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Ciro Ciliberto

An Undergraduate Primer in Algebraic Geometry

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An Undergraduate Primer in Algebraic Geometry

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Preface

This book arouse from notes of courses in Algebraic Geometry that I gave at the University of Roma Tor Vergata during several years. It basically consists of two parts. The first includes Chapters 1–14, and it is devoted to an introduction to basic concepts in Algebraic Geometry. The main objects of interest in this part are affine and projective varieties, some of their main attributes (like irreducibility, dimension, regular and rational functions, morphisms, products, degree, etc.) and basic examples (hypersurfaces, Veronese varieties, Segre varieties, blow-ups, Grassmannians, etc.). One of the leading themes in this first part is elimination theory, to which several sections of the book are dedicated, and which is at the basis of some important applications like Hilbert's Nullstellensatz and basic intersection theory of varieties with Bezout's Theorem. The second part of the book, which includes Chaps. 15–20, is devoted to the theory of curves. A basic preliminary is in Chap. 15 with formal power series, which constitute the main tool for the study of local properties of curves. Then Chaps. 16 and 17 are devoted to the study of affine and projective plane curves, respectively. Chapter 18 contains the proof of resolution of singularities of curves. Chapter 19 is devoted to the classical theory of linear equivalence of divisors and linear series on a curve. Finally, Chap. 20 contains the Riemann–Roch and Riemann–Hurwitz Theorems.

The approach in this book is purely algebraic, so no analysis or differential geometry is needed. The main tool is commutative algebra from which we recall the main results we need, in most cases with proofs. The prerequisites consist in the knowledge of basics in affine and projective geometry (in particular, conics and quadrics in three-dimensional space), basic algebraic concepts regarding rings, modules, fields, linear algebra and basic notions in the theory of categories. A few elementary facts of topology are needed in Chap. 4.

The book can be used as a textbook for a basic undergraduate course in Algebraic Geometry. The users of the book are not necessarily intended to become algebraic geometers but may be simply interested students or researchers who want to have a first smattering in the topic. Chapter 14 is not essential for the rest and can be skipped in a first reading. For a short introductory course, one can focus on the first thirteen chapters only.

The book contains several exercises, in which there are more examples and parts of the theory which are not fully developed in the text. Some exercises are marked with an asterisk, which means either that they are a bit more difficult than the average, or that they are needed for the sequel of the book. Of some exercises, there are the solutions at the end of each chapter.

What readers will not find in this book are (at least) two main things. The first is sheaf theory, cohomology, schemes, etc. For this, the classical references are [4] and, in part, the second volume of [7]. The second is a computational approach to Algebraic Geometry, which is a very interesting topic for which I recommend [2].

I am indebted for inspiration to several sources, for instance [3, 6, 7, 8].

Rome, Italy
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Ciro Ciliberto

Contents

Chapter 1
Affine and Projective Algebraic Sets

1.1 Affine Algebraic Sets

Let $\mathbb{K}$ be a field that throughout the whole book will be assumed to be algebraically closed. This will be the *base field* over which we will consider all the geometric objects we will construct in this book.

We will denote by $\mathbb{A}^n_{\mathbb{K}}$, or simply by $\mathbb{A}^n$, the *n–dimensional numerical affine space* on $\mathbb{K}$, i.e., the set $\mathbb{K}^n$ of all ordered n–tuples of elements of $\mathbb{K}$. An element $P = (p_1, \ldots, p_n)$ of $\mathbb{A}^n$ will be called a *point* and $p_1, \ldots, p_n$ will be called the *coordinates* of P. The *numerical vector* $\mathbf{p} = (p_1, \ldots, p_n)$ on $\mathbb{K}$ will be called the *coordinate vector* of P and we may write $P = (\mathbf{p})$. The point O with zero coordinate vector $\mathbf{0}$ is called the *origin* of $\mathbb{A}^n$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n–tuple of variables on $\mathbb{K}$. We will denote by $A_{\mathbb{K},n}$, or simply by A_n, the polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n]$.

Any element $f \in A_n$ can be regarded as an application $f : \mathbb{A} \to \mathbb{K}$. The subset $Z_a(f) = f^{-1}(0)$ of $\mathbb{A}^n$ will be called the *zero set* of f. More generally, if $F \subseteq A_n$, the subset

$$Z_a(F) := \bigcap_{f \in F} Z_a(F)$$

is called the *zero set* of F. The subscript a in $Z_a(F)$ stays for *affine*, in order to distinguish this notion from the analogous *projective* version which will be introduced in Sect. 1.4. If there is no danger of confusion we will write $Z(f)$ instead of $Z_a(F)$.

Note that

$$F \subseteq G \Longrightarrow Z(G) \subseteq Z(F). \tag{1.1}$$

Hence, if (F) is the ideal of A_n generated by F, one has $Z((F)) \subseteq Z(F)$. It is easy to see that actually $Z(F) = Z((F))$ (see Exercise 1.1.2). Moreover, since A_n is a Noetherian ring, the *Hilbert basis theorem* holds in A_n, namely every ideal of A_n is finitely generated. Therefore there are finitely many $f_1, \ldots, f_m \in F$ such that $(F) = (f_1, \ldots, f_m)$. Then (see Exercise 1.1.3)

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$$Z(F) = Z(f_1, \ldots, f_m) \tag{1.2}$$

where $Z(f_1, \ldots, f_m)$ stays for $Z(\{f_1, \ldots, f_m\})$. If (1.2) holds, one says that

$$f_i(x_1, \ldots, x_n) = 0, \quad 1 \leqslant i \leqslant m$$

is a *system of equations* of $Z(F)$.

A subset Z of $\mathbb{A}^n$ is called an *affine algebraic set* if there is a subset F of A_n such that $Z = Z(F)$. This is equivalent to say that there is an ideal $I \subseteq A_n$ such that $Z = Z(I)$. We will denote by $\mathcal{A}_n$ the set of all affine algebraic sets of $\mathbb{A}^n$. We leave as an exercise to the reader (see Exercise 1.1.4) to prove the following

Proposition 1.1.1 *$\mathcal{A}_n$ is the set of all closed sets of a topology of $\mathbb{A}^n$.*

The topology whose closed sets are the elements of $\mathcal{A}_n$ is called the *Zariski topology* of $\mathbb{A}^n$. If X is a non–empty subset of X we will think of it as endowed with the induced topology, which will be called the *Zariski topology* of X.

Exercise 1.1.2 Prove (1.2). Prove that for any subset $F \subseteq A_n$ one has $Z_a(F) = Z_a((F))$.

Exercise 1.1.3 Prove that if B is a *basis* of the ideal I of A_n (i.e., it is a system of generators), then $Z(I) = Z(B)$.

Exercise 1.1.4 Prove Proposition 1.1.1.

1.2 Projective Spaces

Let V be a vector space of finite dimension $n+1$ on $\mathbb{K}$. Define the following *proportionality relation* on $V \setminus \{\mathbf{0}\}$:

$$\mathbf{x} \sim \mathbf{y} \iff \exists t \in \mathbb{K}^* := \mathbb{K} \setminus \{0\} : \mathbf{y} = t\mathbf{x}$$

This is an equivalence relation. We denote by $[\mathbf{x}]$ the proportionality equivalence class of the vector $\mathbf{x} \in V \setminus \{\mathbf{0}\}$. The quotient set $\mathbb{P}(V) = V \setminus \{\mathbf{0}\}/\sim$ is called the *projective space* associated to the vector space V and $n = \dim(V) - 1$ is called the *dimension* of $\mathbb{P}(V)$, denoted by $\dim(\mathbb{P}(V))$. The empty set is the projective space of dimension -1 associated to the zero vector space. The elements of $\mathbb{P}(V)$ are called *points*. A projective space of dimension 1 is called a *(projective) line*, a projective space of dimension 2 is called a *(projective) plane*. We will use the notation $p_V : \mathbf{x} \in V \setminus \{\mathbf{0}\} \to [\mathbf{x}] \in \mathbb{P}(V)$ and we may write p instead of p_V if no confusion arises.

In particular we can consider the numerical vector space $\mathbb{K}^{n+1}$ on $\mathbb{K}$. The associated vector space is denoted by $\mathbb{P}^n$ and is called the *numerical projective space of dimension n* on $\mathbb{K}$. If $\mathbf{x} = (x_0, \ldots, x_n)$ is a non–zero numerical vector, its proportionality class is denoted by $[\mathbf{x}]$ or by $[x_0, \ldots, x_n]$. We will, say that $(x_0, \ldots, x_n)$ is a vector of *homogeneous coordinates* of the point $[\mathbf{x}]$. The homogeneous coordinates of a point are not all zero and defined up to a non–zero numerical factor. The points P_i of $\mathbb{P}^n$ whose homogeneous coordinates are all zero except the one at place i, with $0 \leqslant i \leqslant n$, are called the *vertices of the fundamental pyramid* of $\mathbb{P}^n$. The point P_{n+1} with homogeneous coordinates $[1, \ldots, 1]$ is called the *unitary point* of $\mathbb{P}^n$. The points P_i with $0 \leqslant i \leqslant n+1$ are called the *fundamental points* of $\mathbb{P}^n$.

A map $\varphi : \mathbb{P}(V) \to \mathbb{P}(W)$ is called a *projectivity* if there is an injective linear map $f : V \to W$ such that $p_W \circ f = \varphi \circ p_V$, i.e., if for every $\mathbf{x} \in V \setminus \{\mathbf{0}\}$ one has $\varphi([\mathbf{x}]) = [f(\mathbf{x})]$. Note that if there is a projectivity $\varphi : \mathbb{P}(V) \to \mathbb{P}(W)$, then $\dim(\mathbb{P}(V)) \leqslant \dim(\mathbb{P}(W))$. Moreover the composition of two projectivities is still a projectivity.

If one wants to stress that the projectivity φ depends on the linear map f, one writes $\varphi = \varphi_f$. Denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps from V to W. It is easy to verify that

$$\varphi_f = \varphi_g \Longleftrightarrow [f] = [g] \quad \text{in} \quad \mathbb{P}(\mathrm{Hom}(V, W)) \tag{1.3}$$

(see Exercise 1.2.1). Hence the set $\mathrm{Pr}(\mathbb{P}(V), \mathbb{P}(W))$ of all projectivities of $\mathbb{P}(V)$ to $\mathbb{P}(W)$ can be identified with the subset of $\mathbb{P}(\mathrm{Hom}(V, W))$ whose points are equivalence classes of injective linear maps. In particular $\mathrm{Pr}(\mathbb{P}(V), \mathbb{P}(V))$ is a group for the composition of applications. It is denoted by $\mathrm{PGL}(V)$ and it is the image of the group $\mathrm{GL}(V)$ of the automorphisms of the vector space V via the map $p_{\mathrm{End}}(V)$, where $\mathrm{End}(V) = \mathrm{Hom}(V, V)$. Of course V and W are isomorphic if and only if there is a surjective projectivity $\varphi : \mathbb{P}(V) \to \mathbb{P}(W)$, then we say that $\mathbb{P}(V)$ and $\mathbb{P}(W)$ are *projectively equivalent*. In that case $\mathrm{GL}(V)$ and $\mathrm{PGL}(V)$ are respectively isomorphic to $\mathrm{GL}(W)$ e $\mathrm{PGL}(W)$.

If $V = \mathbb{K}^{n+1}$, the groups $\mathrm{GL}(V)$ and $\mathrm{PGL}(V)$ are denoted by $\mathrm{GL}(n+1, k)$ and $\mathrm{PGL}(n+1, k)$. The former can be identified with the group of square matrices of order and rank $n+1$, the latter with the quotient of the former by the subgroup of *scalar matrices*, i.e., matrices of the form tI_{n+1} where $t \in \mathbb{K}^*$ and I_{n+1} is the unitary matrix of order $n+1$.

If V has dimension $n+1$, a projectivity $\varphi : \mathbb{P}^n \to \mathbb{P}(V)$ is bijective. It assignes to a point $P \in \mathbb{P}(V)$ a proportionality class $[p_0, \ldots, p_n]$ of numerical vectors. We can think of $\varphi : \mathbb{P}^n \to \mathbb{P}(V)$ as a way of introducing a system of homogeneous coordinates in $\mathbb{P}(V)$. In this system the *fundamental points* of $\mathbb{P}(V)$ are the images of the fundamental points of $\mathbb{P}^n$ via φ. In order to denote that $P \in \mathbb{P}(V)$ has homogeneous coordinates $[p_0, \ldots, p_n]$ in this system, we write $P = [p_0, \ldots, p_n]$. Of course $\mathbb{P}^n$ has a natural system of coordinates induced by the identity map $\mathrm{id} : \mathbb{P}^n \to \mathbb{P}^n$.

If we introduce two systems of coordinates $\varphi : \mathbb{P}^n \to \mathbb{P}(V)$ and $\psi : \mathbb{P}^n \to \mathbb{P}(V)$ in $\mathbb{P}(V)$, there is a square matrix $\mathbf{A}$ of order and rank $n+1$ such that for every point $P \in \mathbb{P}(V)$ which has in the two systems of coordinates the coordinate vectors $\mathbf{x}$ e $\mathbf{y}$, one has

$$\mathbf{y} = \mathbf{x} \cdot \mathbf{A}$$

and this is called the *formula for the change of coordinates* in passing from one to the other of the two systems. The matrix **A** is defined up to a non–zero scalar factor and $[A]$ determines the projectivity $\psi^{-1} \circ \varphi \in \mathrm{PGL}(n+1, k)$.

For our future purposes we will consider as equivalent the consideration of two projective spaces if they are projectively equivalent. Therefore in what follows we will mainly focus on numerical projective spaces $\mathbb{P}^n$.

Exercise 1.2.1 Prove (1.3).

Exercise 1.2.2 Let $\phi : \mathbb{P}(V) \to \mathbb{P}(W)$ be a projectivity. Assume that $\dim(\mathbb{P}(V)) = n, \dim(\mathbb{P}(W)) = m$. Introduce systems of coordinates in $\mathbb{P}(V)$ and $\mathbb{P}(W)$. Prove that there is a matrix **A**, of type $(n+1) \times (m+1)$ and rank $m+1$, defined up to a non–zero factor, such that $\phi(P) = P'$ if and only if $P = [\mathbf{x}]$ e $P' = [\mathbf{y}]$ in the two systems and $\mathbf{y} = \mathbf{x} \cdot A$. This is called an *equation of the projectivity* in the given coordinate systems.

1.3 Graded Rings

Let S be a ring which, as all the rings we will consider in this text, is commutative and with unity. Moreover let $G(+)$ be an abelian group. The ring S will be said to be *G–graded* or endowed with a *G–grading* (or simply a *graded ring*, or a ring with a *grading*, when $G = \mathbb{Z}$) if there is a decomposition

$$S = \bigoplus_{g \in G} S_g$$

as a direct sum of abelian subgroups of the additive group of S, such that $1 \in S_0$ and for any pair $(g, h) \in G \times G$ one has $S_g \cdot S_h \subseteq S_{g+h}$, where, if A and B are subsets of S, we set

$$A \cdot B = \{ab : a \in A, b \in B\}$$

and similarly for $A + B$. The group S_g is called the *part of degree* g of S. The elements of S_g are called the *homogeneous elements* of degree g of S. If F is a non–empty subset of S, we set $F_g = F \cap S_g$ for all $g \in G$ and we denote by $H(F)$ the set $\bigcup_{g \in G} F_g$ of all homogeneous elements in F.

The following properties are an immediate consequence of the definition:

(a) every $f \in S$ can be written in a unique way as a finite sum

$$f = f_{g_1} + \cdots + f_{g_n} \tag{1.4}$$

with $f_{g_i} \in S_{g_i}$ for $1 \leqslant i \leqslant n$ and $g_1, \ldots, g_n \in G$ distinct. One says that $f_{g_1}, \ldots, f_{g_n}$ are the *homogeneous components* of f and (1.4) is called the *decomposition of f in homogeneous components*;

(b) S_0 is a subring of S and S_g is an S_0–module for all $g \in G$. In particular, if S_0 is a field, then S_g is a S_0–vector space;
(c) if $G = \mathbb{Z}$, we set, for every integer n, $S_{>n} = \bigoplus_{d>n} S_d$. If $S_d = \{0\}$ for all $d < 0$, then $S_{>n}$ is an ideal of S for all integer n, moreover $S = S_{>-1}$, $\bigcap_{n\in\mathbb{N}} S_{>n} = \{0\}$ and $S_{>0}$ will be denoted with the symbol S_+ and is called the *irrelevant ideal* of S.

An ideal I of S is said to be *homogeneous* if $I = \bigoplus_g I_g$, i.e., if $f \in I$ if and only if all homogeneous components of f are in I.

The proof of the following propositions are left as exercises to the reader (see Exercises 1.3.5 and 1.3.7).

Proposition 1.3.1 *If S is a G–graded ring and I is an ideal of S, then I is homogeneous if and only if it is generated by a set of homogeneous elements.*

Proposition 1.3.2 *If S is a G–graded ring and I_1, I_2 are homogeneous ideals of S, then $I_1 \cdot I_2$, $I_1 \cap I_2$, $I_1 + I_2$, and the ideal generated by $I_1 \cup I_2$, are homogeneous ideals. If moreover $G = \mathbb{Z}$ and I is a homogeneous ideal of S, then:*

(i) the radical *of I, i.e.,*

$$\operatorname{rad}(I) = \{f \in S : \exists r \in \mathbb{N} : f^r \in I\}$$

is homogeneous;

(ii) I is prime if and only if for every pair $(f, g) \in H(S) \times H(S)$ such that $fg \in I$, either $f \in I$ or $g \in I$.

Let S be a G–graded ring and S' a H–graded ring, and suppose we have a homomorphism $\phi : G \to H$. A homomorphism $f : S \to S'$ is said to be *ϕ–homogeneous*, if for all $g \in G$ one has $f(S_g) \subseteq S'_{\phi(g)}$. If f and ϕ are isomorphisms then the inverse of f is still a homomorphism of graded rings, hence f is a isomorphism of graded rings. If $G = H$, a id_G–homogeneous homomorphism $f : S \to S'$ will be said to be *homogeneous of degree* 0. A homogeneous isomorphism of degree 0 will be simply called a *isomorphism*.

If $G = H = \mathbb{Z}$ and $f : S \to S'$ is homogeneous, then $\phi : \mathbb{Z} \to \mathbb{Z}$ is the multiplication by an integer d, hence $f(S_g) \subseteq S'_{dg}$ for all $g \in G$. In this case we will say that f is *homogeneous of weight d*. A homogeneous homomorphism of weight 1 is of degree 0.

One gives in a similar manner the definition of a *graded module* over a graded ring. If S is a G–graded ring, an S–module M will be said to be *graded* if there is a decomposition

$$M = \bigoplus_{g\in G} M_g$$

as a direct sum of abelian subgroups of the additive group of M, such that for all pairs $(g, h) \in G \times G$ one has $S_g \cdot M_h \subset M_{g+h}$, where, if $A \subseteq S$ and $B \subseteq M$, we set

$$A \cdot B = \{ab : a \in A, b \in B\}.$$

For example a homogeneous ideal of S is a graded module on S. We use for graded modules definitions, terminology and symbols analogous to the ones we introduced for graded rings.

Given S a G–graded ring, M and N graded S–modules and an element $h \in G$, a homomorphism $f : M \to N$ is said to be *homogeneous of degree* h, if for all $g \in M$ one has $f(M_g) \subseteq N_{g+h}$.

Note that we can change the grading of a S–module M in the following way. Fix and $h \in G$ and set

$$M(h) = \bigoplus_{g \in G} M(h)_g \quad \text{where} \quad M(h)_g := M_{h+g}.$$

It is clear that $M(h)$ is still a graded S–module isomorphic to M as an S–module, but in general not isomorphic to M as a graded S–module (see Exercise 1.3.14).

Example 1.3.3 Let V be a $\mathbb{K}$–vector space of dimension $n + 1 \geqslant 1$. The *symmetric algebra* on V

$$\mathrm{Sym}(V) := \bigoplus_{d \in \mathbb{N}} \mathrm{Sym}^d(V)$$

is a graded ring which we will denote by $S(V)$, and we have

$$N_{n,d} := \dim(\mathrm{Sym}^d(V)) = \binom{n+d}{d}.$$

The grading is in $\mathbb{Z}$ but $S(V)_d = 0$ for all $d < 0$. In this case $S(V)_0 \cong \mathbb{K}$ and $S(V)$ is generated as a $\mathbb{K}$–algebra by $S(V)_1$. If $f \in S(V)_d$, we write $d = \deg(f)$.

Let us fix a *reference system* of V, i.e., an order basis $(\mathbf{e}_0, \ldots, \mathbf{e}_n)$. One has the *dual reference system* $(\mathbf{e}^0, \ldots, \mathbf{e}^n)$ of $\check{V} \cong \mathrm{Hom}(V, \mathbb{K})$, where

$$\mathbf{e}^i(\mathbf{e}_j) = \mathbf{e}_j(\mathbf{e}^i) = \delta_{ij}$$

where δ_{ij} is the *Kronecker symbol*. We set $\mathbf{e}^i = x_i$ and $\mathbf{e}_i = \partial_i$, for $0 \leqslant i \leqslant n$. Then $S(\check{V})$ can be identified with $S_{\mathbb{K},n} = \mathbb{K}[x_0, \ldots, x_n]$, also denoted by S_n if $\mathbb{K}$ is understood. By denoting it with S_n instead of A_{n+1}, we want to stress its structure of graded ring, in which the homogeneous part $S_{n,d}$ of degree d is the vector space of *homogeneous polynomials*, or *forms*, of degree d, i.e., those polynomials in which appear only monomials of degree d (the 0 polynomial is considered to be homogeneous of every degree).

The ring $S(V)$ can be identified with the ring of *differential operators* $D_n = \mathbb{K}[\partial_0, \dots, \partial_n]$, which is isomorphic to S_n as a graded ring. The homogenous part $D_{n,d}$ of degree d is the vector space of the *homogeneous differential operators* of degree d, i.e., those operators in which appear only monomials of degree d in $\partial_0, \dots, \partial_n$ (again the 0 operator is considered to be homogeneous of any degree). Note that $S(V)$ is somehow the *dual* of $S(\check{V})$, in the sense that $D_{n,d}$ and $S_{n,d}$ are dual to each other.

We set $\mathbf{x} = (x_0, \dots, x_n)$. Let $\mathbf{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$ be a *multiindex*. We will denote by $|\mathbf{i}| = i_0 + \cdots + i_n$ the *length* of the multiindex. We will set $\mathbf{x}^{\mathbf{i}} = x_0^{i_0} \cdots x_n^{i_n}$. Hence a homogeneous polynomial of degree d in $x_0, \dots, x_n$ can be written as

$$f(\mathbf{x}) = \sum_{|\mathbf{i}|=d} f_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$$

with $f_{\mathbf{i}} \in \mathbb{K}$ the *coefficient* of the monomial $\mathbf{x}^{\mathbf{i}}$. Similar notation can be used for differential operators.

A polynomial $f \in S_n$ is homogeneous of degree d if and only if

$$f(t\mathbf{x}) = t^d f(\mathbf{x}), \quad \forall t \in \mathbb{K} \tag{1.5}$$

(see Exercise 1.3.9). Differentiating (1.5) with respect to t and then setting $t = 1$ one has

$$df(\mathbf{x}) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}) \tag{1.6}$$

The Eq. (1.6) is called the *Euler formula*.

Example 1.3.4 Let V_i, for $1 \leqslant i \leqslant h$, $\mathbb{K}$–vector spaces of dimensions $n_i + 1$. Set $V = \bigotimes_{i=1}^{h} V_i$. The ring $S(V)$ has the grading introduced in Example 1.3.3. It has however also another grading in $\mathbb{Z}^h$, in which the graded part of degree $\mathbf{d} = (d_1, \dots, d_h)$ is $\bigotimes_{i=1}^{h} \mathrm{Sym}(V_{d_i})$, if $d_i \geqslant 0$ for all $i = 1, \dots, h$, and it is $\{0\}$ otherwise. So actually the grading is in $\mathbb{N}^h$. When we want to stress that this is the grading, we write $S(V_1, \dots, V_h)$ rather than $S(V)$ and the part of degree $\mathbf{d}$ is denoted by $S(V_1, \dots, V_h)_{\mathbf{d}}$.

If, as in Example 1.3.3, we introduce reference systems in V_i, for $1 \leqslant i \leqslant h$, then $S(\check{V}_1, \dots, \check{V}_h)$ may be indentified with the ring of polynomials $\mathbb{K}[\mathbf{x}_1, \dots, \mathbf{x}_h]$, where $\mathbf{x}_i = (x_{i0}, \dots, x_{in_i})$ for $i = 1, \dots, h$. This ring is denoted by $S_{n_1,\dots,n_h}$ or by $S_{\mathbf{n}}$ with $\mathbf{n} = (n_1, \dots, n_h)$. Its homogeneous part $S_{\mathbf{n},\mathbf{d}}$ of degree $\mathbf{d} = (d_1, \dots, d_h)$ is the vector space of *plurihomogeneous polynomials* of degree $\mathbf{d}$ in the variables $\mathbf{x}_1, \dots, \mathbf{x}_h$, i.e., those homogeneous polynomials of degree d_i in the variables $\mathbf{x}_i = (x_{i0}, \dots, x_{in_i})$, for all $i \in \{1, \dots, h\}$. One has

$$N_{\mathbf{n},\mathbf{d}} := \dim(S_{\mathbf{n},\mathbf{d}}) = \prod_{i=1}^{h} \binom{n_i + d_i}{d_i}.$$

Similarly $S(V_1, \dots, V_h)$ may be identified with the ring of differential operators $\mathbb{K}[\boldsymbol{\partial}_1, \dots, \boldsymbol{\partial}_h]$ (where $\boldsymbol{\partial}_i = (\partial_{i0}, \dots, \partial_{in_i})$), which is denoted by $D_{n_1,\dots,n_h}$ or by $D_{\mathbf{n}}$, whose part of degree $\mathbf{d}$ is denoted by $D_{\mathbf{n},\mathbf{d}}$ and it is the vector space of plurihomogeneous differential operators of degree $\mathbf{d} = (d_1, \dots, d_h)$ acting on the variables $\mathbf{x}_1, \dots, \mathbf{x}_h$.

With the notation introduced in Example 1.3.3, a plurihomogeneous polynomial in $S_{\mathbf{n},\mathbf{d}}$ may be written as

$$f = \sum_{|\mathbf{i}_1|=d_1,\dots,|\mathbf{i}_h|=d_h} f_{\mathbf{i}_1,\dots,\mathbf{i}_h} \prod_{j=1}^{h} \mathbf{x}_j^{\mathbf{i}_j}$$

with an obvious meaning of the symbols. Analogous notation for the plurihomogeneous differential operators.

Exercise 1.3.5 * Prove Proposition 1.3.1.

Exercise 1.3.6 Verify that the radical of an ideal is also an ideal.

Exercise 1.3.7 * Prove Proposition 1.3.2.

Exercise 1.3.8 * Let S be a G–graded ring, let I be an ideal and consider the canonical surjective morphism $\pi : S \to S/I$. Prove that for all $g \in G$ one has $\pi(S_g) \cong S_g/I_g$, hence $S/I \cong +_{g\in G} S_g/I_g$. Prove that this sum is a direct sum if and only if I is homogeneous. In that case, if we set $(S/I)_g = S_g/I_g$ for all $g \in G$, then also S/I has a G–grading *induced* by the one of S.

Exercise 1.3.9 * Prove that $f \in S_n$ is homogeneous of degree d if and only if (1.5) holds.

Exercise 1.3.10 Prove that a polynomial $f(\mathbf{x}_1, \dots, \mathbf{x}_h)$ in the variables $\mathbf{x}_i = (x_{i0}, \dots, \mathbf{x}_{in_i})$, with $i = 1, \dots, h$, is plurihomogeneous of degrees $\mathbf{d} = (d_1, \dots, d_h)$ if and only if

$$f(t_1\mathbf{x}_1, \dots, t_h\mathbf{x}_h) = t_1^{d_1} \cdots t_h^{d_h} f(\mathbf{x}_1, \dots, \mathbf{x}_h)$$

for all $t_1, \dots, t_h \in \mathbb{K}$.

Exercise 1.3.11 * Let $g_i(\mathbf{y}) \in S_{m,d}$, $0 \leqslant i \leqslant n$, be homogeneous polynomials of degree d. Set $\mathbf{g}(\mathbf{y}) := (g_0(\mathbf{y}), \dots, g_n(\mathbf{y}))$. Prove that the map

$$f(\mathbf{x}) \in S_n \to f(\mathbf{g}(\mathbf{y})) \in S_m$$

is a homogeneous homomorphism of weight d. We will say that it is obtained by the *homogeneous substitution of variables* $\mathbf{x} = \mathbf{g}(\mathbf{y})$ of degree d.

Exercise 1.3.12 * Prove that a homogeneous substitution of variables of degree d as in Exercise 1.3.11 is an isomorphism if and only if $n = m$, $d = 1$, and $g_i(\mathbf{y}) \in S_{n,1}$, $0 \leqslant i \leqslant n$, are linearly independent. Prove that all homogeneous isomorphisms $S_n \to S_n$ are of this form.

Exercise 1.3.13 * Prove that if $f, g \in S_n$, if f is homogeneous and g divides f, then g is also homogeneous.

Exercise 1.3.14 Let M be a graded S–module. Consider $M(h)$ with $h \in G$. Prove that $M(h)$ is still a graded S–module, which is isomorphic to M as an S–module. Give an example in which $M(h)$ is not isomorphic to M as a graded S–module.

Exercise 1.3.15 Let M, N be graded $\mathbb{Z}$–modules and let $f : M \to N$ be a homogeneous homomorphism of degree d, i.e., $f(M_n) \subseteq N_{n+d}$ for all $n \in \mathbb{Z}$. Prove that $f : M(-d) \to N$ and $f : M \to N(d)$ are of degree 0.

Exercise 1.3.16 Let $f \in S_{n,d}$ be a non–zero homogeneous polynomial of degree d, so that (f) is a homogeneous ideal. Prove that the map $g \in S_n \to gf \in (f)$ is a degree d isomorphism of S_n–modules hence $S_n(-d) \to (f)$ is a degree 0 isomorphism.

1.4 Projective Algebraic Sets

Consider the projective space $\mathbb{P}^n$ of dimension n and let us fix a homogeneous element $f \in S_{n,d}$. If $P = [\mathbf{p}] \in \mathbb{P}^n$, one has that $f(\mathbf{p}) = 0$ if and only if $f(t\mathbf{p}) = 0$ for all $t \in \mathbb{K}^*$. Hence, although f cannot be considered as a function on $\mathbb{P}^n$, it makes however sense to say that f *vanishes* at a point $P = [\mathbf{p}] \in \mathbb{P}^n$: this is the case if and only if f vanishes on any vector of homogeneous coordinates of P. In this case we say that P is a *zero* of f.

Hence, given $f \in S_{n,d}$, it makes sense to consider the set $Z_p(f)$ of zeroes of f. If $F \subseteq S_n$, the subset

$$Z_p(F) := \bigcap_{f \in H(F)} Z_p(F)$$

is called the *zero set* of F. The subscript p for $Z_p(F)$ stays for *projective*, in order to distinguish this notion from the analogous affine one introduced in Sect. 1.1. However, if there is no danger of confusion, we may write $Z(f)$ rather than $Z_p(F)$.

Of course $Z(F) = Z(H(F))$. Moreover (1.1) holds. Hence $Z((H(F))) \subseteq Z(F)$ and it is easy to see that $Z(F) = Z((H(F)))$. Note that $(H(F))$ is a homogeneneous ideal (see Proposition 1.3.1). Moreover there are finitely many $f_1, \ldots, f_m \in H(F)$ such that $(H(F)) = (f_1, \ldots, f_m)$, hence (1.2) holds, and

$$f_i(x_0, \ldots, x_n) = 0, \quad 1 \leqslant i \leqslant m$$

is called a *system of equations* of $Z(F)$.

A subset Z of $\mathbb{P}^n$ is called an *algebraic projective set* if there is a subset F of S_n such that $Z = Z(F)$. We will denote by $\mathcal{P}_n$ the set of all algebraic projective sets of $\mathbb{P}^n$.

As in Proposition 1.1.1, one proves that:

Proposition 1.4.1 *$\mathcal{P}_n$ is the set of closed subsets of a topology.*

The topology of $\mathbb{P}^n$ whose closed sets are the elements of $\mathcal{P}_n$ is called the *Zariski topology* of $\mathbb{P}^n$. If X is a non–empty subset of $\mathbb{P}^n$ we will think of X as a topological space with the induced topology, called the *Zariski topology* of X.

Proposition 1.4.2 *A projectivity $\phi : \mathbb{P}^n \to \mathbb{P}^m$ is continuous for the Zariski topologies of $\mathbb{P}^n$ and $\mathbb{P}^m$. In particular, if ϕ is bijective, it is a homeomorphism.*

Proof Let $f : \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ be an injective linear map which determines ϕ. Then there is a rank $n+1$ matrix $\mathbf{A}$ of type $(n+1) \times (m+1)$ such that if $\mathbf{x} \in \mathbb{K}^{n+1}$ and $\mathbf{y} = f(\mathbf{x})$, then

$$\mathbf{y} = \mathbf{x} \cdot \mathbf{A} = (f_0(\mathbf{x}), \ldots, f_m(\mathbf{x})),$$

where $f_0(\mathbf{x}), \ldots, f_m(\mathbf{x})$ are independent forms of degree 1 spanning $S_{n,1}$. Then we have the homogeneous substitution of variables

$$\mathbf{y} = (f_0(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

of degree 1, which determines the degree 0 homomorphism

$$\tau_f : g(\mathbf{y}) \in S_m \to g(f_0(\mathbf{x}), \ldots, f_m(\mathbf{x})) \in S_n$$

(see Exercise 1.3.11). For each homogeneous polynomial $g \in S_m$ one has $\phi^{-1}(Z(g)) = Z(\tau_f(g))$, and this implies that ϕ is continuous. □

1.5 Projective Closure of Affine Sets

For all $i \in \{0, \ldots, n\}$, consider in $\mathbb{P}^n$ the closed subset $H_i = Z_p(x_i)$. We denote by U_i the open subset $\mathbb{P}^n \setminus H_i$. For all $i \in \{0, \ldots, n\}$, consider the well defined map

$$\phi_i : P = [p_0, \ldots, p_n] \in U_i \to (\frac{p_0}{p_i}, \ldots, \frac{p_{i-1}}{p_i}, \frac{p_{i+1}}{p_i}, \ldots, \frac{p_n}{p_i}) \in \mathbb{A}^n.$$

We introduce the following maps

$$\alpha : f(x_0, \ldots, x_n) \in S_n \to f(1, x_1, \ldots, x_n) \in A_n$$

which is called the *dehomogenizing operator*, and

$$\beta : g(x_1, \ldots, x_n) \in A_n \to x_0^d g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in S_n \quad \text{where} \quad d := \deg(g)$$

which is called the *homogenizing operator*.

Lemma 1.5.1 *One has:*

(i) *α is a homomorphism, whereas β is not, but β is* multiplicative*, i.e., $\beta(gh) = \beta(g)\beta(h)$ and it is also* additive*, i.e., $\beta(g+h) = \beta(g) + \beta(h)$ if g, h and $g+h$ have the same degree;*
(ii) *for any $g \in A_n$ of degree d, $\beta(g)$ is homogeneous of degree d;*
(iii) $\alpha \circ \beta = \mathrm{id}_{A_n}$*;*

(iv) x_0 *does not divide* f *if and only if* $\alpha(f)$ *has the same degree of* f*;*
(v) *for every homogeneous polynomial* $f \in S_n$*, if* x_0^m *is the maximal power of* x_0 *dividing* f*, then* $\beta(\alpha(f)) = \frac{f}{x_0^m}$.

Proof Parts (i), (iii) and (iv) are obvious. As for part (ii) apply (1.5). To prove (v), it suffices to do it when x_0 does not divide f, verifying that in this case $\beta(\alpha(f)) = f$. To prove this, taking into account (iv), it is enough to observe that for all monomials h of degree d, one has $x_0^d h(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = h$, and then apply this to every monomial appearing in f. □

Proposition 1.5.2 *The map* ϕ_i *is an homeomorphism for all* $i \in \{0, \dots, n\}$.

Proof We treat the case $i = 0$, the other cases being analogous. Set $\phi_0 = \phi$ and $U_0 = U$. It is clear that ϕ is bijective. We have to prove that ϕ and ϕ^{-1} are closed.

To prove that ϕ is closed, it suffices to prove that $\phi(Z_p(f) \cap U)$ is closed for all $f \in H(S_n)$. This is clear, because $\phi(Z_p(f) \cap U) = Z_a(\alpha(f))$. Similarly, $\phi^{-1}(Z_a(g)) = Z_p(\beta(g)) \cap U$, hence also ϕ^{-1} is closed. □

For every subset $X \subset \mathbb{P}^n$, we will set $X_i = X \cap U_i$, for $0 \leqslant i \leqslant n$. Then $\{X_i\}_{0 \leqslant i \leqslant n}$ is an open covering of X. If X is closed, then this is a covering of X with open subsets each homeomorphic to closed affine sets.

Often we will identify $\mathbb{A}^n$ with U_0 via the map ϕ_0. In this case H_0 is called the *hyperplane at infinity* and its points are called *points at infinity* of $\mathbb{A}^n$. If $X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n$, its closure $\bar{X}_p$ in $\mathbb{P}^n$ will be called the *projective closure* of X. One sets $X_\infty = \bar{X}_p \cap H_0$ and this is called the *set of points at infinity* of X.

Remark 1.5.3 One has

$$\bar{X}_p = \bigcap_{f \in H(S) : X \subset Z_a(\alpha(f))} Z_p(f).$$

Hence $\mathbb{A}^n$ is dense in $\mathbb{P}^n$. Indeed, $\mathbb{A}^n \subset Z_a(\alpha(f))$ if and only if $\alpha(f) = 0$. We can write

$$f(x_0, \dots, x_n) = f_0 x_0^d + f_1 x_0^{d-1} + \dots + f_d \quad \text{with} \quad f_i \in S_{n-1,i}$$

hence

$$\alpha(f) = f_0 + f_1 + \dots + f_d$$

where $f_0, \dots, f_d$ are the homogeneous components of $\alpha(f)$. Thus $\alpha(f) = 0$ implies $f_0 = \dots = f_d = 0$ and so $f = 0$.

Exercise 1.5.4 Prove that, if $Z \subset \mathbb{A}^n$ is closed, then $\bar{Z}_p = Z \cup Z_\infty$. Hence every closed set of $\mathbb{A}^n$ is the intersection of a closed set and an open set of $\mathbb{P}^n$. The intersection of a closed set and an open set in a topological space is called a *locally closed* set.

1.6 Examples

1.6.1 Points

In the Zariski topology of $\mathbb{A}^n$ the points are closed. In fact, if $P = (p_1, \ldots, p_n)$, then $\{P\} = Z(x_1 - p_1, \ldots, x_n - p_n)$.

Similarly, the points of $\mathbb{P}^n$ are closed. For example, if $P = [p_0, \ldots, p_n] \in \mathbb{P}^n$, and if $p_i \neq 0$ for $i = 0, \ldots, n$, then $\{P\} = Z(p_i x_0 - p_0 x_i, \ldots, p_i x_n - p_n x_i)$.

Thus, in the Zariski topology, all finite sets are closed. These are the only proper closed subsets of $\mathbb{A}^1$ and of $\mathbb{P}^1$. To prove this, it suffices to prove that every subset of the type $Z_a(f)$ [resp. $Z_p(f)$] with $f \in A_1$ [resp. with $f \in H(S_1)$] not zero is finite.

Since $\mathbb{K}$ is algebraically closed, by *Ruffini's theorem*, any non–zero $f \in A_1 = \mathbb{K}[x]$ of degree d can be written in a unique way as

$$f(x) = c \prod_{i=1}^{h} (x - p_i)^{m_i}$$

where $c \in \mathbb{K}^*$ is the *leading coefficient* of f, $p_1, \ldots, p_h$ are the distinct *roots* of f, and $m_1, \ldots, m_h$ are the corresponding *multiplicities*, and one has

$$d = m_1 + \ldots + m_h.$$

In particular, if $d = 0$, one has $f = c$ and $Z(f) = \emptyset$. If $d > 0$ one has $Z_a(F) = \{p_1, \ldots, p_h\} \subset \mathbb{A}^1$.

In the projective case, consider a non–zero $f \in S_1$, that is homogeneous of degree d. If $d = 0$, again $Z_p(f) = \emptyset$. If $d > 0$, f can be written as

$$f(x_0, x_1) = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \ldots + a_d x_1^d.$$

If $a_d \neq 0$, i.e., if f is not divisible by x_0, then $P_0 = [0, 1] \notin Z_p(f)$, and for any $[p_0, p_1] \in Z_p(f)$ one has $p_0 \neq 0$. Hence we may assume $p_0 = 1$ and p_1 is a solution of the equation

$$f(1, x) = a_0 + a_1 x + \ldots + a_d x^d = 0.$$

Since this equation has only finitely many solutions, then also $Z_p(f)$ is finite. If $a_d = 0$, f is divisible by x_0, and we can write $f = x_0^m g$ with $m > 0$ and g homogeneous of degree $d - m$, not divisible by x_0. Then $Z_p(f) = \{P_0\} \cup Z_p(g)$. Since $Z_p(g)$ is finite, then also $Z_p(f)$ is finite.

Remark 1.6.1 It is convenient to make a further remark. One has

$$f(x_0, x_1) = x_0^d f\left(1, \frac{x_1}{x_0}\right).$$

The polynomial $f(1,x)$ has degree $d' \leqslant d$ and the equality holds if and only if $a_d \neq 0$. The polynomial

$$g(x_0, x_1) = x_0^{d'} f\left(1, \frac{x_1}{x_0}\right)$$

is homogeneous of degree d', as we see by applying (1.5). Moreover we have

$$f(1,x) = c\prod_{i=1}^{h}(x - p_i)^{m_i}$$

with $c \in \mathbb{K}^*$ and $p_1, \ldots, p_h$ distinct roots of $f(1,x)$, with their multiplicities $m_1, \ldots, m_h$ so that $d' = m_1 + \ldots + m_h$. Thus

$$g(x_0, x_1) = cx_0^{d'}\prod_{i=1}^{h}\left(\frac{x_1}{x_0} - p_i\right)^{m_i} = c\prod_{i=1}^{h}(x_1 - p_i x_0)^{m_i}.$$

In conclusion we have

$$f(x_0, x_1) = cx_0^{d-d'}\prod_{i=1}^{h}(x_1 - p_i x_0)^{m_i}.$$

Hence every homogeneous polynomial of degree d in x_0, x_1 on $\mathbb{K}$ can be written as

$$f(x_0, x_1) = \prod_{i=1}^{h}(q_i x_1 - p_i x_0)^{m_i}$$

with $d = m_1 + \ldots + m_h$, and this expression is unique up to a non–zero multiplicative constant. The non–zero solutions of the equation $f = 0$ are (q_i, p_i), for $1 \leqslant i \leqslant h$, up to a proportionality factor, and $m_1, \ldots, m_h$ are called their *multiplicities*. The set $Z_p(f)$ consists of the points with homogeneous coordinates $[q_i, p_i]$, for $1 \leqslant i \leqslant h$.

Exercise 1.6.2 Let k be any field and $f, g \in A_{k,1}$ polynomials of degree at most d. Prove that if there are $d+1$ elements of $\mathbb{K}$ where f and g take the same value, then $f = g$.

Exercise 1.6.3 * Let k be any infinite field and $f \in A_{k,n}$ a polynomial. Let $\Sigma \subseteq \mathbb{A}_k^1$ be an infinite subset and assume that f is zero on $\Sigma^n \subseteq \mathbb{A}_k^n$. Then f is the zero polynomial.

1.6.2 Projective Subspaces

A subset $\mathbb{P}(W)$ of $\mathbb{P}(V)$, with W a vector subspace of dimension $m+1$ of the vector space V of dimension $n+1$, is called a *linear* or *projective subspace*, or simply a *subspace*, of $\mathbb{P}(V)$ of *dimension* m (in symbols $\dim(\mathbb{P}(W)) = m$) and *codimension* $c = n - m$ (in symbols $\mathrm{codim}(\mathbb{P}(W)) = n - m$). The empty set, corresponding to $W = (\mathbf{0})$, is the unique subspace of dimension -1. The points are the subspaces of dimension 0. The subspaces of codimension 1 are called *hyperplanes*.

