mathcal{O}}(M,N)$ *is a functor contravariant in* M *and covariant in* N.

Next, we note that the definitions of Ext and Tor are not symmetric in M and N. For Tor this may be rectified as follows: Take projective resolutions $E_.(M)$ and $F_.(N)$ for M and N and consider the double complex

$$(E_.(M)\otimes F_.(N), \partial_M \otimes 1 \pm 1 \otimes \partial_N).$$

Recall from Chapter 3 that there are two spectral sequences with

$$'E_1 = H_*(E_.(M)\otimes_{\mathcal{O}} F_.(N), 1\otimes\partial_N),$$
$$''E_1 = H_*(E_.(M)\otimes_{\mathcal{O}} F_.(N), \partial_M\otimes 1),$$

both of which abut to the hypercohomology

$$H_*(E_.(M)\otimes_{\mathcal{O}} F_.(N), \partial_M\otimes 1 \pm 1\otimes\partial_N).$$

Since tensoring with a free module preserves exact sequences, $'E_1^{p,q}=0$ for $q>0$ and $''E_1^{p,q}=0$ for $p>0$. Thus both spectral sequences are trivial, and we deduce that

$$H_*(E_.(M)\otimes_{\mathcal{O}} N) \cong \operatorname{Tor}^{\mathcal{O}}_*(M,N) \cong H_*(M\otimes_{\mathcal{O}} F_.(N)).$$

For Ext the situation is more complicated and necessitates discussing injective resolutions for the second factor. Since this will not be required for our discussion, we won't get into these matters.

Next, we observe that

$$(*) \qquad \operatorname{Ext}^0_{\mathcal{O}}(M,N) \cong \operatorname{Hom}_{\mathcal{O}}(M,N).$$

Proof. If $E_1 \to E_0 \to M \to 0$ is exact, then so is

$$0 \to \mathrm{Hom}_{\mathcal{O}}(M,N) \to \mathrm{Hom}_{\mathcal{O}}(E_0,N) \to \mathrm{Hom}_{\mathcal{O}}(E_1,N). \qquad \text{Q.E.D.}$$

The main property of the Ext functor is:

Short exact sequences of $\mathcal{O}$*-modules*

$$\begin{cases} 0 \to M' \to M \to M'' \to 0, \\ 0 \to N' \to N \to N'' \to 0, \end{cases}$$

induce long exact sequences

$$\begin{cases} \cdots \to \mathrm{Ext}^n_{\mathcal{O}}(M,N) \to \mathrm{Ext}^n_{\mathcal{O}}(M',N) \to \mathrm{Ext}^{n+1}_{\mathcal{O}}(M'',N) \to \cdots, \\ \cdots \to \mathrm{Ext}^n_{\mathcal{O}}(M,N) \to \mathrm{Ext}^n_{\mathcal{O}}(M,N'') \to \mathrm{Ext}^{n+1}_{\mathcal{O}}(M,N') \to \cdots, \end{cases}$$

of Ext's.

Proof. First we note that a short exact sequence of free $\mathcal{O}$-modules splits, as indicated by the dotted arrow in the diagram

$$0 \to E' \to E \to E'' \to 0.$$

Thus $E \cong E' \oplus E''$, and consequently

$$0 \to \mathrm{Hom}_{\mathcal{O}}(E'',N) \to \mathrm{Hom}_{\mathcal{O}}(E,N) \to \mathrm{Hom}_{\mathcal{O}}(E',N) \to 0$$

is exact for any $\mathcal{O}$-module N. Choosing projective resolutions so that

$$0 \to E_{.}(M') \to E_{.}(M) \to E_{.}(M'') \to 0$$

is exact, it follows that

$$0 \to \mathrm{Hom}_{\mathcal{O}}(E_{.}(M''),N) \to \mathrm{Hom}_{\mathcal{O}}(E_{.}(M),N) \to \mathrm{Hom}_{\mathcal{O}}(E_{.}(M'),N) \to 0$$

is an exact sequence of complexes, and this gives the first long exact sequence. The second one is even simpler. Q.E.D.

For example, given $0 \to N' \to N \to N'' \to 0$, we obtain

(**) $0 \to \mathrm{Hom}_{\mathcal{O}}(M,N') \to \mathrm{Hom}_{\mathcal{O}}(M,N) \to \mathrm{Hom}_{\mathcal{O}}(M,N'') \to \mathrm{Ext}^1_{\mathcal{O}}(M,N')$,

so that $\mathrm{Ext}^1_{\mathcal{O}}(M,\cdot)$ measures the extent to which $\mathrm{Hom}_{\mathcal{O}}(M,\cdot)$ fails to be right-exact.

We next shall prove:

$\mathrm{Ext}^q_{\mathcal{O}}(M,N)=0$ *for* $q>0$ *and every* $\mathcal{O}$*-module* $N \Leftrightarrow M$ *is projective.*

Proof. Clearly if M is projective, the higher Ext's are zero.

Conversely, suppose that $\mathrm{Ext}^1_{\mathcal{O}}(M,N)=0$ for all N, and consider a diagram

$$\begin{array}{ccccccccc} & & & & M & & & & \\ & & & & {\scriptstyle\beta}\downarrow & \searrow{\scriptstyle\alpha} & & & \\ 0 & \to & N & \to & P & \to & Q & \to & 0\end{array}$$

in which the solid arrows are given. Applying (**) above and $\operatorname{Ext}^1_{\mathcal{O}}(M,N) = 0$ gives

$$\operatorname{Hom}_{\mathcal{O}}(M,P) \to \operatorname{Hom}_{\mathcal{O}}(M,Q) \to 0,$$

so that the dotted arrow β can be filled in. Consequently M is projective. Q.E.D.

Finally we shall refine this to:

$\operatorname{Ext}^1_{\mathcal{O}}(\mathrm{M},\mathrm{E}) = 0$ *for all projective modules* $\mathrm{E} \Leftrightarrow \mathrm{M}$ *is projective.*

Proof. Suppose that $\operatorname{Ext}^1_{\mathcal{O}}(M,E)$ for all projective (=free) modules E. Choosing generators for M, we obtain a short exact sequence

$$0 \to R \overset{\pi}{\rightleftarrows} E \to M \to 0$$

with E free. Applying the other long exact sequence of Ext's gives

$$\operatorname{Hom}_{\mathcal{O}}(E,E) \to \operatorname{Hom}_{\mathcal{O}}(R,E) \to \operatorname{Ext}^1_{\mathcal{O}}(M,E) = 0.$$

Consequently the dotted arrow π exists, and $M \cong \ker \pi$, $E \cong R \oplus M$. Since a direct summand of a projective module is again projective, we are done. Q.E.D.

In closing we should like to comment that the name "Ext" suggests extensions, and it was in this context that Ext^1 was first defined. We shall discuss this in Section 4, where it will arise quite naturally in context.

Another possible interpretation of Ext is pertaining to some sort of duality, since it reflects the properties of passing from an $\mathcal{O}$-module M to the "dual" $\mathcal{O}$-module $\operatorname{Hom}_{\mathcal{O}}(M,\cdot)$. This will be made precise in the next section, where in particular the local duality theorem will be put in intrinsic form.

Thus Ext has two quite different faces, each interesting in its own right.

The Koszul Complex and Applications

Koszul Complex. We continue using the notation $\mathcal{O}$ for the local ring of germs of analytic functions defined in some neighborhood of the origin in $\mathbb{C}^n$. Suppose $f_1,\dots,f_r \in \mathcal{O}$; denote by $I_k = \{f_1,\dots,f_k\}$ the ideal generated by the first k functions, and set $I = I_r$.

DEFINITION. $(f_1,\ldots,f_r)$ is a *regular sequence* if f_k is not a zero divisor in $\mathcal{O}/I_{k-1}$ for $k=1,\ldots,r$.

We recall the geometric interpretation, mentioned above and proved in the case $n=2$, that this is equivalent to $\operatorname{codim} V_k = k$, where $V_k = \{f_1(z) = \cdots = f_k(z) = 0\}$.

Given a regular sequence, the Koszul complex will give a particularly nice projective resolution of the $\mathcal{O}$-module I. It is modeled on the well-known fact from linear algebra that, for an n-dimensional vector space V and nonzero vector $v^* \in V^*$, the *contraction operator*

$$i(v^*):\ \wedge^k V \to \wedge^{k-1} V$$

induces an exact sequence of vector spaces

$$0 \to \wedge^n V \to \wedge^{n-1} V \to \cdots \to \wedge^2 V \to V \to \mathbb{C} \to 0$$

$(\mathbb{C} = \Lambda^0 V)$. The basis for our intrinsic formulation of local duality is that, under the identifications

$$\operatorname{Hom}(\wedge^k V, \mathbb{C}) \cong \wedge^k V^* \cong \wedge^n V^* \otimes \wedge^{n-k} V,$$

the above sequence is *self-dual* in the sense that the diagram

$$\begin{array}{ccc}
\operatorname{Hom}(\wedge^k V, \mathbb{C}) & \xrightarrow{\sim} & \wedge^n V^* \otimes \wedge^{n-k} V \\
\downarrow{\scriptstyle i(v^*)^*} & & \downarrow{\scriptstyle 1 \otimes i(v^*)} \\
\operatorname{Hom}(\wedge^{k+1} V, \mathbb{C}) & \xrightarrow{\sim} & \wedge^n V^* \otimes \wedge^{n-(k+1)} V
\end{array}$$

is commutative.

Now to construct the Koszul complex let $e_1,\ldots,e_k$ be the standard basis for $\mathbb{C}^r$ and set

$$\begin{cases} E_k = \mathcal{O} \otimes_{\mathbb{C}} \wedge^k \mathbb{C}^r, \\ e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}, \quad J = (j_1,\ldots,j_k) \subset (1,\ldots,r). \end{cases}$$

Then E_k is a free $\mathcal{O}$-module with basis $\{e_J\}$, and we define

$$E_k \xrightarrow{\partial} E_{k-1}$$

by $\mathcal{O}$-linearity and the usual boundary formula

$$\partial(e_J) = \sum_{\nu=1}^{k} (-1)^{\nu-1} f_{j_\nu} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_\nu} \wedge \cdots \wedge e_{j_k}.$$

For $k=1$ we set $E_0 = \mathcal{O}$ and $\partial(e_i) = f_i$. This defines the *Koszul complex* $E_\cdot(f)$ for any set of functions $f = (f_1,\ldots,f_r)$, and we have the

Lemma. *In case* $(f_1,\ldots,f_r)$ *is a regular sequence,*

$$H_q(E_\cdot(f)) = 0 \qquad \textit{for } q > 0$$

and

$$H_0(E_.(f)) \cong \mathcal{O}/I.$$

Consequently, $E_.(f)$ *gives a projective resolution of* $\mathcal{O}/I$.

Proof. It is clear that the image of $E_1 \xrightarrow{\partial} E_0 = \mathcal{O}$ is just the ideal I, so that $H_0(E_.(f)) \cong \mathcal{O}/I$. We shall prove by induction on r that the higher homology is zero.

In case $r=1$ the Koszul complex is $0 \to \mathcal{O} \xrightarrow{f_1} \mathcal{O}$ since $f_1 \neq 0$, and the result is clear.

Now we assume the result for $r-1$, and let $F_k \subset E_k$ be induced by the inclusion $\wedge^k \mathbb{C}^{r-1} \subset \wedge^k \mathbb{C}^r$, where $\mathbb{C}^{r-1}$ is spanned by $e_1, \ldots, e_{r-1}$. There results a big commutative diagram:

$$\begin{array}{lccccccccc}
 & & & 0 & & 0 & & 0 & & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_.: & & 0 \to & F_{r-1} & \to & F_{r-2} & \to \cdots \to & F_1 & \to & I_{r-1} \to 0 \\
 & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_.: & 0 \to & E_r & \to E_{r-1} & \to & E_{r-2} & \to \cdots \to & E_1 & \to & I_r \to 0 \\
 & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Q_.: & 0 \to & Q_r & \to Q_{r-1} & \to & Q_{r-2} & \to \cdots \to & Q_1 & \xrightarrow{\alpha} & I_r/I_{r-1} \to 0 \\
 & & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & 0 & & 0 & & 0 & & 0
\end{array}$$

We make the identification

$$Q_k \cong \mathcal{O}\left(e_r \otimes \wedge^{k-1}\mathbb{C}^{r-1}\right).$$

This being done, for $J=(j_1,\ldots,j_{k-1}) \subset (1,\ldots,r-1)$.

$$\begin{aligned} \partial(e_r \otimes e_J) &= f_r e_J \pm e_r \otimes \partial e_J \\ &\equiv e_r \otimes \partial e_J \text{ modulo } F_k. \end{aligned}$$

Thus $Q_.$ is again a Koszul complex, to which the induction assumption applies.

We now examine the lower right-hand corner

$$\begin{array}{ccc}
 & E_1 \to & I_r \to 0 \\
 & \downarrow & \downarrow \\
Q_2 \xrightarrow{\partial} & Q_1 \xrightarrow{\alpha} & I_r/I_{r-1} \to 0 \\
 & \downarrow & \downarrow \\
 & 0 & 0
\end{array}$$

Under the identifications

$$Q_2 \cong \mathcal{O}(e_r \otimes \mathbb{C}^{r-1}), \qquad \partial(e_r \otimes e_j) = f_j e_r,$$
$$Q_1 \cong \mathcal{O} \cdot e_r, \qquad \alpha(g e_r) = g f_r,$$

the diagram is commutative. If $\alpha(ge_r)=0$, then $gf_r \in \{f_1,\ldots,f_{r-1}\}$, and so $g\in\{f_1,\ldots,f_{r-1}\}$ by the regular sequence assumption. Thus $ge_r \in \partial Q_2$, and so the big diagram is commutative and exact. Since $H_*(F_.)=0=H_*(Q_.)$, we deduce that $H_*(E_.)=0$ as desired. Q.E.D.

Intrinsic Form of Local Duality. We shall use the Koszul complex to compute $\mathrm{Ext}^*_{\mathcal{O}}(\mathcal{O}/I,\mathcal{O})$, and then interpret the result as an intrinsic form of the duality theorem from Section 2 above. In fact, we shall reprove the duality theorem in this new form.

The following plays an analogous role to the $*$-operator in Hodge theory:

Lemma. *There are isomorphisms* $Hom_{\mathcal{O}}(E_k,\mathcal{O})\cong E_{r-k}$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}}(E_k,\mathcal{O}) & \xrightarrow{\sim} & E_{r-k} \\ \downarrow{\scriptstyle \partial^*} & & \downarrow{\scriptstyle \partial} \\ \mathrm{Hom}_{\mathcal{O}}(E_{k+1},\mathcal{O}) & \xrightarrow{\sim} & E_{r-k-1} \end{array}$$

is commutative.

Proof. For an index set $J\subset(1,\ldots,r)$, we let $J^0=(1,\ldots,r)-J$ be the complementary index set and define

$$e_{J*} = \pm e_{J^0} \in E_{r-k}.$$

The sign is chosen to make $e_J\wedge e_{J*}=e_1\wedge\cdots\wedge e_r$. Then we define $\hat{e}_J\in \mathrm{Hom}_{\mathcal{O}}(E_k,\mathcal{O})$ by

$$\hat{e}_J(e_{J'}) = \begin{cases} 0, & J\neq J', \\ 1, & J=J'. \end{cases}$$

The isomorphism in the lemma is given by

$$\hat{e}_J \to e_{J*}.$$

It is a direct computation that the diagram is commutative. Q.E.D.

Applying this lemma gives the first part of the

Proposition. *Suppose that* $\mathrm{f}=(\mathrm{f}_1,\ldots,\mathrm{f}_r)$ *is a regular sequence generating an ideal* $\mathrm{I}=\mathrm{I(f)}$. *Then*

$$(*)\qquad \mathrm{Ext}^k{}_{\mathcal{O}}(\mathcal{O}/I,\mathcal{O}) = 0,\quad k<r;\qquad \mathrm{Ext}^r_{\mathcal{O}}(\mathcal{O}/I,\mathcal{O})\cong \mathcal{O}/I.$$

The second isomorphism has the following functoriality property: Suppose that $\mathrm{I}'=\mathrm{I(f')}$ *is a regular ideal contained in* I, *so that*

$$f_i' = \sum_j a_{ij} f_j.$$

Denote by $\Delta = det(a_{ij})$ *the determinant of the matrix* (a_{ij}). *Then the diagram*

$$(**) \qquad \begin{array}{ccc} \operatorname{Ext}^r_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O}) & \xrightarrow{\sim} & \mathcal{O}/I \\ \downarrow & & \downarrow \Delta \\ \operatorname{Ext}^r_{\mathcal{O}}(\mathcal{O}/I', \mathcal{O}) & \xrightarrow{\sim} & \mathcal{O}/I' \end{array}$$

is commutative. Moreover, the vertical map is injective.

Proof. The computation of the $\operatorname{Ext}^k_{\mathcal{O}}(\mathcal{O}/I, \mathcal{O})$ follows from the previous lemma. To prove the transformation formula (**), we shall define mappings

$$\begin{array}{cccccccccccccc} E.(f): 0 \to & E_r & \to & E_{r-1} & \to \cdots \to & E_k & \to \cdots \to & E_1 & \to & E_0 & \to & \mathcal{O}/I & \to 0 \\ & \uparrow A_r & & \uparrow A_{r-1} & & \uparrow A_k & & \uparrow A_1 & & \uparrow A_0 & & \uparrow & \\ E.(f'): 0 \to & E'_r & \to & E'_{r-1} & \to \cdots \to & E'_k & \to \cdots \to & E'_1 & \to & E'_0 & \to & \mathcal{O}/I' & \to 0 \end{array}$$

between the Koszul complexes as follows: The map $\mathcal{O}/I' \to \mathcal{O}/I$ is the natural map induced by the inclusion $I' \subset I$, and A_0 is the identity under the identifications $E_0 \cong \mathcal{O} \cong E'_0$. $A_1 : E'_1 \to E_1$ is defined by

$$A_1(e'_i) = \sum_j a_{ij} e_j,$$

so that

$$\partial A_1(e'_i) = \sum a_{ij} f_j = f'_i = A_0(\partial e'_i).$$

The remaining maps $A_k : E'_k \to E_k$ are the kth exterior powers of A_1. The diagram is then commutative. Under the identifications

$$E_r \cong \mathcal{O}, \qquad E'_r \cong \mathcal{O},$$

A_r is just the determinant Δ.

It remains to prove the injectivity. Before doing this, it might be instructive to verify directly that

$$\Delta \cdot I \subset I',$$

so that multiplication by Δ induces a map $\mathcal{O}/I \xrightarrow{\Delta} \mathcal{O}/I'$ going in the nonobvious direction. By Cramer's rule

$$\Delta \delta_{ij} = \sum_k a_{kj} A_{ik},$$

where A_{ik} is $\pm$ the (i,k)th cofactor of (a_{ij}). Then

$$\Delta f_i = \sum_j \Delta \delta_{ij} f_j = \sum_{j,k} a_{kj} f_j A_{ik} = \sum_k A_{ik} f'_k,$$

so that $\Delta I \subset I'$ as desired.

Consider now the exact sequence

$$0 \to I/I' \to \mathcal{O}/I' \to \mathcal{O}/I \to 0.$$

The long exact sequence of Ext's gives

$$\to \mathrm{Ext}^k_{\mathcal{O}}(\mathcal{O}/I', \mathcal{O}) \to \mathrm{Ext}^k_{\mathcal{O}}(I/I', \mathcal{O}) \to \mathrm{Ext}^{k+1}(\mathcal{O}/I, \mathcal{O}) \to \cdots.$$

Our desired injectivity thus follows from

$$\mathrm{Ext}^{r-1}_{\mathcal{O}}(I/I', \mathcal{O}) = 0,$$

which we now prove.

Set $I'_k = \{f'_1, \ldots, f'_k\}$ and consider the following array of sequences of $\mathcal{O}$-modules, which are exact by the regular sequence property of the f'_i:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{O} & \xrightarrow{f'_1} & \mathcal{O} & \longrightarrow \mathcal{O}/I'_1 \longrightarrow 0 \\
0 \longrightarrow & \mathcal{O}/I'_1 & \xrightarrow{f'_2} & \mathcal{O}/I'_1 & \longrightarrow \mathcal{O}/I'_2 \longrightarrow 0 \\
 & \vdots & & \vdots & \vdots \\
0 \longrightarrow & \mathcal{O}/I'_{r-1} & \xrightarrow{f'_r} & \mathcal{O}/I'_{r-1} & \longrightarrow \mathcal{O}/I' \longrightarrow 0
\end{array}$$

Applying the exact sequences of Ext in the second variable gives

$$\begin{array}{ccccc}
\mathrm{Ext}^{r-2}_{\mathcal{O}}(I/I', \mathcal{O}/I'_1) \to & \mathrm{Ext}^{r-1}_{\mathcal{O}}(I/I', \mathcal{O}) & \xrightarrow{f'_1} & \mathrm{Ext}^{r-1}_{\mathcal{O}}(I/I', \mathcal{O}) \\
\mathrm{Ext}^{r-3}_{\mathcal{O}}(I/I', \mathcal{O}/I'_2) \to & \mathrm{Ext}^{r-2}_{\mathcal{O}}(I/I', \mathcal{O}/I'_1) & \xrightarrow{f'_2} & \mathrm{Ext}^{r-2}_{\mathcal{O}}(I/I', \mathcal{O}/I'_1) \\
\vdots & \vdots & & \vdots \\
\mathrm{Ext}^{0}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{r-1}) \to & \mathrm{Ext}^{1}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{r-2}) & \xrightarrow{f'_{r-1}} & \mathrm{Ext}^{1}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{r-2})
\end{array}$$

Now, and this is the point, the maps

$$\mathrm{Ext}^{r-k}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{k-1}) \xrightarrow{f'_k} \mathrm{Ext}^{r-k}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{k-1})$$

are all zero, since these maps are $\mathcal{O}$-linear, and therefore the multiplication by f'_k can be moved from the factor $\mathcal{O}/I'_{k-1}$ to I/I', where it is zero. Putting this all together, we obtain a surjective map

$$\mathrm{Hom}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{r-1}) \to \mathrm{Ext}^{r-1}_{\mathcal{O}}(I/I', \mathcal{O}) \to 0.$$

If $\varphi \in \mathrm{Hom}_{\mathcal{O}}(I/I', \mathcal{O}/I'_{r-1})$, then for any $g \in I/I'$

$$f'_r \varphi(g) = \varphi(f'_r g) = 0 \Rightarrow \varphi(g) = 0$$
$$\Rightarrow \mathrm{Ext}^{r-1}_{\mathcal{O}}(I/I', \mathcal{O}) = 0. \qquad \text{Q.E.D.}$$

We now return to the local duality theorem. Let $I=\{f_1,\ldots,f_n\}$ and $I'=\{f_1',\ldots,f_n'\}$ be regular ideals with $I'\subset I$, and denote by Ω^n the stalk at the origin of the sheaf of holomorphic n-forms. A choice of coordinates $z_1,\ldots,z_n$ near the origin induces an isomorphism

$$\mathcal{O} \cong \Omega^n$$

given by

$$g(z) \to g(z)\,dz_1 \wedge \cdots \wedge dz_n.$$

We recall that the pairing

$$\mathrm{res}_f\colon\ \mathcal{O}/I \otimes \mathcal{O}/I \to \mathbb{C}$$

defined by

$$\mathrm{res}_f(h,g) = \mathrm{Res}_{\{0\}}\left(\frac{g(z)h(z)\,dz_1\wedge\cdots\wedge dz_n}{f_1(z)\cdots f_n(z)}\right)$$

depends on the choice of generators f_i for I and local coordinates $z_1,\ldots,z_n$. The behavior of res_f under changing generators for I or changing local coordinates is given by the transformation formula. This brings us to the

Local Duality Theorem II. *For regular ideals* $\mathrm{I}=\{\mathrm{f}_1,\ldots,\mathrm{f}_n\}$ *the pairing*

$$\mathrm{res}\colon\ \mathcal{O}/I \otimes \mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I,\Omega^n) \to \mathbb{C}$$

defined by the isomorphism (*) *in the previous proposition and residue* $\mathrm{res}_{\mathrm{f}}$ *is nondegenerate, independent of the choice of local coordinates and generators of* I, *and functorial in the sense that the diagram*

$$\begin{array}{ccc} \mathcal{O}/I \otimes \mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I,\Omega^n) & \xrightarrow{\mathrm{res}} & \mathbb{C} \\ \uparrow\pi \qquad\qquad \downarrow\pi^* & & \| \\ \mathcal{O}/I' \otimes \mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I',\Omega^n) & \xrightarrow{\mathrm{res}} & \mathbb{C} \end{array}$$

is commutative for regular ideals $\mathrm{I}'\subset\mathrm{I}$.

Proof. The independence of the pairing "res" from choice of local coordinates and generators for I, together with the functoriality, all follow from the commutative diagram (**) and transformation formula. To show that the pairing $\mathcal{O}/I\otimes\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I,\Omega^n)\xrightarrow{\mathrm{res}}\mathbb{C}$ is nondegenerate, it will suffice to find a regular ideal $I'\subset I$ for which the pairing is nondegenerate, and then use the functoriality together with the facts that $\mathcal{O}/I'\xrightarrow{\pi}\mathcal{O}/I$ is surjective (obvious) and $\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I,\Omega^n)\xrightarrow{\pi^*}\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{O}/I',\Omega^n)$ is injective, which was proved in the previous proposition. Appealing to nullstellensatz in Section 1, we may take $I'=\{z_1^d,\ldots,z_n^d\}$, where $d=\dim_{\mathbb{C}}(\mathcal{O}/I)$. Q.E.D.

Tor and the Syzygy Theorem. Having used Ext to put the local duality theorem in final form, we shall use Tor to prove the

Syzygy Theorem. *Let* M *be a finitely generated* $\mathcal{O}$*-module and*

$$0 \to F \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$$

an exact sequence of $\mathcal{O}$*-modules where the* E_k *are projective (=free). Then* F *is also projective.*

Let $m=(z_1,\ldots,z_n)$ be the maximal ideal and $\mathbb{C}=\mathcal{O}/m$ considered as an $\mathcal{O}$-module. We begin by proving:

Lemma. $\mathrm{Tor}_1^{\mathcal{O}}(\mathbb{C},N)=0 \Rightarrow N$ *is a free* $\mathcal{O}$*-module.*

Proof. We first remark that

$$\mathrm{Tor}_0^{\mathcal{O}}(M,N) \cong M \otimes_{\mathcal{O}} N.$$

This is because $\otimes$ is right-exact, so that $E_1 \to E_0 \to M$ gives $E_1 \otimes_{\mathcal{O}} N \to E_0 \otimes_{\mathcal{O}} N \to M \otimes_{\mathcal{O}} N \to 0$.

Next, we note that short exact sequences

$$\begin{cases} 0 \to M' \to M \to M'' \to 0 \\ 0 \to N' \to N \to N'' \to 0 \end{cases}$$

give rise to long exact sequences of Tor's

$$\begin{cases} \cdots \to \mathrm{Tor}_k^{\mathcal{O}}(M,N) \to \mathrm{Tor}_k^{\mathcal{O}}(M'',N) \to \mathrm{Tor}_{k-1}^{\mathcal{O}}(M',N) \to \cdots \\ \cdots \to \mathrm{Tor}_k^{\mathcal{O}}(M,N) \to \mathrm{Tor}_k^{\mathcal{O}}(M,N'') \to \mathrm{Tor}_{k-1}^{\mathcal{O}}(M,N') \to \cdots \end{cases}$$

for the same reason as for Ext.

Now to prove the lemma. Observe that

$$M \otimes_{\mathcal{O}} \mathbb{C} \cong M/mM = M_0$$

is the fiber of M. Choose a free $\mathcal{O}$-module E such that $E_0 \cong M_0$. By the Nakayama lemma, we may extend this isomorphism to a surjective map $E \to M \to 0$. Let R be the module of relations defined by

$$0 \to R \to E \to M \to 0.$$

By the exact sequence of $\mathrm{Tor}_k^{\mathcal{O}}(\mathbb{C},\cdot)$,

$$\begin{array}{ccccccc} \mathrm{Tor}_1^{\mathcal{O}}(\mathbb{C},M) & \to & \mathbb{C}\otimes_{\mathcal{O}} R & \to & \mathbb{C}\otimes_{\mathcal{O}} E & \to & \mathbb{C}\otimes_{\mathcal{O}} M \to 0 \\ \| & & \| & & \| & & \| \\ 0 & \to & R_0 & \to & E_0 & \xrightarrow{\sim} & M_0 \end{array}$$

$$\Rightarrow R_0 = 0$$

$$\Rightarrow R = 0$$

by Nakayama again. Thus $E \cong M$ and the lemma is proved.

Now we can prove the syzygy theorem. For $0 \leqslant k \leqslant n$, we define R_k by

$$\begin{cases} R_0 = M, \\ R_k = \text{image } E_k \to E_{k-1} \quad (0 < k < n), \\ R_n = F. \end{cases}$$

Then we have short exact sequences

$$\begin{cases} 0 \to R_1 \to E_0 \to R_0 \to 0, \\ 0 \to R_{k+1} \to E_k \to R_k \to 0 \quad (0 < k < n), \\ 0 \to R_n \to E_{n-1} \to R_{n-1} \to 0. \end{cases}$$

Since higher Tor's are zero if one of the modules is free, the long exact sequence in the second factor gives

$$\operatorname{Tor}^{\mathcal{O}}_{q+1}(\mathbb{C}, R_{k-1}) \cong \operatorname{Tor}^{\mathcal{O}}_q(\mathbb{C}, R_k), \qquad q \geqslant 1.$$

In particular

$$\operatorname{Tor}^{\mathcal{O}}_1(\mathbb{C}, R_n) \cong \operatorname{Tor}^{\mathcal{O}}_{n+1}(\mathbb{C}, M).$$

To show that the right-hand side is zero, we let

$$K_. : 0 \to K_n \to K_{n-1} \to \cdots \to K_0 \to \mathcal{O}/m = \mathbb{C} \to 0$$

be the Koszul complex associated to the maximal ideal $m = \{z_1, \ldots, z_n\}$. Since

$$\operatorname{Tor}^{\mathcal{O}}_*(\mathbb{C}, M) = H_*(K_. \otimes_{\mathcal{O}} M),$$

it follows that $\operatorname{Tor}^{\mathcal{O}}_{n+k}(\mathbb{C}, M) = 0$ for $k \geqslant 1$. Q.E.D.

A Brief Tour Through Coherent Sheaves

Definitions and Elementary Properties. On an open set $U \subset \mathbb{C}^n$ we now denote by $\mathcal{O}$ the sheaf of holomorphic functions and by $\mathcal{O}_z = \lim_{z \in V} \mathcal{O}(V)$ the stalk of $\mathcal{O}$ at a point $z \in U$. A sheaf mapping $\mathcal{O}^{(p)} \xrightarrow{F} \mathcal{O}^{(q)}$ is given by a $(p \times q)$ matrix of holomorphic functions defined on U. We define the sheaf of $\mathcal{O}$-modules $\mathcal{R}$ by

$$0 \to \mathcal{R} \to \mathcal{O}^{(p)} \xrightarrow{F} \mathcal{O}^{(q)}.$$

In the discussion of local rings, we pointed out that, because of the Noetherian property of $\mathcal{O}_{z_0}$, $\mathcal{R}_{z_0}$ was a finitely generated $\mathcal{O}_{z_0}$-module. The following fundamental lemma is due to Oka:

Oka's Lemma. *The sheaf $\mathcal{R}$ is locally finitely generated as a sheaf of $\mathcal{O}$-modules. More precisely, if* $\mathrm{r}_1, \ldots, \mathrm{r}_m$ *are sections of $\mathcal{R}$ in a neighborhood* *of* z_0 *that generate the* $\mathcal{O}_{z_0}$*-module* $\mathcal{R}_{z_0}$*, then they generate the* $\mathcal{O}_z$*-modules* $\mathcal{R}_z$ *for* $\|z-z_0\|<\varepsilon$.

We shall not give here the proof of this lemma, which is found in the references given at the end of this chapter.

Taking $q=1$, $F=(f_1,\ldots,f_p)$ generates an ideal sheaf $I\subset\mathcal{O}$, and $\mathcal{R}$ is the sheaf of relations $\sum r_i f_i=0$ among the generators. Oka's lemma is therefore a sort of Noetherian property of $\mathcal{O}$, not just in each stalk but, so to speak, spread out over sufficiently small open sets. We note the similarity to the following, which was proved in Section 1 of Chapter 0:

If f *and* g *are holomorphic in* U *and are relatively prime in the local ring* $\mathcal{O}_{z_0}$*, then they are relatively prime in* $\mathcal{O}_z$ *for* $\|z-z_0\|<\varepsilon$.

The proof of this result used the Weierstrass division theorem, and the same is true of the proof of Oka's lemma.

Here is the basic

DEFINITION. Let M be a complex manifold with structure sheaf $\mathcal{O}$ and $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules. Then $\mathcal{F}$ is *coherent* if locally it has a presentation

$$\mathcal{O}^{(p)}\to\mathcal{O}^{(q)}\to\mathcal{F}\to 0.$$

In other words, $\mathcal{F}$ is coherent if, given any point $z_0\in M$, there is a neighborhood U of z_0 and finitely many sections of $\mathcal{F}|U$ that generate each $\mathcal{O}_z$-module $\mathcal{F}_z$ $(z\in U)$, and if moreover the relations among these generators are finitely generated over U.

Here are some remarks and examples. The gist throughout is that Oka's lemma allows properties in the local ring $\mathcal{O}_{z_0}$ to propagate to the same properties in nearby local rings $\mathcal{O}_z$. We shall refer to this as the *propagation principle*; it gives rise to the name *coherent*.

We begin by noting that:

Coherent sheaves admit local syzygies

$$0\to\mathcal{O}^{(k_n)}\to\mathcal{O}^{(k_{n-1})}\to\cdots\to\mathcal{O}^{(k_0)}\to\mathcal{F}\to 0.$$

Proof. By definition we have

$$\mathcal{O}^{(p)}\to\mathcal{O}^{(q)}\to\mathcal{F}\to 0$$

in a neighborhood U of z_0. Applying Oka's lemma, we find

$$\mathcal{O}^{(r)}\to\mathcal{O}^{(p)}\to\mathcal{O}^{(q)}\to\mathcal{F}\to 0$$

in a possibly smaller neighborhood U' of z_0. Applying it again, we obtain

$$\mathcal{O}^{(s)}\to\mathcal{O}^{(r)}\to\mathcal{O}^{(p)}\to\mathcal{O}^{(q)}\to\mathcal{F}\to 0$$

in $U'' \subset U'$. After at most n steps, the syzygy theorem assures us that the kernel on the left will have as stalk at z_0 a free $\mathcal{O}_{z_0}$-module, and this then gives our local syzygy. Q.E.D.

As an application we have:

For a coherent sheaf $\mathcal{F}$, *the cohomology sheaves*

$$\mathcal{H}^q(\mathcal{F}) = 0 \qquad \text{for } q > 0.$$

Proof. By the $\bar{\partial}$-Poincaré lemma, $\mathcal{H}^q(\mathcal{O}) = 0$ for $q > 0$. Arguing by induction on the length of a local syzygy, we may assume that we have

$$0 \to \mathcal{R} \to \mathcal{O}^{(k)} \to \mathcal{F} \to 0,$$

where $\mathcal{H}^q(\mathcal{R}) = 0$ for $q > 0$. Our result then follows by the exact sequence of cohomology. Q.E.D.

A further property of coherent sheaves we shall repeatedly use is:

Given an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of sheaves of $\mathcal{O}$*-modules in which two of the three are coherent, then the remaining one is also.*

The proof is just tedious checking of details and will be omitted.

Now we come to some examples. The simplest are those $\mathcal{F}$ such that locally $\mathcal{F} \cong \mathcal{O}^{(r)}$. Then $\mathcal{F}$ is said to be *locally free of rank* r, and such $\mathcal{F}$ are exactly the sheaves $\mathcal{O}(E)$, where $E \to M$ is a holomorphic vector bundle with fiber $\mathbb{C}^r$.

A subsheaf $I \subset \mathcal{O}$ that is locally finitely generated is called an *ideal sheaf* or *sheaf of ideals*. By Oka's lemma, these are always coherent, and because of the exact sequence

$$0 \to I \to \mathcal{O} \to \mathcal{O}/I \to 0$$

the same is true for $\mathcal{O}/I$. If locally $I = \{f_1, \ldots, f_m\}$ is generated by holomorphic functions $f_1, \ldots, f_m$, then the *support* of $\mathcal{O}/I$ is defined as

$$\begin{aligned} Z &= \operatorname{supp}(\mathcal{O}/I) \\ &= \{z \in M : I_z \neq \mathcal{O}_z\} \\ &= \{z \in M : f_1(z) = \cdots = f_m(z) = 0\}. \end{aligned}$$

As a point set Z is an analytic variety. However, this should be refined, and the pair $(Z, \mathcal{O}_Z)$ should be thought of as a space whose support is an analytic variety but whose *structure sheaf* $\mathcal{O}_Z = \mathcal{O}/I$ is a sheaf of rings possibly with nilpotent elements. These objects are called *Schemes*.

An ideal sheaf I that locally has a single generator f is locally free of rank one, and hence is of the form

$$I \cong \mathcal{O}(L^*)$$

for some holomorphic line bundle $L^* \to M$. Denoting by $D = (f)$ the divisor of f, we have previously used the notations $L^* = [-D]$ and $L = [D]$ for this line bundle and its dual. The sheaf $\mathcal{O}(D)$ is then the sheaf of meromorphic functions g with $(g) + D \geqslant 0$. In general, sheaves $\mathcal{O}(D)$ for a not necessarily effective divisor D are said to be *invertible*. The multiplicative group of invertible sheaves on an algebraic variety M is just $H^1(M, \mathcal{O}^*) = \operatorname{Pic}(M)$.

An ideal sheaf I is a *sheaf of regular ideals* if locally $I = \{f_1, \ldots, f_r\}$, where the f_i define a regular sequence in the local rings $\mathcal{O}_z$. For sheaves of regular ideals the Koszul complex from the preceding section provides an especially nice local syzygy. Later on we shall be especially concerned with the codimension-2 case. If locally $I = \{f_1, f_2\}$, then the Koszul complex gives the local syzygy

$$0 \to \mathcal{O} \xrightarrow{\lambda} \mathcal{O} \oplus \mathcal{O} \xrightarrow{\eta} \mathcal{O} \to \mathcal{O}/I \to 0,$$

where

$$\lambda(g) = (-f_2 g) \oplus (f_1 g),$$
$$\eta(g_1 \oplus g_2) = f_1 g_1 + f_2 g_2.$$

To illustrate how nilpotents arise geometrically, suppose that $Z \subset M$ is an irreducible subvariety defined by a sheaf of prime ideals $I \subset \mathcal{O}$. Then the ideal sheaves $I^{\mu+1}$ define spaces $Z_\mu = (Z, \mathcal{O}/I^{\mu+1})$, which may be thought of as the *μth infinitesimal neighborhood* of Z.

In this context let us reexamine the Reiss relation discussed in Section 2. Here $M = \mathbb{P}^2$ and $Z = L$ is a line. We denote by $\mathcal{O}_{(\mu)} = \mathcal{O}/I^{\mu+1}$ the structure sheaf of the μth infinitesimal neighborhood. The data of a second-order element of arc crossing L is equivalent to locally giving a section of $\mathcal{O}_{(2)}$ defined up to multiplication by units in $\mathcal{O}^*_{(2)}$.

Explicitly, let $\{U_\alpha\}$ be an open covering of a neighborhood of L in $\mathbb{P}^2$ such that in U_α we have holomorphic coordinates (z_α, w_α) with $L \cap U_\alpha$ defined by $w_\alpha = 0$. In $U_\alpha \cap U_\beta$, $z_\alpha = z_\alpha(z_\beta, w_\beta)$ and $w_\alpha = w_\alpha(z_\beta, w_\beta)$, where $w_\beta(z_\beta, 0) = 0$. In U_α the sections of the sheaf $\mathcal{O}_{(\mu)}$ are just the holomorphic functions $f_\alpha(z_\alpha, w_\alpha) = f_0(z_\alpha) + f_1(z_\alpha) w_\alpha + \cdots + f_\mu(z_\alpha) w_\alpha^\mu$ taken modulo $w_\alpha^{\mu+1}$. The data in the Reiss relation are given by $f_\alpha \in \mathcal{O}_{(2)}(U_\alpha)$ with $f_\alpha / f_\beta = f_{\alpha\beta} \in \mathcal{O}^*_{(2)}(U_\alpha \cap U_\beta)$. *Thus, giving the second-order elements of arc is the same as giving an invertible sheaf* $\mathcal{L}_{(2)} \in \mathrm{H}^1(\mathrm{L}_{(2)}, \mathcal{O}^*_{(2)})$ *and section* $\sigma_{(2)} \in \mathrm{H}^0(\mathcal{L}_{(2)})$.