Let us focus on the case of $\mathbb{P}^n$. The following properties are applications of basic notions of linear algebra:

1. $Z \subseteq \mathbb{P}^n$ is a subspace if and only if $Z = Z(f_1, \dots, f_h)$, with $f_1, \dots, f_h$ linear forms;
2. if $f, f_1, \dots, f_h$ are linear forms, then $Z(f_1, \dots, f_h) \subseteq Z(f)$ if and only if f linearly depends from $f_1, \dots, f_h$;
3. $Z(f_1, \dots, f_h) = Z(g_1, \dots, g_k)$ if and only if $f_1, \dots, f_h$ and $g_1, \dots, g_k$ span the same vector subspace of $S_{n,1}$;
4. $Z(f_1, \dots, f_h) = Z(f_{i_1}, \dots, f_{i_c})$, where $(f_{i_1}, \dots, f_{i_c})$ is a basis of the vector subspace of $S_{n,1}$ spanned by $f_1, \dots, f_h$ and c is the codimension of the subspace;
5. the intersection of a family of subspaces is a subspace;
6. if Z is a subset of $\mathbb{P}^n$, it makes sense to consider the minimum subspace of $\mathbb{P}^n$ containing Z. It is denoted by $\langle Z \rangle$ and it is called the *subspace spanned* or *generated* by Z. One says that Z is *non–degenerate* if $\langle Z \rangle = \mathbb{P}^n$, otherwise it is called *degenerate*. If $Z_1, \dots, Z_h$ are subspaces, one writes $Z_1 \vee \dots \vee Z_h$ instead of $\langle Z_1 \cup \dots \cup Z_2 \rangle$;
7. the *Grassmann formula* holds, i.e., if Z_1, Z_2 are subspaces of $\mathbb{P}^n$, one has
$$\dim(Z_1) + \dim(Z_2) = \dim(Z_1 \vee Z_2) + \dim(Z_1 \cap Z_2);$$
8. $S_{n,1}$ can be interpreted as the dual of $\mathbb{K}^{n+1}$ and the points of $\mathbb{P}(S_{n,1})$, which is also denoted by $\check{\mathbb{P}}^n$ and is called the *dual* of $\mathbb{P}^n$, can be interpreted as the hyperplanes of $\mathbb{P}^n$;
9. if $Z \subseteq \mathbb{P}^n$ is a subspace, one sets $Z^\perp = \{[f] \in \check{\mathbb{P}}^n : Z \subseteq Z(f)\}$. This is a subspace of $\check{\mathbb{P}}^n$, that is called the *orthogonal* of Z, and its dimension equals the codimension of Z;
10. one has
$$(Z^\perp)^\perp = Z, \quad (Z_1 \vee Z_2)^\perp = Z_1^\perp \cap Z_2^\perp, \qquad (Z_1 \cap Z_2)^\perp = Z_1^\vee \cap Z_2^\perp.$$

Proposition 1.6.4 *A projectivity $\phi : \mathbb{P}^n \to \mathbb{P}^m$ is a homeomorphism of $\mathbb{P}^n$ onto its image which is a subspace of dimension n of $\mathbb{P}^m$.*

Proof Let $f : \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ be an injective linear map determining ϕ. Set $V = f(\mathbb{K}^{n+1})$, which is a subspace of dimension $n+1$ of $\mathbb{K}^{m+1}$. Then the image of ϕ is the subspace $Z = \mathbb{P}(V)$ of $\mathbb{P}^m$. By proposition 1.4.2, it suffices to prove that ϕ is closed. Consider the map τ_f introduced in the proof of the Proposition 1.4.2. It is easy to see, and left to the reader as an exercise, that τ_f is surjective. If $h \in H(S_n)$ there is a $g \in H(S_m)$ such that $\tau_f(g) = h$. Since $\phi^{-1}(Z(g)) = Z(\tau_f(g)) = Z(h)$ (see the proof of Proposition 1.4.2), we have $\phi(Z(h)) = Z(g) \cap Z$, and this implies that ϕ is closed. □

If $Z = \mathbb{P}(W)$ is a subspace of dimension m of $\mathbb{P}^n$, we can construct a projectivity $\phi : \mathbb{P}^m \to Z$ as follows. Consider a basis $\mathbf{v}_0, \ldots, \mathbf{v}_m$ of W. Then the points $P_i = [\mathbf{v}_i] \in \mathbb{P}^n$, for $0 \leqslant i \leqslant m$, are said to be *linearly independent*, and this definition is well posed. Moreover $Z = P_0 \vee \ldots \vee P_m$. More precisely we can consider the map

$$\phi : [\lambda_0, \ldots, \lambda_m] \in \mathbb{P}^m \to [\lambda_0 \mathbf{v}_0 + \cdots + \lambda_m \mathbf{v}_m] \in Z.$$

This is a projectivity, and it sends the fundamental points of $\mathbb{P}^m$ to the points $P_0, \ldots, P_m$ and the unity point to the point $[\mathbf{v}_0 + \cdots + \mathbf{v}_m]$. This projectivity is also called a *parametric representation* of the subspace Z.

Exercise 1.6.5 * Let U by a non–empty open subset of $\mathbb{A}_{\mathbb{K}}{}^n$ or of $\mathbb{P}_{\mathbb{K}}{}^n$. Prove that if $U \subset Z(f)$ then f is the zero polynomial.

Exercise 1.6.6 Prove that projectivities send homeomorphically subspaces to subspaces of the same dimension.

Exercise 1.6.7 Prove the properties listed on Sect 1.6.2.

Exercise 1.6.8 Let $\phi : \mathbb{P}(V) \to \mathbb{P}(W)$ be a bijective projectivity determined by the isomorphism $f : V \to W$. The projectivity $\phi^t : \mathbb{P}(\check{W}) \to \mathbb{P}(\check{V})$ determined by the transpose map $f^t : \check{W} \to \check{V}$ is called the *transpose* of ϕ. Prove that for any subspace Z of $\mathbb{P}(V)$, one has $\phi^t(Z^{\perp}) = (\phi^{-1}(Z))^{\perp}$.

Exercise 1.6.9 * Prove the *fundamental theorem of projectivities*, which says the following. Let $\mathbb{P}_1$, $\mathbb{P}_2$ be projective spaces of the same dimension n. Let $(P_0, \ldots, P_{n+1})$ e $(Q_0, \ldots, Q_{n+1})$ two $(n+2)$–tuples of distinct points of $\mathbb{P}_1$ e $\mathbb{P}_2$ respectively, and suppose they are in *general position*, i.e., any $(n+1)$–tuple of points contained in them consists of linearly independent points, namely, they span $\mathbb{P}_1$ e $\mathbb{P}_2$ respectively. Then there is a unique projectivity $\phi : \mathbb{P}_1 \to \mathbb{P}_2$ such that $\phi(P_i) = Q_i$, for $0 \leqslant i \leqslant n+2$.

Exercise 1.6.10 Let $P_1, \ldots, P_4$ be distinct points on a projective line $\mathbb{P}(V)$. By Exercise 1.6.9, there is a unique homogeneous coordinate system on $\mathbb{P}(V)$ such that $P_1 = [1, 0]$, $P_2 = [0, 1]$, $P_3 = [1, 1]$. In this system one has $P_4 = [p, q]$, where p and q are not zero and $q \neq p$. The *cross ratio* (P_1, P_2, P_3, P_4) of $P_1, \ldots, P_4$ is, by definition $[p, q]$ or $\frac{q}{p} \in \mathbb{K}$. Note that the cross ratio is never 0 or 1. The 4–uple (P_1, P_2, P_3, P_4) will be said to be *harmonic* if $(P_1, P_2, P_3, P_4) = -1$.

Prove that two quadruples of distinct points P_1, P_2, P_3, P_4 of the line $\mathbb{P}(V)$ and Q_1, Q_2, Q_3, Q_4 of the line $\mathbb{P}(W)$ are *projective*, i.e., there is a projectivity $\phi : \mathbb{P}(V) \to \mathbb{P}(W)$ such that $\phi(P_i) = Q_i$, for $1 \leqslant i \leqslant 4$, if and only if $(P_1, P_2, P_3, P_4) = (Q_1, Q_2, Q_3, Q_4)$.

Exercise 1.6.11 Suppose we have four distinct points $P_i = [p_i, q_i]$ on $\mathbb{P}^1$, with $i = 1, \ldots, 4$. Prove that

$$(P_1, P_2, P_3, P_4) = \frac{(p_1q_4 - p_4q_1)(p_2q_3 - p_3q_2)}{(p_2q_4 - p_4q_2)(p_1q_3 - p_3q_1)}.$$

Exercise 1.6.12 * Let $\mathbb{P}(V)$ be a projective space of dimension n and Z a subspace of codimension c. Prove that, if we introduce a homogeneous coordinate system in $\mathbb{P}(V)$, there is a matrix $\mathbf{A}$ on $\mathbb{K}$ of type $(n+1) \times c$ and rank c such that Z is the set of points of $\mathbb{P}(V)$ with homogeneous coordinates $[\mathbf{x}]$ verifying the matrix equation $\mathbf{x} \cdot \mathbf{A} = \mathbf{0}$, which is equivalent to a system of c independent homogeneous linear equations in the indeterminates $\mathbf{x}$. This is called a *system of equations* of Z in the given coordinate system. Conversely, every set of this type is a subspace of codimension c.

If $\mathbb{P}(V) = \mathbb{P}^n$, one usually considers the natural coordinate system.

Exercise 1.6.13 * Let $\mathbb{P}(V)$ be a projective space of dimension n and Z a subspace of dimension m. Prove that, if we introduce a homogeneous coordinate system in $\mathbb{P}(V)$, there is a matrix $\mathbf{A}$ on $\mathbb{K}$ of type $(m+1) \times (n+1)$ of rank $m+1$ such that Z is the set of points of $\mathbb{P}(V)$ with homogeneous coordinates $[\mathbf{x}]$ of the form $\mathbf{x} = \boldsymbol{\lambda} \cdot \mathbf{A}$, with $[\lambda]$ variable in $\mathbb{P}^m$. Conversely any set of this form is a subspace of dimension m.

Exercise 1.6.14 * Consider a hyperplane H in an n–dimensional projective space $\mathbb{P}$. Let $U = \mathbb{P} \setminus H$. Prove that U is homeomorphic to $\mathbb{A}^n$.

1.6.3 Affine Subspaces

Let us think to $\mathbb{A}^n$ as the open subset U_0 of $\mathbb{P}^n$. A non–empty subset Z of $\mathbb{A}^n$ is called an *affine subspace*, or simply a *subspace* of $\mathbb{A}^n$, of *dimension* m and *codimension* $c = n - m$ if there is a projective subspace Z' of $\mathbb{P}^n$ of dimension m such that $Z = Z' \cap \mathbb{A}^n = Z' \cap U_0$. The empty set is considered as the only subspace of dimension -1. The subspaces of codimension 1 are called *hyperplanes*, the subspaces of dimension 1 *lines*, those of dimension 2 *planes*.

If Z' has equations

$$\begin{gathered} a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots \\ a_{c0}x_0 + \ldots + a_{cn}x_n = 0 \end{gathered} \tag{1.7}$$

(see Exercise 1.6.12), then the independent equations of Z are

$$\begin{gathered} a_{11}x_1 + \ldots + a_{1n}x_n + a_{10} = 0 \\ \ldots \\ a_{c1}x_0 + \ldots + a_{cn}x_n + a_{c0} = 0 \end{gathered} \tag{1.8}$$

that, in matrix form can be written as

$$\mathbf{x} \cdot \mathbf{A} + \mathbf{a} = \mathbf{0}$$

where $\mathbf{A} = (a_{ij})_{i=1,\ldots,n;\, j=1,\ldots,c}$ is a matrix of type $n \times c$ on $\mathbb{K}$ with rank c and $\mathbf{a} = (a_{10}, \ldots, a_{c0})$.

The system obtained by adding the equation $x_0 = 0$ to the system (1.7) defines the subspace $Z' \cap H_0$ of dimension $m - 1$ which is called the *direction space* of Z. Let $\boldsymbol{\xi}_0$ be a solution of the system (1.8), i.e., the coordinate vector of a point P_0 of Z. Let $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_m$ be independent solutions of the homogeneous system associated to (1.8). We can consider the bijective map

$$\phi_0 : (\lambda_1, \ldots, \lambda_m) \in \mathbb{A}^m \to \boldsymbol{\xi}_0 + \lambda_1 \boldsymbol{\xi}_1 + \cdots + \lambda_m \boldsymbol{\xi}_m \in Z.$$

This is the restriction to $\mathbb{A}^m$ of the map $\phi : \mathbb{P}^m \to Z'$ obtained as in Sect. 1.6.2 by choosing in Z' the $m+1$ points $P_0, P_1, \ldots, P_m$ with the homogeneous coordinates $[1, \boldsymbol{\xi}_0], [0, \boldsymbol{\xi}_i]$, with $1 \leqslant i \leqslant m$. Hence ϕ_0 is a homeomorphism of $\mathbb{A}^m$ on Z. It is called a *parametric representation* of Z, and can be interpreted as a way of introducing a *coordinate system* in Z, with the origin at the point P_0.

By Remark 1.5.3 it follows that Z' is the projective closure of Z, hence $Z_\infty = Z' \cap H_0$ is the set of points at infinity of Z.

Exercise 1.6.15 Let $n \geqslant m$, consider m integers $0 < i_1 < i_2 < \ldots < i_m \leqslant n$, and the map

$$\phi_{i_1,\ldots,i_m} : (x_1, \ldots, x_m) \in \mathbb{A}^m \to x_i \mathbf{e}_{i_1} + \cdots + x_m \mathbf{e}_{i_m} \in \mathbb{A}^n.$$

where $\mathbf{e}_i \in \mathbb{K}^n$ is the numerical vector with all entries 0 except the i–th entry which is 1, for $i = 1, \ldots, n$. Prove that $\phi_{i_1,\ldots,i_m}$ is the parametric representation of a subspace of $\mathbb{A}^n$ of dimension m. This is called the $(x_{i_1}, \ldots, x_{i_m})$*–coordinate subspace* (*coordinate axis* if $m = 1$).

Exercise 1.6.16 If Z_1, Z_2 are affine subspaces of $\mathbb{A}^n$, prove that $Z_1 \cap Z_2$ is again a subspace. If $Z_i = Z_i' \cap \mathbb{A}^n$, with Z_i' projective subspaces of $\mathbb{P}^n$, with $1 \leqslant i \leqslant m$, one defines the *joining subspace* or their *span* as $Z_1 \vee \ldots \vee Z_m = (Z_1' \vee \ldots \vee Z_m') \cap \mathbb{A}^n$. Prove that in general Grassmann formula does not hold in the affine setting and give conditions in order that it holds.

Exercise 1.6.17 * A map $\psi : \mathbb{A}^n \to \mathbb{A}^m$, con $n \leqslant m$, is called an *affinity*, if there is a projectivity $\psi : \mathbb{P}^n \to \mathbb{P}^m$ such that its restriction to $\mathbb{A}^n$ coincides with ψ. Prove that an affinity is a homeomorphism on its image and carries affine subspaces into affine subspaces of the same dimension.

Exercise 1.6.18 Prove that $\psi : \mathbb{A}^n \to \mathbb{A}^m$ is an affinity if and only if there is a matrix $\mathbf{A}$ on $\mathbb{K}$ of type $n \times m$ and rank n, and there is a point $\mathbf{a} \in \mathbb{A}^m$ such that for every $\mathbf{x} \in \mathbb{A}^n$ one has

$$\psi(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A} + \mathbf{a}. \tag{1.9}$$

Exercise 1.6.19 * Fix a matrix $\mathbf{A}$ on $\mathbb{K}$ of type $n \times m$ and a point $\mathbf{a} \in \mathbb{A}^m$. Consider the map $\psi : \mathbb{A}^n \to \mathbb{A}^m$ defined via (1.9). It is called an *affine map*. Prove that such a map is continuous and it is injective if and only if it is an affinity. Prove that it carries subspaces to subspaces, but in general it does not preserve the dimension.

Exercise 1.6.20 Let $n \geqslant m$ and consider m integers $0 < i_1 < i_2 < \ldots < i_m \leqslant n$, and the map

$$\pi^n_{i_1,\ldots,i_m} : (x_1, \ldots, x_n) \in \mathbb{A}^n \to (x_{i_1}, \ldots, x_{i_m}) \in \mathbb{A}^m.$$

Prove that this is a surjective affine map, it is called the *projection* of $\mathbb{A}^n$ onto $\mathbb{A}^m$ from the space at infinity of the variables $j_1, \ldots, j_{n-m}$ where $\{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. What are the counterimages of the points of $\mathbb{A}^m$?

Exercise 1.6.21 Prove that the affinities of $\mathbb{A}^n$ in itself form a group, called he *affine group* of $\mathbb{A}^n$.

Exercise 1.6.22 Prove that an affine map $\psi : \mathbb{A}^n \to \mathbb{A}^m$ is a homomorphism of $\mathbb{K}^n$ to $\mathbb{K}^m$ if and only if $\psi(\mathbf{0}) = \mathbf{0}$.

1.6.4 Hypersurfaces

Let $f \in A_n$ be a non–zero polynomial. The set $Z(f) \subset \mathbb{A}^n$ is called a *hypersurface* and $f = 0$ is an *equation* of it. Note that the empty set is a hypersurface with equation $1 = 0$, which is called the 0*–hypersurface*.

Two non–zero polynomials $f_1, f_2 \in A_n$ are said to be *essentially distinct* if there is no element $t \in \mathbb{K}^*$ such that $f_1 = tf_2$, and are called *essentially equal* otherwise. If f_1 and f_2 are essentially equal one has $Z(f_1) = Z(f_2)$. Let us see when it is the case that $Z(f_1) = Z(f_2)$.

Since A_n is a unique factorization domain (abbreviated in UFD), for every non–zero polinomial $f \in A_n$, one has

$$f = f_1^{r_1} \cdots f_h^{r_h} \tag{1.10}$$

where $f_1, \ldots, f_h$ are irreducible polynomials, pairwise essentially distinct, $r_1, \ldots, r_h$ are positive integers, and (1.10) is *essentially unique*, i.e., in two expressions of f of this sort the f_i can change only by the product by an element of $\mathbb{K}^*$, but the integers $r_1, \ldots, r_h$ cannot change. The polynomials $f_1, \ldots, f_h$ are called the *irreducible factors* of f and $r_1, \ldots, r_h$ are the corresponding *multiplicities*. The Eq. (1.10) is called the *decomposition* of f into irreducible factors, and we have

$$\deg(f) = r_1 \deg(f_1) + \cdots + r_h \deg(f_h). \tag{1.11}$$

One has

$$Z(f) = \bigcup_{i=1}^{h} Z(f_i)$$

hence if $s_1, \ldots, s_h$ are positive integers, we still have $Z(f_1^{s_1} \cdots f_h^{s_h}) = Z(f)$. In particular $f_1 \cdots f_h = 0$ is still an equation of $Z(f)$, which is called the *reduced equation* of $Z(f)$ and the polynomial $f_1 \cdots f_h$ is said to be *reduced*.

Hilbert's Nullstellensatz, which we will prove later (see Theorem 2.5.2 below), implies the:

Proposition 1.6.23 (Study's Principle) *If $f \in A_n$ is an irreducible, non–zero polynomial and $g \in A_n$ is a polynomial such that $Z(f) \subseteq Z(g)$, then f divides g.*

Therefore $Z(f) = Z(g)$ if and only if they have the same irreducible factors, which can differ only for the multiplicities. Hence the reduced equation of a hypersurface $Z(f)$, with f as in (1.10), is essentially unique. The hypersurfaces $Z(f_i)$ are called the *irreducible components* of the hypersurface $Z(f)$, and this is said to be *irreducible* if it has a unique irreducible component. The degree of the reduced equation of the hypersurface Z is called the *degree* of Z, it is denoted by $\deg(Z)$, and it is the sum of the degrees of its irreducible components. The hypersurfaces of degree 1 are the hyperplanes (points if $n = 1$, *lines* if $n = 2$, *planes* if $n = 3$). The hypersurfaces of degree 2 are called *quadrics* (pairs of points if $n = 1$, *conics* if $n = 2$). The hypersurfaces of degree 3, 4, etc. are called *cubics*, *quartics*, etc. For $n = 2$ the hypersurfaces are called *curves*, for $n = 3$ *surfaces*.

Suppose now that $f \in S_n$ is homogeneous and non–zero. The closed subset $Z(f) \subset \mathbb{P}^n$ is called a *hypersurface* of $\mathbb{P}^n$. Note that, for a change of variables, this polynomial changes for an invertible linear substitution of variables, which is a homogeneous isomorphism of S_n into itself.

Consider now that factorization (1.10). Since f is homogeneous, the factors $f_1, \ldots, f_h$ are homogeneous as well (see Exercise 1.3.13). From this it follows that all what we said about the hypersurfaces in $\mathbb{A}^n$ can be repeated verbatim for the hypersurfaces in a projective space, and for them we will use an analogous terminology.

1.6.5 Divisors

Let X be an affine space $\mathbb{A}^n$ or a projective space $\mathbb{P}^n$. We will denote by $\mathrm{Div}(X)$ the free abelian group generated by the set $\mathcal{H}$ of irreducible hypersurfaces of X. Every element of $\mathrm{Div}(X)$ is of the form $D = \sum_{Z\in\mathcal{H}} r_Z Z$ where the r_Z are integers that are different from 0 only for a finite number of elements $Z \in \mathcal{H}$. Such a D is called a *divisor* of X, r_Z is called the *multiplicity* of Z in D and the Z such that $r_Z \neq 0$ are called the *irreducible components* of D. The hypersurface $\mathrm{Supp}(D) := \bigcup_{r_Z\neq 0} Z$ is called the *support* of D and it is sometimes identified with the divisor $\sum_{r_Z\neq 0} Z$. If $r_Z \in \{-1, 0, 1\}$ for all $Z \in \mathcal{H}$, then D is called *reduced*. We define the *degree* of D as $\deg(Z) = \sum_{Z\in\mathcal{H}} r_Z \deg(Z)$. The divisor $D = \sum_{Z\in\mathcal{H}} r_Z Z$ is called *effective*, or a *generalised hypersurface* (or simply a *hypersurface* if no confusion arises), if $r_Z \geqslant 0$ for all $Z \in \mathcal{H}$.

An effective divisor has non–negative degree and it has degree 0 if and only if it is the *zero divisor*, i.e., the 0 element of the group $\mathrm{Div}(X)$. An effective divisor D consisting of a unique $Z \in \mathcal{H}$ is said to be *irreducible*.

If $f_Z = 0$, with f_Z irreducible, is an equation of $Z \in \mathcal{H}$, which is uniquely defined up to a constant factor, then given the effective divisor $D = \sum_{Z\in\mathcal{H}} r_Z Z$ and set $f_D = \prod_{Z\in\mathcal{H}} f_Z^{r_Z}$, we say that $f_D = 0$ is an equation of D. By the Nullstellensatz (see Theorem 2.5.2 below), one has that if $f = 0$ and $g = 0$ are equations of D then f and g are essentially equal.

Let us now focus on the case $X = \mathbb{P}^n$. Let D be an effective divisor and let Π be a subspace of dimension m of $\mathbb{P}^n$. Suppose that D has equation $f = 0$ and that Π has a parametric representation as $\mathbf{x} = \boldsymbol{\lambda} \cdot \mathbf{A}$, where $\mathbf{A}$ is a matrix of type $(m+1) \times (n+1)$ and rank $m+1$, $\mathbf{x} = (x_0, \ldots, x_n)$ and $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_m)$ (see Exercise 1.6.13). The polynomial $f(\lambda \cdot \mathbf{A})$ is zero if and only if $\Pi \subseteq \mathrm{Supp}(D)$. In this case we say that D contains Π. If this is not the case, then the equation $f(\lambda \cdot \mathbf{A}) = 0$ defines a divisor D_Π in Π, which is called *intersection* of Π with D, and one has $\deg(D_\Pi) = \deg(D)$. The definition of D_Π is well posed (see Exercise 1.6.28).

In the case $X = \mathbb{A}^n$, we can make the same considerations and give the same definitions. The only difference is that $\deg(D_\Pi) \leqslant \deg(D)$ and strict inequality can hold.

Hence we have:

Proposition 1.6.24 (Bézout Theorem for linear sections) *Let D be an effective divisor in an affine or projective space, and let Π be a subspace. Then either Π is contained in D or D intersects Π in an effective divisor of degree at most equal to* $\deg(D)$*, and exactly equal to* $\deg(D)$ *in the projective case.*

Example 1.6.25 If we are in the projective case, D has degree d and ℓ is a line not contained in D, then $D_\ell = m_1 P_1 + \cdots + m_h P_h$, with P_i distinct points of ℓ and m_i positive integers, with $1 \leqslant i \leqslant h$. One has $d = m_1 + \cdots + m_h$. The positive integer m_i is called the *intersection multiplicity* of ℓ and D at P_i, and it is denoted by $m_{P_i}(D, \ell)$, for $i = 1, \ldots, h$. We set $m_P(D, \ell) = 0$ if $P \notin \mathrm{Supp}(D_\ell)$ and $m_P(D, \ell) = \infty$ for all $P \in \ell$, if ℓ is contained in D.

Let us consider again the case $X = \mathbb{P}^n$. Fix a positive integer d and let us consider the set $\mathcal{L}_{n,d}$ of all effective divisors of degree d of $\mathbb{P}^n$. One has $\mathcal{L}_{n,d} = \mathbb{P}(S_{n,d})$, hence

$$\dim(\mathcal{L}_{n,d}) = \binom{n+d}{n} - 1.$$

If the divisor $D \in \mathcal{L}_{n,d}$ has equation $f = 0$ with f a form, uniquely determined up to a non–zero constant factor, of the type

$$f(\mathbf{x}) = \sum_{|\mathbf{i}|=d} f_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$$

then $[f_{\mathbf{i}}]_{|\mathbf{i}|=d}$ (the indices are lexicographically ordered) are in a natural way homogeneous coordinates of D in $\mathcal{L}_{n,d}$.

A subspace of dimension r of $\mathcal{L}_{n,d}$ is called a *linear system* of divisors, or simply of hypersurfaces, of degree d in $\mathbb{P}^n$. A linear system of dimension 0 is a unique divisor, a system of dimension 1 is called a *pencil*, a system of dimension 2 is called a *net*. The empty system has dimension -1.

Example 1.6.26 One has $\dim(\mathcal{L}_{n,1}) = n$ and the points of $\mathcal{L}_{n,1}$ are in 1:1 correspondence with the hyperplanes of $\mathbb{P}^n$. Then $\mathcal{L}_{n,1}$ is denoted by $\check{\mathbb{P}}^n$ and it is called the *dual* of $\mathbb{P}^n$. A line in $\check{\mathbb{P}}^n$ corresponds to the set of hyperplanes containing a fixed subspace of codimension 2. This is called a pencil of hyperplanes. Similarly, a plane in $\check{\mathbb{P}}^n$ corresponds to a net of hyperplanes containing a subspace of codimension 3.

Let $\mathcal{L} \subseteq \mathcal{L}_{n,d}$ be a linear system of dimension r. If $D_0, \ldots, D_r \in \mathcal{L}$ are linearly independent divisors, with equations $f_i = 0$, for $0 \leqslant i \leqslant r$, in the given coordinate system, then a divisor D sits in $\mathcal{L}$ if and only if it has equation of the form $\lambda_0 f_0 + \cdots + \lambda_r f_r = 0$, with $[\lambda_0, \ldots, \lambda_r] \in \mathbb{P}^r$. Note that $Z(f_0, \ldots, f_r) = \bigcap_{Z \in \mathcal{L}} Z$. This is called the *base locus* of the linear system $\mathcal{L}$ and it is denoted by $\mathrm{Bs}(\mathcal{L})$. If $P \in \mathbb{P}^n$, we will denote by $\mathcal{L}(-P)$ the set of all divisors in $\mathcal{L}$ with support containing P. One has $\mathcal{L}(-P) = \mathcal{L}$ if and only if $P \in \mathrm{Bs}(\mathcal{L})$. If $P \notin \mathrm{Bs}(\mathcal{L})$, then $\mathcal{L}(-P)$ is a linear system of codimension 1 in $\mathcal{L}$ (see Exercise 1.6.31).

Example 1.6.27 A linear system $\mathcal{L}$ of dimension n of $\mathcal{L}_{1,d}$ is called a *linear series* of *degree* d and *dimension* n and it is denoted with the symbol g^n_d. The series is *complete* if $\mathcal{L} = \mathcal{L}_{1,d}$, i.e., if $n = d$.

There is a unique complete g^1_1, formed by all effective divisors of degree 1 on the line, and they can be identified with the points of the line.

Let us determine all linear series g^1_2. There are two homogeneous, non–proportional degree 2 forms f_0, f_1 in x_0, x_1 such that the elements of the g^1_2 are precisely the divisors defined by an equation of the form $\lambda_0 f_0 + \lambda_1 f_1 = 0$, with λ_0, λ_1 not both zero.

The two divisors D_i with equations $f_i = 0$, for $0 \leqslant i \leqslant 1$, cannot have the same support, otherwise the polynomials f_0, f_1 would be proportional. A first case is the one in which D_1 and D_2 have a common point P. Then P is a *base point* of the g^1_2, and the divisors of the g^1_2 are the ones of the form $P + Q$ with Q varying on $\mathbb{P}^1$.

Suppose next that D_1, D_2 have disjoint supports, i.e., the g^1_2 has no base points. Then the g^1_2 is formed by divisors of the form $P_1 + P_2$ such that for any point P of $\mathbb{P}^1$ there is a unique point Q such that $P + Q \in g^1_2$. Then the g^1_2 determines a map $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ which sends a point $P \in \mathbb{P}^1$ to the aforementioned point Q. The map σ is bijective and it is an *involution*, i.e., $\sigma^{-1} = \sigma$. For this reason the base point free linear series g^1_2 are called *involutions*.

The map σ is a projectivity and conversely any involutory projectivity of $\mathbb{P}^1$ is of this type (see Exercise 1.6.33).

Exercise 1.6.28 Prove that the definition of D_Π does neither depend on the parametric representation of Π nor on the homogeneous coordinate system.

Exercise 1.6.29 Give an example of an effective divisor D in $\mathbb{A}^n$ such that its intersection with a subspace Π has degree strictly smaller than $\deg(D)$.

Exercise 1.6.30 Prove that the linear systems of dimension m of hyperplanes in a projective space of dimension n are precisely the sets of all hyperplanes containing a given projective subspace of dimension $n - m - 1$.

Exercise 1.6.31 * Prove that if $P \notin \mathrm{Bs}(\mathcal{L})$, then $\mathcal{L}(-P)$ is a linear system of codimension 1 in $\mathcal{L}$.

Exercise 1.6.32 * Let $\mathcal{L}$ be a linear system of dimension r of divisors of $\mathbb{P}^n$. Let Z be a subset of $\mathbb{P}^n$. Set $\mathcal{L}(-Z) = \{D \in \mathcal{L} : Z \subset \mathrm{Supp}(D)\}$. Prove that $\mathcal{L}(-Z)$ is a sublinear system of $\mathcal{L}$ and that $\mathcal{L}(-Z) = \mathcal{L}$ if and only if $Z \subseteq \mathrm{Bs}(\mathcal{L})$.

Prove that if $Z = \{P_1, \ldots, P_h\}$, then $\dim(\mathcal{L}(-Z)) \geqslant r - h$ ($\mathcal{L}(-Z)$ is also denoted by $\mathcal{L}(-P_1 - \cdots - P_h)$). Prove that for all positive integers h there are distinct points $P_1, \ldots, P_h$ such that $\dim(\mathcal{L}(-P_1 - \cdots - P_h)) = \max\{-1, r - h\}$.

Exercise 1.6.33 * Considering the Example 1.6.27, prove that the map σ determined by a base point free g^1_2 is an involutory projectivity and that all involutory projectivities are of this form.

Deduce that, if $\mathrm{char}(\mathbb{K}) \neq 2$, there are exactly two distinct points P_1, P_2 on $\mathbb{P}^1$ such that $2P_i \in g^1_2$, for $1 \leqslant i \leqslant 2$. They are called *ramification points* of the g^1_2.

Prove also that $Q = \sigma(P)$ if and only if $(P_1, P_2, P, Q) = -1$, i.e., if and only if P_1, P_2, P, Q is a harmonic quadruple.

1.6.6 Product Topology

The affine space $\mathbb{A}^{n+m}$ coincides with $\mathbb{A}^n \times \mathbb{A}^m$ (with $n, m > 0$). Hence $\mathbb{A}^{n+m}$ has the Zariski topology but also the product topology of the Zariski topologies of $\mathbb{A}^n$ and $\mathbb{A}^m$. The latter topology is less fine than the former. Indeed, if $Z_1 \subseteq \mathbb{A}^n$ and $Z_2 \subseteq \mathbb{A}^m$ are closed subsets, then $Z_1 \times Z_2$ is closed in $\mathbb{A}^{n+m}$. In fact, suppose that $Z_1 = Z(F_1)$, with $F_i \in A_n = \mathbb{K}[x_1, \ldots, x_n]$ and $Z_2 = Z(F_2)$, with $F_2 \in A_m = \mathbb{K}[y_, \ldots, y_m]$. We have the inclusions

$$A_n = \mathbb{K}[x_1, \ldots, x_n] \to A_{n+m} = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

$$A_n = \mathbb{K}[y_1, \ldots, y_n] \to A_{n+m} = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$$

hence $F_1 \cup F_2$ can be considered as a subset of A_{n+m} and we have $Z_1 \times Z_2 = Z(F_1 \cup F_2)$.

If $Z \subset \mathbb{A}^n$ is a closed subset, $Z \times \mathbb{A}^m$ is called the *cylinder* with *directrix* Z.

Exercise 1.6.34 Prove that the Zariski topology on $\mathbb{A}^2$ is strictly finer than the product topology. Similarly for $\mathbb{A}^{n+m}$.

Exercise 1.6.35 Prove that if $Z_1 \subseteq \mathbb{A}^n$ and $Z_2 \subseteq \mathbb{A}^m$, the topology induced by the product topology on $Z_1 \times \{P\} = Z_1$, with $P \in Z_2$ is the Zariski topology on Z_1.

1.7 Solutions of Some Exercises

1.1.4 One has $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$. If $Z_i = Z(F_i)$, for $1 \leqslant i \leqslant 2$, then $Z_1 \cup Z_2 = Z(F_1 \cdot F_2)$ where $F_1 \cdot F_2 := \{f_1 f_2 : f_1 \in F_1, f_2 \in F_2\}$. If $Z_i = Z(F_i)$ is a family of closed subsets depending on $i \in \mathcal{I}$, then $\bigcap_{i \in \mathcal{I}} Z_i = Z(\bigcup_{i \in \mathcal{I}} Z_i)$.

1.3.5 One of the implications is obvious. As for the other, let $\{f_\ell\}_{\ell \in L}$ be a family of homogeneous generators of I. If $f \in I$ then $f = \sum_{\ell_1, \ldots, \ell_m} f_{\ell_1, \ldots, \ell_m} f_{\ell_1} \cdots f_{\ell_m}$ with $f_{\ell_1, \ldots, \ell_m} \in S$. Consider the decomposition in homogeneous components $f_{\ell_1, \ldots, \ell_m} = \sum_i f_{\ell_1, \ldots, \ell_m, i}$. Then $f = \sum_{\ell_1, \ldots, \ell_m} \sum_i f_{\ell_1, \ldots, \ell_m, i} f_{\ell_1} \cdots f_{\ell_m}$ is the decomposition in homogeneous components of f, and each of such components is in I.

1.3.7 The first part of the assertion follows from Proposition 1.3.1. As for (i), suppose that $f^r \in I$ with r a positive integer. Consider the decomposition in homogeneous components $f = f_\ell + o(\ell)$, with $o(\ell) \in S_{>\ell}$. Then $f^r = f_\ell^r + o(r\ell)$. Hence f_ℓ^r is a homogeneous component of f^r. Since I is homogeneous, one has $f_\ell^r \in I$, thus $f_\ell \in \mathrm{rad}(I)$. The assertion is proved by iterating this argument. The proof of (ii) is similar.

1.3.9 One implication is obvious. As for the other, suppose that (1.5) holds. Let $f = f_{d_1} + \ldots + f_{d_h}$ be the decomposition in homogeneous components of f, with $f_{d_i} \in S_{d_i} \setminus \{0\}$ for $1 \leqslant i \leqslant h$ and $d_1, \ldots, d_h \in \mathbb{N}$ distinct. Then $t^d f(\mathbf{x}) = f(t\mathbf{x}) = f_{d_1}(t\mathbf{x}) + \ldots + f_{d_h}(t\mathbf{x}) = t^{d_1} f_{d_1}(\mathbf{x}) + \ldots + t^{d_h} f_{d_h}(\mathbf{x})$ for $t \in \mathbb{K}$. This is a polynomial identity in $x_0, \ldots, x_n$ and t. The assertion follows.

1.3.13 If $f = gh$, write $g = g_a + m(a)$ and $h = h_b + m(b)$, where g_a [resp. h_b] is the homogeneous component of maximum degree of g [resp. of h] and $m(a)$ [resp. $m(b)$] stays for a polynomial of degree smaller than a [resp. than b]. Then $f = g_a h_b + m(a + b)$. Since f is homogeneous, one has $f = g_a h_b$ hence $m(a) = m(b) = 0$.

1.3.14 The first part of the exercise is trivial. As for the required example, consider S_n as a module over itself. Consider $S_n(1)$. One has $S_n(1)_d = S_{n,d+1}$. Then S_n is not isomorphic to $S_n(1)$ because $\dim(S_{n,d}) \neq \dim(S_n(1)_d)$.
1.5.4 One has $Z = \bar{Z}_p \cap \mathbb{A}^n$.
1.6.3 Proceed by induction on the number n of variables. If $n = 1$ the assertion follows by Ruffini's Theorem. Suppose $n > 1$. If f does not depend on any of the variables $x_1, \dots, x_n$, the assertion is clear. So we may assume that f depends of the variable x_1. Fix $c \in \Sigma$. The polynomial $f(c, x_2, \dots, x_n)$ vanishes on Σ^{n-1} and, by induction, it is zero. Then $f(x_1, x_2, \dots, x_n)$, as a polynomial in x_1 over the field $k(x_2, \dots, x_n)$ has infinitely many solutions, so it is zero, as wanted.
1.6.5 Consider the affine case, the projective one is analogous. If $U = \mathbb{A}^n$ the assertion follows by the Identity Principle of polynomials. If $U \subset \mathbb{A}^n$ is a proper open subset, it suffices to show that $U \subseteq Z(f)$ implies that $\mathbb{A}^n \subseteq Z(f)$. If $P \in \mathbb{A}^n - U$, consider a line ℓ passing through P which intersects U. Then the restriction of f to ℓ vanishes on a non–empty open subset of ℓ which consists of infinitely many points, hence it vanishes on ℓ and therefore on P. The assertion follows.
1.6.11 Consider the projectivity $\omega : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\omega([x_0, x_1]) = [y_0, y_1]$, with $y_0 = (p_2x_1 - x_0q_2)(p_1q_3 - p_3q_1)$, $y_1 = (p_1x_1 - x_0q_1)(p_2q_3 - p_3q_2)$. Remark that ω sends P_1 to $[1, 0]$, P_2 to $[0, 1]$, P_2 to $[1, 1]$, and therefore P_4 to (P_1, P_2, P_3, P_4).
1.6.33 We treat the case $\mathrm{char}(\mathbb{K}) \neq 2$, the case $\mathrm{char}(\mathbb{K}) = 2$ can be treated in a similar way, but it requires a bit more care, and is left to the reader. Suppose a g^1_2 is determined by the two polynomials $f_0(x_0, x_1) = a_0x_0^2 + a_1x_0x_1 + a_2x_1^2$, $f_1(x_0, x_1) = b_0x_0^2 + b_1x_0x_1 + b_2x_1^2$ which have no common solution. So the g^1_2 is the family of divisors with equations

$$(\lambda a_0 + \mu b_0)x_0^2 + (\lambda a_1 + \mu b_1)x_0x_1 + (\lambda a_2 + \mu b_2)x_1^2 = 0 \tag{1.12}$$

with $[\lambda, \mu] \in \mathbb{P}^1$. Among these divisors there is certainly at least one which is non–reduced. In fact such a non–reduced divisor corresponds to $[\lambda, \mu]$ such that (1.12) has a solution with multiplicity 2, and this happens if and only if

$$(\lambda a_1 + \mu b_1)^2 - 4(\lambda a_0 + \mu b_0)(\lambda a_2 + \mu b_2) = 0 \tag{1.13}$$

which certainly has some solution in λ, μ. So, up to a change of coordinates, we may assume that for instance $f_0 = x_0^2$, so that (1.13) becomes $\mu(\mu(b_1^2 - 4b_0) - 4\lambda) = 0$. This has the solution $\mu = 0$, which corresponds to f_0 and the solution $\mu = 4$, $\lambda = b_1^2 - 4b_0$ (up to a factor), which corresponds to another polynomial. So, up to a new change of coordinates, we may assume that $f_1 = x_1^2$, and the g^1_2 is the set of divisors defined by the equation

$$\lambda x_0^2 + \mu x_1^2 = 0. \tag{1.14}$$

In this situation it is immediate to see that the map σ sends the point $P = [p, q]$ to the point $Q = [-p, q]$, which proves that σ is an involutory projectivity.

Conversely, let σ be an involutory projectivity different from the identity. Then if $\sigma([x_0, x_1]) = [y_0, y_1]$, there are $a, b, c, d \in \mathbb{K}$ such that $y_0 = ax_0 + bx_1$, $y_1 = cx_0 + dx_1$, with $ad - bc \neq 0$. It is easy to see that the involutory condition is equivalent to $a + d = 0$. Using this, one sees that for an involutory projectivity on $\mathbb{P}^1$ there are exactly two distinct points P such that $\sigma(P) = P$. By changing coordinates one may assume that these points are $P_1 = [1, 0]$ and $P_2 = [0, 1]$, in which case the projectivity has the form $\sigma([x_0, x_1]) = [-x_0, x_1]$. Then the divisors of the type $P + \sigma(P)$ have Eq. (1.14) with $[\lambda, \mu] \in \mathbb{P}^1$ varying, so they form a g^1_2.

The final assertion is a direct verification with $P_1 = [1, 0]$, $P_2 = [0, 1]$, $P = [p, q]$, $Q = [-p, q]$.

1.6.34 The line with equation $x_1 + x_2 = 0$ is not closed in the product topology.

Chapter 2
Basic Notions of Elimination Theory and Applications

2.1 The Resultant of Two Polynomials

Let k be any, not necessarily algebraically closed, field. We will denote by $\mathbb{K}$ its algebraic closure. A system of algebraic equations

$$f_i(x_1 \dots, x_n) = 0, \quad f_i \in A_{k,n}, \quad i \in \mathcal{I}$$

is said to be *compatible* if $\bigcap_{i \in \mathcal{I}} Z(f_i) \neq \emptyset$ in $\mathbb{A}^n_{\mathbb{K}}$.

Let

$$f(x) = a_0 x^n + \cdots + a_n, \quad g(x) = b_0 x^m + \cdots + b_m \tag{2.1}$$

be non–zero polynomials on k. The system

$$f = 0, \quad g = 0 \tag{2.2}$$

is compatible if and only if the greatest common divisor D of f and g has positive degree. Since $D \in k[x]$, we have:

Lemma 2.1.1 *The system (2.2) is compatible if and only if f and g have some common divisor of positive degree in $k[x]$.*

This implies the:

Lemma 2.1.2 (Euler Lemma) *Suppose that either $a_0 \neq 0$ or $b_0 \neq 0$. Then the system* (2.2) *is compatible if and only if there are non–zero polynomials $p(x), q(x) \in k[x]$, with* $\deg(p) < m$ *and* $\deg(q) < n$*, such that*

$$pf = qg. \tag{2.3}$$

Proof If (2.2) is compatible, one has

$$f = \phi q, \quad g = \phi p$$

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
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with $\phi \in k[x]$ of positive degree. Hence p, q verify the assertion.

Conversely, suppose that (2.3) holds with $\deg(p) < m$ and $\deg(q) < n$. Suppose, for example, $a_0 \neq 0$. If f has no common factor of positive degree with g, then f has to divide q, and this is impossible because $\deg(q) < n = \deg(f)$. □

The existence of the polynomials p, q verifying (2.3) is equivalent to the existence of $n + m$ elements of k

$$c_i, \quad 0 \leqslant i \leqslant m-1, \quad \text{not all zero}, \quad d_i, \quad 0 \leqslant i \leqslant n-1, \quad \text{not all zero}$$

such that

$$(c_0 x^{m-1} + \cdots + c_{m-1}) f(x) = (d_0 x^{n-1} + \cdots + d_{n-1}) g(x), \tag{2.4}$$

i.e., satisfying the system of $n + m$ equations

$$\begin{aligned} &a_0 c_0 = b_0 d_0 \\ &a_1 c_0 + a_0 c_1 = b_1 d_0 + b_0 d_1 \\ &\dots \\ &a_n c_{m-1} = b_m d_{n-1} \end{aligned} \tag{2.5}$$

Note that, if $c_0, \dots, c_{m-1}, d_0, \dots, d_{n-1}$ verify (2.4), or equivalently, (2.5), and are not all zero, then neither $c_0, \dots, c_{m-1}$, nor $d_0, \dots, d_{n-1}$, are all zero. In conclusion, if either $a_0 \neq 0$ or $b_0 \neq 0$, the system (2.2) is compatible if and only if the homogeneous linear system (2.5) of $n + m$ equations in the $n + m$ indeterminates $c_0, \dots, c_{m-1}, d_0, \dots, d_{n-1}$ has a non–trivial solution. This happens if and only if the matrix of the system has zero determinant, i.e., if and only if

$$\begin{vmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_n & 0 & \dots & 0 \\ & \dots & & & & & & \\ 0 & \dots & 0 & 0 & a_0 & a_1 & \dots & a_n \\ b_0 & b_1 & \dots & b_m & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_m & 0 & \dots & 0 \\ & \dots & & & & & & \\ 0 & \dots & 0 & 0 & b_0 & b_1 & \dots & b_m \end{vmatrix} = 0 \tag{2.6}$$

where the block where $a_0, \dots, a_n$ appear consists of m rows and the one in which $b_0, \dots, b_m$ appear consists of n rows. The first member of (2.6) is called the *Sylvester determinant* of f and g. Its vanishing is equivalent either to $a_0 = b_0 = 0$ or to the system (2.2) being compatible.

Consider now $a_0, \dots, a_n, b_0, \dots, b_m$ as indeterminates on the *fundamental field* $\mathbb{F}$ of k, which is $\mathbb{Q}$ if $\mathrm{char}(k) = 0$, and the finite field $\mathbb{F}_p$ with p elements if $\mathrm{char}(k) = p$. Then the Sylvester determinant can be considered as a polynomial

$R(a_0, \dots, a_n, b_0, \dots, b_m) = R(\mathbf{a}, \mathbf{b})$ in $\mathbb{F}[\mathbf{a}, \mathbf{b}]$. This polynomial is called the *resultant polynomial* of type (n, m).

Proposition 2.1.3 *The resultant polynomial $R(\mathbf{a}, \mathbf{b})$ belongs to the ideal generated by f and g. Precisely, there are polynomials $A, B \in \mathbb{F}[\mathbf{a}, \mathbf{b}, x]$ with degrees at most $m-1$ and $n-1$ in x respectively, such that*

$$Af + Bg = R \tag{2.7}$$

Proof One has the relations

$$\begin{array}{lcl}
x^{m-1} f(x) & = & a_0 x^{n+m-1} + \cdots + a_n x^{m-1} \\
x^{m-2} f(x) & = & \quad a_0 x^{n+m-2} + \cdots + a_n x^{m-2} \\
\dots & & \\
f(x) & = & \qquad\qquad a_0 x^n + \cdots + a_n \\
x^{n-1} g(x) & = & b_0 x^{n+m-1} + \cdots + b_m x^{n-1} \\
x^{n-2} g(x) & = & \quad b_0 x^{n+m-2} + \cdots + b_m x^{n-2} \\
\dots & & \\
g(x) & = & \qquad\qquad b_0 x^m + \cdots + b_m
\end{array}$$

By multiplying each of these by the cofactor of the corresponding element of the last column of R, and then adding up, one obtains (2.7). □

Exercise 2.1.4 Consider two homogeneous polynomials on k

$$f(x_0, x_1) = a_0 x_0^n + \ldots + a_n x_1^n, \quad g(x_0, x_1) = b_0 x_0^m + \ldots + b_m x_1^m.$$

Prove that $R(\mathbf{a}, \mathbf{b}) = 0$ if and only if $Z_p(f, g) \neq \emptyset$ in $\mathbb{P}^1_{\mathbb{K}}$.

Exercise 2.1.5 Prove that $R(\mathbf{a}, \mathbf{b})$ is bihomogeneous of degree m in $\mathbf{a}$ and of degree n in $\mathbf{b}$.

Exercise 2.1.6 * Consider the polynomials (2.1) with indeterminate coefficients on the fundamental field $\mathbb{F}$ and let $\alpha_1, \dots, \alpha_n$ [resp. $\beta_1, \dots, \beta_m$] be the roots of f [resp. of g] in the algebraic closure of $\mathbb{F}[\mathbf{a}]$, [resp. in the algebraic closure of $\mathbb{F}[\mathbf{b}]$]. Consider

$$S := a_0^m b_0^n \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j) = a_0^m \prod_{i=1}^n g(\alpha_i) = (-1)^{nm} b_0^n \prod_{j=1}^m f(\beta_j)$$

which can be considered as a polynomial on $\mathbb{F}[a_0, \alpha_1, \dots, \alpha_n]$ in the variables $\mathbf{b}$, or as a polynomial on $\mathbb{F}[b_0, \beta_1, \dots, \beta_m]$ in the variables $\mathbf{a}$.

Prove that S belongs to $\mathbb{F}[\mathbf{a}, \mathbf{b}]$ and it is homogeneous of degree m in the $\mathbf{a}$ and of degree n in the $\mathbf{b}$.

Exercise 2.1.7 * Continuing the previous exercise, prove that S is prime with $a_0 b_0$.

Exercise 2.1.8 * Continuing the previous exercise, prove that S is irreducible.

Exercise 2.1.9 * Continuing the previous exercise, prove that if $T \in \mathbb{F}[\mathbf{a}, \mathbf{b}]$ is a bihomogeneous polynomial which vanishes if (2.2) is compatible, then S divides T.

Exercise 2.1.10 * Continuing the previous exercise, prove that $R = S$.

Exercise 2.1.11 * Prove that R is *isobaric* of *weight nm*, i.e., if $a_0^{i_0} \dots a_n^{i_n} b_0^{j_0} \dots b_m^{j_m}$ is any monomial appearing in R with a non–zero coefficient, then $\sum_{h=1}^{n} hi_h + \sum_{h=1}^{m} hj_h = nm$.

Exercise 2.1.12 Let p, q be positive integers. Prove that if we make in $R(\mathbf{a}, \mathbf{b})$ a substitution of the variables a_i with homogeneous polynomials of degree $i + p$ in some variables $\mathbf{y}$ and of the variables b_j with homogeneous polynomials of degree $j + q$ in the variables $\mathbf{y}$, then $R(\mathbf{a}(\mathbf{y}), \mathbf{b}(\mathbf{y}))$ is a polynomial of degree $nm + pm + qn$ in the variables $\mathbf{y}$.

2.2 The Intersection of Two Plane Curves

Consider two affine or projective curves. We make here a first treatment of the problem of determining their intersection. We will prove the following:

Theorem 2.2.1 *The intersection of two plane affine or projective curves is the union of a (may be empty) affine or projective curve and of a finite set of points.*

An immediate consequence is:

Corollary 2.2.2 *In the Zariski topology, the proper closed subsets of the affine or projective plane are the finite unions of curves and points.*

We start by considering the affine case. Consider two curves $Z(f)$, $Z(g)$ with non–constant $f, g \in A_2$. If f and g have a common factor h of positive degree, the curve $Z(h)$ is contained in the intersection $Z(f, g)$ of the two curves. Hence we can assume that f and g have no non–constant common factor. We will prove that in this case $Z(f, g)$ is a finite set, and this will prove Theorem 2.2.1 in the affine case.

If f and g have degrees n and m respectively in x_2, we have:

$$f(x_1, x_2) = a_0(x_1)x_2^n + \cdots + a_n(x_1), \quad g(x_1, x_2) = b_0(x_1)x_2^m + \cdots + b_m(x_1),$$

with $a_i(x_1), b_j(x_1) \in A_1$, for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Consider the polynomial

$$R(x_1) = R(a_0(x_1), \dots, a_n(x_1), b_0(x_1), \dots, b_m(x_1)).$$

Lemma 2.2.3 *The polynomial* $R(x_1)$ *is not zero.*

Suppose, for the time being, that this Lemma holds. Then if $(p, q) \in Z(f, g)$, we have $R(p) = 0$. Since R is not zero, p can assume at most finitely many values. Exchanging the roles of x_1 and x_2, we see that also q can assume at most finitely many values, and this implies that $Z(f, g)$ is finite.

To prove Lemma 2.2.3, we need a preliminary. Let A be a UFD and let $\mathbb{Q}(A)$ be its quotient field. If $f \in \mathbb{Q}(A)[x]$ is non–zero, then we can write f as $f = c_f \cdot f_1$, with $c_f \in \mathbb{Q}(A)$ and $f_1 \in A[x]$, with the coefficients of f_1 having invertible greatest common divisor, and this expression is unique up to multiplying c_f for an invertible element in A (see Exercise 2.2.7). Moreover c_{f_1} is invertible in A. If $f \in A[x]$ and c_f is invertible, one says that f is *primitive*.

Lemma 2.2.4 *If $f, g \in \mathbb{Q}(A)[x]$ are not zero, then $c_{fg} = c_f c_g$.*

Proof One has $fg = c_f c_g f_1 g_1$, and we may assume $c_{f_1} = c_{g_1} = 1$. Hence it suffices to prove that if f and g (polynomials in $A[x]$) are primitive, also fg is primitive. Let

$$f(x) = a_0 x^n + \cdots + a_n, \quad g(x) = b_0 x^m + \cdots + b_m,$$

with $a_0 b_0 \neq 0$. For every $p \in A$ non–invertible, p does not divide all the coefficients of f. Hence we can consider the minimum integer r such that p does not divide a_r. Similarly, let s be the minimum integer such that p does not divide b_s. Consider the coefficient of $x^{n+m-r-s}$ in fg, which is

$$c = a_r b_s + a_{r+1} b_{s-1} + \cdots + a_{r-1} b_{s+1} + \cdots$$

Then p does not divide $a_r b_s$ but it divides all the other summands in c, hence p does not divide c. This proves the assertion. □

Theorem 2.2.5 (Gauss Lemma) *If a polynomial $f \in A[x]$ factors as $f = gh$, with $g, h \in \mathbb{Q}(A)[x]$, then it factors in $A[x]$ as $f = c_f g_1 h_1$, with $c_f \in A$,* $\deg(g) = \deg(g_1)$ *and* $\deg(h) = \deg(h_1)$.

Proof By Lemma 2.2.4, one has $f = gh = c_g c_h g_1 h_1 = c_f g_1 h_1$ and $c_f \in A$. □

We can now give the:

Proof (of Lemma 2.2.3) Suppose, by contradiction, that $R(x_2)$ is zero. Then $f(x_1, x_2)$ and $g(x_1, x_2)$ would have a common factor of positive degree in $k(x_1)[x_2]$. But then, by Gauss Lemma 2.2.5, f, g would have a common factor in A_2, against the hypothesis. □

This ends the proof of Theorem 2.2.1 in the affine case. The projective case is a consequence of the affine case, and can be left as an exercise (see Exercise 2.2.9).

We explicitely remark that, as a consequence of the results of this section, we have the:

Theorem 2.2.6 (Bézout Theorem, weak form) *Let Z_1, Z_2 be plane affine or projective curves, with equations $f = 0, g = 0$ respectively. If ϕ is the greatest common divisor of f and g, then $Z_1 \cap Z_2 = Z(\phi) \cup Z_3$ where Z_3 is a finite set.*

Exercise 2.2.7 * Suppose A is a UFD. Prove existence and uniqueness (up to the product with a non–zero element in $\mathbb{Q}(A)$) of the expression $f = c_f f_1$ for any non–zero $f \in \mathbb{Q}(A)[x]$, with $c_f \in \mathbb{Q}(A)$ and $f_1 \in A[x]$, and the coefficients of f_1 with invertible greatest common divisor.

Exercise 2.2.8 * Prove that if $Z_a(f)$ is an affine curve, its projective closure is $Z_p(\beta(f))$.

Exercise 2.2.9 * Prove Theorem 2.2.1 in the projective case.

2.3 Kronecker Elimination Method: One Variable

Consider the polynomials

$$f_i(x) = a_{i0}x^{n_i} + \cdots + a_{in_i}, \quad i = 1, \ldots, m \tag{2.8}$$

on the field k and the system

$$f_i(x) = 0, \quad i = 1, \ldots, m. \tag{2.9}$$

Let us set $\boldsymbol{\alpha} = (\alpha_{ij})_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n_i}$, where α_{ij} are indeterminates on k and $\mathbf{a} = (a_{ij})_{1 \leqslant i \leqslant m; 1 \leqslant j \leqslant n_i}$,

Consider the polynomials

$$\phi_i(x) = \alpha_{i0}x^{n_i} + \cdots + \alpha_{in_i}, \quad i = 1, \ldots, m \tag{2.10}$$

on $F = k(\boldsymbol{\alpha})$. The polynomials in (2.10) are polynomials on the field $\mathbb{F}(\boldsymbol{\alpha})$, where $\mathbb{F}$ is the fundamental subfield of k and reduce to the polynomials (2.8) if $\boldsymbol{\alpha} = \mathbf{a}$, hence the system

$$\phi_i(x) = 0, \quad i = 1, \ldots, m \tag{2.11}$$

reduces to the system (2.9) for $\boldsymbol{\alpha} = \mathbf{a}$.

Let us set $n = \max\{n_i, 1 \leqslant i \leqslant m\}$. The system

$$x^{n-n_i}\phi_i(x) = 0, \quad (1-x)^{n-n_i}\phi_i(x) = 0, \quad i = 1, \ldots, m \tag{2.12}$$

is *equivalent* to the system (2.11), i.e., the two systems have the same solutions, and, for $\boldsymbol{\alpha} = \mathbf{a}$, it reduces to the system

$$x^{n-n_i}f_i(x) = 0, \quad (1-x)^{n-n_i}f_i(x) = 0, \quad i = 1, \ldots, m \tag{2.13}$$

which is equivalent to the system (2.9).

We denote, for brevity, by

$$g_i(x) = 0, \quad i = 1, \ldots, h \tag{2.14}$$

the system (2.12). One has:

(i) the coefficients of the polynomials $g_i \in \mathbb{F}[x]$, with $i = 1, \ldots, h$, are linear combinations of the coefficients of the polynomials in (2.10) with coefficients in the fundamental subfield;
(ii) the polynomials g_i, $i = 1, \ldots, h$, have the same degree n in x, and the set of their *leading coefficients* (i.e., of the coefficients of x^n) coincides with the one $\{\alpha_{i0}, 1 \leqslant i \leqslant m\}$ of the polynomials (2.10);
(iii) the system (2.14), for $\boldsymbol{\alpha} = \mathbf{a}$, is equivalent to the system (2.9).

Let us set $\mathbf{u} = (u_{ts})_{1\leqslant t\leqslant 2, 1\leqslant s\leqslant h}$ and $\mathbf{u}_t = (u_{ts})_{1\leqslant s\leqslant h}$, $1 \leqslant t \leqslant 2$, where u_{ts} are indeterminates on F, and consider the two polynomials of degree n

$$U_t(x) = \sum_{s=1}^{h} u_{ts} g_i(x) = \sum_{\ell=0}^{n} \lambda_{t\ell} x^{n-\ell}, 1 \leqslant t \leqslant 2$$

on the field $F(\mathbf{u})$. Note that:

(iv) the coefficients $\lambda_{t\ell}$ of U_t are elements of $\mathbb{F}[\boldsymbol{\alpha}, \mathbf{u}]$;
(v) the leading coefficients of U_1, U_2 are linear combinations, with coefficients entries of $\mathbf{u}$, of the leading coefficients of the polynomials $g_i(x)$, $i = 1, \ldots, h$, i.e., of the polynomials (2.10).

Set $\boldsymbol{\lambda} = (\lambda_{t\ell})_{1\leqslant t\leqslant 2, 1\leqslant \ell\leqslant n}$. Consider the Sylvester determinant $R(\boldsymbol{\lambda})$ of U_1, U_2. By (iv), R can be interpreted as an element in $k[\boldsymbol{\alpha}, \mathbf{u}]$. Let $\bar{R}(\mathbf{u})$ be the polynomial obtained by setting $\boldsymbol{\alpha} = \mathbf{a}$ in R.

Lemma 2.3.1 *$\bar{R}$ is zero if and only if either the system (2.9) is compatible or $a_{i0} = 0$, $1 \leqslant i \leqslant m$.*

Proof The polynomial $\bar{R}$ is zero if and only if either the system

$$\bar{U}_1 = \bar{U}_2 = 0 \tag{2.15}$$

obtained by $U_1 = U_2 = 0$ for $\boldsymbol{\alpha} = \mathbf{a}$, is compatible, or the leading coefficients of the two polynomials appearing in it are zero. By (v), this happens if and only if $a_{i0} = 0$, for $1 \leqslant i \leqslant m$ (see (ii)). On the other hand if (2.9) is compatible, then (2.15) is compatible as well, because of (iii). Conversely, if (2.15) is compatible, let ξ be a solution, which is an element of the algebraic closure of $k(\mathbf{u})$. Of course ξ belongs to the intersection of the algebraic closures of $k(\mathbf{u}_t)$, for $1 \leqslant t \leqslant 2$, and this is $\mathbb{K}$. Hence (2.9) is compatible. □

Let

$$R_q(\boldsymbol{\alpha}) \in k[\boldsymbol{\alpha}], \quad 1 \leqslant q \leqslant N \tag{2.16}$$

be the coefficients of R as a polynomial in the variables $\mathbf{u}$. The polynomials (2.16) are called the *resultant polynomials* of the polynomials (2.10). One has

$$R_q(\mathbf{a}) = 0, \quad 1 \leqslant q \leqslant N \tag{2.17}$$

if and only if either (2.9) is compatible, or $a_{i0} = 0$, for $1 \leqslant i \leqslant m$. The expressions $R_q(\mathbf{a}) \in k$ appearing in (2.17) are called the *resultants* of the system (2.9).

Proposition 2.3.2 *The resultant polynomials (2.16) belong to the ideal generated by the polynomials (2.10) in $k[\boldsymbol{\alpha}]$.*

Proof The assertion holds for $m = 2$ (see Proposition 2.1.3). If $m > 2$, again by Proposition 2.1.3 we have a relation of the form $R = AU_1 + BU_2$, with $A, B \in k[\boldsymbol{\alpha}, \mathbf{u}]$. The assertion follows by equating the coefficients of the entries of $\mathbf{u}$ in such a relation. □

2.4 Kronecker Elimination Method: More Variables

Next we extend the considerations of Sect. 2.4 to the case of more variables. Consider a system of equations

$$f_i(x_1, \dots, x_n) = 0, \quad 1 \leqslant i \leqslant m \tag{2.18}$$

with $f_i \in k[x_1, \dots, x_n]$ not zero and $n \geqslant 2$.

Let us set $\mathfrak{R} = k[x_2, \dots, x_n]$ and let us consider the polynomials appearing in (2.18) as elements in $\mathfrak{R}[x_1]$. Hence the polynomials f_i are of the form (2.8), where $x = x_1$, $a_{ij} \in \mathfrak{R}$ and the leading coefficients a_{i0} are non–zero in $\mathfrak{R}$. Hence the resultant polynomials $R_q(\mathbf{a})$, $1 \leqslant q \leqslant N$, can be regarded as elements $R_q(x_2, \dots, x_n)$ of $\mathfrak{R}$. They are called *resultants* of the polynomials appearing in (2.18) by the *elimination of the variable* x_1 and the system

$$R_q(x_2, \dots, x_n) = 0, \quad 1 \leqslant q \leqslant N \tag{2.19}$$

is called the *resultant system* of (2.18) obtained by *eliminating the variable* x_1.

Lemma 2.4.1 *One has $R_q = 0$ for $1 \leqslant q \leqslant N$, if and only if the polynomials appearing in* (2.18) *have a greatest common divisor of positive degree in x_1.*

Proof Let

$$D(x_1, \dots, x_n) = b_0 x_1^h + \dots + b_h, \quad b_i \in \mathfrak{R}, \quad 0 \leqslant i \leqslant h \tag{2.20}$$

be the greatest common divisor of the polynomials in (2.18), and assume that $h > 0$ and $b_0 \neq 0$. Let U be the non–empty set of $\mathbb{A}^{n-1}_{\mathbb{K}}$ which is the complement of $Z(b_0, a_{i0})_{1 \leqslant i \leqslant m}$. For any $(a_2, \dots, a_n) \in U$, there are solutions of the equation $D(x_1, a_2, \dots, a_n) = 0$ in x_1, and these are also solution of the system

$$f_i(x_1, a_2 \dots, a_n) = 0, \quad 1 \leqslant i \leqslant m. \tag{2.21}$$

But then the $R_q(a_2, \dots, a_n)$, $1 \leqslant q \leqslant N$, that are the resultants of the system (2.21), are all zero. Thus $U \subseteq Z(R_q)_{1 \leqslant q \leqslant N}$, hence the polynomials $R_q(x_2, \dots, x_n)$ are zero (see Exercise 1.6.5).

Conversely, if the polynomials $R_q(x_2, \dots, x_n)$ are zero, the system (2.18) is compatible over $\mathbb{Q}(\mathfrak{R})$, hence the polynomials appearing in (2.18) have a greatest common divisor of positive degree in x_1 on $\mathbb{Q}(\mathfrak{R})[x_1]$. The assertion follows by Gauss Lemma (see Theorem 2.2.5). □

The above Lemma enables us to discuss the compatibility of the system (2.18). Let D be the greatest common divisor of the polynomials appearing in (2.18), and let us set $f_i = Dg_i$, $1 \leqslant i \leqslant m$. In $\mathbb{A}^n_{\mathbb{K}}$ we have $Z(f_1, \dots, f_m) = Z(D) \cup Z(g_1, \dots, g_m)$.

As for the solutions of $D = 0$, if D is a constant, there is no solution. Then we may assume that D depends on x_1, i.e., that D is as in (2.20). Let U be the non–empty subset of $\mathbb{A}^{n-1}_{\mathbb{K}}$ which is the complement of $Z = Z(b_i)_{0\leqslant i\leqslant h-1}$. For every $(a_2, \dots, a_n) \in U$, there are solutions in x_1 of the equation $D(x_1, a_2, \dots, a_n) = 0$, hence this determines the points $(a_1, a_2, \dots, a_n) \in Z(D)$ with $(a_2, \dots, a_n) \in U$. If $(a_2, \dots, a_n) \in Z \cap Z(b_h)$, then the polynomial $D(x_1, a_2, \dots, a_n)$ is zero, hence any $(a_1, a_2, \dots, a_n)$ with $(a_2, \dots, a_n) \in Z \cap Z(b_h)$ sits in $Z(D)$. Finally, if $(a_2, \dots, a_n) \in Z \setminus (Z \cap Z(b_h))$, then $D(x_1, a_2, \dots, a_n)$ is a non–zero constant and there is no point $(a_1, a_2, \dots, a_n)$ in $Z(D)$, with $(a_2, \dots, a_n) \in Z \setminus (Z \cap Z(b_h))$. In conclusion, the search for the solutions of $D = 0$ reduces to the search of the solutions of equations with a lower set of variables.

It remains to determine $Z(f_1, \dots, f_m)$, and we may assume that $f_1, \dots, f_m$ are coprime. Consider the resultant system (2.19) obtained from (2.18) by eliminating the variable x_1. By Lemma 2.4.1 this system is non–trivial, i.e., not all polynomials appearing in it are zero. If $(a_1, a_2, \dots, a_n)$ is a solution of (2.18) then $(a_2, \dots, a_n)$ is a solution of (2.19). Conversely, if $(a_2, \dots, a_n)$ is a solution of (2.19), then:

(i) either $(a_2, \dots, a_n)$ does not belong to $Z(a_{i0})_{1\leqslant i\leqslant m}$ and then there is a finite number of solutions $(a_1, a_2, \dots, a_n)$ of (2.18), whose first coordinate can be obtained by solving equations in one variable;
(ii) or $(a_2, \dots, a_n) \in Z(a_{i0})_{1\leqslant i\leqslant m}$, then one has to solve the system in one variable $f_i(x_1, a_2, \dots, a_n) = 0$, $1 \leqslant i \leqslant m$, which could be non–compatible.

In any event, in order to understand $Z(f_1, \dots, f_m)$ one has to solve equations in one variable and determine $Z(R_q)_{1\leqslant q\leqslant N}$, where we have a similar problem, but with one less variable. Proceeding inductively, we see that in order to solve the system (2.18) one has to compute resultants and solve equations in one variable.

Remark 2.4.2 A polynomial $f \in k[x_1, \dots, x_n]$ of degree d is said to be *regular* in the variable x_i if in f the monomial x_i^d appears with a non–zero coefficient. It is clear that f is regular in x_i if and only if its homogeneous component of degree d is regular in x_i, and this happens if and only if $f_d(0, \dots, 0, 1, 0, \dots, 0) \neq 0$ (here 1 appears only in the i–th component).

Suppose f of degree $d > 0$ is non–regular in one of the variables, for instance in x_1, so that $f_d(1, 0, \dots, 0) = 0$. Suppose that k is an infinite field. Then there is some point $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{A}^n_k$ such that $a_1 \neq 0$ and $f_d(\mathbf{a}) \neq 0$ (apply Exercise 1.6.3).

Consider the affinity

$$\sigma : (x_1, \dots, x_n) \in \mathbb{A}^n_k \to (a_1x_1, a_2x_1 + x_2, \dots, a_nx_1 + x_n) \in \mathbb{A}^n_k,$$

that is also an automorphism of k–vector spaces, and the homogeneous linear substitution of variables, which is a homogenous automorphism (see Exercises 1.3.11, 1.3.12)

$$\tau : g(\mathbf{x}) \in k[x_1, \dots, x_n] \to g(\sigma(\mathbf{x})) \in k[x_1, \dots, x_n].$$

One has $Z(f) = \sigma^{-1}(Z(\tau(f)))$. Hence studying the equation $f = 0$ is equivalent to studying the equation $\tau(f) = 0$. Moreover $\tau(f)$ is regular in x_1, since $\tau(f)_d(1, 0. \dots, 0) = \tau(f_d)(1, 0. \dots, 0) = f_d(\mathbf{a}) \neq 0$.

In conclusion: in order to study the compatibility of the system (2.18), we can replace it with a system of equations on the same field k, such that at least one of the polynomials appearing in it is regular with respect to at least one variable. Then the system is said to be *regular* with respect to that variable.

Assuming the system (2.18) regular with respect to a variable, e.g., respect to x_1, simplifies the solution of the system. In fact in this case also the greatest common divisor D of the polynomials in (2.18) is regular with respect to x_1, hence the solution of the equation $D = 0$ simplifies. For all $(a_2, \dots, a_n) \in \mathbb{A}_k^{n-1}$, there is a finite number of solutions in x_1 of the equation $D(x_1, a_2, \dots, a_n) = 0$, and this determines all the points of $Z(D)$.

Moreover, if $(a_2, \dots, a_n)$ is a solution of the resultant system (2.19), then there is a finite number of solutions $(a_1, a_2, \dots, a_n)$ of (2.18), whose first coordinate can be computed by solving equations in one variable, and all solutions of (2.18) can be obtained in this way.

If k is not infinite, we can still proceed in the same way, substituting to k its algebraic closure $\mathbb{K}$, which is infinite. In this case we can still replace (2.18) with a regular system in one variable, but it is now defined, in general, on $\mathbb{K}$ and not on k.

Exercise 2.4.3 Interpret geometrically the Kronecker elimination process in more variables over an algebraically closed field.

2.5 Hilbert Nullstellensatz

Here we assume that the field $k = \mathbb{K}$ is algebraically closed. The crucial step for the proof of the Hilbert Nullstellensatz is the following:

Theorem 2.5.1 (Incompatibility Criterion) *The system* (2.18) *is incompatible if and only if* $A_{\mathbb{K},n} = (f_1, \dots, f_m)$.

Proof We prove only the non–trivial implication. By the discussion in Sect. 2.4, $f_1, \dots, f_m$ are coprime and the resultant system (2.19) is also incompatible.

If $n = 1$, the polynomials in (2.19) are constant and at least one of these constants is not zero. The assertion follows from Proposition 2.3.2. Let us now proceed by induction on n. Since (2.19) is incompatible, the polynomials appearing in it generate A_{n-1}. By Proposition 2.3.2 these polynomials sit in the ideal $(f_1, \dots, f_m)$. The assertion follows. □

We can now prove the:

Theorem 2.5.2 (Hilbert Nullstellensatz) *If $f \in A_n$ and $I \subseteq A_n$ is an ideal, such that $Z(I) \subseteq Z(f)$ then $f \in \mathrm{rad}(I)$.*

Proof We can assume that f is not zero. Moreover we can assume that $I = (f_1, \dots, f_m)$. Let z be an indeterminate on $\mathbb{Q}(A_n)$. Consider the system obtained by adding to (2.18) the equation $1 - zf(\mathbf{x}) = 0$. This system is incompatible, and, for the Incompatibility Criterion 2.5.1, we have an expression of the sort

$$1 = A(\mathbf{x}, z)(1 - zf(\mathbf{x})) + \sum_{i=1}^{m} A_i(\mathbf{x}, z) f_i(\mathbf{x}).$$

By setting $z = \frac{1}{f(\mathbf{x})}$ and eliminating the denominators, which are powers of $f(\mathbf{x})$, we get the assertion. □

As a consequence we have the:

Theorem 2.5.3 (Homogeneous Nullstellensatz) *If $f \in S_n$, if f is homogeneous, not constant and if $I \subseteq S_n$ is a homogeneous ideal such that $Z_p(I) \subseteq Z_p(f)$ then $f \in \mathrm{rad}(I)$.*

Proof By the hypotheses, one has also $Z_a(I) \subseteq Z_a(f)$ and the assertion follows by applying the Hilbert Nullstellensatz. □

Exercise 2.5.4 An ideal of a ring is said to be *radical* if it coincides with its radical. Prove that a prime ideal is radical.

Exercise 2.5.5 * Prove that a maximal ideal is radical.

Exercise 2.5.6 * Prove that the radical of an ideal I of the ring A is the intersection of all prime ideals of A that contain I.

Exercise 2.5.7 Let A be a ring. Prove that $\mathrm{rad}((0))$ is the set of all *nilpotent* elements of A. This ideal is called the *nilradical* of A and is denoted by $\mathrm{nilrad}(A)$.

2.6 Solutions of Some Exercises

2.1.6 Looking at the righmost expression of S, it is clear that S is a homogeneous polynomial of degree m in the variables $\mathbf{a}$, with coefficients polynomials in b_0 and $\beta_1, \dots, \beta_m$. We prove that these coefficients can be expressed as homogeneous polynomials of degree n in the variables $\mathbf{b}$, with coefficients in $\mathbb{F}$. The number of these coefficients is $N = \binom{n+m}{n}$. To compute them we proceed in the following way. Consider the identity

$$a_0^m \prod_{i=1}^{n} g(\alpha_i) = (-1)^{nm} b_0^n \prod_{j=1}^{m} f(\beta_j). \tag{2.22}$$

Give arbitrary values at the variables $\mathbf{a}$ in $\mathbb{F}$, and accordingly to $\alpha_1, \ldots, \alpha_n$ in the algebraic closure if $\mathbb{F}$. Then we get, on the right hand side of (2.22), a linear combination of the coefficients in question, which equals the left hand side of (2.22) that is a homogeneous polynomial of degree n in the variables $\mathbf{b}$. Consider, as it is certainly possible, N linearly independent such relations. By solving the corresponding linear system of equations we obtain the required expressions of the coefficients in question.

2.1.7 We have

$$S(a_0, 0, \ldots, 0; b_0, \ldots, b_m) = a_0^m \prod_{i=1}^{n} g(0) = a_0^m b_m^n,$$

so in S there is the monomial $a_0^m b_m^n$ and this is the only one which contains a_0^m. Similarly S contains the monomial $(-1)^{nm} b_0^n a_n^m$, which is the only one that contains b_0^n. Therefore S is prime with $a_0 b_0$.

2.1.8 By Exercise 2.1.7, S is irreducible if and only if

$$S' = \frac{S}{a_0^m b_0^n} = S(1, a_1', \ldots, a_n'; 1, b_1', \ldots, b_m')$$

is irreducible in the variables $a_i' = \frac{a_i}{a_0}$, $b_j' = \frac{b_j}{b_0}$, with $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Suppose that $S' = P' \cdot Q'$, with P' a non–constant polynomial in $a_1', \ldots, a_n', b_1', \ldots, b_m'$. Express $a_1', \ldots, a_n', b_1', \ldots, b_m'$ as elementary symmetric functions in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$. Then P' is a symmetric function in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$. Since P' divides

$$S' = \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j)$$

then P' contains one of the factors $\alpha_i - \beta_j$. But, since P' is a symmetric function in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$, it contains all the factors $\alpha_i - \beta_j$, which implies that Q' is constant, hence S' is irreducible, so it is S.

2.1.9 Suppose T is a bihomogeneous polynomial of degree p in the variables $\mathbf{a}$ and of degree q in the variables $\mathbf{b}$, which vanishes if $\alpha_i = \beta_j$, with $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$. Then $T' = \frac{T}{a_0^p b_0^q}$ is a polynomial in $a_1', \ldots, a_n', b_1', \ldots, b_m'$, hence it is a symmetric function in $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$. So T' is divisible by all factors $\alpha_i - \beta_j$, hence it is divisible by S'. This implies that S divides T.

2.1.10 By Exercise 2.1.9, S divides R, and since S and R have the same degree in the variables $\mathbf{a}$ and $\mathbf{b}$, they can differ only by a non–zero constant factor. This factor is 1, because both polynomials contain the monomial $a_0^m b_m^n$ with coefficient 1.

2.1.11 In $R = S$ we make the following substitution $\bar{\alpha}_i = t\alpha_i$ and $\bar{\beta}_j = t\beta_j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This implies that $a_0, \ldots, a_n$ and $b_0, \ldots, b_m$ are substituted by $\bar{a}_h = t^h a_h$ and $\bar{b}_l = t^l b_l$, with $h = 0, \ldots, n$ and $l = 0, \ldots, m$. With these substitutions R becomes

$$\bar{R} = t^{nm} R$$

and we must have identically

$$R(a_0, ta_1, \ldots, t^n a_n; b_0, tb_1, \ldots, t^m b_m) = t^{nm} R(a_0, a_1, \ldots, a_n; b_0, b_1, \ldots, b_m).$$

By comparing the degrees in t the assertion follows.

2.1.12 If we make the substitution indicated in the exercise, then R becomes a form in the new indeterminates $\mathbf{y}$ of degree

$$\sum_{h=0}^{n}(h+p)i_h+\sum_{l=0}^{m}(l+q)j_l=$$
$$=\sum_{h=0}^{n}hi_h+p\sum_{h=0}^{n}i_h+\sum_{l=0}^{m}lj_l+q\sum_{l=0}^{m}j_l=mn+pm+qn.$$

2.2.7 Find the minimum common denominator a of the coefficients of f. Then $c_f=\frac{b}{a}$ where b is the greatest common divisor of the numerators of the coefficients of $af\in A[x]$.
2.2.8 It suffices to reduce to the case in which f is irreducible. One has $Z_a(f)\subseteq Z_p(\beta(f))$, hence $\overline{Z_a(f)}\subseteq Z_p(\beta(f))$. If g is a homogeneous polynomial such that $Z_a(f)\subseteq Z_p(g)$, then $Z_a(f)\subseteq Z_p(\alpha(g))$. Then f divides $\alpha(g)$, thus $\beta(f)$ divides $\beta(\alpha(g))$, that in turn divides g. In conclusion $Z_p(\beta(f))\subseteq Z_p(g)$, hence $\overline{Z_a(f)}=Z_p(\beta(f))$.
2.2.9 If Z is a closed subset of $\mathbb{P}^2$ we have $Z=Z_1\cup Z_2$, with $Z_1=\overline{Z\cap U_0}$ and $Z_2=Z\cap H_0$. Then Z_2 is a closed subset of $\mathbb{P}^1=H_0$ hence either it is the whole of H_0 or it is a finite set. Moreover $Z\cap U_0$ is a closed subset of $U_0=\mathbb{A}^2$, hence it is the union of a finite subset and of an affine curve. Then apply Exercise 2.2.8 to conlcude.
2.5.6 It is clear that $\mathrm{rad}(I)$ is contained in any prime ideal containing I. Conversely, let x be an element contained in all prime ideals containing I. Consider the multiplicatively closed set $S=\{x^n, n\in\mathbb{N}-\{0\}\}$. If $S\cap I\neq\emptyset$, then $x\in\mathrm{rad}(I)$ and we are done. So we argue by contradiction and assume $S\cap I=\emptyset$. Let $\mathcal{J}$ be the set of all ideals J such that $I\subseteq J$ and $S\cap J=\emptyset$. The set $\mathcal{J}$ is non–empty, because $I\in\mathcal{J}$. Moreover $\mathcal{J}$ is partially ordered by the inclusion. If $\{J_h\}_{n\in\mathbb{N}}$ is a chain of ideals in $\mathcal{J}$, then $\bigcup_h J_h\in\mathcal{J}$. By Zorn's lemma, there is an ideal $P\in\mathcal{J}$ which is maximal by the inclusion. We claim that P is prime. Indeed, suppose the contrary holds, i.e., we have $ab\in P$ but $a,b\notin P$. Then (a,P) and (b,P) properly contain P, so they do not lie in $\mathcal{J}$. Since they both contain I, this means that $(a,P)\cap S\neq\emptyset$ and $(b,P)\cap S\neq\emptyset$. This means we have relations of the sort
$$x^n=ya+p,\quad x^m=zb+q,\quad \text{with}\quad p,q\in P.$$
Multiplying the above relations and taking into account that $ab\in P$, we deduce that $x^{n+m}\in P$, a contradiction. Then P is prime, but this implies that $x\in P$, again a contradiction.

Chapter 3
Zariski Closed Subsets and Ideals in the Polynomials Ring

3.1 Ideals and Coordinate Rings

Let X be a subset of $\mathbb{A}^n$. We will denote by $\mathcal{I}_a(X)$ the ideal of A_n of all the polynomials $f \in A_n$ such that $X \subseteq Z_a(f)$. Then $\mathcal{I}_a(X)$ is called the *ideal* of X. The ring $A(X) := A_n/\mathcal{I}_a(X)$ is called the *(affine) coordinate ring* of X. Similarly, if X is a subset of $\mathbb{P}^n$ we define the *ideal* of X to be the homogeneous ideal $\mathcal{I}_p(X)$ of S_n which is generated by all homogeneous polynomials $f \in S_n$ such that $X \subseteq Z_p(f)$. The ring $S(X) := S_n/\mathcal{I}_p(X)$ is called the *(homogeneous) coordinate ring* of X.

Proposition 3.1.1 *One has:*

(a) if F_1, F_2 are subsets of A_n such that $F_1 \subseteq F_2$ then $Z_a(F_2) \subseteq Z_a(F_1)$;
(b) if X_1, X_2 are subsets of $\mathbb{A}^n$ such that $X_1 \subseteq X_2$ then $\mathcal{I}_a(X_2) \subseteq \mathcal{I}_a(X_1)$;
(c) if X_1, X_2 are subsets of $\mathbb{A}^n$, one has $\mathcal{I}_a(X_1 \cup X_2) = \mathcal{I}_a(X_1) \cap \mathcal{I}_a(X_2)$;
(d) For all subsets X of $\mathbb{A}^n$, one has $Z_a(\mathcal{I}_a(X)) = \bar{X}$.

Analogous properties hold for subsets of $\mathbb{P}^n$ and of S_n.

Proof Properties (a), (b) and (c) are obvious and we leave the proof to the reader. As for (d), note that $X \subseteq Z_a(\mathcal{I}_a(X))$, hence $\bar{X} \subseteq Z_a(\mathcal{I}_a(X))$. Let now $W = Z_a(\mathcal{I})$ be a closed subset of $\mathbb{A}^n$, with $\mathcal{I}$ an ideal of A_n, and suppose that $X \subseteq W$. From (b) we have $\mathcal{I}_a(W) = \mathcal{I}_a(Z_a(\mathcal{I})) \subseteq \mathcal{I}_a(X)$, hence $\mathcal{I} \subseteq \mathcal{I}_a(Z_a(\mathcal{I})) \subset \mathcal{I}_a(X)$. By (a) we have $W = Z_a(\mathcal{I}) \supseteq Z_a(\mathcal{I}_a(X))$, whence the assertion follows. □

The following is an immediate consequence of Hilbert Nullstellensatz 2.5.2:

Corollary 3.1.2 *If $\mathcal{I}$ is an ideal of A_n, then* $\mathrm{rad}(I) = \mathcal{I}_a(Z_a(\mathcal{I}))$.

Hence the map

$$\mathcal{I}_a : X \in \mathcal{A}_n \to \mathcal{I}_a(X) \in \mathfrak{R}(A_n),$$

is a bijection of $\mathcal{A}_n$ on the set $\mathfrak{R}(A_n)$ of radical ideals of A_n. The same does not hold in the projective case, as the following remark shows.

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
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Remark 3.1.3 The ambient is the projective space $\mathbb{P}^n$. The irrelevant ideal $S_{>0}$ of S_n is maximal, hence it is radical (see Exercise 2.5.5). However $\mathcal{I}_p(Z_p(S_{>0})) = \mathcal{I}_p(\emptyset) = S_n$. Hence Corollary 3.1.2 does not hold in the projective setting and the map

$$\mathcal{I}_p : X \in \mathcal{P}_V \to \mathcal{I}_p(X) \in \mathfrak{R}(S_n)$$

is not surjective on the set of radical ideals.

Lemma 3.1.4 *Let $\mathcal{I} \subseteq S_n$ be a homogeneous ideal. The following are equivalent:*

(a) $Z_p(\mathcal{I}) = \emptyset$;
(b) *either* $\mathrm{rad}(\mathcal{I}) = S_n$ *or* $\mathrm{rad}(\mathcal{I}) = S_{>0}$;
(c) *there is a positive integer d such that $S_{n,d} \subseteq \mathcal{I}$.*

Proof It is clear that $Z_p(\mathcal{I}) = p(Z_a(\mathcal{I} \setminus \{0\})$, where $p : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ is, as usual, the natural projection. Hence if (a) holds, then either $Z_a(\mathcal{I}) = \emptyset$ or $Z_a(\mathcal{I}) = \{\mathbf{0}\}$. From this, and from Corollary 3.1.2, (b) follows. Suppose then (b) holds. If $\mathrm{rad}(\mathcal{I}) = S_n$ then $\mathcal{I} = S_n$ and (c) holds. If $\mathrm{rad}(\mathcal{I}) = S_{>0}$, there are positive integers $i_0, \ldots, i_n$ such that $x_j^{i_j} \in \mathcal{I}$ for $j = 0, \ldots, n$. If $d \geqslant i_0 + \cdots + i_n$ then any monomial of degree d in $x_0, \ldots, x_n$ belongs to $\mathcal{I}$, so (c) holds. It is finally clear that (c) implies (a).

In order to obtain, in the projective case, a result similar to Corollary 3.1.2, one has to use the Homogeneous Nullstellensatz 2.5.3, which implies the following:

Corollary 3.1.5 *If $\mathcal{I} \subseteq S_n$ is a homogeneous ideal with $Z_p(\mathcal{I}) \neq \emptyset$, then* $\mathrm{rad}(\mathcal{I}) = \mathcal{I}_p(Z_p(\mathcal{I}))$.

From this it follows that the map $\mathcal{I}_p$ is a bijection between $\mathcal{P}_V$ and the set of radical homogeneous ideals of S_n which are different from the irrelevant ideal $S_{>0}$.

Exercise 3.1.6 Let f be a non–constant polynomial which has the distinct irreducible factors $f_1, \ldots, f_h$. Prove that $\mathrm{rad}(f) = (f_1 \cdots f_h)$. Hence the ideal of $Z = Z_a(f)$ is $(f_1 \cdots f_h)$. The same in the projective setting.

Exercise 3.1.7 Let $Z \subseteq \mathbb{A}^n$ be a closed set and let $\bar{Z}$ be its projective closure. Prove that $\mathcal{I}_p(\bar{Z})$ is the homogeneous ideal generated by $\beta(\mathcal{I}_a(Z))$. In particular if $Z = Z_a(f) \subset \mathbb{A}^n$ is a hypersurface, its projective closure $\bar{Z}$ is the hypersurface with equation $\beta(f) = 0$.

Exercise 3.1.8 Let Z be a subspace of $\mathbb{P}^n$ of codimension $c < n + 1$ and let $Z = Z_p(f_1, \ldots, f_c)$ with $f_1, \ldots, f_c$ independent linear forms. Prove that $\mathcal{I}_p(Z) = (f_1, \ldots, f_c)$. Prove an analogous result for affine subspaces of $\mathbb{A}^n$.

3.2 Examples

3.2.1 Maximal Ideals

Let $\mathfrak{m}$ be a maximal ideal of A_n. By Proposition 3.1.1, $Z_a(\mathfrak{m})$ is a minimal closed subset of $\mathbb{A}^n$, which is not empty by Corollary 3.1.2. Hence $Z_a(\mathfrak{m})$ is a point $P =$

$(a_1, \ldots, a_n)$. Since $\mathfrak{m}$ is radical (see Exercise 2.5.5), one has $\mathfrak{m} = \mathcal{I}_a(Z_a(\mathfrak{m})) = \mathcal{I}_a(\{P\}) = (x_1 - a_1, \ldots, x_n - a_n)$. The last equality follows from the fact that, for every polynomial $f \in A_n$, one has $f(\mathbf{x}) = f(\mathbf{a} + (\mathbf{x} - \mathbf{a})) = f(\mathbf{a}) + g(\mathbf{x})$, where $g(\mathbf{x}) \in (x_1 - a_1, \ldots, x_n - a_n)$.

Exercise 3.2.1 Consider the two polynomials

$$f(x_1, x_2) = x_1^2 + x_2^2 - 1, \quad g(x_1, x_2) = x_1 - 1.$$

Prove that $\mathcal{I}_a(Z_a(f, g)) \neq (f, g)$.

3.2.2 The Twisted Cubic

Let $Z \subset \mathbb{A}^3$ be the subset $Z = \{(t, t^2, t^3), t \in \mathbb{K}\}$, i.e., the image of the map

$$\phi : t \in \mathbb{A}^1 \to (t, t^2, t^3) \in \mathbb{A}^3.$$

This application is clearly a homeomorphism of $\mathbb{A}^1$ onto Z. Since $Z = Z_a(x_1^2 - x_2, x_1^3 - x_3)$, Z is a closed subset of $\mathbb{A}^3$ that is called the *affine twisted cubic*. Set $f(x_1, x_2, x_3) = x_1^2 - x_2$, $g(x_1, x_2, x_3) = x_1^3 - x_3$. One has $A_3/(f, g) \cong A_1$, the isomorphism being the following

$$\varphi : [h(x_1, x_2, x_3)] \in A_3/(f, g) \to h(x, x^2, x^3) \in A_1$$

where we denoted by $[h(x_1, x_2, x_3)]$ the class of $h(x_1, x_2, x_3) \in A_3$ in $A_3/(f, g)$: the reader will verify that φ is well defined and is indeed an isomorphism (see Exercise 3.2.2). Then the ideal (f, g) is prime, hence it is radical (see Exercise 2.5.5), and therefore $\mathcal{I}_a(Z) = (f, g)$.

Consider now the map

$$\psi : [\lambda, \mu] \in \mathbb{P}^1 \to [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3] \in \mathbb{P}^3$$

which is a homeomorphism of $\mathbb{P}^1$ onto its image. If we identify, as usual, $\mathbb{A}^3$ with the open subset U_0 of $\mathbb{P}^3$ (see Sect. 1.5), we have $\psi(\mathbb{P}^1) = Z \cup \{P\}$, where $P = [0, 0, 0, 1]$. If $h \in S_3$ is a homogeneous polynomial such that $Z \subseteq Z_p(h)$, one has $h(1, t, t^2, t^3) = 0$ for all $t \in \mathbb{K}$, hence one has $h(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0$ for all $\mu \in \mathbb{K}$ and $\lambda \in \mathbb{K} \setminus \{0\}$, thus for all $\lambda \in \mathbb{K}$. Hence $Z \cup \{P\} \subseteq \bar{Z}$. On the other hand $\psi(\mathbb{P}^1) = Z \cup \{P\} = Z_p(f_0, f_1, f_2)$, with

$$f_0 = x_1x_3 - x_2^2, \quad f_1 = x_1x_2 - x_0x_3, \quad f_2 = x_0x_2 - x_1^2,$$

is a projective closed subset, that is called the *projective twisted cubic*. It follows that $\psi(\mathbb{P}^1) = Z \cup \{P\} = \bar{Z}$.

The equations that define $\bar{Z}$ are obtained by equating to zero the order two minors of the matrix

$$\mathbf{A} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}. \tag{3.1}$$

One also says that $\bar{Z}$ is defined by the *matrix equation* $\mathrm{rank}(\mathbf{A}) < 2$.

Note that, though f and g generate $\mathcal{I}_a(Z)$, $\beta(f)$ and $\beta(g)$ do not generate $\mathcal{I}_p(\bar{Z})$, actually $\bar{Z} \neq Z_p(\beta(f), \beta(g))$. Indeed it is easy to check that $Z_p(\beta(f), \beta(g)) = \bar{Z} \cup Z_p(x_0, x_1)$.

More generally, one defines *affine twisted cubic* [resp. *projective twisted cubic*] any image of Z [resp. of $\bar{Z}$] via an affinity [resp. a projectivity].

Exercise 3.2.2 Let $f(x_1, x_2, x_3) = x_1^2 - x_2$, $g(x_1, x_2, x_3) = x_1^3 - x_3$. Prove that the map

$$\varphi : [h(x_1, x_2, x_3)] \in A_3/(f, g) \to h(x, x^2, x^3) \in A_1$$

is well defined and it is an isomorphism.

Exercise 3.2.3 * Prove that the minors of the matrix $\mathbf{A}$ in (3.1) generate the ideal of the projective twisted cubic.

Exercise 3.2.4 Prove that the (affine or projective) twisted cubic is *non–degenerate*, namely it is not contained in any plane (of $\mathbb{A}^3$ or $\mathbb{P}^3$).

Exercise 3.2.5 Prove that any plane of $\mathbb{P}^3$ intersects the twisted cubic in at most three distinct points.

Exercise 3.2.6 Prove that there is no line in $\mathbb{P}^3$ intersecting the twisted cubic in more than two distinct points.

Exercise 3.2.7 Prove that an affine twisted cubic of $\mathbb{A}^3$ is the set of points of $\mathbb{A}^3$ of the form

$$x_i = a_i + a_{i1}t + a_{i2}t^2 + a_{i3}t^3, \quad t \in \mathbb{K}, \quad i = 1, 2, 3,$$

where the matrix $(a_{ij})_{i,j=1,2,3}$ is of maximal rank.

Exercise 3.2.8 Prove that a projective twisted cubic of $\mathbb{P}^3$ is the set of points of $\mathbb{P}^3$ of the form

$$x_i = a_{i0}\lambda^3 + a_{i1}\lambda^2\mu + a_{i2}\lambda\mu^2 + a_{i3}\mu^3, \quad [\lambda, \mu] \in \mathbb{P}^1, \quad i = 0, 1, 2, 3,$$

where the matrix $(a_{ij})_{i,j=0,1,2,3}$ is of maximal rank, defined up to a multiplicative constant.