We now ask when there is an invertible sheaf $\mathcal{L} \in H^1(\mathcal{O}^*_{\mathbb{P}^2})$ that restricts to $\mathcal{L}_{(2)}$. For this we consider the exact sequence

$$0 \to 1 + I^3 \to \mathcal{O}^*_{\mathbb{P}^2} \to \mathcal{O}^*_{(2)} \to 0,$$

where $I \cong \mathcal{O}_{\mathbb{P}^2}(-L)$ is the ideal sheaf of the line $L \subset \mathbb{P}^2$ and $1+I^3$ denotes the multiplicative sheaf of functions $1+f$, where f vanishes to third order along L. Clearly $1+I^3 \cong I^3$, and since $I^3 \cong \mathcal{O}_{\mathbb{P}^2}(-3)$,

$$H^1(\mathcal{O}_{\mathbb{P}^2}(-3L))=0 \qquad \text{by Kodaira vanishing,}$$

$$H^2(\mathcal{O}_{\mathbb{P}^2}(-3L)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C} \qquad \text{by Kodaira-Serre duality,}$$

$$H^2(\mathcal{O}_{\mathbb{P}^2}^*)=0,$$

where the last step follows from the cohomology sequence of $0 \to \mathbb{Z} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}^* \to 1$. Thus we obtain

$$0 \to H^1(\mathcal{O}_{\mathbb{P}^2}^*) \to H^1(\mathcal{O}_{(2)}^*) \to \mathbb{C} \to 0,$$

and consequently: *There is one condition on* $\mathcal{L}_{(2)} \in \mathrm{H}^1(\mathcal{O}_{(2)}^*)$ *to be the restriction of some* $\mathcal{L} \in \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^2}^*)$.

Assuming now this to be the case, we have $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(n)$ $(n>0)$, and the exact sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(n-3) \to \mathcal{O}_{\mathbb{P}^2}(n) \to \mathcal{O}(\mathcal{L}_{(2)}) \to 0$$

together with $H^1(\mathcal{O}_{\mathbb{P}^2}(n-3)) \cong H^1(\mathcal{O}_{\mathbb{P}^2}(-n))=0$ gives in cohomology

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^2}(n-3)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(n)) \to H^0(\mathcal{O}(\mathcal{L}_{(2)})) \to 0.$$

Combining this with the previous paragraph, we conclude: *There is exactly one condition that the second-order arcs* $\mathrm{C_i}$ $(\mathrm{i}=1,\ldots,\mathrm{n})$ *be cut out by an algebraic curve* C *in* $\mathbb{P}^2$. The Reiss relation gives this condition explicitly; this alternate approach illustrates the use of nilpotents.

As another example of nilpotents, we assume that $I \subset \mathcal{O}$ is a sheaf of regular ideals whose support Z consists of a finite set of points. Thus, locally $I=\{f_1,\ldots,f_n\}$, where $n=\dim_{\mathbb{C}} M$. Setting $\mathcal{O}_Z=\mathcal{O}/I$ the ringed space $(Z,\mathcal{O}_Z)$ consists of the points $P \in Z$ together with a finite- dimensional $\mathbb{C}$-algebra $\mathcal{O}_{Z,P}=\mathcal{O}_P/I_P$. The associated zero-cycle is $\sum_{P \in Z} \dim_{\mathbb{C}}(\mathcal{O}_{Z,P})P$. The space $(Z,\mathcal{O}_Z)$ contains more information than just the set of points $P \in Z$, even if we include the multiplicities $\dim_{\mathbb{C}}(\mathcal{O}_{Z,P})$.

Now we come to the question of *sheafifying* Ext and Tor. Recall from the section on homological algebra that we proved a proposition giving four properties of projective resolutions of $\mathcal{O}_z$-modules. The definition and basic properties of Ext and Tor for modules over local rings were formal consequences of this proposition. The point we wish to make here is this: *By the propagation principle, the same four properties in that proposition are valid locally for coherent sheaves.* For example, the first one, that projective resolutions exist, becomes that local syzygies exist, which we have already checked.

As a consequence, if one thinks matters through, the following conclusion emerges: *Given coherent sheaves* $\mathcal{F}$ *and* $\mathcal{G}$, *we may define sheaves*

$\underline{\mathrm{Ext}}^k_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ *and* $\underline{\mathrm{Tor}}^{\mathcal{O}}_k(\mathcal{F},\mathcal{G})$ *with the properties*:

1. $\begin{cases} \underline{\mathrm{Ext}}^k_{\mathcal{O}}(\mathcal{F},\mathcal{G})_x \cong \underline{\mathrm{Ext}}^k_{\mathcal{O}_x}(\mathcal{F}_x,\mathcal{G}_x), \\ \underline{\mathrm{Tor}}^{\mathcal{O}}_k(\mathcal{F},\mathcal{G})_x \cong \underline{\mathrm{Tor}}^{\mathcal{O}_x}_k(\mathcal{F}_x,\mathcal{G}_x); \end{cases}$
2. $\begin{cases} \underline{\mathrm{Ext}}^0_{\mathcal{O}}(\mathcal{F},\mathcal{G}) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{F},\mathcal{G}), \\ \underline{\mathrm{Tor}}^{\mathcal{O}}_0(\mathcal{F},\mathcal{G}) \cong \mathcal{F}\otimes_{\mathcal{O}}\mathcal{G}; \end{cases}$
3. *The exact sequences of* Ext *and* Tor *are valid*; *and*
4. $\underline{\mathrm{Ext}}^*_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ *and* $\underline{\mathrm{Tor}}^{\mathcal{O}}_*(\mathcal{F},\mathcal{G})$ *are coherent sheaves.*

The last property is because $\underline{\mathrm{Ext}}$ and $\underline{\mathrm{Tor}}$ fit into exact sequences where two out of three terms are coherent.

As an illustration of property 3, given an exact sequence $0\to\mathcal{G}'\to\mathcal{G}\to\mathcal{G}''\to 0$ of coherent sheaves, application of $\otimes_{\mathcal{O}}\mathcal{F}$ gives

$$\cdots\to\underline{\mathrm{Tor}}^{\mathcal{O}}_1(\mathcal{G},\mathcal{F})\to\underline{\mathrm{Tor}}^{\mathcal{O}}_1(\mathcal{G}'',\mathcal{F})\to\mathcal{G}'\otimes_{\mathcal{O}}\mathcal{F}\to\mathcal{G}\otimes_{\mathcal{O}}\mathcal{F}\to\mathcal{G}''\otimes_{\mathcal{O}}\mathcal{F}\to 0.$$

This sequence will prove useful in a little while.

Cohomology of Coherent Sheaves. Suppose now that M is a compact complex manifold and $\mathcal{F}$ is a coherent sheaf on M. The fundamental global fact is: *The cohomology* $H^*(M,\mathcal{F})$ *is a finite-dimensional vector space.*

Again, this is proved in the references listed at the end of this chapter. We shall not prove it here but will show how the finite dimensionality may be used to draw consequences in case M is a smooth algebraic variety.

Suppose then that $L\to M$ is a positive line bundle, which we may as well take to be the hyperplane bundle relative to a smooth projective embedding $M\subset\mathbb{P}^N$. We let $\mathcal{L}=\mathcal{O}(L)$ and set $\mathcal{F}(k)=\mathcal{F}\otimes_{\mathcal{O}}\mathcal{L}^k$. The sections of $\mathcal{F}(k)$ may be thought of as sections of $\mathcal{F}$ having poles of order k along a hyperplane. Consider the following two assertions:

Theorem A. *The global sections* $H^0(M,\mathcal{F}(k))$ *generate each* $\mathcal{O}_x$*-module* $\mathcal{F}(k)_x$ *for* $k\geq k_0$ and $x\in M$; *i.e.*,

$$H^0(M,\mathcal{F}(k))\to\mathcal{F}(k)_x/m_x\mathcal{F}(k)\to 0,$$

where $m_x\subset\mathcal{O}_x$ *is the maximal ideal.*

(The equivalence of the two statements in this theorem is the Nakayama lemma. In general, $\mathcal{F}_x/m_x\mathcal{F}_x$ may be called the *fiber* of the coherent sheaf $\mathcal{F}$ at $x\in M$.)

Theorem B. $H^q(M,\mathcal{F}(k))=0$ *for* $q>0$, $k\geq k_0$.

We have proved the finite dimensionality and also Theorems A and B in case $\mathcal{F}=\mathcal{O}(E)$ is locally free so that potential-theoretic methods may be used. Now we prove the assertions:

1. *Theorem A ⇒ Theorem B.*
2. *Theorem B ⇒ Theorem A.*
3. *dim* $H^1(M,\mathfrak{F})<\infty$ *for all coherent sheaves* $\mathfrak{F}$ ⇒ *Theorem A.*

Proof of 1. Assuming Theorem A, we have

$$0\to\mathcal{G}'\to\mathcal{O}^{(p)}\to\mathfrak{F}(k)\to 0$$

for some large k. Applying $\otimes\mathcal{L}^{-k}$ to this exact sequence and setting $\mathcal{G}=\mathcal{G}'(-k)$, we obtain

$$0\to\mathcal{G}\to\mathcal{E}_0\to\mathfrak{F}\to 0.$$

Now apply the same procedure to $\mathcal{G}$, and keep on going. After at most $n=\dim_{\mathbb{C}}M$ steps we arrive at a global syzygy

$$(*)\qquad 0\to\mathcal{E}_n\to\mathcal{E}_{n-1}\to\cdots\to\mathcal{E}_0\to\mathfrak{F}\to 0,$$

where each $\mathcal{E}_l$ is locally free. Note that for $0\leqslant l\leqslant n-1$, $\mathcal{E}_k$ is a direct sum of copies of $\mathcal{L}^{m_k}$, but all we can say about the last term is that $\mathcal{E}_n=\mathcal{O}(E)$ for some holomorphic vector bundle $E\to M$. The existence of such global syzygies for coherent sheaves is of fundamental importance.

Returning to the proof of 1, we have proved in Section 4 of Chapter 1 that $H^q(M,\mathcal{E}(k))=0$ for $\mathcal{E}$ locally free, $q>0$, and $k\geqslant k_0$. Applying this inductively on the length of a global syzygy gives Theorem B.

Proof of 2. For each $x_0\in M$, we have

$$0\to m_{x_0}\mathfrak{F}(k)\to\mathfrak{F}(k)\to\mathfrak{F}(k)_{x_0}/m_{x_0}\mathfrak{F}(k)\to 0,$$

where $m_{x_0}\subset\mathcal{O}$ is the sheaf of ideals given by the maximal ideal m_{x_0} at x_0. The fiber $\mathfrak{F}(k)_{x_0}/m_{x_0}\mathfrak{F}(k)$ is an example of a coherent sheaf supported at a point—these are sometimes called *skyscraper sheaves*. Now $m_{x_0}\mathfrak{F}(k)=(m_{x_0}\mathfrak{F})(k)=\mathcal{G}(k)$, where $\mathcal{G}=m_{x_0}\mathfrak{F}$ is a coherent sheaf. Using $H^1(M,\mathcal{G}(k))=0$ for $k\geqslant k_0$, we deduce that the global sections $H^0(M,\mathfrak{F}(k))$ generate the fiber of $\mathfrak{F}(k)$ at x_0 for $k\geqslant k_0$. By the Nakayama lemma, these global sections generate the $\mathcal{O}_{x_0}$-module $\mathfrak{F}(k)_{x_0}$ for $k\geqslant k_0$. By Oka's lemma they generate the $\mathcal{O}_x$-modules $\mathfrak{F}(k)_x$ for x near to x_0. The result now follows by the compactness of M.

Proof of 3. The proof is by induction on $n=\dim_{\mathbb{C}}M$. Given a point $x\in M$ and hyperplane ξ in the tangent space $T'_x(M)$, we may find a nonsingular hypersurface passing through x and with tangent plane ξ. Replacing $\mathcal{L}$ by $\mathcal{L}^k$, we may assume that this hypersurface is a hyperplane H. Then we have an exact sequence

$$0\to\mathcal{O}_M(-1)\xrightarrow{\sigma}\mathcal{O}_M\to\mathcal{O}_H\to 0,$$

where $\sigma\in H^0(M,\mathcal{O}(1))$ defines H and $\mathcal{O}_H$ is the usual structure sheaf on the complex manifold H. Applying $\otimes\mathcal{F}$ to this sequence gives ($\mathcal{O}=\mathcal{O}_M$)

$$0\to \underline{\mathrm{Tor}}_1^{\mathcal{O}}(\mathcal{O}_H,\mathcal{F})\to\mathcal{F}(-1)\to\mathcal{F}\to\mathcal{F}_H\to 0,$$

where $\mathcal{F}_H=\mathcal{F}\otimes_{\mathcal{O}_M}\mathcal{O}_H$ is a coherent sheaf of $\mathcal{O}_H$-modules, and where we used that $\underline{\mathrm{Tor}}_1^{\mathcal{O}}(\mathcal{O}_M,\mathcal{F})=0$. The sheaf $\mathcal{G}=\underline{\mathrm{Tor}}_1^{\mathcal{O}}(\mathcal{O}_H,\mathcal{F})$ is a coherent sheaf of $\mathcal{O}_M$-modules. Multiplying a section of $\mathcal{G}$ by σ gives zero, since $\sigma\cdot\mathcal{O}_H=0$, and so $\mathcal{G}=\mathcal{G}_H$ is a coherent sheaf of $\mathcal{O}_H$-modules.

Now apply $\otimes^k\mathcal{L}$ to this sequence. Since locally $\mathcal{L}\cong\mathcal{O}$, exactness is preserved and we obtain

$$(**)\qquad 0\to\mathcal{G}_H(k)\to\mathcal{F}(k-1)\to\mathcal{F}(k)\to\mathcal{F}_H(k)\to 0.$$

The induction hypothesis applies to give Theorem A, and hence Theorem B, for $\mathcal{G}_H(k)$ and $\mathcal{F}_H(k)$. Thus $H^q(\mathcal{G}_H(k))=H^q(\mathcal{F}_H(k))=0$ for $q>0$ and $k\geqslant k_0$. From the exact cohomology sequence of $(**)$ we obtain *surjective* maps

$$H^1(\mathcal{F}(k))\twoheadrightarrow H^1(\mathcal{F}(k+1))\twoheadrightarrow H^1(\mathcal{F}(k+2))\twoheadrightarrow\cdots$$

for $k\geqslant k_0$. Since $H^1(\mathcal{F}(k))$ is finite dimensional, we must have isomorphisms

$$H^1(\mathcal{F}(k))\xrightarrow{\sim}H^1(\mathcal{F}(k+1))\xrightarrow{\sim}H^1(\mathcal{F}(k+2))$$

for $k\geqslant k_1$. But then the cohomology sequence of $(**)$ gives

$$H^0(\mathcal{F}(k))\to H^0(\mathcal{F}_H(k))\to 0$$

for $k\geqslant k_1$. Now $H^0(\mathcal{F}_H(k))$ generates $\mathcal{F}_H(k)_x$ as an $\mathcal{O}_{H,x}$-module for $k\geqslant k_2$ and any $x\in H$. Since the tangent space to H at x was assigned arbitrarily, it follows easily that $H^0(\mathcal{F}(k))$ generates $\mathcal{F}(k)_x$ as an $\mathcal{O}_{M,x}$-module. Q.E.D.

Noether's "AF+BG" Theorem. As an illustration of a particular global syzygy and application of the local residue theorem, we shall discuss a classical result of Max Noether, which is traditionally used as a cornerstone in the algebraic treatment of plane curves.

In $\mathbb{P}^2$ with homogeneous coordinates $X=[X_0,X_1,X_2]$ let $F(X)$ and $G(X)$ be homogeneous polynomials of respective degrees m and n whose divisors are plane curves C and D, which we assume to have no common component. Given a homogeneous polynomial $H(X)$ of degree $d=m+k=n+l$ with $k,l\geqslant 0$, we ask when there is a relation

$$(*)\qquad H=AF+BG.$$

An obvious necessary condition is that $(*)$ should hold locally. This has the following meaning: Let $P\in C\cap D$ and suppose that P is contained in the affine coordinate system $(x,y)=[1,x,y]$. Then $f(x,y)=F(1,x,y)$ and $g(x,y)=G(1,x,y)$ generate an ideal I_P in the local ring $\mathcal{O}_P$ of germs of holomorphic functions defined in a neighborhood of P. The obvious necessary local condition is that, setting $h(x,y)=H(1,x,y)$,

$$(**) \qquad h(x,y)\in I_P \subset \mathcal{O}_P$$

for each $P\in C\cap D$. Conversely, we shall prove

Noether's AF + BG Theorem. *If the local conditions* $(**)$ *are satisfied, then there is a global relation* $(*)$.

Proof. We let $I\subset\mathcal{O}$ be the sheaf of ideals generated by the various localizations f and g as above. Then I is coherent and $\operatorname{supp}(\mathcal{O}/I)=C\cap D$. Setting $d=m+k=n+l$ and $r=k-n=l-m$, the Koszul complex gives the global syzygy (cf. p. 698)

$$0\to\mathcal{O}(r)\to\mathcal{O}(m)\oplus\mathcal{O}(n)\to I(d)\to 0,$$

where $I(d)=I\otimes_{\mathcal{O}}\mathcal{O}(d)$ and the maps in this sequence are

$$\eta\to\eta G\oplus-\eta F,\qquad \eta\in\mathcal{O}(r),$$

$$\xi\oplus\psi\to F\xi+G\psi,\qquad \xi\in\mathcal{O}(k),\quad \psi\in\mathcal{O}(l),$$

where F,G are considered as global sections of $\mathcal{O}(m),\mathcal{O}(n)$, respectively.

Next we recall that $H^1(\mathbb{P}^2,\mathcal{O}(r))=0$ for all r, since first $H^1(\mathbb{P}^2,\mathcal{O}(r))=0$ for $r<0$ by the Kodaira vanishing theorem, and second

$$H^1(\mathbb{P}^2,\mathcal{O}(r))\cong H^1(\mathbb{P}^2,\mathcal{O}(-r-3))=0$$

for $r\geqslant 0$ by Kodaira-Serre duality. The exact cohomology sequence then gives

$$H^0(\mathbb{P}^2,\mathcal{O}(m))\oplus H^0(\mathbb{P}^2,\mathcal{O}(n))\to H^0(\mathbb{P}^2,I(d))\to 0.$$

Our local assumptions $(**)$ exactly mean that

$$H\in H^0(\mathbb{P}^2,I(d))\subset H^0(\mathbb{P}^2,\mathcal{O}(d)),$$

and this proves the theorem. Q.E.D.

In order to apply Noether's theorem, it is useful to have numerical criteria for when the local conditions $(**)$ are satisfied. It is pretty clear that the local duality theorem is relevant to this question, and we shall pursue this lead in one rather simple case here.

Suppose that $f(z,w)$ is holomorphic in a neighborhood of the origin and has divisor a nonsingular curve C passing through the origin. If $g(z,w)$ is holomorphic near the origin, then we define $\operatorname{Ord}_C(g)$ to be the order of vanishing of $g|_C$ at the origin. Suppose now that $g(z,w)$ has divisor D and that the set-theoretic intersection $C\cap D=\{0\}$. Denote by $I\subset\mathcal{O}$ the ideal $\{f,g\}$ in the local ring at the origin.

Lemma. *Given* $h(z,w)\in\mathcal{O}$, *if* $\mathrm{Ord}_C(h)\geqslant\mathrm{Ord}_C(g)$, *then* $h\in I$.

Proof. According to the local duality theorem we must prove that

$$\mathrm{Res}_{\{0\}}\left(\frac{hk\,dz\wedge dw}{fg}\right)=0$$

for all $k\in\mathcal{O}$. Since $\mathrm{Ord}_C(hk)\geqslant\mathrm{Ord}_C(h)$ this will follow from showing that

$$\mathrm{Res}_{\{0\}}\left(\frac{h\,dz\wedge dw}{fg}\right)=0$$

whenever $\mathrm{Ord}_C(h)\geqslant\mathrm{Ord}(g)$. We may choose local coordinates so that $f(z,w)=z$. Then, by iteration of the residue integral

$$\begin{aligned}\mathrm{Res}_{\{0\}}\left(\frac{h\,dz\wedge dw}{fg}\right)&=\left(\frac{1}{2\pi\sqrt{-1}}\right)\int_{|g|=\varepsilon}\left(\int_{|z|=\varepsilon}\frac{h(z,w)\,dz}{g(z,w)z}\right)dw\\&=\frac{1}{2\pi\sqrt{-1}}\int_{|g(0,w)|=\varepsilon}\frac{h(0,w)\,dw}{g(0,w)}\\&=0,\end{aligned}$$

since $\mathrm{Ord}_C(h)\geqslant\mathrm{Ord}_C(g)$. Q.E.D.

Of course this particular lemma may be proved directly, but the method of using residues and local duality will work in a variety of circumstances. Using the Max Noether theorem and this lemma, we shall reprove the result on cubics encountered in Section 4:

> *Suppose that* C, D, E *are cubics in* $\mathbb{P}^2$ *and that each point* $P\in C\cap D$ *is a simple point on* C. *Suppose that for all but one such point* $\mathrm{Ord}_C(E)_P\geqslant\mathrm{Ord}_C(D)_P$, *and at the remaining one, say* Q, *we have* $\mathrm{Ord}_C(E)_Q\geqslant\mathrm{Ord}_C(D)_Q-1$. *Then* $\mathrm{Ord}_C(E)_Q\geqslant\mathrm{Ord}_C(D)_Q$. *Briefly stated: any cubic* E *passing through eight of the nine points of* $C\cap D$ *passes through the remaining point also.*

Proof. Let F,G,H be homogeneous cubic polynomials defining C,D,E, respectively. Suppose L is a linear form vanishing at Q and at two points R_1,R_2 on C but not on D. Applying the lemma and Max Noether theorem to HL, we have

$$HL=AF+BG.$$

The linear form B vanishes at R_1 and R_2, and so $B=\beta L$. It follows that L divides AF, and since the line $L=0$ is not a component of C, it follows that $A=\alpha L$. Then $H=\alpha F+\beta G$, and so $\mathrm{Ord}_C(H)_Q\geqslant\mathrm{Ord}_C(G)_Q$. Q.E.D.

These methods may be generalized to prove the Cayley-Bacharach theorem from Section 4.

4. GLOBAL DUALITY

Global Ext

Let M be a compact, complex manifold and $\mathcal{L}$ the invertible sheaf associated to a positive line bundle $L \to M$. Given coherent sheaves $\mathcal{F}, \mathcal{G}$ on M, and using Theorem A discussed in the previous section, we may find a *global syzygy*

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

for $\mathcal{F}$. This gives rise to a complex of sheaves $\mathrm{Hom}_{\mathcal{O}}(\mathcal{E}_{.}(\mathcal{F}), \mathcal{G})$, whose associated hypercohomology we take as the definition of global Ext, written

$$\mathrm{Ext}(M; \mathcal{F}, \mathcal{G}) = \mathbb{H}^*(M, \mathrm{Hom}_{\mathcal{O}}(E_{.}(\mathcal{F}), \mathcal{G})).$$

Of course, we must prove that the right-hand side is independent of the choice of global syzygy. Moreover, we would like global Ext to have functorial properties analogous to those enjoyed by Ext for local rings and the sheaf $\underline{\mathrm{Ext}}$.

As was the case for the sheaf $\underline{\mathrm{Ext}}$, these matters will fall into place if we have at hand some global analogue of the four properties of projective resolutions given in the section on homological algebra. To achieve this, we recall the notation $\mathcal{F}(k) = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{L}^k$ and note that since $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}(k), \mathcal{G}(k)) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$, we may replace $\mathcal{F}$ by $\mathcal{F}(k)$ when convenient. Thus, suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{E}' & & & & \\ & & & \swarrow & \downarrow & & & & \\ 0 & \to & \mathcal{R} & \to & \mathcal{E} & \to & \mathcal{F} & \to & 0 \end{array}$$

of coherent sheaves on M where $\mathcal{E}$ and $\mathcal{E}'$ are locally free. It may not be possible to fill in the dotted arrow as it stands, but we can do the following: A section $s \in H^0(M, \mathcal{L}^k)$ gives an inclusion $\mathcal{E}'(-k) \subset \mathcal{E}'$, and we claim that, for $k \geqslant k_0$, the dotted arrow in the diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{E}'(-k) & & & & \\ & & & \swarrow & \downarrow & & & & \\ 0 & \to & \mathcal{R} & \to & \mathcal{E} & \to & \mathcal{F} & \to & 0 \end{array}$$

may be filled in.

Proof. Since $\mathcal{E}'(-k)$ is locally free, $\underline{\mathrm{Ext}}^1_{\mathcal{O}}(\mathcal{E}'(-k), \cdot) = 0$, and so the exact sequence of $\underline{\mathrm{Ext}}$'s gives

$$0 \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}'(-k), \mathcal{R}) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}'(-k), \mathcal{E}) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}'(-k), \mathcal{F}) \to 0.$$

By Theorem B,

$$H^1(M, \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}'(-k), \mathcal{R})) = H^1(M, \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}', \mathcal{R})(k)) = 0$$

for $k \geqslant k_0$. Consequently, we obtain a surjection

$$H^0(M, \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}'(-k), \mathcal{E})) \to H^0(M, \mathrm{Hom}(\mathcal{E}'(-k), \mathcal{F})) \to 0. \qquad \text{Q.E.D.}$$

From this we may draw the following conclusion: *When working globally with coherent sheaves on* M, *the four properties of projective resolutions of* $\mathcal{O}$*-modules carry over to global syzygies, at least provided we allow ourselves to tensor with* $\mathcal{L}^{-k}$. As a consequence, $\mathrm{Ext}(M; \mathcal{F}, \mathcal{G})$ is well-defined and has functorial properties analogous to those of local Ext. The most important of these are the two long exact sequences.

In order to calculate global Ext, a main tool are the two spectral sequences of hypercohomology. The first of these is a spectral sequence $\{'E_r\}$ with

$$'E_2^{p,q} = H^p(M, \underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})),$$
$$'E_\infty^{p,q} \Rightarrow \mathrm{Ext}^{p+q}(M; \mathcal{F}, \mathcal{G}).$$

Two applications of this spectral sequence will be useful. The first one is: *For* $\mathcal{E}$ *a locally free sheaf on* M,

$$\mathrm{Ext}^q(M; \mathcal{E}, \mathcal{G}) \cong H^q(M, \mathcal{E}^* \otimes_{\mathcal{O}} \mathcal{G}).$$

In particular, for any coherent sheaf $\mathcal{F}$,

$$\mathrm{Ext}^q(M, \mathcal{O}, \mathcal{F}) \cong H^q(M, \mathcal{F}).$$

This is clear, since for $\mathcal{E}$ locally free, $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{E}, \mathcal{G}) = 0$ for $q > 0$ and $\underline{\mathrm{Ext}}^0_{\mathcal{O}}(\mathcal{E}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{E}, \mathcal{G}) \cong \mathcal{E}^* \otimes_{\mathcal{O}} \mathcal{G}$.

The second property is: *Suppose* $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$ *for* $0 \leqslant q < k$. *Then*

$$\mathrm{Ext}^k(M; \mathcal{F}, \mathcal{G}) \cong H^0(M, \underline{\mathrm{Ext}}^k_{\mathcal{O}}(\mathcal{F}, \mathcal{G})).$$

Proof. The E_2 term of the spectral sequence has only zeros below the horizontal line passing through $(0, k)$, and this gives the result. Q.E.D.

We now are in a position to globalize the local duality theorem. Suppose that $I \subset \mathcal{O}$ is a sheaf of regular ideals such that the support $Z = \mathrm{supp}(\mathcal{O}/I)$ has dimension zero. Equivalently, locally $I = \{f_1, \ldots, f_n\}$, where the f_i form a regular sequence and $n = \dim_{\mathbb{C}} M$. We consider Z as a ringed space with structure sheaf $\mathcal{O}_Z = \mathcal{O}/I$.

We now refer to the intrinsic form of the local duality theorem in Section 3 above. According to the proposition in that section, the sheaves $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{O}_Z, \Omega^n) = 0$ for $q < n$. Moreover, since the sheaves $\mathcal{O}_Z$ and

$\underline{\mathrm{Ext}}^n_{\mathcal{O}}(\mathcal{O}_Z,\Omega^n)$ are skyscraper sheaves,

$$\begin{cases} H^0(M,\mathcal{O}_Z) \cong \bigoplus_{P\in Z} \mathcal{O}_{Z,P}, \\ H^0(M, \underline{\mathrm{Ext}}^n_{\mathcal{O}}(\mathcal{O}_Z,\Omega^n)) \cong \bigoplus_{P\in Z} \underline{\mathrm{Ext}}^n_{\mathcal{O}_P}(\mathcal{O}_{Z,P},\Omega^n_P), \\ H^q(M,\mathcal{O}_Z) = H^q(M, \underline{\mathrm{Ext}}^n_{\mathcal{O}}(\mathcal{O}_Z,\Omega^n)) = 0 \qquad \text{for } q>0 \end{cases}$$

Adding up the local duality theorems in each point $P\in Z$ gives the

Global Duality Theorem I. *Let* $\mathrm{I}\subset\mathcal{O}$ *be a sheaf of regular ideals such that* $Z=\mathrm{supp}(\mathcal{O}/\mathrm{I})$ *has dimension zero. Then there is a nondegenerate pairing*

$$H^q(M,\mathcal{O}_Z)\otimes \mathrm{Ext}^{n-q}(M;\mathcal{O}_Z,\Omega^n)\to\mathbb{C}$$

that is functorial in the sheaf of ideals I.

Explanation of the General Global Duality Theorem

We have now found duality theorems for the coherent sheaf cohomology $H^q(M,\mathcal{F})$ in the two cases where $\mathcal{F}\cong\mathcal{O}(F)$ is locally free and $\mathcal{F}=\mathcal{O}_Z$ with $\dim Z=0$ and $\mathcal{O}_Z=\mathcal{O}/I$ with I a sheaf of regular ideals. These represent the two extremes of a general duality theorem for $H^q(M,\mathcal{F})$, which will now be explained.

The steps are the following:

1. Given modules L,M,N over the local ring $\mathcal{O}=\mathcal{O}_n$, the pairing

$$\mathrm{Hom}_{\mathcal{O}}(L,M)\otimes_{\mathcal{O}}\mathrm{Hom}_{\mathcal{O}}(M,N)\to\mathrm{Hom}_{\mathcal{O}}(L,N)$$

induces a pairing, called the *Yoneda pairing*

$$\mathrm{Ext}^p_{\mathcal{O}}(L,M)\otimes_{\mathcal{O}}\mathrm{Ext}^q_{\mathcal{O}}(M,N)\to\mathrm{Ext}^{p+q}_{\mathcal{O}}(L,N)$$

having associativity and graded commutativity properties analogous to the usual cup product. This is a formal exercise using the four-part proposition.

2. Applying the propogation principle, if $\mathcal{F},\mathcal{G},\mathcal{H}$ are coherent sheaves on M, then there is

$$\underline{\mathrm{Ext}}^p_{\mathcal{O}}(\mathcal{F},\mathcal{G})\otimes_{\mathcal{O}}\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{G},\mathcal{H})\to\underline{\mathrm{Ext}}^{p+q}_{\mathcal{O}}(\mathcal{F},\mathcal{H})$$

inducing the previous pairing in each stalk.

3. This procedure globalizes to give a pairing

$$\mathrm{Ext}^p(M;\mathcal{F},\mathcal{G})\otimes\mathrm{Ext}^q(M;\mathcal{G},\mathcal{H})\to\mathrm{Ext}^{p+q}(M;\mathcal{F},\mathcal{H}).$$

Taking

$$q=n-p,$$
$$\mathcal{F}=\mathcal{O},\quad \mathcal{G}=\mathcal{F},\quad \mathcal{H}=\Omega^n,$$

this pairing is

$$(*) \qquad H^p(M,\mathcal{F})\otimes \operatorname{Ext}^{n-p}(M;\mathcal{F},\Omega^n)\to H^n(M,\Omega^n),$$

since $\operatorname{Ext}^*(M;\mathcal{O},\mathcal{F})\cong H^*(M,\mathcal{F})$.

All this can be defined for any complex manifold M. For M an algebraic variety the proofs have essentially been given. In case M is compact and connected, $H^n(M,\Omega^n)\cong\mathbb{C}$ and $(*)$ becomes

$$H^p(M,\mathcal{F})\otimes \operatorname{Ext}^{n-p}(M;\mathcal{F},\Omega^n)\to\mathbb{C}.$$

Global Duality Theorem II. *The above pairing is nondegenerate and is functorial in the following sense: A sheaf mapping* $\rho:\mathcal{F}\to\mathcal{G}$ *induces* $\rho_*:H^*(M,\mathcal{F})\to H^*(M,\mathcal{G})$ *and* $\rho^*:\operatorname{Ext}^*(M;\mathcal{G},\Omega^n)\to\operatorname{Ext}^*(M;\mathcal{F},\Omega^n)$ *such that the diagram*

$$\begin{array}{ccccc} H^p(M,\mathcal{F}) & \otimes & \operatorname{Ext}^{n-p}(M;\mathcal{F},\Omega^n) & \to & \mathbb{C} \\ \rho_*\downarrow & & \uparrow\rho^* & & \| \\ H^p(M,\mathcal{G}) & \otimes & \operatorname{Ext}^{n-p}(M;\mathcal{G},\Omega^n) & \to & \mathbb{C} \end{array}$$

is commutative.

As mentioned before, we have proved this in the two extreme cases $\mathcal{F}\cong\mathcal{O}(E)$ and $\mathcal{F}=\mathcal{O}_Z$, which is all that we shall have geometric applications for. The general result can also be proved without too much additional effort—and most of this in the nature of formalism—from our local duality theorem.

Finally, there is an even more general duality theorem dealing with a map—cf. the reference to Hartshorne's notes at the end of this chapter.

Global Ext and Vector Fields with Isolated Zeros

We shall prove a recent theorem, due to Carrell and Liebermann,* which will illustrate several of the techniques developed above, and which also will tie in with several previous results in the book.

Let M be a compact Kähler manifold and v a holomorphic vector field having a set Z of isolated zeros.

Theorem. *If Z is nonempty, then*

$$H^{p,q}(M)=0 \qquad \text{for } p\neq q.$$

*J. Carrell and D. Liebermann, Holomorphic vector fields and Kähler manifolds, *Inventiones Math.*, Vol. 21 (1973), pp. 303–309.

Actually, Carrell and Lieberman proved the more general statement

$$H^{p,q}(M)=0 \qquad \textit{for } |p-q|>\dim_{\mathbf{C}} Z,$$

where Z is the zero set of any holomorphic vector field on M. Granted the general duality theorem, the proof of this stronger assertion runs about the same as the one we shall now give, which proceeds in two steps.

Step One in the Proof. We denote by $\iota(v)$ the operation of contraction of a differential form with the vector field v. This operator was already encountered in the proof of the Bott residue theorem. If locally

$$\begin{cases} v=\sum_i v_i(z)\dfrac{\partial}{\partial z_i} & \text{and} \\ \varphi=\dfrac{1}{p!q!}\sum_{I,J}\varphi_{IJ}dz_I\wedge d\bar z_J \end{cases}$$

is a (p,q) form, then

$$\iota(v)\varphi=\frac{1}{(p-1)!q!}\sum_{I,J}\left(\sum_{i\in I}\pm v_i\varphi_{IJ}dz_{I-\{i\}}\wedge d\bar z_J\right).$$

From this the formal rules

$$\begin{cases} \iota(v):\ A^{p,q}(M)\to A^{p-1,q}(M), \\ \iota(v)^2=0, \\ \iota(v)\bar\partial+\bar\partial\iota(v)=0, \\ \iota(v)(\varphi\wedge\psi)=\iota(v)\varphi\wedge\psi+(-1)^{\deg\varphi}\varphi\wedge\iota(v)\psi, \end{cases}$$

are easily verified.

In particular, contraction with v gives the complex of sheaves

$$\Omega^n\xrightarrow{\iota(v)}\Omega^{n-1}\to\cdots\to\Omega^1\xrightarrow{\iota(v)}\mathcal{O}.$$

We note that the image of $\Omega^1\xrightarrow{\iota(v)}\mathcal{O}$ is the ideal sheaf I of Z; in fact, near a zero of v the above sequence is the Koszul complex associated to regular ideal $\{v_1(z),\dots,v_n(z)\}$. Consequently we have a very natural global syzygy for $\mathcal{O}_Z=\mathcal{O}/I$, one which will be used to calculate global Ext.

For this we observe the commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{O}}(\Omega^p,\Omega^n) & \xrightarrow{\sim} & \Omega^{n-p} \\ {\scriptstyle\iota(v)^*}\downarrow & & \downarrow{\scriptstyle\iota(v)} \\ \underline{\mathrm{Hom}}_{\mathcal{O}}(\Omega^{p+1},\Omega^n) & \xrightarrow{\sim} & \Omega^{n-p-1} \end{array}$$

similar to the one encountered in the lemma in the discussion concerning the intrinsic form of local duality in Section 3 above. Recall that $\mathrm{Ext}^*(M;\mathcal{O}_Z,\Omega^n)$ is the hypercohomology of the complex of sheaves

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O},\Omega^n)\to\underline{\mathrm{Hom}}_{\mathcal{O}}(\Omega^1,\Omega^n)\to\cdots.$$

Using the identifications provided by the commutative diagram, we write this complex of sheaves as $\Omega^{n-\cdot}$. Thus

$$\mathrm{Ext}^*(M;\mathcal{O}_Z,\Omega^n)\cong\mathbb{H}^*(M,\Omega^{n-\cdot})$$

We note that the differential used to calculate the hypercohomology on the right is $\delta\pm\iota(v)$, where δ is the Čech coboundary mapping.

Now, according to the general discussion of hypercohomology there are two spectral sequences $\{''E_r\}$ and $\{'E_r\}$ abutting to $\mathbb{H}^*(M,\Omega^{n-\cdot})$. One of them has

$$\begin{aligned}''E_2^{p,q}&=H^p\big(M,\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{O}_Z,\Omega^n)\big)\\&=0,\qquad\text{unless } p=0\text{ and } q=n.\end{aligned}$$

The other spectral sequence has

$$'E_1^{p,q}=H^q(M,\Omega^{n-p}).$$

The differentials $d_1',d_2',\cdots$ are induced from $\iota(v)$. If we can show that $d_1'=d_2'=\cdots=0$, then

$$\begin{aligned}'E_1^{p,q}&\cong{}'E_2^{p,q}\cong\cdots\cong{}'E_\infty^{p,q}\\&=0,\qquad\text{unless } p+q=n\end{aligned}$$

by the previous spectral sequence. This proves the theorem.

Step Two. Let $L\in H^{1,1}(M)$ be the cohomology class of a Kähler metric ω. The proof that $d_1'=d_2'=\cdots=0$ will use the hard Lefschetz theorem

$$L^k\colon H^{n-k}(M)\xrightarrow{\sim}H^{n+k}(M)$$

and primitive decomposition

$$\left\{\begin{aligned}&H^l(M)=\bigoplus_{j\le[l/2]}L^jP^{l-2j}(M),\qquad\text{where}\\&P^{n-k}(M)=\ker\{L^{k+1}\colon H^{n-k}(M)\to H^{n+k+2}(M)\}\end{aligned}\right\},$$

both of which were proved in Section 6 of Chapter 0. We recall also that the primitive decomposition is compatible with the Hodge (p,q) decomposition.

Now, as noted above, the diagram

$$\begin{array}{ccc}'E_1^{p,q}&\xrightarrow{d_1}&'E_1^{p+1,q}\\ \wr\|&&\wr\|\\ H^{n-p,q}(M)&\xrightarrow{\iota(v)}&H^{n-p-1,q}(M)\end{array}$$

is commutative, at least up to ± 1. Thus, we must prove that $\iota(v)$ induces zero as a map on cohomology. For a holomorphic 1-form $\varphi \in H^{1,0}(M)$, $\iota(v)\varphi$ is a holomorphic function on M that vanishes on $Z \neq \varphi$. Thus $\iota(v)\varphi = 0$. Using this, we shall prove

Lichnerowicz' Lemma. $\iota(\mathrm{v})\omega = 0$ *in* $\mathrm{H}^{0,1}(\mathrm{M})$.

Proof. By hard Lefschetz and Kodaira-Serre duality, the pairing

$$H^{1,0}(M) \otimes H^{0,1}(M) \to \mathbb{C}$$

given by

$$\varphi \otimes \psi \to \int_M \omega^{n-1} \wedge \varphi \wedge \psi$$

is nondegenerate. According to the formal rules for $\iota(v)$ listed above, for all $\varphi \in H^{1,0}(M)$

$$\begin{aligned} 0 &= \iota(v)(\omega^n \wedge \varphi) && \text{(since } \omega^n \wedge \varphi \equiv 0 \text{ for trivial reasons)} \\ &= n\omega^{n-1} \wedge \iota(v)\omega \wedge \varphi && \text{(since as observed above } \iota(v)\varphi \equiv 0\text{)} \\ &\Rightarrow \int_M \omega^{n-1} \wedge \iota(v)\omega \wedge \varphi = 0 \\ &\Rightarrow \iota(v)\omega = 0 \text{ in } H^{0,1}(M) \end{aligned}$$

by the nondegeneracy of the pairing. Q.E.D. for lemma.

In particular, $d_1' L = 0$, so that L defines a class in $'E_2^{n-1,1}$. Since

$$d_r' : {}'E_r^{n-1,1} \to {}'E_r^{n-1+r,2-r} = 0$$

for $r \geqslant 2$, it follows that L defines a class in $'E_r^{n-1,1}$ for all r. Moreover, multiplication by L^k commutes with the differentials in the spectral sequence, as follows by the formal rules for calculating with $\iota(v)$. Therefore, to prove that $d_1' = 0$, by using the primitive decomposition and hard Lefschetz it will suffice to prove that this is the case on primitive cohomology. Let

$$\psi \in P^{n-q-k,q}(M) \subset {}'E^{q+k,q}.$$

Then, essentially repeating the argument from the degeneration of the Leray spectral sequence in Section 6 of Chapter 3,

$$0 = \iota(v)(\omega^{k+1}\psi) = \omega^{k+1}\iota(v)\psi$$

in cohomology. But $\iota(v)\psi \in H^{n-k-1}(M)$, and so $L^{k+1}\iota(v)\psi = 0 \Rightarrow \iota(v)\psi = 0$ by hard Lefschetz. Thus $d_1' = 0$.