3.2.3 Cones

Let $Z \subseteq \mathbb{P}^n$ be a non–empty closed subset. The subset $C(Z) = p^{-1}(Z) \cup \{\mathbf{0}\}$ of $\mathbb{A}^{n+1}$ is called the *affine cone* on Z with vertex $\mathbf{0}$. Note that $C(Z)$ is a closed subset of $\mathbb{A}^{n+1}$. Indeed, if $Z = Z_p(f_1, \ldots, f_m)$, with $f_1, \ldots, f_m \in S_n$ homogeneous, then $C(Z) = Z_a(f_1, \ldots, f_m)$. It is clear that $\mathcal{I}_p(Z) \subseteq \mathcal{I}_a(C(Z))$, but since $Z_a(\mathcal{I}_p(Z)) = C(Z)$ and since $\mathcal{I}_p(Z)$ is radical, we actually have $\mathcal{I}_p(Z) = \mathcal{I}_a(C(Z))$.

Now think of $\mathbb{A}^{n+1}$ as embedded in $\mathbb{P}^{n+1}$, so that $\mathbb{P}^n$ can be regarded as the hyperplane at infinity of $\mathbb{A}^{n+1}$. Then we can consider the projective closure $\overline{C(Z)}$, which is closed in $\mathbb{P}^{n+1}$ and is called the *projective cone* on Z with vertex $\mathbf{0}$. Of course one has $\overline{C(Z)} \cap \mathbb{P}^n = C(Z)_\infty = Z$. Moreover $\mathcal{I}_p(\overline{C(Z)}) = \mathcal{I}_p(Z)$, where the latter is considered as an ideal of S_{n+1}.

More generally, any transformed of an affine or projective cone as above via an affinity or a projectivity respectively, is still called a *cone*.

Let finally Z be any subset of $\mathbb{P}^n$. Set $C(Z) = p^{-1}(Z) \cup \{\mathbf{0}\}$. If $C(Z)$ is closed in $\mathbb{A}^{n+1}$, then Z is closed in $\mathbb{P}^n$. In fact, it is clear that $Z = C(Z)_\infty$.

From the above considerations it follows that the map $p : \mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$ is continuous, and actually the Zariski topology of $\mathbb{P}^n$ is the quotient topology of the Zariski topology of $\mathbb{A}^{n+1} \setminus \{\mathbf{0}\}$ with respect to the equivalence relation of proportionality.

Exercise 3.2.9 * Consider in $\mathbb{P}^n$ a hypersurface V with equation $f(x_1, \ldots, x_{n-1}) = 0$ with f not depending on x_n. Prove that it is a cone with vertex $P = [0, \ldots, 0, 1]$.

Exercise 3.2.10 * Consider any quadric X in $\mathbb{P}^n$ whose matrix $\mathbf{A}$ has rank smaller than $n + 1$. Prove that X is a cone with vertex any point $P = [\mathbf{p}]$ such that $\mathbf{p} \cdot \mathbf{A} = \mathbf{0}$.

3.3 Solutions of Some Exercises

3.1.7 If $f \in \mathcal{I}_p(\bar{Z})$, one has $\alpha(f) \in \mathcal{I}_a(Z)$. On the other hand $\beta(\alpha(f))$ divides f.
3.1.8 It suffices to prove that $(f_1, \ldots, f_c)$ is a radical ideal. Complete $f_1, \ldots, f_c$ to a basis $f_1, \ldots, f_{n+1}$ of $S_{n,1}$. Consider the automorphism $f : S_n \to S_n$ of a $\mathbb{K}$–algebra, which is obtained by extending by linearity the automorphism f of S_1 such that $f(x_i) = f_i$, for $i = 1, \ldots, n+1$. Then f maps the radical ideal $(x_1, \ldots, x_c)$ to $(f_1, \ldots, f_c)$, and this proves that $(f_1, \ldots, f_c)$ is radical.

The affine case is analogous.
3.2.3 What one has to prove is that, for every integer $d \geqslant 2$, the map of $\mathbb{K}$–vector spaces

$$\phi_d : (g_0, g_1, g_2) \in S_{3,d-2} \oplus S_{3,d-2} \oplus S_{3,d-2} \to g_0 f_0 + g_1 f_1 + g_2 f_2 \in \mathcal{I}_p(\bar{Z})_d$$

is surjective.

Let K_{d-2} be the kernel of ϕ_d: the elements in K_{d-2} are called *syzygies* of degree $d-2$ of (f_0, f_1, f_2). It is clear that $\mathbf{a}_1 = (x_0, x_1, x_2)$ and $\mathbf{a}_2 = (x_1, x_2, x_3)$ are syzygies of degree 1. Hence for every positive integer d we have a linear map

$$\psi_d : (a, b) \in S_{3,d-3} \oplus S_{3,d-3} \to a\mathbf{a}_1 + b\mathbf{a}_2 \in K_{d-2}.$$

Let us prove that ψ_d is an isomorphism. Let $(g_0, g_1, g_2) \in K_{d-2}$. Then we have

$$\begin{vmatrix} g_0 & g_1 & g_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0 \tag{3.2}$$

hence the rows of the determinant appearing in (3.2) are linearly dependent on $\mathbb{K}(x_0, x_1, x_2, x_3)$.

Since the rows of **A** are linearly independent, because f_0, f_1, f_2 are not zero, there are rational functions $\frac{a_1}{a_0}, \frac{b_1}{b_0} \in \mathbb{K}(x_0, x_1, x_2, x_3)$, with $\mathrm{GCD}(a_0, a_1) = \mathrm{GCD}(b_0, b_1) = 1$, such that

$$
\begin{aligned}
g_0 &= \frac{a_1}{a_0} x_0 + \frac{b_1}{b_0} x_1 = \frac{a_1 b_0 x_0 + a_0 b_1 x_1}{a_0 b_0} \\
g_1 &= \frac{a_1}{a_0} x_1 + \frac{b_1}{b_0} x_2 = \frac{a_1 b_0 x_1 + a_0 b_1 x_2}{a_0 b_0} \\
g_2 &= \frac{a_1}{a_0} x_2 + \frac{b_1}{b_0} x_3 = \frac{a_1 b_0 x_2 + a_0 b_1 x_3}{a_0 b_0}.
\end{aligned}
$$

Since the left hand sides are polynomials, then in the right hand sides the numerator is divided by the denominator. If p is a prime divisor of a_0, then p has to divide $b_0 x_0, b_0 x_1, b_0 x_2$, hence p divides b_0. By iterating this argument and exchanging the roles of a_0 and b_0, we see that we may assume that $a_0 = b_0$. Hence we have

$$
\begin{aligned}
g_0 &= \frac{a_1 x_0 + b_1 x_1}{b_0} \\
g_1 &= \frac{a_1 x_1 + b_1 x_2}{b_0} \\
g_2 &= \frac{a_1 x_2 + b_1 x_3}{b_0}
\end{aligned}
$$

thus b_0 divides

$$
\begin{aligned}
\alpha_0 &= a_1 x_0 + b_1 x_1 \\
\alpha_1 &= a_1 x_1 + b_1 x_2 \\
\alpha_2 &= a_1 x_2 + b_1 x_3.
\end{aligned}
$$

Then b_0 divides

$$
\begin{aligned}
x_1 \alpha_0 - x_0 \alpha_1 &= -b_1 f_2 \\
x_2 \alpha_0 - x_0 \alpha_1 &= b_1 f_1 \\
x_2 \alpha_1 - x_1 \alpha_2 &= -b_1 f_0
\end{aligned}
$$

and since f_0, f_1, f_2 are irreducible and distinct, we have that b_0 divides b_1, hence we may assume that $b_0 = a_0 = 1$.

Set now $a = a_1$ and $b = b_1$ and let us prove that $(a, b) \in S_{3,d-3} \oplus S_{3,d-3}$. Indeed, if $i \neq d-3$ and a_i, b_i are the homogeneous components of degree i of a and b, from the relations

$$
\begin{aligned}
g_0 &= a x_0 + b x_1 \\
g_1 &= a x_1 + b x_2 \\
g_2 &= a x_2 + b x_3
\end{aligned}
$$

we get that

$$
\begin{aligned}
a_i x_0 + b_i x_1 &= 0 \\
a_i x_1 + b_i x_2 &= 0 \\
a_i x_2 + b_i x_3 &= 0
\end{aligned}
$$

which, arguing as above, implies that $a_i, b_i = 0$. All this proves the bijectivity of the map ψ_d. Then we have

$$
\dim(\mathrm{im}(\psi_d)) = 3\binom{d+1}{3} - 2\binom{d}{3}.
$$

Let us compute the dimension of $\mathcal{I}_p(\bar{Z})_d$. Consider, for every positive integer, the $\mathbb{K}$–linear map

$$
r_d : f(x_0, x_1, x_2, x_3) \in S_{3,d} \to f(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) \in S_{1,3d}
$$

which is easily proven to be surjective and its kernel is $\mathcal{I}_p(\bar{Z})_d$. One has then

$$\dim(\mathcal{I}_p(\bar{Z})_d) = \binom{d+3}{3} - (3d+1) = 3\binom{d+1}{3} - 2\binom{d}{3}$$

whence the surjectivity of ψ_d follows.

Chapter 4
Some Topological Properties

4.1 Irreducible Sets

Let X be a topological space. A subset of X is said to be *irreducible* if it cannot be expressed as the union of two proper closed subsets. A subset of X which is not irreucible is said to be *reducible*. The empty set is considered to be reducile.

Example 4.1.1 Every non–empty subset U of $\mathbb{P}^1$ is irreducible: indeed, U is infinite and the only closed subsets of $\mathbb{P}^1$ are finite. The only irreducible proper closed subsets of $\mathbb{P}^1$ are the points. The same for $\mathbb{A}^1$.

Example 4.1.2 If $Z_1 \subseteq \mathbb{A}^r$ and $Z_2 \subseteq \mathbb{A}^s$ are closed irreducible subsets, then $Z_1 \times Z_2$ is closed (see Sect. 1.6.6) and irreducible.

Indeed, suppose we have $Z_1 \times Z_2 = W_1 \cup W_2$, with W_1, W_2 closed subsets. For every point $P \in Z_1$ we have that $\{P\} \times Z_2$ is homeomorphic to Z_2 (see Exercise 1.6.35), so it is closed and irreducible. Then either $\{P\} \times Z_2 \subseteq W_1$ or $\{P\} \times Z_2 \subseteq W_2$. Let us set

$$Z_{1,i} = \{P \in Z_1 : \{P\} \times Z_2 \subseteq W_i\}, \quad \text{for} \quad i = 1, 2,$$

and let us prove that $Z_{1,1}, Z_{1,2}$ are closed subsets of Z_1. For every point $Q \in Z_2$, set

$$Z_1^i(Q) = \{P \in Z_1 : (P, Q) \in W_i\}, \quad \text{for} \quad i = 1, 2.$$

We have

$$(Z_1 \times \{Q\}) \cap W_i = Z_1^i(Q) \times \{Q\}, \quad \text{for} \quad i = 1, 2,$$

hence $Z_1^i(Q)$ is closed for $i = 1, 2$. Since

$$Z_{1,i} = \bigcap_{Q \in Z_2} Z_1^i(Q), \quad \text{for} \quad i = 1, 2,$$

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
https://doi.org/10.1007/978-3-030-71021-7_4

we have that $Z_{1,i}$ is closed, for $i = 1, 2$. Since Z_1 is irreducible, we have either $Z_1 = Z_{1,1}$ or $Z_1 = Z_{1,2}$, and therefore, either $Z_1 \times Z_2 = W_1$ or $Z_1 \times Z_2 = W_2$.

Proposition 4.1.3 *Let X be a topological space and let Y be a subset of X. Then:*

(a) Y is irreducible if and only if for every pair of distinct points P_1, P_2 of Y, there is an irreducible subset of Y containing P_1 and P_2;
(b) if Y is irreducible and U is a non–empty open subset of Y, then U is dense in Y;
(c) Y is irreducible if and only if $\bar{Y}$ is irreducible;
(d) Y is irreducible if and only if every non–empty open subset of Y is irreducible.

Proof Part (a) is obvious and can be left to the reader.

Let us prove (b). If U were not dense, we would have $Y = \bar{U} \cup (Y - U)$, with $\bar{U}$ and $Y - U$ proper closed subsets of Y, a contradiction.

Let us prove (c). Suppose first Y irreducible. Assume $\bar{Y} = Y_1 \cup Y_2$, with Y_1, Y_2 closed subsets. Since $Y = (Y \cap Y_1) \cup (Y \cap Y_2)$, we must have either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ and therefore either $\bar{Y} \subseteq Y_1$ or $\bar{Y} \subseteq Y_2$. Suppose, conversely, that $\bar{Y}$ is irreducible, and assume $Y = Y_1 \cup Y_2$, with Y_1, Y_2 proper closed subsets of Y. Then there are closed subsets X_1, X_2 of X such that $Y_i = Y \cap X_i$, for $i = 1, 2$. Then $Y \subseteq X_1 \cup X_2$. Thus $\bar{Y} \subseteq \overline{X_1 \cup X_2} = X_1 \cup X_2$. This implies that either $\bar{Y} \subseteq X_1$ or $\bar{Y} \subseteq X_2$, hence either $Y \subseteq X_1$ or $Y \subseteq X_2$ and therefore either $Y = Y_1$ or $Y = Y_2$.

Finally, let us prove (d). One implication is trivial. As for the other, assume Y is irreducible and let $U \subseteq Y$ be a non–empty open subset. Let Y_1, Y_2 closed subsets of Y such that $U = (U \cap Y_1) \cup (U \cap Y_2)$. Then we have $Y = (Y \setminus U) \cup (Y_1 \cup Y_2)$ and, by the irreducibility of Y, we have either $Y = Y_1$ or $Y = Y_2$, hence either $U \subseteq Y_1$ or $U \subseteq Y_2$. □

Example 4.1.4 Every non–empty open subset of $\mathbb{P}^n$ is irreducible. By (d) of Proposition 4.1.3, it suffices to prove that $\mathbb{P}^n$ is irreducible. To prove this apply (a) of Proposition 4.1.3: given two distinct points P_1, P_2 of $\mathbb{P}^n$, there is the line $P_1 \vee P_2$, homeomorphic to $\mathbb{P}^1$ hence irreducible by Example 4.1.1, containing P_1 and P_2.

The following proposition gives an irreducibility criterion for affine or projective closed subsets:

Proposition 4.1.5 *One has:*

(a) $Z \subseteq \mathbb{A}^n$ is a closed irreducible non–empty subset if and only if $\mathcal{I}_a(Z)$ is a prime proper ideal, i.e., if and only if $A(Z)$ is a domain;
(b) $Z \subseteq \mathbb{P}^n$ is a closed irreducible non–empty subset if and only if $\mathcal{I}_p(Z)$ is a prime proper ideal, i.e., if and only if $S(Z)$ is a domain.

Proof We prove only (a), the proof of (b) being analogous. Let $Z \subseteq \mathbb{A}^n$ be a closed irreducible non–empty subset. If $fg \in \mathcal{I}_a(Z)$, then $(fg) \subseteq \mathcal{I}_a(Z)$ and therefore $Z = Z_a(\mathcal{I}_a(Z)) \subseteq Z_a(fg) = Z_a(f) \cup Z_a(g)$, hence either $Z \subseteq Z_a(f)$ or $Z \subseteq Z_a(g)$, thus either $f \in \mathcal{I}_a(Z)$ or $g \in \mathcal{I}_a(Z)$.

Conversely, if $\mathcal{I}_a(Z)$ is a prime proper ideal and $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed subsets, then $\mathcal{I}_a(Z) = \mathcal{I}_a(Z_1) \cap \mathcal{I}_a(Z_2)$, so either $\mathcal{I}_a(Z) = \mathcal{I}_a(Z_1)$ or $\mathcal{I}_a(Z) = \mathcal{I}_a(Z_2)$, hence either $Z = Z_1$ or $Z = Z_2$. □

In what follows we will call *quasi–projective variety* (defined over the field $\mathbb{K}$), or simply *variety*, every locally closed irreducible subset of a projective space, i.e., an irreducible set which is the intersection of a closed and an open subset of a projective space. We will call *projective variety* an irreducible closed subset of a projective space, *affine variety* an irreducible closed subset of an affine space.

Exercise 4.1.6 Prove that X is irreducible if and only if there is no expression of the sort $U_1 \cap U_2 = \emptyset$, with U_1, U_2 open, non–empty subsets of X.

Exercise 4.1.7 * Let X, Y be topological spaces, assume X is irreducible and that $f : X \to Y$ is a continuous surjective map. Prove that Y is irreducible.

Exercise 4.1.8 Prove that the projective [resp. affine] subspaces of a projective space [resp. of an affine space] are irreducible.

Exercise 4.1.9 Prove that an affine or projective twisted cubic is irreducible.

Exercise 4.1.10 Let $Z \subseteq \mathbb{P}^n$ be a closed set. Prove that Z is irreducible if and only if $C(Z)$ [resp. $\overline{C(Z)}$] is irreducible.

Exercise 4.1.11 Let $Z \subseteq \mathbb{P}^n$ be a hypersurface of $\mathbb{A}^n$ or of $\mathbb{P}^n$, with reduced equation $f_1 \cdots f_h = 0$. Prove that Z is irreducible if and only if $h = 1$. The hypersurfaces with equations $f_i = 0$, for $i = 1, \ldots, h$, are called the *irreducible components* of Z.

4.2 Noetherian Spaces

A topological space X is called a *noetherian space* if it verifies the condition of descending chains of closed subsets, i.e., for every chain of closed subsets

$$X_1 \supseteq X_2 \supseteq \ldots$$

there is an integer r such that $X_n = X_r$ for all $n \geqslant r$.

The following proposition relates the notion of irreducibility with noetherianity, and extends to any noetherian space the decomposition in irreducible components that we saw for affine and projective hypersurfaces in Exercise 4.1.11:

Theorem 4.2.1 *Let X be a noetherian topological space and let Y be a non–empty closed subset of X. Then Y can be expressed as a finite union $Y = Y_1 \cup \ldots \cup Y_h$ of closed irreducible subsets. This decomposition is unique under the condition of being* irredundant, *i.e., one has $Y_j \not\subseteq Y_i$, for every $i \neq j$ and $i, j \in \{1, \ldots, h\}$. In this case $Y_1, \ldots, Y_h$ are called the* irreducible components *of Y.*

Proof First we prove the existence of the decomposition of Y. If the assertion were false, then Y would be reducible and therefore we would have $Y = Y_1 \cup Y_1'$. with Y_1, Y_1' proper closed subsets of Y and the assertion would be false either for Y_1 or for Y_1'. Let us suppose it would be false for Y_1. By repeating the argument we would construct a sequence of closed subsets $Y \supsetneq Y_1 \supsetneq Y_2 \supsetneq \ldots$, and this is in contradiction with the noetherianity of X.

Next let us prove the uniqueness. Let $Y = \bigcup_{i=1}^{h} Y_i$ and $Y = \bigcup_{j=1}^{k} Y'_j$ be two irredundant decompositions. One has

$$Y_i = Y_i \cap Y = \bigcup_{j=1}^{k} (Y'_j \cap Y_i),$$

hence there exists a $j \in \{1, \ldots, k\}$ such that $Y_i \subseteq Y'_j$. Similarly there exists a $i' \in \{1, \ldots, h\}$ such that $Y'_j \subseteq Y_{i'}$, hence $Y_i \subseteq Y_j \subseteq Y_{i'}$. Then the irredundancy of $Y = \bigcup_{i=1}^{h} Y_i$ implies $i = i'$ and $Y_i = Y'_j$. This proves the assertion. □

Exercise 4.2.2 Prove that X is noetherian if and only if it verifies the condition of ascending open subsets.

Exercise 4.2.3 Prove that affine and projective spaces are noetherian.

Exercise 4.2.4 Prove that any subspace of a noetherian space is noetherian.

Exercise 4.2.5 Prove that any noetherian space is compact.

Exercise 4.2.6 Prove that the irreducible components of $Z_p(x_1^2 - x_0x_2, x_1^3 - x_0x_3)$ are the skew cubic and the line $x_0 = x_1 = 0$ (see Sect. 3.2.2). This example shows that, in general, the intersection of two varieties is not a variety. Indeed, the two polynomials $x_1^2 - x_0x_2, x_1^3 - x_0x_3$ are irreducible.

4.3 Topological Dimension

Let X be a topological space. We define the *topological dimension*, denoted by $\dim_{\mathrm{top}}(X)$, of X as the supremum of the integers n such that there is a chain

$$Z_0 \subsetneqq Z_1 \subsetneqq \ldots \subsetneqq Z_n \tag{4.1}$$

of distinct closed irreducible subsets of X. It is clear that if X is noetherian, $\dim_{\mathrm{top}}(X)$, is the maximum of the topological dimension of its irreducible components. A noetherian space X is said to be *pure* if all of its irreducible components have the same topological dimension. It is also clear that if X is irreducible and noetherian and every point of X is closed then:

(a) $\dim_{\mathrm{top}}(X) = 0$ if and only if X consists of only one point;
(b) $\dim_{\mathrm{top}}(X) = 1$ if and only if the proper closed subsets are its finite subsets.

We will call *curve* any pure closed algebraic set of topological dimension 1. For example the twisted cubics are irreducible curves.

Proposition 4.3.1 *Let X be a topological space and Y a subset of X. Then:*

(a) $\dim_{\mathrm{top}}(Y) \leqslant \dim_{\mathrm{top}}(X)$*; hence if* $\dim_{\mathrm{top}}(X)$ *is finite, the same happens for* $\dim_{\mathrm{top}}(Y)$*, and* $\dim_{\mathrm{top}}(X) - \dim_{\mathrm{top}}(Y) \geqslant 0$ *is called the* (topological) codimension *of Y in X and it is denoted by* $\mathrm{codim}_{\mathrm{top,X}}(Y)$*;*

(b) if $\{U_i\}_{i\in I}$ *is an open covering of* X*, then* $\dim_{\text{top}}(X) = \sup_{i\in I}\{\dim_{\text{top}}(U_i)\}$*;*
(c) if X *is irreducible,* $\dim_{\text{top}}(X)$ *is finite,* Y *is closed in* X *and* $\operatorname{codim}_{\text{top},X}(Y) = 0$*, then* $X = Y$*.*

Proof Part (a) can be left as an exercise for the reader (look at Exercise 4.2.4: with the notation of its solution, note that if Y_i is irreducible, we can assume that also X_i is irreducible, for $i = 1, 2$).

Let us prove (b). It follows from (a) and from the fact that if (4.1) is a chain of distinct irreducible closed subsets of X and if $P \in Z_0$, there is an $i \in I$ such that $P \in U_i$. Then for all $j \in \{0, \dots, n\}$, one has $U_i \cap Z_j \neq \emptyset$. Moreover by the irreducibility of Z_{j+1}, one has $U_i \cap Z_j \subsetneqq U_i \cap Z_{j+1}$, because $\overline{U_i \cap Z_j} = Z_j$.

Let us prove (c). If (4.1) is a maximal chain of distinct irreducible closed subsets of Y, it is also a maximal chain of distinct irreducible closed subsets of X. Then $Z_n = X \subseteq Y$ and $Y = X$. □

Let A be a ring and $\mathcal{I} \subseteq A$ a prime ideal. We call *height* of $\mathcal{I}$, denoted by $\text{height}(I)$, the supremum of the integers n such that there is a chain

$$\mathcal{I}_0 \subsetneqq \mathcal{I}_1 \subsetneqq \dots \subsetneqq \mathcal{I}_n = \mathcal{I}$$

of distinct prime ideals of A. One calls *Krull dimension* of A, denoted by $\dim_K(A)$, the supremum of heights of its prime ideals. From Propositions 3.1.1 and 4.1.5 it follows that if Z is an affine closed subset one has $\dim_{\text{top}}(Z) = \dim_K(A(Z))$.

Exercise 4.3.2 Prove that affine and projective lines have topological dimension 1.

Exercise 4.3.3 Prove that (affine or projective) plane curves have topological dimension 1.

Exercise 4.3.4 Prove that (affine or projective) planes have topological dimension 2.

Exercise 4.3.5 Prove that $\dim_{\text{top}}(\mathbb{A}^n) = \dim_{\text{top}}(\mathbb{P}^n)$ and that $\dim_{\text{top}}(\mathbb{A}^n) \geqslant n$.

Exercise 4.3.6 Prove that any bijection between two curves is a homeomorphism.

4.4 Solutions of Some Exercises

4.2.3 By Proposition 3.1.1, the noetherianity of affine and projective spaces is equivalent to the noetherianity of the ring of polynomials.
4.2.4 It follows from the following simple remark. Let X be a topological space and Y a subset of X. Let $Y_1 \supsetneqq Y_2$ be closed subsets of Y. Then there are closed subsets $X_1 \supsetneqq X_2$ of X, such that $Y_i = Y \cap X_i$, for $i = 1, 2$.
4.2.5 Let $\{U_i\}_{i\in I}$ be an open covering of X and suppose we cannot extract from it a finite covering. Then there would be a sequence $\{i_n\}_{n\in\mathbb{N}}$ of elements of I and a sequence $\{P_n\}_{n\in\mathbb{N}}$ of points of X such that $P_n \notin \bigcup_{h=1}^n U_{i_h}$, whereas $P_{n-1} \in \bigcup_{h=1}^n U_{i_h}$. This contradicts the condition of ascending chains of open subsets.
4.3.4 Apply Corollary 2.2.2.
4.3.5 To prove that $\dim_{\text{top}}(\mathbb{A}^n) = \dim_{\text{top}}(\mathbb{P}^n)$ apply Proposition 4.3.1, (b). To prove that $\dim_{\text{top}}(\mathbb{A}^n) \geqslant n$, note that there is a chain of length $n + 1$ of distinct closed irreducible subsets of $\mathbb{A}^n$ formed by affine subspaces.

Chapter 5
Regular and Rational Functions

5.1 Regular Functions

Let $V \subseteq \mathbb{P}^n$ be a locally closed subset. Let $f : V \to \mathbb{K}$ be a function and let P be a point of V. We will say that f is *regular* at P if there is an open neighborhood U of P in V and there are homogeneous polynomials of the same degree $g, h \in S_n$, with $Z_p(h) \cap U = \emptyset$, such that the restriction of f to U coincides with the restriction of $\frac{g}{h}$ to U (note that $\frac{g}{h}$, as a function of $\mathbb{P}^n \setminus Z_p(h)$ to $\mathbb{K}$, is well defined). We will say that f is *regular* in V if it is regular at any point of V. Note that any set V which is locally closed in $\mathbb{A}^n$ is also locally closed in $\mathbb{P}^n$, if we consider $\mathbb{A}^n$ as identified with the open set U_0 of $\mathbb{P}^n$. The reader will verify that in this case a function $f : V \to \mathbb{K}$ is regular at $P \in V$ if and only if there is an open neighborhood U of P in V and there are polynomials $g, h \in A_n$, with $Z_a(h) \cap U = \emptyset$ such that f coincides with the restriction of $\frac{g}{h}$ to U (see Exercise 5.1.2). Note that constant functions are regular.

Let $V \subseteq \mathbb{P}^n$ be a locally closed subset and let U be a non–empty open subset of V. We will denote by $\mathcal{O}_V(U)$ (or simply by $\mathcal{O}(U)$ if V is intended), the set of regular functions on U. If $f, g \in \mathcal{O}(U)$, the functions

$$f + g : P \in U \to f(P) + g(P) \in \mathbb{K}, \quad fg : P \in U \to f(P) \cdot g(P) \in \mathbb{K},$$

are regular. Then $\mathcal{O}(U)$, with the above two operations, is a $\mathbb{K}$–algebra which is called the *algebra of regular functions* in U. Let $U' \subseteq U$ be two open subsets of V. There is a natural map

$$r^U_{U'} : f \in \mathcal{O}(U) \to f_{|U'} \in \mathcal{O}(U'),$$

where $f_{|U'}$ is the restriction of f to U'. This map is called the *restriction map* and it is a homomorphism of $\mathbb{K}$–algebras. If $f \in \mathcal{O}(U)$ we will write $Z_U(f)$ (or $Z(f)$ if U is intended) to denote $f^{-1}(0)$, and we will call $Z_U(f)$ the *zero locus* of f in U.

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
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Proposition 5.1.1 *One has:*

(a) if $V \subseteq \mathbb{P}^n$ *is a locally closed subset and* $f \in \mathcal{O}(V)$*, then* f *is continuous in the Zariski topologies of* V *and of* $\mathbb{K} = \mathbb{A}^1$*;*

(b) if V *is irreducible and* $f, g \in \mathcal{O}(V)$ *are such that there is a non–empty open subset of* V *such that* $f_{|U} = g_{|U}$*, then* $f = g$*.*

Proof To prove (a) it suffices to prove that for all $a \in \mathbb{K}$, $f^{-1}(a) = Z_V(f - a)$ is closed in V. This can be verified locally (see Exercise 5.1.3). Let $P \in V$ and let U be an open neighborhood of P in V such that $f - a = \frac{g}{h}$ on U, with $g, h \in S_n$ of the same degree and $Z_p(h) \cap U = \emptyset$. Then $Z_V(f - a) \cap U = Z_p(g) \cap U$, which is closed in U.

As for (b), note that, by part (a), $Z_V(f - g)$ is closed and it contains the open dense subset U (see Proposition 4.1.3, (b)). □

By Proposition 5.1.1, if $f \in \mathcal{O}(V)$ then $Z_V(f)$ is closed and $U_V(f) = Z \setminus Z_V(f)$ (also denoted by $U(f)$ if V is intended) is an open subset, which is called the *principal open set* associated to f. In $U(f)$ the function $\frac{1}{f}$ is well defined and regular.

Exercise 5.1.2 Let V be a locally closed subset of $\mathbb{A}^n$. Prove that a function $f : V \to \mathbb{K}$ is regular in $P \in V$ if and only if there is an open neighborhood U of P in V and there are polynomials $g, h \in A_n$, with $Z_a(h) \cap U = \emptyset$ such that f coincides with the restriction of $\frac{g}{h}$ to U.

Exercise 5.1.3 * Let X be a topological space and $Y \subseteq X$. Prove that Y is closed if and only if for any point $P \in X$ there is an open neighborhood U of P in X such that $U \cap Y$ is closed in U.

Exercise 5.1.4 Let V be a quasi–projective variety and W a subvariety of V. Prove that if $f \in \mathcal{O}(V)$, then $f_{|W} \in \mathcal{O}(W)$. Prove that the map $f \in \mathcal{O}(V) \to f_{|W} \in \mathcal{O}(W)$ is a homomorphism of $\mathbb{K}$–algebras called *restriction map*.

Exercise 5.1.5 Let V be a quasi–projective variety. Prove that $\mathcal{O}(V)$ is a domain.

Exercise 5.1.6 * Let V be a quasi–projective variety. Prove that the principal open subsets of V are a basis for the Zariski topology of V.

5.2 Rational Functions

Let V be a quasi–projective variety. Let us consider the set $\mathcal{K}(V)$ formed by all pairs (U, f) where U is a non–empty open subset of V and $f \in \mathcal{O}(U)$. Let us define in $\mathcal{K}(V)$ the following relation $\mathcal{R}$

$$(U, f)\mathcal{R}(U', f') \quad \text{if and only if} \quad f_{|U\cap U'} = f'_{|U\cap U'}.$$

Note that, since V is irreducible, $U \cap U' \neq \emptyset$. The relation $\mathcal{R}$ is an equivalence relation (see Exercise 5.2.1). We will denote by $K(V)$ the quotient set $\mathcal{K}(V)/\mathcal{R}$. Any element of $K(V)$ is called a *rational function* on V. The $\mathcal{R}$–equivalence class of the pair (U, f) is denoted by $[U, f]$, or with f if U is intended.

Now we will endow $K(V)$ with the structure of a field, which will be called the *field of rational functions* of V. Let $[U, f]$ and $[U', f']$ be elements of $K(V)$. We define

$$[U, f] + [U', f'] = [U \cap U', f_{U\cap U'} + f'_{U\cap U'}], \ [U, f] \cdot [U', f'] = [U \cap U', f_{U\cap U'} \cdot f'_{U\cap U'}].$$

By Proposition 5.1.1, (b), these definitions are well posed and $K(V)(+, \cdot)$ is an extension of $\mathbb{K}$. The immersion of $\mathbb{K}$ in $K(V)$ is given by

$$a \in \mathbb{K} \to [V, a] \in K(V).$$

The inverse of $[U, f] \neq 0$ is given by $[U, f]^{-1} = [U \setminus Z_U(f), \frac{1}{f}]$. Note that for every non–empty open subset U of V, $K(V)$ is an extension of the algebra $\mathcal{O}(U)$, where the immersion of $\mathcal{O}(U)$ in $K(V)$ is given by

$$r_U : f \in \mathcal{O}(U) \to [U, f] \in \mathcal{K}(V).$$

Exercise 5.2.1 * Prove that the relation $\mathcal{R}$ in $\mathcal{K}(V)$ is an equivalence relation.

Exercise 5.2.2 * Let V be a quasi–projective variety and let $[U, f]$ be a rational function on V. Prove that there exists a pair $(\tilde{U}, \tilde{f}) \in \mathcal{K}(V)$ such that $[\tilde{U}, \tilde{f}] = [U, f]$ and for each pair $(U', f') \in \mathcal{K}(V)$ such that $[U', f'] = [U, f]$, one has $U' \subseteq \tilde{U}$. The open set $\tilde{U}$ is called the *definition set* of $[U, f]$.

5.3 Local Rings

In this section we introduce the concept of a *local ring*, which will play an important role in the sequel.

A ring A is called a local ring if it has a unique maximal ideal $\mathfrak{m}$, and we will express this by saying that $(A, \mathfrak{m})$ is a local ring. The ring $A/\mathfrak{m}$ has no non–trivial ideals, hence it is a field, called the *residue field* of $(A, \mathfrak{m})$.

Proposition 5.3.1 *Let A be a ring and $\mathfrak{m} \subset A$ be an ideal. Then $(A, \mathfrak{m})$ is a local ring if and only if $A \setminus \mathfrak{m}$ coincides with the set of invertible elements in A.*

Proof Suppose $(A, \mathfrak{m})$ is local. Then if $x \in A$ is invertible, then $(x) = A$ and therefore $x \notin \mathfrak{m}$. On the other hand, if $x \notin \mathfrak{m}$, then (x) is not contained in $\mathfrak{m}$ hence $(x) = A$ and x is invertible.

Conversely, assume that $A \setminus \mathfrak{m}$ coincides with the set of invertible elements in A. If $\mathcal{I} \subsetneq A$ is an ideal of A, no element of $\mathcal{I}$ is invertible, hence $\mathcal{I} \subseteq \mathfrak{m}$, hence $\mathfrak{m}$ is the maximal ideal of A. □

There is a standard construction, called *localization*, producing local rings starting with any ring.

Let A be a ring and let $\mathfrak{S}$ be a subset of A which is *multiplicatively closed*, i.e., if $s, t \in \mathfrak{S}$, then $st \in \mathfrak{S}$. Moreover assume that $\mathfrak{S}$ contains 1 and does not contain 0. In $A \times \mathfrak{S}$ define the following relation:

$$(a, s) \equiv (b, t) \quad \text{if and only if there is a} \quad u \in \mathfrak{S} \quad \text{such that} \quad u(at - bs) = 0.$$

This is an equivalence relation. The equivalence class of (a, s) is usually denoted by $\frac{a}{s}$ and the set $A \times \mathfrak{S}/\equiv$ is denoted by $A_{\mathfrak{S}}$. In $A_{\mathfrak{S}}$ one introduces the following operations

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}, \quad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st},$$

which are easily seen to be well defined and $A_{\mathfrak{S}}$, with these two operations, is a unitary, commutative ring as well as A. The ring $A_{\mathfrak{S}}$ is called the *localization* of A with respect to $\mathfrak{S}$.

There is a natural homomorphism

$$j : a \in A \to \frac{a}{1} \in A_{\mathfrak{S}}$$

which in general is not injective, but it is so if A is a domain. In this case we will identify A with $j(A)$ and $A_{\mathfrak{S}}$ will be identified with a subring of the quotient field $\mathbb{Q}(A)$ of A.

If A is a graded ring and $\mathfrak{S}$ is, as above, a multiplicatively closed set such that $\mathfrak{S}$ contains 1 and does not contain 0, then we will denote by $A_{(\mathfrak{S})}$ the subset of $A_{\mathfrak{S}}$ consisting of fractions $\frac{a}{s}$ such that a, s are homogeneous of the same degree. Then $A_{(\mathfrak{S})}$ is a subring of $A_{\mathfrak{S}}$, called the *homogeneous localization* of A with respect to $\mathfrak{S}$.

Example 5.3.2 Let A be a ring and $\mathcal{I}$ a proper prime ideal of A. Then $\mathfrak{S} = A \setminus \mathcal{I}$ is multiplicatively closed, contains 1 and does not contain 0. Then we can consider the localization $A_{\mathfrak{S}}$, which is also denoted by $A_{\mathcal{I}}$ and it is called the *localization* of A with respect to $\mathcal{I}$. Consider $\mathfrak{m}$ the ideal generated by $j(\mathcal{I})$ in $A_{\mathcal{I}}$. It is clear that $\frac{a}{s} \notin \mathfrak{m}$ if and only if $a \notin \mathcal{I}$, thus if and only if $\frac{s}{a} = (\frac{a}{s})^{-1} \in A_{\mathcal{I}}$. It follows that $(A_{\mathcal{I}}, \mathfrak{m})$ is a local ring (see Proposition 5.3.1). The residue field of $(A_{\mathcal{I}}, \mathfrak{m})$ is $\mathbb{Q}(A/\mathcal{I})$.

In the above setting, if A is graded, we can consider $A_{(\mathfrak{S})}$, which is also denoted by $A_{(\mathcal{I})}$, and it is called the *homogeneous localization* of A with respect to $\mathcal{I}$. It is clear that $A_{(\mathcal{I})}$ is local with maximal ideal $\mathfrak{m} \cap A_{(\mathcal{I})}$.

If A is a domain and $\mathcal{I} = (0)$, one has $A_{\mathcal{I}} = \mathbb{Q}(A)$. If A is graded, $A_{(\mathcal{I})}$ is a subfield of $\mathbb{Q}(A)$ which is denoted by $\mathbb{Q}(A)_0$.

Example 5.3.3 Let A be a ring and $f \in A$ a *non–nilpotent* element, i.e., f is such that for all positive integers i, one has $f^i \neq 0$. Then $\mathfrak{S} = \{f^i\}_{i \in \mathbb{N}}$ is multiplicatively closed, contains $1 = f^0$ and does not contain 0. We can consider $A_{\mathfrak{S}}$, which is also denoted by A_f. Note that

$$A_f = A\left[\frac{1}{f}\right] = A[x]/(fx - 1).$$

If A is graded and f is homogeneous, we write $A_{(f)}$ instead of $A_{(\mathfrak{S})}$.

If $\mathcal{I}$ is an ideal of A, we will denote by $\mathcal{I}^e := \mathcal{I}A_{\mathfrak{S}}$ the ideal generated by $j(\mathcal{I})$ in $A_{\mathfrak{S}}$, i.e., the ideal formed by all fractions of the form $\frac{a}{s}$ with $a \in \mathcal{I}$. It is called the *extension* of $\mathcal{I}$ to $A_{\mathfrak{S}}$. Since $\mathcal{I} = j^{-1}(j(A) \cap \mathcal{I}^e)$, it is clear that all ideals in $A_{\mathfrak{S}}$ are extended ideals. Moreover it is easy to see that $\mathcal{I}^e$ is a proper ideal of $A_{\mathfrak{S}}$ if and only if $\mathcal{I} \cap \mathfrak{S} = \emptyset$ (see Exercise 5.3.7).

Let now $\mathcal{J}$ be an ideal of $A_{\mathfrak{S}}$. We set

$$\mathcal{J}^c = \left\{a \in A : \frac{a}{s} \in \mathcal{I} \quad \text{for some} \quad s \in \mathfrak{S}\right\}.$$

This is an ideal of A, called the *contraction* of $\mathcal{J}$ to A. It is clear that:

(a) for every ideal $\mathcal{I}$ of A one has $\mathcal{I} \subseteq (\mathcal{I}^e)^c$;
(b) for every ideal $\mathcal{J}$ of $A_{\mathfrak{S}}$ one has $\mathcal{J} = (\mathcal{J}^c)^e$.

Proposition 5.3.4 *Let A be a ring and let $\mathfrak{S}$ be a multiplicatively closed subset of A containing 1 and not containing 0. One has:*

(a) if $\mathcal{I}$ is a prime ideal of A such that $\mathcal{I} \cap \mathfrak{S} = \emptyset$, then $\mathcal{I}^e$ is a prime ideal of $A_{\mathfrak{S}}$;
(b) if $\mathcal{I}$ is a prime ideal of A such that $\mathcal{I} \cap \mathfrak{S} = \emptyset$, then $(\mathcal{I}^e)^c = \mathcal{I}$;
(c) there is a 1:1 correspondence between prime ideals of $A_{\mathfrak{S}}$ and prime ideals of A with empty intersection with $\mathfrak{S}$, given by contraction and extension of ideals.

Proof Let us prove (a). If $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st} \in \mathcal{I}^e$, then there is $u \in \mathfrak{S}$ such that $uab \in \mathcal{I}$. Since $u \notin \mathcal{I}$ and $\mathcal{I}$ is prime, one has $ab \in \mathcal{I}$. Since $\mathcal{I}$ is prime, we conclude that either $\frac{a}{s}$ or $\frac{b}{t}$ is in $\mathcal{I}^e$, proving that $\mathcal{I}^e$ is prime.

As for (b), if $b \in (\mathcal{I}^e)^c$, there are $a \in \mathcal{I}$, $s, t \in \mathfrak{S}$, such that $\frac{a}{s} = \frac{b}{t}$. Then there is $u \in \mathfrak{S}$ such that $uta = usb$. Since $uta \in \mathcal{I}$ and $us \in \mathfrak{S}$ is not in $\mathcal{I}$, then $b \in \mathcal{I}$, proving the assertion.

Finally, it is clear that if $\mathcal{J}$ is a prime ideal of $A_{\mathfrak{S}}$, then $\mathcal{J}^c$ is a prime ideal of A. Whence (c) follows, by taking into account (a) and (b). □

Exercise 5.3.5 * Prove that the relation $\equiv$ in $A \times \mathfrak{S}$ is an equivalence relation.

Exercise 5.3.6 * Prove that the operations $+, \cdot$ in $A_{\mathfrak{S}}$ are well defined and $A_{\mathfrak{S}}$, with these two operations, is a unitary, commutative ring.

Exercise 5.3.7 Prove that $\mathcal{I}^e$ is a proper ideal of $A_{\mathfrak{S}}$ if and only if $\mathcal{I} \cap \mathfrak{S} = \emptyset$.

Exercise 5.3.8 Prove that:

(a) for every ideal $\mathcal{I}$ of A one has $\mathcal{I} \subseteq (\mathcal{I}^e)^c$;
(b) for every ideal $\mathcal{J}$ of $A_{\mathfrak{S}}$ one has $\mathcal{J} = (\mathcal{J}^c)^e$.

Exercise 5.3.9 Let A be a ring and $\mathcal{I}$ a prime ideal of A. Prove that extension and contraction provide a 1:1 correspondence between the prime ideals of $A_{\mathcal{I}}$ and the prime ideals of A contained in $\mathcal{I}$.

5.4 Integral Elements over a Ring

Let A be a domain and B a domain containing A. An element $x \in B$ is said to be *integral* on A if x is a root of a monic polynomial with coefficients in A, i.e., if there are elements $a_1, \ldots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0.$$

Lemma 5.4.1 *The following are equivalent:*

(a) $x \in B$ *is integral on* A;
(b) $A[x]$ *is a finitely generated* A*–module.*

Proof It is clear that (a) implies (b). Conversely, assume that $A[x]$ is generated by $y_1, \ldots, y_n$ as an A–module. Then we have $xy_i = \sum_{j=1}^n a_{ij} y_j$, with $a_{ij} \in A$ and $i = 1, \ldots, n$. Hence the homogenous linear system

$$\begin{aligned} (a_{11} - x)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - x)x_n &= 0 \end{aligned}$$

in $x_1, \ldots, x_n$ with coefficients in B has some non–trivial solution. Thus the determinant of the matrix of the system is zero, which implies that x is integral on A. □

If $x, y \in B$ are integral on A, then $A[x, y]$ is a finitely generated $A[x]$–module, which in turn is a finitely generated A–module. Hence $A[x, y]$ is a finitely generated A–module and $A[x \pm y]$ and $A[xy]$ are contained in $A[x, y]$. By applying Exercise 5.4.5, one sees that $x \pm y$ and xy are integral on A. Hence the set of elements of B which are integral on A is a subring of B (which contains A), that is called the *integral closure* of A in B. The integral closure of A in $\mathbb{Q}(A)$ is called the *integral closure* of A and A is called *integrally closed* if it coincides with its integral closure.

Exercise 5.4.2 Let k be a field and let $a \in k$. Prove that the ring $k[x]_{(x-a)}$ is integrally closed.

Exercise 5.4.3 * Let A be a UFD. Prove that A is integrally closed.

Exercise 5.4.4 * Let A be a ring, M a finitely generated A–module, $\mathcal{I}$ an ideal of A and $\phi : M \to M$ an endomorphism of A–modules such that $\phi(M) \subseteq \mathcal{I}M$. Prove that ϕ verifies an equation of the form

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0, \quad \text{with} \quad a_1, \ldots, a_n \in \mathcal{I}.$$

Exercise 5.4.5 * Let A be a ring. An A–module M is said to be *faithful* if $\mathrm{Ann}(M) = (0)$, i.e., if for any $a \in A$ such that $aM = (0)$, one has $a = 0$.

Let A, B be domains, with $A \subseteq B$. Prove that the following propositions are equivalent:

(a) $x \in B$ is integral over A;
(b) $A[x]$ is contained in a subring C of B such that C is a finitely generated A–module;

(c) there is a faithful $A[x]$–module M which is finitely generated as an A–module.

Exercise 5.4.6 * Let A, B be domains, with $A \subseteq B$. Prove that if B is a finitely generated as an A–module, then any element $x \in B$ is integral over A.

Exercise 5.4.7 * Let A, B be finitely generated $\mathbb{K}$–algebras, with $A \subseteq B$ and B integral over A. Prove that B is finitely generated as an A–module.

Exercise 5.4.8 * Let $A \subseteq B \subseteq C$ be noetherian domains, and assume that B is integral over B and C integral over B. Prove that C is integral over A. Deduce that if $A \subseteq B$ are domains and C is the integral closure of A in B, then C is integrally closed in B.