The argument for $d_2' = d_3' = \cdots = 0$ is the same. Q.E.D.

We note that this proof has not used the full strength of duality. For example, the equality

$$\dim \operatorname{Ext}^*(M; \mathcal{O}_Z, \Omega^n) = \deg(Z) = \dim H^0(M, \mathcal{O}_Z)$$

gives

$$\sum_p h^{p,p}(M) = \deg(Z).$$

Since $h^{p,q}(M)=0$ for $p \neq q$ the left-hand side is just the topological Euler characteristic, what we have is just a special case of the Hopf index theorem. More substantial applications, including a proof of Bott's residue formula, arise by keeping track of the filtrations induced by the spectral sequences. These are given in the paper of Carrell and Liebermann.

Global Duality and Superabundance of Points on a Surface

Let $L \to S$ be a holomorphic line bundle over an algebraic surface. In the Riemann-Roch theorem for surfaces

$$\chi(\mathcal{O}_S(L)) = \tfrac{1}{2}(L \cdot L - K \cdot L) + \chi(\mathcal{O}_S)$$

the terms

$$h^0(L), \quad h^2(L) = h^0(K-L), \quad p_g, \quad q, \quad L \cdot L, \quad K \cdot L$$

all have immediate geometric interpretations, at least in case $L=[D]$ for some effective divisor D on S. The Italian algebraic geometers first wrote this formula as

$$\dim|L| + h^0(K-L) = \tfrac{1}{2}(L \cdot L - K \cdot L) + p_g - q + \omega,$$

and then proved directly that the quantity ω defined by this equation was nonnegative, which they then called the *superabundance*. The reader should keep in mind that the dual of $H^1(\mathcal{O}_S(L))$ is $H^1(\mathcal{O}_S(K-L))$, and sheaf cohomology was 50 years in the future. Working backward historically, we shall use our global duality theorem for coherent sheaves to geometrically interpret the superabundance in some cases. We begin with an example; the final result is the Reciprocity Formula II on page 716.

Suppose that Γ_0 is a set of distinct points in $\mathbb{P}^2$ and $|C_0| = |\mathcal{I}_{\Gamma_0}(n)|$ is the linear system of curves of degree n passing through Γ_0. Let S be the quadratic transform of $\mathbb{P}^2$ along Γ_0 and $|C|$ the linear system on S of proper transforms of curves $C \in |\mathcal{I}_{\Gamma_0}(n)|$. Denote by $\pi: S \to \mathbb{P}^2$ the projection map and $E = \pi^{-1}(\Gamma_0)$ the exceptional curve. Then $|C|$ is the complete linear system $|L|$ where

$$L = \pi^* H^n - E$$

for $H \to \mathbb{P}^2$ the hyperplane bundle. The canonical bundle of S is

$$\begin{aligned} K_S &= \pi^*(K_{\mathbb{P}^2}) + E \\ &= \pi^* H^{-3} + E, \end{aligned}$$

and the numerical characters for $L \to S$ in the Riemann-Roch formula are

$$h^2(L) = h^0(K_S - L) = 0, \qquad p_g = q = 0,$$
$$C \cdot C = C_0 C_0 - d = n^2 - d, \qquad \text{where } d = \deg \Gamma_0,$$
$$C \cdot K_S = C \cdot (\pi^* H^{-3} + E) = -3n + d.$$

Consequently, by the Riemann-Roch formula,

$$r = \dim|L| = \tfrac{1}{2}(C \cdot C - C \cdot K_S) + \omega$$
$$= \frac{n(n+3)}{2} - d + \omega.$$

On the other hand from the exact cohomology sequence of

$$0 \to \mathcal{I}_{\Gamma_0}(n) \to \mathcal{O}_{\mathbf{P}^2}(n) \to \mathcal{O}_{\Gamma_0}(n) \to 0$$

and $h^1(\mathcal{O}_{\mathbf{P}^2}(n)) = 0$, we obtain

$$r = \dim|\mathcal{I}_{\Gamma_0}(n)|$$
$$= \frac{n(n+3)}{2} - d + h^1(\mathcal{O}_{\mathbf{P}^2}(n)),$$

so that the superabundance

$$\omega = h^1(\mathcal{O}_S(L)) = h^1(\mathcal{I}_{\Gamma_0}(n)).$$

It is now clear that $\omega = 0$ *if and only if the points in* Γ_0 *impose independent conditions on the linear system* $|\mathcal{O}_{\mathbf{P}^2}(n)|$. As was seen in the Cayley-Bacharach theorem in Section 2 of this chapter, it may well happen that $h^1(\mathcal{I}_{\Gamma_0}(n)) > 0$—indeed, this is frequently the interesting case.

Returning now to a general surface S with line bundle $L \to S$, we will show: *Suppose that* $\dim|L| > 0$ *and that the complete linear system* $|L|$ *has no base curves. Assume moreover that the irregularity* $q = 0$. *Then the superabundance* $\omega \doteq h^1(L)$ is given by

$$\omega = \dim|\mathcal{I}_\Gamma(K_S + L)| - 2p_g + \dim|K_S - L| + 2,$$

where $\Gamma = C \cdot C'$ *for general curves* $C, C' \in |L|$.

Proof. Let $s, s' \in H^0(\mathcal{O}_S(L))$ define C, C', respectively, and consider the Koszul complex

$$0 \longrightarrow \mathcal{O}(K_S - L) \longrightarrow \mathcal{O}(K_S) \oplus \mathcal{O}(K_S) \longrightarrow \mathcal{I}_\Gamma(K_S + L) \longrightarrow 0.$$

By our assumption

$$h^1(\mathcal{O}(K_S)) = h^{2,1}(S) = h^{0,1}(S) = 0,$$

so we find

$$h^1(\mathcal{O}(L)) = h^1(\mathcal{O}(K_S - L)) \qquad \text{(by duality)}$$
$$= h^0(\mathcal{I}_\Gamma(K_S + L)) - 2h^0(\mathcal{O}(K_S)) + h^0(\mathcal{O}(K_S - L)),$$

which gives our assertion. Q.E.D.

In case $p_g=0$, $\dim|K_S-L|=-1$, and the formula simplifies to

$$\omega = \dim|\mathcal{I}_\Gamma(K_S+L)|+1.$$

As an application, suppose that Γ_0 is a set of $d<n(n+3)/2$ points in $\mathbb{P}^2$. Then the linear system $|\mathcal{I}_{\Gamma_0}(n)|$ of curves of degree n passing through Γ_0 contains at least a pencil, and either

1. this linear system has a fixed curve of degree less than n; or
2. general curves $C,C'\in|\mathcal{I}_{\Gamma_0}(n)|$ will have intersection

$$C\cdot C' = \Gamma_0+\Gamma,$$

where Γ is a set of n^2-d points that we call a *residue* of Γ_0 with respect to curves of degree n. We shall prove the

Reciprocity Formula, I

$$\dim|\mathcal{I}_{\Gamma_0}(n)| = \left\{\frac{n(n+3)}{2}-d\right\}+h^0(\mathcal{I}_\Gamma(n-3)).$$

Thus, the superabundance of Γ_0 *relative to the linear system* $|\mathcal{O}_{\mathbb{P}^2}(n)|$ *is given by*

$$\omega = h^0(\mathcal{I}_\Gamma(n-3)).$$

Proof. Let S be the blow-up of $\mathbb{P}^2$ along Γ_0 considered above and $L=\pi^*H^n-E$. Then $K_S+L=\pi^*H^{n-3}$ and by the result of p. 713

$$\begin{aligned}\dim|\mathcal{I}_{\Gamma_0}(n)| &= \dim|L|\\ &= \left\{\frac{n(n+3)}{2}-d\right\}+\omega,\end{aligned}$$

where $\omega=h^0(\mathcal{I}_\Gamma(K_S+L))=\dim|\mathcal{I}_\Gamma(n-3)|+1.$ Q.E.D.

As a first illustration we shall show how the reciprocity formula may be used to derive the properties of linear systems of cubics that arose in Section 1 of Chapter 4 in our study of the cubic surface. We begin with:

A set Γ_0 *of seven points imposes independent conditions on* $|\mathcal{O}_{\mathbb{P}^2}(3)|$, *unless five of the seven are collinear.*

Proof. If there are two cubics $C,C'\in|\mathcal{I}_{\Gamma_0}(3)|$ without a common component, then for the residual set Γ we have $h^0(\mathcal{I}_\Gamma)=0$, and by (the most trivial case of) the reciprocity formula the points Γ_0 impose independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(3)|$.

So we may assume that $\dim|\mathcal{I}_{\Gamma_0}(3)|\geqslant 3$ and that any two cubics in this linear system have a common component C_0, which must be a line or a conic. Since the linear system of lines has dimension 2, C_0 cannot be a conic and so must be a line. If C_0 contains $\leqslant 4$ points from Γ_0, then there will be a set Γ_0' of $\geqslant 3$ points left over. These will impose independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$ of plane conics, and consequently

$$\dim|\mathcal{I}_{\Gamma_0}(3)| = \dim|\mathcal{I}_{\Gamma_0'}(2)| \leqslant 5 - 3 = 2,$$

which is a contradiction. Q.E.D.

A set Γ_0 of eight points imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(3)|$, unless five are on a line or all eight are on a conic.

Proof. If we assume that Γ_0 fails to impose independent conditions, then $\dim|\mathcal{I}_{\Gamma_0}(3)| \geqslant 2$ and, as before, we conclude that any two cubics from this linear system have a common component C_0. If C_0 is a conic, then since $\dim|\mathcal{I}_{\Gamma_0}(3)| \geqslant 2$, all the points of Γ_0 must lie on C_0. If C_0 is a line, then the previous argument shows that at most three points of Γ_0 can fail to lie on C_0. Q.E.D.

The result we needed in the section on cubic surfaces now follows easily:

Let Δ be six points in $\mathbb{P}^2$, no three of which are on a line and which are not on a conic. Then $\Gamma_0 = \Delta + p + q$ imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(3)|$ for any $p, q \in \mathbb{P}^2$.

Proof. If not, then either five points from Γ_0 must be collinear or all eight must be on a conic—this contradicts the assumption on Δ. Q.E.D.

Here is one more illustration of the reciprocity formula.

Let Γ_0 be a set of 12 points in $\mathbb{P}^2$ that fails to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(4)|$. Then either

$$\begin{cases} \Gamma_0 = C_4 \cdot C_3 \text{ is a complete intersection, or} \\ 10 \text{ points from } \Gamma_0 \text{ are on a conic, or} \\ \text{six points from } \Gamma_0 \text{ are collinear.} \end{cases}$$

Proof. We assume that $\dim|\mathcal{I}_{\Gamma_0}(4)| \geqslant 3$. Recalling that $\dim|\mathcal{O}_{\mathbb{P}^2}(4)| = 14$, if there are two curves C, C' from this linear system having no common component, then

$$C \cdot C' = \Gamma_0 + \Gamma.$$

By the reciprocity formula, Γ consists of four collinear points, and this implies that $\Gamma_0 = C_4 \cdot C_3$.

Now assume that any two curves from $|\mathcal{I}_{\Gamma_0}(4)| = |C|$ have a common component C_0. It is not possible that C_0 is a cubic, since otherwise the residual system $|C - C_0|$ would consist of lines and have dimension $\geqslant 3$. Suppose next that C_0 is a conic containing $\leqslant 9$ points from Γ_0. Then the linear system of conics $|C - C_0|$ will pass through $\geqslant 3$ points and have dimension at least 3, which is a contradiction. So 10 or more points from Γ_0 lie on a conic.

If, finally, C_0 is a line containing $\leqslant 5$ points from Γ_0, then $|C - C_0|$ will be a linear system of $\infty^{3+\rho}(\rho \geqslant 0)$ cubics passing through $\geqslant 7$ points. By our previous result, either five will be on a line—in which case 10 points from Γ_0 are on a (degenerate) conic—or eight will be on a conic. But then this conic must be a fixed component of the $\infty^{3+\rho}$ cubics $C - C_0$, which is a contradiction. Q.E.D.

Now the Reciprocity Formula I was proved for $\mathbb{P}^2$ under the assumption that $C \cdot C' = \Gamma_0 + \Gamma$ where the 0-cycle $\Gamma_0 + \Gamma$ consists of distinct points of transverse intersection. It is easy to extend the formula to a general surface, but relaxing the restriction on $\Gamma_0 + \Gamma$ is more difficult by the previous method, which was to convert the ideal sheaf of Γ_0 into a locally free one by a blowing up. In practice it is desirable to have a more general reciprocity formula, and as an application of global duality we shall give this extension.

Suppose that S is a regular algebraic surface—thus $h^{1,0}(S) = h^{2,1}(S) = 0$ —and $L \to S$ is a holomorphic line bundle with two sections s, s' such that the simultaneous equations $s = 0, s' = 0$ define a zero-dimensional subscheme Z of S. There is an ideal sheaf $\mathcal{I} \subset \mathcal{O}$ with $\mathcal{O}_Z = \mathcal{O}/\mathcal{I}$; in fact, $\mathcal{I}$ is the image under the mapping

$$\mathcal{O}(L^*) \oplus \mathcal{O}(L^*) \to \mathcal{O}$$

given by $(f, f') \to fs + f's'$. Suppose that we decompose Z into two disjoint sets Γ_0 and Γ; we may think of Γ_0 as part of the base of the pencil $|s + \lambda s'| \subset |L|$, and shall refer to Γ as the *residue* of Γ_0.

Reciprocity Formula (II). *With the above notations,*

$$h^1(\mathcal{I}_{\Gamma_0}(L)) = h^0(\mathcal{I}_\Gamma(K + L)) - 2p_g + h^0(\mathcal{O}(K - L)).$$

In particular, if both $q = p_g = 0$, *then*

$$h^1(\mathcal{I}_{\Gamma_0}(L)) = h^0(\mathcal{I}_\Gamma(K + L)).$$

Proof. We consider the two exact sheaf sequences

$$\begin{cases} 0 \to \mathcal{I}(L) \to \mathcal{I}_{\Gamma_0}(L) \to \mathcal{O}_\Gamma(L) \to 0, \\ 0 \to \mathcal{O}(L^*) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{I}(L) \to 0, \end{cases}$$

where $\mathcal{O}_\Gamma = \mathcal{O}/\mathcal{I}_\Gamma$ in the first, and the second is the Koszul resolution. By the duality theorem, $H^1(\mathcal{I}_{\Gamma_0}(L))$ and $\mathrm{Ext}^1(S; \mathcal{I}_{\Gamma_0}(L), \Omega^2)$ are cononically dual vector spaces. We shall calculate the latter by applying the exact sequence of global Ext and interpreting the maps.

First we observe from $\underline{\mathrm{Ext}}^0_{\mathcal{O}}(\mathcal{O}_\Gamma(L), \Omega^2) = \underline{\mathrm{Ext}}^1_{\mathcal{O}}(\mathcal{O}_\Gamma(L), \Omega^2) = 0$ and the spectral sequence

$$\begin{array}{c} E_2^{p,q} \Rightarrow \mathrm{Ext}^*(S;\mathcal{O}_\Gamma(L),\Omega^2) \\ \| \\ H^p(S,\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{O}_\Gamma(L),\Omega^2)) \end{array}$$

for global Ext that $\mathrm{Ext}^1(S;\mathcal{O}_\Gamma(L),\Omega^2)=0$. Thus we have

$$(*)\quad 0\to \mathrm{Ext}^1(S;\mathcal{I}_{\Gamma_0}(L),\Omega^2)\to \mathrm{Ext}^1(S;\mathcal{I}(L),\Omega^2)\xrightarrow{\rho}\mathrm{Ext}^2(S;\mathcal{O}_\Gamma(L),\Omega^2).$$

For the middle term we use the second exact sheaf sequence, noting that $\mathrm{Ext}^1(S;\mathcal{O}\oplus\mathcal{O},\Omega^2)\cong H^1(S;\Omega^2\oplus\Omega^2)=0$ since S is regular, and also that

$$\begin{cases} \mathrm{Ext}^0(S;\mathcal{O}(L^*),\Omega^2)\cong H^0(\mathcal{O}(K+L)), \\ \mathrm{Ext}^0(S;\mathcal{O}\oplus\mathcal{O},\Omega^2)\cong H^0(\mathcal{O}(K))\oplus H^0(\mathcal{O}(K)), \\ \mathrm{Ext}^0(S;\mathcal{I}(L),\Omega^2)\cong H^0(\underline{\mathrm{Hom}}(\mathcal{I}(L),\Omega^2))\cong H^0(\mathcal{O}(K-L)) \end{cases}$$

to obtain

$$\begin{aligned} 0&\to H^0(\mathcal{O}(K-L))\to H^0(\mathcal{O}(K))\oplus H^0(\mathcal{O}(K)) \\ &\to H^0(\mathcal{O}(K+L))\to \mathrm{Ext}^1(S;\mathcal{I}(L),\Omega^2)\to 0. \end{aligned}$$

This gives the interpretation

$$(**)\quad \mathrm{Ext}^1(S;\mathcal{I}(L),\Omega^2)\cong\left\{\begin{array}{l} H^0(\mathcal{O}(K+L))/\{s\omega+s'\omega'\},\ \text{where} \\ \omega,\omega'\in H^0(\mathcal{O}(K)). \end{array}\right\}.$$

Now since $\underline{\mathrm{Ext}}^0_{\mathcal{O}}(\mathcal{O}_\Gamma(L),\Omega^2)=\underline{\mathrm{Ext}}^1_{\mathcal{O}}(\mathcal{O}_\Gamma(L),\Omega^2)=0$ and $\underline{\mathrm{Ext}}^2_{\mathcal{O}}(\mathcal{O}_\Gamma(L),\Omega^2)$ is a skyscraper sheaf concentrated at points $p\in\Gamma$ and with stalks *canonically* isomorphic to $(\mathcal{O}_\Gamma(L)_p)^*$,

$$\mathrm{Ext}^2(S;\mathcal{O}_\Gamma(L),\Omega^2)\cong\bigoplus_{p\in\Gamma}(\mathcal{O}_\Gamma(L)_p)^*.$$

Combining this with $(*)$ and $(**)$ yields

$$0\to\mathrm{Ext}^1(S;\mathcal{I}_{\Gamma_0}(L),\Omega^2)\to\{H^0(\mathcal{O}(K+L))/\{s\omega+s'\omega'\}\}\xrightarrow{\rho}\bigoplus_{p\in\Gamma}(\mathcal{O}_\Gamma(L)_p)^*.$$

To interpret the mapping ρ, we suppose that $\psi\in H^0(\mathcal{O}(K+L))$ and $\eta\in\mathcal{O}_\Gamma(L)_p$. Then

$$\mathrm{Res}_p\left(\frac{\eta\psi}{s\cdot s'}\right)$$

has intrinsic meaning since $\eta\psi\in\mathcal{O}(K+2L)_p$ and $s\cdot s'\in\mathcal{O}(2L)_p$. Because duality is functorial, we deduce that

$$\rho(\psi)(\eta)=\mathrm{Res}_p\left(\frac{\eta\psi}{s\cdot s'}\right).$$

From the local duality theorem it follows that

$$\ker\rho \cong H^0(\mathcal{I}_\Gamma(K+L))/\{s\omega+s'\omega'\}.$$

We conclude from (*) and (**) that

$$\begin{aligned} h^1(\mathcal{I}_{\Gamma_0}(L)) &= \dim \operatorname{Ext}^1\big(S;\mathcal{I}_{\Gamma_0}(L),\Omega^2\big) \\ &= \dim(\ker\rho) \\ &= h^0(\mathcal{I}_\Gamma(K+L)) - 2p_g + h^0(\mathcal{O}(K-L)), \end{aligned}$$

which establishes the reciprocity formula. Q.E.D.

We now illustrate how the singularities enter in a special case. Suppose that C and C' are two irreducible plane quartic curves having three ordinary double points p_i $(i=1,2,3)$ in common. We assume that at each of these points the four tangent lines to the two curves are distinct. These curves then define an ideal $\mathcal{I}_{p_i}\subset\mathcal{O}_{p_i}$, which is contained in but not equal to the square m_i^2 of the maximal ideal, and which we now shall describe: Choose local coordinates (x,y) relative to which C and C' have respective equations

$$\begin{cases} xy=0, \\ (x-y)(x-\gamma y)=0. \end{cases}$$

This is possible by identifying the directions through p_i with $\mathbb{P}^1$ and noting that any four points of $\mathbb{P}^1$ may be projectively transformed to $\{0,1,\gamma,\infty\}$. So in fact γ is the *cross-ratio* associated to these four tangent lines in some order. Functions $f(x,y)$ in the ideal have the form

$$f(x,y) = \alpha xy + \beta(x-y)(x-\gamma y).$$

In general β will be a unit; then we may assume $\beta=1$ and

$$f(x,y) = (x-\mu y)(x-\lambda y) + (\text{higher-order terms})$$

where

$$\mu\lambda = \gamma, \quad \lambda+\mu = 1+\gamma-\alpha.$$

Having fixed the tangent directions to C to correspond to the points $0,\infty\in\mathbb{P}^1$, the tangent directions to the curve C_f defined by f will be μ,λ and the condition $\mu\lambda=\gamma$ has intrinsic meaning and defines the ideal $\mathcal{I}_{p_i}\subset\mathcal{O}_{p_i}$. *Geometrically, the curve* C_f *must have an ordinary double point at* p_i *and the cross-ratio of its tangents together with those of* C *is prescribed.*

We now write

$$C\cdot C' = \Gamma_0+\Gamma,$$

where Γ_0 and Γ are zero-dimensional schemes with $\mathcal{I}_{\Gamma_0}=\mathcal{I}_{p_1}\cap\mathcal{I}_{p_2}\cap\mathcal{I}_{p_3}$ and Γ is the residue of Γ_0 relative to the pencil $|C+\lambda C'|$. We note that

$\deg \Gamma_0 = 12$ and consequently $\deg \Gamma = 4$. The Reciprocity Formula II gives

$$h^1(\mathcal{I}_{\Gamma_0}(4)) = h^0(\mathcal{I}_\Gamma(1)) \leqslant 1,$$

where $\mathcal{I}_{\Gamma_0}(4) = \mathcal{I}_{\Gamma_0} \otimes \mathcal{O}_{\mathbb{P}^2}(4)$, and where equality holds on the right if and only if the points of Γ are distinct and collinear. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(4)) = 15$ and $h^1(\mathcal{O}_{\mathbb{P}^2}(4)) = 0$, it follows that: *The linear system of plane quartics through* Γ_0 *has dimension given by*

$$\dim|\mathcal{I}_{\Gamma_0}(4)| = 2 \text{ or } 3$$

where the second possibility holds exactly when the points of Γ *are distinct and collinear.*

To see when this happens we consider the triangle Δ with vertices p_1, p_2, p_3. This is a plane cubic with double points at the p_i, and we shall say that our configuration is in *special position* in case the defining equation of Δ is in the ideal $\mathcal{I}_{p_i}$ at each vertex. We now prove that: *The configuration is in special position if, and only if, the points of* Γ *are distinct and collinear.*

Proof. If the points of Γ are distinct and collinear, then some member E of the pencil $|C + \lambda C'|$ will have this line L_0 as tangent, from which it follows that

$$E = \Delta + L_0,$$

and so $\Delta \in |\mathcal{I}_{\Gamma_0}(3)|$ and the configuration is in special position. If, conversely, $\Delta \in |\mathcal{I}_{\Gamma_0}(3)|$, then the curves $\Delta + L$, L a line in $\mathbb{P}^2$, give a $\mathbb{P}^2$ in the projective space $|\mathcal{I}_{\Gamma_0}(4)|$. Since not all curves in this linear system are reducible, we deduce that $\dim|\mathcal{I}_{\Gamma_0}(4)| = 3$, which gives our conclusion. Q.E.D.

It is interesting to investigate the rational map

$$f\colon \mathbb{P}^2 \to \mathbb{P}^3$$

defined by the linear system $|\mathcal{I}_{\Gamma_0}(4)|$ when the configuration is in special position. The image is a surface S of degree four with the property that there are ∞^2 reducible hyperplane sections.

To see what the image S of f is, we observe first that f is well-defined on the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at the points p_1, p_2, p_3 of Γ_0. (See Figures 2 and 3.) If E_i is the exceptional divisor in $\tilde{\mathbb{P}}^2$ over p_i, then the proper transform $\tilde{D}$ of a generic element $D \in |\mathcal{I}_{\Gamma_0}(4)|$ is given by

$$\tilde{D} \sim \pi^* D - 2E_1 - 2E_2 - 2E_3.$$

The degree of the image $S = f(\tilde{\mathbb{P}}^2)$ in $\mathbb{P}^3$ is thus

$$\begin{aligned}\tilde{D} \cdot \tilde{D} &= D \cdot D - 4E_1 \cdot E_1 - 4E_2 \cdot E_2 - 4E_3 \cdot E_3 \\ &= 16 - 4 - 4 - 4 \\ &= 4.\end{aligned}$$

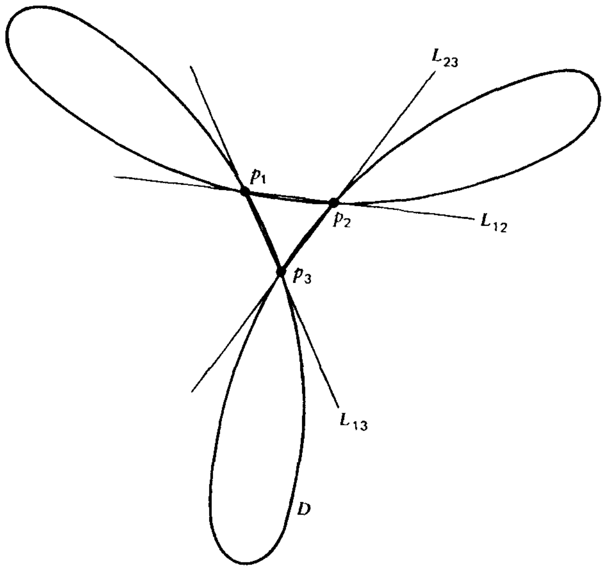


Figure 2

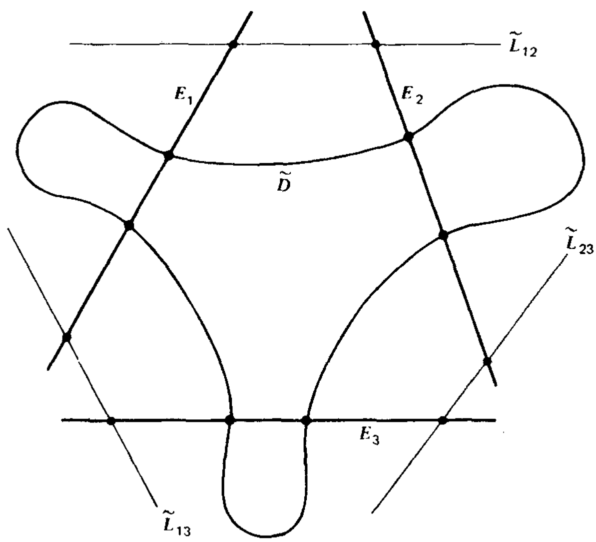


Figure 3

Now consider the map f: $\mathbb{P}^2 \to \mathbb{P}^3$. Since the linear system $|\mathcal{I}_{\Gamma_0}(4)|$ contains, as a subsystem, the triangle Δ plus the linear system of lines, f will be one-to-one and smooth away from the inverse image in $\tilde{\mathbb{P}}^2$ of the triangle. The proper transforms $\tilde{L}_{ij}$ of the lines $L_{ij} = \overline{p_i p_j}$, on the other hand, are blown down to points: any curve $D \in |\mathcal{I}_{\Gamma_0}(4)|$ containing a point $q \in L_{ij}$ other than p_i and p_j has two double points and one single point of intersection with L_{ij}, and so contains L_{ij}. Also, while the proper transform of the linear system $|\mathcal{I}_{\Gamma_0}(4)|$ in $\tilde{\mathbb{P}}^2$ has intersection number 2 with each E_i, it does not cut out a complete linear system in E_i: fixing one tangent line to a curve $D \in |\mathcal{I}_{\Gamma_0}(4)|$ at p_i determines the other, and so f maps E_i two-to-one onto a line in $\mathbb{P}^3$. These lines are double lines of the image S. Finally, since the triangle Δ is in the ideal $\mathcal{I}_{\Gamma_0}$, the points of intersection of the proper transforms $\tilde{L}_{ij}$ and $\tilde{L}_{ik}$ with E_i, while distinct on $\tilde{\mathbb{P}}^2$, are identified under the map $\tilde{f}$. In other words, after blowing down the lines $\tilde{L}_{ij}$ on $\tilde{\mathbb{P}}^2$ the divisors E_i form a triangular configuration. (See Figure 4.) The map f then folds each side of this triangle over so that the vertices are identified; the resulting configuration is shown in Figure 5. *The surface* $S \subset \mathbb{P}^3$ *is thus a quartic with three double lines meeting in a point.*

In fact, we have already encountered this surface, in Section 5 of Chapter 4; we saw there that S is the image under projection to $\mathbb{P}^3$ of the Veronese surface in $\mathbb{P}^5$. Indeed, we can see this directly in the present context: Since the transformation φ of $\mathbb{P}^2$ which blows up p_1, p_2, p_3 and blows down the lines $\tilde{L}_{ij}$ is given by the linear system $|\mathcal{I}_{p_1} \otimes \mathcal{I}_{p_2} \otimes \mathcal{I}_{p_3}(2)|$ of conics through the points p_i, the composition $i_{|\mathcal{O}(2)|} \circ \varphi : \mathbb{P}^2 \to \mathbb{P}^5$ of φ with the map is given by the linear system $|\mathcal{I}_{p_1}^2 \otimes \mathcal{I}_{p_2}^2 \otimes \mathcal{I}_{p_3}^2(4)|$ of quartics with double points at the points p_i; the map f, given by the sublinear system $|\mathcal{I}_{\Gamma_0}(4)| \subset |\mathcal{I}_{p_1}^2 \otimes \mathcal{I}_{p_2}^2 \otimes \mathcal{I}_{p_3}^2(4)|$, is just the composition of $i_{|\mathcal{O}(2)|} \circ \varphi$ with a projection to $\mathbb{P}^3$.

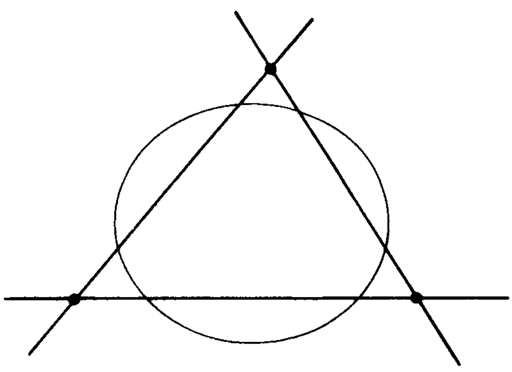


Figure 4

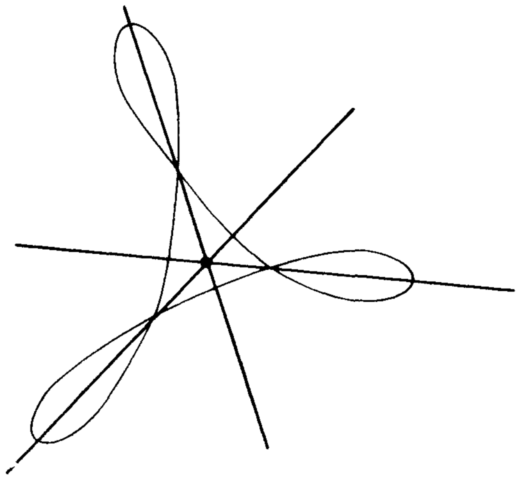


Figure 5

Extensions of Modules

Ext^1 *and Extensions—Local Case.* We consider again the local ring $\mathcal{O} = \mathbb{C}\{z_1, \ldots, z_n\}$ and finitely generated modules over $\mathcal{O}$. An *extension* of M by N is given by a short exact sequence

$$(E) \qquad 0 \to N \to E \to M \to 0.$$

The *trivial* or *split extension* is $M \oplus N$, and two extensions are *equivalent* in case there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & E & \to & M & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & N & \to & E' & \to & M & \to & 0 \end{array}$$

The name "Ext" is derived from the following

Lemma. *There is a bijective correspondence between equivalence classes of extensions and* $\mathrm{Ext}^1_{\mathcal{O}}(\mathrm{M},\mathrm{N})$, *with zero corresponding to the trivial extension.*

Proof. Given an extension (E) as above, we have

$$\mathrm{Hom}_{\mathcal{O}}(M,E) \to \mathrm{Hom}(M,M) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{O}}(M,N).$$

The obstruction to splitting the sequence (E) is $\partial(1_M) \in \mathrm{Ext}^1_{\mathcal{O}}(M,N)$, where

1_M is the identity map from M to itself. This gives the map from extensions to $\mathrm{Ext}^1_{\mathcal{O}}(M,N)$.

Before giving the inverse map, we need one remark. Given the data

$$\begin{cases} 0 \to R \xrightarrow{i} S \xrightarrow{\pi} M \to 0, \\ \quad R \xrightarrow{j} N, \end{cases}$$

we may construct an extension

$$(F) \qquad 0 \to N \to F \to M \to 0$$

as follows: Define $\mu = j \oplus i : R \to N \oplus S$ and $F = N \oplus S/\mu(R)$. Then $n \oplus s \to \pi(s)$ and $n \to n \oplus (0)$ gives the exact sequence (F).

To construct $\mathrm{Ext}^1_{\mathcal{O}}(M,N)$, we start with part of a projective resolution

$$E_2 \to E_1 \to E_0 \to M \to 0$$

and take kernel/image in

$$\mathrm{Hom}_{\mathcal{O}}(E_0,N) \to \mathrm{Hom}_{\mathcal{O}}(E_1,N) \to \mathrm{Hom}_{\mathcal{O}}(E_2,N).$$

Thus a cycle gives a map $E_1/E_2 \to N$, and so a class in $\mathrm{Ext}^1_{\mathcal{O}}(M,N)$ gives the data

$$\begin{cases} 0 \to E_1/E_2 \to E_0 \to M \to 0, \\ \quad E_1/E_2 \to N \end{cases}.$$

Applying the discussion of the previous paragraph gives an extension.

We leave it as an exercise to check that the two mappings

$$\left\{ \begin{matrix} \text{equivalent classes} \\ \text{of extensions} \end{matrix} \right\} \rightleftarrows \mathrm{Ext}^1_{\mathcal{O}}(M,\mathcal{O})$$

are well-defined and inverse to one another. Q.E.D.

Now suppose that $\mathcal{O} = \mathbb{C}\{z_1,z_2\}$ is the local ring in *two* variables and $I = \{f_1,f_2\}$ is a regular ideal. From the exact sequence

$$0 \to I \to \mathcal{O} \to \mathcal{O}/I \to 0$$

and computation of Ext's in the section on Koszul complexes we have

$$\mathrm{Ext}^1_{\mathcal{O}}(I,\mathcal{O}) \cong \mathrm{Ext}^2_{\mathcal{O}}(\mathcal{O}/I,\mathcal{O}) \cong \mathcal{O}/I.$$

The second isomorphism depends on the choice of generators for I, but the assertion:

$e \in \mathrm{Ext}^1_{\mathcal{O}}(\mathrm{I},\mathcal{O}) \cong \mathcal{O}/\mathrm{I}$ *is a unit—i.e.*, $e(0) \neq 0$,

has intrinsic meaning, since if also $I = \{f_1',f_2'\}$, then

$$f_i' = \sum_j a_{ij} f_j \quad \text{and} \quad e = \Delta e',$$

where $\Delta=\det(a_{ij})$ is a unit. For our construction of rank-two vector bundles on a surface, we will use the

Lemma. *Suppose that* $e \in \mathrm{Ext}^1_{\mathcal{O}}(I, \mathcal{O})$ *gives an extension*

$$0 \to \mathcal{O} \to E \to I \to 0.$$

Then E *is projective* $\Leftrightarrow$ e *is a unit.*

Proof. The exact sequence of Ext gives

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{O}}(I, \mathcal{O}) \to \mathrm{Ext}^1_{\mathcal{O}}(E, \mathcal{O}) \to 0.$$

By definition, $\partial(1)=e$, where 1 means the constant function "one" under the identification $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O},\mathcal{O}) \cong \mathcal{O}$. Identifying $\mathrm{Ext}^1_{\mathcal{O}}(I,\mathcal{O})$ with $\mathcal{O}/I$ and using that ∂ is $\mathcal{O}$-linear, if e is a unit,

$$\partial(e^{-1}) = 1 \in \mathcal{O}/I.$$

It follows that ∂ is surjective, and so $\mathrm{Ext}^1_{\mathcal{O}}(E,\mathcal{O})=0$. It is trivially the case that $\mathrm{Ext}^q_{\mathcal{O}}(E,\mathcal{O})=0$ for $q \geq 2$.

We use this to prove that $\mathrm{Ext}^q_{\mathcal{O}}(M,N)=0$ for any $\mathcal{O}$-module N and $q \geq 1$. The argument is by induction on the length of a projective resolution of N. Thus we may assume given

$$0 \to R \to F \to N \to 0,$$

where F is free and $\mathrm{Ext}^q_{\mathcal{O}}(M,R)=0$ for $q \geq 1$. The exact sequence of $\mathrm{Ext}^*_{\mathcal{O}}(M,\cdot)$ then gives the result.

From our original discussion of Ext, it follows from the vanishing of $\mathrm{Ext}^1_{\mathcal{O}}(E,N)$ for all N implies that E is projective (= free); this happens if e is a unit. If e is not a unit, then $\mathrm{Ext}^1_{\mathcal{O}}(E,\mathcal{O}) \neq 0$ and E is not projective. Q.E.D.

Ext^1 *and Extensions—Global Case.* Let M be an algebraic variety and $\mathcal{F},\mathcal{G}$ coherent sheaves on M. We may speak of *global extensions*

$$(\mathcal{E}) \qquad 0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0,$$

by which we mean an exact sequence of sheaves of $\mathcal{O}$-modules—then $\mathcal{E}$ is necessarily coherent—and with the equivalence relation and notion of trivial extension as in the local case. One's first guess might be that such $(\mathcal{E})$'s are in bijective correspondence with $H^0(M, \underline{\mathrm{Ext}}^1_{\mathcal{O}}(\mathcal{F},\mathcal{G}))$. This is not quite correct for the following reason:

Given a global section of $H^0(M, \underline{\mathrm{Ext}}^1_{\mathcal{O}}(\mathcal{G},\mathcal{F}))$, choose a sufficiently fine covering $\underline{\mathrm{U}} = \{\mathrm{U}_\alpha\}$ and corresponding local extensions

$$(\mathcal{E}_\alpha) \qquad 0 \to \mathcal{F}|\mathrm{U}_\alpha \to \mathcal{E}_\alpha \to \mathcal{G}|\mathrm{U}_\alpha \to 0.$$

In $\mathrm{U}_\alpha \cap \mathrm{U}_\beta$ there will be a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathcal{F}|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to & \mathcal{E}_\alpha|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to & \mathcal{G}|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to 0 \\ & \| & & \downarrow \varphi_{\alpha\beta} & & \| & \\ 0 \to & \mathcal{F}|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to & \mathcal{E}_\beta|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to & \mathcal{G}|\mathrm{U}_\alpha \cap \mathrm{U}_\beta & \to 0, \end{array}$$

but it need *not* be the case that in $U_\alpha \cap U_\beta \cap U_\gamma$ the triangle

$$\begin{array}{ccc} & \mathcal{E}_\alpha & \\ \swarrow{\scriptstyle \varphi_{\alpha\beta}} & & \searrow{\scriptstyle \varphi_{\gamma\beta}} \\ \mathcal{E}_\beta & \underset{\varphi_{\beta\gamma}}{\longrightarrow} & \mathcal{E}_\gamma \end{array}$$

is commutative—i.e., the "transition functions" for gluing the local extensions $(\mathcal{E}_\alpha)$ may not satisfy the cocycle rule, and thus may not patch together to give a global extension. What is true is the

Lemma. *The equivalence classes of global extensions* $(\mathcal{E})$ *are in bijective correspondence with* $\mathrm{Ext}^1(M;\mathcal{G},\mathcal{F})$.

Proof. Given $(\mathcal{E})$, the exact sequence of global Ext's gives

$$\begin{array}{ccccc} \mathrm{Ext}^0(M;\mathcal{G},\mathcal{E}) & \to & \mathrm{Ext}^0(M;\mathcal{G},\mathcal{G}) & \xrightarrow{\partial} & \mathrm{Ext}^1(M;\mathcal{G},\mathcal{F}) \to \cdots \\ \| & & \| & & \\ H^0(M, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{G},\mathcal{E})) & \to & H^0(M, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{G},\mathcal{G})) & & \end{array}$$

and the obstruction to splitting the sequence $(\mathcal{E})$ is just $\partial(1_{\mathcal{G}})$ as in the local case.

The converse is more interesting. Let $\mathcal{E}_\cdot(\mathcal{G}): \cdots \to \mathcal{E}_2 \xrightarrow{\partial} \mathcal{E}_1 \xrightarrow{\partial} \mathcal{E}_0 \to \mathcal{G} \to 0$ be the global syzygy for $\mathcal{G}$, and $\underline{U} = \{U_\alpha\}$ a sufficiently fine covering of M so that a class $e \in \mathrm{Ext}^1(M;\mathcal{G},\mathcal{F})$ is given by a cocycle in the hypercohomology group

$$\mathbb{H}^1(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_\cdot(\mathcal{G}),\mathcal{F})).$$

In the diagram

$$\begin{array}{ccccc} & C^0(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_1,\mathcal{F})) & \oplus & C^1(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_0,\mathcal{F})) & \\ \swarrow{\scriptstyle \partial^*} & \searrow{\scriptstyle \delta} & & \swarrow{\scriptstyle \partial^*} & \searrow{\scriptstyle \delta} \\ C^0(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_2,\mathcal{F})) & \oplus & C^1(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_1,\mathcal{F})) & \oplus & C^2(\underline{U}, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_0,\mathcal{F})), \end{array}$$

The cocycle e is given by $\varphi \oplus \eta$, where

$$\varphi = \{\varphi_\alpha\} \quad \text{with } \varphi_\alpha \in H^0(U_\alpha, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_1,\mathcal{F})),$$
$$\eta = \{\eta_{\alpha\beta}\} \quad \text{with } \eta_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_0,\mathcal{F})).$$

Writing out the conditions that e be a cocycle gives the relations:

(1) $\quad \partial^*\varphi = 0 \Rightarrow \varphi_\alpha \in H^0(U_\alpha, \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}_1/\mathcal{E}_2,\mathcal{F}))$

$\qquad \Rightarrow \varphi_\alpha$ defines an extension

$(\mathcal{E}_\alpha)$ $\quad 0 \to \mathcal{F}|U_\alpha \to \mathcal{E}_\alpha \to \mathcal{G}|U_\alpha \to 0$

by the same argument as in the proof of the first lemma in the preceding section,

(2) $\delta\varphi = \partial^*\eta \Rightarrow \varphi_\alpha - \varphi_\beta = \partial^*\eta_{\alpha\beta}$

$\Rightarrow$ the local extensions $(\mathcal{E}_\alpha)$ given above patch together in double intersections $U_\alpha \cap U_\beta$; and

(3) $\delta\eta = 0$ $\Rightarrow$ the cocycle rule for the patchings of the local extensions in double intersections.

Admittedly, step 3 needs some amplification, but the details to be checked are straightforward enough. Q.E.D.

Points on a Surface and Rank-Two Vector Bundles

As an application of the global duality theorem (I), we shall discuss the following question:* *Given an algebraic surface* S *and sheaf of regular ideals* $I \subset \mathcal{O}$ *with* $\text{supp}(\mathcal{O}/I)$ *a set of points* Z, *we define* $\mathcal{O}_Z = \mathcal{O}/I$ *and ask whether there exists a rank-two holomorphic vector bundle* $E \to S$ *with given first Chern class* $c_1(E)$ *and section* $s \in H^0(S, \mathcal{O}(E))$ *whose divisor* (s) *is* Z *ideal-theoretically*?

To answer this question, suppose first that $E \to S$ is the rank-two bundle and $s \in H^0(S, \mathcal{O}(E))$ the holomorphic section with divisor $(s) = Z$ that we are trying to construct. We may consider s as a sheaf mapping $\mathcal{E}^* \xrightarrow{s} \mathcal{O}$, $\mathcal{E}^* = \mathcal{O}(E^*)$, and what we are asking for is a short exact sequence

$$(*) \qquad 0 \to \mathcal{L} \xrightarrow{i} \mathcal{E}^* \xrightarrow{s} I \to 0$$

where $\mathcal{L}$ is locally free of rank one.

Proof. Locally, $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$ and $s = (f_1, f_2)$. The map $\mathcal{E}^* \xrightarrow{s} I$ is given by

$$(g_1, g_2) \to f_1 g_1 + f_2 g_2,$$

and by comparison with the Koszul complex—of which this is the first step—$\mathcal{L}$ is locally isomorphic to $\mathcal{O}$ in such a way that $i(h) = (-f_2 h, f_1 h)$. Q.E.D.

The relation between the line bundle $\mathcal{L}$ and vector bundle $\mathcal{E}$ is

$$c_1(\mathcal{L}) = -c_1(\mathcal{E}).$$

*This was first considered by R. L. E. Schwarzenberger, Vector bundles on algebraic surfaces, *Proc. London Math. Soc.*, Vol. 11 (1961), pp. 601–622, and Vector bundles on the projective plane, *loc. cit.*, Vol. 11 (1961), pp. 623–640.

Proof. On $S^* = S - Z$ we have

$$
\begin{aligned}
&0 \to \mathcal{L}|S^* \to \mathcal{E}^*|S^* \to \mathcal{O}_{S^*} \to 0 \\
&\Rightarrow c_1(\mathcal{L}) = c_1(\mathcal{E}^*) \qquad \text{in } H^2(S^*, \mathbb{Z}) \\
&\Rightarrow c_1(\mathcal{L}) = c_1(\mathcal{E}^*) \qquad \text{in } H^2(S, \mathbb{Z}),
\end{aligned}
$$

since in the exact cohomology sequence

$$H^2(S, S^*; \mathbb{Z}) \to H^2(S, \mathbb{Z}) \to H^2(S^*, \mathbb{Z})$$

we have by excision

$$
\begin{aligned}
H^2(S, S^*, \mathbb{Z}) &\cong \sum_{p \in Z} H^2(B_p, B_p^*; \mathbb{Z}) \\
&= 0,
\end{aligned}
$$

where B_p is a ball around p. Q.E.D.

Actually this makes sense, since, very roughly speaking, giving $\mathcal{E}$ is the same as giving its Chern classes $c_1(E)$ and $c_2(E)$, and $c_2(E)$ is just Z. The assertion $c_1(\mathcal{L}) = -c_1(\mathcal{E})$ may be refined to

$$\mathcal{L} = \wedge^2 \mathcal{E}^* \quad \text{in} \quad \text{Pic}(S) = H^1(S, \mathcal{O}^*),$$

which follows from the Levi Extension Theorem given in Section 2 of Chapter 3.

Referring to (*), we may rephrase the problem as follows: *Given* $(Z, \mathcal{O}_Z)$ *and* $\mathcal{L} \in \text{Pic}(S)$, *we seek*

$$
(**) \qquad \begin{cases} e \in \text{Ext}^1(S; I, \mathcal{L}), & \textit{such that } e_p \textit{ is a unit in} \\ \underline{\text{Ext}}^1(I, \mathcal{L})_p \cong \mathcal{O}_{Z,p} \textit{ for each point } p \in Z. \end{cases}
$$

Explanation. For any open set $U \subset S$, there is a restriction mapping

$$\text{Ext}^*(S; \mathcal{F}, \mathcal{G}) \to \text{Ext}^*(U; \mathcal{F}, \mathcal{G})$$

of global Ext's. For U sufficiently small so that $H^q(U, \underline{\text{Ext}}^*_{\mathcal{O}}(\mathcal{F}, \mathcal{G})) = 0$ for $q > 0$,

$$\text{Ext}^*(U; \mathcal{F}, \mathcal{G}) \cong H^0(U, \underline{\text{Ext}}^*_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$$

by the spectral sequence relating local and global Ext's. Thus $e \in \text{Ext}^1(S; I, \mathcal{L})$ induces e_p in each stalk $\underline{\text{Ext}}^1_{\mathcal{O}}(I, \mathcal{L})_p$ for any point $p \in S$.

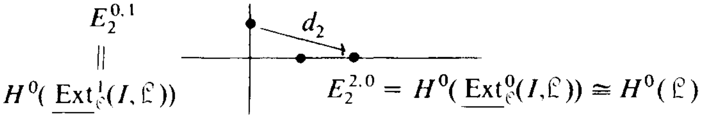
By the discussion in the preceding section, a class $e \in \text{Ext}^1(S; I, \mathcal{L})$ defines a global extension

$$0 \to \mathcal{L} \to \mathcal{E}^* \to I \to 0$$

over S, and by the lemma on p. 724 the coherent sheaf $\mathcal{E}^*$ will be locally free if each e_p is a unit for $p \in Z$. *Thus, solving* (**) *and finding* (*) *are entirely equivalent.*

The first approximation to understanding (**) is to look into the spectral sequence relating local and global Ext's.

Using that $E_2^{p,q} = H^p(S, \underline{\mathrm{Ext}}_{\mathcal{O}}^q(I, \mathcal{L}))$, the picture of E_2 is



where the isomorphism results from

$$\begin{cases} 0 \to I \to \mathcal{O} \to \mathcal{O}_Z \to 0, & \text{and} \\ \underline{\mathrm{Ext}}_{\mathcal{O}}^0(\mathcal{O}_Z, \mathcal{L}) = 0 \end{cases}$$
$$\Rightarrow \mathcal{L} \cong \underline{\mathrm{Ext}}_{\mathcal{O}}^0(\mathcal{O}, \mathcal{L}) \cong \underline{\mathrm{Ext}}_{\mathcal{O}}^0(I, \mathcal{L}),$$

using the exact sequence of Ext.

In particular, if $E_2^{2,0} = 0$, then $\mathrm{Ext}^1(S; I, \mathcal{L}) \cong H^0(S, \underline{\mathrm{Ext}}_{\mathcal{O}}^1(I, \mathcal{L}))$, and (**) may be solved. Thus: *If* $\mathrm{H}^2(\mathrm{S}, \mathcal{L}) = 0$, *then we may find a rank-two holomorphic vector bundle* $\mathrm{E} \to \mathrm{S}$ *and section* $\mathrm{s} \in \mathrm{H}^0(\mathrm{S}, \mathcal{E})$ *such that* $\mathrm{c}_1(\mathcal{L}) = -\mathrm{c}_1(\mathcal{E})$ *and* s *defines* Z *ideal-theoretically. In particular, we may take* $\mathcal{L} = \mathcal{O}$ *in case* $\mathrm{p_g(S)} = 0$. *Taking* $\mathcal{L}$ *to be sufficiently ample, we may always arrange that* $\mathrm{H}^2(\mathrm{S}, \mathcal{L}) = 0$, *so that our original problem will have at least one solution.*

Example

Take $S = \mathbb{P}^2$, so that the $p_g = 0$ condition is satisfied. Then there exists a rank-two holomorphic vector bundle $E \to \mathbb{P}^2$ and section $s \in H^0(\mathbb{P}^2, \mathcal{O}(E))$ that defines any given Z. If Z is nonempty, then we claim that s is unique up to a constant, and the vector bundle E is *not* a global extension

$$0 \to L \to E \to L' \to 0$$

of line bundles on $\mathbb{P}^2$. Thus, in this manner we have found a whole collection of "new" vector bundles over $\mathbb{P}^2$.

Proof. If $s' \in H^0(\mathbb{P}^2, \mathcal{O}(E))$ also defines Z, then $s \wedge s' \in H^0(\mathbb{P}^2, \mathcal{O}(\Lambda^2 E)) = H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C}$, since $\wedge^2 E$ is a trivial line bundle because $c_1(E) = 0$. Thus either $s \wedge s' \equiv 0$, in which case s' is a constant multiple of s, or else $s \wedge s'$ is nowhere zero, which is excluded by the assumption that Z is nonempty.

If E is a global extension of line bundles, then $\mathcal{L} \cong \mathcal{O}(n)$ and $\mathcal{L}' \cong \mathcal{O}(n')$, since $\mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$. Now $\mathrm{Ext}^1(\mathbb{P}^2; \mathcal{L}', \mathcal{L}) \cong H^1(\mathbb{P}^2, \mathcal{O}(n - n')) = 0$. Also, $n + n' = 0$, since $c_1(E) = 0$. Thus

$$\mathcal{E} \cong \mathcal{O}(n) \oplus \mathcal{O}(-n),$$

where $n \geqslant 0$, and this is a contradiction, since any section of $\mathcal{E}$ is either nowhere zero ($n = 0$) or else vanishes on a curve ($n > 0$). Q.E.D.

We still have not found necessary and sufficient conditions that (**) may be solved. To do this, we assume for simplicity that

$$\begin{cases} \mathcal{L} = \mathcal{O}, & \text{and} \\ I_p = m_p & \text{is the maximal ideal for each point } p \in Z. \end{cases}$$

The exact sequence of global Ext applied to

$$0 \to I \to \mathcal{O} \to \mathcal{O}_Z \to 0$$

gives

$$\cdots \to H^1(S,\mathcal{O}) \to \mathrm{Ext}^1(S;I,\mathcal{O}) \to \mathrm{Ext}^2(S;\mathcal{O}_Z,\mathcal{O}) \to H^2(S,\mathcal{O}) \to \cdots .$$

Since $\underline{\mathrm{Ext}}^q_{\mathcal{O}}(\mathcal{O}_Z,\mathcal{O})=0$ for $q\neq 2$ while $\underline{\mathrm{Ext}}^2_{\mathcal{O}}(\mathcal{O}_Z,\mathcal{O})$ is a skyscraper sheaf concentrated on Z with stalks

$$\underline{\mathrm{Ext}}^2_{\mathcal{O}}(\mathcal{O}_Z,\mathcal{O})_p \cong \Lambda^2 T_p'(S),$$

the above exact sequence is the top row in the diagram

$$(***)\qquad \begin{array}{ccccc} \mathrm{Ext}^1(S;I,\mathcal{O}) & \to & \bigoplus_{p\in Z} \wedge^2 T_p'(S) & \to & H^2(S,\mathcal{O}) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathrm{Ext}^1(S;I,\mathcal{O})^* & \leftarrow & \bigoplus_{p\in Z} \wedge^2 T_p'^*(S) & \overset{\rho}{\leftarrow} & H^0(S,\Omega^2) \end{array}$$

The bottom row are the dual vector spaces, where the dual of $\mathrm{Ext}^2(S,\mathcal{O}_Z,\mathcal{O})$ is $H^0(S,\mathcal{O}_Z\otimes\Omega^2)$ by the duality theorem (I). By functorality, the mapping ρ is simply the restriction of a global holomorphic 2-form on S to each point $p\in Z$.

Now what we are seeking is

$$e\in \mathrm{Ext}^1(S;I,\mathcal{O}) \qquad \text{with } e_p\neq 0 \text{ in each } \wedge^2 T_p'(S) \quad (p\in Z).$$

Applying the duality in (***), we have the following result:

Given a set of points $Z\subset S$, *there is a rank-two holomorphic vector bundle* $E\to S$ *with* $\wedge^2\mathcal{E}\cong\mathcal{O}$ *and section* $s\in H^0(S,\mathcal{O}(\mathcal{E}))$ *that defines* Z $\Leftrightarrow$ *there are bivectors* $0\neq\tau_p\in\wedge^2 T_p'(S)$ $(p\in Z)$ *such that*

$$\sum_{p\in Z}\langle\psi,\tau_p\rangle = 0$$

for all $\psi\in H^0(S,\Omega^2)$. *In particular, if* $\deg Z > p_g(S)$, *then* (E,s) *always exists.*

It remains to intrinsically interpret this relation, which we shall do in the next section.

Residues and Vector Bundles

We will interpret the relation at the end of the preceding section as a residue theorem for general vector bundles and will then put our conclusions in a more geometric form.

Suppose that M is a compact, complex manifold of dimension n and

$$\mathbb{C}^n \to E \to M$$

is a rank-n vector bundle having a holomophic section $s \in H^0(M, \mathcal{O}(E))$ with a set Z of isolated zeros. Using the notations

$$\begin{cases} \bigwedge^q \mathcal{E}^* = \mathcal{O}(\bigwedge^q E^*), \\ \mathcal{I}_Z = \text{ideal sheaf of } Z \text{ and } \mathcal{O}_Z = \mathcal{O}/I_Z, \end{cases}$$

the sequence

$$(*) \qquad 0 \to \bigwedge^n \mathcal{E}^* \xrightarrow{s} \cdots \to \bigwedge^2 \mathcal{E}^* \xrightarrow{s} \mathcal{E}^* \to \mathcal{I}_Z \to 0$$

localizes to the Koszul complex, and therefore gives a global projective resolution of both $\mathcal{O}_Z$ and $\mathcal{I}_Z$. In particular (*) gives an element

$$e \in \operatorname{Ext}^{n-1}(M; \mathcal{I}_Z, \bigwedge^n \mathcal{E}^*).$$

In the spectral sequence relating global and local Ext's we consider

$$d_{n-1}: H^0\big(M; \underline{\operatorname{Ext}}^{n-1}_{\mathcal{O}}(\mathcal{I}_Z, \bigwedge^n \mathcal{E}^*)\big) \to H^n(M, \bigwedge^n \mathcal{E}^*),$$

where we have used the isomorphism

$$\underline{\operatorname{Ext}}^0_{\mathcal{O}}(\mathcal{I}_Z, \bigwedge^n \mathcal{E}^*) \cong \bigwedge^n \mathcal{E}^*$$

from the section on Koszul complexes. For each $p \in Z$ there is an induced local extension class

$$e_p \in \underline{\operatorname{Ext}}^{n-1}_{\mathcal{O}}(\mathcal{I}_Z, \bigwedge^n \mathcal{E}^*)_p,$$

where $\bigoplus_{p \in Z} e_p \in H^0(M, \underline{\operatorname{Ext}}^{n-1}_{\mathcal{O}}(\mathcal{I}_Z, \bigwedge^n \mathcal{E}^*))$ is the image of e, and therefore satisfies

$$d_{n-1}\Big(\bigoplus_{p \in Z} e_p\Big) = 0 \qquad \text{in } H^n(M, \bigwedge^n \mathcal{E}^*).$$

We will interpret this relation as a residue theorem.

For this we consider the vector space $H^0(M, \mathcal{O}(K \otimes \det E))$ dual to $H^n(M, \bigwedge^n \mathcal{E}^*)$. In terms of a local holomorphic frame $e_1, \ldots, e_n$ for E and local holomorphic coordinates $z_1, \ldots, z_n$ on M, a section

$$\psi \in H^0(M, \mathcal{O}(K \otimes \det E))$$

is

$$\psi = h(z)(dz_1 \wedge \cdots \wedge dz_n) \otimes (e_1 \wedge \cdots \wedge e_n).$$

Writing

$$s = s_1(z)e_1 + \cdots + s_n(z)e_n,$$

we consider the form

$$\frac{\psi}{s} = \frac{h(z)dz_1 \wedge \cdots \wedge dz_n}{s_1(z) \cdots s_n(z)}.$$

Of course the right-hand side is not well-defined, but by the transformation formula the *residue* at a point $p \in Z$

$$(**) \qquad \operatorname{Res}_p\left(\frac{\psi}{s}\right) = \operatorname{Res}_p\left\{\frac{h(z)dz_1\wedge\cdots\wedge dz_n}{s_1(z)\cdots s_n(z)}\right\}$$

is independent of choices. Because of the functorial property of duality we have

$$\begin{aligned} 0 &= \langle d_{n-1}\Big(\bigoplus_{p\in Z} e_p\Big),\psi\rangle \\ &= \sum_{p\in Z}\operatorname{Res}_p\left(\frac{\psi}{s}\right), \end{aligned}$$

which we may state formally as the

Residue Theorem for Vector Bundles. *Given a rank-*n *holomorphic vector bundle* $E\to M$ *over a compact, complex* n*-manifold and holomorphic section* $s\in H^0(M,\mathcal{O}(E))$ *having a set* Z *of isolated zeros, if for each* $\psi\in H^0(M,\mathcal{O}(K\otimes \det E))$ *and* $p\in Z$ *we define the residue*

$$\operatorname{Res}_p\left(\frac{\psi}{s}\right)$$

by (**) *above, then*

$$\sum_{p\in Z}\operatorname{Res}_p\left(\frac{\psi}{s}\right) = 0.$$

Corollary (Cayley-Bacharach for Vector Bundles). *If* Z *consists of distinct simple points, then each* $D\in|K\otimes\det E|$ *that passes through all but one point of* Z *necessarily contains that remaining point.*

The result at the end of the preceding section may be rephrased as:

Corollary. *On an algebraic surface* S *given a set* Z *of isolated points and holomorphic line bundle* L, *there exists a rank-two holomorphic vector bundle* $\mathbb{C}^2\to E\to S$ *with* $\det E = L$ *and having a section* $s\in H^0(S,\mathcal{O}(E))$ *with* $(s)=Z$ *if, and only if,* Z *has the Cayley-Bacharach property relative to the linear system* $|K\otimes L|$.

Finally, the Cayley-Bacharach property may be given a nice geometric interpretation in case the linear system $|K\otimes L|$ gives a base-point-free mapping

$$S\to\mathbb{P}^r.$$

We denote by $\bar{Z}$ the linear span in $\mathbb{P}^r$ of a set Z of d distinct points on S. For generic Z, $\dim\bar{Z} = d-1$; the Cayley-Bacharach property implies that

$$\dim\bar{Z} = d-2-\rho \qquad (\rho\geq 0),$$

so that configurations Z satisfying that property may be roughly thought of as "multisecant planes" such as trichords, etc.

REFERENCES

We give a few sources to assist the reader in amplifying the discussion in this chapter. These will also serve as a guide to the literature.

Section 1

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Section 2

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Section 5

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6
THE QUADRIC LINE COMPLEX

This chapter occupies a somewhat anomalous position in the book: it falls, in fact, somewhere between a chapter and a protracted exercise. No new ground is broken: even the discussion of quadrics in Section 1 represents a gap in the previous material rather than further development. There are three reasons for its inclusion:

First of all in Chapters 2 and 4 we have discussed in some detail the theory of curves and surfaces; it is natural now to look at varieties of higher dimension, such as threefolds. Unfortunately, there is for threefolds no systematic body of knowledge comparable to what we have for curves and surfaces. Whatever the reason, the fact is that the only wholly successful treatment of threefolds has been in special cases; this is one such.

Second, while we have tried to provide applications of the theory and techniques developed in this book, we have not yet encountered a problem broad enough to bring to bear the full range of our techniques. The quadric line complex is just such a problem: in the course of our analysis of it we will have occasion to call upon results from Hodge theory, curves, Abelian varieties, surfaces, Chern classes, and the Schubert calculus.

The third and final reason for the inclusion of this chapter is simply the subject itself. The quadric line complex is an object of long-standing attraction: much of the material that follows was developed in the mid-nineteenth century and is still of interest today. It is a subject full of intricate symmetries and surprises; we hope the reader will find it as delightful to study as we did.

1. PRELIMINARIES: QUADRICS

Rank of a Quadric

A quadric hypersurface $F \subset \mathbb{P}^n$ may be represented as the locus of a quadratic form

$$Q(X,X) = \sum_{i,j=0}^{n} q_{ij} X_i X_j$$

with the matrix $Q=(q_{ij})$ symmetric. The *rank* of the quadric F is defined to be the rank of the matrix Q; since the only invariant of a symmetric quadratic form over $\mathbb{C}$ is its rank, two quadrics $F, F' \subset \mathbb{P}^n$ will be projectively isomorphic if and only if they have the same rank. Now, taking partials,

$$\frac{\partial}{\partial X_i} Q(X,X) = 2\sum_j q_{ij} X_j,$$

we deduce that the singular locus of F is just the linear subspace of $\mathbb{P}^n$ corresponding to the kernel of the matrix Q on $\mathbb{C}^{n+1}$; thus

A quadric $F \subset \mathbb{P}^n$ *is smooth if and only if it has maximal rank* $n+1$,

and more generally,

A quadric $F \subset \mathbb{P}^n$ *of rank* $n-k$ *is singular along a* k*-plane* $\Lambda \subset F \subset \mathbb{P}^n$.

Indeed, we can be more explicit in our description: suppose $F \subset \mathbb{P}^n$ is a quadric of rank k with singular set $\Lambda \cong \mathbb{P}^{n-k}$, and take V_{k-1} a generic $(k-1)$-plane complementary to, i.e., disjoint from, Λ; $\tilde{F} = F \cap V_{k-1}$ is then a smooth quadric of dimension $k-2$. Now if L is any line in $\mathbb{P}^n$ meeting both Λ and $\tilde{F}$, L meets F three times and so is contained in F. Conversely, if $p \in F$ is any point lying off Λ, the $(n-k+1)$-plane spanned by p and Λ must meet V_{k-1} in a point q. The line $L=\overline{pq}$ then meets F at p and twice again in Λ, and so lies in F; in particular $q \in \tilde{F}$, so p lies on a line joining Λ and $\tilde{F}$. Consequently

A quadric $F \subset \mathbb{P}^n$ *of rank* k *is the cone through an* $(n-k)$*-plane* $\Lambda \subset F \subset \mathbb{P}^n$ *over a quadric of rank* k *in* $\mathbb{P}^{k-1}$.

Note, incidentally, that since F contains all lines joining any point $p \in F$ to Λ, the tangent plane to F at any point contains Λ. Thus, any plane in $\mathbb{P}^n$ disjoint from Λ intersects F smoothly.

We can see most of this in terms of the Gauss map

$$\mathcal{G} : F \to \mathbb{P}^{n*}$$

defined by sending a point $p \in F$ to its tangent plane $T_p(F) \in \mathbb{P}^{n*}$. Since the tangent plane to the quadric given by Q above at a point $p = [a_0, \dots, a_n]$ is

$$T_p(F) = \left(\sum_{i,j} q_{ij} a_j X_i = 0 \right),$$

we see that the Gauss map on F is just the restriction to F of the rational map $\mathbb{P}^n \to \mathbb{P}^{n*}$ given by the matrix Q. If F is smooth, $\mathcal{G}$ is an isomorphism, and the dual variety $F^* = \mathcal{G}(F) \subset \mathbb{P}^{n*}$ is again a smooth quadric. Note that in this case no hyperplane in $\mathbb{P}^n$ will be tangent to F more than once; so every tangent hyperplane section $T_p(F) \cap F$ of a smooth quadric in $\mathbb{P}^n$ has rank $n-1$, i.e., is the cone through p over a smooth quadric in $\mathbb{P}^{n-2}$. In general, if F has rank k and singular set Λ_{n-k}, then every tangent hyperplane to F contains Λ, and $\mathcal{G}$ maps F to a smooth quadric in the subspace $\mathbb{P}^{k-1*} \subset \mathbb{P}^{n*}$ of hyperplanes containing Λ.

Linear Spaces on Quadrics

One fascinating aspect of quadrics is the behavior of the linear spaces lying on them. This is described in the

Proposition. *A smooth quadric* F *of dimension* m *contains no linear spaces of dimension strictly greater than* m/2; *on the other hand*

1. *If* $m = 2n+1$ *is odd, then* F *contains an irreducible* $(n+1)(n+2)/2$-*dimensional family of* n-*planes*; *while*
2. *If* $m = 2n$ *is even, then* F *contains two irreducible*, $n(n+1)/2$-*dimensional families of* n-*planes and moreover for any two* n-*planes* $\Lambda, \Lambda' \subset F$,

$$\dim(\Lambda \cap \Lambda') \equiv n \ (2)$$

if and only if Λ *and* Λ' *belong to the same family.*

Before we prove this, note that we have already observed this phenomenon in the case $m=2$: on a quadric surface in $\mathbb{P}^3$ there are two one-dimensional families of lines; and two lines of opposite families always meet in a point, while lines of the same family are either disjoint or meet in a line. The reader is also referred to the discussion in Section 2 of the geometry of the Grassmannian $G(2,4)$ to see in detail the behavior of 2-planes on a quadric in $\mathbb{P}^5$.

The first statement of the proposition is readily verified: since the Gauss map $\mathcal{G}$ on a smooth quadric $F \subset \mathbb{P}^{m+1}$ is the restriction of a linear isomorphism $\mathbb{P}^{m+1} \rightarrow \mathbb{P}^{m+1*}$, the family of tangent planes to F along a linear subspace $\Lambda_k \subset F$ forms a k-dimensional linear subspace of $\mathbb{P}^{m*}$. Since the tangent space to F at any point of Λ contains Λ, moreover, the image $\mathcal{G}(\Lambda)$ lies entirely in the $(m-k)$-dimensional subspace of $\mathbb{P}^{m+1*}$ of planes through Λ; thus

$$k \leqslant m - k$$

i.e., $k \leqslant m/2$.

To prove the remainder of the proposition we use an induction on n. Let $\Sigma'_n \subset G(n+1, 2n+3)$ be the set of n-planes Λ lying on a smooth quadric F of odd dimension $2n+1$ in $\mathbb{P}^{2n+2}$, and $\Sigma_n \subset G(n+1, 2n+2)$ the family of n-planes on a smooth $F_{2n} \subset \mathbb{P}^{2n+1}$. Assume the statement of the proposition for $m < n$ (it is trivially true for $n=0$), and for $F \subset \mathbb{P}^{2n+2}$ a smooth quadric consider the incidence correspondence

$$I \subset F \times G(n+1, 2n+3)$$

defined by

$$I = \{(p, \Lambda_n): \ p \in \Lambda \subset F\}.$$

The projection map $\pi_2: I \rightarrow G(n+1, 2n+3)$ maps I onto Σ'_n, with fibers isomorphic to $\mathbb{P}^n$. On the other hand, consider the fibers of the projection $\pi_1: I \rightarrow F$, that is, the n-planes on F passing through a point p. Clearly, any such n-plane Λ lies in the tangent plane to F at p, and hence in the intersection $F \cap T_p(F)$. But we have seen that $F \cap T_p(F)$ is just the cone through p over a smooth quadric $\tilde{F}_{2n-1} \subset \mathbb{P}^{2n}$, and so the n-planes in F through p are exactly the n-planes spanned by p together with $(n-1)$-planes in $\tilde{F}$. The fibers of π_1 are therefore isomorphic to Σ'_{n-1}, which by hypothesis is irreducible of dimension $n(n+1)/2$; it follows that I itself is irreducible of dimension $n(n+1)/2 + 2n + 1$. Finally, since the map $\pi_2: I \rightarrow \Sigma'_n$ has fiber dimension n, we see that Σ'_n is irreducible of dimension

$$\frac{n(n+1)}{2} + 2n + 1 - n = \frac{(n+1)(n+2)}{2},$$

and part 1 of our proposition is proved.

Now let $F_{2n} \subset \mathbb{P}^{2n+1}$ be a smooth quadric; again, we set

$$I = \{(p, \Lambda_n): P \in \Lambda \subset F\} \subset F \times G(n+1, 2n+2).$$

As before, the fibers of the projection map $\pi_2: I \rightarrow \Sigma_n \subset G(n+1, 2n+2)$ are isomorphic to $\mathbb{P}^n$, and the fibers of $\pi_1: I \rightarrow F$ isomorphic to Σ_{n-1}. In this case, however, by induction Σ_{n-1} is the disjoint union of two irreducible varieties of dimension $n(n-1)/2$. The connected components of the fibers of π_1 thus constitute an unbranched 2-sheeted cover of F, which, since F is rational *and hence simply connected*, must be disconnected. It follows that I has two connected components, each mapping via π_1 onto F with fibers isomorphic to one irreducible component of Σ_{n-1}; as in the last argument, each of the connected components of I is irreducible of dimension $n(n-1)/2+2n$. Since the fibers of the projection map $\pi_2: I \to \Sigma_n$ are irreducible of dimension n, we see that Σ_n has two connected components Σ_n^1 and Σ_n^2, each irreducible of dimension

$$\frac{n(n-1)}{2} + 2n - n = \frac{n(n+1)}{2}.$$

Finally, it remains to show that for any two n-planes $\Lambda, \Lambda' \subset F$, the dimension of their intersection is congruent to n mod 2 if and only if they belong to the same family. Again, we proceed by induction: the statement is trivially true for $n=0$ (and more visibly for $n=1$); assume it for all $m<n$. Suppose first that Λ and Λ' intersect, and let p by any point of $\Lambda \cap \Lambda'$. Let $\mathbb{P}^{2n}$ be any hyperplane in $\mathbb{P}^{2n+1}$ not containing p; by what we have seen, the intersection $F \cap T_p(F)$ of F with its tangent plane at p is just the cone through p over the smooth, $(2n-2)$-dimensional quadric $\tilde{F} = F \cap T_p(F) \cap \mathbb{P}^{2n}$. Set

$$\tilde{\Lambda} = \Lambda \cap \mathbb{P}^{2n} \quad \text{and} \quad \tilde{\Lambda}' = \Lambda' \cap \mathbb{P}^{2n};$$

$\tilde{\Lambda}$ and $\tilde{\Lambda}'$ are then $(n-1)$-planes in $\tilde{F}$, and by our previous argument Λ and Λ' belong to the same family on F if and only if $\tilde{\Lambda}$ and $\tilde{\Lambda}'$ belong to the same family on $\tilde{F}$. But the intersection $\Lambda \cap \Lambda'$ is just the plane spanned by the intersection $\tilde{\Lambda} \cap \tilde{\Lambda}'$ together with p. By the induction hypothesis we have

$$\begin{aligned}
&\Lambda, \Lambda' \text{ belong to the same family of } n\text{-planes on } F \\
\Leftrightarrow\, &\tilde{\Lambda}, \tilde{\Lambda} \text{ belong to the same family of } (n-1)\text{-planes on } \tilde{F} \\
\Leftrightarrow\, &\dim(\tilde{\Lambda} \cap \tilde{\Lambda}') \equiv n - 1\,(2) \\
\Leftrightarrow\, &\dim(\Lambda \cap \Lambda') = \dim(\tilde{\Lambda} \cap \tilde{\Lambda}') + 1 \equiv n\,(2),
\end{aligned}$$

and we are done. Suppose on the other hand that Λ and Λ' are disjoint. In this case, take any point $p \in \Lambda$ and set

$$\Lambda'' = \overline{\Lambda' \cap T_p(F), p}\,.$$

Now $T_p(F)$ cannot contain Λ'—all n-planes in $T_p(F) \cap F$ contain p and hence meet Λ—so Λ'' is an n-plane and

$$\dim(\Lambda' \cap \Lambda'') = n - 1,$$

and we deduce from our first argument that Λ' and Λ'' belong to opposite families. We also see that Λ meets Λ'' only in the point p—if $\Lambda \cap \Lambda''$ contained a line, Λ would necessarily meet the hyperplane $\Lambda' \cap T_p(F) \subset \Lambda''$. Thus, by our first argument, Λ and Λ'' belong to the same family on F if and only if $n \equiv 0(2)$; it follows that

$$\begin{aligned}
&\Lambda \text{ and } \Lambda' \text{ belong to the same family} \\
\Leftrightarrow\, &n \equiv -1 = \dim(\Lambda \cap \Lambda')(2).
\end{aligned}$$

This completes the proof of the proposition.

We can write down explicitly the two families of n-planes on the smooth $2n$-dimensional quadric $F \subset \mathbb{P}^{2n+1}$ given by

$$Q(X) = \sum_{i=0}^{n} X_i X_{n+i+1}.$$

In this case for B any $(n+1)\times(n+1)$ matrix the n-plane Λ_B spanned by the row vectors $e_i=(0,\ldots,1,0,\ldots,0,b_{i,0},\ldots,b_{i,n})$ of the $2n \times n$ matrix (I,B) lies in F if and only if

$$Q(e_i,e_j) = b_{ij} + b_{ji} = 0$$

for all i and j, i.e., if and only if B is skew-symmetric. The n-planes $\{\Lambda_B\}$ form an open set Γ_0 in one of the two families of n-planes on F. (Note that

$$\dim(\Lambda_B \cap \Lambda_{B'}) = n - \operatorname{rank}(B-B') \equiv n(2)$$

for any B,B' skew-symmetric.) More generally, if $I=\{i_1,\ldots,i_m\}$ is any subset of $\{0,\ldots,n\}$, then the automorphism φ_I of F defined by

$$\varphi_I[X]=[X'], \quad X'_i=\begin{cases} X_{n+i+1}, & i\in I,\\ X_i, & i\notin I,\end{cases}$$

$$X'_{n+1+i}=\begin{cases} X_i, & i\in I,\\ X_{n+1+i}, & i\notin I,\end{cases}$$

carries the set $\Gamma_0=\{\Lambda_B\}$ of n-planes into another set Γ_I; Γ_I will be of the same family as Γ_0 if and only if $m={}^{\#}I$ is even. In this way, we represent all n-planes on F.

We consider now the family of k-planes on a smooth quadric F in $\mathbb{P}^{n+1}$. The dimension of this family is easy to compute: we let $|F|\cong \mathbb{P}^{(n+2)(n+3)/2-1}$ denote the linear system of all quadrics in $\mathbb{P}^{n+1}$ and consider the incidence correspondence

$$I \subset |F| \times G(k+1,n+2)$$

given by

$$I = \{(F,\Lambda):\ \Lambda\subset F\}.$$

The linear system $|F|$ cuts out on any k-plane Λ the complete $(k+1)(k+2)/2-1$-dimensional linear series of quadrics in Λ, so the fibers of the projection map $\pi_2: I\to G(k+1,n+2)$ have dimension

$$\frac{(n+2)(n+3)}{2} - \frac{(k+1)(k+2)}{2} - 1,$$

and I has dimension

$$(k+1)(n-k+1) + \frac{(n+2)(n+3)}{2} - \frac{(k+1)(k+2)}{2} - 1;$$

it follows that a fiber of $\pi_1: I\to|F|$—the family of k-planes on a quadric F

— has dimension

$$(k+1)(n-k+1)-\frac{(k+1)(k+2)}{2}$$

or, alternatively, codimension $(k+1)(k+2)/2$ in the Grassmannian $G(k+1,n+2)$.

Let us now determine the class on $G(k+1,n+2)$ of the cycle $\Sigma_{k,n}$ of k-planes on a smooth quadric $F\subset\mathbb{P}^{n+1}$. Recall from Section 6 of Chapter 1 that for any flag $V_0\subset V_1\subset\cdots\subset V_{n+1}\subset\mathbb{C}^{n+2}$ the cohomology group $H^{(k+1)(k+2)}(G(k+1,n+2))$ is generated by the classes of the Schubert cycles

$$\sigma_{a_1,\ldots,a_{k+1}}=\left\{\Lambda_{k+1}\colon\ \dim\left(\Lambda\cap V_{n-k+1+i-a_i}\right)\geqslant i\right\}$$

for all sequences

$$n-k+1\geqslant a_1\geqslant\cdots\geqslant a_{k+1}\geqslant 0$$

with $\Sigma a_i=(k+1)(k+2)/2$. The cohomology in complementary dimension is likewise generated by Schubert cycles σ_b with $\Sigma b_i=(k+1)(n-k+1)-(k+1)(k+2)/2$; the intersection pairing is

$$^{\#}\left(\sigma_{a_1,\ldots,a_{k+1}}\cdot\sigma_{b_1,\ldots,b_{k+1}}\right)=\begin{cases}1, & \text{if } a_i+b_{k+2-i}=n-k+1 \text{ for all } i,\\ 0, & \text{otherwise.}\end{cases}$$

To find the class of $\Sigma_{k,n}$, accordingly, we have to evaluate the intersection numbers $^{\#}(\Sigma_{k,n}\cdot\sigma_b)$ for all such $b=(b_1,\ldots,b_{k+1})$ with $\Sigma b_i=(k+1)(n-k-1)-(k+1)(k+2)/2$. We start by noting that if the flag $\{V_\alpha\}$ is generically chosen, each subspace $\overline{V}_\alpha\subset\mathbb{P}^{n+1}$ will intersect F in a smooth $(\alpha-2)$-dimensional quadric $F_{\alpha-2}$. Now, as we have seen, $F_{\alpha-2}$ cannot contain any linear subspaces of projective dimension greater than $(\alpha-2)/2$; thus, no k-plane Λ_k lying on F can meet $\overline{V}_\alpha$ in a space of projective dimension $>(\alpha-2)/2$. If $^{\#}(\Sigma_{k,n}\cdot\sigma_b)$ is to be nonzero, then we must have

$$n-k+1+i-b_i\geqslant 2i\qquad\text{for all } i;$$

i.e.,

$$b_i\leqslant n-k-i+1.$$

But from

$$\begin{aligned}(n-k+1)(k+1)-\frac{(k+1)(k+2)}{2}&=\sum_{i=1}^{k+1}b_i\\&\leqslant\sum_{i=1}^{k+1}n-k-i+1\\&=(n-k+1)(k+1)-\sum_{i=1}^{k+1}i\\&=(n-k+1)(k+1)-\frac{(k+1)(k+2)}{2}\end{aligned}$$

we deduce that $b_i = n-k-i+1$, i.e.,

The cycle $\Sigma_{k,n} \subset G(k+1, n+2)$ *has intersection number* 0 *with all Schubert cycles of complementary dimension except* $\sigma_{n-k,n-k-1,n-k-2,\ldots}$.