5.5 Subvarieties and Their Local Rings

Let V be a quasi–projective variety and let W be an irreducible, locally closed subset of V. Then W, as well as V, is a quasi–projective variety, that we will call a *subvariety* of V. Given a rational function $[U, f] \in K(V)$, we will say that it is *defined* on W, if there is $(U', f') \in \mathcal{K}(V)$ such that $(U', f')\mathcal{R}(U, f)$ and $U' \cap W \neq \emptyset$. Then $[U' \cap W, f'_{|U' \cap W}]$ is a rational function on W that is uniquely determined by $[U, f]$ (see Exercise 5.5.4) and it is called the *restriction* of $[U, f]$ to W.

We will denote by $\mathcal{O}_{V,W}$ the set of rational functions on V that are defined on W. Note that $\mathcal{O}_{V,V} = K(V)$. The irreducibility of W implies that $\mathcal{O}_{V,W}$ is a subring of $K(V)$. Moreover $\mathcal{O}(V)$ is a subring of $\mathcal{O}_{V,W}$. Let us consider the subset $\mathfrak{m}_{V,W}$ of $\mathcal{O}_{V,W}$ (also denoted by $\mathfrak{m}_W$ if V is intended) formed by the rational functions whose restrictions to W are zero. It is clear that $\mathfrak{m}_{V,W}$ is an ideal of $\mathcal{O}_{V,W}$.

Proposition 5.5.1 *Let V be a quasi projective variety and let W be a subvariety of V. Then $(\mathcal{O}_{V,W}, \mathfrak{m}_{V,W})$ is a local ring with residue field $K(W)$.*

Proof Let $[U, f] \in \mathcal{O}_{V,W} \setminus \mathfrak{m}_{V,W}$, with $U \cap W \neq \emptyset$. Then $Z_U(f) \cap W$ is a proper closed subset of $U \cap W$. Set $U' = U \setminus Z_U(f)$ and consider in U' the function, which is there well defined, $f' = \frac{1}{f}$. It is clear that $f' \in \mathcal{O}(U')$ and that

$$U' \cap W = (U \setminus Z_U(f)) \cap W = (U \cap W) \setminus (Z_U(f) \cap W) \neq \emptyset,$$

so that $[U, f]^{-1} = [U', f'] \in \mathcal{O}_{V,W}$, i.e., $[U, f]$ is invertible. By Proposition 5.3.1, this proves that $(\mathcal{O}_{V,W}, \mathfrak{m}_{V,W})$ is a local ring.

Let us consider the residue field $K_{V,W}$ of $\mathcal{O}_{V,W}$, and consider the map

$$\phi_{V,W} : [U, f] + \mathfrak{m}_{V,W} \in K_{V,W} \to [U \cap W, f_{|U \cap W}] \in K(W).$$

It is clear that this map is well defined and it is injective. Let us prove it is surjective. Let $[U', f'] \in K(W)$ and let $P \in U'$. Then there is an open neighborhood U'' of P in U', such that in U'' one has $f = \frac{g}{h}$ with g, h homogeneous polynomials of the same degree and h non–zero on U''. Let $\tilde{U}$ be an open subset of $\mathbb{P}^n$ such that $\tilde{U} \cap W = U''$

and $\tilde{U} \cap Z_p(h) = \emptyset$. Let us set $U = \tilde{U} \cap V$ which is not empty, and in U consider the regular function $f = \frac{g}{h}$. It is clear that $\phi_{V,W}([U, f] + \mathfrak{m}_{V,W}) = [U', f']$. This proves the assertion. □

The local ring $(\mathcal{O}_{V,W}, \mathfrak{m}_{V,W})$ is called the *local ring of W in V*.

Let V and W be as above, and let U be an open subset of V such that $W' := U \cap W \neq \emptyset$. Then it makes sense to consider the local ring $\mathcal{O}_{U,W'}$.

Lemma 5.5.2 *In the above setting, one has $\mathcal{O}_{V,W} \cong \mathcal{O}_{U,W'}$. In particular $K(V) \cong K(U)$.*

Proof If $[U', f'] \in \mathcal{O}_{U,W'}$, then $[U', f']$ can be also considered as an element of $\mathcal{O}_{V,W}$, hence we have an injective homomorphism

$$[U', f'] \in \mathcal{O}_{U,W'} \to [U', f'] \in \mathcal{O}_{V,W}.$$

It is also surjective. In fact, if $[U'', f''] \in \mathcal{O}_{V,W}$, it comes from $[U \cap U'', f''_{|U \cap U''}] \in \mathcal{O}_{U,W'}$. □

Before proceeding, we introduce a notation which will be useful in the sequel. If $V \subseteq \mathbb{A}^n$ is an affine variety, we will abuse notation and we will still denote by $x_1, \ldots, x_n$ the images in $A(V)$ of $x_1, \ldots, x_n$ via the canonical epimorphism $A_n \to A(V)$. Then $A(V)$ is generated, as a $\mathbb{K}$–algebra, by $x_1, \ldots, x_n$. Similarly, if $V \subseteq \mathbb{P}^n$ is a projective variety, we will denote by $x_0, \ldots, x_n$ the images in $S(V)$ of $x_0, \ldots, x_n$ via the canonical epimorphism $S_n \to S(V)$. Then $S(V) = \bigoplus_{d \in \mathbb{N}} S(V)_d$ and $S(V)_d$ is generated, as a $\mathbb{K}$–vector space, by the monomials of degree d in $x_0, \ldots, x_n$. So, if $f(x_1, \ldots, x_n) \in A_n$, its image in $A(V)$ will still be denoted by $f(x_1, \ldots, x_n)$, and similarly in the projective case.

We can now prove the basic:

Theorem 5.5.3 *Let $V \subset \mathbb{A}^n$ be an affine variety. Then:*

(a) $\mathcal{O}(V) = A(V)$;
(b) if W is a subvariety of V, then

$$\mathcal{I}_W(V) = \{f \in \mathcal{O}(V) : W \subseteq Z_V(f)\}$$

is a prime ideal of $\mathcal{O}(V)$ and every prime ideal of $\mathcal{O}(V)$ is obtained in this way; moreover $\mathcal{I}_W(V)$ is maximal if and only if W is a point;
(c) if W is a subvariety of V, then $\mathcal{O}_{V,W} = \mathcal{O}(V)_{\mathcal{I}_W(V)}$; in particular $K(V) = \mathbb{Q}(A(V))$.

If $V \subseteq \mathbb{P}^n$, then:

(d) $\mathcal{O}(V) = \mathbb{K}$;
(e) if W is a subvariety of V and $\mathcal{I}_{p,W}(V)$ is the image ideal of $\mathcal{I}_p(W)$ via the canonical epimorphism $\pi : S_n \to S(V)$, then $\mathcal{I}_{p,W}(V)$ is a homogeneous prime ideal of $S(V)$ and every homogeneous non–irrelevant prime ideal of $S(V)$ is obtained in this way; moreover $\mathcal{I}_{p,W}(V)$ is maximal if and only if W is a point;

(f) $\mathcal{O}_{V,W} = S(V)_{(\mathcal{I}_{p,W}(V))}$.

Proof Every polynomial $f \in A_n$ defines a regular function on V, so there is a natural homomorphism $A_n \to \mathcal{O}(V)$, whose kernel is $\mathcal{I}_a(V)$. Hence there is an injective homomorphism $\alpha : A(V) \to \mathcal{O}(V)$. From Sect. 3 we know that there is a 1:1 correspondence between maximal ideals of $A(V)$ and points of V. Precisely, by identifying elements of $A(V)$ with regular functions via α, the ideal corresponding to a point P, is $\mathfrak{m}_P = \{f \in A(V) : f(P) = 0\}$. Note now that there is a natural homomorphism

$$\alpha_P : A(V)_{\mathfrak{m}_P} \to \mathcal{O}_{V,P}$$

defined in the following way. If $\frac{f}{g} \in A(V)_{\mathfrak{m}_P}$, let U be the principal open neighborhood of P in V associated to g. Then we set

$$\alpha_P\left(\frac{f}{g}\right) = \left[U, \frac{f}{g}\right],$$

where f, g are considered as polynomials in A_n. Since α is injective, then also α_P is injective. Moreover α_P is also surjective, because every regular function is locally of the form $\frac{f}{g}$, with f, g polynomials. Hence we have $A(V)_{\mathfrak{m}_P} \cong \mathcal{O}_{V,P}$. Note now that $\mathcal{O}(V) \subseteq \bigcap_{P\in V} \mathcal{O}_{V,P}$, and this implies that

$$A(V) \subseteq \mathcal{O}(V) \subseteq \bigcap_{P\in V} \mathcal{O}_{V,P} = \bigcap_{\mathfrak{m}} A(V)_{\mathfrak{m}} \tag{5.1}$$

where the last intersection is over all maximal ideals of $A(V)$. Now the rightmost and leftmost terms in (5.1) are equal (see Exercise 5.5.9). Then (a) follows.

Part (b) follows by the results of Sect. 3. Part (c) has been proved if W is a point. The general case is completely analogous.

Next let us move to the projective case. Let $i \in \{0, \dots, n\}$ be such that $V_i = V \cap U_i \neq \emptyset$ (notation as in Sect. 1.5). Let us start by proving that $A(V_i) \cong S(V)_{(x_i)}$. Let us assume, to fix the ideas, that $i = 0$, and consider the homomorphism

$$\phi : f(x_1, \dots, x_n) \in A_n \to f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in (S_n)_{(x_0)}.$$

It is clear that ϕ is an isomorphism that sends $\mathcal{I}_a(V_0)$ to $\mathcal{I}_p(V)(S_n)_{(x_0)}$, hence, passing to the quotient, ϕ induces an isomorphism $\bar{\phi} : A(V_0) \to S(V)_{(x_0)}$.

Next let W be a closed subvariety of V and let us choose an $i \in \{0, \dots, n\}$ such that $W_i = W \cap U_i \neq \emptyset$. Assume again $i = 0$. Then $\mathcal{O}_{V,W} \cong \mathcal{O}_{V_0,W_0}$ by Lemma 5.5.2. Moreover, by part (c) we have $\mathcal{O}_{V_0,W_0} \cong \mathcal{O}(V_0)_{\mathcal{I}_{W_0}(V_0)}$. Also $A(V_0) = \mathcal{O}(V_0)$, by part (a) and $\mathcal{I}_{W_0}(V_0)$ corresponds to the ideal $\mathcal{I}_{a,W_0}(V_0)$ of $A(V_0)$, which is the image of $\mathcal{I}_a(W_0)$ via the canonical epimorphism of A_n to $A(V_0)$. Finally $\bar{\phi}$ maps $\mathcal{I}_{a,W_0}(V_0)$ to $\mathcal{I}_{p,W}(S_n)_{(x_0)}$. In conclusion

$$\mathcal{O}_{V,W} \cong \mathcal{O}_{V_0,W_0} \cong \mathcal{O}(V_0)_{\mathcal{I}_{W_0}(V_0)} \cong A(V_0)_{\mathcal{I}_{a,W_0}(V_0)} \cong \left((S_n)_{(x_0)}\right)_{\mathcal{I}_{p,W}(V)},$$

and, since $x_0 \notin \mathcal{I}_{p,W}(V)$, one has

$$\left((S_n)_{(x_0)}\right)_{\mathcal{I}_{p,W}(V)} = (S_n)_{(\mathcal{I}_{p,W}(V))},$$

proving (f). Part (e) is an immediate consequence of the results on Sect. 3.

Finally we are left to prove (d). Let $f \in \mathcal{O}(V)$. For every $i \in \{0, \ldots, n\}$ such that $V_i \neq \emptyset$, we have $f_i := f_{|V_i} \in \mathcal{O}(V_i) \cong A(V_i) \cong S(V)_{(x_i)}$. Hence $f_i = \frac{g_i}{x_i^{m_i}}$, where $g_i \in S(V)$ is homogeneous of degree $m_i \geqslant 0$. Let us look at $\mathcal{O}(V)$, $K(V)$, $S(V)$ as subrings of $\mathbb{Q}(S(V))$. There one has $f = f_i$ and $x_i^{m_i} f = x_i^{m_i} f_i \in S(V)_{m_i}$. Such a relation trivially holds even if $V_i = \emptyset$. Now, let us choose an integer $m \geqslant \sum_{i=0}^n m_i$. Then $S(V)_m$ is generated as a $\mathbb{K}$–vector space, by the monomials of degree m in $x_0, \ldots, x_n$, and each such monomial has to contain at least a x_i raised to a power with exponent at least m_i. Hence $S(V)_m \cdot f \subseteq S(V)_m$. By iterating, we have $S(V)_m \cdot f^q \subseteq S(V)_m$, for all integers $q \geqslant 1$. In particular, if $x_0 \neq 0$ on V we have $x_0^m f^q \in S(V)$ for all positive integers q, and this proves that $S(V)[f]$ is contained in $x_0^{-m} \cdot S(V)$, which is a finitely generated $S(V)$–module. Since $S(V)$ is noetherian, $S(V)[f]$ is a finitely generated $S(V)$–module (see [1, Prop. 6.5, p. 76]), hence f is integral over $S(V)$ by Lemma 5.4.1. So there are $a_1, \ldots, a_k \in S(V)$ such that

$$f^k + a_1 f^{k-1} + \cdots + a_k = 0.$$

Recall that $f \in K(V) = S(V)_{((0))}$, hence $f = \frac{g}{h}$, with $g, h \in S(V)$ homogeneous of the same degree. Then we have

$$g^k + a_i g^{k-1} h + \cdots + a_k h^k = 0$$

and in such a relation we can replace $a_1, \ldots, a_k$ with their homogenous components of degree 0. Since $S(V)_0 = \mathbb{K}$, it follows that f is algebraic over $\mathbb{K}$ and, since $\mathbb{K}$ is algebraically closed, one has $f \in \mathbb{K}$. □

A first important consequence of Theorem 5.5.3 is that if V is a quasi–projective variety, $K(V)$ is a finite type extension of $\mathbb{K}$. Indeed, $K(V) = K(\bar{V})$, where $\bar{V}$ is the projective closure of V, and $K(\bar{V})$ is contained $\mathbb{Q}(S(V))$, which is generated over $\mathbb{K}$ by $x_0, \ldots, x_n$. The transcendence degree of $K(V)$ over $\mathbb{K}$ is called the *transcendent dimension* of V and it is denoted by $\dim_{\mathrm{tr}}(V)$. If U is a non–empty subset of V, one has $\dim_{\mathrm{tr}}(U) = \dim_{\mathrm{tr}}(V)$. Varieties with transcendent dimension 0 are points. Varieties with transcendent dimension 1 are called *curves*, those with transcendent dimension 2 are called *surfaces*.

Exercise 5.5.4 Let V be a quasi–projective variety and let W be a subvariety. Prove that if $[U, f] \in \mathcal{O}_{V,W}$ then its restriction to W is well defined.

Exercise 5.5.5 Let V be a quasi–projective variety and let W be a subvariety. Prove that $\mathcal{O}_{V,W}$ is a subring of $K(V)$.

Exercise 5.5.6 Let V be a quasi–projective variety, let W be a subvariety and let U be an open subset of V such that $U \cap W \neq \emptyset$. Consider the map

$$\rho_U : f \in \mathcal{O}(U) \to [U, f] \in \mathcal{O}_{V,W}.$$

Prove that ρ_U is an injective homomorphism of $\mathbb{K}$ algebras and that $\mathcal{I}_U(W) := \rho_U^{-1}(\mathfrak{m}_{V,W})$ is the ideal of $\mathcal{O}(U)$ formed by all functions $f \in \mathcal{O}(U)$ such that $U \cap W \subseteq Z_U(f)$.

Exercise 5.5.7 * Let V be a quasi–projective variety, let W be a subvariety. Prove that there is a 1:1 correspondence between the prime ideals of $\mathcal{O}_{V,W}$ and the closed subvarieties of V containing W.

Exercise 5.5.8 Let V be a quasi–projective variety, let W be a subvariety. Prove that $\mathbb{Q}(\mathcal{O}_{V,W}) = K(V)$.

Exercise 5.5.9 * Let A be a domain. Prove that $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$, where the intersection is over all maximal ideals of A, and all $A_{\mathfrak{m}}$ are contained in $\mathbb{Q}(A)$.

Exercise 5.5.10 Prove that if the variety V consists of one point, one has $\mathcal{O}(V) = K(V) = \mathbb{K}$. Prove that if V is a variety and $P \in V$ a point, then the residue field of $\mathcal{O}_{V,P}$ is $\mathbb{K}$.

Exercise 5.5.11 Prove that $\mathcal{O}(\mathbb{A}_n) = A_n$ and $K(\mathbb{A}_n) = K(\mathbb{P}^n) = \mathbb{Q}(A_n) = \mathbb{K}(x_1, \ldots, x_n)$, hence $\dim_{\text{tr}}(\mathbb{A}_n) = \dim_{\text{tr}}(\mathbb{P}^n) = n$.

Exercise 5.5.12 Consider the affine plane curve V with equation $x_1x_2 = 1$, which is clearly irreducible. Prove that $A(V) = \mathbb{K}[x_1, x_1^{-1}] = (A_1)_{x_1}$, and that $K(V) = \mathbb{K}(x_1)$.

Exercise 5.5.13 Consider the affine plane curve V with equation $x_1^3 = x_2^2$, which is irreducible. Then $A(V) = \mathbb{K}[x_1, x_2]/(x_1^3 - x_2^2)$. Prove that every element $f \in A(V)$ can be written in a unique way as $f = P(x_1) + Q(x_1)x_2$, with $P, Q \in A_1$.

Exercise 5.5.14 (*Hilbert Nullstellensatz for affine varieties*) Let V be an affine variety, let $f, g_1, \ldots, g_m \in \mathcal{O}(V)$ and suppose that $\bigcap_{i=1}^{g} Z_V(g_i) \subseteq Z_V(f)$. Prove that $f \in \text{rad}(g_1, \ldots, g_m)$. Make a similar statement for projective varieties.

Exercise 5.5.15 Prove that if V is an irreducible (projective or affine) plane curve, then $\dim_{\text{tr}}(V) = 1$.

Exercise 5.5.16 Let V be a quasi–projective variety and W a subvariety of V. Prove that $\mathcal{O}_{V,W} = \mathcal{O}_{V,\bar{W}}$, where $\bar{W}$ is the closure of W in V.

Exercise 5.5.17 Prove that if $P \in \mathbb{P}^1$ is any point, then the ring $\mathcal{O}_{\mathbb{P}^1,P}$ is integrally closed.

Exercise 5.5.18 Assume $\text{char}(\mathbb{K}) = 0$. Let K be an extension of $\mathbb{K}$ of finite transcendence degree on $\mathbb{K}$. Prove that there is an affine variety V such that $K(V) = K$.

5.6 Product of Affine Varieties

Here we prove the following:

Proposition 5.6.1 *Let $V_1 \subseteq \mathbb{A}^r$ and $V_2 \subseteq \mathbb{A}^s$ be affine varieties. Then $A(V_1 \times V_2) \cong A(V_1) \otimes_{\mathbb{K}} A(V_2)$.*

Proof Consider the bilinear map

$$a : (f, g) \in A_r \times A_s \to fg \in A_{r+s}$$

which verifies that $a(\mathcal{I}_a(V_1) \times A_s)$ and $a(A_r \times \mathcal{I}_a(V_2))$ are both contained in $\mathcal{I}_a(V_1 \times V_2)$. For this reason, a induces a bilinear map

$$b : A(V_1) \times A(V_2) \to A(V_1 \times V_2).$$

Note that the minimal subalgebra of A_{r+s} containing $a(A_r \times A_s)$ is A_{r+s}, thus every element of $A(V_1 \times V_2)$ is of the form

$$h = \sum_{ij} c_{ij} f_i g_j, \quad \text{with} \quad c_{ij} \in \mathbb{K}, f_i \in A(V_1), g_j \in A(V_2). \tag{5.2}$$

We claim that h as in (5.2) is zero if and only if we can re–arrange the expression of h in (5.2) so that for every pair (i, j), one has either $c_{ij} = 0$, or $f_i = 0$ in $A(V_1)$, or $g_j = 0$ in $A(V_2)$. One implication is clear, let us prove the other. We argue by contradiction and suppose that $h = 0$ but that there is in the expression (5.2) some pair (i, j), for which $c_{ij} \neq 0$, $f_i \neq 0$ and $g_j \neq 0$. We can extract a maximal system of linearly independent elements on $\mathbb{K}$ from the set $\{f_i\}$ of elements of $A(V_1)$, then we can express all the elements of $\{f_i\}$ as linear combinations of the elements of such a system and substitute in the expression of h. This re–arranges the expression in (5.2) so that the set $\{f_i\}$ consists of linearly independent elements. The same for the set $\{g_j\}$. If in the new expression as in (5.2) we have found, for every pair (i, j) one has either $c_{ij} = 0$, or $f_i = 0$ or $g_j = 0$ we are done. Otherwise we have some pair (i, j), for which $c_{ij} \neq 0$, $f_i \neq 0$ and $g_j \neq 0$, and we can consider in the summation only these pairs, because the others give 0 contribution. One has

$$h = \sum_i \left(\sum_j c_{ij} g_j \right) f_i = 0.$$

For all i, $\sum_j c_{ij} g_j \in A(V_2)$ is non–zero, hence there is a point $Q \in V_2$ such that

$$\sum_j c_{ij} g_j(Q) \neq 0$$

for some i. On the other hand we have

$$\sum_i \left(\sum_j c_{ij} g_j(Q) \right) f_i = 0,$$

hence we have a contradiction. This proves our claim.

The claim implies that if $h = 0$, we can re–arrange the expression of h as in (5.2) so that for each pair (i, j) such that $c_{ij} \neq 0$, one has either $f_i = 0$ or $g_j = 0$. This is the same as saying that $\mathcal{I}_a(V_1 \times V_2)$ is generated by $\alpha(\mathcal{I}_a(V_1) \times A_s)$ and by $\alpha(A_r \times \mathcal{I}_a(V_2))$.

Let now M be a $\mathbb{K}$–vector space and let $\gamma : A(V_1) \times A(V_2) \to M$ be a bilinear map. There is then a unique homomorphism of $\mathbb{K}$–vector spaces $\delta : A(V_1 \times V_2) \to M$ such that $\gamma = \delta \circ b$. It is defined in the following way. If $h = \sum_{ij} c_{ij} f_i g_j \in A(V_1 \times V_2)$, one sets $\delta(h) = \sum_{ij} c_{ij} \gamma(f_i, g_j)$. From the above discussion, it follows that δ is well defined, that it is linear, and it is uniquely determined by b and γ. By the universal property of the tensor product we conclude that $A(V_1 \times V_2) \cong A(V_1) \otimes_{\mathbb{K}} A(V_2)$. □

As a consequence, we have:

Proposition 5.6.2 *Let $V_1 \subseteq \mathbb{A}^r$ and $V_2 \subseteq \mathbb{A}^s$ be affine varieties. Then* $\dim_{tr}(V_1 \times V_2) = \dim_{tr}(V_1) + \dim_{tr}(V_2)$.

Proof The ring $A(V_1)$ is a quotient of $A_r = \mathbb{K}[x_1, \ldots, x_r]$, so it is generated as a $\mathbb{K}$–algebra by $x_1, \ldots, x_r$, hence $K(V_1)$ is generated on $\mathbb{K}$ by $x_1, \ldots, x_r$, and we can extract from them a maximal system of algebraically independent elements, say $x_1, \ldots, x_n$, with $n = \dim_{tr}(V_1)$. Similarly, $A(V_2)$ is a quotient of $A_s = \mathbb{K}[y_1, \ldots, y_s]$, and we can assume that $y_1, \ldots, y_m$, with $m = \dim_{tr}(V_2)$, is a maximal system of algebraically independent elements of $y_1, \ldots, y_s$ on $\mathbb{K}$. By Proposition 5.6.1, we have that $x_1, \ldots, x_r, y_1, \ldots, y_s$ generate $A(V_1 \times V_2)$ and this implies that $K(V_1 \times V_2)$ is generated on $\mathbb{K}$ by $x_1, \ldots, x_n, y_1, \ldots, y_m$. We are left to prove that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are algebraically independent. Suppose, by contradiction, that there is a non–zero polynomial $F(t_1, \ldots, t_n, u_1, \ldots, u_m)$ with coefficients in $\mathbb{K}$ such that $F(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0$. By the algebraic independence of $y_1, \ldots, y_m$, we have that any coefficient $a(x_1, \ldots, x_n)$ of $F(x_1, \ldots, x_n, u_1, \ldots, u_m) = 0$, as a polynomial in $u_1, \ldots, u_m$, is zero. On the other hand, by the algebraic independence of $x_1, \ldots, x_n$, we have that $a(t_1, \ldots, t_n)$ is identically zero. This implies that F is zero, contrary to the assumption. □

5.7 Solutions of Some Exercises

5.2.2 The assertion is a consequence of the following remark. Let $[U', f'] = [U'', f'']$. Then we can consider the function f''' defined in $U' \cup U''$ such that $f'''(P) = f'(P)$ if $P \in U'$, and $f'''(P) = f''(P)$ if $P \in U''$. This function is well defined because f' and f'' coincide on $U' \cap U''$. Moreover it is regular, because so it is in U' and U''.

5.4.3 Let $\frac{f}{g} \in \mathbb{Q}(A)$ be integral over A with f, g coprime. One has a relation of the form

$$\left(\frac{f}{g}\right)^n + a_1\left(\frac{f}{g}\right)^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n \in A$. Eliminating the denominators, we see that g divides f^n, hence it divides f, so that g is invertible.

5.4.4 The proof is similar to the one of Lemma 5.4.1. Let $y_1, \ldots, y_n$ be a set of generators of M as an A–module. Then $\phi(y_i) = \sum_{j=1}^n a_{ij} y_j$, with $a_{ij} \in \mathcal{I}$ and $i = 1, \ldots, n$. So

$$\sum_{j=1}^{n} (\delta_{ij}\phi - a_{ij}) y_j = 0, \quad \text{for} \quad i = 1, \ldots, n.$$

Fix any $h = 1, \ldots, n$. If we multiply the ith of the above relations for the cofactor of the element of place (i, h) of the matrix $(\delta_{ij}\phi - a_{ij})_{i,j=1,\ldots,n}$ and then add up on i, we see that $D = \det(\delta_{ij}\phi - a_{ij})_{i,j=1,\ldots,n}$ annihilates y_h for $h = 1, \ldots, n$, so multiplication by D is the 0–endomorphism of M. By expanding the determinant, one finds the required expression.

5.4.5 (a) implies (b): by Lemma 5.4.1 it suffices to take $C = A[x]$.

(b) implies (c): it suffices to take $M = C$. Indeed C is faithful because $yC = (0)$, implies $y = y \cdot 1 = 0$.

(c) implies (a): apply Exercise 5.4.4 taking $\phi : M \to M$ equal to the multiplication by x (note that $xM \subseteq M$ because M is an $A[x]$ module) and $\mathcal{I} = A$. Since M is faithful, we have a relation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \quad \text{with} \quad a_1, \ldots, a_n \in A,$$

which proves that x is integral over A. Then (a) follows by Lemma 5.4.1.

5.4.6 It suffices to apply Exercise 5.4.5: for all $x \in B$, $A[x]$ is contained in B which is a finitely generated as an A–module; then (b) of Exercise 5.4.5 is verified and therefore x is integral over A.

5.4.7 Let $b_1, \ldots, b_n$ be a set of generators of B as a $\mathbb{K}$–algebra. By Lemma 5.4.1, $A[b_1]$ is a finitely generated A–module. Similarly, $A[b_1, b_2]$ is a finitely generated $A[b_1]$–module, hence it is a finitely generated A–module. By iterating this argument, we see that $A[b_1, \ldots, b_n]$ is a finitely generated A–module. But $A[b_1, \ldots, b_n] = B$ and we are done.

5.4.8 Let x be an element of C. Then we have a relation of the form

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0, \quad \text{with} \quad b_i, \ldots, b_n \in B.$$

Then x is integral over $B' = A[b_1, \ldots, b_n]$, so $B'[x]$ is finitely generated over B'. On the other hand, B' is finitely generated over A, so $B'[x]$ is finitely generated over A and therefore also $A[x]$ is finitely generated over A (see [1, Prop. 6.5, p. 76]). This implies that x is integral over A by Lemma 5.4.1.

5.5.7 First of all reduce to the case that V is affine and W is closed in V. Then apply Theorem 5.5.3, the properties of localization and the results of Sect. 3.

5.5.8 Reduce to the affine case and apply Theorem 5.5.3.

5.5.9 It is clear that $A \subseteq \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$. Let us prove the opposite inclusion. Let $x \in \mathbb{Q}(A)$. We consider the *ideal of denominators* of x so defined

$$D(x) = \{a \in A : ax \in A\}.$$

One has $x \in A$ if and only if $D(x) = A$ and $x \in A_{\mathfrak{m}}$, with $\mathfrak{m}$ a maximal ideal, if and only if $D(x)$ is not contained in $\mathfrak{m}$. Hence, if $x \notin A$, then $D(x)$ is a proper ideal of A and therefore there is a maximal ideal $\mathfrak{m}$ of A such that $D(x) \subseteq \mathfrak{m}$, so that $x \notin A_{\mathfrak{m}}$. The assertion follows.

5.5.14 Suppose that $\mathcal{I}_a(V) = (f_1, \ldots, f_h)$. By the Nullstellensatz, there is a positive integer r such that $f^r \in (g_1, \ldots, g_m, f_1, \ldots, f_h)$ in A_n. The assertion follows by modding out by $\mathcal{I}_a(V)$.

5.5.15 We can reduce to the affine case. Then V has an equation of the form $f(x_1, x_2) = 0$, with $f \in A_2$ irreducible. Then $A(V) = A_2/(f)$ and $K(V) = \mathbb{K}(x_1, x_2)$, with x_1, x_2 linked by the relation $f(x_1, x_2) = 0$. It is moreover clear that x_1, x_2 are not both algebraic over $\mathbb{K}$, otherwise they would be constant and V would be a point. The assertion follows.

5.5.17 It suffices to prove that if $P \in \mathbb{A}^1$ is any point, then $\mathcal{O}_{\mathbb{A}^1,P}$ is integrally closed. This follows from Exercise 5.4.2.

Chapter 6
Morphisms

6.1 The Definition of Morphism

Let V, W be quasi-projective varieties. A map $\phi : V \to W$ is called a *morphism* if it is continuous and if, for every open subset $U \subseteq W$ such that $\phi^{-1}(U)$ is not empty, and for every regular function $\mathcal{O}(U)$, the function $f_\phi = f \circ \phi$ is regular on $\phi^{-1}(U)$. We will denote by $M(V, W)$ the set of all morphisms from V to W. It is clear that the identity is a morphism and the composition of two morphisms is a morphism. So it makes sense to consider the *category* in which the objects are the quasi-projective varieties and the morphisms are the ones we defined above. In this category a morphism $\phi : V \to W$ is an *isomorphism* if it has the inverse, i.e., if and only if there is a morphism $\psi : W \to V$ such that $\psi \circ \phi = \mathrm{id}_V$, $\phi \circ \psi = \mathrm{id}_W$.

If $\phi \in M(V, W)$, for every open subset $U \subseteq W$ such that $\phi^{-1}(U)$ is not empty, one has a map

$$\phi^U : f \in \mathcal{O}(U) \to f_\phi \in \mathcal{O}(\phi^{-1}(U))$$

which is easily seen to be a homomorphism of $\mathbb{K}$-algebras. If ϕ is a isomorphism, ϕ^U is a isomorphism of $\mathbb{K}$-algebras, its inverse being $(\phi^{-1})^{\phi^{-1}(U)}$. In particular, if V and W are isomorphic, then $\mathcal{O}(V) \cong \mathcal{O}(W)$.

If $\phi \in M(V, W)$ and $\overline{\phi(V)} = W$, we say that ϕ is *dominant*. In this case we have a natural map

$$\phi' : (U, f) \in \mathcal{K}(W) \to (\phi^{-1}(U), f_\phi) \in \mathcal{K}(V).$$

In fact, if ϕ is dominant, for any non-empty open subset U of W, then $\phi^{-1}(U)$ is also non-empty, hence it is dense in V. It is clear that ϕ' is compatible with the equivalence relations $\mathcal{R}$ in $\mathcal{K}(V)$ and in $\mathcal{K}(W)$, hence ϕ' induces a field homomorphism

$$\phi^* : [U, f] \in K(W) \to [\phi^{-1}(U), f_\phi] \in K(V).$$

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The map ϕ^* is not zero because it is the identity on $\mathbb{K}$, hence it is injective. In particular, if V and W are isomorphic, we have $K(V) \cong K(W)$.

Let again $\phi \in M(V, W)$. Let V' be a subvariety of V and W' a subvariety of W, such that $\phi(V') \subseteq W'$. Consider the map $\phi' = \phi_{|V'} : V' \to W'$. It is easy to check that $\phi' \in M(V', W')$. Moreover, as we constructed the map ϕ^* above, we have a homomorphism of local rings

$$\phi^*_{V'} : [U, f] \in \mathcal{O}_{W,W'} \to [\phi^{-1}(U), f_\phi] \in \mathcal{O}_{V,V'}$$

in the hypothesis that $\phi(V')$ is dense in W', i.e., if $\phi' : V' \to W'$ is dominant. In particular, if V and W are isomorphic and if this isomorphism induces a isomorphism of V' onto W', then $\mathcal{O}_{V,V'} \cong \mathcal{O}_{W,W'}$.

Exercise 6.1.1 Prove that any constant map is a morphism.

Exercise 6.1.2 Let V be a quasi-projective variety. Prove that regular functions on V coincide with morphisms of V to $\mathbb{A}^1 = \mathbb{K}$.

Exercise 6.1.3 Let V_1, V_2 be affine varieties. Consider the projection maps $p_i : V_1 \times V_2 \to V_i$, with $i = 1, 2$. Prove they are morphisms.

Exercise 6.1.4 Let $\phi \in M(V, W)$. Let V' be a subvariety of V and W' a subvariety of W, such that $\phi(V') \subseteq W'$. Prove that $\phi' = \phi_{|V'} : V' \to W'$ is a morphism.

Exercise 6.1.5 Let V be a quasi-projective variety and W be a subvariety of V. Prove that the inclusion $i : W \to V$ is a morphism, called the *immersion* of W in V.

Exercise 6.1.6 *Let V, W, Z be quasi-projective varieties, with Z a subvariety of W. Prove that a map $\phi : V \to Z$ is a morphism if and only if $\phi : V \to W$ is a morphism.

6.2 Which Maps Are Morphisms

It is useful to have criteria for maps between varieties to be morphisms. Here is one:

Proposition 6.2.1 *Let V be a quasi-projective variety. A map $\phi : V \to \mathbb{A}^n$ is a morphism if and only if for all $i \in \{1, \dots, n\}$, the composite map $\phi_i = p_i \circ \phi$ is regular, where $p_i : \mathbb{A}^n \to \mathbb{A}^1 = \mathbb{K}$ is the projection on the ith factor.*

Proof We prove the only non-trivial implication. If $f \in A_n$, then $f' = f(\phi_1, \dots, \phi_n) \in \mathcal{O}(V)$, hence $Z_V(f')$ is closed in V. This proves that ϕ is continuous. Let $U \subseteq \mathbb{A}^n$ be an open subset, and let $f \in \mathcal{O}(U)$. For each point $P \in U$, there is an open neighborhood U' of P in U where $f = \frac{P}{Q}$, where $P, Q \in A_n$ and $Z_{U'}(Q) = \emptyset$. Hence $f \circ \phi = \frac{P(\phi_1, \dots, \phi_n)}{Q(\phi_1, \dots, \phi_n)}$ is a regular function on $\phi^{-1}(U')$. This proves that ϕ is a morphism. □

As an application of the previous proposition we have the following important:

Theorem 6.2.2 *Let V be a quasi-projective variety and W an affine variety. The map*

$$\alpha : M(V, W) \to \phi^W \in \mathrm{Hom}(A(W), \mathcal{O}(V))$$

is bijective (here $\mathrm{Hom}(A(W), \mathcal{O}(V))$ *denotes the set of* $\mathbb{K}$*-algebra homomorphisms).*

Proof Let $h \in \mathrm{Hom}(A(W), \mathcal{O}(V))$. Suppose that $W \subseteq \mathbb{A}^n$. Then $A(W) = \mathbb{K}[x_1, \dots, x_n]/\mathcal{I}_a(W)$, hence $\xi_i := h(x_i) \in \mathcal{O}(V)$, for all $i = 1, \dots, n$. Consider the map

$$\phi_h : P \in V \to (\xi_1(P), \dots, \xi_n(P)) \in \mathbb{A}^n,$$

which is a morphism by Proposition 6.2.1. Let us prove that $\phi_h(V) \subseteq W$, and this will imply that $\phi_h \in M(V, W)$ (see Exercise 6.1.6). Indeed, if $f \in \mathcal{I}_a(W)$, for all $P \in V$ one has

$$f(\phi_h(P)) = f(\xi_1(P), \dots, \xi_n(P)) = f(h(x_1)(P), \dots, h(x_n)(P)).$$

Since h is a $\mathbb{K}$-algebras homomorphism, one has

$$f(h(x_1), \dots, h(x_n)) = h(f(x_1, \dots, x_n))(P) = 0$$

because $f = 0$ in $A(W)$, hence $h(f) = 0$. This proves that $\phi_h(V) \subseteq W$, as wanted. Finally, the map

$$h \in \mathrm{Hom}(A(W), \mathcal{O}(V)) \to \phi_h \in M(V, W)$$

is clearly the inverse of α. □

As a consequence we have:

Corollary 6.2.3 *One has:*

(a) if V, W are affine varieties, V is isomorphic to W if and only if $A(V) \cong A(W)$ as $\mathbb{K}$-algebras;
(b) the contravariant functor $V \to \mathcal{O}(V)$, from the category of quasi-projective varieties to the category of $\mathbb{K}$-algebras with no zero divisors, induces an equivalence of categories between the category of affine varieties and the category of finitely generated $\mathbb{K}$-algebras with no zero divisor.

Proof The only thing that we are left to prove is that any finitely generated $\mathbb{K}$-algebra with no zero divisors A is the coordinate ring of an affine variety. Let $t_1, \dots, t_n$ be generators of A. The map

$$f \in A_n \to f(t_1, \dots, t_n) \in A$$

is a surjective $\mathbb{K}$-algebras homomorphism, whose kernel is a prime ideal $\mathcal{I}$ of A_n. By the results of Sect. 3, 4, there is a variety $V \subseteq \mathbb{A}^n$ such that $\mathcal{I}_a(V) = \mathcal{I}$. The assertion follows. □

Next we can give a criterion for maps between quasi-projective varieties to be morphisms. We start by proving the:

Lemma 6.2.4 *For all $i = 0, \dots, n$, the map $\phi_i : U_i \to \mathbb{A}^n$ (see Sect. 1.5) is an isomorphism.*

Proof We already know that ϕ_i is a homeomorphism, so the only thing to prove is that the regular functions are the same on $\mathbb{A}^n$ and U_i, which follows from the very definition of regular functions (details are left to the reader). □

Proposition 6.2.5 *Let $V \subseteq \mathbb{P}^n$ be a quasi-projective variety. A map $\phi : V \to \mathbb{P}^m$ is a morphism if and only if for every point $P \in V$ there is an open neighborhood U of P in V and $m+1$ homogeneous polynomials of the same degree $f_0, \dots, f_m \in S_n$ such that for every point $P' \in U$ there is an $i \in \{0, \dots, m\}$ such that $f_i(P') \neq 0$ and $\phi(P') = [f_0(P'), \dots, f_m(P')]$.*

Proof Let $\phi : V \to \mathbb{P}^m$ be a morphism. Given $P \in V$, set $Q = \phi(P)$. Then there is an $i \in \{0, \dots, m\}$ such that $Q \in U_i$: to fix ideas, let us assume that $i = 0$. Set $U := \phi^{-1}(U_0)$, which is an open neighborhood of P in V. Then ϕ induces a morphism $\phi' : U \to U_0$. By identifying U_0 with $\mathbb{A}^m$ via the map ϕ_0 in Lemma 6.2.4, applying Proposition 6.2.1 and may be restricting U, we can ensure the existence of m pairs of homogenous polynomials of the same degree $(f_1, f_{1,0}), \dots, (f_m, f_{m,0})$, such that for every point $P' \in U$, $f_{i,0}(P') \neq 0$ for all $i \in \{1, \dots, m\}$, and such that

$$\phi'(P) = \Big(\frac{f_1(P')}{f_{1,0}(P')}, \dots, \frac{f_m(P')}{f_{m,0}(P')}\Big).$$

By reducing the fractions $\frac{f_1}{f_{1,0}}, \dots, \frac{f_m}{f_{m,0}}$ to minimum common denominator, we may assume that $f_{1,0} = \cdots = f_{m,0} = f_0$, that $f_0, \dots, f_m$ have the same degree, and that $f_0(P') \neq 0$ for all $P' \in U$. Since $\phi'(P') = [f_0(P'), \dots, f_m(P')]$, we have the assertion.

Let us prove that the condition on ϕ is sufficient for ϕ to be a morphism. First we note that, up to restricting U, we may assume that there is an $i \in \{0, \dots, m\}$ such that $f_i(P') \neq 0$. To fix ideas, suppose that $i = 0$. Then $\phi_{|U}$ maps U to U_0 and, by Proposition 6.2.1, it determines a morphism ϕ', because for every $P' \in U$, one has

$$\phi'(P') = \Big(\frac{f_1(P')}{f_0(P')}, \dots, \frac{f_m(P')}{f_0(P')}\Big).$$

If j is the immersion of U_0 in $\mathbb{P}^m$, one has that $\phi_{|U} = j \circ \phi'$, so it is a morphism. Since the notion of morphism is clearly local, the assertion is proved. □

Proposition 6.2.6 *Let V, W be quasi-projective varieties and $\phi, \psi \in M(V, W)$, such that there is a non-empty open subset U of V such that $\phi_{|U} = \psi_{|U}$. Then $\phi = \psi$.*

Proof If $U = V$ there is nothing to prove. So we assume $U \neq V$. Suppose, by contradiction that there is a point $P \in V \setminus U$ such that $\phi(P) \neq \psi(P)$. We can consider W as embedded in a projective space $\mathbb{P}^n$ and actually we can reduce ourselves to the case $W = \mathbb{P}^n$ (see Exercise 6.1.6). Moreover, up to a projectivity (see Exercise 6.2.12), we may assume that both $\phi(P)$ and $\psi(P)$ belong to the affine open subset $U_0 \cong \mathbb{A}^n$. Finally, working in $\phi^{-1}(U_0) \cap \psi^{-1}(U_0) \cap U$, that is an open neighborhood of P in U, we can even assume $W = \mathbb{A}^n$. Then we can apply Proposition 6.2.1 and Proposition 5.1.1 and conclude that $\phi = \psi$, reaching a contradiction. □

Exercise 6.2.7 *Prove that affine maps are morphisms. Prove that affinities are isomorphisms onto their images. Affinities between affine spaces of the same dimension can be considered as changes of coordinates.

Exercise 6.2.8 Prove that all automorphism of $\mathbb{A}^1$ are affinities.

Exercise 6.2.9 Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be a morphism. Prove that there are polynomials $P_1, \ldots, P_n \in A_n$, such that $\phi(\mathbf{x}) = (P_1(\mathbf{x}), \ldots, P_n(\mathbf{x}))$. Prove that if ϕ is an automorphism, then the *jacobian determinant*

$$J_\phi(\mathbf{x}) = \det \left(\frac{\partial P_i}{\partial x_j} \right)_{i=1, j=1, \ldots, n}$$

is an element J_ϕ of $\mathbb{K} \setminus \{0\}$.

Exercise 6.2.10 Consider the group G_n of all automorphisms of $\mathbb{A}_n$. Prove that the map $J : \phi \in G_n \to J_\phi \in \mathbb{K} \setminus \{0\}$ is a group homomorphism.

Exercise 6.2.11 Give an example of an automorphism of $\mathbb{A}^n$, with $n \geqslant 2$, which is not an affinity.

Exercise 6.2.12 *Prove that projectivities are isomorphisms onto their images. Prove that all projective spaces of the same dimension are isomorphic. Projectivities of $\mathbb{P}^n$ to itself can be considered as changes of homogeneous coordinates.

Exercise 6.2.13 *Let V be a quasi-projective variety. Prove that a map $f : V \to \mathbb{P}^n$ is a morphism if and only if for every point $P \in V$ there is an open neighborhood U of P in V and there are regular functions $f_0, \ldots, f_n \in \mathcal{O}(U)$, such that for any point $Q \in U$ one has $(f_0(Q), \ldots, f_n(Q)) \neq \mathbf{0}$ and $f(Q) = [f_0(Q), \ldots, f_n(Q)]$.

Exercise 6.2.14 Prove that the affine [resp. projective] twisted cubic is isomorphic to $\mathbb{A}^1$ [resp. to $\mathbb{P}^1$].

Exercise 6.2.15 Consider the affine conic V with equation $x_2 = x_1^2$, called a *parabola*. Consider the homomorphism of $\mathbb{K}$-algebras

$$\pi^* : x_1 \in A_1 \to x_1 \in A(V) = \mathbb{K}[x_1, x_2]/(x_2 - x_1^2)$$

corresponding to the restriction π to V of the projection from the point at infinity of the x_2-axis of $\mathbb{A}^2$ on $\mathbb{A}^1$, identified with the x_1-axis. Prove that π^* is an isomorphism, so that π is an isomorphism.

Exercise 6.2.16 Consider the affine conic W with equation $x_1 x_2 = 1$, called a *hyperbola*. Prove that $A(W) \cong (A_1)_{x_1}$. Prove that $A(W)$ is not isomorphic to A_1, hence W is not isomorphic to $\mathbb{A}^1$. Prove that W is isomorphic to $\mathbb{A}^1 \setminus \{0\}$.

Exercise 6.2.17 *Let Z be any irreducible conic in $\mathbb{A}^2_{\mathbb{K}}$ with $\mathbb{K}$ of characteristic different from 2. Prove that there is an affinity of $\mathbb{A}^2$ which either maps Z to $V = Z_a(x_2 - x_1^2)$ or maps Z to $W = Z_a(x_1x_2 - 1)$: in the former case Z is called a *parabola*, in the latter case it is called and *hyperbola*.

Exercise 6.2.18 Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Let $f \in A_2$ be an irreducible polynomial of degree 2. Prove that either $A_2/(f)$ is isomorphic to A_1 or it is isomorphic to $(A_1)_{x_1}$.

Exercise 6.2.19 *Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Prove that given two irreducible conics in $\mathbb{P}^2$, there is a projectivity of $\mathbb{P}^2$ which maps the former to the latter.

Exercise 6.2.20 *Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Prove that any irreducible conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

Exercise 6.2.21 Let $Z_1 \subseteq \mathbb{A}^r$, $Z_2 \subseteq \mathbb{A}^s$ be affine varieties. Let $P \in Z_1$ and $Q \in Z_2$. Consider the maps

$$P' \in Z_1 \to (P', Q) \in Z_1 \times \{Q\}, \quad Q' \in Z_2 \to (P, Q') \in \{P\} \times Z_2.$$

Prove they are isomorphisms of Z_1, Z_2 respectively to $Z_1 \times \{Q\}$, $\{P\} \times Z_2$, which are subvarieties of $Z_1 \times Z_2$ (intersections of $Z_1 \times Z_2$ with the subspaces $\mathbb{A}^r \times \{Q\}$, $\{P\} \times \mathbb{A}^s$).

Exercise 6.2.22 Let Z be a subvariety of $\mathbb{A}^n$. Consider the subset $\Delta_Z = \{(P, P) : P \in Z\} \subset Z \times Z$, called the *diagonal* of $Z \times Z$. Prove that Δ_Z is a subvariety of $Z \times Z$ isomorphic to Z.

Exercise 6.2.23 *Let Z_1, Z_2 be subvarieties of $\mathbb{A}^n$. Then $\Delta_{\mathbb{A}^n} \cap (Z_1 \times Z_2)$ is a closed subset in $Z_1 \times Z_2$. Consider the map

$$\iota : P \in Z_1 \cap Z_2 \to (P, P) \in \Delta_{\mathbb{A}^n} \cap (Z_1 \times Z_2).$$

Prove that ι is an isomorphism of any irreducible component of $Z_1 \cap Z_2$ onto its image.

Exercise 6.2.24 *Let V, W be affine varieties, let $\phi : V \to W$ be a morphism and $\phi^* : A(W) \to A(V)$ the corresponding $\mathbb{K}$-algebras homomorphism. Prove that ϕ is dominant if and only if ϕ^* is injective.

Exercise 6.2.25 Let $U = \mathbb{A}^1 \setminus \{a_1, \ldots, a_n\}$, with $a_1, \ldots, a_n$ distinct, be a proper open subset of $\mathbb{A}^1$. Prove that U is an affine variety, which is not isomorphic to $\mathbb{A}^1$, but it is homeomorphic to $\mathbb{A}^1$.

Exercise 6.2.26 Let $V \subseteq \mathbb{A}^n$ be an affine variety. Prove that V is isomorphic to a projective variety if and only if V is a point.

Exercise 6.2.27 Prove that any morphism of a projective variety to an affine variety is constant.

Exercise 6.2.28 Consider the morphism

$$\phi : t \in \mathbb{A}^1 \to (t^2, t^3) \in V \subset \mathbb{A}^2,$$

where $V \subset \mathbb{A}^2$ is the curve with equation $x_1^3 = x_2^2$. Prove that it is an homeomorphism, but is is not an isomorphism. Prove however that its restriction to $\mathbb{A}^1 \setminus \{0\}$ is an isomorphism onto $V \setminus \{\mathbf{0}\}$.

Exercise 6.2.29 *Suppose $\mathrm{char}(\mathbb{K}) = p > 0$. Consider the morphism

$$F : [x_0, \ldots, x_n] \in \mathbb{P}^n \to [x_0^p, \ldots, x_n^p] \in \mathbb{P}^n$$

which is well defined and induces a morphism

$$F : (x_1, \ldots, x_n) \in \mathbb{A}^n \to (x_1^p, \ldots, x_n^p) \in \mathbb{A}^n.$$

These morphisms are called *Frobenius morphisms*. Prove that they are homeomorphisms, but not isomorphisms.

Exercise 6.2.30 *Prove that $\mathbb{A}^2 \setminus \{\mathbf{0}\}$ is not an affine variety.

Exercise 6.2.31 *Let H be a subspace of $\mathbb{P}^n$ of codimension 2. Prove that $\mathcal{O}(\mathbb{P}^n \setminus H) = \mathbb{K}$.

Exercise 6.2.32 Corollary 6.2.3, (b) characterizes the $\mathbb{K}$-algebras that are the coordinate rings of affine varieties. Prove that a $\mathbb{K}$-algebra is the coordinate ring of an affine closed subset if and only if it is finitely generated and with no nilpotent elements.

6.3 Affine Varieties

Lemma 6.2.4 says that $\mathbb{P}^n$ has an open covering of varieties isomorphic to affine varieties. This is a particular case of a more general situation, which we will now explain. First of all let us give a definition: given a quasi-projective variety V we will say that V is *affine*, if it is isomorphic to an affine variety. An open subset U of a quasi-projective variety V is said to be *affine* if it is itself an affine variety.

Lemma 6.3.1 *Let Z be the hypersurface of $\mathbb{A}^n$ with equation $f = 0$, with $f(x_1, \dots, x_n) \in A_n$ a non-constant polynomial. Then the open set $\mathbb{A}^n \setminus Z$ is isomorphic to the irreducible hypersurface of $\mathbb{A}^{n+1}$ with equation*

$$x_{n+1} f(x_1, \dots, x_n) - 1 = 0.$$

In particular $\mathbb{A}^n \setminus Z$ is affine and $\mathcal{O}(\mathbb{A}^n \setminus Z) \cong (A_n)_f$.

Proof It is clear that $x_{n+1} f(x_1, \dots, x_n) - 1$ is an irreducible polynomial. Moreover the map

$$\phi : (a_1, \dots, a_{n+1}) \in Z_a(x_{n+1} f - 1) \to (a_1, \dots, a_n) \in \mathbb{A}^n \setminus Z \subset \mathbb{A}^n$$

is a morphism because of Proposition 6.2.1, and it is clearly bijective. Its inverse

$$\phi^{-1} : (a_1, \dots, a_n) \in \mathbb{A}^n \setminus Z \to \left(a_1, \dots, a_n, \frac{1}{f(a_1, \dots, a_n)}\right) \in Z_a(x_{n+1} f - 1)$$

is again a morphism by Proposition 6.2.1. Thus ϕ is an isomorphism. One has

$$\mathcal{O}(\mathbb{A}^n \setminus Z) \cong A(Z_a(x_{n+1} f - 1)) \cong (A_{n+1})/(x_{n+1} f - 1) \cong (A_n)_f,$$

so the assertion is proved. □

Now we are able to prove the:

Proposition 6.3.2 *Let V be any quasi-projective variety. There is a basis for the Zariski topology of V consisting of affine open subsets.*

Proof We have to prove that for any point $P \in V$ and for any open neighborhood U of P in V, there is an affine open neighborhood U' of P in V such that $U' \subseteq U$. Since

U is a quasi-projective variety, we may actually assume $U = V$. Moreover since any quasi-projective variety is covered by quasi-affine open subsets, we can assume that V is quasi-affine in $\mathbb{A}^n$. Let us denote by $\bar{V}$ the closure of V in $\mathbb{A}^n$. Then $Z = \bar{V} \setminus V$ is a closed subset in $\mathbb{A}^n$. Since $P \notin Z$, there is a polynomial $f \in \mathcal{I}_a(Z)$ such that $f(P) \neq 0$ so that $P \in V \setminus V \cap Z_a(f)$. Since $Z \subseteq \bar{V} \cap Z_a(f)$, we have that $V \setminus V \cap Z_a(f)$ is closed in $\mathbb{A}^n \setminus Z_a(f)$: indeed, on one side we have $V \setminus V \cap Z_a(F) \subseteq \bar{V} \cap (\mathbb{A}^n \setminus Z_a(f))$; on the other side $\bar{V} \cap (\mathbb{A}^n \setminus Z_a(f)) = \bar{V} \setminus (\bar{V} \cap Z_a(f)) \subseteq \bar{V} \setminus Z = V$, so that $\bar{V} \cap (\mathbb{A}^n \setminus Z_a(f)) \subseteq V \setminus (V \cap Z_a(f))$, hence $V \setminus (V \cap Z_a(f)) = \bar{V} \cap (\mathbb{A}^n \setminus Z_a(f))$. Finally the open set $V \setminus (V \cap Z_a(f))$ of V is also a closed subset of the affine variety $\mathbb{A}^n \setminus Z_a(f)$, hence it is affine. □

Finally we give a useful characterization of the isomorphisms:

Proposition 6.3.3 *Let $\phi : V \to W$ be a morphism between quasi-projective varieties. Then:*

(a) ϕ is dominant if and only if for all $P \in V$, the map $\phi_P^* : \mathcal{O}_{W,\phi(P)} \to \mathcal{O}_{V,P}$ is injective;
(b) ϕ is an isomorphism if and only if it is a homeomorphism and for all $P \in V$, the map $\phi_P^* : \mathcal{O}_{W,\phi(P)} \to \mathcal{O}_{V,P}$ is an isomorphism.

Proof Let us prove (a). Suppose ϕ is dominant. Let $P \in V$ and set $Q = \phi(P)$. Let $[U, f] \in \mathcal{O}_{W,Q}$ be such that $\phi_P^*[U, f] = [\phi^{-1}(U), f \circ \phi] = 0$. Then we have

$$\phi(V) = \phi(\overline{\phi^{-1}(U)}) \subseteq \overline{\phi(\phi^{-1}(U))} \subseteq \overline{Z_U(f)}.$$

Since ϕ is dominant, we have $\overline{Z_U(f)} = W$. But then $Z_U(f) = \overline{Z_U(f)} \cap U = W \cap U = U$ and $f = 0$ proving that ϕ_P^* is injective.

Suppose, conversely, that there is a $P \in V$ such that ϕ_P^* is injective. By Proposition 6.3.2 we may assume that both V and W are affine varieties. Hence we have $\mathcal{O}_{V,P} \cong A(V)_{\mathfrak{m}_P}$, $\mathcal{O}_{W,Q} \cong A(W)_{\mathfrak{m}_Q}$, where $Q = f(P)$ and $\mathfrak{m}_P$ and $\mathfrak{m}_Q$ are the maximal ideals in $A(V)$ and $A(W)$ respectively, corresponding to the points P and Q. Then the homomorphism $\phi_P^* : A(W)_{\mathfrak{m}_Q} \to A(V)_{\mathfrak{m}_P}$ induces the homomorphism $\phi^* : A(W) \to A(V)$ and, if ϕ_P^* is injective, then also ϕ^* is injective. Then ϕ is dominant (see Exercise 6.2.24).

Let us prove (b). Suppose ϕ is an isomorphism. Then it is a homeomorphism. Moreover if $f(P) = Q$, then $(\phi_P^*)^{-1} = (\phi^{-1})_Q^*$. This proves the assertion. Conversely, suppose ϕ is a homeomorphism and ϕ_P^* is an isomorphism for all $P \in V$. Let $P \in V$ and set again $Q = \phi(P)$. Let $[U, f] \in \mathcal{O}_{V,P}$. By the surjectivity of ϕ_P^*, there is $[U', g] \in \mathcal{O}_{W,Q}$ such that

$$[U, f] = \phi_P^*[U', g] = [\phi^{-1}(U'), g \circ \phi].$$

Hence there is an open neighborhood U'' of P in $U \cap \phi^{-1}(U')$ such that $f_{|U''} = g \circ \phi_{|U''}$, i.e., $f_{|U''} \circ \phi^{-1}_{|\phi(U'')} = g_{|U''}$. This proves that ϕ^{-1} is a morphism, hence ϕ is an isomorphism. □

Remark 6.3.4 In proving (a) of Proposition 6.3.3, we proved that:

(a) if ϕ is dominant, then ϕ_P^* is injective for all $P \in V$;
(b) if there is a $P \in V$ such that ϕ_P^* is injective, then ϕ is dominant.

So there is a $P \in V$ such that ϕ_P^* is injective if and only if ϕ_P^* is injective is injective for all $P \in V$.

Exercise 6.3.5 Prove that the morphism ϕ in the proof of Lemma 6.3.1, interpreted as a morphism of $Z_a(x_{n+1}f - 1)$ to $\mathbb{A}^n$, corresponds to the inclusion $A_n \to (A_n)_f$.

Exercise 6.3.6 *Let V be an affine variety and $f \in A(V) \setminus \{0\}$ a regular function. Prove that $U_V(f)$ is an affine open subset of V and $\mathcal{O}(U_V(f)) = A(V)[\frac{1}{f}] = A(V)_f$.

6.4 The Veronese Morphism

Let n, d be positive integers and set $N(n, d) = \binom{n+d}{d} - 1$. Then $N(n, d) + 1$ is the number of distinct monomials $x_0^{i_0} \cdots x_n^{i_n}$, with $i_0 + \cdots + i_n = d$, of degree d in $n + 1$ variables $x_0, \ldots, x_n$. We will consider the projective space $\mathbb{P}^{N(n,d)}$, whose points have homogeneous coordinates which we will denote by $[v_{i_0 \ldots i_n}]_{i_0+\cdots+i_n=d}$, that we can considered as order with the lexicographic order. Then we can consider the morphism

$$v_{n,d} : [x_0, \ldots, x_n] \in \mathbb{P}^n \to [x_0^{i_0} \cdots x_n^{i_n}]_{i_0+\cdots+i_n=d} \in \mathbb{P}^{N(n,d)}$$

that is called the *Veronese morphism* of type (n, d). We will set $V_{n,d} = v_{n,d}(\mathbb{P}^n)$.

Proposition 6.4.1 *One has:*

(a) $V_{n,d}$ *is a subvariety of* $\mathbb{P}^{N(n,d)}$ *called the* Veronese variety *of type* (n, d)*;*
(b) $v_{n,d}$ *is an isomorphism between* $\mathbb{P}^n$ *and* $V_{n,d}$.

Proof Consider the $\mathbb{K}$-algebras homomorphism

$$\vartheta_{n,d} : f(v_{i_0 \ldots i_n}) \in \mathbb{K}[v_{i_0 \ldots i_n}]_{i_0+\cdots+i_n=d} \to f(x_0^{i_0} \cdots x_n^{i_n}) \in \mathbb{K}[x_0, \ldots, x_n]$$

and interpret $\mathbb{K}[v_{i_0 \ldots i_n}]_{i_0+\cdots+i_n=d}$ as the coordinate ring of $\mathbb{P}^{N(n,d)}$ and $\mathbb{K}[x_0, \ldots, x_n]$ as the coordinate ring of $\mathbb{P}^n$. Note that $\mathrm{im}(\vartheta_{n,d})$ is the graded subring $[S_n]_d := \bigoplus_{a=0}^{\infty}(S_n)_{da}$ of $S_n = \mathbb{K}[x_0, \ldots, x_n]$. Moreover $\vartheta_{n,d}$ is a homogeneous homomorphism of weight d (recall Sect. 1.3). This implies that $\mathcal{I}_{n,d} := \ker(\vartheta_{n,d})$ is a homogeneous, prime ideal of $\mathbb{K}[v_{i_0 \ldots i_n}]_{i_0+\cdots+i_n=d}$. Let us prove that $V_{n,d} = Z_p(\mathcal{I}_{n,d})$.

It is clear that $V_{n,d} \subseteq Z_p(\mathcal{I}_{n,d})$. On the other hand $\mathcal{I}_{n,d}$ contains all polynomials of the type

$$v_{i_{00} \ldots i_{0n}}^{\alpha_0} \cdots v_{i_{\ell 0} \ldots i_{\ell n}}^{\alpha_\ell} = v_{j_{00} \ldots j_{0n}}^{\beta_0} \cdots v_{j_{\ell 0} \ldots j_{\ell n}}^{\beta_\ell}$$

with

$$\alpha_0 i_{0\mu} + \cdots + \alpha_\ell i_{\ell\mu} = \beta_0 j_{0\mu} + \cdots + \beta_\ell j_{\ell\mu}, \quad \text{for all} \quad \mu = 0, \ldots, n. \tag{6.1}$$

From this follows that if $[v_{i_0 \ldots i_n}] \in Z_p(\mathcal{I}_{n,d})$, then at least one of the coordinates $v_{0\ldots 0d0\ldots 0}$ is not zero. Indeed, from (6.1), one has

$$v^d_{i_0\ldots i_n} = v^{i_0}_{d0\ldots 0} v^{i_1}_{0d\ldots 0} \cdots v^{i_n}_{0\ldots 0d}.$$

If for example $v_{d0\ldots 0} \neq 0$, set

$$x_0 = 1, x_1 = \frac{v_{d-1,10\ldots 0}}{v_{d0\ldots 0}}, \ldots, x_n = \frac{v_{d-1,0\ldots 01}}{v_{d0\ldots 0}}. \tag{6.2}$$

By (6.1) we have

$$v_{i_0\ldots i_n} v^{d-1}_{d0\ldots 0} = v^{i_0}_{d0\ldots 0} v^{i_1}_{d-1,10\ldots 0} \cdots v^{i_n}_{d-10\ldots 01},$$

hence

$$v_{i_0\ldots i_n} = v_{d0\ldots 0} x_0^{i_0} \cdots x_n^{i_n}$$

so that

$$[v_{i_0\ldots i_n}] = v_{n,d}([x_0, \ldots, x_n]).$$

This proves part (a), and shows also that $S(V_{n,d}) = [S_n]_d$.

Let us now prove part (b). It is clear that $v_{n,d}$ is a bijective map from $\mathbb{P}^n$ to $V_{n,d}$, whose inverse is still a morphism, since it is locally defined by formula (6.2) or by analogous formulae. This proves the assertion. □

We will still call Veronese variety of type (n, d) every variety which is projectively equivalent to $V_{n,d}$. We already met some Veronese varieties, for instance the projective twisted cubics are projectively equivalent to $V_{1,3}$, the irreducible conics are projectively equivalent to $V_{1,2}$. The varieties $V_{1,n}$ are called *rational normal curves*, the varieties $V_{2,n}$ are called *Veronese surfaces* of type n, in particular $V_{2,2}$ is simply called *Veronese surface*.

Next we want to give a geometric interpretation of the construction of Veronese varieties. Consider, for every positive integers n, d, the projective space $\mathcal{L}_{n,d} = \mathbb{P}(S_{n,d})$ of dimension $N(n, d) = \binom{n+d}{d} - 1$, whose points, as we saw in Sect. 1.6.5, are in 1:1 correspondence with effective divisors of degree d in $\mathbb{P}^n$. Recall that there is a natural system of coordinates in $\mathcal{L}_{n,d}$ by assigning to the class of a polynomial $\sum_{i_0+\cdots+i_n=d} v_{i_0\ldots i_n} x_0^{i_0} \cdots x_n^{i_n}$ the homogeneous coordinates $[v_{i_0\ldots i_n}]_{i_0+\cdots+i_n=d}$ lexicographically ordered. This way $\mathcal{L}_{n,d}$ can be identified with $\mathbb{P}^{N(n,d)}$, which we will constantly do in what follows. Recall that $\mathcal{L}_{n,1}$ identifies with the dual $\check{\mathbb{P}}^n$ of $\mathbb{P}^n$.

Consider the following map

$$\tilde{v}_{n,d} : H \in \check{\mathbb{P}}^n \to dH \in \mathcal{L}_{n,d}$$

which associates to any hyperplane H of $\mathbb{P}^n$ the degree d divisor dH, consisting of H with multiplicity d. This map is called the *Veronese map* of type (n, d).

Lemma 6.4.2 *The Veronese map $\tilde{v}_{n,d}$ is a morphism.*

Proof Suppose $H = [u_0, \dots, u_n] \in \check{\mathbb{P}}^n$, i.e., H has equation $u_0x_0 + \cdots + u_nx_n = 0$. Then dH has equation

$$(u_0x_0 + \cdots + u_nx_n)^d = \sum_{i_0+\cdots+i_n=d} \frac{d!}{i_0! \cdots i_n!} u_0^{i_0} \cdots u_n^{i_n} x_0^{i_0} \cdots x_n^{i_n} = 0.$$

Hence

$$\tilde{v}_{n,d}([u_0, \dots, i_n]) = \left[\frac{d!}{i_0! \cdots i_n!} u_0^{i_0} \cdots u_n^{i_n}\right]_{i_0+\cdots+i_n=d},$$

which proves the assertion. □

Now we consider two cases:

(a) $\mathrm{char}(\mathbb{K}) = 0$. Consider the projectivity of $\mathbb{P}^{N(n,d)}$ given by

$$\omega : [v_{i_0 \dots i_n}]_{i_0+\cdots+i_n=d} \in \mathbb{P}^{N(n,d)} \to \left[\frac{d!}{i_0! \cdots i_n!} v_{i_0 \dots i_n}\right]_{i_0+\cdots+i_n=d} \in \mathbb{P}^{N(n,d)}$$

whose matrix is diagonal with non-zero entries. One has $\tilde{v}_{n,d} = \omega \circ v_{n,d}$. Hence in this case $\tilde{v}_{n,d}$ is an isomorphism of $\check{\mathbb{P}}^n$ onto its image which is a Veronese variety of type (n, d);

(b) $\mathrm{char}(\mathbb{K}) = p \neq 0$. Then it can happen that some of the integers $\frac{d!}{i_0! \cdots i_n!}$ are 0 modulo p. In this case what we said in (a) is no longer valid, and one has to examine the situation case by case.

We can consider one more map, which is called the *dual Veronese map*

$$\check{v}_{n,d} : \mathbb{P}^n \to \check{\mathcal{L}}_{n,d}$$

which sends a point $P \in \mathbb{P}^n$ to the hyperplane of $\mathcal{L}_{n,d}$ consisting of all divisors of degree d in $\mathbb{P}^n$ whose support contains P.

Lemma 6.4.3 *The dual Veronese map coincides with the Veronese morphism of $\mathbb{P}^n$ to $\check{\mathcal{L}}_{n,d}$.*

Proof Let $P = [p_0, \dots, p_n]$ and consider an effective divisor D of degree d in $\mathbb{P}^n$, which has equation $\sum_{i_0+\cdots+i_n=d} v_{i_0 \dots i_n} x_0^{i_0} \cdots x_n^{i_n} = 0$. Then D contains P if and only if $\sum_{i_0+\cdots+i_n=d} v_{i_0 \dots i_n} p_0^{i_0} \cdots p_n^{i_n} = 0$, and this equation defines $\check{v}_{n,d}(P) \in \check{\mathcal{L}}_{n,d}$. So in conclusion $\check{v}_{n,d}(P) = \check{v}_{n,d}([p_0, \dots, p_n]) = [p_0^{i_0} \cdots p_n^{i_n}]_{i_0+\cdots+i_n=d}$, as wanted. □

Exercise 6.4.4 Give another proof of (b) of Proposition 6.4.1 based on Proposition 6.3.3, (b).

Exercise 6.4.5 *(Steiner construction) Consider two distinct points $P, Q \in \mathbb{P}^2$. Let (P) [resp. (Q)] be the pencil of lines passing through P [resp. through Q]. Consider a projectivity $\omega : (P) \to (Q)$, such that $\omega(P \vee Q) \neq P \vee Q$. Consider the following set of points of $\mathbb{P}^2$

$$V = \{P = r \cap \omega(r), r \in (P)\}.$$

Prove that V is an irreducible conic.

Exercise 6.4.6 Prove that Veronese varieties are non-degenerate.

Exercise 6.4.7 Consider in $\mathbb{P}^n$ the set V of points $[x_0, \dots, x_n]$ such that

$$\operatorname{rank}\begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} \\ x_1 & x_2 & \dots & x_n \end{pmatrix} = 1.$$

Prove that V is a rational normal curve.

Exercise 6.4.8 Prove that a hyperplane intersects a rational normal curve $V_{1,n}$ in no more than n points, and there are hyperplanes which intersect $V_{1,n}$ in exactly n distinct points.

Exercise 6.4.9 Prove that a rational normal curve V_n is the image of a morphism

$$[x_0, x_1] \in \mathbb{P}^1 \to [f_0(x_0, x_1), \dots, f_n(x_0, x_1)] \in \mathbb{P}^n \tag{6.3}$$

where $f_0, \dots, f_n$ is a basis of $S_{1,n}$.

Exercise 6.4.10 Let $g(x_0, x_1) = \prod_{i=0}^{n}(\mu_i x_0 - \nu_i x_1)$ be a homogeneous polynomial of degree $n + 1$ in x_0, x_1, which has $n + 1$ distinct roots (up to a proportionality factor), i.e., the points $[\nu_i, \mu_i] \in \mathbb{P}^1$ are distinct, for $i = 0, \dots, n$. Prove that the polynomials $f_i(x_0, x_1) = \frac{g(x_0,x_1)}{\mu_i x_0 - \nu_i x_1}$, for $i = 0, \dots, n$, form a basis of $S_{1,n}$.

Exercise 6.4.11 Continuing the Exercise 6.4.10, consider the rational normal curve V given by (6.3) with $f_i(x_0, x_1)$, for $i = 0, \dots, n$, as in Exercise 6.4.10. Assume that $\mu_i \neq 0$ and $\nu_i \neq 0$, for $i = 0, \dots, n$. Then prove that V contains the vertices of the fundamental pyramid and the two distinct points $[\frac{1}{\mu_0}, \dots, \frac{1}{\mu_n}]$ and $[\frac{1}{\nu_0}, \dots, \frac{1}{\nu_n}]$. On the whole these are $n + 3$ distinct points of $\mathbb{P}^n$ in general position (see Exercise 1.6.9).

Exercise 6.4.12 *Consider $n + 3$ distinct points of $\mathbb{P}^n$ in general position. Prove that there is a rational normal curve containing them.

Exercise 6.4.13 Let V be a divisor of degree d in $\mathbb{P}^n$. Prove that there is a hyperplane $H \subset \mathbb{P}^{N(n,d)}$ such that $v_{n,d}(V) = H \cap V_{n,d}$.

Exercise 6.4.14 Let V be a hypersurface of degree d in $\mathbb{P}^n$. Prove that $\mathbb{P}^n \setminus V$ is an affine variety.

Exercise 6.4.15 *Let $X \subseteq \mathbb{P}^n$ be a variety which is not a point and let $V \subset \mathbb{P}^n$ be a hypersurface. Prove that $X \cap V \neq \emptyset$.

Exercise 6.4.16 Prove the following complement to the weak formulation of Bézout Theorem 2.2.6: two projective plane curves have non-empty intersection.

Exercise 6.4.17 Prove that $\mathbb{A}^2$ and $\mathbb{P}^2$ are not homeomorphic.

Exercise 6.4.18 *Prove that the coordinate ring of a projective variety is not invariant under isomorphisms.

6.5 Solutions of Some Exercises

6.2.8 An automorphism of $\mathbb{A}^1$ is a map $\phi : x \in \mathbb{A}^1 \to P(x) \in \mathbb{A}^1$, where $P \in A_1$. Then $\phi^{-1} : x \in \mathbb{A}^1 \to Q(x) \in \mathbb{A}^1$ with $Q \in A_1$, and one has the identity $x = Q(P(x))$. This implies that P and Q have degree 1.

6.2.11 Consider the map

$$\phi : (x_1, x_2) \in \mathbb{A}^2 \to (x_1, x_2 + P(x_1)) \in \mathbb{A}^2$$

where $P \in A_1$ is any polynomial. These are automorphisms of $\mathbb{A}^2$ which form a group isomorphic to the additive group of A_1.

6.2.14 We consider the projective case only, the affine one being trivial. Let $\bar{Z}$ be the projective twisted cubic. We have the homeomorphism $\psi : \mathbb{P}^1 \to \bar{Z}$ considered in Sect. 3.2.2, which is a morphism. Its inverse sends $[x_0, x_1, x_2, x_3] \in \bar{Z}$ to $[x_0, x_1]$ if $x_0 \neq 0$, or to $[x_2, x_3]$ if $x_3 \neq 0$, hence it is also a morphism.

6.2.17 This is a standard theorem in elementary affine geometry.

6.2.19 This is a standard theorem in elementary projective geometry, or, if you wish, in the classification of quadratic forms over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) \neq 2$.

6.2.20 It suffices to prove the assertion for the conic Z with equation $x_2^2 = x_0 x_1$. In this case an isomorphism with $\mathbb{P}^1$ is given by the map

$$\phi : [\lambda, \mu] \in \mathbb{P}^1 \to [\lambda^2, \lambda\mu, \mu^2] \in Z.$$

6.2.24 If $f \in \ker(\phi^*)$ is non-zero, then $f \circ \phi$ is zero in $A(V)$. Then $\phi(V) \subseteq Z_W(f) \subsetneq W$ and $\overline{\phi(V)}$ is a proper closed subset of W. Conversely, if $\overline{\phi(V)}$ is a proper closed subset of W, there is a polynomial f such that $\overline{\phi(V)} \subseteq Z_a(f)$, with $f \notin \mathcal{I}_a(W)$. Then f determines a non-zero function such that $f \in \ker(\phi^*)$.

6.2.25 If $f \in \mathcal{O}(U) \subseteq K(\mathbb{A}^1) \cong \mathbb{K}(x_1)$, then $f = \frac{h(x_1)}{g(x_1)}$, where $Z_a(g) \subseteq \{a_1, \ldots, a_n\}$. In particular $x_1 - a_i$ and $(x_1 - a_i)^{-1}$ are both in $\mathcal{O}(U)$; for all $i = 1, \ldots, n$. Since $x_1 - a_i$ is non-constant, this proves that U is not isomorphic to $\mathbb{A}^1$.

6.2.26 Since V is isomorphic to a projective variety, then $\mathcal{O}(V) = A(V) = \mathbb{K}$. This implies that $\mathcal{I}_a(V)$ is a maximal ideal, hence V is a point.

6.2.27 Let V be a projective variety and W an affine variety. Then $\phi \in M(V, W)$ corresponds to the $\mathbb{K}$-algebras homomorphism $\phi^W : A(W) \to \mathcal{O}(V) = \mathbb{K}$. If $A(W) = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_a(W)$, set $\phi^W(x_i) = a_i \in \mathbb{K}$, for $i = 1, \ldots, n$. Then for all $P \in V$ one has $\phi(P) = (a_1, \ldots, a_n) \in W$.

6.2.28 The morphism ϕ corresponds to the $\mathbb{K}$-algebras homomorphism

$$\phi^* : f(x_1, x_2) \in A(V) \to f(t^3, t^2) \in A_1$$

which is not surjective because $t \notin \text{im}(\phi^*)$. The inverse of ϕ restricted to $V \setminus \{\mathbf{0}\}$ is the map $(x_1, x_2) \mapsto \frac{x_2}{x_1}$, so its is a morphism.

6.2.29 Consider the map

$$\phi : x \in \mathbb{K} \to x^p \in \mathbb{K}.$$

This is an isomorphism, called the *Frobenius isomorphism*. It fixes all the points of the *fundamental field* $\mathbb{F}_p \subset \mathbb{K}$. If $f(x_1, \ldots, x_n) \in A_n$, we denote by $f_\phi(x_1, \ldots, x_n)$ the polynomial obtained by applying ϕ to all the coefficients of f. In this way we get a map

$$\bar{\phi} : f \in A_n \to f_\phi \in A_n$$

which is a ring (but not a $\mathbb{K}$-algebra) isomorphism. Moreover $\bar{\phi}$ preserves degrees and homogeneous polynomials. Since ϕ is bijective, also the Frobenius morphisms F are bijective. Moreover, if $f \in S_n$, one has

$$f(x_1^p, \dots, x_n^p) = \left(\bar{\phi}^{-1}(f)(x_1, \dots, x_n)\right)^p,$$

hence, if f is homogeneous, one has $F^{-1}(Z_p(f)) = Z_p(\bar{\phi}^{-1}(f))$ and $F(Z_p(f)) = Z_p(\bar{\phi}(f))$. So F is a homeomorphism. To see that F is not an isomorphism, it suffices to see it is not an isomorphism on $\mathbb{A}^n$. The map F corresponds to the homomorphism

$$F^* : f(x_1, \dots, x_n) \in A_n \to f(x_1^p, \dots, x_n^p) \in A_n$$

which is clearly not surjective.

6.2.30 By Corollary 6.2.3, (b), it suffices to show that $\mathcal{O}(\mathbb{A}^2 \setminus \{\mathbf{0}\}) = A_2$. To see this, remark that the inclusion $\mathbb{A}^2 \setminus \{\mathbf{0}\} \subset \mathbb{A}^2$, determines an inclusion $A_2 \subseteq \mathcal{O}(\mathbb{A}^2 \setminus \{\mathbf{0}\})$. We want to see that also the opposite inclusion holds. One has $\mathcal{O}(\mathbb{A}^2 \setminus \{\mathbf{0}\}) \subseteq K(\mathbb{A}^2) = \mathbb{Q}(A_2)$. So, if $f \in \mathcal{O}(\mathbb{A}^2 \setminus \{\mathbf{0}\})$, we can write $f = \frac{g}{h}$, with $g, h \in A_2$ which we can assume to be coprime. If $h \in \mathbb{K} \setminus \{0\}$, then $f \in A_2$ as wanted. Assume $h \notin \mathbb{K} \setminus \{0\}$. If $P \in Z_a(h) \setminus \{\mathbf{0}\}$, there is an open neighborhood U of P in $\mathbb{A}^2 \setminus \{\mathbf{0}\}$, such that in U the function f writes as $f = \frac{g'}{h'}$, with $h'(Q) \neq 0$, for all $Q \in U$. In $U \setminus (U \cap Z_a(h))$ one has $\frac{g}{h} = \frac{g'}{h'}$, hence we have there also $gh' = hg'$, and then this relation holds in the whole of $\mathbb{A}^2$. Since g is prime with h, then h divides h', hence $h'(P) = 0$, a contradiction. This prove that $h \in \mathbb{K} \setminus \{0\}$ and we are done.

6.2.31 The argument is very similar to the one for the solution of Exercise 6.2.30.

6.2.32 If $Z \subseteq \mathbb{A}^n$ is a closed subset, then $A(Z) = A_n/\mathcal{I}_a(Z)$ is finitely generated with no nilpotent elements because $\mathcal{I}_a(Z)$ is radical. The converse is proved as in Corollary 6.2.3, (b).

6.3.6 Imitate the proof of Lemma 6.3.1.

6.4.4 If $f(v_{i_0\dots i_n}) \in \mathbb{K}[v_{i_0\dots i_n}]_{i_0+\dots+i_n=d}$ is a homogeneous polynomial, then $f^d \in [S_n]_d$, hence there is a polynomial $g(v_{i_0\dots i_n}) \in \mathbb{K}[v_{i_0\dots i_n}]_{i_0+\dots+i_n=d}$ such that $f^d = \vartheta_{n,d}(g)$. Then $v_{n,d}(Z_p(f)) = V_{n,d} \cap Z_p(g)$, and this proves that $v_{n,d}$ is closed, so it is a homeomorphism. It remains to prove that for any point $P \in \mathbb{P}^n$, the map $(v_{n,d})^*_P$ is an isomorphism, i.e., by Proposition 6.3.3, (a), that it is surjective. If $[U, f] \in \mathcal{O}_{\mathbb{P}^n,P}$, we may assume that $f = \frac{g}{h}$, with g, h homogenous polynomials of the same degree a and h non-zero in U. If k is a homogenous polynomial of degree $\alpha d - a$, with $\alpha \gg 0$, non-zero in P, in a suitable neighborhood of P we have $\frac{g}{h} = \frac{gk}{hk}$. On the other hand there are polynomials $G, H \in \mathbb{K}[v_{i_0\dots i_n}]_{i_0+\dots+i_n=d}$, both homogenous of degree α, such that $gk = \vartheta_{n,d}(G)$ and $hk = \vartheta_{n,d}(H)$, so that $[U, f] = (v_{n,d})^*_P[V_{n,d} \setminus (V_{n,d} \cap Z_p(H)), \frac{G}{H}]$, which proves the required surjectivity.

6.4.10 Suppose we have a relation $\sum_{j=0}^n \lambda_j f_j = 0$. Computing at (ν_i, μ_i) we get $\lambda_i \prod_{j\neq i}(\mu_j\nu_i - \nu_j\mu_i) = 0$, which implies $\lambda_i = 0$.

6.4.12 By applying the fundamental theorem of projectivities (see Exercise 1.6.9) we may assume that the points in question are the vertices of the fundamental pyramid, the unitary point and another point $[p_0, \dots, p_n]$, with $p_0, \dots, p_n$ non-zero and not equal. Then the rational normal curve in question is the one constructed as in Exercise 6.4.12, where $\mu_0 = \ldots = \mu_n = 1$ and $\nu_i = \frac{1}{p_i}$, for $i = 0, \dots, n$.

6.4.14 One has that $\mathbb{P}^n \setminus V$ is isomorphic to $V_{n,d} \setminus (V_{n,d} \cap V') \subseteq \mathbb{P}^{N(n,d)} \setminus V' \cong \mathbb{A}^{N(n,d)}$. (see Exercise 6.4.14).

6.4.16 If $X \cap V = \emptyset$ then, by Exercise 6.4.14, X would be isomorphic to an affine variety, and then it would be a point by Exercise 6.2.26.

6.4.17 Remark that there are affine plane curves (which are closed subsets with topological dimension 1) which do not intersect. Then apply Exercise 6.4.16.

6.4.18 For example $\mathbb{P}^n$ and $V_{n,d}$, with $d > 1$, are isomorphic but $S(\mathbb{P}^n) = S_n$ whereas $S(V_{n,d}) = [S_n]_d$ are not isomorphic as $\mathbb{K}$-algebras, since they have a different minimal number of generators as $\mathbb{K}$-algebras.

Chapter 7
Rational Maps

7.1 Definition of Rational Maps and Basic Properties

Let V, W be quasi-projective varieties. Let us denote by $\mathcal{K}(V, W)$ the set of all pairs (U, ϕ), where U is a non-empty open subset of V and $\phi \in M(U, W)$. We define the following relation $\mathcal{R}$ in $\mathcal{K}(V, W)$:

$$(U, \phi)\mathcal{R}(U', \phi') \quad \text{if and only if} \quad \phi_{|U\cap U'} = \phi'_{|U\cap U'}.$$

It is easy to verify, taking into account Proposition 6.2.6, that $\mathcal{R}$ is an equivalence relation. The set $\mathcal{K}(V, W)/\mathcal{R}$ is denoted by $K(V, W)$ and its elements are called *rational maps* of V in W. The equivalence class of (U, ϕ) is denoted by $[U, \phi]$. Proceeding as in Exercise 5.2.2, one sees that for any pair $(U, \phi) \in \mathcal{K}(V, W)$, there is a pair $(\tilde{U}, \tilde{\phi}) \in \mathcal{K}(V, W)$, such that $(U, \phi)\mathcal{R}(\tilde{U}, \tilde{\phi})$, and for any pair (U', ϕ') such that $(U, \phi)\mathcal{R}(U', \phi')$, one has that $U' \subseteq \tilde{U}$. Then $\tilde{U}$ is called the *definition set* of the rational map. The rational map determined by the pair $(U, \phi) \in \mathcal{K}(V, W)$ is often denoted as $\phi : V \dashrightarrow W$.

There is an obvious injective map

$$\theta : \phi \in M(V, W) \to [V, \phi] \in K(V, W).$$

The image of this map is the set of rational maps whose definition set coincides with V. We will identify these maps with the morphisms of which they are the images via θ.

A rational map $[U, \phi] \in K(V, W)$ is said to be *dominant* if $\phi : U \to W$ is dominant. This definition is well posed (see Exercise 7.1.6). Similarly one sees that given $[U, \phi] \in K(V, W)$, then $\overline{\phi(U)}$ is a closed subvariety of W which depends uniquely from $[U, \phi]$ and not from the pair (U, ϕ). We say that $\overline{\phi(U)}$ is the *image* of the rational map $\phi : V \dashrightarrow W$, and it is denoted by $\mathrm{im}(\phi)$. Of course $\phi : V \dashrightarrow W$ is dominant onto $\mathrm{im}(\phi)$.

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If $V' \subseteq V$ is a subvariety, the rational map $[U, \phi] \in K(V, W)$ is said to be *defined* on V', if there is a pair (U', ϕ') such that $(U', \phi')\mathcal{R}(U, \phi)$ and $V' \cap U' \neq \emptyset$. In this case the rational map $[V' \cap U', \phi'_{|V' \cap U'}] \in K(V', W)$ is well defined and it is called the *restriction* of $[U, \phi]$ to V'. If $K(V', V, W)$ is the set of all rational maps of V to W which are defined on V', we have the obvious *restriction map*

$$r_{V'} : K(V', V, W) \to K(V', W)$$

which, for morphisms, coincides with the restriction.

Let Z be again a variety and consider $[U, \phi] \in K(V, W), [U', \psi] \in K(\text{im}(\phi), W, Z)$. Since $\overline{\phi(U)} \cap U' \neq \emptyset$, one has also $\phi(U) \cap U' \neq \emptyset$, hence $\phi^{-1}(\phi(U) \cap U') = U''$ is a non-empty open subset of U, where the morphism $\psi \circ \phi$ is defined. It is easy to check that $[U'', \psi \circ \phi]$ is a well defined element of $K(V, Z)$ which is called the rational map *composed* of $[U, \phi]$ and $[U', \psi]$. Note that dominant rational maps can be always composed. Hence it makes sense to consider the category whose objects are quasi-projective varieties and the morphism are dominant rational maps. Two varieties which are isomorphic in this category are said to be *birationally equivalent* or simply *birational* and the isomorphisms in this category are called *birational maps* or *birational transformations*. So a birational transformation between the quasi-projective varieties V and W is a dominant rational map $\phi : V \dashrightarrow W$ such that there is a dominant rational map $\psi : W \to V$ such that $\psi \circ \phi = \text{id}_V$ and $\phi \circ \psi = \text{id}_W$. Then one writes $\psi = \phi^{-1}$.

Let again V, W be quasi-projective varieties and let V' [resp. W'] a subvariety of V [resp. of W]. Let $\phi : V \dashrightarrow W$ which is defined in V' and the image of $\phi_{|V'} = r_{V'}(\phi)$ contains W'. Let $f \in \mathcal{O}_{W,W'} \subseteq K(W)$. We can consider the composed rational function of ϕ and f, which we denote as $f \circ \phi$. Then $f \circ \phi \in K(V)$, but it is clear that it is defined on V', so that $f \circ \phi \in \mathcal{O}_{V,V'}$. This way we have a map

$$\phi^* : \mathcal{O}_{W,W'} \to \mathcal{O}_{V,V'}$$

and it is easy to see that this is a $\mathbb{K}$-algebras homomorphism. If $V = W$ and $V' = W'$ and $\phi = \text{id}_V$ then $\phi^* = \text{id}_{\mathcal{O}_{V,V'}}$. Finally, if Z is a third variety, Z' is a subvariety of Z and $\psi : W \dashrightarrow Z$ is such that

$$\psi^* : \mathcal{O}_{Z,Z'} \to \mathcal{O}_{W,W'}$$

can be considered, then, as it is easy to check, one has $\phi^* \circ \phi^* = (\psi \circ \phi)^*$.

Lemma 7.1.1 *In the above setting, if $\overline{W'}$ coincides with the image of $\phi_{|V'}$, then*

$$\phi^*(\mathfrak{m}_{W,W'}) \subseteq \mathfrak{m}_{V,V'}.$$

Proof By Lemma 5.5.2 and Proposition 6.3.2 we may assume that V, V', W, W' are all affine and ϕ is a morphism. Then ϕ corresponds to a $\mathbb{K}$-algebras homomorphism $\phi^* : A(W) \to A(V)$ and $\phi_{|V'}$ corresponds to the homomorphism $(\phi_{|V'})^* : A(W) \to$

$A(V') = A(V)/\mathcal{I}_V(V')$, which is composed of ϕ^* and of the canonical map $\pi : A(V) \to A(V')$. On the other hand, since $\overline{\phi(V')} = W'$, we have that $\phi_{|V'}$ can be interpreted as a dominant morphism of V' onto W', and this implies that $(\phi_{|V'})^*$ factors through the canonical map $\pi' : A(W) \to A(W') = A(W)/\mathcal{I}_W(W')$. Hence $\phi^*(\mathcal{I}_W(W')) \subseteq \mathcal{I}_V(V')$. Since

$$\phi^* : \mathcal{O}_{W,W'} = A(W)_{\mathcal{I}_W(W')} \to \mathcal{O}_{V,V'} = A(V)_{\mathcal{I}_V(V')}$$

is induced by $\phi^* : A(W) \to A(V)$, the assertion follows. □

In particular, if ϕ is dominant and $V = V'$ and $W = W'$, one has a homomorphism of fields $\phi^* : K(W) \to K(V)$, which, being non-zero, is injective. If ϕ is birational, then $\phi^* : K(W) \to K(V)$ is an isomorphism, hence the field of rational functions is invariant under birational transformations.

Theorem 7.1.2 *Let V, W be quasi-projective varieties, V', W' subvarieties of V, W respectively, and let*

$$\alpha : \mathcal{O}_{W,W'} \to \mathcal{O}_{V,V'}$$

be a $\mathbb{K}$-algebra homomorphism such that

$$\alpha(\mathfrak{m}_{W,W'}) \subseteq \mathfrak{m}_{V,V'}. \tag{7.1}$$

Then there is a unique rational map $\phi : V \dashrightarrow W$, defined on V' and inducing on V' a dominant rational map onto W' such that $\alpha = \phi^*$. Moreover α is injective if and only if ϕ is dominant.

Proof Let U' be an affine open subset of W such that $U' \cap W' \neq \emptyset$. Then $A(U')$ is a finitely generated $\mathbb{K}$-algebra, and let $\xi_1, \ldots, \xi_n$ be a set of generators of $A(U')$. Since $A(U') \subseteq \mathcal{O}_{W,W'} \cong \mathcal{O}_{U',U'\cap W'}$, we may interpret $\alpha(\xi_1), \ldots, \alpha(\xi_n)$ as rational functions on V defined on V'. Let U be an open affine subset of V such that $U \cap V' \neq \emptyset$, on which $\alpha(\xi_1), \ldots, \alpha(\xi_n)$ are all defined. So α determines a $\mathbb{K}$-algebras homomorphism $\alpha' : A(U') \to A(U)$, hence a morphism $\phi : U \to U'$ (see Theorem 6.2.2), thus we have a rational map $\phi : V \dashrightarrow W$ defined on V'.

Note that

$$\mathcal{O}_{W,W'} \cong A(U')_{\mathcal{I}_{U'}(\overline{W'\cap U'})}, \quad \mathcal{O}_{V,V'} \cong A(U')_{\mathcal{I}_U(\overline{V'\cap U})},$$

and α comes, as an obvious extension, from α' and it is just equal to ϕ^*. Hence if α is injective, so is also α', thus ϕ is dominant (see Exercise 6.2.24); conversely, if ϕ is dominant, then α' is injective and α is injective as well.

Let now $f \in \mathfrak{m}_{W,W'}$. By (7.1), α' sends f to a function regular on U, which vanishes on $U \cap V'$: this implies that $\phi(V' \cap U) \subseteq \overline{W' \cap U'}$. Let us prove that the restriction of ϕ to V' is dominant onto W'. For this, note that $\phi_{|V'\cap U}$ corresponds a homomorphism $\phi^*_{|V'\cap U} : A(U' \cap W') \to A(U \cap V')$, where we can assume that both V', W' are closed. On the other hand, by (7.1), α induces an injective homomorphism of the residue field $K(W')$ of $\mathcal{O}_{W,W'}$ to the residue field $K(V')$ of $\mathcal{O}_{V,V'}$. This homomorphism extends $\phi^*_{|V'\cap U}$, which is therefore injective, and this proves that $\phi_{|V'}$ is dominant onto W'.