To compute the intersection number of $\Sigma_{k,n}$ with

$$\sigma_{n-k,n-k-1,\ldots} = \{\Lambda_k : \dim(\Lambda_k \cap V_{2i}) \geq i\}$$

we read off the defining conditions for $\sigma_{n-k,n-k-1,\ldots}$ one by one. The first condition says that any $\Lambda \in \Sigma_{k,n} \cap \sigma_{n-k,n-k-1,\ldots}$ must meet a line $\bar{V}_2 \subset \mathbb{P}^{n+1}$ in a point; since $\Lambda \subset F$, this point must be one of the two points p_1, p_2 of intersection of $\bar{V}_2$ with F. For each $i = 1,2$, let $\mathbb{P}_i^{n-1}$ be an $(n-1)$-plane contained in the tangent plane $T_{p_i}(F)$ to F at p_i and not containing p_i; $\mathbb{P}_i^{n-1}$ then intersects F in a smooth quadric F_i, with $F \cap T_{p_i}(F)$ the cone through p_i over F_i. Now the second condition on $\sigma_{n-k,n-k-1,\ldots}$ says that any $\Lambda \in \Sigma_{k,n} \cap \sigma_{n-k,n-k-1,\ldots}$ must meet the 3-plane $\bar{V}_4$ in a line. But $\bar{V}_4$ will meet $\mathbb{P}_i^{n-1}$ in a line, and F_i in a pair of points p_{i1}, p_{i2}; and writing any k-plane $\Lambda \subset F$ through p_i as

$$\Lambda = \overline{p_i, \Lambda \cap \mathbb{P}_i^{n-1}},$$

we see that any $\Lambda \in p_i$ meets $\bar{V}_3$ in a line if and only if Λ contains either of the points p_{i1} or p_{i2}.

Consider now the set of k-planes on F passing through the points p_i and p_{ij}. Take $\mathbb{P}_{ij}^{n-3}$ an $(n-3)$-plane lying in the intersection $T_{p_i}(F) \cap T_{p_{ij}}(F)$ and missing the line $\overline{p_i p_{ij}}$; $\mathbb{P}_{ij}^{n-3}$ intersects F in a smooth quadric F_{ij}, with $F \cap T_{p_i}(F) \cap T_{p_{ij}}(F)$ the cone through the line $\overline{p_i p_{ij}}$ over F_{ij}. The third condition on $\sigma_{n-k,n-k-1,\ldots}$ says that any $\Lambda \in \Sigma_{k,n} \cap \sigma_{n-k,n-k-1,\ldots}$ meets the 5-plane $\bar{V}_6$ in a 2-plane. But $\bar{V}_6$ meets $\mathbb{P}_{ij}^{n-3}$ in a line, and F_{ij} in a pair of points p_{ij1} and p_{ij2}; and writing any k-plane $\Lambda \subset F$ through p_i and p_{ij} as

$$\Lambda = \overline{p_i, p_{ij}, \Lambda \cap \mathbb{P}_{ij}^{n-3}},$$

we see that Λ satisfies this condition if and only if it contains either p_{ij1} or p_{ij2}.

The process is now clear. Defining inductively a collection of points $p_{i_1}, p_{i_1 i_2}, \ldots, p_{i_1,\ldots,i_k}$ by letting $p_{i_1,\ldots,i_{m-1},1}$ and $p_{i_1,\ldots,i_{m-1},2}$ be the two points of intersection of $\bar{V}_{2m}$ with a chosen $(n-2m+1)$-plane in the intersection of the tangent spaces to F at $p_{i_1}, p_{i_1, i_2}, \ldots, p_{i_2,\ldots,i_{m-1}}$, we find that the k-planes $\Lambda \subset F$ lying in $\sigma_{n-k,n-k-1,\ldots}$ are exactly the planes

$$\overline{p_{i_1}, p_{i_1,i_2}, p_{i_1,i_2,i_3}, \ldots, p_{i_1,i_2,\ldots,i_{k+1}}},$$

and there are 2^{k+1} of these. Consequently

$$^{\#}(\Sigma_{k,n} \cdot \sigma_{n-k,n-k-1,\ldots}) = 2^{k+1},$$

and we have

$$\Sigma_{k,n} \sim 2^{k+1} \cdot \sigma_{k+1,k,k-1,\ldots,1}$$

in the cohomology of $G(k+1,n+2)$.

Note that since the two families of n-planes on a $2n$-dimensional quadric in $\mathbb{P}^{2n+1}$ may be taken into one another by an automorphism of $\mathbb{P}^{2n+1}$, they represent the same class on $G(n+1,2n+1)$, and hence each represents the class $2^n \cdot \sigma_{n+1,n,\ldots,1}$. Also, in case the quadric has odd dimension $n=2m+1$ and $k=m$, the codimension $(m+1)(m+2)/2$ of the cycle $\Sigma_{m,2m+1}$ is exactly half the dimension of the Grassmannian $G(m+1,2m+3)$, and so we may expect there to be a finite number of m-planes in the generic intersection of two quadrics in $\mathbb{P}^{2m+2}$; indeed, by our calculation this number is

$$\begin{aligned} {}^{\#}(\Sigma_{m,2m+1} \cdot \Sigma_{m,2m+1}) &= 2^{2m+2} \cdot (\sigma_{m+1,m,\ldots,1} \cdot \sigma_{m+1,\ldots,1}) \\ &= 2^{2m+2}. \end{aligned}$$

We have already verified this in case $n=3$.

Linear Systems of Quadrics

Thus far, we have examined the geometry of a single quadric hypersurface in $\mathbb{P}^n$. We would now like to consider linear systems of quadrics; specifically we will study linear systems of quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$.

We begin with $\mathbb{P}^2$. In the complete system $W \cong \mathbb{P}^5$ of conic plane curves, let $W_1 \subset W$ be the subvariety of conics of rank two or less and $W_2 \subset W_1$ the set of conics of rank one. W_1 is a hypersurface in W; we first ask for its degree. This question may be answered in four ways:

1. Suppose $L = \{F_\lambda\}$ is a generic line in W, that is, a generic pencil of quadrics in $\mathbb{P}^2$. The conics F of the pencil L may be given as the zero loci of the quadratic forms

$$Q^\lambda(X) = \sum q_{ij}^\lambda X_i X_j,$$

where

$$Q^\lambda = (q_{ij}^\lambda) = Q^0 + \lambda Q^\infty$$

for suitable choice of nonsingular symmetric matrices Q^0 and Q^∞. F_λ will then be singular exactly when the determinant $|Q^0 + \lambda Q^\infty|$ vanishes; since this determinant is a cubic polynomial in λ, this will occur for three values of λ. L thus intersects W_1 three times, so $\deg(W_1) = 3$. Note that in general by this argument the singular quadrics in $\mathbb{P}^n$ form a hypersurface of degree $n+1$ in the system of all quadrics.

2. Letting L be, as above, a generic pencil of conics, we have by the

formula of p. 509

$$\chi(\mathbb{P}^2) = 2\chi(F_\lambda) + \mu - n,$$

where F_λ is a generic element of L, $n = F_\lambda \cdot F_\lambda$ the number of base points of L, and μ the number of singular conics in L. Since $\chi(F_\lambda) = 2$ and $n = 4$, this yields

$$3 = 2 \cdot 2 + \mu - 4,$$

i.e., $\mu = 3$, and W_1 is cubic.

3. Letting L again be a generic pencil of W, L will have four base points p_1, p_2, p_3, p_4 and will consist of all conics in $\mathbb{P}^2$ passing through the points $\{p_i\}$. (See Figure 1.) But since no three of the points p_i are collinear, if F is a conic consisting of two lines l, l' and containing $\{p_i\}$, then l and l' must each contain two of the points $\{p_i\}$. The singular conics passing through $\{p_i\}$ are therefore

$$\overline{p_1p_2} + \overline{p_3p_4}, \qquad \overline{p_1p_3} + \overline{p_2p_4} \quad \text{and} \quad \overline{p_1p_4} + \overline{p_2p_3}.$$

So we see again that L contains three singular conics.

4. Alternatively, note that $W_1 \subset W$ is the image of $\mathbb{P}^{2*} \times \mathbb{P}^{2*}$ under the map f sending a pair of lines (l_1, l_2) to the conic $l_1 + l_2 \in W$. Now the cohomology ring of $\mathbb{P}^{2*} \times \mathbb{P}^{2*}$ is generated by the classes ω_1 and ω_2, where

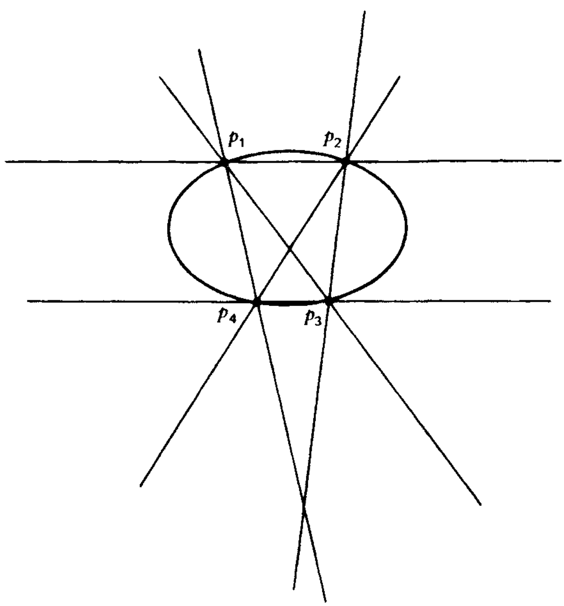

Figure 1

ω_1 and ω_2 are the pullbacks of the hyperplane class in $\mathbb{P}^{2*}$ via the two projection maps, with the relations $\omega_1^3=\omega_2^3=0$, $\omega_1^2\omega_2^2=1$. If $H\subset W$ is the hyperplane in W consisting of conics containing a point $p\in\mathbb{P}^2$, then the pullback f^*H is the divisor of pairs (l_1,l_2) with either $p\in l_1$ or $p\in l_2$, and so it represents the class $\omega_1+\omega_2$. Consequently, since f is two-to-one,

$$\begin{aligned}\deg W_1 &= (H)^4_{W_1}\\ &= \tfrac{1}{2}(\omega_1+\omega_2)^4_{\mathbb{P}^{2*}\times\mathbb{P}^{2*}}\\ &= \tfrac{1}{2}\cdot 6 = 3.\end{aligned}$$

Note that the subvariety $W_2\subset W_1\subset W$ is just the image under f of the diagonal $\Delta\cong\mathbb{P}^{2*}$ in $\mathbb{P}^{2*}\times\mathbb{P}^{2*}$, which is the branch locus of f. Since the series $|\omega_1+\omega_2|$ cuts out the complete series $|\mathcal{O}_{\mathbb{P}^2}(2H)|$ on Δ, *W_2 is the Veronese surface $\iota_{2H}(\mathbb{P}^2)$ in* $W\cong\mathbb{P}^5$.

We note that W_1 is smooth away from W_2: if $F\in W_1$ is a conic consisting of two distinct lines, we can find another conic G meeting F transversely so that the pencil L generated by F and G will have four distinct base points. By argument 3, L will meet W_1 in three distinct points, so that $m_F(L,W_1)=1$ and F is a smooth point of W_1. On the other hand, if F is a double line and L a generic line through F, we see (Figure 2) that the pencil L will consist of all conics passing through the points p,p' of intersection of F with a second conic G of L, and tangent to (i.e., having intersection multiplicity >1 with) G at those points. The only singular conic of L other than F is thus the sum of the tangent lines to C at p and p'; so $m_F(L,W_1)=2$ and F is a double point of W_1. The reader may check that the tangent space to W_1 at a smooth point $F=l_1+l_2$ is just the plane $H\subset W$ of conics passing through the point $p=l_1\cap l_2\in\mathbb{P}^2$, while the tangent cone to W_1 at a double point $F=2l$ is the locus of conics tangent

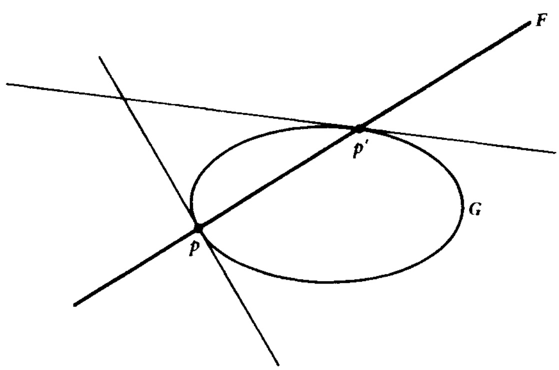

Figure 2

to l. (Note that since W_2 is the double locus of the cubic W_1, any line meeting W_2 twice must lie in W_1; this provides another proof that the chordal variety of the Veronese surface is a cubic hypersurface.)

Finally, it is interesting to observe that there are two distinct four-dimensional families of lines on the variety W_1—that is, two kinds of pencils of singular conics. First, there are the pencils formed by a fixed line l_0 plus a pencil l_λ of lines; for example,

$$L = \{(\lambda X_0 X_1 + X_0 X_2)\}_\lambda \qquad (l_0 = (X_0 = 0)).$$

(See Figure 3.) Such a pencil will either miss W_2 altogether or meet it in a single point, depending on whether the base point of the pencil l_λ lies on l_0. Second, there are the chords to W_2 in W_1; such a pencil, containing two distinct double lines, will have only a single point p as its base locus and so will be just the pullback, via the rational projection $\pi_p : \mathbb{P}^2 \to \mathbb{P}^1$, of a pencil on $\mathbb{P}^1$. (See Figure 4.) For example,

$$L = \{(\lambda X_0^2 + X_1^2)\}$$

is such a pencil.

We now turn to quadrics in $\mathbb{P}^3$. Let $W \cong \mathbb{P}^9$ be the complete linear system of all quadrics, $W_1 \subset W$ the locus of quadrics of rank three or less, $W_2 \subset W_1 \subset W$ the locus of quadrics of rank two or less and W_3 the set of rank-one quadrics. By the first of our previous arguments, W_1 is a hypersurface of degree 4 in W. Again, W_1 is smooth away from W_2: if F is

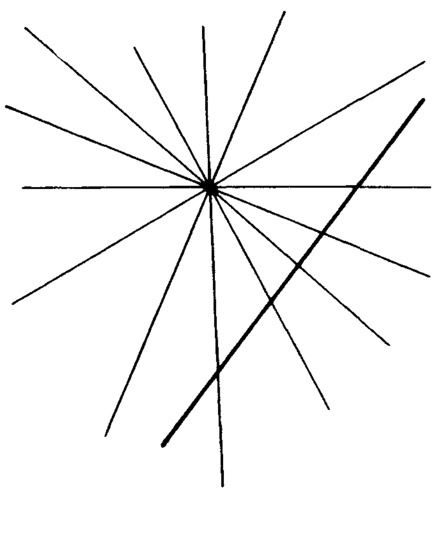

Figure 3

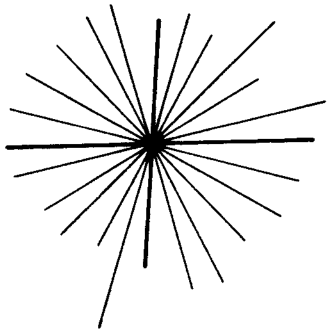

Figure 4

any quadric of rank three—which we may take to be given by the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and L is the pencil generated by F and a generic quadric G, given by the matrix Q', then the polynomial

$$|\lambda Q + Q'|$$

has degree 3 in λ—i.e., L will contain three singular quadrics other than F, so $m_F(L\cdot W_1) = 1$ and F is a smooth point of W_1. Note that the polynomial $|\lambda Q + Q'|$ will fail to have degree 3 exactly when the upper left-hand entry of Q' is zero, i.e., when the quadric G contains the point $[1,0,0,0]$. The tangent plane to W_1 at F is thus the space of quadrics containing the singular point of F.

Similarly, a quadric $F \in W_2 - W_3$ of rank two may be represented by the matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

for generic Q', then, the polynomial $|\lambda Q + Q'|$ will have degree 2; that is, a generic pencil L in W containing F will meet W_1 in only two other points. Thus F is a double point of W_1; indeed, since the polynomial $|\lambda Q + Q'|$ will have degree <2 exactly when the determinant of the upper left-hand 2×2 minor of Q' is zero, we deduce that the tangent cone to W_1 at a point $F \in W_2$ is just the locus of quadrics tangent to the singular line of F.

To find the degree of W_2, we proceed as in 4 above: W_2 is the image in W of $\mathbb{P}^{3*} \times \mathbb{P}^{3*}$ by the map f sending a pair (H_1, H_2) of hyperplanes in $\mathbb{P}^{3*}$

to the quadric H_1+H_2. The cohomology of $\mathbb{P}^{3*}\times\mathbb{P}^{3*}$ is generated by the pullbacks ω_1 and ω_2 of the hyperplane class in $\mathbb{P}^{3*}$ via the two projection maps; as before, since f is 2-sheeted, we obtain

$$\begin{aligned}\deg W_2 &= \tfrac{1}{2}(\omega_1+\omega_2)^6\\ &= \tfrac{1}{2}\cdot 20\omega_1^3\omega_2^3 = 10.\end{aligned}$$

Lines on Linear Systems of Quadrics

Earlier in this section, we found the class on the Grassmannian of the cycle of k-planes lying on a smooth quadric in $\mathbb{P}^n$. It is interesting to try and answer the same question for the cycle of k-planes lying on a linear system of quadrics; we will discuss here the case of lines on quadrics in $\mathbb{P}^3$.

To begin with, recall that the integral homology of the Grassmannian $G(2,4)$ is generated by the Schubert cycles

$$\begin{aligned}\sigma_1(l_0) &= \{x\in G:\ l_x\cap l_0\neq\emptyset\},\\ \sigma_2(p_0) &= \{x\in G:\ l_x\ni p_0\},\\ \sigma_{1,1}(h_0) &= \{x\in G:\ l_x\subset h_0\},\end{aligned}$$

and

$$\sigma_{2,1}(p_0,h_0) = \{x\in G:\ p_0\in l_x\subset h_0\}$$

for $p_0\in l_0\subset h_0$ any choice of point, line, and hyperplane in $\mathbb{P}^3$. The intersections of these Schubert cycles are

$$\begin{aligned}\sigma_1\cdot\sigma_1 &= \sigma_2+\sigma_{1,1},\\ \sigma_1\cdot\sigma_2 &= \sigma_1\cdot\sigma_{1,1} = \sigma_{2,1},\\ \sigma_2\cdot\sigma_2 &= \sigma_{1,1}\cdot\sigma_{1,1} = \sigma_1\cdot\sigma_{2,1} = 1,\\ \sigma_2\cdot\sigma_{1,1} &= 0.\end{aligned}$$

We have seen in the preceding discussion that the variety $V(F)$ of lines lying on a smooth quadric $F\subset\mathbb{P}^3$ is homologous to $4\sigma_{2,1}$: consider now the variety $V_0(L)$ of lines in $\mathbb{P}^3$ lying on *some* quadric of a generic pencil $L=\{F_\lambda\}$ of quadrics.

The base locus of the pencil L, being a smooth intersection of two quadrics, is an elliptic curve C of degree 4, and in fact it is not hard to see that $V_0(L)$ is just the set of chords to C: on the one hand, if $l\subset F_\lambda$ for some λ, then $C\cap l=F_\lambda\cap F_{\lambda'}\cap l=F_{\lambda'}\cap l$ consists of two points; on the other hand, if l meets C in two points p and q, then for any third point $r\in l$ some quadric $F_\lambda\in L$ contains r, and so contains l.

This being established, it is easy to compute the class of $V_0(L)\subset G(2,4)$: First, since a generic hyperplane $H\subset\mathbb{P}^3$ meets C in four points, $\binom{4}{2}=6$

chords of C will lie in H, so

$$^{\#}(V_0(L)\cdot\sigma_{1,1}) = 6.$$

Second, projection of C from a generic point $p\in\mathbb{P}^3$ to a plane maps C to a plane quartic, which by the genus formula will have two double points; consequently p lies on two chords of C and

$$^{\#}(V_0(L)\cdot\sigma_2) = 2.$$

In sum,

$$V_0(L)\sim 2\sigma_2+6\sigma_{1,1}.$$

We proceed to a generic net $N=\{F_{\lambda,\mu}\}_{(\lambda,\mu)\in\mathbb{P}^2}$ of quadrics. In this case we may associate to N two varieties of lines: the set $V_0(N)=\cup_{\mu,\lambda}V(F_{\mu,\lambda})$ of all lines contained in some quadric of N, and the set $V_1(N)$ of lines that lie on a pencil of quadrics in N. The latter is readily described: if $l\subset\mathbb{P}^3$ lies on a pencil $\overline{F_{\mu,\lambda},F_{\mu',\lambda'}}$ of quadrics in N, then the intersection of l with the base locus of N consists of two points:

$$l\cap F_{\mu,\lambda}\cap F_{\mu',\lambda'}\cap F_{\mu'',\lambda''}=L\cap F_{\mu'',\lambda''}$$

for any third quadric $F_{\mu'',\lambda''}$ in N. Conversely, if l contains two base points p,q, of N, choose a third point $r\in l$; r will lie on a pencil of quadrics from N that, containing p, q, and $r\in l$, contain l. Since N has eight base points, no three collinear, $V_1(N)$ will consist of the $\binom{8}{2}=28$ lines joining these points.

The class of $V_0(N)\subset G(2,4)$ may be determined as follows: let p be a generic point of $\mathbb{P}^3$, H a generic plane containing p. Then the restriction to H of the set of quadrics in N containing p is a pencil L of conics with p as one base point, and by argument 3 above, p will lie on exactly three lines of L. Thus

$$^{\#}(V_0(N)\cdot\sigma_{2,1}) = 3$$

and hence

$$V_0(N)\sim 3\sigma_1.$$

Finally let W be a generic web of quadrics. Since in this case the set $V_0(W)$ of all lines on W is all of $G(2,4)$, we will be concerned with the variety $V_1(W)\subset G(2,4)$ of lines lying on a pencil of quadrics from W. To begin with, if $p\in\mathbb{P}^3$ is a generic point, then the set of quadrics in W through p forms a generic net N_p, with p as a base point. By the argument given in the count for $V_1(N)$, the lines through p lying on a pencil of quadrics from W will just be the lines joining p to the other seven base points of N_p. Consequently

$$^{\#}(V_1(W)\cdot\sigma_2) = 7.$$

There is another way to make the count, based on the fact that a smooth point p on a quadric F will lie on two lines of F if F is smooth, but only one if F is singular. Let $\mathbb{P}^2 \subset \mathbb{P}^3$ be a plane not containing p_1 and $I \subset \mathbb{P}^2 \times N_p$ the incidence correspondence defined by

$$I = \{(q,F): \overline{pq} \subset F\}.$$

The projection of I on the second factor expresses I as a double cover of $N_p \cong \mathbb{P}^2$, branched over the locus of singular quadrics in N_p. But by our analysis of the linear system of quadrics in $\mathbb{P}^3$, the locus of singular quadrics in N_p is a smooth quartic curve. Using the discussion of Section 4 of Chapter 4, I has Euler characteristic 10. On the other hand, the map $I \to \mathbb{P}^2$ expresses I as the blow-up of $\mathbb{P}^2$ at the points of intersection of $\mathbb{P}^2$ with the lines $l \in V_1(W)$ through p; there are thus $\chi(I) - \chi(\mathbb{P}^2) = 10 - 3 = 7$ such lines.

The calculation for $^{\#}(V_1(W) \cdot \sigma_{1,1})$ is somewhat more difficult. Let H be a generic plane in $\mathbb{P}^3$, and let X be the restriction to H of the web W. If $l \subset H \subset \mathbb{P}^3$ lies on a pencil $\{F_\lambda\}$ of quadrics in W, then the conics $\{C_\lambda = F_\lambda \cap H\}$ in X are all singular; thus we may ask for the number of pencils of singular conics in X having a fixed line. Now, by our discussion of the system of conics, the locus of singular conics in X is a cubic surface with four double points, these corresponding to the double lines in X. Such a surface has, we have seen in Section 6 of Chapter 4, nine lines on it—but of these nine, six comprise the edges of the tetrahedron whose vertices are the double points of the surface, and the pencils corresponding to these lines are of the second type (p. 744). Of the nine pencils of singular conics in X, then, only three have fixed lines. Thus

$$^{\#}(V_1(W) \cdot \sigma_{1,1}) = 3$$

and finally

$$V_1(W) \sim 7\sigma_2 + 3\sigma_{1,1}.$$

We may check this calculation as follows: let N_1, N_2 be two generic nets in the web W, $L = N_1 \cap N_2$ their common pencil, and consider the intersection $V_0(N_1) \cap V_0(N_2)$. If a line l lies on a quadric $F_1 \in N_1$ and a quadric $F_2 \in N_2$, then either

(1) $F_1 \neq F_2$, so l lies on the pencil $\overline{F_1F_2} \subset W$, and hence $l \in V_1(W)$; or
(2) $F_1 = F_2 \in L$, i.e., $l \in V_0(L)$.

Since, conversely, both $V_0(L)$ and $V_1(W)$ are contained in $V_0(N_1) \cap V_0(N_2)$,

$$V_0(N_1) \cap V_0(N_2) = V_1(W) \cup V_0(L).$$

Now $V_0(N_1) \sim V_0(N_2) \sim 3\sigma_1$, so the intersection $V_0(N_1) \cap V_0(N_2)$ has class

$$(3\sigma_1)^2 = 9\sigma_{1,1} + 9\sigma_2;$$

on the other hand, $V_0(L) \sim 2\sigma_2 + 6\sigma_{1,1}$, so we find again that

$$V_1(W) \sim 7\sigma_2 + 3\sigma_{1,1}.$$

The reader may find it an interesting exercise to prove that, for a generic web W, the surface $V_1(W)$ is an Enriques surface.

The Problem of Five Conics

To conclude this section, we will use the computation of Section 6, Chapter 4, for the cohomology ring of a blow-up to solve a classical problem in enumerative geometry:

Given $C_1, \ldots, C_5 \subset \mathbb{P}^2$ five smooth conic curves chosen generically, how many smooth conic curves in $\mathbb{P}^2$ are tangent to all five?

To answer this question, consider first the linear system

$$W = |2H| \cong \mathbb{P}^5$$

of all conic curves in $\mathbb{P}^2$. For any smooth conic curve C, let $V_C \subset W$ be the set of conic curves tangent to C (that is, having a point of intersection multiplicity $\geqslant 2$ with C); V_C is a hypersurface of some degree d in $\mathbb{P}^5$. If we could show that for a generic choice of five conics the hypersurfaces $V_{C_1}, \ldots, V_{C_5} \subset W$

1. met transversely away from the subvariety of singular conics, and
2. contained no singular conics in common,

the answer to our question would be easy: it would just be the fivefold self-intersection $(\deg V_C)^5$ of V_C in $W \cong \mathbb{P}^5$. Unfortunately, matters are not so simple: while assertion 1 above is the case, and half of assertion 2, namely

2′. for $C_1, \ldots, C_5$ generically chosen, no conic consisting of two distinct lines will be tangent to all five

holds, the problem is that all the hypersurfaces $V_C \subset W$ will contain the subvariety

$$W_2 = \{2L\}_{L \subset \mathbb{P}^2}$$

of double lines.

We can overcome this difficulty by blowing up. Precisely, let

$$\pi: \tilde{W} \longrightarrow W$$

be the blow-up of W along the variety W_2 of double lines; for C any smooth conic, denote by $\tilde{V}_C$ the proper transform of the subvariety

$V_C \subset \mathbb{P}^5$. Then, once we verify assertions 1 and 2 above and the additional assertion that for $C_1, \ldots, C_5$ generically chosen,

3. the proper transforms $\tilde{V}_{C_1}, \ldots, \tilde{V}_{C_5}$ have no common points in the exceptional divisor of $\tilde{W}$,

the answer to our original question will be simply the fivefold self-intersection of the divisor $\tilde{V}_C$ on $\tilde{W}$, and readily calculable. We will proceed with the computation, leaving the proof of assertions 1, 2, and 3 until later.

We first compute the degree of the hypersurface $V_C \subset \mathbb{P}^5$ for smooth C; to do this, let $L \subset W$ be a generic line and $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ the pencil of conics it represents. The curves $\{C_\lambda\}$ then cut out on C a linear system of degree 4 without base points. The corresponding map expresses C as a 4-sheeted cover of $\mathbb{P}^1$; and by the Riemann-Hurwitz formula, the number of branch points of this map is

$$\begin{aligned} b &= 2g(C) - 2 - 4(2g(\mathbb{P}^1) - 2) \\ &= 6. \end{aligned}$$

The pencil $\{C_\lambda\}$ therefore contains six conics tangent to C, and consequently

$$\deg V_C = 6.$$

In fact, this argument tells us a bit more. Suppose that C' is a smooth point of V_C and that C' is simply tangent to C at a single point $p \in C$. If $L \subset W$ is a generic line through C' lying in the hyperplane $H_p \subset W$ of conics containing p, the corresponding pencil $\{C'_\lambda\}_{\lambda \in \mathbb{P}^1}$ will cut out on C a linear system of degree 4, *with* p *as a base point*. The corresponding map then expresses C as a 3-sheeted cover of $\mathbb{P}^1$, and so has only

$$\begin{aligned} b &= 2(g(c) - 2) - 3(2g(\mathbb{P}^1) - 2) \\ &= 4 \end{aligned}$$

branch points—i.e., the pencil $\{C'_\lambda\}$ can contain at most four conics tangent to C other than C'. It follows that $\mathrm{H_p}$ *is the tangent plane to* $\mathrm{V_C}$ *at* C′, and conversely if C' is simply tangent to C at only one point, then C' is a smooth point of V_C.

Next, we compute the multiplicity of the locus W_2 of double lines in the generic divisor V_C. This is not hard: for C a conic, $2L$ a generic double line, and $\{C_\lambda\}$ a generic pencil of conics containing the double line $2L$ as an element, $\{C_\lambda\}$ again cuts out on C a pencil of degree 4, without base points. The corresponding map then has six branch points as before—but two of these are just the points of intersection of L with C. $\{C_\lambda\}$ thus has four points of intersection with V_C other than $2L$; it follows that

$$\mathrm{mult}_{2L}(\{C_\lambda\}, V_C) = 2,$$

and so, for generic C,

$$\mathrm{mult}_{W_2}(V_C) = 2.$$

We can thus write

$$\tilde{V}_C \sim 6\tilde{\omega} - 2e \in H^2(\tilde{W}, \mathbb{Z}),$$

where $\tilde{\omega} = \pi^*\omega$ is the pullback to $\tilde{W}$ of the class ω of the hyperplane in W, and e the class of the exceptional divisor E.

Now, to determine the fivefold self-intersection

$$(6\tilde{\omega} - 2e)^5$$

of V_C in $\tilde{W}$, recall from our discussion in Section 1 that the surface W_2 is the Veronese surface $\iota_{2H}(\mathbb{P}^2)$. Let l and $p = l^2$ denote the classes of a point and a line in $W_2 \cong \mathbb{P}^2$; let $\tilde{p} = \pi^* p$ and $\tilde{l} = \pi^* l$ be the pullback classes in $E \subset \tilde{W}$. We have

$$\omega|_{W_2} = 2l,$$

and so

$$\omega^2|_{W_2} = (2l)^2 = 4p.$$

Now by the computation for the Chern classes of projective space,

$$\begin{aligned} c(T(W)|_{W_2}) &= (1 + 6\omega + 15\omega^2)|_{W_2} \\ &= 1 + 12l + 60l^2 \end{aligned}$$

and

$$c(T(W_2)) = 1 + 3l + 3l^2.$$

From the C^∞ decomposition of vector bundles

$$T(W)_{W_2} = T(W_2) \oplus N_{W_2/W}$$

we obtain

$$c(N_{W_2/W}) = \frac{c(T(W))}{c(T(W_2))},$$

and performing the divison

$$\begin{array}{r@{\,}l} & 1 + 9l + 30l^2 \\ 1 + 3l + 3l^2 \big) & 1 + 12l + 60l^2 \\ & \underline{1 + 3l + 3l^2} \\ & 9l + 57l^2 \\ & \underline{9l + 27l^2} \\ & 30l^2 \\ & \underline{30l^2} \end{array}$$

we find that

$$c(N_{W_2/W}) = 1 + 9l + 30l^2.$$

Thus if $\zeta \in H^2(E, \mathbb{Z})$ denotes the Chern class of the tautological bundle on $E \cong \mathbb{P}(N_{W_2/W})$, our general relation (p. 606) reads

$$(*) \qquad \zeta^3 - 9\tilde{l}\cdot\zeta^2 + 30\tilde{l}^2\cdot\zeta = 0.$$

Now we have seen that the tautological bundle restricts to the universal bundle $[-H]$ on each fiber E_p of $E \to W_2$, and so

$$\zeta^2\cdot\tilde{l}^2 = c_1\left(T|_{E_p}\right)^2 = 1.$$

Multiplying the basic relation $(*)$ by $\tilde{l}$—and recalling that $\tilde{l}^3 = 0$—we have

$$\tilde{l}\cdot\zeta^3 - 9\tilde{l}^2\cdot\zeta^2 = 0$$

and hence

$$\tilde{l}\cdot\zeta^3 = 9.$$

Finally, multiplying $(*)$ by ζ,

$$\begin{aligned} \zeta^4 - 9\tilde{l}\,\zeta^3 + 30\tilde{l}^2\zeta^2 &= 0 \\ \Rightarrow \zeta^4 \quad &= 9\tilde{l}\,\zeta^3 - 30\tilde{l}^2\zeta^2 \\ &= 51. \end{aligned}$$

It is now possible to calculate $(6\tilde{\omega} - 2e)^5$. First, since the class ω of a hyperplane in $\mathbb{P}^5$ restricts to the class $2l$ on W_2

$$\tilde{\omega}|_E = 2\tilde{l},$$

and the tautological bundle

$$T = N_{E/\mathbb{P}^5},$$

we obtain

$$e|_E = c_1(T) = \zeta.$$

Also,

$$\begin{aligned} (\tilde{\omega}^5)_{\tilde{W}} &= (\omega^5)_W = 1. \\ \tilde{\omega}^5 &= 1, \\ \tilde{\omega}^4\cdot e &= \left((2\tilde{l}\ )^4\right)_E = 0, \\ \tilde{\omega}^3\cdot e^2 &= \left((2\tilde{l}\ )^3\cdot\zeta\ \right)_E = 0, \\ \tilde{\omega}^2\cdot e^3 &= \left((2\tilde{l}\ )^2\cdot\zeta^2\right)_E = 4\left(\tilde{l}^2\zeta^2\right)_E = 4, \\ \tilde{\omega}\cdot e^4 &= \left(2\tilde{l}\cdot\zeta^3\right)_E = 18, \\ e^5 &= \zeta^4 = 51, \end{aligned}$$

and so

$$
\begin{aligned}
(6\tilde{\omega}-2e)^5 &= 6^5\tilde{\omega}^5 - 5\cdot 6^4\cdot 2\cdot\tilde{\omega}^4\cdot e + 10\cdot 6^3\cdot 2^2\cdot\tilde{\omega}^3\cdot e^2 - 10\cdot 6^2\cdot 2^3\cdot\tilde{\omega}^2\cdot e^3 \\
&\quad + 5\cdot 6\cdot 2^4\cdot\tilde{\omega}\cdot e^4 - 2^5\cdot e^5 \\
&= 6^5 - 10\cdot 6^2\cdot 2^3\cdot 4 + 5\cdot 6\cdot 2^4\cdot 18 - 2^5\cdot 51 \\
&= 7776 - 11520 + 8640 - 1632 \\
&= 3264.
\end{aligned}
$$

The answer, then, is that

For a generic choice of five conic curves in $\mathbb{P}^2$, there will be exactly 3264 *smooth conics tangent to all five.*

We now go back and verify the transversality assertions 1, 2, and 3. For assertion 1, note that for C smooth, the divisor V_C is irreducible: to see this, let

$$I' \subset V_C \times C$$

be the incidence correspondence given by

$$I' = \{(C_0, p):\ C_0 \text{ is tangent to } C \text{ at } p\}.$$

Since C is irreducible and the fibers of the projection map

$$\pi_2:\ I' \to C$$

are linear subspaces of W, I' is irreducible. This implies that V_C is irreducible. Now let $U \subset W$ be the open set of smooth conics and denote by I the incidence correspondence

$$I \subset (W)^5 \times U$$

defined by

$$I = \{(C_1,\ldots,C_5; C'):\ C' \in V_{C_i} \text{ for all } i\};$$

let $J \subset I$ be the closed subvariety of I consisting of $(C_1,\ldots,C_5; C')$ such that C' is a nontransverse point of intersection of $V_{C_1},\ldots,V_{C_5}$. The fibers of the projection

$$\pi_2:\ I \to U$$

on the last factor are isomorphic to $(V_{C'})^5$, and so irreducible; consequently I is irreducible. Since the map $\pi_1: I \to (W)^5$ is generically finite-to-one, then, we see that assertion 1 can fail to hold—i.e., J can map surjectively onto $(W)^5$—only if $J = I$. To verify this assertion it will suffice to exhibit a point of $I - J$; that is, six conics $C_1,\ldots,C_5$ and C' such that $V_{C_1},\ldots,V_{C_5}$ meet transversely at C'. But this is clear: if C' is any smooth conic, $C_1,\ldots,C_5$ conics simply tangent to C' at distinct points $p_1,\ldots,p_5$, then the tangent hyperplanes $T_{C_i}(V_{C'}) = H_{p_i}$ are independent.

Assertions 2′ and 3 are easier. Note first that in general *if* $\{D_\mu\}$ *is any family of divisors without base points on an* n*-dimensional variety* V, *the generic choice of* n+1 *divisors* $D_{\mu_1},\ldots,D_{\mu_{n+1}}$ *of the family have no points in common*. This follows by an induction argument: if we assume the result for varieties of dimension $n-1$, then by restricting the divisors $\{D_\mu\}$ to a hyperplane section of V the generic choice of n divisors $D_{\mu_1},\ldots,D_{\mu_n}$ will have only finitely many points in common. Since the family $\{D_\mu\}$ has no base points, for generic $D_{\mu_{n+1}}$

$$D_{\mu_1}\cap\cdots\cap D_{\mu_n}\cap D_{\mu_{n+1}}=\varnothing.$$

Now since the locus $W_1\subset W$ of conics of rank two has dimension 4, to prove assertion 2′ we need only check that the family $\{V_C\}_{C\in W}$ has no base points on this locus. This is immediate: for any conic of rank two, we can obviously find a conic not tangent to it.

Assertion 3 remains. We must prove that the family $\{\tilde{V}_C\}$ has no base points in E. To do this, note that for any point $2L\in W_2$ and a normal vector v to W_2 at $2L$ represented by a line $\{C_\lambda\}$ in W, the proper transform $\tilde{V}_C$ will contain the point of E corresponding to v if and only if the line $\{C_\lambda\}$ has intersection multiplicity 3 or more with V_C at $2L$; it will thus suffice to show that for any point $2L\in W_2$ and any line $\{C_\lambda\}$ through $2L$ but not tangent to W_2 at $2L$, there exists a conic C such that

$$\operatorname{mult}_{2L}(V_C,\{C_\lambda\})=2.$$

Now, if any pencil of conics contains two double lines $2L$ and $2L'$, it has a single base point of order 4, and so must consist entirely of singular conics. In the limiting case, then, we see that any pencil tangent to W_2 at $2L$ consists entirely of singular conics; the tangent plane $T_{2L}(W_2)$ to W_2 at a point $2L$ is therefore contained in—hence equal to—the 2-plane

$$\{L+L'\}_{L'\in\mathbb{P}^{2*}}\subset W.$$

If $\{C_\lambda\}$ *is any pencil through* 2L *but not in the tangent space* $T_{2L}(W_2)$, *then, it can have only finitely many base points*. Choosing the conic C to miss these base points, the same argument as before shows that $\{C_\lambda\}$ meets V_C with multiplicity 2 at $2L$.