Let us prove finally the uniqueness of ϕ. If $\phi' : V \dashrightarrow W$ is such that $\alpha = (\phi')^*$, then, repeating the above argument, we see that ϕ' coincides with ϕ on U and therefore $\phi = \phi'$. □

Corollary 7.1.3 *Let V, W, V', W' be as in the statement of Theorem 7.1.2. The following are equivalent:*

(a) there is a birational map $\phi : V \dashrightarrow W$, defined on V' such that $\phi_{|V'}$ induces a birational transformation of V' onto W';
(b) $\mathcal{O}_{V,V'}$ and $\mathcal{O}_{W,W'}$ are isomorphic as $\mathbb{K}$-algebras;
(c) there is an open subset U of V such that $U \cap V' \neq \emptyset$, an open subset U' of W such that $U' \cap W' \neq \emptyset$, and an isomorphism $\phi : U \to U'$ such that $\phi(U \cap \bar{V}') = U' \cap \bar{W}'$.

Proof It is clear that (a) implies (b). Theorem 7.1.2 implies that (b) implies (a). It is again obvious that (c) implies (a). Let us prove that (a) implies (c).

Let $[U_0, \phi]$ be a birational map of V onto W, defined on V', such that $U_0 \cap V' \neq \emptyset$ and such that $[U_0 \cap V', \phi_{|U_0 \cap V'}]$ is a birational map of V' onto W'. Let $[U_0', \psi]$ be the inverse of ϕ, so that $\psi \circ \phi$ is represented by $[U_0 \cap \phi^{-1}(U_0'), \psi \circ \phi]$ and $\phi \circ \psi$ by $[U_0' \cap \psi^{-1}(U_0), \phi \circ \psi]$. Since $\psi \circ \phi = \mathrm{id}_V$ and $\phi \circ \psi = \mathrm{id}_W$, then $\psi \circ \phi$ is the identity on $U_0 \cap \phi^{-1}(U_0')$ and $\phi \circ \psi$ is the identity on $U_0' \cap \psi^{-1}(U_0)$. Let us set $U = \phi^{-1}(U_0' \cap \psi^{-1}(U_0))$ and $U' = \psi^{-1}(U_0 \cap \phi^{-1}(U_0'))$. One has $\phi(U) \subseteq U_0' \cap \psi^{-1}(U_0)$. If $P \in U_0' \cap \psi^{-1}(U_0)$ one has $\phi(\psi(P)) = P$, hence $P \in \psi^{-1}(\phi^{-1}(U_0'))$, and therefore

$$P \in \psi^{-1}(\phi^{-1}(U_0')) \cap \psi^{-1}(U_0) = \psi^{-1}(\phi^{-1}(U_0') \cap U_0) = U'.$$

So $\phi(U) \subseteq U'$. Similarly $\psi(U') \subseteq U$, and clearly ϕ and ψ induce isomorphisms between U and U'. Moreover $\phi_{|U \cap V'}$ is a birational morphism of $U \cap V'$ to $U' \cap \bar{W}'$. Since $\phi(U \cap V')$ is dense in $U' \cap \bar{W}'$ and ϕ is closed on U, the final part of (c) follows. □

Corollary 7.1.4 *The correspondence that to any variety V associates the field $K(V)$ and to any dominant rational map $\phi : V \dashrightarrow W$ associates the $\mathbb{K}$-algebra injective homomorphism $\phi^* : K(W) \to K(V)$, is a contravariant functor, which is an equivalence of categories between the category of varieties and the category of fields which are finitely generated extensions of $\mathbb{K}$.*

Proof By Theorem 7.1.2 and Corollary 7.1.3 applied to $V = V'$ and $W = W'$, hence $\mathcal{O}_{V,V'} = K(V)$ and $\mathcal{O}_{W,W'} = K(W)$, it suffices to prove that the functor in question is surjective. Let K be a finitely generated extension of $\mathbb{K}$, and let $\xi_1, \ldots, \xi_n$ be a set of generators of K. Then $K = \mathbb{Q}(\mathbb{K}[\xi_1, \ldots, \xi_n])$, and on the other hand $\mathbb{K}[\xi_1, \ldots, \xi_n]) = A(V)$ for some affine variety $V \subseteq \mathbb{A}^n$. The assertion follows by Theorem 5.5.3, (c). □

The birational transformations of a variety V to itself form, with the operation of composition, a group which is denoted by $\mathrm{Bir}(V)$, and it contains, as a subgroup, the group of *automorphisms* of V, i.e., the isomorphisms of V to itself. In particular $\mathrm{Bir}(\mathbb{P}^n)$ is called the *Cremona group* of $\mathbb{P}^n$ and its elements are called *Cremona transformations* of $\mathbb{P}^n$.

Exercise 7.1.5 Prove that $\mathcal{R}$ is an equivalence relation in $\mathcal{K}(V, W)$.

Exercise 7.1.6 Prove that the definition of dominant rational map is well posed.

Exercise 7.1.7 *Let V be a quasi-projective variety. Prove that a rational function $f \in K(V)$ can be interpreted as a rational function $f : V \dashrightarrow \mathbb{P}^1$.

Exercise 7.1.8 *Prove that any quasi-projective variety is birationally equivalent to any of its non-empty open subsets.

Exercise 7.1.9 Let V, W, V', W' be quasi-projective varieties, $\alpha : V \dashrightarrow V'$, $\beta : W \dashrightarrow W'$ birational maps, $\phi : V \dashrightarrow W$ a dominant rational map. Prove that there is a unique dominant rational map $\phi' : V' \to W'$ such that $\phi' \circ \alpha = \beta \circ \phi$.

Exercise 7.1.10 *Prove that any rational map $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ is everywhere defined, hence it is a morphism.

Exercise 7.1.11 *Let $\mathbf{A}$ be a non-zero matrix on $\mathbb{K}$ of type $(n+1) \times (m+1)$ of rank r with $1 \leqslant r \leqslant n+1$. The set of points $[\mathbf{x}] \in \mathbb{P}^n$ such that $\mathbf{x} \cdot \mathbf{A} = \mathbf{0}$ is a subspace of dimension $n - r$ that we will denote by $\mathbb{P}_{\mathbf{A}}$. The map

$$\tau_{\mathbf{A}} : [\mathbf{x}] \in \mathbb{P}^n \setminus \mathbb{P}_A \to [\mathbf{x} \cdot \mathbf{A}] \in \mathbb{P}^m$$

is well defined and it is a morphism from the open set $U_{\mathbf{A}} = \mathbb{P}^n \setminus \mathbb{P}_{\mathbf{A}}$ to $\mathbb{P}^n$, that determines a rational map $\tau_{\mathbf{A}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, which is called a *generalized projectivity*. This is an actual projectivity if $r = n + 1$, in which case $\mathbb{P}_{\mathbf{A}} = \emptyset$. If $r \leqslant n$, the map is also called a *degenerate projectivity with centre* $\mathbb{P}_{\mathbf{A}}$.

Prove that a degenerate projectivity is dominant onto a subspace of $\mathbb{P}^m$ of dimension $r - 1$.

Prove that if $r = 1$ then $\tau_{\mathbf{A}}$ is constant, hence it is everywhere defined. Prove that if $r > 1$ the definition set of $\tau_{\mathbf{A}}$ is $U_{\mathbf{A}}$.

Exercise 7.1.12 *Let $\mathbb{P}_1$ be a subspace of dimension $n - r$, with $r \geqslant 1$ of $\mathbb{P}_n$, and let $\mathbb{P}_2$ be another subspace of $\mathbb{P}^n$ of dimension $r - 1$, which are *skew*, i.e., such that $\mathbb{P}_1 \cap \mathbb{P}_2 = \emptyset$. By Grassmann formula, one has then $\mathbb{P}_1 \vee \mathbb{P}_2 = \mathbb{P}^n$. For every point $P \in \mathbb{P}^n \setminus \mathbb{P}_1$, the subspace $\mathbb{P}_1 \vee P$ has dimension $n - r + 1$ which, again by Grassmann formula, intersects $\mathbb{P}_2$ in a point P'. Consider the map

$$\tau : P \in \mathbb{P}^n \setminus \mathbb{P}_1 \to P' = (\mathbb{P}_1 \vee P) \cap \mathbb{P}_2 \in \mathbb{P}_2$$

called *projection* of $\mathbb{P}^n$ onto $\mathbb{P}_2$ with centre $\mathbb{P}_1$. Prove that this is a degenerate projectivity which is surjective onto $\mathbb{P}_2$. Prove that given two points $P, Q \in \mathbb{P}^n \setminus \mathbb{P}_1$, then $\tau(P) = \tau(Q) = P'$ if and only if $P \vee \mathbb{P}_1 = Q \vee \mathbb{P}_1 = P' \vee \mathbb{P}_1$.

Exercise 7.1.13 *Prove that if $\tau_{\mathbf{A}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is a degenerate projectivity with $\mathbf{A}$ of rank r, then there are two projectivities $\alpha : \mathbb{P}^n \to \mathbb{P}^n$ and $\beta : \mathbb{P}^m \to \mathbb{P}^m$, and a projection $\sigma : \mathbb{P}^n \dashrightarrow \mathbb{P}_2 \cong \mathbb{P}^{r-1} \subseteq \mathbb{P}^m$ such that $\tau = \beta \circ \sigma \circ \alpha$.

Exercise 7.1.14 *Consider $\mathbb{A}^n$ as embedded in $\mathbb{P}^n$ and consider the projection τ of $\mathbb{P}^n$ onto $\mathbb{P}_2$, of dimension $r - 1 \geqslant 0$, from a centre $\mathbb{P}_1$ of dimension $n - r$. Suppose that $\mathbb{P}_2$ is not contained in the hyperplane at infinity of $\mathbb{A}^n$, so that $\mathbb{A}_2 := \mathbb{P}_2 \setminus (\mathbb{P}_2)_\infty$ is an affine subspace of dimension $r - 1$ of $\mathbb{A}^n$. The restriction of τ to $\mathbb{A}^n \setminus (\mathbb{A}^n \cap \mathbb{P}_1)$ is called *projection of* $\mathbb{A}^n$ on $\mathbb{A}_2$ *with centre* $\mathbb{A}_1 = \mathbb{A}^n \cap \mathbb{P}_1$ if $\mathbb{A}_1 \neq \emptyset$, or *projection of* $\mathbb{A}^n$ on $\mathbb{A}_2$ *parallel to the direction of* $\mathbb{P}_1$, if $\mathbb{A}_1 = \emptyset$ (see Exercise 1.6.20 for a special case of this situation). Prove that the latter projection is a surjective morphisms onto $\mathbb{A}_2$.

Exercise 7.1.15 *Prove that any affine map between affine spaces is the composite of parallel projections and affinities.

Exercise 7.1.16 *Consider a projection of $\mathbb{A}^n$ on $\mathbb{A}_2$ with centre $\mathbb{A}_1 = \mathbb{A}^n \cap \mathbb{P}_1$ with $\mathbb{A}_1 \neq \emptyset$. Prove that this is a rational map which is not a morphism, unless $r = 1$, in which case it is constant.

Exercise 7.1.17 Consider the rational function $f = \frac{x_1}{x_0}$ on $\mathbb{P}^2$. What is its definition set? We can consider $\frac{x_1}{x_0}$ also as a rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, by writing it as $f[x_0, x_1, x_2] = [x_0, x_1]$. As a rational map of $\mathbb{P}^2$ to $\mathbb{P}^1$, what is its definition set?

Exercise 7.1.18 Consider V an affine or projective variety. The restriction to V of a projection on a given subspace P_2 from a certain centre P_1 not containing V is a rational map of V in P_2, still called *projection* of V to P_2 with centre P_1. Prove with an example that the definition set of such a projection can be bigger than $V \setminus P_1$.

Exercise 7.1.19 Consider the following rational map

$$\phi : [x_0, x_1, x_2] \in \mathbb{P}^2 \dashrightarrow [x_1x_2, x_0x_2, x_0x_1] = \left[\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}\right] \in \mathbb{P}^2.$$

Prove that this is a involutory Cremona transformation, i.e., $\phi = \phi^{-1}$. It is called a *standard quadratic transformation* of $\mathbb{P}^2$, together with all its composition with a projectivity of $\mathbb{P}^2$.

Exercise 7.1.20 Continue Exercise 7.1.19. Set $A = [1, 0, 0]$, $B = [0, 1, 0]$, $C = [0, 0, 1]$ and $a = B \vee C$, $b = A \vee C$, $c = A \vee B$. Set $U = \mathbb{P}^2 \setminus \{a, b, c\}$. Prove that ϕ is an isomorphism of U to itself, and $\phi^2 = \mathrm{id}_U$.

Exercise 7.1.21 Continue Exercise 7.1.20. Prove that $\phi(a \setminus \{B, C\}) = A$, $\phi(b \setminus \{A, C\}) = B$, $\phi(c \setminus \{A, B\}) = C$. This is expressed by saying that ϕ *contracts* the lines a, b, c to the points A, B, C respectively, and the points A, B, C are *blown-up* to the lines a, b, c respectively. Set $U' = \mathbb{P}^2 \setminus \{A, B, C\}$. Prove that U' is the set of definition of ϕ.

Exercise 7.1.22 Continue Exercise 7.1.21. Consider a line r containing the point A different from b and c, with equation $\lambda x_1 + \mu x_2 = 0$. Prove that $\overline{\phi(r \setminus \{A\})}$ is the line with equation $\mu x_1 + \lambda x_2 = 0$, which cuts the line a in the point $R = [0, -\lambda, \mu]$. Prove that the map which sends the line r different from b and c through A to the point R in a, and the lines b, c to the points B, C respectively, is a projectivity from the pencil (A) of lines through A (considered as a line in $\check{\mathbb{P}}^2$), and the line a.

Similar discussion can be made for the point B [resp. C] in relation with the line b [resp. c].

Exercise 7.1.23 Continue Exercise 7.1.22. Consider a line r which does not pass through any of the points A, B, C, so that it has equation of the form $\alpha x_0 + \beta x_1 + \gamma x_2 = 0$, with α, β, γ not zero. Prove that the image of r is the conic Γ with equation $\alpha x_1x_2 + \beta x_0x_2 + \gamma x_0x_1 = 0$ passing through the points A, B, C. Prove that the tangent lines to the conic Γ at A, B, C correspond in the projectivities considered in Exercise 7.1.22 to the intersection points of r with the lines a, b, c.

Exercise 7.1.24 Continue Exercise 7.1.23. Prove that any irreducible conic Γ passing through the points A, B, C, hence with equation $\alpha x_1 x_2 + \beta x_0 x_2 + \gamma x_0 x_1 = 0$ is such that $\overline{\phi(\Gamma \setminus \{A, B, C\})}$ is the line r with equation $\alpha x_0 + \beta x_1 + \gamma x_2 = 0$.

Exercise 7.1.25 Prove that $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1)$.

Exercise 7.1.26 *Prove that $\mathrm{Aut}(\mathbb{P}^1)$ coincides with the group of projectivities of $\mathbb{P}^1$ to itself.

Exercise 7.1.27 *Prove that any automorphism of the field $\mathbb{K}(x)$ as a $\mathbb{K}$-algebra is obtained by extending by $\mathbb{K}$-linearity the map which sends x to an element of the type $\frac{a+bx}{c+dx}$, with $ad - cb \neq 0$.

Exercise 7.1.28 Prove that if $\phi : \mathbb{P}^1 \setminus \{P_1, \ldots, P_r\} \to \mathbb{P}^1 \setminus \{Q_1, \ldots, Q_s\}$ is an isomorphism, with $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_s$ distinct, then $s = r$.

Exercise 7.1.29 Prove that there is always an isomorphism $\phi : \mathbb{P}^1 \setminus \{P_1, \ldots, P_r\} \to \mathbb{P}^1 \setminus \{Q_1, \ldots, Q_r\}$, with $P_1, \ldots, P_r$ and $Q_1, \ldots, Q_r$ distinct, if $r \leqslant 3$. Prove that this is not always the case if $r \geqslant 4$.

Exercise 7.1.30 Let $\mathcal{L} \subseteq \mathcal{L}_{n,d}$ be a linear system of dimension m with base locus not containing any divisor. Consider the map

$$\phi_{\mathcal{L}} : \mathbb{P}^n \setminus \mathrm{Bs}(\mathcal{L}) \to \check{\mathcal{L}} \cong \mathbb{P}^m$$

that sends a point $P \in \mathbb{P}^n \setminus \mathrm{Bs}(\mathcal{L})$ to the hyperplane $\mathcal{L}(-P)$ of $\mathcal{L}$ (notation as in Sect. 1.6.5). Prove that $\phi_{\mathcal{L}}$ determines a rational map $\phi_{\mathcal{L}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ such that the image of $\phi_{\mathcal{L}}$ is non-degenerate. Prove that for any rational map $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ such that the image of ϕ is non-degenerate there is a positive integer d and a linear system $\mathcal{L} \subseteq \mathcal{L}_{n,d}$ of dimension m such that $\phi = \phi_{\mathcal{L}}$.

7.2 Birational Models of Quasi-projective Varieties

In this section we will prove a basic theorem which says any any quasi-projective variety is birationally equivalent to a hypersurface in affine or projective space. Before that we need the following results of algebra:

Lemma 7.2.1 *Let k be an infinite field and let $k(\alpha_1, \alpha_2)$ be an algebraic extension with α_2 separable on k. Then there exists an $\alpha \in k(\alpha_1, \alpha_2)$ such that $k(\alpha_1, \alpha_2) = k(\alpha)$.*

Proof Let $f_1(x), f_2(x)$ be the minimal polynomials, of degrees n, m of α_1, α_2 respectively, and let $\alpha_{11} = \alpha_1, \ldots, \alpha_{1n}, \alpha_{21} = \alpha_2, \ldots, \alpha_{2m}$, be the roots of the two polynomials in the algebraic closure of k. If $m = 1$ there is nothing to prove. So we assume $m > 1$. Since α_2 is separable, $\alpha_{21} = \alpha_2, \ldots, \alpha_{2m}$ are all distinct, so it makes sense to consider the elements

$$\frac{\alpha_{1i} - \alpha_{11}}{\alpha_{21} - \alpha_{2j}}, \quad \text{for} \quad i = 1, \ldots, n,\ j = 2, \ldots, m,$$

and we can choose $a \in k$ distinct from all these elements. So one has

$$\alpha_{1i} + a\alpha_{2j} \neq \alpha_{11} + a\alpha_{21} = \alpha_1 + a\alpha_2, \quad \text{for} \quad i = 1, \ldots, n,\ j = 2, \ldots, m.$$

Set $\alpha = \alpha_1 + a\alpha_2$ and let us prove that $k(\alpha_1, \alpha_2) = k(\alpha)$. It is clear that $k(\alpha) \subseteq k(\alpha_1, \alpha_2)$ so we need to prove that $k(\alpha_1, \alpha_2) \subseteq k(\alpha)$. And it suffices to prove that $\alpha_2 \in k(\alpha)$, because $\alpha_1 = \alpha - a\alpha_2$.

Note that $f_2(\alpha_2) = 0$ and $f_1(\alpha_1) = f_1(\alpha - a\alpha_2) = 0$, so that the two polynomials $f_2(x)$ and $f_1(\alpha - ax)$ have a common root, hence they have a greatest common divisor $f(x)$ of positive degree in $k(\alpha)[x]$, which we can suppose to be monic. Now $f(x)$ has the root α_2 and it does not have multiple roots, because α_2 is separable. Let us prove that $f(x)$ has the only root α_2 which will prove the assertion. If $f(x)$ had another root, this would be one among $\alpha_{22}, \dots, \alpha_{2m}$, but none of these is also a root of $f_1(\alpha - ax)$, because $\alpha - a\alpha_{2j} \neq \alpha_{1i}$, for $i = 1, \dots, n$, $j = 2, \dots, m$. □

As an immediate consequence we have the:

Theorem 7.2.2 (Abel's Theorem of the Primitive Element) *If k is an infinite field and $k \subseteq k'$ is a separable, finite extension, then it is* simple*, i.e., there is an element $\alpha \in k'$ such that $k' = k(\alpha)$.*

Now we can prove the announced result:

Theorem 7.2.3 *Every quasi-projective variety of transcendent dimension n is birationally equivalent to an irreducible hypersurface in $\mathbb{A}^{n+1}$ or in $\mathbb{P}^{n+1}$.*

Proof By taking into account Exercise 7.1.8 it suffices to prove the assertion for affine varieties, proving that such a variety is birationally equivalent to an affine hypersurface.

So, let $V \subseteq \mathbb{A}^r$ be an affine variety with $\dim_{\text{tr}}(V) = n$ and consider $K(V)$ which is finitely generated on $\mathbb{K}$, with system of generators $x_1, \dots, x_r$, from which we can extract a maximal system of algebraically independent elements over $\mathbb{K}$, which we may suppose to be $x_1, \dots, x_n$. We may also assume $r > n$, otherwise the assertion is trivially true. Then every element $y \in K(V)$ algebraically depends on $\mathbb{K}$ from $x_1, \dots, x_n$, i.e., there is a non-zero, irreducible polynomial $f(t_1, \dots, t_{n+1}) \in \mathbb{K}[t_1, \dots, t_{n+1}]$ such that $f(x_1, \dots, x_n, y) = 0$. Let $y = x_{n+1}$. Let us prove that for the corresponding polynomial f, there is an $i = 1, \dots, n+1$, such that $\frac{\partial f}{\partial t_i} \neq 0$. Indeed, if this were not the case, then the polynomial f would be of the form

$$f(t_1, \dots, t_{n+1}) = \sum_{i_1 \dots i_{n+1}} a_{i_1 \dots i_{n+1}} t_1^{p i_1} \dots t_{n+1}^{p i_{n+1}}$$

with $p = \text{char}(\mathbb{K})$. Then, if we set $a_{i_1 \dots i_{n+1}} = b^p_{i_1 \dots i_{n+1}}$ and

$$g(t_1, \dots, t_{n+1}) = \sum_{i_1 \dots i_{n+1}} b_{i_1 \dots i_{n+1}} t_1^{i_1} \dots t_{n+1}^{i_{n+1}}$$

we would have $f = g^p$ against the irreducibility of f.

Suppose now that $\frac{\partial f}{\partial t_i} \neq 0$. Then $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}$ are algebraically independent on $\mathbb{K}$. Note in fact that x_i algebraically depends on $x_1, \dots, x_{i-1}, x_{i+1},$

$\dots, x_{n+1}$ because t_i actually appears in f. Then if $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}$ were not algebraically independent on $\mathbb{K}$, then the transcendence degree of $K(V)$ on $\mathbb{K}$ would be smaller than n, a contradiction. By changing name to the variables, we may assume that $\frac{\partial f}{\partial t_{n+1}} \neq 0$, which shows that x_{n+1} is separable over $\mathbb{K}(x_1, \dots, x_n)$. Since x_{n+2} is algebraic over this field, Theorem 7.2.2 implies that there is a $y \in \mathbb{K}(x_1, \dots, x_{n+2}) \subseteq K(V)$ such that $\mathbb{K}(x_1, \dots, x_{n+2}) = \mathbb{K}(x_1, \dots, x_n, y)$. Iterating this argument we see that finally $K(V) = \mathbb{K}(z_1, \dots, z_{n+1})$, where $z_1, \dots, z_n$ are algebraically independent over $\mathbb{K}$ and there is an irreducible polynomial $f \in \mathbb{K}[t_1, \dots, t_{n+1}]$, with $\frac{\partial f}{\partial t_{n+1}} \neq 0$, such that $f(z_1, \dots, z_{n+1}) = 0$. If $W = Z_a(f) \subset \mathbb{A}^{n+1}$, it is now clear that $K(W) = \mathbb{K}(z_1, \dots, z_{n+1}) = K(V)$, so that, by Corollary 7.1.4, V is birational to W, as wanted. □

Exercise 7.2.4 *Let $V \subseteq \mathbb{A}^r$ be an affine variety with $\dim_{\text{tr}}(V) = n$. Prove that there is a projection τ of $\mathbb{A}^r$ onto $\mathbb{A}^{n+1}$ such that the restriction of τ to V is a birational morphism of V onto a hypersurface of $\mathbb{A}^{n+1}$.

7.3 Unirational and Rational Varieties

A variety V of transcendent dimension n is said to be *unirational* if there is a dominant rational map $\phi : \mathbb{P}^n \dashrightarrow V$. This is equivalent to say that there is an algebraic extension $K(V) \subseteq \mathbb{K}(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are transcendent over $\mathbb{K}$. The variety V of transcendent dimension n is said to be *rational* if it is birationally equivalent to $\mathbb{P}^n$, i.e., if $K(V) \cong \mathbb{K}(x_1, \dots, x_n)$ with $x_1, \dots, x_n$ transcendent over $\mathbb{K}$. Every rational variety is unirational, but the converse is not always true. A classical problem in algebraic geometry is the one of understanding if and when unirationality implies rationality. This problem is called *Lüroth's problem*, in honor of J. Lüroth who in 1880 proved the following important theorem which we will prove in a while:

Theorem 7.3.1 (Lüroth Theorem) *Any unirational curve is also rational.*

There is a similar, much more difficult, result by Castelnuovo (1894) for surfaces in characteristic zero: any unirational surface is rational. Only more recently it has been proved that in general unirationality does not imply rationality for varieties of transcendent dimension $n \geqslant 3$.

Let us focus on the proof of Lüroth Theorem 7.3.1. First we need the following algebraic lemma:

Lemma 7.3.2 *Let x be transcendent over a field k. Let $\theta = \frac{g(x)}{h(x)} \in k(x) \setminus k$, with* $\text{GCD}(g, h) = 1$. *Then x is algebraic over $k(\theta)$ of degree d* $= \max\{\deg(g), \deg(h)\}$.

Proof Let z be transcendent over $k(x)$. Consider the polynomial

$$F(z) = h(z) - \theta g(z) \in k[\theta, z]$$

which has degree d in z. One has $F(x) = 0$. Let us prove that $F(z)$ is irreducible over $k(\theta)$, which will imply the assertion. Note that θ is transcendent over k. Moreover

$c_F = 1$, with F as polynomial in z (see Sect. 2.1 for the definition of c_F). Hence by Gauss Lemma 2.2.5, we have that $F(z)$ is irreducible over $k(\theta)$ if and only if it is irreducible over $k[\theta, z]$. Suppose $F(z)$ is reducible in $k[\theta, z]$. Since $F(z)$ has degree 1 in θ, we have $F(z) = p(\theta, z) \cdot q(z)$, hence $q(z)$ divides both h and g hence it is a constant. □

We can now prove Lüroth Theorem in the following algebraic formulation:

Theorem 7.3.3 (Lüroth Theorem, algebraic formulation) *Let x be transcendent over a field k. Let L be a field transcendent over k and such that $k \subset L \subseteq k(x)$. Then there is an element $\theta \in L$, transcendent over k, such that $L = k(\theta)$.*

Proof The assertion is trivial if $L = k(x)$. So we assume $L \neq k(x)$. First of all we remark that $k(x)$ is algebraic over L. Indeed, if $\beta \in L \setminus k$, then $\beta \in k(x) \setminus k$ hence $k(x)$ is algebraic over $k(\beta) \subseteq L$ (see Lemma 7.3.2). Let now z be an indeterminate over L and consider the minimal polynomial of x over L

$$f(z) = a_0 + a_1 z + \ldots + z^n \in L[z].$$

We have $a_i = \frac{g_i(x)}{h_i(x)}$, with $g_i, h_i \in k[x]$ such that $h_i \neq 0$ (and also $g_i \neq 0$ if $a_i \neq 0$), and g_i, h_i coprime, with $i = 0, \ldots, n-1$. Since x is transcendent over k, at least one a_i is not in k, for $i = 0, \ldots, n-1$. Let $\theta = \frac{g(x)}{h(x)}$ such a coefficient of $f(z)$. We will show that $L = k(\theta)$.

Let us multiply $f(z)$ for the least common multiple of $h_0, \ldots, h_{n-1}$. Then we get a polynomial $f(x, z) \in k[x, z]$, and one has $c_f = 1$, with f as a polynomial in $(k[x])[z]$. Consider the polynomial

$$p(z) = g(z) - \theta h(z) \in k[\theta, z] \subseteq k(x)[z].$$

One has $p(x) = 0$, hence $p(z)$ is divided by $f(z)$ in $k(\theta)[z]$, hence also in $k(x)[z]$. Then the polynomial

$$\ell(x, z) = h(x)g(z) - g(x)h(z) \in k[x, z]$$

is divided by $f(x, z)$ in $k(x)[z]$. Since $c_f = 1$, by Gauss Lemma 2.2.5 the polynomial $f(x, z)$ divides $\ell(x, z)$ even in $k[x, z]$, i.e., we have $\ell(x, z) = f(x, z)q(x, z)$ with $q(x, z) \in k[x, z]$. Now note that $\ell(x, z) = -\ell(z, x)$, hence ℓ has the same degree in x and in z. This is also the degree of $p(z)$, and it is $d = \max\{\deg(g), \deg(h)\}$.

The degree of $f(x, z)$ in x is at least d, because g and h are factors of coefficients of $f(x, z)$ of different powers of z. Since f divides ℓ, we have that $q \in k[z]$. On the other hand, since $\mathrm{GCD}(g, h) = 1$, we have $q \in k$, thus $\ell = f$ up to a constant factor. Therefore $\deg(\ell) = \deg(f)$ both with respect to x and z. But $[k(x) : L] = \deg(f)$, $[k(x) : k(\theta)] \leqslant \deg(p) = \deg(\ell) = \deg(f)$. In conclusion we have $[L : k(\theta)] = 1$, as wanted. □

Remark 7.3.4 We notice that in the proof of Lüroth Theorem we do not need the base field k to be algebraically closed.

Exercise 7.3.5 *Consider in $\mathbb{A}^n$, with $n \geqslant 2$, an irreducible hypersurface Z of degree $d \geqslant 2$ with reduced equation of the form

$$f_{d-1}(x_1, \ldots, x_n) - f_d(x_1, \ldots, x_n) = 0$$

where f_{d-1}, f_d are non-zero homogeneous polynomials of degrees $d-1, d$ respectively. Note that, being Z irreducible, f_{d-1}, f_d are coprime. Then we say that Z is a *monoid* with *vertex* $\mathbf{0}$. Any hypersurface which is transformed of Z by an affinity is still called a monoid. Also the projective closure of a monoid is called a (projective) monoid. Prove that every monoid is rational.

Exercise 7.3.6 Prove that the projection of a projective monoid Z in $\mathbb{P}^n$ from its vertex to a hyperplane is a morphism (i.e., it is everywhere defined) if Z is an irreducible conic. The projection of a projective monoid from its vertex to a hyperplane is called *stereographic projection* of the monoid.

Exercise 7.3.7 Continue Exercise 7.3.5. Prove that if $n \geqslant 3$ then the stereographic projection of a projective monoid Z from its vertex is not defined at the vertex.

Exercise 7.3.8 Consider a projective monoid Z with vertex P. Consider two distinct points $P_1, P_2 \in Z \setminus \{P\}$. Let ϕ be the stereographic projection of Z from P. Then $\phi(P_1) = \phi(P_2)$ if and only if the line $r = P_1 \vee P_2$ contains P and is contained in Z, and for all points $Q \in r \setminus \{P\}$, one has $\phi(Q) = \phi(P_1) = \phi(P_2)$.

Exercise 7.3.9 Continue Exercise 7.3.8. Suppose the projective monoid Z has equation

$$x_0 f_{d-1}(x_1, \ldots, x_n) - f_d(x_1, \ldots, x_n) = 0.$$

and vertex $P = [1, 0, \ldots, 0]$. Prove that if $n \geqslant 3$, the union of lines in Z through P is non-empty and it has equation $f_d = f_{d-1} = 0$. Prove that if $n \geqslant 3$ the inverse of the stereographic projection from P is not a morphism.

Exercise 7.3.10 Prove that any irreducible quadric in $\mathbb{P}^n$ is a monoid, and it is therefore rational.

7.4 Solutions of Some Exercises

7.1.6 Suppose $[U, \phi] \in K(V, W)$ is such that $\phi : U \to W$ is dominant. Assume $[U, \phi] = [U', \phi']$. Then

$$\overline{\phi'(U')} \supseteq \overline{\phi'(U' \cap U)} = \overline{\phi(U' \cap U)} \supseteq \phi(\overline{(U \cap U')} \cap U) = \phi(U).$$

The assertion follows.

7.1.7 We can assume that $V \subseteq \mathbb{A}^n$ is affine. Let $f \in K(V)$. Then $f \in \mathbb{Q}(A(V))$ (see Theorem 5.5.3), so $f = \frac{g}{h}$ with g, h the classes in $A(V)$ of polynomials in A_n, with $h \notin \mathcal{I}_a(V)$. Let $U = V \setminus Z_V(h)$ which is a non-empty open subset of V. Then f determines the morphism $P \in U \to [h(P), g(P)] \in \mathbb{P}^1$, which in turn determines a rational function, still denoted by f, of V in $\mathbb{P}^1$.

7.1.9 This follows from Corollary 7.1.4. In fact, ϕ corresponds to an injective homomorphism $\phi^* : K(W) \to K(V)$, α and β to isomorphisms $\alpha^* : K(V') \to K(V)$ and $\beta^* : K(W') \to K(W)$, so ϕ' is uniquely determined since it corresponds to the injective homomorphism $\alpha^* \circ \phi^* \circ (\beta^*)^{-1} : K(W') \to K(V')$.

7.1.10 There is a non-empty open subset U of $\mathbb{P}^1$ in which ϕ is of the form $\phi([x_0, x_1]) = [f_0(x_0, x_1), \ldots, f_n(x_0, x_1)]$, with $f_0, \ldots, f_n \in S_2$ homogeneous polynomials of the same degree which we can assume to be coprime and for all $[x_0, x_1] \in U$, one has $(f_0(x_0, x_1), \ldots, f_n(x_0, x_1)) \neq$

0. Let now $P = [y_0, y_1]$ be any point of $\mathbb{P}^1$. Then there is an $i = 0, \ldots, n$ such that $f_i(y_0, y_1) \neq 0$, otherwise for all $i = 0, \ldots, n$, we have that f_i is divisible by $x_0 y_1 - x_1 y_0$, contrary to the assumption that $f_0, \ldots, f_n$ are coprime. This implies that ϕ is everywhere defined.

7.1.11 The dominance onto a subspace of $\mathbb{P}^m$ of dimension $r-1$ is clear. The cases $r = 1, n+1$ are trivial. Assume $1 < r < n+1$. If $\tau_{\mathbf{A}}$ is defined in $P \in \mathbb{P}^n$, then there exists an open neighborhood U of P in $\mathbb{P}^n$, and there are homogeneous polynomials of the same degree $f_0, \ldots, f_m \in S_n$, not all zero, such that

$$\tau_{\mathbf{A}}(Q) = [f_0(Q), \ldots, f_m(Q)], \quad \text{for all} \quad Q \in U.$$

If $\mathbf{a}^0, \ldots, \mathbf{a}^m$ are the columns of $\mathbf{A}$, we set $g_i(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a}^i$, for $i = 0, \ldots, m$. Then in $U \cap U_{\mathbf{A}}$ we have also

$$\tau_{\mathbf{A}}(Q) = [g_0(Q), \ldots, g_m(Q)],$$

hence in $U \cap U_{\mathbf{A}}$ we have

$$\operatorname{rank} \begin{pmatrix} f_0 & f_1 & \cdots & f_m \\ g_0 & g_1 & \cdots & g_m \end{pmatrix} = 1. \tag{7.2}$$

Since $U \cap U_{\mathbf{A}}$ is a non-empty open subset of $\mathbb{P}^n$, (7.2) holds everywhere in $\mathbb{P}^n$. Fix now an $i = 0, \ldots, m$. Since $r \geqslant 2$, there is a $j = 0, \ldots, m$ such that $\lambda g_i \neq g_j$ for all $\lambda \in \mathbb{K}$. Since $f_i g_j = f_j g_i$, if $g_i = 0$ then also $f_i = 0$. If $g_i \neq 0$ then f_i is divisible by g_i. In any event, for all $i = 0, \ldots, m$ if $g_i(Q) = 0$ then also $f_i(Q) = 0$. This proves the assertion.

7.1.12 We can change coordinates and assume that $\mathbb{P}_1$ has equations $x_0 = \cdots = x_{r-1} = 0$ and $\mathbb{P}_2$ has equations $x_r = \cdots = x_n = 0$, so that $\mathbb{P}_2$ can be identified with $\mathbb{P}^{r-1}$ with coordinates $[x_0, \ldots, x_{r-1}]$. Then $\tau[x_0, \ldots, x_n] = [x_0, \ldots, x_{r-1}]$ and the assertions follow.

7.1.13 Given $\tau : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, one can change coordinates so that it has equations of the form $\tau[x_0, \ldots, x_n] = [x_0, \ldots, x_{r-1}]$.

7.1.14 We can change coordinates in $\mathbb{P}^n$ and assume that $\mathbb{P}_1$ has equations $x_0 = \cdots = x_{r-1} = 0$, $\mathbb{P}_2$ has equations $x_r = \cdots = x_n = 0$, and $\mathbb{A}^n$ is the open set U_0. Then $\mathbb{A}_2$ is still defined by the equations $x_r = \cdots = x_n = 0$ and the parallel projection is defined as

$$(x_1, \ldots, x_n) \in \mathbb{A}^n \to (x_1, \ldots, x_{r-1}, 0, \ldots, 0) \in \mathbb{A}_2.$$

7.1.15 The affine map is of the form

$$\tau : \mathbf{x} \in \mathbb{A}^n \to \mathbf{a} + \mathbf{x} \cdot \mathbf{A} \in \mathbb{A}^m$$

with $\mathbf{A}$ a matrix of type $n \times m$ and rank ℓ. It suffices to remark that we can change coordinates in $\mathbb{A}^n$ and $\mathbb{A}^m$ and assume that $\mathbf{A}$ is of the form

$$\begin{pmatrix} I_\ell & 0_{\ell, m-\ell} \\ 0_{n-\ell, n} & 0_{n-\ell, m-\ell} \end{pmatrix}$$

with obvious meaning of the symbols.

7.1.16 We can find explicit equations for this projection. First of all change coordinates in $\mathbb{A}^n$ so that $\mathbb{A}_2$ is the subspace with equations $x_r = \ldots = x_n = 0$. Note that there is a unique hyperplane H_r containing $\mathbb{A}_1$ and *parallel* to $\mathbb{A}_2$, i.e., such that its projective closure intersects $\bar{\mathbb{A}}_2 = \mathbb{P}_2$ in $(\mathbb{A}_2)_\infty$. Suppose that this hyperplane has equation $f_r(x_1, \ldots, x_n) = 0$. Note that this hyperplane does not pass through the origin, hence in f_r there is a constant term, which we can suppose to be 1, in this way f_r is uniquely determined. Note also that the system $f_r = x_r = \ldots = x_n = 0$ is incompatible, so in f_r the variables $x_1, \ldots, x_{r-1}$ do not appear.

Consider now, for any $i = 1, \ldots, r-1$, the unique hyperplane H_i containing $\mathbb{A}_1$ and the subspace with equations $x_i = x_r = \ldots = x_n = 0$, and let $f_i(x_1, \ldots, x_n) = 0$ be an equation of H_i. The hyperplane H_i passes through the origin, so f_i is homogeneous of degree 1. Moreover the variable x_i appears in f_i, whereas the variables x_j with $j < r$ and $j \neq i$ do not appear in f_i.

Indeed, $f_i(x_1, \dots, x_{r-1}, 0, \dots, 0) = 0$ has the same solutions as $x_i = x_r = \dots = x_n = 0$. Hence we may assume that $f_i = x_i - g_i(x_r, \dots, x_n)$, with g_i homogeneous, for $i = 1, \dots, r-1$, and in this way also f_i is uniquely determined. Now $H_1, \dots, H_r$ are linearly independent hyperplanes which intersect in $\mathbb{A}_1$. Moreover, if $P = (p_1, \dots, p_n) \notin H_r$, then $f_r(p_1, \dots, p_n) \neq 0$ and we can consider the system of linear equations

$$x_i - g_i(x_r, \dots, x_n) - \frac{p_i - g_i(p_r, \dots, p_n)}{f_r(p_1, \dots, p_n)} f_r(x_1, \dots, x_n) = 0, \quad i = 1, \dots, r-1.$$

They are linearly independent, vanish on P and on $\mathbb{A}_1$ and define the affine subspace which joins P with $\mathbb{A}_1$. Its intersection with $\mathbb{A}_2$ is the image of P, which therefore has coordinates

$$\frac{f_i(p_1, \dots, p_n)}{f_r(p_1, \dots, p_n)} \quad \text{for} \quad i = 1, \dots, r-1, \quad \text{and} \quad 0 \quad \text{for} \quad i = r, \dots, n.$$

Hence the projection is defined in the following way

$$(x_1, \dots, x_n) \in \mathbb{A}^n \setminus H_r \to \left(\frac{f_1(x_1, \dots, x_n)}{f_r(x_1, \dots, x_n)}, \dots, \frac{f_{r-1}(x_1, \dots, x_n)}{f_r(x_1, \dots, x_n)}, 0, \dots, 0 \right) \in \mathbb{A}_2$$

and it is not defined on H_r.

7.1.17 As a rational map on $\mathbb{P}^2$, the definition set of f is $\mathbb{P}^2 \setminus Z_p(x_0)$. As a rational map of $\mathbb{P}^2$ to $\mathbb{P}^1$ this can be interpreted as the projection from $P = [0, 0, 1]$, hence the definition set is $\mathbb{P}^2 \setminus \{P\}$.

7.1.18 Consider the projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ of $\mathbb{P}^n$ to an hyperplane from a point P. Consider a rational normal curve V containing P. The restriction of the projection to V is defined in P. Indeed, V is isomorphic to $\mathbb{P}^1$ and then apply Exercise 7.1.10.

7.1.25 This immediately follows by Exercise 7.1.10.

7.1.26 Suppose $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ is an automorphism. Then there are two non-constant, coprime, homogeneous polynomials $f_0(x_0, x_1), f_1(x_0, x_1)$ of the same degree $n > 0$ such that $\phi([x_0, x_1]) = [f_0(x_0, x_1), f_1(x_0, x_1)]$. There are also two non-constant, coprime, homogeneous polynomials $g_0(x_0, x_1), g_1(x_0, x_1)$ of the same degree $m > 0$ such that $\phi^{-1}([x_0, x_1]) = [g_0(x_0, x_1), g_1(x_0, x_1)]$. Consider the two homogeneous polynomials

$$P_i(x_0, x_1) = f_i(g_0(x_0, x_1), g_1(x_0, x_1)), \quad i = 0, 1$$

of the same degree nm. Since $\phi^{-1} \circ \phi = \mathrm{id}_{\mathbb{P}^1}$, we have

$$\det \begin{pmatrix} P_0 & P_1 \\ x_0 & x_1 \end{pmatrix} = 0$$

which implies that there is a homogeneous polynomial $P(x_0, x_1)$ of degree $nm - 1$ such that $P_i = x_i P$, for $i = 0, 1$. We claim that P is constant. Suppose in fact P is of positive degree and let (a_0, a_1) be a non-trivial solution of the equation $P(x_0, x_1) = 0$. Then

$$P_i(a_0, a_1) = f_i(g_0(a_0, a_1), g_1(a_0, a_1)), \quad i = 0, 1.$$

However $(g_0(a_0, a_1), g_1(a_0, a_1)) \neq (0, 0)$ because g_0, g_1 have no common factor. Moreover it cannot be the case that $f_i(g_0(a_0, a_1), g_1(a_0, a_1)) = 0$, for $i = 0, 1$, because also f_0, f_1 have no common factor. This proves that P is constant and therefore $nm = 1$, hence $n = m = 1$, which implies that ϕ is a projectivity.

7.1.27 By Exercise 7.1.27 any birational transformation of $\mathbb{P}^1$ is a projectivity, which is of the type

$$\phi : [x_0, x_1] \in \mathbb{P}^1 \to [cx_0 + dx_1, ax_0 + b_1x_1] \in \mathbb{P}^1$$

with

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

By interpreting, as usual, $\mathbb{A}^1$ as the open subset U_0 of $\mathbb{P}^1$, the map ϕ corresponds to the birational transformation

$$\phi : x \in \mathbb{A}^1 \dashrightarrow \frac{a+bx}{c+dx} \in \mathbb{A}^1$$

and this in turn corresponds to the automorphism of $\mathbb{K}(x) \cong K(\mathbb{A}^1)$ which is $\mathbb{K}$-linear and maps x to $\frac{a+bx}{c+dx}$. Conversely, any automorphism of $\mathbb{K}(x)$ as a $\mathbb{K}$-algebra, corresponds to a birational transformation of $\mathbb{A}^1$, that determines a birational transformation of $\mathbb{P}^1$, which is a projectivity as above.

7.2.4 Keep the notation of the proof of Theorem 7.2.3. We have that $K(V) = \mathbb{K}(x_1, \dots, x_r)$ is isomorphic to $K(W) = \mathbb{K}(z_1, \dots, z_{n+1})$. By the proof of the Theorem of the Primitive Element, we have relations of the type $z_i = \sum_{j=1}^r c_{ij} x_j$, for $i = 1, \dots, n+1$. Hence the isomorphism $K(W) \to K(V)$ is induced by the $\mathbb{K}$-algebras homomorphism

$$\tau^* : \mathbb{K}[z_1, \dots, z_{n+1}] \to \mathbb{K}[x_1, \dots, x_r]$$

which associates to z_i the polynomial $\sum_{j=1}^r c_{ij} x_j$, for $i = 1, \dots, n+1$. This in turn corresponds to an affine map $\mathbb{A}^r \to \mathbb{A}^{n+1}$ that, up to a change of coordinates, is a projection.

7.3.5 Consider $\bar{Z}$ the projective closure of Z. The projection of $\bar{Z} \subset \mathbb{P}^n$ to the hyperplane $H_0 = Z_p(x_0)$ from the point $P = [1, 0, \dots, 0]$ is birational. To see this suffices to prove that there is a dominant rational map $\psi : H_0 \dashrightarrow \bar{Z}$ which is the inverse of the projection

$$\phi : [x_0, \dots, x_n] \in \bar{Z} \setminus \{P\} \to [x_1, \dots, x_n] \in H_0.$$

Consider the closed subsets $Z_1 = Z_p(f_{d-1})$, $Z_2 = Z_p(f_d)$ of $H_0 = \mathbb{P}^{n-1}$, and the non-empty open subset $U = H_0 \setminus (Z_1 \cap Z_2)$. Define ψ as follows:

$$\psi : [x_1, \dots, x_n] \in U \to [f_d(\mathbf{x}), x_1 f_{d-1}(\mathbf{x}), \dots, x_n f_{d-1}(\mathbf{x})] \in \bar{Z}.$$

It is easy to verify that ϕ and ψ are inverse of each other.