A note: The problem of determining the number of conics tangent to five conics is of some historical importance, being one of the first problems requiring nontrivial intersection theory; it is interesting to see how it may be solved without explicit reference to abstract blow-ups or cohomology. One argument proceeds as follows: let I_p and $I_l\subset W$ be, respectively, the variety of conics passing through the point p, and tangent to the line l; let $\tilde{I}_p$ and $\tilde{I}_l$ be their proper transforms in the blow-up $\tilde{W}$ of W along W_2. Then it is easy to see that, in the cohomology ring of $\tilde{W}$,

$$\tilde{I}_p\sim\tilde{\omega}\quad\text{and}\quad\tilde{I}_l\sim 2\tilde{\omega}-e,$$

so

$$\tilde{V}_C \sim 2\tilde{I}_p + 2\tilde{I}_l.$$

Without reference to blow-ups or cohomology, then, one could make the statement: "the condition that a conic be tangent to a conic C is equivalent to the condition that it contain either of two points, or be tangent to either of two lines"; this can be seen by noting that, as the conic C degenerates into a pair of lines $l_1 + l_2$, the variety V_C degenerates into the variety $I_{l_1} + I_{l_2} + I_{l_1 \cdot l_2}$, the latter component occurring with multiplicity 2. (In fact, the blow-up W may be constructed geometrically as follows: let W^* denote the linear system of conics in $\mathbb{P}^{2*}$, W_1^* W^* the locus of singular conics, and take the closure in $W \times W^*$ of the locus

$$\{(C,D):\ D = C^*\}\ (W - W_1) \times (W^* - W_1^*).$$

A pair (C,D) in this closure was classically called a *complete conic*). Now, the product

$$\begin{aligned} \tilde{V}_C^5 &= 32(\tilde{I}_p + \tilde{I}_l)^5 \\ &= 32(\tilde{I}_p^5 + 5\tilde{I}_p^4\tilde{I}_l + 10\tilde{I}_p^3\tilde{I}_l^2 + 10\tilde{I}_p^2\tilde{I}_l^3 + 5\tilde{I}_p\tilde{I}_l^4 + \tilde{I}_l^5) \end{aligned}$$

can be evaluated by elementary geometry: since there is a unique conic in the plane through five generically chosen points,

$$\tilde{I}_p^5 = 1.$$

Likewise, the conics through four generic points cut out on a generic line l a pencil of degree 2, which then has two branch points; so

$$\tilde{I}_p^4\tilde{I}_l = 2.$$

Next, the quadratic transformation of $\mathbb{P}^2$ based at three points p_1, p_2, p_3 transforms the net of conics through p_1, p_2, p_3 into the complete series of lines in $\mathbb{P}^2$, and the generic lines in $\mathbb{P}^2$ into conics; the number of conics through p_1, p_2, p_3 tangent to two lines is just the number of lines in $\mathbb{P}^2$ tangent to two conics. Since the tangent lines to a conic in $\mathbb{P}^2$ form a conic curve in $\mathbb{P}^{2*}$, this number is

$$\tilde{I}_p^3 \cdot \tilde{I}_l^3 = 4.$$

The remaining three products of $\tilde{I}_p$ and $\tilde{I}_l$ are dual to the ones above—e.g., a conic $C \subset \mathbb{P}^2$ will be tangent to five lines $l_1, \ldots, l_5 \subset \mathbb{P}^2$ if the dual conic $C^* \subset \mathbb{P}^{2*}$ of tangent lines to C contains the five points $l_1, \ldots, l_5 \in \mathbb{P}^{2*}$—so we have

$$\tilde{I}_p^2\tilde{I}_l^3 = \tilde{I}_p^3\tilde{I}_l^2 = 4, \quad \tilde{I}_p\tilde{I}_l^4 = \tilde{I}_p^4\tilde{I}_l = 2, \quad \tilde{I}_l^5 = \tilde{I}_p^5 = 1.$$

The answer to the problem—modulo the checking of transversality assumptions—is then

$$\tilde{V}_C^5 = 32(1+5\cdot 2+10\cdot 4+10\cdot 4+5\cdot 2+1)$$
$$= 32 \cdot 102 = 3264.$$

2. THE QUADRIC LINE COMPLEX: INTRODUCTION

Geometry of the Grassmannian $G(2,4)$

First we will discuss the geometry of the Grassmannian $G(2,4)$ of 2-planes in $\mathbb{C}^4$, viewed primarily as the set of lines in $\mathbb{P}^3$. Recall from Section 5 of Chapter 1 that the Plücker embedding

$$G(2,4) \longrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4) = \mathbb{P}^5$$

is given by mapping the 2-plane Λ spanned by vectors $v_1, v_2 \in \mathbb{C}^4$ into the wedge product $v_1 \wedge v_2 \in \Lambda^2\mathbb{C}^4$. As was proved there, a general multivector ω will be decomposable—that is, of the form $v_1 \wedge v_2$—exactly when

$$\omega \wedge \omega = 0.$$

This is a quadratic relation; the image of $G(2,4)$ under the Plücker embedding is therefore a quadric hypersurface in $\mathbb{P}^5$, which we will henceforth denote by G. The reader is referred to p. 746 for the definition and intersection numbers of the Schubert cycles $\sigma_1(l_0)$, $\sigma_2(p_0)$, $\sigma_{1,1}(h_0)$, and $\sigma_{2,1}(p_0, h_0)$ on G.

Now, since the wedge product

$$\wedge: \Lambda^2\mathbb{C}^4 \times \Lambda^2\mathbb{C}^4 \to \Lambda^4\mathbb{C}^4 \cong \mathbb{C}$$

is a nondegenerate pairing, every hyperplane in $\mathbb{P}(\Lambda^2\mathbb{C}^4)$ is of the form

$$H_{\omega_0} = \{\omega: \omega \wedge \omega_0 = 0\}.$$

In particular, if $\omega_0 = v_1 \wedge v_2$ the hyperplane section $H_{\omega_0} \cap G$ of G consists of the Schubert cycle $\sigma_1(l_0)$ of lines in $\mathbb{P}^3$ meeting the line $l_0 = \overline{v_1, v_2}$ spanned by v_1 and v_2. Thus

Every Schubert cycle $\sigma_1(l_0) \subset G$ is a hyperplane section of G.

Since the Schubert cycle $\sigma_{2,1}(p,h) \subset G$ has intersection number 1 with the hyperplane class σ_1, it follows that

Every Schubert cycle $\sigma_{2,1}(p,h) \subset G$ is a line in $\mathbb{P}^5$.

Similarly, since

$$\sigma_1^2 \cdot \sigma_{1,1} = \sigma_1^2 \cdot \sigma_2 = 1,$$

Every Schubert cycle $\sigma_2(p)$ or $\sigma_{1,1}(h) \subset G$ is a 2-plane in $\mathbb{P}^5$.

To prove the converse of the last two statements, let $x \in G$ be any point. Since $G \subset \mathbb{P}^5$ is a quadric, by what we have seen the intersection $T_x(G) \cap G$ is just the locus of lines in G through x. But if $x' \in G$ is any point whose corresponding line in $\mathbb{P}^3$ $l_{x'}$ meets l_x, then x and x' both lie on the Schubert cycle $\sigma_{2,1}(p,h)$ of lines in $\mathbb{P}^3$ through the point $p = l_x \cap l_{x'}$ and contained in the hyperplane $h = \overline{l_x, l_{x'}}$. Since $\sigma_{2,1}(p,h)$ is a line, it follows that $\sigma_{2,1}(p,h)$—and hence x'—lies in the locus $T_x(G) \cap G$. The hyperplane section $T_x(G) \cap G$ thus contains the Schubert cycle $\sigma_1(l_x)$ of lines meeting l_x—but $\sigma_1(l_x)$ is itself a hyperplane section of G, and so we have:

For any x∈G

$$T_x(G) \cap G = \sigma_1(l_x).$$

It follows that for any $x, x' \in G$,

$$\begin{aligned} l_x \cap l_{x'} \neq \phi &\Leftrightarrow x' \in T_x(G) \\ &\Leftrightarrow \overline{x, x'} \subset G. \end{aligned}$$

We see from this that

Any line L *lying on the Grassmannian is a Schubert cycle* $\sigma_{2,1}(\mathrm{p},\mathrm{h})$.

For any two points $x \neq x' \in L$, let $p = l_x \cap l_{x'}$ be the point of intersection of the corresponding lines and $h = \overline{l_x, l_{x'}}$ the plane they span; the line $\sigma_{2,1}(p,h)$ in G then contains x and x', and so equals L.

Finally, to see that

Every 2-plane $\mathrm{V}_2 \subset \mathbb{P}^5$ *contained in* G *is a Schubert cycle* $\sigma_2(\mathrm{p})$ *or* $\sigma_{1,1}(\mathrm{h})$.

Observe that for any point $x \in V_2$ the tangent plane section $T_x(G) \cap G$ contains V_2; thus for x_1, x_2, x_3 any three noncollinear points of V_2,

$$V_2 \subset G \cap T_{x_1}(G) \cap T_{x_2}(G) \cap T_{x_3}(G) = \{x \in G:\ l_x \cap l_{x_i} \neq \varnothing, i = 1,2,3\}.$$

But the line $\overline{x_i x_j}$ lies in $V_2 \subset G$, and so the corresponding lines l_{x_i} and l_{x_j} must have a point p_{ij} in common. Since by hypothesis x_1, x_2, and x_3 do not all lie on a Schubert cycle $\sigma_{2,1}(p,h)$, we must have either

1. p_{12}, p_{23}, and p_{13} are distinct, in which case a line $l \subset \mathbb{P}^3$ will meet l_{x_1}, l_{x_2}, and l_{x_3} if and only if l lies in the hyperplane $h = \overline{p_{12}, p_{23}, p_{13}} = \overline{l_{x_1}, l_{x_2}, l_{x_3}}$; or
2. $p_{12} = p_{23} = p_{13}$, in which case, since l_{x_1}, l_{x_2}, and l_{x_3} cannot be coplanar, a line $l \subset \mathbb{P}^3$ will meet l_{x_1}, l_{x_2}, and l_{x_3} if and only if it passes through the point $p = p_{12}$.

In the first case, V_2 is contained in—hence equal to—the Schubert cycle $\sigma_{1,1}(h)$ of lines lying in h; in the second case V_2 is contained in, and so equal to, the Schubert cycle $\sigma_2(p)$ of lines through p.

We will henceforth write the Schubert cycles on G simply as $\sigma(p)$, $\sigma(h)$, $\sigma(l)$, and $\sigma(p,h)$. In particular, the Schubert cycle $L=\sigma(p,h)$ of lines through a point p and lying in a hyperplane $h\subset\mathbb{P}^3$ is called a *pencil* of lines. The common point $p=\bigcap_{x\in L} l_x$ of a pencil L is called its *focus* and will be denoted p_L; the plane $h=\bigcup_{x\in L} l_x$ swept out by the lines of the pencil is called simply its *plane* and will be denoted h_L.

Note that we can write, for any $x\in G$,

$$T_x(G)\cap G=\sigma(l_x)=\bigcup_{p\in l_x}\sigma(p)=\bigcup_{h\supset l_x}\sigma(h),$$

and conversely, for any line $L\subset G$,

$$G\cap\bigcap_{x\in L}T_x(G)=\sigma(p_L)\cup\sigma(h_L).$$

We can get a nice picture of the relations among the Schubert cycles on G by considering again the locus $T_x(G)\cap G$. As we have seen, if $V_3\subset T_x(G)$ is any 3-plane not containing x

$$G\cap T_x(G)=\bigcup_{y\in V_3\cap G}\overline{xy},$$

i.e., $G\cap T_x(G)$ is the cone over the smooth quadric surface $Q=V_3\cap G$. (See Figure 5.) Now, Q has two families $\{L_\lambda\}_{\lambda\in\mathbb{P}^1}$ and $\{L'_\lambda\}_{\lambda\in\mathbb{P}^1}$ of lines on it, with two lines meeting if and only if they are of different families. Let L be any line of the first family. Then the 2-plane $\overline{x,L}$ spanned by x and L lies in G, and so must be of the form

$$\sigma(p),\qquad \text{for some } p\in l_x,$$

or

$$\sigma(h),\qquad \text{for some } h\supset l_x.$$

Indeed, since two Schubert cycles $\sigma(p),\sigma(p')$ intersect only in one point, while for $p\in l_x\subset h$ the Schubert cycles $\sigma(p)$ and $\sigma(h)$ intersect in a line, we see that the 2-planes $\{\overline{x,L_\lambda}\}_{\lambda\in\mathbb{P}^1}$ spanned by x and the lines of one ruling

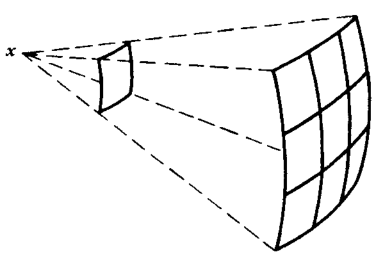

Figure 5. $T_x(G)\cap G$.

must be all the Schubert cycles $\{\sigma(p)\}_{p\in l_x}$, while the planes $\{\overline{x, L'_\lambda}\}_{\lambda\in\mathbb{P}^1}$ associated to lines of the second ruling must be the Schubert cycles $\{\sigma(h)\}_{h\supset l_x}$. Note that since the tangent plane $T_y(Q)$ to Q at any point $y\in Q$ meets Q in the sum of two lines, one from each family, the 3-plane $\overline{x, T_y(Q)}$ meets G in the sum of a $\sigma(p)$ and a $\sigma(h)$, showing directly that

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2.$$

Line Complexes

We have given, above and in Section 1 of this chapter, accounts of various cycles in the Grassmannian $G(2,4)$ arising from the geometry of $\mathbb{P}^3$. Of interest classically was the converse problem: to describe the geometry of the family of lines in $\mathbb{P}^3$ cut out in $G(2,4)\subset\mathbb{P}^5$ by hypersurfaces in $\mathbb{P}^5$. In particular, we define

DEFINITION. A *line complex* of degree d in $\mathbb{P}^3$ is the three-parameter family of lines in $\mathbb{P}^3$ corresponding to the intersection of the Grassmannian $G(2,4)\subset\mathbb{P}^5$ with a hypersurface of degree d in $\mathbb{P}^5$.

We consider first *linear line complexes*, that is, line complexes $X = G\cap H$ given as the intersection of G with a hyperplane $H\subset\mathbb{P}^5$. If X is singular—i.e., if $H = T_x(G)$ is the tangent plane to G at some point x—then, as we have seen, the complex X is the Schubert cycle $\sigma(l_x)$ of lines in $\mathbb{P}^3$ meeting l_x. Suppose on the other hand that X is smooth. For each $p\in\mathbb{P}^3$, then, the set

$$X_p = \sigma(p)\cap H$$

of lines of the complex X passing through p is either all of $\sigma(p)$, or a line in $\sigma(p)$. But the set of tangent planes

$$\{T_x(G)\}_{x\in\sigma(p)}$$

to G at points of $\sigma(p)$ form the linear system of all hyperplanes containing $\sigma(p)$, i.e., any hyperplane containing $\sigma(p)$ is tangent to G. Thus X_p must be a line, that is,

For each $p\in\mathbb{P}^3$, *the lines of* X *through* p *form a pencil* $\sigma(p,h)$.

Likewise, H cannot contain the 2-plane $\sigma(h)$ for any hyperplane $h\subset\mathbb{P}^3$, and so

For each hyperplane $h\subset\mathbb{P}^3$, *the lines of* X *lying in* h *form a pencil* $\sigma(p,h)$.

Here is another way to view this: any element ω of $\Lambda^2\mathbb{C}^4$ corresponds to a skew-symmetric quadratic form

$$\Gamma_\omega(v,v') = \omega \wedge v \wedge v' \in \Lambda^4\mathbb{C}^4 \cong \mathbb{C};$$

the corresponding linear line complex $X = H_\omega \cap G$ is then given by

$$X = \left\{ l = \overline{v,v'} : \Gamma_\omega(v,v') = 0 \right\}.$$

If $\omega = v \wedge v'$ is decomposable, then H_ω is tangent to G at $l = \overline{v,v'}$, and $X = H_\omega \cap G$ is the Schubert cycle $\sigma(l)$; if, on the other hand, ω is indecomposable, then the form Γ_ω is nondegenerate and for any $p = [v] \in \mathbb{P}^3$

$$X_p = \sigma(p,h),$$

where the hyperplane $h \subset \mathbb{P}^3$ is the kernel of the linear functional $\Gamma_\omega(v,\cdot)$ on $\mathbb{C}^4$.

An amusing construction associated to a nonsingular linear complex $X = G \cap H$ is the *configuration of Möbius*, defined as follows: Let T be any tetrahedron in $\mathbb{P}^3$, with sides h_1, h_2, h_3, h_4 and vertices

$$p_i = \bigcap_{j \neq i} h_j.$$

For each i, let h_i' be the plane of the pencil $X_{p_i} = \sigma(p_i) \cap H$ of lines of X through p_i and p_i' the focus of the pencil $X_{h_i} = \sigma(h_i) \cap H$ of lines of X lying in h_i. Note first that the planes h_i' are linearly independent: if all four contained a point q, then all four points p_i would have to lie in the plane swept out by the pencil X_q of lines of X through q; dually, the points $\{p_i'\}$ are independent. Next, we observe that for any $i \neq j$ the line $h_i \cap h_j'$ is a line of the complex X, lying in h_j' and passing through the point p_i'. Thus

$$p_i' = \bigcap_{j \neq i} h_j',$$

i.e., *the points* $\{p_i'\}$ *are the vertices of the tetrahedron* T' *having sides* $\{h_i'\}$. (See Figure 6.)

The line complex X *thus associates to any tetrahedron* T *in* $\mathbb{P}^3$ *a "dual" tetrahedron* T^* *both inscribed in and circumscribed about* T.

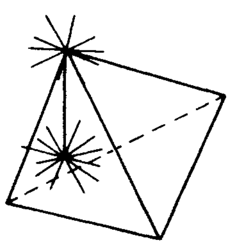

Figure 6

This process, moreover, is self-dual: the tetrahedron associated to T^* is T again.

We now claim that conversely any such configuration of two tetrahedrons T and T' inscribed in and circumscribed about each other determines uniquely a smooth linear line complex in $\mathbb{P}^3$: if T has sides $\{h_i\}$ and vertices $\{p_i\}$, T' sides $\{h_i'\}$ and vertices $\{p_i'\}$ as above, then T' will be the dual tetrahedron of T with respect to the complex X exactly when the lines

$$L_i = \sigma(p_i, h_i'), \qquad i = 1,2,3,4,$$

and

$$L_i' = \sigma(p_i', h_i), \qquad i = 1,2,3,4,$$

in G all lie in X. But we have

$$L_i \cap L_j' \neq \emptyset \qquad \text{for } i \neq j;$$

and

$$L_i \cap L_i' = \emptyset.$$

The lines $\{L_i\}, \{L_i'\}$ in $\mathbb{P}^5$ thus form the configuration shown in Figure 7, and so all lie in the 4-plane spanned by the points $L_1' \cap L_3$, $L_1' \cap L_4$, $L_2' \cap L_3$, $L_2' \cap L_4$, and $L_3' \cap L_2$. On the other hand, no quadric surface $Q = G \cap V_3$ in $\mathbb{P}^3$ can contain such a configuration of lines: if Q were smooth, then clearly the lines $\{L_i\}$ and $\{L_i'\}$ would belong to opposite families—but in that case L_1 and L_1' would meet; if Q had rank three, all lines on Q would meet, and if Q were the union of two planes, any hyperplane containing V_3 would be tangent to G. Consequently the lines $\{L_i, L_i'\}$ lie in a unique 4-plane. In sum, we have proved the rather amusing result:

The set of nondegenerate skew-symmetric quadratic forms on $\mathbb{C}^4$, up to multiplication by scalars, is in one-to-one correspondence with the set of tetrahedra inscribed in and circumscribed about a given tetrahedron T_0 *in* $\mathbb{P}^3$.

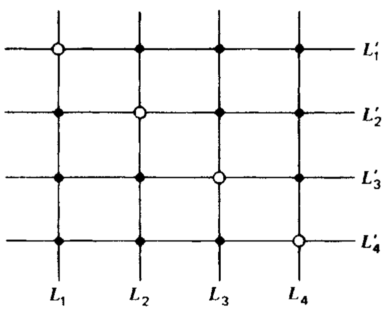

Figure 7

The Quadric Line Complex and Associated Kummer Surface I

We come now to the main object of study in this chapter: the *quadric line complex*, defined to be the family of lines in $\mathbb{P}^3$ corresponding to the smooth intersection $X = G \cap F$ of the Grassmannian $G \subset \mathbb{P}^5$ with a quadric hypersurface F. As in the case of the linear complex, our initial problem in regard to the quadric line complex is to identify the pencils of lines in X and to determine, for any point p and any hyperplane h in $\mathbb{P}^3$, the locus of lines in our complex passing through p or contained in h. We first check that

Lemma. *No 2-plane* $\sigma(p)$ *or* $\sigma(h)$ *lies in the quadric line complex* $X = F \cap G$.

Proof. We will give two proofs of this fact. First, in an elementary but rather special vein, we can argue as follows: if the quadrics F and G contained a 2-plane $V_2 \subset \mathbb{P}^5$ in common, then the Gauss maps

$$\mathcal{G}_F : F \to \mathbb{P}^{5*} \quad \text{and} \quad \mathcal{G}_G : G \to \mathbb{P}^{5*}$$

would each map V_2 isomorphically onto the set V_2^* of hyperplanes containing V_2. But then the isomorphism

$$\mathcal{G}_F^{-1} \circ \mathcal{G}_G : V_2 \to V_2$$

would have a fixed point—i.e., for some $x \in V_2$ we would have $T_x(F) = T_x(G)$, contradicting the assumption that F and G meet transversely.

Alternatively, we see by the Lefschetz theorem on hyperplane sections that the generator of

$$H^2(X,\mathbb{Z}) \cong H^2(G,\mathbb{Z}) \cong H^2(\mathbb{P}^5,\mathbb{Z})$$

is the restriction to X of the hyperplane class ω in $\mathbb{P}^5$; in particular, that every surface on X has even degree. Note that this argument may be used in general to show that a smooth nondegenerate complete intersection of dimension n in $\mathbb{P}^N$ cannot contain a linear subspace of dimension $> n/2$.

Now, we deduce from the lemma that for each $p \in \mathbb{P}^3$ the set

$$X_p = X \cap \sigma(p)$$

of lines in the complex X passing through p forms a conic curve in $\sigma(p)$. There are three possible cases:

1. F meets $\sigma(p)$ transversely, i.e., X_p is a smooth conic curve. The locus of lines in X through p will then be a cone through p over a smooth conic curve (Figure 8). As we shall see, this is the generic case.
2. F is tangent to $\sigma(p)$ at a point, i.e., X_p consists of two pencils with focus p. In this case the locus of the lines in X_p will be two hyperplanes (Figure 9).

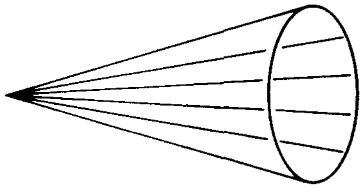

Figure 8

3. F is tangent to $\sigma(p)$ along a line, i.e., X_p consists of one double line. In this case, the locus of X_p will be a single hyperplane (Figure 10).

Dually, for every hyperplane $h \subset \mathbb{P}^3$ the set $X_h = X \cap \sigma(h)$ of lines of X lying in h is a conic curve; again, there are three possible cases:

1′. F meets $\sigma(h)$ transversely, so that $X_h \subset \sigma(h)$ is a smooth conic curve. The lines of X lying in h are thus the set of tangent lines to a smooth conic curve in h (Figure 11).

2′. F is tangent to $\sigma(h)$ at one point, so that X_h consists of two pencils with plane h (Figure 12).

3′. F is tangent to $\sigma(h)$ along a line. In this case, X_h will consist of one pencil in h (Figure 13).

Let $S \subset \mathbb{P}^3$ be the locus of points $p \in \mathbb{P}^3$ such that X_p is singular, i.e., such that case 2 or 3 above occurs. S is called the *associated Kummer surface* of the quadric line complex X; it may be thought of, in slightly different terms, as the set of foci of pencils of lines in the complex X. We denote by $R \subset S$ the locus of points $p \in S$ such that case 3 occurs. We define the *dual Kummer surface* $S^* \subset \mathbb{P}^{3*}$ to be the locus of hyperplanes $h \in \mathbb{P}^{3*}$ such that X_h is singular, i.e., the set of planes in $\mathbb{P}^3$ swept out by the pencils of X; let $R^* \subset S^*$ be the set of hyperplanes $h \in \mathbb{P}^{3*}$ such that case 3′ above occurs. Inasmuch as the set of singular plane conic curves has codimension 1 in the linear system of all conics, and the set of double lines codimension 3, we would expect the varieties S and R to be a surface and a finite collection of points, respectively. That S is indeed a surface will be apparent from the following computations; that R is finite will emerge later.

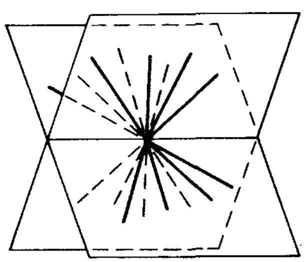

Figure 9

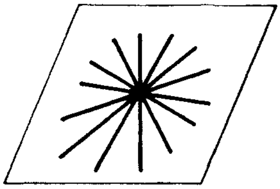

Figure 10

Our first task will be to determine the degree of S. To this end, we offer a computation and a proof, as follows.

1. Let $l_x \subset \mathbb{P}^3$ be a generic line of the complex X—which we will assume is not tangent to S—and consider the locus $l_x \cap S$. For every point $p \in l_x \cap S$, the line l_x will be an element of one or both of the two pencils in our complex with focus p; in other words, x will lie on one or both of the lines of $F \cap \sigma(p)$. Conversely, of course, any pencil of lines in X containing l_x has its focus on l_x, and hence in $l_x \cap S$. Thus, if we make the assumption that the generic line l_x does not lie on two confocal pencils of X, the points of intersection of l_x with S correspond exactly to the lines L on X through x. But we have seen that the locus of lines in G (resp. F) through any point x is just the intersection $T_x(G) \cap G$ (resp. $T_x(F) \cap F$), so the locus of lines in $X = F \cap G$ through x is

$$T_x(X) \cap X = T_x(F) \cap F \cap T_x(G) \cap G.$$

$T_x(X) \cap X$ has degree 4, and—making the final assumption that it contains no multiple components—it must consist of four lines. We thus have

$$\deg S = {}^{\#}(l_x \cap S) = 4.$$

Now, all the assumptions made about the generic line l_x of our complex are in fact the case, but their verifications are best left until we know more about the complex. There is one point worth mentioning now, which will emerge from this computation once we have established that S is quartic: Since $T_x(X) \cap X$ can never contain more than four lines, for any point $p \in S - R$ the line l_x held in common by the two pencils in X through p—that is, the line of intersection of the two hyperplanes comprising the

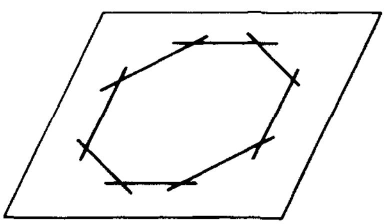

Figure 11

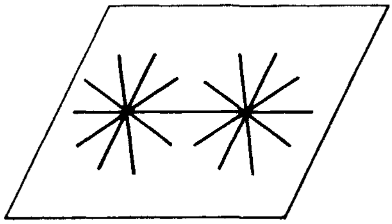

Figure 12

locus of X_p—can lie on at most two lines in X not on $\sigma(p)$. Thus l_x meets S in at most three points and so must be a tangent line to S.

2. A more conclusive argument for the degree of S goes as follows: we first claim that for a generic $x \in G$, the surface

$$U = T_x(G) \cap X \subset \mathbb{P}^5$$

is smooth—this fact will emerge in a moment. Granting this, we recall

$$G \cap T_x(G) = \bigcup_{p \in l_x} \sigma(p)$$

so that the curves

$$X_p = \sigma(p) \cap F \subset U$$

form a linear system on U without base point. In fact, we see that

$$\deg S = {}^\#(l_x \cap S) = {}^\#\{p : X_p \text{ is singular}\}$$

is just the number μ of singular curves in this pencil. Now the generic curve X_p is a smooth conic, with Euler characteristic 2, and if we take l_x disjoint from R, all the singular curves X_p in our pencil will consist of two distinct lines, i.e., the pencil will be Lefschetz. By the general formula

$$\chi(S) = 2\chi(C_\lambda) - n + \mu$$

of Section 2, Chapter 4, we have

$$\chi(U) = 4 + \mu.$$

But U, being the smooth intersection of two quadrics in $\mathbb{P}^4$, is biholomorphic to $\mathbb{P}^2$ blown up five times (Section 4, Chapter 4) and so has Euler

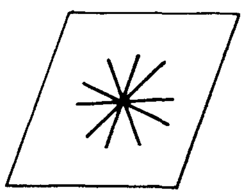

Figure 13

characteristic 8. Consequently

$$\deg S = \mu = 4.$$

In fact, this last argument gives us something more: it tells us that S *is smooth away from the locus* R. To see this, let q be any point of $S-R$. Then the hyperplane sections

$$\{T_x(G)\cap X\}_{x\in\sigma(q)}$$

form a linear system on X; by Bertini's theorem, for generic $x\in\sigma(q)$ the surface $U_x = T_x(G)\cap X$ will be smooth away from the base locus $X_q = \sigma(q)\cap X$ of the linear system. If $q\notin R$, moreover, then each U_x can be singular only at the singular point of X_q—but the hyperplanes $\{T_x(G)\}_{x\in\sigma(q)}$ are exactly all the hyperplanes in $\mathbb{P}^5$ containing $\sigma(q)$, and so the generic one will not contain the tangent space to X at the singular point of X_q. We have thus shown that for l_x a generic line in $\mathbb{P}^3$ through $q\in S-R$, the surface U_x is smooth. The argument above then shows that l_x meets S in four distinct points, and hence meets S transversely; a fortiori, it shows that q is a smooth point of S.

Note that all three of these arguments apply as well to show that the dual Kummer surface S^* is a quartic surface smooth away from the locus R^*. In the first argument, we observe that the points of intersection of S^* with the pencil $l_x^*\subset\mathbb{P}^{3*}$ of hyperplanes in $\mathbb{P}^3$ containing a line l_x of our complex again correspond to the pencils in X containing l_x, and hence to the lines L on X containing x. Likewise, the second argument goes over, and indeed establishes an important point: given any line $l_x\subset\mathbb{P}^3$ not passing through any points of R or lying in any hyperplanes of R^*, we have two pencils on the surface $U = T_x(G)\cap X$:

$$\{X_p = \sigma(p)\cap U\}_{p\in l_x} \quad\text{and}\quad \{X_h = \sigma(h)\cap U\}_{h\supset l_x}.$$

Both are Lefschetz, and so the number of singular fibers in each is $\chi(U)-4$. But while the singular fibers of the pencil $\{X_p\}$ correspond to points of intersection of l_x with S, singular fibers of $\{X_h\}$ correspond to points of intersection of the dual line $l_x^*\subset\mathbb{P}^{3*}$ of hyperplanes containing l_x with S^*. In particular, $^{\#}(l_x\cap S)<4 \Leftrightarrow {}^{\#}(l_x^*\cap S^*)<4$, i.e., l_x *is tangent to* S *if and only if* l_x^* *is tangent to* S^*. Now suppose $p\in S$ is any point, $h = T_p(S)$ its tangent plane, and let p^* and h^* be the hyperplane and point in $\mathbb{P}^3$ dual to p and h, respectively. The dual lines $\{l_x^*\}$ to the pencil of lines $\{l_x\}$ in $\mathbb{P}^3$ through p and lying in h form the pencil of lines in $\mathbb{P}^{3*}$ containing h^* and lying in p^*, and they are all tangent to S^*. Every element of the pencil $\{l_x^*\cdot S^*\}$ they cut out on the curve $p^*\cap S^*$ is therefore singular, and so by Bertini they are all singular at the base locus h^* of $\{l_x^*\}$, i.e., $h^*\in S^*$ and

$p^* = T_{h^*}(S^*)$. We see, then, that

S *and* S* *are dual surfaces,*

that is, S^* is the locus of tangent planes to S and vice versa.

Singular Lines of the Quadric Line Complex

The next step in our study of X is to introduce a subvariety $\Sigma \subset X$ closely related to the Kummer surface S.

DEFINITION. For any $x \in X$, the line l_x is called a *singular line* of the complex X if it is an element of two confocal pencils of X—in other words, if $\sigma(p)$ is tangent to F at x for some point $p \in l_x$.

For $p \in S - R$, of course, there is a unique singular line through p: the line of intersection of the two hyperplanes comprising the locus of X_p; for $p \in R$, any line l_x of X through p is singular. We denote by $\Sigma \subset X$ the set of $x \in X$ such that l_x is singular.

We first check that no line l_x is singular at more than one point, i.e., that if $\sigma(p)$ is tangent to F at x, then for $q \neq p \in l_x$, $\sigma(q)$ cannot also be tangent to F at x. But $\sigma(p) \cap \sigma(q) = \{x\}$, so the linear span of $\sigma(p)$ and $\sigma(q)$ in $\mathbb{P}^5$ is all of $T_x(G)$; thus $\sigma(p)$ and $\sigma(q)$ cannot both be contained in $T_x(F) \neq T_x(G)$. We can therefore define a map

$$\pi: \Sigma \to S$$

sending each $x \in \Sigma$ to the unique $p \in l_x$ for which $\sigma(p)$ is tangent to F at x. By what was said above, π is one-to-one and surjective over $S - R$, with $\pi^{-1}(p) = X_p \cong \mathbb{P}^1$ for $p \in R$.

Σ is easy to describe, once we have the following characterization.

Lemma. *For* x $\in$ X,

$$x \in \Sigma \Leftrightarrow T_x(F) \text{ is tangent to } G.$$

Proof. Say $T_x(F)$ is tangent to G at x'. Then $x \in T_{x'}(G)$, and so l_x meets $l_{x'}$ at a point $p \in \mathbb{P}^3$. The plane $\sigma(p)$ is then contained in $T_{x'}(G) = T_x(F)$, i.e., is tangent to F at x; thus $x \in \Sigma$.

Conversely, if $\sigma(p) \subset T_x(F)$, then the quadric threefold $T_x(F) \cap G$ contains a 2-plane and so by our earlier argument must be singular; thus $T_x(F)$ must be tangent to G somewhere. Q.E.D.

This argument will become clearer if we refer back to our picture of the locus $T_x(G) \cap G$ as the cone over a quadric $Q = T_x(G) \cap G \cap H$, H a hyperplane disjoint from x. (See Figure 14.) Recall that the 2-planes

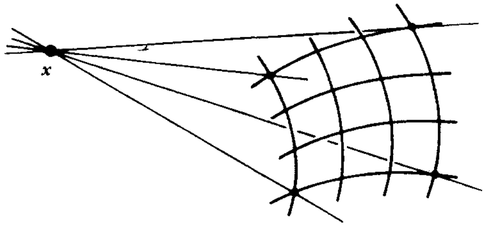

Figure 14. $T_x(G) \cap G$.

$\{\sigma(p)\}_{p \in l_x}$ lying in $T_x(G) \cap G$ are spanned by x together with the lines of one of the rulings of Q, while the planes $\{\sigma(h)\}_{h \supset l_x}$ are spanned by x and the lines of the other ruling of Q. Now if $T_x(F)$ is tangent to G at some point y, we may take $y \in H$, so that the locus $T_x(F) \cap T_x(G) \cap G$ consists of the two 2-planes $\sigma(p)$ and $\sigma(h)$ spanned by x and the two lines of intersection $Q \cap T_x(F)$. (See Figure 15.) Of the four lines of $T_x(X) \cap X = T_x(F) \cap T_x(G) \cap G \cap F$, then, two will lie on the 2-plane $\sigma(p)$ and two on $\sigma(h)$. Conversely, if $T_x(F)$ is nowhere tangent to G, then the locus $T_x(F) \cap T_x(G) \cap G$ will just be the cone over the smooth conic $T_x(F) \cap Q$, and no two of the lines of $T_x(X) \cap X$ will lie on the same 2-plane $\sigma(p)$—unless, of course, F is tangent to $T_x(F) \cap Q$, i.e., $T_x(X) \cap X$ contains a multiple line. (See Figure 16.)

One corollary of our lemma implied by this picture is that the locus $T_x(X) \cap X$ will contain two lines from the same $\sigma(p)$ if and only if it contains two lines from the same $\sigma(h)$: in other words,

A line l_x *of our complex is singular—i.e., lies on two confocal pencils—if and only if it lies on two coplanar pencils.*

We can now give an explicit description of $\Sigma \subset X$. Let $X = [x_0, \ldots, x_5]$ be homogeneous coordinates on $\mathbb{P}^5$, and suppose that G and F are given as the loci

$$(Qx, x) = 0 \quad \text{and} \quad (Q'x, x) = 0,$$

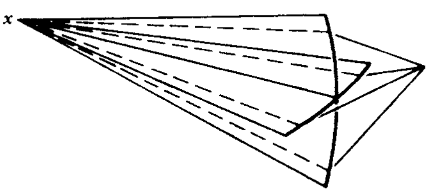

Figure 15. $T_x(G) \cap G \cap T_x(F)$ if $x \in \Sigma$.

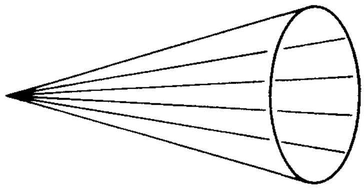

Figure 16. $T_x(G) \cap G \cap T_x(F)$ if $x \notin \Sigma$.

respectively. Then, in terms of dual coordinates x^* on $\mathbb{P}^{5*}$, the Gauss maps of G and F are given by

$$x^* = Qx \quad \text{and} \quad x^* = Q'x;$$

the dual hypersurfaces G^* and $F^* \subset \mathbb{P}^{5*}$ of tangent hyperplanes to G and F are thus

$$G^* = \left((x^*, Q^{-1}x^*) = 0\right)$$

and

$$F^* = \left((x^*, Q'^{-1}x^*) = 0.\right)$$

We see from this that for $x \in F$, $T_x(F)$ will be tangent to G if and only if

$$\mathcal{G}_F(x) \in G^*,$$

i.e., when

$$(Q'x, Q^{-1}Q'x) = (Q'Q^{-1}Q'x, x) = 0.$$

The surface $\Sigma \subset X$ is thus cut out by the quadric hypersurface

$$H = \left((Q'Q^{-1}Q'x, x) = 0\right).$$

We claim now that in fact the intersection

$$\Sigma = F \cap G \cap H$$

is everywhere transverse. To see this, suppose that for some $x \in F \cap G \cap H$ the hyperplanes $T_x(F), T_x(G)$, and $T_x(H)$ were linearly dependent, i.e., that the points

$$\mathcal{G}_G(x) = Qx, \qquad \mathcal{G}_F(x) = Q'x, \quad \text{and} \quad \mathcal{G}_H(x) = Q'Q^{-1}Q'x$$

in $\mathbb{P}^{5*}$ lay on a line. The three points

$$x, \qquad x' = Q^{-1}Q'x, \quad \text{and} \quad x'' = (Q^{-1}Q')^2 x$$

would then likewise be collinear in $\mathbb{P}^5$; since all three lie on G, the line L they span would lie on G. But now the linear transformation

$$M: \; x \mapsto Q^{-1}Q'x$$

taking G into G takes x and x' (distinct, since by hypothesis $Qx \neq Q'x$ for any $x \in F \cap G$) into L, and so takes L into itself; thus $L \subset F \cap G$. M must have a fixed point y somewhere on L, i.e., for some $y \in L$,

$$Qy = Q'y.$$

But since $L \subset F \cap G$, this implies that F and G are tangent at y, a contradiction.

Now that we have described Σ as the smooth intersection of three quadrics in $\mathbb{P}^5$, the reader will recognize Σ as a K-3 surface (Section 5, Chapter 4); in particular, Σ has numerical invariants

$$K_\Sigma \equiv 0, \quad q(\Sigma) = 0, \quad p_g(\Sigma) = 1, \quad c_1^2(\Sigma) = 0, \quad c_2(\Sigma) = 24.$$

Inasmuch as Σ is minimal and smooth, moreover, the map

$$\Sigma \xrightarrow{\pi} S$$

is the minimal desingularization os S; and since the inverse images $\pi^{-1}(p) = X_p$ of the singular points $p \in S$ in Σ are all smooth rational curves, having by adjunction self-intersection -2 on Σ, we see from our discussion of isolated singularities of surfaces that *the points of* R *are all ordinary double points of* S.

It remains to determine the number $^{\#}R$ of double points on S. We will do this first by an Euler characteristic argument, as follows: Let $\{h_\lambda\}$ be a generic pencil of hyperplanes in $\mathbb{P}^3$—specifically, one such that for each $p \in R$, p lies on a unique H_λ and H_λ is generic among hyperplanes containing p; and such that the pencil $\{H_\lambda \cap S\}$ is Lefschetz on $S - R$. Let

$$\{C_\lambda = \pi^{-1}(H_\lambda)\}$$

be the corresponding pencil of curves on Σ. The generic curve C_λ is isomorphic to a smooth plane quartic, hence has genus 3 and Euler characteristic -4; C_λ will be singular if H_λ either contains a point $p \in R$ or is tangent to S. (See Figure 17.) In the first case, we can write

$$C_\lambda = \tilde{C}_\lambda + X_p$$

with—by taking H_λ generic—$\tilde{C}_\lambda$ a smooth curve meeting X_p in two distinct points. Now $\tilde{C}_\lambda$ is the disingularization of the plane quartic $H_\lambda \cap S$ having one double point at p, and so has genus 2; since X_p is a line and meets $\tilde{C}_\lambda$ in two points,

$$\chi(C_\lambda) = \chi(\tilde{C}_\lambda) = -2.$$

In the latter case—when H_λ is simply tangent to S—C_λ is isomorphic to a plane quartic with one ordinary double point and

$$\chi(C_\lambda) = -3.$$

Thus if ν is the number of tangent hyperplanes to S in a pencil, the pencil

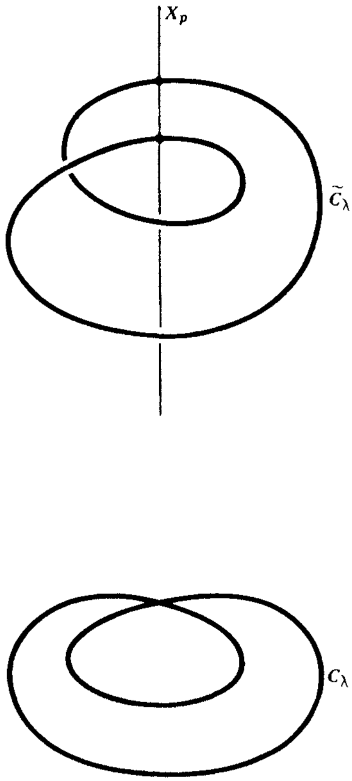

Figure 17

$\{C_\lambda\}$ on Σ has exactly

$$\mu = \nu + {}^{\#}R$$

singular elements. By the formula of p. 509, then, we see that

$$\chi(\Sigma) = 2 \cdot -4 + \nu + 2{}^{\#}R$$

But as we have seen the dual Kummer surface S^* is the dual surface to S, so that

$$\nu = \deg S^* = 4,$$

and, since $\chi(\Sigma) = 24$, this yields

$${}^{\#}R = \tfrac{1}{2}(24+8) = 16.$$

Another way to compute the number of double points of S is by Schubert calculus, inasmuch as ${}^{\#}R$ will be just the number of points of intersection of the three- and six-dimensional cycles

$$\tau = \{\sigma(p)\}_{p \in \mathbb{P}^3}$$

and

$$\omega_F = \{\Lambda_2 \subset \mathbb{P}^5 : \; V_2 \cdot F \text{ is a double line}\}$$

in the Grassmannian $G(3,6)$ of 2-planes in $\mathbb{P}^5$. We have seen that

$$\tau \sim 4\sigma_{3,2,1},$$

where

$$\sigma_{3,2,1} = \{\Lambda_2 \subset \mathbb{P}^5 : \; \Lambda \ni p, \dim(\Lambda \cap V_2) \geq 1, \Lambda \subset V_4\}$$

for any point, 2-plane, and hyperplane $p \in V_2 \subset V_4$. To compute

$$^{\#}(\tau \cdot \omega_F) = 4 \cdot {}^{\#}(\sigma_{3,2,1} \cdot \omega_F)$$

let p, V_2, and V_4 be generic, so that $p \notin F$, V_2 intersects F in a smooth conic curve C, and V_4 intersects F in a smooth quadric threefold Q. Say $\Lambda \in \omega_F \cap \sigma_{3,2,1}$, i.e., Λ is a 2-plane containing p, having a line in common with V_2, lying in V_4, and meeting F in a line. (See Figure 18.) Then the line $\Lambda \cap V_2$ can meet C in only one point, hence must be one of the two tangent lines L_1, L_2 to C through p in V_2. Let x_1, x_2 denote the points of tangency of L_1, L_2 with C; the locus of lines on $Q = F \cap V_4$ through x_i is

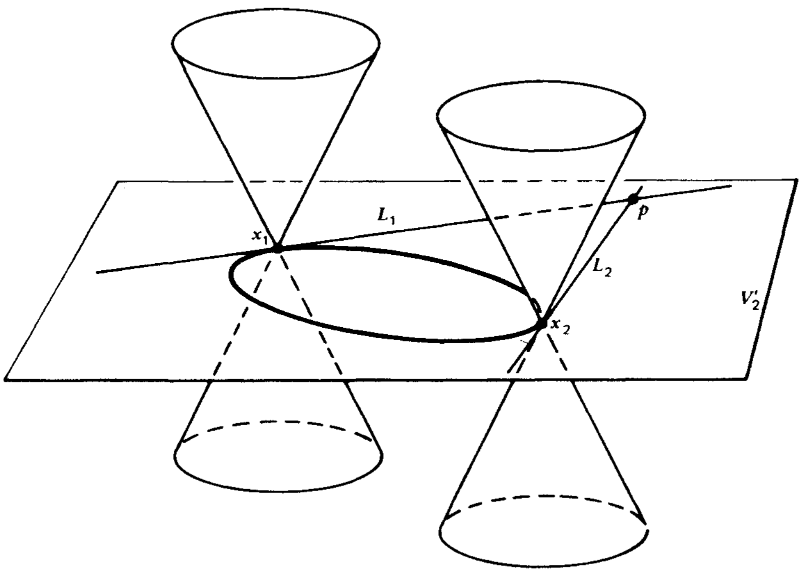

Figure 18

just $T_{x_i}(Q) \cap Q$. Let U_i be a 2-plane in $T_{x_i}(Q)$ containing p and not containing x_i; then $F \cap U_i$ is a smooth conic curve C_i and $T_{x_i}(Q) \cap Q$ is just the locus of lines through x_i meeting C_i. Λ must therefore intersect U_i, $i=1$ or 2, in one of the two tangent lines L_{i_1}, L_{i_2} to C_i through p; i.e., Λ must be one of the four 2-planes Λ_{ij} spanned by L_i and L_{ij}. Clearly all four 2-planes Λ_{ij} lie in $\omega_F \cap \sigma_{3,2,1}$, so we see that

$$(\omega_F \cdot \sigma_{3,2,1}) = 4$$

and finally

$$(\tau \cdot \omega_F) = 16.$$

Note that, since any line of the complex X lies in two confocal pencils of X if and only if it lies in two coplanar pencils, we can also define a map

$$\pi' : \Sigma \longrightarrow S^*$$

by sending any point $x \in \Sigma$ to the common plane $h \in S^*$ of the two coplanar pencils containing l_x, or equivalently to the unique plane $h \supset l_x$ for which $\sigma(h)$ is tangent to F at x. The map π' is, by virtue of the same arguments, the desingularization of S^*; note, however, that the lines $\{X_h = \pi'^{-1}(h)\}_{h \in R^*}$ of Σ lying over the double points of S^* are not the lines X_p of Σ lying over the double points of S.

For later use, we compute the Euler characteristic $\chi(S)$, as follows. Take a triangulation of Σ that extends a triangulation of $\pi^{-1}(R) = \cup X_p$. Then the images of the simplices in Σ not in $\pi^{-1}(R)$, together with the points $p \in R$ as vertices, form a cell decomposition of S. Since, in the course of passing from Σ to S, we lose all the simplices in $\cup X_p$ and gain one new vertex for each p, we have

$$\begin{aligned}\chi(S) &= \chi(\Sigma) - \sum_{p \in R} (\chi(X_p) - 1) \\ &= \chi(\Sigma) - 16 = 8.\end{aligned}$$

Two Configurations

There are two classical configurations associated to the Kümmer surface $S \subset \mathbb{P}^3$ and its desingularization $\Sigma \subset \mathbb{P}^5$. The first has to do with the 16 double points of S and may be described as follows. Let $p_0 \in R$ be any of the double points of S, let $\tilde{S}$ be the blow-up of S at p_0, and consider the map

$$r: \tilde{S} \longrightarrow \mathbb{P}^2$$

obtained by projection from p_0 onto a hyperplane. The generic hyperplane section C_h of S through p_0 is a plane quartic curve with one double point at

p_0, its proper transform $\tilde{C}_h \subset \tilde{S}$ its desingularization. $\tilde{C}_h$ thus has genus 2, and since r expresses $\tilde{C}_h$ as a 2-sheeted cover of its image $L = h \cap \mathbb{P}^2 \cong \mathbb{P}^1$, r must be branched at exactly six points over L. The branch locus $F \subset \mathbb{P}^2$ of r is thus a sextic plane curve without multiple components. On the other hand, if h is a generic hyperplane passing through p_0 and another double point p_i of S, then the curve C_h, having two double points, is elliptic, and the map $r: \tilde{C}_h \to \mathbb{P}^1$, expressing $\tilde{C}_h$ as a 2-sheeted cover of $\mathbb{P}^1$, can be branched at at most four points other than $r(p_i)$. The generic line $L \subset \mathbb{P}^2$ through $r(p_i)$ thus meets F at most four times away from $r(p_i)$, and so we see that the images $\bar{p}_i = r(p_i)$ of the double points of S are double points of F.

Now, suppose the curve F has irreducible components F_i of degree d_i. Singular points of F then arise in two ways: either as points of intersection of components F_i, F_j or as singular points of a component F_i. There are, of course, at most $\sum_{i \neq j} d_i d_j$ singular points of F of the former kind and, by the result of Section 2, Chapter 4, at most $\sum_i ((d_i - 1)(d_i - 2)/2)$ of the latter. But we know that $\sum d_i = \deg F = 6$, and we have seen that F has at least the 15 singular points $r(p_i)$, $i = 1, \ldots, 15$. From the chain of inequalities

$$\begin{aligned} 15 &\leqslant \sum_{i \neq j} d_i d_j + \sum \frac{(d_i - 1)(d_i - 2)}{2} \\ &= \tfrac{1}{2}\left(\sum d_i\right)^2 - \sum \frac{d_i}{2} - \sum (d_i - 1) \\ &= 18 - 3 - \sum (d_i - 1) \end{aligned}$$

we conclude that $d_i = 1$ for all i, i.e., that *F consists of the sum of six distinct lines L_i*. F then has exactly the 15 double points $L_i \cdot L_j$; these must, of course, be the images $r(p_i)$. It follows that each of the lines L_i contains exactly five of the points $r(p_i)$, and correspondingly that the plane $\overline{p_0, L_i} \subset \mathbb{P}^3$ contains exactly six of the double points p_i of S. Our first observation, then, is that

Through each double point p *of* S *there pass six hyperplanes, each containing six of the points* p_i.

Let us consider in more detail one of the hyperplanes $h = \overline{p_0, L}$ found in the last argument. We note first that, inasmuch as L is part of the branch locus of the map r, every line through p_0 in the plane h meets S in exactly one more point and is tangent to S there. It follows from Bertini's theorem that h *is tangent to* S *at every point* $p \in S \cap h$, since otherwise the pencil cut out on C_h by the lines in h through p would be generically singular away from its base locus p. The curve C_h is thus a plane conic, counted with multiplicity 2 in the intersection $S \cdot h$. We see from this that the hyperplane h is a double point of the dual Kummer surface $S^* \subset \mathbb{P}^{3^*}$: clearly $h \in S^*$, so that if h were not in R^*, then X would contain two pencils lying in h, and through a generic point $p \in C_h$ there would pass two distinct lines of the complex. But the common line of the two pencils of X through each $p \in C_h$, we have seen, is tangent to S at p and so lies in h; if X contained a secant line through p in h, it would follow that X contained the pencil $\sigma(p,h)$, hence all of $\sigma(h)$. Thus, X can contain a priori only one pencil in $\sigma(h)$, and so $h \in R^*$.

Finally, applying the same arguments to the dual Kummer surface $S^* \subset \mathbb{P}^{3^*}$, we see that every point $h^* \in R^*$ lies on six of the hyperplanes $p^* \in R$, or in other words every hyperplane h in $\mathbb{P}^3$ corresponding to a point of R^* contains six of the points p_i; in sum, then, we have that:

Every hyperplane $\mathrm{h} \in \mathrm{R}^*$ *contains exactly six of the* 16 *double points of* S *and every double point of* S *lies on exactly six of the* 16 *hyperplanes* $\mathrm{h} \in \mathrm{R}^*$.

This configuration of 16 points and 16 hyperplanes is called the (16_6) configuration.

Now consider the K-3 surface $\Sigma \subset \mathbb{P}^5$. Σ contains 32 lines: the 16 lines $\{X_p\}_{p \in R}$ forming the inverse image $\pi^{-1}(R)$ of the double points of S, and likewise the 16 lines $\{X_h\}_{h \in R^*}$; the latter may be thought of either the exceptional divisors of the desingularization $\pi' : \Sigma \to S^*$ or as the inverse images $\{\pi^{-1}(C_h)\}_{h \in R^*}$ of the 16 double hyperplane sections of S. The lines $\{X_p\}$ are, of course, all disjoint, as are the lines $\{X_h\}$; and from our last argument we see that each line X_p on Σ meets exactly six of the lines $\{X_h\}$, and vice versa.

Note that these are all the lines on Σ: if $L \subset \Sigma$ is any line, $\sigma(p,h)$ the corresponding pencil, then by definition every line $l \in \sigma(p,h)$ belongs to two confocal pencils of X. If the common focus of these two pencils is p for every l, then clearly $\sigma(p,h) = X_p$, while if for generic $l \in \sigma(p,h)$ the common focus of the pencils containing l is a point $q \neq p \in C_h$, then clearly h cannot contain two pencils, and so $\sigma(p,h) = X_h$.

We wish now to describe a set of special hyperplane sections of Σ. To do this, we go back to the picture of the (16_6) configuration obtained by projection from a point $p_0 \in R$. We saw that under such a projection, the 15 remaining points of R were mapped to the points of intersection of six lines $L_1, \ldots, L_6 \subset \mathbb{P}^2$; let $\overline{p_{ij}} = L_i \cdot L_j$ and let p_{ij} be the point of R lying over $\overline{p_{ij}}$. Choose three of the lines L_i, L_j, and L_k, and consider the lines on Σ corresponding to the points p_0, p_{ij}, p_{jk}, and $p_{ik} \in R$, and the hyperplanes $h_i = \overline{p_0 L_i}$, $h_j = \overline{p_0 L_j}$, and $h_k = \overline{p_0 L_k} \in R^*$; these form on Σ a configuration as

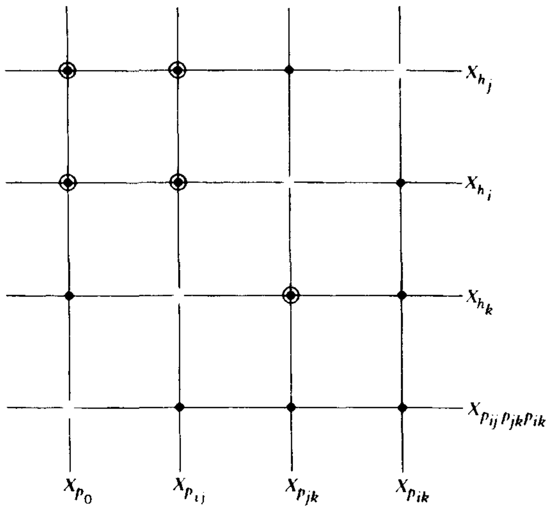

Figure 19

shown in Figure 19. Note that these seven lines lie in the hyperplane in $\mathbb{P}^5$ spanned by the points that are circled. Now in that hyperplane, $X_{p_{ik}}$ and $X_{p_{jk}}$ span a 3-plane, which must then meet $X_{p_{ij}}$ in a point; thus there is a line $L \subset \mathbb{P}^5$ meeting $X_{p_{ij}}$, $X_{p_{ik}}$, and $X_{p_{jk}}$. But since $\Sigma \subset \mathbb{P}^5$ is cut out by quadrics, the line L, meeting Σ in three points, must lie in Σ; since L meets lines of the form X_p on Σ, we must have $L = X_h$ for some $h \in R^*$; and since L meets $X_{p_{ij}}$, $X_{p_{ik}}$, and $X_{p_{jk}}$, we must have $h = \overline{p_{ij}p_{jk}p_{ik}}$. Thus all four faces of the tetrahedron in $\mathbb{P}^3$ with vertices p_0, p_{ij}, p_{jk}, and p_{ik} are hyperplanes $h \in R^*$. Such a tetrahedron will be called *special*; corresponding to a special tetrahedron we have a hyperplane section of Σ consisting of eight lines forming the configuration of Figure 19. Indeed, since we have one special tetrahedron passing through p_0 for every choice of three lines L_i, L_j, L_k out of the six $\{L_i\}$, every line X_p (and likewise every line X_h) on Σ lies on 20 such hyperplanes. Finally, since we have 16 points $p \in R$, 20 special tetrahedra containing each p as a vertex, and four vertices on each tetrahedron, we see that there are exactly 80 such hyperplane sections of Σ. In sum, then,

The surface $\Sigma \subset \mathbb{P}^5$ contains 32 lines, forming two families of 16 disjoint lines, with each line meeting exactly six members of the opposite family. There are 80 hyperplanes in $\mathbb{P}^5$ intersecting Σ in the sum of eight lines—four from each family—forming the configuration of Figure 19; and every line in Σ lies on 20 such hyperplane sections of Σ.

This configuration of 32 lines and 80 hyperplanes in $\mathbb{P}^5$ we will call the $(32_{20}80_8)$ configuration.

This last discussion sheds some additional light on the (16_6) configuration. In terms of our description of the 16 points of R above, we could a priori identify only six of the 16 hyperplanes of R^*: the planes $h_i = \overline{p_0, L_i}$, containing the points p_0 and $\{p_{ij}\}_j$. We can now describe the remaining 10: as we saw, for each triple L_i, L_j, L_k of lines, the hyperplane $h_{ijk} = \overline{p_{ij}p_{ik}p_{jk}} \in R^*$; we want now to identify the remaining three points q_1, q_2, q_3 of R lying on h_{ijk}. (See Figure 20.) To do this, we recall that the points p_{ij}, p_{jk}, p_{ik}, q_1, q_2, and q_3 all lie on a conic curve, and hence so do their images $\overline{p_{ij}}$, $\overline{p_{jk}}$, $\overline{p_{ik}}$, $\overline{q_1}$, $\overline{q_2}$, and $\overline{q_3}$. In particular, this means that no three of these points are collinear, i.e., that $\overline{q_1}$, $\overline{q_2}$, and $\overline{q_3}$ must lie off the lines L_i, L_j, and L_k. These three lines, however, account for 12 of the 15 points $\{p_{ij}\}$; consequently the points q_1, q_2, and q_3 can only be the points p_{lm}, p_{mn}, and p_{ln}. Thus, if we label the 16 double points of S by $\{p_0, p_{ij}\}$ and the 16 double points of S^* as $\{h_i, h_{ijk} = h_{lmn}\}$, the incidence relationships are

$$h_i \supset \{p_0, p_{ij}\}, \qquad j \neq i,$$
$$h_{ijk} \supset \{p_{ij}, p_{ik}, p_{jk}, p_{lm}, p_{mn}, p_{nl}\},$$
$$p_0 \in h_i, \qquad i = 1,\ldots,6,$$

and

$$p_{ij} \in h_i, h_j, h_{ijk}, \qquad k \neq i,j.$$

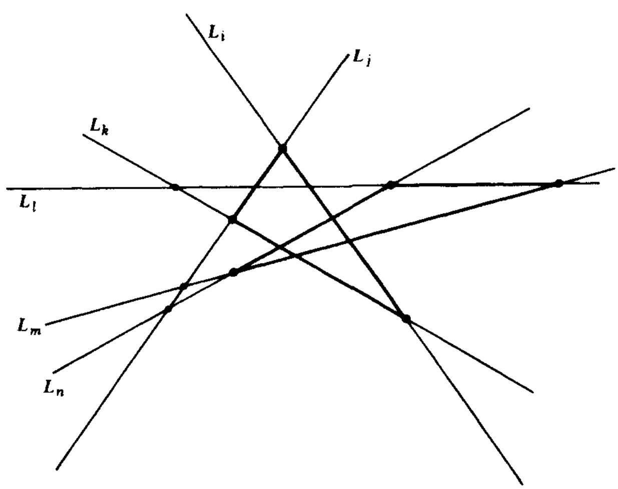

Figure 20

3. LINES ON THE QUADRIC LINE COMPLEX

The Variety of Lines on the Quadric Line Complex

We now introduce a central variety in our study: the variety $A = \{L \subset \mathbb{P}^5 : L \subset X\} \subset G(2,6)$ of lines lying on the quadric line complex X. To show that A is smooth, we first compute its cohomology class in $G(2,6)$. Recalling from Section 1 of Chapter 6 that the cycle $\tau(F) \subset G(2,6)$ of lines in $\mathbb{P}^5$ lying on a quadric hypersurface is homologous to the Schubert cycle

$$\tau(F) \sim 4 \cdot \sigma_{2,1},$$

we see that the variety $A = \tau(F) \cdot \tau(G)$ represents the cycle

$$A \sim 16(\sigma_{2,1} \cdot \sigma_{2,1});$$

in particular, the intersection number of A with the Schubert cycles

$$\sigma_{1,1}(V_4) = \{L \subset \mathbb{P}^5 : L \subset V_4\}$$

and

$$\sigma_2(V_2) = \{L \subset \mathbb{P}^5 : L \cap V_2 \neq \emptyset\}$$

in $G(2,6)$ is given, according to our reduction formulas, by

$$\begin{aligned} {}^{\#}(A \cdot \sigma_{1,1}) &= 16(\sigma_{2,1} \cdot \sigma_{2,1} \cdot \sigma_{1,1})_{G(2,6)} \\ &= 16(\sigma_1 \cdot \sigma_1)_{G(2,3)} \\ &= 16 \end{aligned}$$

and

$$\begin{aligned} {}^{\#}(A \cdot \sigma_2) &= 16 \cdot (\sigma_{2,1} \cdot \sigma_{2,1} \cdot \sigma_2)_{G(2,6)} \\ &= 16 \cdot (\sigma_1 \cdot \sigma_1 \cdot \sigma_2)_{G(2,4)} \\ &= 16. \end{aligned}$$

Thus we can write

$$A \sim 16 \cdot \sigma_{4,2} + 16 \cdot_{3,3}.$$

Now, for any point $a \in A$, we can find a hyperplane $V_4 \subset \mathbb{P}^5$ containing the corresponding line $L_a \subset X$ and intersecting X transversely (by Bertini, the generic V_4 containing L_a meets $X - L_a$ transversely, and by direct examination we see that $V_4 \cap X$ is smooth along L_a for a generic such 4-plane). But we have seen in Section 4 of Chapter 4 that any smooth intersection of two quadrics in $\mathbb{P}^4$ contains exactly 16 lines, so that the Schubert cycle $\sigma_{1,1}(V_4) \subset G(2,6)$ will meet A in 16 distinct points, including a. Since ${}^{\#}(A \cdot \sigma_{1,1}) = 16$, it follows that A has intersection multiplicity 1 with $\sigma_{1,1}(V_4)$ at every point of $\sigma_{1,1}(V_4) \cap A$, and hence that a is a smooth point of A.

For every pencil $L \subset X$ in the complex X, the focus p_L of L is by definition a point of the Kummer surface S and the plane h_L of L likewise a point of the dual Kummer surface S^*; thus we have natural maps

$$j: A \to S \quad \text{and} \quad j': A \to S^*$$

defined by

$$j: L \mapsto p_L \quad \text{and} \quad j': L \mapsto h_L.$$

For $p \in S - R$, X contains two pencils with focus p, while for $p \in R$, X contains a single such pencil p; thus j expresses A as a double cover of S branched in the 16 points of R. Similarly, a hyperplane $h \in S^* - R^*$ contains two pencils of X while a hyperplane $h \in R^*$ contains only one; thus $j': A \to S^*$ is a double cover of S^* branched at R^*. Let

$$\iota: A \to A$$

be the involution of A that exchanges the sheets of $j: A \to S$, sending each pencil $L \subset X$ to the unique other pencil of X confocal with L; let

$$\iota': A \to A$$

similarly be the involution exchanging sheets of $j': A \to S^*$, sending each $L \subset X$ to the other pencil of X coplanar with L.

We can now describe A intrinsically. First, from the expression of A as a double cover of S branched in the 16 points of R, we see that

$$\chi(A) = 2\chi(S) - 16 = 2 \cdot 8 - 16 = 0.$$

We have seen that $K_\Sigma = 0$; let ω be a holomorphic nonzero 2-form on Σ. Let $\pi^{-1}(\omega)$ denote the corresponding 2-form on $(S - R) \cong \Sigma - \cup_{p \in R} X_p$. Then $j^* \pi^{-1}(\omega)$ is a holomorphic nonzero 2-form on $A - j^{-1}R$ and by Hartogs' theorem it extends to a global nonzero holomorphic 2-form on A; so

$$K_A = 0.$$

By Riemann-Roch,

$$\chi(\mathcal{O}_A) = \frac{c_1^2 + c_2}{12} = 0,$$

so $q(A) = 2$; and from the classification theorem of Section 5, Chapter 4, we recognize that

A *is an Abelian variety.*

The involutions ι and ι' are readily identified: Let z_1, z_2 be Euclidean coordinates on $\mathbb{C}^2$, and consider the holomorphic 1-forms dz_1, dz_2 on A: the forms

$$\omega_i = dz_i + \iota^* dz_i$$

are invariant under ι^*, and so we can write

$$\omega_i = j^*\tilde{\omega}_i$$

for $\tilde{\omega}_1, \tilde{\omega}_2$ holomorphic 1-forms on $S-R$. $\{\pi^*\tilde{\omega}_i\}_{i=1,2}$ are then bounded holomorphic 1-forms on $\Sigma - \cup_{p\in R} X_p$; by Riemann's theorem they extend to all of Σ, and, since Σ is simply connected, it follows that $\omega_i \equiv 0$. Thus $\iota^* dz_i = -dz_i$, i.e., *ι is the standard involution of the Abelian variety* $A = \mathbb{C}^2/\Lambda$ *induced by the map* $(z_1, z_2) \mapsto (-z_1, -z_2)$ *on* $\mathbb{C}^2$; precisely the same argument shows that the involution ι' is likewise induced by the involution $z \mapsto -z$ on $\mathbb{C}^2$, but with a different choice of base point.

Curves on the Variety of Lines

We wish now to consider curves on the Abelian variety A. To start with, recall that the Schubert cycle σ_1 on $G(2,6)$ is given by

$$\sigma_1(V_3) = \{L \subset \mathbb{P}^5 : L \cap V_3 \neq \emptyset\},$$

and that σ_1 is the hyperplane section of $G(2,6)$ under the Plücker embedding $G(2,6) \to \mathbb{P}(\Lambda^2\mathbb{C}^6)$. For any 3-plane $V_3 \subset \mathbb{P}^5$ we set

$$\begin{aligned} D_V &= A \cap \sigma_1(V_3) \\ &= \{L \subset X : L \cap V_3 \neq \emptyset\} \subset A. \end{aligned}$$

The self-intersection of D_V on A is given by

$$\begin{aligned} (D_V \cdot D_V)_A &= (A \cdot \sigma_1 \cdot \sigma_1)_{G(2,6)} \\ &= (A \cdot (\sigma_{1,1} + \sigma_2))_{G(2,6)} \\ &= 16 + 16 = 32. \end{aligned}$$

We claim now that for a generic 3-plane V, the curve $D_V \subset A$ is smooth. Note that this does not follow immediately from Bertini's theorem: the divisors $\{D_V\}_{V \subset \mathbb{P}^5}$ are all linearly equivalent, but they do not form a linear system. Indeed, the complete linear system $|\sigma_1|$ of hyperplane sections of the Grassmannian $G(2,6) \subset \mathbb{P}(\Lambda^2\mathbb{C}^6)$ corresponds naturally to the projective space

$$\mathbb{P}(\Lambda^2\mathbb{C}^6)^* = \mathbb{P}(\Lambda^4\mathbb{C}^6);$$

and the map

$$G(4,6) \to \mathbb{P}(\Lambda^4\mathbb{C}^6)$$

given by

$$V_3 \mapsto \sigma_1(V_3) \in |\sigma_1|$$

is just the Plücker embedding of the dual Grassmannian $G(4,6)$ of 3-planes in $\mathbb{P}^5$. However, since the Schubert cycle

$$\sigma_{2,2,2,1}(V_2, V_4) = \{V_3 \subset \mathbb{P}^5 : V_2 \subset V_3 \subset V_4\}$$

in $G(4,6)$ has degree

$$^{\#}(\sigma_{2,2,2,1}\cdot\sigma_1)_{G(4,6)} = 1$$

under the dual Plücker embedding, the family

$$\{D_V\}_{V\in\sigma_{2,2,2,1}(V_2,V_4)}$$

of divisors on A is in fact a pencil. Now, as we have seen, through a generic $x\in X$ there pass four lines in X, comprising the locus $X\cap T_x(X)$. Let V_2 be a generic 2-plane, meeting X in four distinct such points, and let V_4 be a generic hyperplane containing V_2, not containing any line on X meeting V_2; consider the pencil $\{D_V\}_{V:V_2\subset V\subset V_4}$ on A. By Bertini, the generic element of this pencil is smooth away from the base locus. But the base locus of this pencil consists of the 16 lines of X passing through the four points of $V_2\cap X$, and the 16 lines lying in the hyperplane section $V_4\cap X$ of X—32 distinct lines in all. The base points of our pencil are therefore all simple points and hence smooth points of every curve D_V in our pencil. Thus the generic divisor D_V is smooth. Note that the genus of a smooth D_V is then given by

$$\pi(D_V) = \frac{D_V\cdot D_V}{2} + 1 = 17.$$

A second family of curves on A, more fundamental than the curves D_V, are the *incidence divisors* $B_L\subset A$, defined to be the set of lines on X meeting a given line L. More precisely—since it is not a priori clear when L itself is to be counted among the lines meeting L—we will define B_L to be the closure in A of the set of lines $L'\in A-\{L\}$ meeting L; the Lévi theorem assures us that B_L is analytic, and we will see later under what circumstances $L_0\in B_{L_0}$. The curves $\{B_L\}_{L\in A}$ form a continuous, connected family, and so all represent the same homology class on A. Since we can find 3-planes $V_3\subset\mathbb{P}^5$ intersecting X in the sum of four lines L_1, L_2, L_3, L_4—for example, $T_x(X)$—we see from this that

$$\begin{aligned} D_V &= B_{L_1} + B_{L_2} + B_{L_3} + B_{L_4} \\ &\sim 4B_L. \end{aligned}$$

We have then

$$B_L\cdot B_L = \tfrac{1}{16}D_V\cdot D_V = 2$$

and hence the virtual genus

$$\pi(B_L) = \frac{B_L\cdot B_L}{2} + 1 = 2.$$

Note also that since D_V is positive, so is B_L.

We claim now that for any $L\subset X$, the curve $B_L\subset A$ is smooth. To see this, we observe that if two lines L and L' in X meet—i.e., if the corresponding pencils have a line l in common—then the focus $p_{L'}$ of the second pencil must lie on the line l, and hence on the plane h_L of the first pencil. The map

$$j: A \to S$$

sending each line $L \subset X$ to its focus p_L *thus maps the curve* B_L

$$j: B_L \to h_L \cap S$$

onto the hyperplane section $h_L \cap S$ *of* S; $j|_{B_L}$ is clearly generically one-to-one. By the duality of S and S^*, h_L is tangent to S, so that for generic L the curve $C_L = h_L \cap S$ is a plane quartic with one ordinary double point. By the genus formula, then,

$$g(B_L) = g(C_L) = 2,$$

so B_L is smooth. (Note that since $\pi(B_L)=2$ implies a priori that $g(C_L)= g(B_L)$ is less than or equal to 2, this affords another proof that h_L is tangent to S, i.e., that S and S^* are dual.)

Now, since B_L is a positive divisor on A by the Lefschetz theorem the inclusion map on integral homology

$$i_*: H_1(B_L,\mathbb{Z}) \twoheadrightarrow H_1(A,\mathbb{Z})$$

is surjective. But since B_L has genus 2,

$$H_1(B_L,\mathbb{Z}) \cong H_1(A,\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z};$$

so the kernel of i_* must have rank zero; since $H_1(B_L,\mathbb{Z})$ has no torsion, this implies that *the map* i_* *is an isomorphism*. Likewise, by Lefschetz the restriction map

$$H^{1,0}(A) \to H^{1,0}(B_L)$$

is an isomorphism, and so we have

$$A \cong \frac{(H^{1,0}(A))^*}{H_1(A,\mathbb{Z})} = \frac{(H^{1,0}(B_L))^*}{H_1(B_L,\mathbb{Z})} = \mathcal{J}(B_L),$$

i.e.,

The Abelian variety A *is the Jacobian of the curve* B_L.

Note that since the analytic representative B_L of the cohomology class $[B_L] \in H^2(A,\mathbb{Z})$ is unique up to translations, *all the curves* $B_L \subset A$ *are translates of one another*. Hence all the curves B_L are smooth, and $A = \mathcal{J}(B_L)$ for any $L \in A$.

To relate the various curves B_L on A, let L_0 be one of the 16 lines in $j'^{-1}(R^*)$ and take L_0 to be the origin in A. Since clearly

$$\iota'(L) \in B_L \qquad \text{for } L \neq \iota'(L),$$

by continuity,

$$L_0 \in B_{L_0};$$

so we can also take L_0 as base point on the curve B_{L_0}. Now, we saw in Section 6 of Chapter 2 that the theta-divisor of a principally polarized Abelian variety cannot be carried into itself by a translation other than the identity. Thus we may define a map

$$\kappa: A \to A$$

by setting, for each L,

$$B_L = B_{L_0} + \kappa(L);$$

our first problem is to identify κ. This is not hard: since $\kappa(0)=0$, by the result of Section 6 of Chapter 2, κ is a group homomorphism. We have

$$\iota'(L) \in B_L = B_{L_0} + \kappa(L)$$

for each L, and hence

$$\kappa(L) + L \in -B_{L_0} = B_{L_0}$$

for each L. But the map $L \mapsto \kappa(L) + L$ is again a group homomorphism, and since $B_{L_0} + B_{L_0} = A$, this implies that $\kappa(L) + L$ is constant, i.e.,

$$\kappa(L) = -L$$

or in other words

$$B_L = B_{L_0} - L \qquad \text{for all } L.$$

We can now identify the line bundles j^*H and j'^*H associated to the maps j and j'. To begin with, we note that for any hyperplane $h \subset \mathbb{P}^3$, the inverse image j^*h in A is just the set of pencils $L \in A$ with focus lying on the hyperplane section $h \cap S$ of S. In particular, if we take $h \in S^*$—so that h contains two pencils L and $\iota'(L)$ from X—then j^*h will consist simply of the set of pencils having a line in common with either L or $\iota'(L)$—i.e.,

$$j^*h = B_L \cup B_{\iota'(L)}.$$

(To avoid confusion, we will here use the union symbol $\cup$ to denote addition of divisors.) Similarly, for any point $p \in S$, the pullback $j'^*(p^*)$ of the dual hyperplane $p^* \subset \mathbb{P}^{3^*}$ of hyperplanes containing p will consist of pencils whose plane contains p—that is, of pencils having a line in common with either of the pencils L' and $\iota(L')$ with focus p. Thus

$$j'^*P^* = B_L \cup B_{\iota(L)}.$$

Now in general, for any two lines L and L' and any element $\lambda \in A$, the divisors

$$B_L \cup B_{L'} \quad \text{and} \quad (B_L + \lambda) \cup (B_{L'} - \lambda)$$

are linearly equivalent: the map

$$A \to \hat{A} = \mathrm{Pic}^0(A)$$

defined by

$$\lambda \mapsto [(B_L + \lambda) \cup (B_{L'} - \lambda)] - [B_L \cup B_{L'}]$$

sends the points λ and $\lambda' = L - L' - \lambda$ into the same point—but being a group homomorphism, this implies it is constant. Thus we can write

$$\begin{aligned} j^*h &= B_L \cup B_{\iota(L)} \\ &= (B_{L_0} - L) \cup (B_{L_0} + L) \\ &= 2B_{L_0} \end{aligned}$$

and

$$\begin{aligned} j'^*H &= B_L \cup B_{\iota'(L)} \\ &= (B_{L_0} - L) \cup (B_{L_0} + L + \mu) \\ &= 2B_{L_0} + \tfrac{1}{2}\mu, \end{aligned}$$

for some $\mu \in A$, i.e., *the line bundles* j^*H' *and* j'^*H *differ by translation.* Since by the theorem of p. 317,

$$h^0(2B_{L_0}) = h^0(2B_{L_0} + \tfrac{1}{2}\mu) = 4,$$

we see that both j and j' are given by complete linear systems, it follows that

The Kummer surfaces S *and* S* *are projectively isomorphic.*

Combined with the fact that $S^* \subset \mathbb{P}^{3*}$ is the dual variety of $S \subset \mathbb{P}^3$, this proves that

The Kummer surface S *is self-dual.*

Two Configurations Revisited

We may, by considering the Kummer surface $S \subset \mathbb{P}^3$ and its desingularization $\Sigma \subset \mathbb{P}^5$ as the images of the Abelian variety A, get a new slant on the configurations associated to these varieties. To see this, think of A as the Jacobian of the curve $B = B_L$, and realize B as the locus of

$$y^2 = \prod_{i=0}^{5} (x - \lambda_i),$$

with $p_i = (\lambda_i, 0)$ the Weierstrass points of B. Then, since the hyperelliptic series on B contains the divisors $2p_i$, the points

$$\mu_i = (p_i - p_0) \in \mathrm{Pic}^0(B) = A, \qquad i = 0, \ldots, 5,$$

are points of order 2 on A, as are the points

$$\mu_{ij} = (p_i + p_j - 2p_0) \in A, \qquad 1 \leqslant i < j \leqslant 5.$$

Inasmuch as the hyperelliptic series on B is unique, no pair $p_i + p_j$ is linearly equivalent to another pair $p_k + p_l$, so the points μ_i, μ_{ij} are all distinct; these, then, are the 16 half-lattice points of A. The group law on the points μ_i, μ_{ij} is easily written down: clearly

$$\mu_i + \mu_j = \mu_{ij},$$

and since the meromorphic function

$$f(x,y) = \frac{y}{(x-\lambda_0)^3}$$

on B has divisor

$$(f) = \sum_{i=0}^{5} p_i - 6p_0,$$

we see that

$$\begin{aligned}\mu_i + \mu_{jk} &\sim (p_i + p_j + p_k - 3p_0) \\ &\sim (-p_l - p_m + 2p_0) \sim -\mu_{lm} = \mu_{lm}\end{aligned}$$

for i,j,k,l,m distinct; and

$$\begin{aligned}\mu_{ij} + \mu_{kl} &\sim (p_i + p_j + p_k + p_l - 4p_0) \\ &\sim (-p_m + p_0) \sim -\mu_m = \mu_m.\end{aligned}$$

Note that the standard theta-divisor

$$\Theta = \{(p - p_0) \colon p \in B\} \subset A$$

of course contains the six half-lattice points $\{\mu_i\}$; likewise its translate

$$\Theta_i = \Theta + \mu_i = \{(p + p_i - 2p_0)\}$$

contains the six points $\mu_0 = 0$, μ_i and $\{\mu_{ij}\}_{j \neq 0,i}$ and

$$\Theta_{ij} = \Theta + \mu_{ij} = \{(p + p_i + p_j - 3p_0)\}$$

contains the six points μ_i, μ_j, μ_{ij}, and $\{\mu_{lm}\}_{l,m \neq i,j}$. Conversely, each of the half-lattice points $\mu_i \mu_{ij}$ lies on exactly six of the divisors Θ_i, Θ_{ij}:

$$\mu_i \in \Theta,\ \Theta_i,\ \text{and}\ \Theta_{ij}\ \text{for}\ j \neq 0, i$$

and

$$\mu_{ij} \in \Theta_i,\ \Theta_j,\ \Theta_{ij},\ \text{and}\ \Theta_{kl}\ \text{for}\ k,l \neq i,j.$$

Now, we have seen that the map $j: A \to S$ from A to the Kummer surface $S \subset \mathbb{P}^3$ is given by some translate $|2\Theta + \lambda|$ of the linear system $|2\Theta|$ on A;

since $|2\Theta+\lambda|$ is invariant under the involution $\mu\mapsto-\mu$ fixing the 16 points μ_i,μ_{ij}, we must have $\lambda=0$. In particular, then, the divisors $2\Theta_i,2\Theta_{ij}$ are all elements of the linear series $|2\Theta|$; and being invariant under the involution $\mu\mapsto-\mu$ they are mapped 2-1 onto hyperplane sections of S, consisting of double conic curves. Each divisor Θ_i,Θ_{ij} contains exactly six of the half-lattice points of A; consequently each of the corresponding hyperplane sections of S will pass through exactly six of the double points of S, and every double point of S is contained in exactly six of these hyperplanes, giving us the (16_6) configuration.

Now consider the map ρ from A to the K-3 surface $\Sigma\subset\mathbb{P}^5$. ρ is given, as the reader may check, by the linear series of curves in the system $|4\Theta|$ passing through the 16 half-lattice points of A, or more properly by the linear system

$$|4\pi^*\Theta-\sum E_i|$$

on the blow-up $\tilde{A}$ of A at the half-lattice points of A. The map is 2-sheeted, branched exactly at the 16 exceptional divisors E_i of the blow-up.

We first locate the 32 lines of Σ. Sixteen are obvious: there are the images X_p of the 16 exceptional divisors E_i, each of which has intersection number 1 with the system $|\pi^*4\Theta-\Sigma E_i|$ and maps 1-1 onto a line in $\mathbb{P}^5$. The other 16 are the images in $\mathbb{P}^5$ of the proper transforms of the theta-divisors Θ_i,Θ_{ij} on A. Each of these has intersection number 8 with $\pi^*4\Theta$, and, meeting six of the exceptional divisors E_i, intersection number 2 with $\pi^*4\Theta-\Sigma E_i$; being invariant under the involution fixing the half-lattice points, it maps 2-1 onto a line in $\mathbb{P}^5$.

Now, by the same argument as before, for any λ_1, λ_2, λ_3, and $\lambda_4\in A$ the divisor

$$(\Theta+\lambda_1)\cup(\Theta+\lambda_2)\cup(\Theta+\lambda_3)\cup(\Theta+\lambda_4)$$

will be in the linear system $|4\Theta|$ if and only if $\Sigma\lambda_i=0$. In particular, we see that the system $|4\Theta|$ contains the 80 divisors

$$\begin{aligned}
\alpha_{ij} &= \Theta\cup\Theta_i\cup\Theta_j\cup\Theta_{ij} && (1\leqslant i<j\leqslant 5),\\
\beta_{ijk} = \alpha_{jk}+\mu_i &= \Theta_i\cup\Theta_{ij}\cup\Theta_{ik}\cup\Theta_{lm} && (1\leqslant i\leqslant 5; 1\leqslant j<k\leqslant 5),\\
\gamma_{ij} &= \Theta_{ij}\cup\Theta_k\cup\Theta_l\cup\Theta_m && (1\leqslant i<j\leqslant 5),\\
\delta_{ij} = \gamma_{ij}+\mu_j &= \Theta_i\cup\Theta_{jk}\cup\Theta_{jl}\cup\Theta_m && (1\leqslant i\leqslant 5, 1\leqslant j\leqslant 5),\\
\varepsilon_{ijk} = \gamma_{lm}+\mu_k &= \Theta\cup\Theta_{ij}\cup\Theta_{ik}\cup\Theta_{jk} && (1\leqslant i<j<k\leqslant 5).
\end{aligned}$$

Each of these divisors contains all 16 half-lattice points, and each contains exactly four of them with multiplicity 3: three of the four components of α_{ij}, for example, pass through each of μ_0, μ_i, μ_j, and μ_{ij}, while three components of γ_{ij} pass through each of μ_{kl}, μ_{lm}, μ_{km}, and μ_0; the

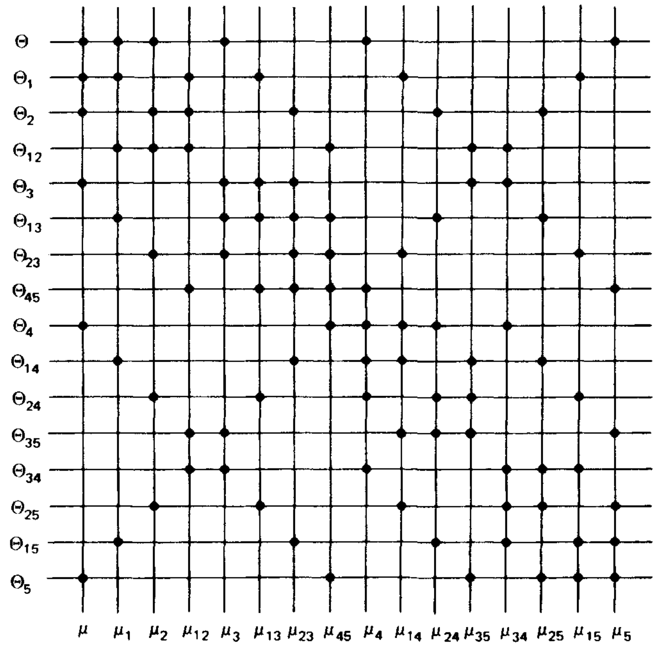

Figure 21

remaining divisors β_{ijk}, δ_{ij}, and ε_{ijk} are all translates of these two types. The corresponding elements of the linear series $|\pi^*4\Theta-\sum E_i|$ on $\tilde{A}$ thus all consist of four of the curves $\tilde{\Theta}_i, \tilde{\Theta}_{ij}$ and four of the exceptional divisors E_i taken with multiplicity 2; and the corresponding hyperplane sections of $\Sigma\subset\mathbb{P}^5$ consist of eight lines forming the configuration of Figure 19. These, then, are the 32 lines and 80 hyperplanes of the $(32_{20}80_8)$ configuration on Σ.

The incidence relations among the 16 theta-divisors $\Theta_0, \Theta_i, \Theta_{ij}$ and the 16 points μ_0, μ_i, μ_{ij} (that is, among the points and planes of the 16_6 configuration, or among the 32 lines on Σ) are diagramed in Figure 21.

The Group Law

We will now give an abstract representation of the curves B_L, which will allow us both to identify B_L (and hence $A=\mathcal{J}(B_L)$) and to describe geometrically the group law on the variety A of lines of X.

First, we consider not just the two quadrics F and G in $\mathbb{P}^5$, but the entire pencil $\{F_\lambda\}$ spanned by F and G, that is, the pencil of all quadrics in $\mathbb{P}^5$ containing X.

We define a map

$$\pi:\ B_L \to \mathbb{P}^1$$

as follows: for any line $L' \subset X$ meeting L, let $\Lambda = \overline{L, L'}$ be the 2-plane spanned by L and L'. There is then a unique quadric $F_{\lambda(L')}$ in the pencil $\{F_\lambda\}$ containing the 2-plane Λ. To see this, let $q \in \Lambda$ be any point of Λ lying off L and L'. (See Figure 22.) q is then contained in some quadric $F_{\lambda(L')}$—but $F_{\lambda(L')}$, containing L, L', and q, has three points of intersection with any line L'' in Λ through q, and so contains L''; thus $F_{\lambda(L')}$ contains Λ. $F_{\lambda(L')}$ is clearly unique; if Λ lay on two quadrics of the pencil F_λ, it would be contained in X; but X, as we saw, contains no 2-planes. We may thus define the map π by sending any line $L' \in B_L$ to $\lambda(L')$.

Now let F_λ be any quadric in our pencil, and consider the inverse image $\pi^{-1}(\lambda)$. If Λ is any 2-plane in F_λ containing L, then the intersection of Λ with X—that is, the intersection of Λ with any second element F_μ of the pencil—will consist of L plus a second line L'; the inverse image $\pi^{-1}(\lambda)$ thus corresponds to the 2-planes in F_λ containing L. There are two possibilities: first, if F_λ is smooth, then, as we have seen, the 2-planes on F_λ fall into two connected three-dimensional components. Now if $p \in L \subset F_\lambda$ is any point of L, the intersection $T_p(F_\lambda) \cap F_\lambda$ will be a cone over the smooth quadric surface $\tilde{F}_\lambda$ cut out on F_λ by any 3-plane in $T_p(F_\lambda)$ missing p, and the 2-planes on F_λ through the point p will be spanned by the lines on $\tilde{F}_\lambda$ together with p. Since there are two lines on $\tilde{F}_\lambda$ containing the point $L \cap \tilde{F}_\lambda$, there will be two 2-planes on F_λ containing L, one from each family. Suppose, on the other hand, that F_λ is singular. Inasmuch as $X = F_\lambda \cap F_\mu$ is smooth, the singular locus of F_λ must lie outside F_μ; in particular, it follows that the singular locus of F_λ is only a point q, and that F_λ is the cone through q over a smooth quadric $\tilde{F}_\lambda$ in a $\mathbb{P}^4 \subset \mathbb{P}^5$. In this

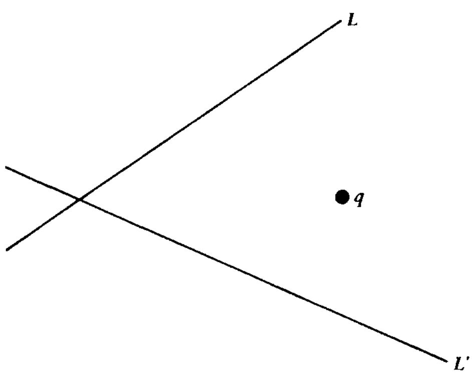

Figure 22

case, the 2-planes on F_λ will form a single irreducible three-dimensional family: namely, the 2-planes spanned by q together with the lines on $\tilde{F}_\lambda$; and the 2-plane $\overline{q,L}$ will clearly be the only 2-plane in F_λ containing L.

We see then that *the map* $\pi: \mathrm{B_L} \to \mathbb{P}^1$ *expresses* $\mathrm{B_L}$ *as a 2-sheeted cover of* $\mathbb{P}^1$, *branched at the points of* $\mathbb{P}^1$ *corresponding to the singular quadrics in the pencil* $\{\mathrm{F}_\lambda\}$; indeed, all the curves B_L may be naturally identified with the abstract curve B of irreducible families of 2-planes in the quadrics of the pencil $\{F_\lambda\}$.

To be explicit, suppose that our original pair of quadrics G and F are given as the locus of two symmetric quadric forms Q and Q'. We can, of course, take Q to be given by the identity matrix, and by standard linear algebra we may at the same time diagonalize Q'; i.e., we may take

$$G = \left(\sum X_i^2 = 0\right) \quad \text{and} \quad F = \left(\sum \lambda_i X_i^2 = 0\right).$$

The singular elements of the pencil

$$F_\lambda = \left(\sum_{i=0}^{5} (\lambda - \lambda_i) X_i^2 = 0\right)$$

are then the six quadrics $F_{\lambda_0}, \ldots, F_{\lambda_5}$. The map π is thus branched at the six points $\lambda_0, \ldots, \lambda_5$; and consequently

The variety A *of lines on the quadric line complex* X *given as the intersection of the two quadrics*

$$G = \left(\sum X_i^2 = 0\right) \quad \textit{and} \quad F = \left(\sum \lambda_i X_i^2 = 0\right)$$

is the Jacobian of the curve expressible as a double cover of $\mathbb{P}^1$ *branched at the six points* $\lambda_0, \ldots, \lambda_5$.

As promised, we can now describe geometrically the group law on A. There are basically two ingredients in this construction. The first is to note that *the sum on* A *of four lines comprising the intersection of* X *with a 3-plane* V *is constant*. This is because if $V \cdot X = L_1 + L_2 + L_3 + L_4$, then by the argument of p. 784 we may write

$$\begin{aligned} D_V &= B_{L_1} \cup B_{L_2} \cup B_{L_3} \cup B_{L_4} \\ &= (B_{L_0} - L_1) \cup (B_{L_0} - L_2) \cup (B_{L_0} - L_3) \cup (B_{L_0} - L_4) \\ &= 3B_{L_0} \cup (B_{L_0} - L_1 - L_2 - L_3 - L_4). \end{aligned}$$

Since all divisors D_V are linearly equivalent, and since no translation of A fixes B_{L_0}, it follows that the sum $L_1 + L_2 + L_3 + L_4$ does not depend on V. Choose as the origin in A a line L_0 with $4L_0 \sim D_V$, so that the sum of any four lines on S lying in a 3-plane is zero on A.

The second point is to identify the isomorphism

$$t_{L_0-L}: B_{L_0} \to B_L$$

of B_{L_0} and B_L given by translation on A. To do this, consider the set of all isomorphisms

$$\varphi_L: B_{L_0} \to B_L;$$

since B_L can have only finitely many automorphisms, the set $\{\varphi_L\}_L$ forms an unbranched covering of A, of which the isomorphisms $\{t_{L_0-L}\}$ form one sheet. But we can also define for each L an isomorphism $\varphi_2: B_{L_0} \to B_L$ via the natural identification of both B_{L_0} and B_L with the abstract curve B introduced above; since $\varphi_{L_0} = t_0 = \text{id.}$, it follows that $\varphi_2 = t_{L_0-L}$ for all L.

Now suppose we are given two lines L_1 and L_2 in X, and we want to find their sum $L_1 + L_2$ in A.

The first step is to express L_1 as the sum of two points on the curve B_{L_0}. This is easy: L_1 and L_0 together span a 3-plane V in $\mathbb{P}^5$, which will intersect X in L_0 and L_1 plus two additional lines M_1 and M_2 meeting L_0 and L_1; we have

$$L_1 = -M_1 - M_2$$

in A. (See Figure 23.) The second step is to translate the points $M_1, M_2 \in A$ by L_2; this is done by identifying the curves B_{L_0} and B_{L_2} via the abstract curve B, as follows: each of the lines M_1 and M_2 determines, together with L_0, a unique quadric F_{λ_i} in the pencil spanned by F and G, and an irreducible family of 2-planes in F_{λ_i}. In that family of 2-planes, moreover,

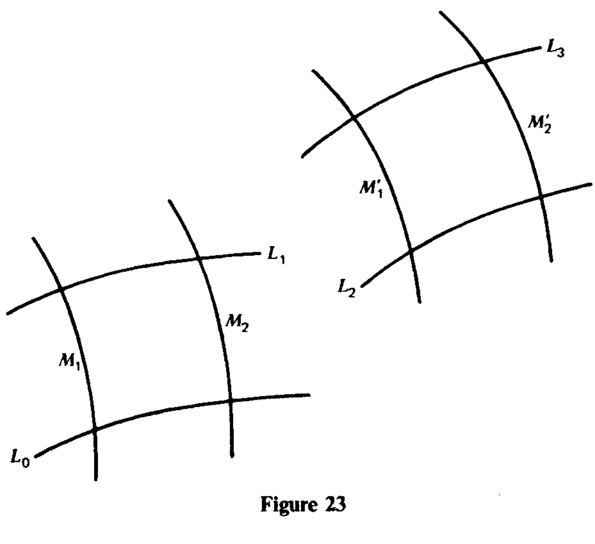

Figure 23

there will be a unique element Λ_i containing L_2; if we let M_i' be the remaining line of intersection of A_i with X, then as we have seen,

$$M_i' = M_i - L_2.$$

Finally, the lines L_2, M_1', and M_2' span a 3-plane $V' \subset \mathbb{P}^5$ that will intersect S in L_2, M_1', M_2', and a fourth line L_3. We have then

$$\begin{aligned} L_3 &= -M_1' - M_2' - L_2 \\ &= -M_1 - M_2 + L_2 \\ &= L_1 + L_2 \end{aligned}$$

in A; this is the group law.

4. THE QUADRIC LINE COMPLEX: REPRISE

The Quadric Line Complex and the Associated Kummer Surface II

We return now to the geometry of the complex X of lines in $\mathbb{P}^3$. Our starting point this time around is the question: which lines l_x of our complex are tangent lines to the Kummer surface S? To answer this, we go back to our initial computation of the degree of the Kummer surface:

Let $l_x, x \in X$, be any line of the complex. Then for any point $p \in l_x \cap S$ of intersection of l_x with S, l_x will be an element of one or both of the pencils of lines in the complex through the point p—that is, the point $x \in X$ will lie on one or both of the lines of $X_p = \sigma(p) \cap F$; and conversely, for any line $L \subset X$ containing x, the focus p_L of the corresponding pencil of lines in $\mathbb{P}^3$ must by definition lie in $l_x \cap S$. Now the locus

$$T_x(X) \cap X = T_x(G) \cap G \cap T_x(F) \cap F$$

of lines on X passing through the point x has degree 4. We concluded, then, if l_x did not lie on any pair of confocal pencils, and if $T_x(X) \cap X$ contained no multiple components, that l_x met S in four distinct points. We have since seen that the degree of S is indeed four, and so we can now invert our argument to obtain the characterization:

For any $x \in X$, *the line* l_x *will be tangent to* S *if and only if either*

1. l_x *is a singular line, i.e., it is held in common by two confocal pencils; or*
2. *the intersection* $T_x(X) \cap X$ *contains a multiple component.*

We have already seen that the locus $\Sigma \subset X$ of singular lines is the smooth intersection of X with a quadric hypersurface in $\mathbb{P}^5$; we turn our attention now to the second possibility. Now we notice something unexpected: the intersection of $T_x(X)$ with X will fail to be transverse—that is, fail to consist of four distinct lines—everywhere along a line $L \subset X$ if the intersection

$$T_x(X) \cap T_y(X) = T_x(F) \cap T_y(F) \cap T_x(G) \cap T_y(G)$$

is two-dimensional for all $y \in L$. But the family of hyperplanes

$$\{T_x(F)\}_{x \in L}$$

forms a pencil, as does the family $\{T_x(G)\}_{x \in L}$; thus for any $x \neq x' \in L$,

$$T_x(G) \cap T_{x'}(G) = \bigcap_{y \in L} T_y(G)$$

and

$$T_x(F) \cap T_{x'}(F) = \bigcap_{y \in L} T_y(F).$$

This says that the intersection

$$T_x(X) \cap T_y(X) = \bigcap_{x \in L} T_x(X)$$

will be two-dimensional for *any* pair of distinct points $x,y \in L$ if and only if it is two-dimensional for *all* pairs $x,y \in L$; in other words, the line L will be a multiple component of the intersection $T_x(X) \cap X$ for some $x \in L$ if and only if it is for all $x \in L$, if and only if the locus

$$\bigcap_{x \in L} T_x(X)$$

contains a 2-plane. In this case, all the lines $\{l_x\}_{x \in L}$ of the corresponding pencil will be tangent to S; thus we see that

> *A line* l_x *of the complex* X, *other than a singular line, is tangent to* S *if and only if it lies on a pencil of lines of* X *all tangent to* S.

Note that if $L \subset X$ is any pencil of lines, all tangent to S, then by Bertini they must all be tangent at the focus p_L of the pencil, i.e., the plane h_L of the pencil must be the tangent plane to S at p_L. We may thus make the following definition:

DEFINITION. A line $L \subset X$ is called *special* if, equivalently,

1. $\dim(\cap_{x \in L} T_x(X)) = 2$; or
2. the locus $T_x(X) \cap X$ of lines in X through a generic point $x \in L$ consists of fewer than four lines;

3. $h_L = T_{p_L}(S)$, i.e., all the lines $\{l_x\}_{x\in L}$ are tangent to S at p_L.

Let $\tilde{D} \subset A$ be the set of special lines of X, and

$$\Delta = \bigcup_{L\in\tilde{D}} L \subset X$$

the locus of all special lines. We can also write

$$\Delta = \{x\in X:\ T_x(X)\cap X \text{ contains fewer than four lines}\}$$

and

$$\{x\in X:\ l_x \text{ is tangent to } S\} = \Sigma\cup\Delta.$$

To find the degree of Δ, we make a second computation for the genus of the curve

$$D_V = \{L\subset X:\ L\cap V_3\neq\varnothing\}\subset A.$$

Note that the generic $V_3\subset\mathbb{P}^5$ meets X in a curve $E\subset V_3$ that is the smooth intersection of two quadrics on $V_3\cong\mathbb{P}^3$. By the adjunction formula, E is an elliptic curve. Since E contains no lines, every line $L\subset X$ meeting E meets E in only one point, and so we may define a map

$$\tau:\ D = \{L\subset X:\ L\cap V_3\neq\varnothing\}\to E = V_3\cap X$$

expressing D as a fourfold branched cover of E. Now

$$K_E = 0$$

and, as we have seen, the generic curve D_V is smooth, so

$$\deg K_D = 32;$$

thus the map τ must have 32 branch points. But the branch locus of τ in E is just the set of points $x\in E$ having fewer than four lines through them; thus

$$\deg\Delta = {}^{\#}(\Delta\cdot V_3)_{\mathbb{P}^5} = 32.$$

Similarly, this argument yields

$${}^{\#}(D_V\cdot\tilde{D})_A = 32,$$

hence

$${}^{\#}(B_L\cdot\tilde{D})_A = \tfrac{1}{4}{}^{\#}(D_V\cdot\tilde{D})_A = 8$$

and likewise

$${}^{\#}(\Delta\cdot L)_X = 8.$$

We are now in a position to sketch a picture of the generic pencil L of the complex X in relation to the Kummer surface S. Assume that neither L

nor its coplanar pencil $L' = \iota'(L)$ is special, and let

$h_L = h_{L'}$ be the plane of the pencil $\{l_x\}_{x \in L}$,
$p_L, p_{L'}$ be the foci of the pencils L and L',
$C_L = h_L \cap S$,

$$j|_{B_L}: B_L \to C_L$$

the map sending a line $M \subset X$ meeting L to the focus p_M of the corresponding pencil, and

$$\gamma: B_L \to L$$

the extension of the map from $B_L - \{L\}$ to L sending each line $M \neq L \in B_L$ to its point of intersection with L. Note that γ can be realized as the composition of $j|_{B_L}$ with the projection map of C_L from the point p_L.

To start, we note that the pencil $\{l_x\}_{x \in L}$ contains 10 tangent lines to S: two singular lines corresponding to points of intersection of L with $\Sigma = X \cap H$, and eight nonsingular tangents, corresponding to the eight points of intersection of L with Δ. (In fact only eight of these lines—namely the eight nonsingular tangent lines—correspond to honest branch points of the map $\gamma: B_L \to L$.) We can locate the two singular lines in the pencil L readily enough: first, the common line of the pencil L and its confocal pencil $\iota(L)$, i.e., the tangent line to C_L at p_L. Second, we have seen that a line l_x of the complex lies in two confocal pencils if and only if it lies on two other coplanar pencils, so the line $\overline{p_L p_{L'}}$ held in common by L and its coplanar pencil L' must be the second singular line of the pencil L. (See Figure 24.) In particular, we see that the line $\overline{p_L p_{L'}}$ meets S in just one point q other than p_L and $p_{L'}$.

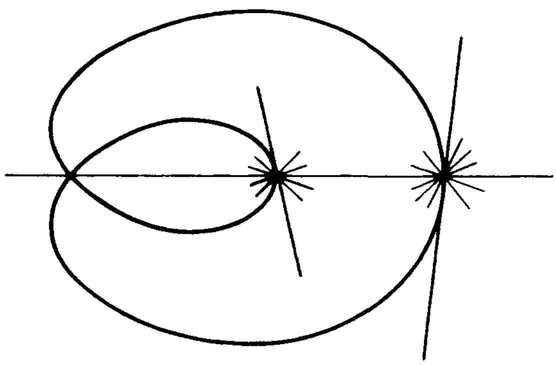

Figure 24

We can now identify the singular point $q' \in C_L$. Clearly, the lines $\overline{p_L, q'}$ and $\overline{p_{L'}, q'}$ are both tangent lines to S; we claim that in fact

$$\overline{p_L, q'} = \overline{p_{L'}}, \overline{q'},$$

i.e.,

$$q = q'$$

is the singular point of C_L. This is clear: if $\overline{p_L, q'}$ and $\overline{p_{L'}, q'}$ were distinct and nonsingular, then they would necessarily lie in one special pencil, which could only be the pencil of lines through q' in h_L—but L and L' are the only pencils of X in h_L. On the other hand, through a point $q' \notin R$ in S there is only one singular line of X, and so we must have $\overline{p_L, q'} = \overline{p_{L'}, q}$. Thus in general, if $h \in S^*$ is any hyperplane and L, L' the two pencils of X in h, h *is tangent to* S *at the focus of the two confocal pencils of* X *containing the singular line* $\overline{\mathrm{p_L}, \mathrm{p_{L'}}}$.

Dually, of course, for $p \in S$ any point, L and $L' = \iota(L)$ the two pencils of X confocal at p,

> *The tangent plane to* S *at* p *is swept out by the two coplanar pencils of* X *containing the singular line* $\mathrm{h_L} \cap \mathrm{h_{L'}}$.

Note in particular that the map

$$j: B_L \to C_L$$

is one-to-one at p_L, and that

$$j(\iota(L)) = p_L.$$

It follows from this that $B_L - \{L\}$ is closed, and hence that

$$L \notin B_L$$

for L a nonspecial line. This, finally, gives us the means to describe the divisor $\tilde{D} \subset A$ of special lines on X. Let

$$\tilde{D}' = \{L \in A: L \in B_L\}.$$

Then we have

$$\begin{aligned} \tilde{D}' &= \{L: L \in B_{L_0} - L\} \\ &= \{L: 2L \in B_{L_0}\} \\ &= m_2^* B_{L_0}, \end{aligned}$$

where $m_2: A \to A$ is the map multiplication by two. In particular,

$$^{\#}(\tilde{D}' \cdot B_{L_0}) = 4 \cdot (B_{L_0} \cdot B_{L_0}) = 8.$$

Now, since no nonspecial line L is an element of $\tilde{D}'$,

$$\tilde{D}' \subset \tilde{D},$$

i.e., $\tilde{D}-\tilde{D}'$ is an effective divisor on A. But we have seen that

$$(\tilde{D}\cdot B_L) = 8,$$

and so

$$((\tilde{D}-\tilde{D}')\cdot B_L) = 0;$$

since B_L is positive and $\tilde{D}-\tilde{D}'$ effective, this implies that

$$\tilde{D}-\tilde{D}' = 0.$$

In sum, then,

> *A line* $L \subset X$ *is special if and only if* $L \in B_L$; *the divisor* $\tilde{D} \subset A$ *of special lines is the pullback* $m_2^* B_{L_0}$ *of* B_{L_0} *under multiplication by two.*

Rationality of the Quadric Line Complex

We now shift our focus to consideration of the quadric line complex $X = F \cap G$ as an abstract variety. In particular, we want to consider, for any line $L \cup X$, the rational map

$$f_L : X - L \to \mathbb{P}^3$$

obtained by projection from L onto a complementary 3-plane $V_3 \subset \mathbb{P}^5$. We claim first that f_L is a birational isomorphism of X with $\mathbb{P}^3$. To see this, simply note that if any 2-plane $V_2 \subset \mathbb{P}^5$ containing L contains two points $p \neq q$ of X not on L, then the line $\overline{pq} \subset V_2$ must meet X in at least three points—p, q, and the point of intersection $\overline{pq} \cup L$—and so must lie in X. (See Figure 25.) Thus *the map* f_L *is one-to-one away from the locus of lines in* X *meeting* L; this is sufficient to establish that f_L is birational.

A closer examination of f_L, in fact, tells us a good deal more about X. To begin with, note that if

$$\pi : \tilde{X}_L \to X$$

is the blow-up of X along L and $F = \pi^{-1}(L) \subset \tilde{X}_L$ the exceptional divisor of the blow-up, then f_L may be extended to a holomorphic map

$$\tilde{f}_L : \tilde{X}_L \to \mathbb{P}^3$$

by sending a point $(p, \eta) \in F$, corresponding to the normal vector η to L at p, to the point of intersection of V_3 with the 2-plane spanned by L and any line through p representing the vector η—since η is defined as a tangent vector to X at p modulo tangent vectors to L at p, this is well-defined.

Now let $E_L \subset \mathbb{P}^3$ be the image under f_L of the locus $\cup_{L' \in B_L} L'$ of lines in X meeting L, and let $Q \subset \mathbb{P}^3$ be the image $\tilde{f}_L(F)$ of the exceptional divisor

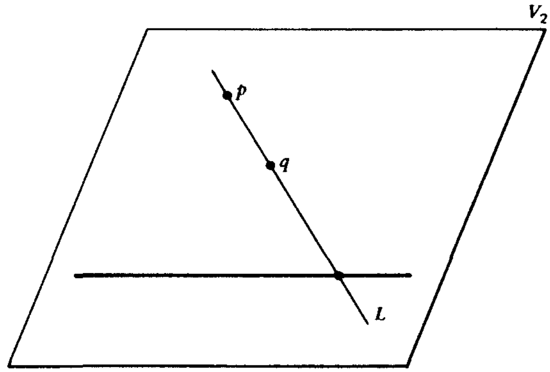

Figure 25

of $\tilde{X}_L$. E_L is a curve naturally isomorphic to B_L, and we can compute its degree as follows: for any hyperplane $V_2 \subset V_3$, the points of intersection $V_2 \cap E_L$ will correspond to the lines in X meeting L and lying in the hyperplane $\overline{L, V_2} \subset \mathbb{P}^5$. But for generic V_2, the hyperplane section $\overline{L, V_2} \cap X$ is the smooth intersection of two quadrics in 4-space, and we have seen that each of the 16 lines on such a surface meets exactly five other lines on the surface. Thus E_L is a quintic space curve. The image Q of F, on the other hand, is readily seen to be a quadric surface—in fact, it is just the intersection of $V_3 \subset \mathbb{P}^5$ with the quadric hypersurface

$$\bigcup_{x \in L} T_x(S).$$

The map $\tilde{f}_L$ is best understood by cases: for each point $r \in V_3$, let $V_2(r)$ be the 2-plane spanned by r and L, and write

$$G \cdot V_2(r) = L + L_1, \qquad F \cdot V_2(r) = L + L_2.$$

There are then a number of possibilities:

1. Generically, L, L_1, and L_2 are all distinct, and L_1 meets L_2 at a point $p \in X$ not on L. (See Figure 26.) In this case $V_2(r)$ will not be tangent to X anywhere along L, so

$$\tilde{f}_L^{-1}(r) = \{p\}.$$

2. In case L_1, L_2, and L are again distinct, but have a point $p \in L$ in common—i.e., $V_2(r) \cap X = L$—we see that $V_2(r)$ is tangent to X exactly at p. (See Figure 27.) The point r is thus the image of the point $r(p) \in F$ on the exceptional divisor of $\tilde{X}_L$ corresponding to the normal vector to $L \subset X$ at p lying in $V_2(r)$.

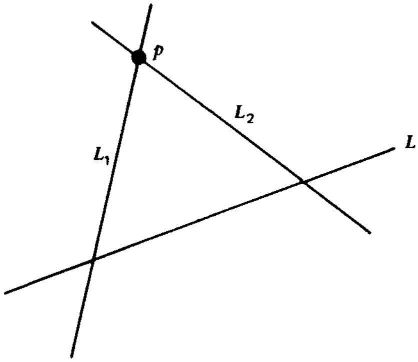

Figure 26

3. In case $L_1 = L \neq L_2$—or similarly $L_2 = L \neq L_1$—we see as in the least case that $V_2(r)$ is tangent to X at the point p of intersection of L with L_2 (resp. L_1). (See Figure 28.) r is thus the image of the normal vectors to $L \subset X$ at p lying in $V_2(r)$.

4. If $L_1 = L_2 \neq L$, then clearly r is the image of the proper transform in $\tilde{X}_L$ of the line $L_1 = L_2 \subset X$. (See Figure 29.)

5. The final possibility is $L_1 = L_2 = L$. This can occur only when L is a special line and

$$V_2(r) = \bigcap_{x \in L} T_x(X).$$

In this case, $V_2(r)$ contains a normal vector to $L \subset X$ at each point of L, and the map $\tilde{f}_L$ sends the curve consisting of these normal vectors to the point r.

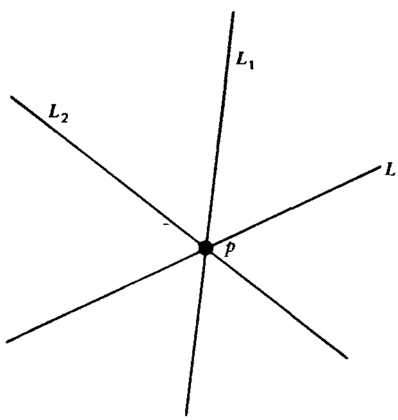

Figure 27

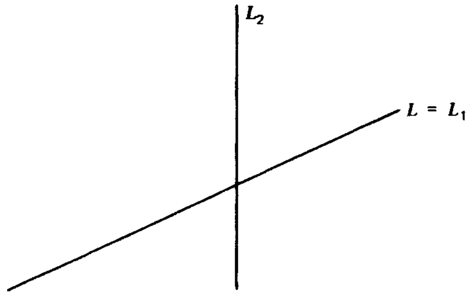

Figure 28

Suppose now that the line L of projection is not a special line. Then by the above, the map $\tilde{f}_L$ is one-to-one away from the proper transforms in $\tilde{X}_L$ of the lines on X meeting L, and maps each of these proper transforms onto the corresponding point of E_L, i.e., that

$$\tilde{f}_L: \tilde{X}_L \to \mathbb{P}^3$$

is the blow-up of the quintic curve $E_L \subset \mathbb{P}^3$. We can identify one of the rulings on the quadric $Q = \tilde{f}_L(F)$: first, for every $x \in L$, the image under $\tilde{f}_L$ of $\pi^{-1}(x) \subset \tilde{X}_L$ is just the line $V_3 \cap T_x(X)$ lying on Q. Note that since L is nonspecial, for $x \neq x' \in L$, $T_x(X)$ meets $T_{x'}(X)$ only in L, so that the corresponding lines $\tilde{f}_L(\pi^{-1}x)$ and $\tilde{f}_L(\pi^{-1}x')$ are disjoint; thus Q is a smooth quadric. Two of the lines in the second ruling are also visible: at each point $p \in L$, one normal vector to $L \subset X$ will lie in the 3-plane $\cap_{x \in L} T_x(G)$, and the images of the points of $F \subset X_L$ corresponding to these normal vectors will be the line $(\cap_{x \in L} T_x(G)) \cap V_3$ in Q; similarly the intersection of V_3 with $\cap_{x \in L} T_x(F)$ will be a line of the second ruling in Q. Note that the

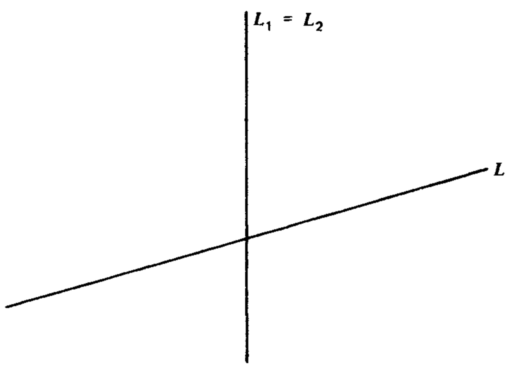

Figure 29

lines of the first ruling—the fibers of the blow-up $\pi:\tilde{X}_L \to X$—each meet E_L three times, while the lines in the second ruling meet E_L twice.

The situation is slightly different in case L is a special line of X. Now all the lines $\{\tilde{f}_L(\pi^{-1}x) = V_3 \cap T_x(X)\}_{x\in L}$ on the quadric Q have in common the point

$$p = V_3 \cap \bigcap_{x \in L} T_x(X)$$

corresponding to the 2-plane tangent to X everywhere along L. Thus Q is singular—it is the cone over a conic curve, with vertex p. Away from the inverse image $\tilde{f}_L^{-1}(p)$, $\tilde{f}_L$ is as before one-to-one except on the locus of lines in X meeting L and as we shall see later, the vertex p of Q lies in the closure E_L of the image of this locus. Thus $\tilde{f}_L : \tilde{X}_L \to \mathbb{P}^3$ *is again the blow-up of the curve* $\mathrm{E_L}$. (Recalling that a special line $L \subset X$ is to be counted among the lines meeting L, we may think of the line $\tilde{f}_L^{-1}(p)$ of points corresponding to normal vectors to $L \subset X$ in $\cap_{x\in L} T_x(X)$ as the "proper transform of L" itself in the blow-up $\tilde{X}_L$ of X).

In either case, then, we have seen that the birational map $f_L : X \to \mathbb{P}^3$ consists of the blow-up of the line $L \subset X$, followed by the blowing-down of the proper transforms in $\tilde{X}_L$ of the lines in X meeting L. In reverse, then, *the quadric line complex* X *is obtained by blowing up the quintic curve* $\mathrm{E_L} \subset \mathbb{P}^3$ *and blowing down the proper transform of the quadric* Q *containing it (more precisely, the proper transforms of the family of trichords to* $\mathrm{E_L}$) *to a curve.*

One question that arises in this context is: what is the hyperplane bundle on the curve E_L? Explicitly, we have seen that for every line $L \in A$ on the quadric line complex, we obtain an embedding $E_L \subset \mathbb{P}^3$ of the curve B. Accordingly, we may define a map

$$\rho:\ A = \mathcal{J}(B) \to \mathcal{J}(B)$$

by sending each line $L \in A$ to the class of the hyperplane bundle on $B \cong E_L \subset \mathbb{P}^3$; we ask now for a description of the map ρ.

To answer this question, we argue as follows. First, we note that the linear system associated to any divisor D of degree 5 on the curve B gives an embedding of B in $\mathbb{P}^3$ $\mathbb{P}^3$ as a quintic curve E_D (Section 1, Chapter 2). Second, since by Riemann-Roch

$$h^0(2D) = 10 - 2 + 1 = 9$$

and the vector space $H^0(\mathbb{P}^3, \mathcal{O}(2H))$ of quadrics on $\mathbb{P}^3$ has dimension 10, the restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}(2H)) \to H^0(B, \mathcal{O}(2D))$$

must have a kernel—i.e., E_D must lie on a quadric surface Q in$\mathbb{P}^3$. Since

E_D is nondegenerate of degree 5, moreover, we see that the quadric Q is uniquely determined by E_D.

Suppose first that Q is a singular quadric, i.e., Q is the cone pC over a smooth conic curve C. Q then contains a single family of lines $\{L_q=\overline{pq}\}_{q\in C}$; since any two of these lines comprise a hyperplane section of Q, it follows that E_D contains the vertex p of Q and every line $L_q\subset Q$ meets E_D in two points other than p. But now the divisors

$$D_q = L_q\cdot E_D - p$$

form a linear system of degree 2, and so D_q must be the standard hyperelliptic divisor D_0 on B. Thus the divisor $D=H\cdot E_D$ on B is of the form

$$D = 2D_0 + p.$$

Conversely, suppose that D is of the form $2D_0+p$ for some $p\in C$. Then the divisors

$$\{p+D_0+D_\lambda\}_{D_\lambda\sim D_0}$$

are all hyperplane sections of $E_D\subset\mathbb{P}^3$, and so p must be collinear with the points of D_0. The lines $L_\lambda=\{\overline{pD_\lambda}\}_{D_\lambda\sim D_0}$, containing three points of E_D, must lie on the quadric Q; it follows that

$$Q = \cup L_\lambda$$

is a singular quadric with singular point p.

We have seen then that among all divisors D of degree 5 on B, the divisors for which $E_D=\iota_D(B)$ lies on a singular quadric are exactly those of the form $2D_0+p$. Now the set of such divisors forms a translate of the theta-divisor Θ on $\mathcal{J}(B)$. But by what we have said, the inverse image $\rho'\Theta$ of lines $L\in A$ such that E_L lies on a singular quadric is just the divisor $\tilde{D}\subset A\cong\mathcal{J}(B)$, i.e., up to translation

$$\rho^*\Theta = \tilde{D} = m_2^*\Theta$$

it follows—at least in case B has no automorphisms other than the hyperelliptic—that, up to translation, *the map ρ is simply multiplication by two.*

One point that emerges from this discussion is this: since ρ is surjective, the quadric line complex X is determined by the curve B. Indeed, we can give an explicit recipe for the reconstruction of X from B: first embed B in $\mathbb{P}^3$ as a quintic E_D—by the above, it will not matter what divisor D we employ for the embedding. Then blow up $\mathbb{P}^3$ along the curve E_D, and blow down the family of proper transforms in $\tilde{\mathbb{P}}^3_{E_D}$ of the trichords of E_D into a curve. (Note that if D and D' are two divisors of degree 5 on B, not linearly equivalent, then $\tilde{\mathbb{P}}^3_{E_D}$ will not in general be isomorphic to $\tilde{\mathbb{P}}^3_{E_{D'}}$; they become isomorphic only after we blow down the trichlords to E_D and $E_{D'}$, respectively). In particular, since B is itself determined by the Abelian variety A,

The quadric line complex X *is determined up to isomorphism by the abstract variety* A *of lines lying on it.*

Note, incidentally, that the preceding gives us another characterization of the special lines on X: for any line $L \subset X$ we have a C^∞ decomposition of vector bundles on L:

$$T(\mathbb{P}^5)|_L = N_{X/\mathbb{P}^5|_L} \oplus N_{L/X} \oplus T(L).$$

Now we have

$$\begin{aligned} c_1(T(\mathbb{P}^5))|_L &= 6, \\ c_1(N_{X/\mathbb{P}^5|_L}) &= c_1(N_{F/\mathbb{P}^5|_L}) + c_1(N_{G/\mathbb{P}^5|_L}) \\ &= 2 + 2 = 4, \end{aligned}$$

and of course

$$c_1(T(L)) = 2;$$

it follows that

$$c_1(N_{L/S}) = 0.$$

Thus, by our classification (Section 3, Chapter 4) of vector bundles on $\mathbb{P}^1$, we can write

$$N_{L/X} = H^n \oplus H^{-n}, \qquad n \geqslant 0,$$

where H is the hyperplane bundle on $L \cong \mathbb{P}^1$. If L is nonspecial, then we have seen that

$$\mathbb{P}(N_{L/X}) = \mathbb{P}^1 \times \mathbb{P}^1$$

and it follows that $n = 0$, i.e., *the normal bundle of* L *in* X *is trivial.* On the other hand, if L is special, then $\mathbb{P}(N_{L/X})$ is the ruled surface $S_{0,2} = \mathbb{P}(H^m \oplus H^{m+2})$, and so

$$N_{L/X} = H \oplus H^{-1}.$$

Thus we see that *the special lines* $L \subset X$ *are exactly the lines in* X *having normal bundle* $H \oplus H^{-1}$ *in* X; *the nonspecial lines in* X *are exactly the lines having trivial normal bundle.*

If we use the intermediate Jacobian $J(X) = H^3(X, \mathbb{R}H^3(X, \mathbb{Z})$ defined on p. 331, then many of the results of this chapter may be summarized as follows:

The intermediate Jacobian J(X), *together with its principal polarization determined by the intersection form on* $H^3(X, \mathbb{Z})$, *is biholomorphic to the* *surface* A *of lines in* X *with the corresponding polarization on* A *being given by the incidence curve* B.

In general, if X is the transverse intersection of two smooth quadrics in $\mathbb{P}^{2n+1}$, then the set A of $\mathbb{P}^{n-1}$'s contain in X has the structure of an Abelian variety which may be identified with the middle intermediate Jacobian of X; a proof of this may be found in Ran Donagi, The variety of linear spaces on the intersection of two quadrics, to appear. The principally polarized Abelian variety determines the variety X so that we have a Torelli theorem—i.e., the Hodge structure of X determines the variety X.

Particular to the case $n=2$ is the Kummer surface S defined by taking the quotient of A by the involution $z \to -z$, and which we have identified geometrically as the surface in $\mathbb{P}^3$ defined by the condition that the conic $X_i p$ of lines in the quadric line complex passing through p should be singular. The Kummer surface S uniquely determines A and hence X: If we desingularize S to obtain a K-3 surface $\tilde{S}$ having a divisor $E=\sum_{i=1}^{16} E_i$ lying over the double points of S, then the class of E in $H^2(\tilde{S},\mathbb{Z})$ is even so that we may construct a two-sheeted covering $\pi:\tilde{A}\to\tilde{S}$ branched over E. The curves $\tilde{C}_i=\pi^{-1}(C_i)$ are rational curves with $\tilde{C}_i^2=-1$; blowing them down gives the Abelian surface A.

REFERENCES

The quadric line complex and its associated Kummer surface were extensively studied in the last century; a classic source is

F. Klein, Zur Theorie der Linencomplexe des ersten und zweiten Grades, *Math. Ann.*, Vol. 2 (1870), pp. 198–226.

It has reappeared in connection with the moduli of stable vector bundles over curves; the reference here is

M. S. Narasimhan and S. Ramanan, Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.*, Vol. 89 (1969), pp. 14–51.

This paper contains the result that the Kummer surface associated to the quadric line is obtained from the Jacobian of the hyperelliptic curve defined by the pencil of quadrics.

Finally, an interesting general reference for threefolds and their intermediate Jacobians is

A. N. Turin, Five lectures on three-dimensional varieties, *Uspeki Math. Nauk.*, Vol. 27 (1972), pp. 4–50 (English translation: *Russian Math. Surveys*, Vol. 27 (1972), 1–53).

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