7.3.6 If $\bar{Z}$ is an irreducible conic, then, up to projectivity, $\bar{Z}$ has equation of the form $x_0 x_1 - x_2^2 = 0$ which is a monoid with vertex $P = [1, 0, 0]$, and the projection from P is a morphism by Exercise 7.1.10 (because the conic is a rational normal curve and, as such, it is isomorphic to $\mathbb{P}^1$).

7.3.7 Keep the notation of the solution to Exercise 7.3.5. If $Q \in X = Z_1 \setminus (Z_1 \cap Z_2)$, then $\psi(Q) = P$. Since f_{d-1}, f_d are coprime, there are certainly points in X and such points are infinitely many if $n > 2$. In this case ϕ cannot be defined at the vertex.

7.3.8 If $\phi(P_1) = \phi(P_2)$ it is clear that $r = P_1 \vee P_2$ contains P. We may assume that $P = [1, 0, \dots, 0]$ and that we are projecting on the hyperplane $H_0 = Z_p(x_0)$. Then if $P_1 = [a_0, \dots, a_n]$, $P_2 = [b_0, \dots, b_n]$, we may assume that $a_i = b_i$ for $i = 1, \dots, n$. Suppose Z has equation

$$x_0 f_{d-1}(x_1, \dots, x_n) - f_d(x_1, \dots, x_n) = 0.$$

We have the relations

$$a_0 f_{d-1}(a_1, \dots, a_n) - f_d(a_1, \dots, a_n) = 0, \quad b_0 f_{d-1}(b_1, \dots, b_n) - f_d(b_1, \dots, b_n) = 0.$$

Multiplying the first relation by λ and the second by μ, and adding up, we obtain

$$(\lambda a_0 + \mu b_0) f_{d-1}(a_1, \dots, a_n) - f_d(a_1, \dots, a_n) = 0$$

which proves that the line r is all contained in Z. The last assertion is obvious.

7.3.9 Any point in $Z_p(f_d, f_{d-1})$ is clearly contained in Z. The same proof of Exercise 7.3.7 implies that the points in $Z_p(f_d, f_{d-1})$ lie on lines in Z passing through the vertex P. As for the final assertion, note that the intersection of the set of points $Z_p(f_d, f_{d-1})$ with the hyperplane $H_0 = Z_p(x_0)$ on which we project, are certainly points where the inverse of the stereographic projection cannot be defined.

7.3.10 First prove that there is a point P on the quadric X such that there is another point $Q \in X$ such that the line $P \vee Q$ is not contained in X. Then change coordinates and assume $P = [1, 0, \ldots, 0]$. Then the equation of X becomes $x_0 f_1(x_1, \ldots, x_n) + f_2(x_1, \ldots, x_n) = 0$, with f_1, f_2 non-zero, coprime homogeneous polynomials of degrees $1, 2$ respectively. Then X is a monoid with vertex P.

Chapter 8
Product of Varieties

8.1 Segre Varieties

The product of two affine spaces is an affine space and the product of affine varieties is in a natural way an affine variety. By contrast, the product of projective spaces is not a projective space. In this chapter we will give a structure of a projective variety on the product of projective spaces, which will make it possible to define the general concept of product of quasi–projective varieties.

Let n, m be non–negative integers, and consider $\mathbb{A}^{(n+1)(m+1)}$, whose points can be identified with matrices $\mathbf{w} = (w_{ij})_{i=0,\dots,n, j=0,\dots,m}$ of type $(n+1) \times (m+1)$ on the field $\mathbb{K}$. Accordingly, the points of $\mathbb{P}^{nm+n+m}$, quotient of $\mathbb{A}^{(n+1)(m+1)} \setminus \{\mathbf{0}\}$ by proportionality, can be identified with proportionality equivalence classes $[\mathbf{w}]$ of non–zero matrices of type $(n+1) \times (m+1)$ on $\mathbb{K}$.

Now consider the set

$$\mathrm{Seg}_{n,m} = \{[\mathbf{w}] \in \mathbb{P}^{nm+n+m} : \mathrm{rank}(\mathbf{w}) = 1\}$$

which is well defined and it is a closed subset of $\mathbb{P}^{nm+n+m}$. Indeed $\mathrm{Seg}_{n,m}$ is the set of points $[\mathbf{w}] \in \mathbb{P}^{nm+n+m}$ such that all order 2 minors of $\mathbf{w}$ vanish, i.e., $\mathrm{Seg}_{n,m}$ is defined by the homogeneous degree 2 equations

$$w_{ij} w_{k\ell} = w_{kj} w_{i\ell}, \quad i, k = 0, \dots, n, \quad j, \ell = 0, \dots, m.$$

Let $[\mathbf{x}] \in \mathbb{P}^n$ and $[\mathbf{y}] \in \mathbb{P}^m$. The matrix $\mathbf{x}^t \cdot \mathbf{y}$ (here $\mathbf{x}^t$ denotes the $(n+1) \times 1$ matrix transpose of the vector $\mathbf{x}$) is of type $(n+1) \times (m+1)$, the point $[\mathbf{x}^t \cdot \mathbf{y}] \in \mathbb{P}^{nm+n+m}$ is well defined (i.e., it depends only on $[\mathbf{x}] \in \mathbb{P}^n$ and $[\mathbf{y}] \in \mathbb{P}^m$), and the rank of $\mathbf{x}^t \cdot \mathbf{y}$ is 1. Hence $[\mathbf{x}^t \cdot \mathbf{y}] \in \mathrm{Seg}_{n,m}$.

Finally, given a matrix $\mathbf{w}$ of type $(n+1) \times (m+1)$ on $\mathbb{K}$ and rank 1, there are vectors $\mathbf{x}, \mathbf{y}$ of length $n+1$ and $m+1$ respectively, such that $\mathbf{w} = \mathbf{x}^t \cdot \mathbf{y}$ (see Exercise 8.1.3). This implies that the map

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$$\sigma_{n,m} : ([\mathbf{x}], [\mathbf{y}]) \in \mathbb{P}^n \times \mathbb{P}^m \to [\mathbf{x}^t \cdot \mathbf{y}] \in \mathrm{Seg}_{n,m}$$

is well defined and surjective. It is also easy to see that it is injective (see Exercise 8.1.4).

In conclusion the closed subset $\mathrm{Seg}_{n,m}$ is in 1:1 correspondence with $\mathbb{P}^n \times \mathbb{P}^m$ via the map $\sigma_{n,m}$. We will often interpret this 1:1 correspondence as an identification.

Proposition 8.1.1 *The closed set* $\mathrm{Seg}_{n,m}$ *is irreducible.*

Proof Consider the $\mathbb{K}$–algebras homomorphism

$$\sigma^*_{n,m} : S_{nm+n+m} \to S_{n+m+1}$$

defined in the following way. Let us interpret S_{nm+n+m} as the coordinate ring of $\mathbb{P}^{nm+n+m}$, so that $S_{nm+n+m} = \mathbb{K}[w_{ij}]_{i=0,\dots,n,j=0,\dots,m}$ and set $S_{n+m+1} = \mathbb{K}[x_0, \dots, x_n, y_0, \dots, y_m]$. Then we define $\sigma^*_{n,m}$ by extending by $\mathbb{K}$–linearity the map which associates to w_{ij} the monomial $x_i y_j$, for $i = 0, \dots, n$, $j = 0, \dots, m$. Note that $\sigma^*_{n,m}$ is a homogeneous homomorphism of weight 2 and $\mathcal{I}_{n,m} := \ker(\sigma^*_{n,m})$ is a homogeneous ideal (see the proof of Proposition 6.4.1). It is moreover clear, by the bijectivity of $\sigma_{n,m}$, that $\mathcal{I}_{n,m} = \mathcal{I}_p(\mathrm{Seg}_{n,m})$.

Let now f, g be homogeneous polynomials in S_{nm+n+m}, with $f \notin \mathcal{I}_{n,m}$ and $fg \in \mathcal{I}_{n,m}$. Then $f(x_i y_j)$ is a polynomial which is not identically zero in S_{n+m+1}, whereas $f(x_i y_j)g(x_i y_j)$ is identically zero. This implies that $g(x_i y_j)$ is identically zero, i.e., $g \in \mathcal{I}_{n,m}$. Hence $\mathcal{I}_{n,m}$ is a prime ideal and accordingly $\mathrm{Seg}_{n,m}$ is irreducible. □

So $\mathrm{Seg}_{n,m}$ is a subvariety of $\mathbb{P}^{nm+n+m}$, which is called the *Segre variety* of type (n, m). Sometimes we denote it by $\mathbb{P}^n \times \mathbb{P}^m$, thinking to the identification given by the map $\sigma_{n,m}$. In this way $\mathbb{P}^n \times \mathbb{P}^m$ is endowed with the Zariski topology. We will also call a Segre variety any transformed of $\mathrm{Seg}_{n,m}$ via a projectivity of $\mathbb{P}^{nm+n+m}$.

We can understand the Zariski topology of $\mathbb{P}^n \times \mathbb{P}^m$ intrinsically, in the following way. Let us take a more general viewpoint. Consider a polynomial $f(\mathbf{x}_1, \dots, \mathbf{x}_h)$ in the variables $\mathbf{x}_i = (x_{i0}, \dots, x_{in_i})$, for all $i \in \{1, \dots, h\}$, which is plurihomogeneous of degree $\mathbf{d} = (d_1, \dots, d_h)$ (see Example 1.3.4). Let $(P_1, \dots, P_h) \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_h}$, with $P_i = [\mathbf{p}_i] = [p_{i0}, \dots, p_{in_i}]$, $i = 1, \dots, h$. We say that $(P_1, \dots, P_h)$ is a *zero* of f, and we write $f(P_1, \dots, P_h) = 0$ if $f(\mathbf{p}_1, \dots, \mathbf{p}_h) = 0$. By Exercise 1.3.14 this definition is well posed. The subset

$$Z_s(f) = \{(P_1, \dots, P_h) \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_h} : f(P_1, \dots, P_h) = 0\}$$

is called the *zero set* of f. If $F \subseteq S_{\mathbf{n}}$, with $\mathbf{n} = (n_1, \dots, n_h)$, one sets

$$Z_s(F) = \bigcap_{f \in F^p} Z_s(f),$$

where F^p is the set of plurihomogeneous polynomials in F. One has $Z_s(F) = Z_s((F^p))$, and there are plurihomogeneous polynomials $f_1, \dots, f_m \in (F^p)$ such that

$$Z_s(F) = Z_s(f_1) \cap \cdots \cap Z_s(f_m).$$

If this holds one says that

$$f_i(\mathbf{x}_1, \ldots, \mathbf{x}_h) = 0, \quad i = 1, \ldots, m$$

is a *system of equations* for $Z_s(F)$.

Consider now the family $\mathcal{C}_s^{\mathbf{n}}$ of subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$ of type $Z_s(F)$, with $F \subseteq S_{\mathbf{n}}$ as above. As we saw in Chap. 1 in the affine and in the projective case, one verifies that $\mathcal{C}_s^{\mathbf{n}}$ can be considered as the family of closed subsets of a topology, which is called the *Zariski topology* of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_h}$.

Proposition 8.1.2 *If we identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\mathrm{Seg}_{n,m}$ via the map $\sigma_{n,m}$ then the Zariski topology of $\mathbb{P}^n \times \mathbb{P}^m$ coincides with the Zariski topology of $\mathrm{Seg}_{n,m}$.*

Proof Let $f \in \mathbb{K}[w_{ij}]_{i=0,\ldots,n,j=0,\ldots,m}$ be a homogeneous polynomial of degree d. Then $f'(\mathbf{x}, \mathbf{y}) = f(x_i y_j) \in \mathbb{K}[x_0, \ldots, x_n, y_0, \ldots, y_m] = S_{(n,m)}$ is a bihomogeneous polynomial of degree (d, d), and one has $Z_s(f') = \sigma^{-1}(Z_p(f))$.

On the other hand, if $f'(\mathbf{x}, \mathbf{y}) = f'(x_0, \ldots, x_n, y_0, \ldots, y_m)$ is bihomogeneous of degree (d_1, d_2) and, for instance, $d_1 \leqslant d_2$, if we set $e = d_2 - d_1$, then $f_i' = x_i^e f'$, with $i = 0, \ldots, n$, is bihomogeneous of degree (d_2, d_2). Since $\sigma_{n,m}^*$ is surjective on the subalgebra $\bigoplus_{d \in \mathbb{N}} S_{(n,m),(d,d)}$ of S_{n+m+1}, there are homogeneous polynomials $f_i \in S_{nm+n+m,d_2}$ such that $f(x_i y_j) = f_i'$, for $i = 0, \ldots, n$. Then one has

$$\sigma_{n,m}(Z_s(f')) = \sigma_{n,m}(Z_s(f_1', \ldots, f_n')) = Z_p(f_1, \ldots, f_n) \cap \mathrm{Seg}_{n,m}.$$

This implies the assertion. □

Exercise 8.1.3 Prove that given a matrix $\mathbf{w}$ of type $(n+1) \times (m+1)$ on $\mathbb{K}$ with rank 1, there are vectors $\mathbf{x}, \mathbf{y}$ of length $n+1$ and $m+1$ respectively, such that $\mathbf{w} = \mathbf{x}^t \cdot \mathbf{y}$.

Exercise 8.1.4 Prove that the map $\sigma_{n,m}$ is injective.

Exercise 8.1.5 Prove that Segre varieties $S_{n,m}$ are non–degenerate, i.e., they are not contained in any hyperplane of $\mathbb{P}^{nm+n+m}$.

Exercise 8.1.6 * Prove that given projectivities $\omega_1 : \mathbb{P}^n \to \mathbb{P}^n$ and $\omega_2 : \mathbb{P}^m \to \mathbb{P}^m$, there is a projectivity $\omega : \mathbb{P}^{nm+n+m} \to \mathbb{P}^{nm+n+m}$ such that for all points $(P, Q) \in \mathrm{Seg}_{n,m}$, one has $\omega(P, Q) = (\omega_1(P), \omega_2(Q))$.

8.2 Products

We start with the following:

Proposition 8.2.1 *If $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ are quasi–projective varieties, then $V \times W \subseteq \mathbb{P}^n \times \mathbb{P}^n$ is also a subvariety of $\mathbb{P}^n \times \mathbb{P}^n$. If V, W are projective varieties, then $V \times W$ is also projective.*

Proof We start by proving the final assertion. Suppose $V = Z_p(f_1, \dots, f_a)$ and $W = Z_p(g_1, \dots, g_b)$, with $f_i(\mathbf{x}) \in S_n$, $g_j(\mathbf{y}) \in S_m$ homogeneous, for $i = 1, \dots a$, $j = 1 \dots, b$. Then we can consider f_i, g_j as homogeneous elements of $S_{(n,m)}$ and $f_i = 0, g_j = 0$, with $i = 1, \dots a$, $j = 1 \dots, b$, is a system of equations for $V \times W$ in $\mathbb{P}^n \times \mathbb{P}^n$. The fact that $V \times W$ is irreducible can be proved as in Example 4.1.2.

Let us now consider the general case. Suppose that $V = V_0 \setminus V_1$, $W = W_0 \setminus W_1$, with V_0, V_1, W_0, W_1 closed subsets with V_0, W_0 irreducible. Then $V \times W = V_0 \times W_0 \setminus (V_0 \times W_1 \cup V_1 \times W_0)$, and, taking into account the first part of the proof, this proves the assertion, since $V_0 \times W_0$ is irreducible. □

If V, W are quasi–projective varieties, the variety $V \times W$ is called the *product* of V and W. We recall that for affine varieties we already considered their product in Example 4.1.2 and in Sect. 8. As we shall soon see, the two notions coincide, up to isomorphism.

Given V, W quasi–projective varieties us consider the two canonical projections

$$p_1 : V \times W \to V, \quad p_2 : V \times W \to W.$$

Lemma 8.2.2 *Given V, W quasi–projective varieties, the two projections p_1, p_2 are morphisms.*

Proof It suffices to reduce to the case $V = \mathbb{P}^n$, $W = \mathbb{P}^m$ and consider the first projection p_1. If $([\mathbf{x}], [\mathbf{y}]) \in \mathbb{P}^n \times \mathbb{P}^m$ and if, for instance $x_0 \neq 0$ and $y_0 \neq 0$, then $([\mathbf{x}], [\mathbf{y}])$, as a point of $\mathrm{Seg}_{n,m}$ has homogeneous coordinates $[\mathbf{w}]$, with $w_{00} \neq 0$. Then $p_1([\mathbf{x}], [\mathbf{y}]) = [w_{00}, w_{10}, \dots, w_{n0}]$. This proves the assertion by Proposition 6.2.5. □

Let Z be a third quasi–projective variety and let $f : Z \to V$ and $g : Z \to W$ be maps. Then we have a unique map

$$f \times g : P \in Z \to (f(P), g(P)) \in V \times W$$

such that $p_1 \circ (f \times g) = f$ and $p_2 \circ (f \times g) = g$.

Lemma 8.2.3 *In the above setting, f, g are morphisms if and only if $f \times g$ is a morphism.*

Proof If $f \times g$ is a morphism, then also f, g are morphisms because of Lemma 8.2.2.

Suppose that f, g are morphisms. Let P be a point of Z. By Proposition 6.2.5 there is an open neighborhood U of P in $Z \subseteq \mathbb{P}^r$, and there are homogeneous polynomials $f_0, \dots, f_n \in S_r$ of the same degree and homogeneous polynomials $g_0, \dots, g_m \in S_r$ of the same degree, the former and the latter not all zero in U, such that for every

$Q \in U$ one has

$$f(Q) = [f_0(Q), \dots, f_n(Q)] \in V \subseteq \mathbb{P}^n, \quad g(Q) = [g_0(Q), \dots, g_m(Q)] \in W \subseteq \mathbb{P}^m.$$

Then for all $Q \in U$ one has

$$f \times g(Q) = [f_i(Q)g_j(Q)]_{i=0,\dots,n, j=0,\dots,m} \in V \times W \subseteq \mathbb{P}^{nm+n+m},$$

where $f_i g_j$ are homogeneous polynomials of the same degree not all zero in U, for $i = 0, \dots, n$, $j = 0, \dots, m$. Again by Proposition 6.2.5, the assertion follows. □

From the two previous lemmas, we have the following:

Theorem 8.2.4 *Let V, W be quasi–projective varieties, and let X be a quasi–projective variety with two morphisms $\pi_1 : X \to V$, $p_2 : X \to W$, such that for any quasi–projective variety Z and for any pair of morphisms $f : Z \to V$, $g : Z \to W$, there is a unique morphism $\phi : Z \to X$ such that $\pi_1 \circ \phi = f$, $\pi_2 \circ \phi = g$, then there is a unique isomorphism $a : X \to V \times W$ such that $\pi_1 = p_1 \circ a$ and $\pi_2 = p_2 \circ a$.*

Proof By Lemma 8.2.3, the morphism $a = \pi_1 \times \pi_2$ is such that $\pi_1 = p_1 \circ a$ and $\pi_2 = p_2 \circ a$. Let us prove that a is an isomorphism. Because of the hypotheses, there is a morphism $b : V \times W \to X$ such that $p_1 = \pi_1 \circ b$ and $p_2 = \pi_2 \circ b$. Then $b \circ a : X \to X$ is such that $\pi_1 = \pi_1 \circ (b \circ a)$ and $\pi_2 = \pi_2 \circ (b \circ a)$. Hence, by the hypotheses, one has $b \circ a = \mathrm{id}_X$. Similarly one verifies that $a \circ b = \mathrm{id}_{V \times W}$, and this proves the assertion. □

The previous theorem expresses the so called *universal property* of the product of two varieties.

Corollary 8.2.5 *The product of affine varieties defined in Example 4.1.2 and in Sect. 8 coincides with the product we defined in this chapter.*

Proof Let V, W be affine varieties and let X be the product $V \times W$ as defined in Example 4.1.2 and in Sect. 8. Recall that X is the affine variety corresponding to the finitely generated $\mathbb{K}$–algebra $A(V) \otimes_{\mathbb{K}} A(W)$. Let $\pi_1 : X \to V$ and $\pi_2 : X \to W$ be the projections, which are the morphisms corresponding to the $\mathbb{K}$–algebras homomorphisms

$$\pi_1^* : f \in A(V) \to f \otimes 1 \in A(V) \otimes_{\mathbb{K}} A(W),$$

$$\pi_2^* : g \in A(W) \to 1 \otimes g \in A(V) \otimes_{\mathbb{K}} A(W).$$

If Z is a quasi–projective variety and $(f, g) \in M(Z, V) \times M(Z, W)$, then f, g correspond to $\mathbb{K}$–algebras homomorphisms

$$f^* : A(V) \to \mathcal{O}(Z), \quad g^* : A(W) \to \mathcal{O}(Z).$$

Then, by the properties of the tensor product, there is a unique homomorphism of $\mathbb{K}$–algebras

$$f^* \otimes g^* : \sum_{i,j} a_i \otimes b_j \in A(V) \otimes_{\mathbb{K}} A(W) \to \sum_{i,j} f^*(a_i) g^*(b_j) \in \mathcal{O}(Z)$$

such that $f^* = (f^* \otimes g^*) \circ \pi_1^*$ and $g^* = (f^* \otimes g^*) \circ \pi_2^*$. Then $f^* \otimes g^*$ corresponds to a unique morphism $\phi : Z \to X$ such that $f = \pi_1 \circ \phi$ and $g = p_2 \circ \phi$. Thus the assertion is a consequence of Theorem 8.2.4. □

Exercise 8.2.6 Prove that if V, W are quasi projective varieties then $\dim_{\text{tr}}(V \times W) = \dim_{\text{tr}}(V) + \dim_{\text{tr}}(W)$.

Exercise 8.2.7 * Prove the following properties of products:

(a) if V, W, V', W' are quasi–projective varieties, with V [resp. W] isomorphic to V' [resp. to W'], then $V \times W$ is isomorphic to $V' \times W'$;
(b) if V, W are quasi–projective varieties, then $V \times W$ is isomorphic to $W \times V$;
(c) if V, W, Z are quasi–projective varieties, then $(V \times W) \times Z$ is isomorphic to $V \times (W \times Z)$; this variety is denoted by $V \times W \times Z$ and is called the *product* of V, W and Z;
(d) more generally, if $V_1, \ldots, V_n$ are quasi–projective varieties, it is well defined their *product* $V_1 \times \cdots \times V_n$ as the variety $(\cdots((V_1 \times V_2) \times V_3) \times \cdots) \times V_n$, which is isomorphic to $V_{i_1} \times \cdots \times V_{i_n}$, where $(i_1, \ldots, i_n)$ is any permutation of $(1, \ldots, n)$; the reader may state and prove the analogue of Theorem 8.2.4 for more than two factors;
(e) if V, W are quasi projective varieties and $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$ are the projections, then for any point $P \in V$ [resp. any point $Q \in W$] one has that $p_1^{-1}(P)$ [resp. $p_2^{-1}(Q)$] is isomorphic to W [resp. to V];
(f) if V, W, Z are quasi–projective varieties and $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$ are the projections, a map $f : Z \to V \times W$ is a morphism if and only if $p_i \circ f$ are morphisms, for $i = 1, 2$.

Exercise 8.2.8 Prove that Segre varieties $S_{n,m}$ are rational.

Exercise 8.2.9 * Consider $\mathbb{P}^n$ with homogeneous coordinates $[x_0, \ldots, x_n]$ and $\mathbb{A}^m$ with coordinates $(y_1, \ldots, y_m)$. Prove that the closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$ are the subsets of $\mathbb{P}^n \times \mathbb{A}^m$ which are formed by the pairs of points $([x_0, \ldots, x_n], (y_1, \ldots, y_m))$ that are solutions of systems of equations of the form

$$f_i(x_0, \ldots, x_n, y_1, \ldots, y_m) = 0, \quad i = 1, \ldots, h$$

with f_i polynomials, which are homogeneous in the variables $x_0, \ldots, x_n$.

Exercise 8.2.10 Let V be a quasi–projective variety. Consider $\Delta_V = \{(P, P) \in V \times V\}$ the *diagonal* of the product. Prove that Δ_V is a closed subset of $V \times V$ isomorphic to V.

Exercise 8.2.11 * Let V, W, Z be quasi–projective varieties and let $f : V \to Z$, $g : W \to Z$ be morphisms. Denote by $V \times_Z W$ the so called *fibred product* of V and W over Z, i.e., the subset of $V \times W$ consisting of all pairs $(P, Q) \in V \times W$ such that $f(P) = g(Q)$. Prove that $V \times_Z W$ is closed in $V \times W$. Give some example in which $V \times_Z W$ is not irreducible.

Exercise 8.2.12 * Continuing Exercise 8.2.11, consider the case in which $W = Z$ and $g = \mathrm{id}_Z$. Then $V \times_Z W$ is the set of pairs $(P, Q) \in V \times W$ such that $Q = f(P)$, i.e., this is the *graph* Γ_f of the function f. Prove that Γ_f is isomorphic to V.

Exercise 8.2.13 Consider the projective spaces $\mathbb{P}^{n_1}, \dots, \mathbb{P}^{n_r}$, in which we have homogeneous coordinates $[\mathbf{x}_i] = [x_{i,0}, \dots, x_{i,n_i}]$, for $i = 1, \dots, r$. Consider the map

$$\sigma_{n_1,\dots,n_r} : ([\mathbf{x}_1], \dots, [\mathbf{x}_r]) \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to$$
$$\to [x_{1,i_1} \cdots x_{r,i_r}]_{i_1=0,\dots,n_1,\dots,i_r=0,\dots,n_r} \in \mathbb{P}^{(n_1+1)\cdots(n_r+1)-1},$$

where the points of $\mathbb{P}^{(n_1+1)\cdots(n_r+1)-1}$ have homogeneous coordinates

$$[w_{i_1,\dots,i_r}]_{i_1=0,\dots,n_1,\dots,i_r=0,\dots,n_r}.$$

Prove that:

(a) the map $\sigma_{n_1,\dots,n_r}$ is injective: via $\sigma_{n_1,\dots,n_r}$ we will identify $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with its image denoted by $\mathrm{Seg}_{n_1,\dots,n_r}$;
(b) $\mathrm{Seg}_{n_1,\dots,n_r}$ is an irreducible closed subset of $\mathbb{P}^{(n_1+1)\cdots(n_r+1)-1}$ called the *Segre variety* of type $(n_1, \dots, n_r)$ (as well as any variety transformed of it via a projectivity);
(c) the Zariski topology in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ coincides with the inverse image via $\sigma_{n_1,\dots,n_r}$ of the Zariski topology on $\mathrm{Seg}_{n_1,\dots,n_r}$;
(d) $\mathrm{Seg}_{n_1,\dots,n_r}$ is non–degenerate in $\mathbb{P}^{(n_1+1)\cdots(n_r+1)-1}$.

Exercise 8.2.14 Continue Exercise 8.2.13. Fix $P_j \in \mathbb{P}^{n_j}$ for a given $j = 1, \dots, r$. Consider the jth projection

$$p_j : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_j}.$$

Prove that $\sigma_{n_1,\dots,n_r}(p_j^{-1}(P_j))$ is a Segre variety of type $(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_r)$.

Exercise 8.2.15 Work out a theory, analogous to the one in Chap. 3, about the relations between the closed subsets of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ and the ideals of $S_{(n_1,\dots,n_r)}$. Prove in particular that a closed subset Z of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is irreducible if and only if the corresponding plurihomogeneous ideal $\mathcal{I}_s(Z)$ generated by the plurihomogeneous polynomials $f \in S_{(n_1,\dots,n_r)}$ such that $Z \subseteq Z_s(f)$, is a prime ideal.

Exercise 8.2.16 Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Prove that $\mathrm{Seg}_{1,1}$ is a *maximal rank* quadric in $\mathbb{P}^3$, i.e., it is such that its matrix has maximal rank 4. It is well known that for any two maximal rank quadrics in $\mathbb{P}^3$ there is a projectivity which sends one to the other. Prove that given any maximal rank quadric X in $\mathbb{P}^3$, there are two distinct morphisms $p_1 : X \to \mathbb{P}^1$, $p_2 : X \to \mathbb{P}^1$, such that their *fibres*, i.e., the counterimages of the points of $\mathbb{P}^1$, are lines in X. We set $\ell_{i,x} = p_i^{-1}(x)$, with $x \in \mathbb{P}^1$, for $i = 1, 2$. Hence X has two families of lines $\mathcal{L}_i = \{\ell_{i,x}\}_{x \in \mathbb{P}^1}$, $i = 1, 2$. Prove that two distinct lines of the same family $\mathcal{L}_i$ do not intersect, whereas any line of $\mathcal{L}_1$ and any line of $\mathcal{L}_2$ intersect at one point. Prove that, although X is rational, it is not isomorphic to $\mathbb{P}^2$.

Exercise 8.2.17 Prove that if Z is a closed subset of $\mathrm{Seg}_{1,1} \subset \mathbb{P}^3$ in general it is not the case that $S(Z)$ is $\mathbb{K}$–isomorphic to $S_{(1,1)}/\mathcal{I}_s(Z)$.

Exercise 8.2.18 Identify $\mathrm{Seg}_{1,1}$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Then find the equations in $\mathbb{P}^1 \times \mathbb{P}^1$ of the conics which are the intersections of $\mathrm{Seg}_{1,1}$ with the planes of $\mathbb{P}^3$.

Exercise 8.2.19 Prove that the Zariski topology of $\mathbb{P}^n \times \mathbb{P}^m$ is strictly finer than the product topology if $n, m \geqslant 1$.

8.3 The Blow–up

Consider $\mathbb{P}^n$ with homogeneous coordinates $[x_0, \dots, x_n]$ and $\mathbb{P}^{n-1}$ with homogeneous coordinates $[y_1, \dots, y_n]$. Consider in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ the closed subset $\tilde{\mathbb{P}}^n$ with equations

$$x_i y_j = y_i x_j, \quad i, j = 1, \dots, n.$$

Consider the map

$$\sigma : \tilde{\mathbb{P}}^n \to \mathbb{P}^n,$$

where σ is the restriction of the projection $p_1 : \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ to $\tilde{\mathbb{P}}^n$. Let $P = [1, 0, \dots, , 0] \in \mathbb{P}^n$. The closed set $\tilde{\mathbb{P}}^n$, endowed with the map σ is called the *blow–up* of $\mathbb{P}^n$ at P. We will study the main properties of the pair $(\tilde{\mathbb{P}}^n, \sigma)$. We set $E := \sigma^{-1}(P)$, that is a closed subset of $\tilde{\mathbb{P}}^n$, called the *exceptional locus* of the blow–up.

Proposition 8.3.1 *$\tilde{\mathbb{P}}^n \setminus E$ is a quasi–projective variety that is isomorphic to $\mathbb{P}^n \setminus \{P\}$ via the map σ.*

Proof Let $Q = [q_0, \dots, q_n] \in \mathbb{P}^n$, with $Q \neq P$. Hence there is an $i \in \{1, \dots, n\}$ such that $q_i \neq 0$. If $[y_1, \dots, y_n] \in \sigma^{-1}(Q)$, one has

$$y_j = y_i \frac{q_j}{q_i}, \quad j = 1, \dots, n, \tag{8.1}$$

whence we deduce $y_i \neq 0$, otherwise we would have $y_j = 0$ for all $j = 1, \dots, n$, a contradiction. So we can take $y_i = q_i \neq 0$, and then from (8.1) we deduce $y_j = q_j$ for all $j = 1, \dots, n$. Hence $\sigma^{-1}(Q) = \{[q_0, \dots, q_n], [q_1, \dots, q_n]\}$. Consider the map

$$\tau : Q \in \mathbb{P}^n \setminus \{P\} \to \sigma^{-1}(Q) \in \mathbb{P}^n \times \mathbb{P}^{n-1}.$$

By Exercise 8.1.6, (f), we have that τ is a morphism. Moreover $\tau(\mathbb{P}^n \setminus \{P\}) = \tilde{\mathbb{P}}^n \setminus E$, hence this set, which is locally closed in $\mathbb{P}^n \times \mathbb{P}^{n-1}$, is also irreducible (see Exercise 4.1.7), thus it is a quasi–projective variety. Finally $\sigma_{|\tilde{\mathbb{P}}^n \setminus E}$ is a morphism and $\tau = (\sigma_{|\tilde{\mathbb{P}}^n \setminus E})^{-1}$, which proves the assertion. □

Next we want to study the set E. Note that $E = \sigma^{-1}(P) = p_1^{-1}(P) \cong \mathbb{P}^{n-1}$. We want to give a geometric interpretation to the isomorphism $E \cong \mathbb{P}^{n-1}$. To do this, consider the set (P) of all lines of $\mathbb{P}^n$ containing P.

Lemma 8.3.2 *The set (P) can be regarded in a natural way as a projective space of dimension $n - 1$.*

Proof Consider in the dual projective space $\check{\mathbb{P}}^n$ the hyperplane H_P which consists of all points of $\check{\mathbb{P}}^n$ corresponding to hyperplanes of $\mathbb{P}^n$ containing P. We claim that there is a natural bijection

$$\varphi : (P) \to \check{H}_P$$

so that, by identifying (P) with $\check{H}_P$ via φ, (P) inherits the structure of projective space of dimension $n - 1$ of $\check{H}_P$. The map φ is so defined. Take a line $r \in (P)$ and send it via φ to the set (r) of the hyperplanes of $\mathbb{P}^n$ that contain r. This is a hyperplane in H_P and so it defines the point $\varphi(r)$ in $\check{H}_P$. The reader will easily check that φ is bijective. □

Let now $r \in (P)$ be a line, locus of points of $\mathbb{P}^n$ with homogeneous coordinates

$$x_0 = \lambda, \quad x_i = \mu q_i, \quad [\lambda, \mu] \in \mathbb{P}^1, \quad i = 1, \ldots, n.$$

The map τ, restricted to $r \setminus \{P\}$, acts in the following way

$$\tau[\lambda, \mu q_1, \ldots, \mu q_n] = ([\lambda, \mu q_1, \ldots, \mu q_n], [q_1, \ldots, q_n]), \quad \text{with} \quad \mu \in \mathbb{K} \setminus \{0\}.$$

So we can extend τ to P on r (see Exercise 7.1.10) by setting

$$\tau_{|r}(P) = ([1, 0, \ldots, 0], [q_1, \ldots, q_n])$$

and in this way we have a morphism $\tau_{|r} : r = \mathbb{P}^1 \to \tilde{\mathbb{P}}^n$. We denote by $\tilde{r}$ the image of r via $\tau_{|r}$. We have $\tau_{|r}(P) = \tilde{r} \cap E$. This way we have a map

$$\omega : r \in (P) \cong \mathbb{P}^{n-1} \to \tau_{|r}(P) \in E \cong \mathbb{P}^{n-1}.$$

Proposition 8.3.3 *The map ω is a projectivity.*

Proof The map ω sends r, that we may assume to have homogeneous coordinates $[q_1, \ldots, q_n] \in (P)$, to the point $\tau_{|r}(P)$ that in E has coordinates $[q_1, \ldots, q_n]$. This proves the assertion. □

The previous proposition gives the geometric interpretation of E we were seeking for: E is in 1:1 projective correspondence with the set of all lines issuing from P.

Proposition 8.3.4 *$\tilde{\mathbb{P}}^n$ is irreducible.*

Proof It suffices to prove that any point on E is in the closure of some subset of $\tilde{\mathbb{P}}^n \setminus E$. Note, indeed, that for any line $r \in (P)$, $\tau(r \setminus \{P\}) \cong r \setminus \{P\}$ is a quasi–projective subvariety of $\tilde{\mathbb{P}}^n \setminus E$, whose closure in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ is $\tilde{r}$. This follows from the fact that $\tau(r \setminus \{P\})$ sits in $p_2^{-1}(\omega(r)) \cong \mathbb{P}^n$, where $p_2 : \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is the projection to the second factor. Hence $\tau(r \setminus \{P\})$ is isomorphic to the line $\tilde{r}$ minus the point $\omega(r)$. □

Remark 8.3.5 In the above setting we may identify $\mathbb{P}^{n-1}$ with (P). Then $\tilde{\mathbb{P}}^n$ can be seen as the set of all pairs $(P', r) \in \mathbb{P}^n \times (P) = \mathbb{P}^n \times \mathbb{P}^{n-1}$ such that $P' \in r$.

More generally, we can fix any point $Q \in \mathbb{P}^n$ and we can consider the set

$$\tilde{\mathbb{P}}^n_Q = \{(P', r) \in \mathbb{P}^n \times (Q) = \mathbb{P}^n \times \mathbb{P}^{n-1} : P' \in r\}.$$

If $\alpha : \mathbb{P}^n \to \mathbb{P}^n$ is a projectivity such that $\alpha(P) = Q$, it is clear that α induces a projectivity

$$\bar{\alpha} : r \in (P) \to \alpha(r) \in (Q).$$

Then we have an isomorphism

$$\alpha \times \bar{\alpha} : \mathbb{P}^n \times (P) = \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n \times (Q) = \mathbb{P}^n \times \mathbb{P}^{n-1}$$

and it is clear that $\alpha \times \bar{\alpha}(\tilde{\mathbb{P}}^n) = \tilde{\mathbb{P}}^n_Q$. So $\tilde{\mathbb{P}}^n_Q$ is a projective variety isomorphic to $\tilde{\mathbb{P}}^n$. We denote by α' the restriction of $\alpha \times \bar{\alpha}$ to $\tilde{\mathbb{P}}^n$. Consider

$$\sigma_Q : \tilde{\mathbb{P}}^n_Q \to \mathbb{P}^n$$

the restriction to $\tilde{\mathbb{P}}^n_Q$ of the projection to the first factor of $\mathbb{P}^n \times (Q)$. Then we have $\sigma_Q \circ \alpha' = \alpha \circ \sigma$. The variety $\tilde{\mathbb{P}}^n_Q$, endowed with the map σ_Q, is called the *blow–up* of $\mathbb{P}^n$ at Q. By the above, it behaves exactly as $\sigma : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$.

Let $V \subseteq \mathbb{P}^n$ be a quasi–projective variety and let Q be a point of V and assume $V \neq \{Q\}$. The locally closed subset $\sigma_Q^{-1}(V)$ of $\tilde{\mathbb{P}}^n$ is called the *total transform* of V on $\tilde{\mathbb{P}}^n_Q$. We denote by V'_Q the closure of $\sigma_Q^{-1}(V \setminus \{Q\})$ in $\tilde{\mathbb{P}}^n_Q$. We set $\tilde{V}_Q = V'_Q \cap \sigma_Q^{-1}(V)$. Since V is locally closed in $\mathbb{P}^n$ and irreducible, and since V'_Q is closed in $\tilde{\mathbb{P}}^n$, then $\tilde{V}_Q$ is locally closed in $\tilde{\mathbb{P}}^n$ and irreducible. Moreover $\tilde{V}_Q \setminus (\tilde{V}_Q \cap E)$ is isomorphic to $V \setminus \{Q\}$ via σ_Q. The variety $\tilde{V}_Q$, endowed with the morphism $\sigma_{V,Q} = \sigma_{Q|\tilde{V}_Q} : \tilde{V}_Q \to V$, is called the *blow–up* of V at Q, and also the *proper transform* or *strict transform* of V on $\tilde{\mathbb{P}}^n$. Note that $\sigma_{V,Q}$ is a birational morphism.

Exercise 8.3.6 Referring to the proof of Lemma 8.3.2, consider any hyperplane H of $\mathbb{P}^n$ not containing P, and consider the map

$$\phi_{H,P} : Q \in H \to P \vee Q \in (P).$$

prove that $\phi_{H,P}$ is a projectivity. Let H_1, H_2 be two such hyperplanes. Consider the map

$$\phi_{H_1,H_2,P} := \phi^{-1}_{H_2,P} \circ \phi_{H_1,P} : H_1 \to H_2.$$

Prove that $\phi_{H_1,H_2,P}$ is a projectivity called the *perspective* of H_1 to H_2 with center P. Prove that the points of $H_1 \cap H_2$ are fixed by $\phi_{H_1,H_2,P}$.

Exercise 8.3.7 Consider $\mathbb{A}^n$ and a point $P \in \mathbb{A}^n$. Denote again by (P) the set of all lines in $\mathbb{A}^n$ containing the point P. Prove that (P) has a natural structure of projective space of dimension $n - 1$. Consider the set

$$\tilde{\mathbb{A}}^n_P = \{(Q, r) \in \mathbb{A}^n \times (P) = \mathbb{A}^n \times \mathbb{P}^{n-1} : Q \in r\}$$

endowed with the map $\sigma_P : \tilde{\mathbb{A}}^n_P \to \mathbb{A}^n$ restriction of the projection to the first factor. Set $E = \sigma_P^{-1}(P)$. Prove that:

(a) $\tilde{\mathbb{A}}^n_P$ is an irreducible, closed subvariety of $\mathbb{A}^n \times (P) = \mathbb{A}^n \times \mathbb{P}^{n-1}$;
(b) $\sigma_{P|\tilde{\mathbb{A}}^n_P \setminus E} : \tilde{\mathbb{A}}^n_P \setminus E \to \mathbb{A}^n \setminus \{P\}$ is an isomorphism;
(c) E is isomorphic to (P) and its points can be identified with the lines in $\mathbb{A}^n$ issuing from P;
(d) identified, as usual, $\mathbb{A}^n$ with the open subset U_0 of $\mathbb{P}^n$, then $\tilde{\mathbb{A}}^n_P$ embeds as an open subset of $\tilde{\mathbb{P}}^n_P$.

Moreover, suppose that $P = (p_1, \dots, p_n)$. Let r be a line in (P) so that r has parametric equations $x_i = p_i + ty_i$, for $i = 1, \dots, n$, where $(y_1, \dots, y_n) \neq \mathbf{0}$. Prove that $[y_1, \dots, y_n]$ can be assumed to be homogeneous coordinates of r in $(P) = \mathbb{P}^{n-1}$. Prove that $\tilde{\mathbb{A}}^n_P$ is the set of points $((x_1, \dots, x_n), [y_1, \dots, y_n]) \in \mathbb{A}^n \times (P)$ such that $(x_i - p_i)y_j = (x_j - p_j)y_i$, for all $i, j = 1, \dots, n$.

The variety $\tilde{\mathbb{A}}^n_P$ endowed with the map σ_P is called the *blow–up* of $\mathbb{A}^n$ at P. The closed subset $E = \sigma_P^{-1}(P) = (P) \cong \mathbb{P}^{n-1}$ is called the *exceptional locus* of the blow–up.

Exercise 8.3.8 Continuing the Exercise 8.3.7, let V be a locally closed subvariety of $\mathbb{A}^n$. Make sense of the notion of proper transform of V on $\tilde{\mathbb{A}}^n_P$ and of the blow–up of V at P.

Exercise 8.3.9 Let us consider the blow–up $\sigma : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ of $\mathbb{A}^2$ at the origin. Prove that $\tilde{\mathbb{A}}^2$ can be covered by two affine subsets each isomorphic to a quadric of $\mathbb{A}^3$ which in turn is isomorphic to $\mathbb{A}^2$.

Exercise 8.3.10 Let us consider the two curves $C_1, C_2 \subset \mathbb{A}^2$ with respective equations

$$x_1^2 = x_2^3, \quad x_2^2 - x_1^2 = x_1^3.$$

Describe the proper transforms of C_1 and C_2 on the blow–up of $\mathbb{A}^2$ at the origin.

Exercise 8.3.11 Let $\Pi \subset \mathbb{P}^n$ be a subspace of dimension $m < n$ and let $P \in \Pi$. Prove that the proper transform of Π in the blow–up of $\mathbb{P}^n$ at P equals the blow–up of Π at P.

Exercise 8.3.12 Let $\Pi \subset \mathbb{P}^n$ be a subspace of dimension $m < n$. Consider the set (Π) of all subspaces of $\mathbb{P}^n$ of dimension $m + 1$ containing Π. Prove that there is a natural identification of (Π) with a projective space of dimension $n - m - 1$.

Exercise 8.3.13 Extend the construction of the blow–up in the following way. Let $\Pi \subset \mathbb{P}^n$ be a subspace of dimension $m < n$. Consider the set

$$\tilde{\mathbb{P}}^n_\Pi = \{(P, \Sigma) \in \mathbb{P}^n \times (\Pi) : P \in \Sigma\}$$

with the projection $\sigma_\Pi : \tilde{\mathbb{P}}^n_\Pi \to \mathbb{P}^n$. Prove that $\tilde{\mathbb{P}}^n_\Pi$ is a closed subvariety of $\mathbb{P}^n \times (\Pi) \cong \mathbb{P}^n \times \mathbb{P}^{n-m-1}$, called the *blow–up* of $\mathbb{P}^n$ along Π. Prove that σ_Π is a morphism. Set $E = \sigma_\Pi^{-1}(\Pi)$, called the *exceptional locus* of the blow–up. Prove that σ_Π induces an isomorphism between $\tilde{\mathbb{P}}^n_\Pi \setminus E$ and $\mathbb{P}^n \setminus \Pi$. Prove that for all points $P \in \Pi$, $\sigma_\Pi^{-1}(P) \cong \mathbb{P}^{n-m-1}$. Make sense of the notions of total transform and strict transform a subvariety V of $\mathbb{P}^n$ not contained in Π.

Exercise 8.3.14 * Consider the linear system $\mathcal{L} \subset \mathcal{L}_{2,2}$ of conics in $\mathbb{P}^2$ containing two distinct points $P, Q \in \mathbb{P}^2$. Prove that $\mathcal{L}$ has dimension 3. Consider the rational map $\phi_\mathcal{L} : \mathbb{P}^2 \dashrightarrow \check{\mathcal{L}} \cong \mathbb{P}^3$ (see Exercise 7.1.30). Let $\pi : \tilde{\mathbb{P}} \to \mathbb{P}^2$ be the blow–up of $\mathbb{P}^2$ at P and Q. Prove that there is a morphism $\phi : \tilde{\mathbb{P}} \to \mathbb{P}^3$ such that $\phi = \phi_\mathcal{L} \circ \pi$. Prove that ϕ is birational onto its image, which is a quadric of rank 4 in $\mathbb{P}^3$.

8.4 Solutions of Some Exercises

8.1.3 There are two square matrices $\mathbf{A}, \mathbf{B}$ of order $n + 1$ and $m + 1$ respectively, and of maximal rank, such that

$$\mathbf{A}\cdot\mathbf{w}\cdot\mathbf{B} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} = (1, 0, \dots, 0)^t \cdot (1, 0, \dots, 0).$$

Then take $\mathbf{x} = (1, 0, \dots, 0) \cdot (\mathbf{A}^{-1})^t$ and $\mathbf{y} = (1, 0, \dots, 0) \cdot (\mathbf{B}^{-1})^t$.

8.1.4 Referring to the solution of Exercise 8.1.3, the assertion follows from the fact that the matrix of type $(n+1) \times (m+1)$

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

can be expressed as the product $\mathbf{x}^t \cdot \mathbf{y}$ only if $\mathbf{x}=(x, 0, \dots, 0)$ and $\mathbf{y}=(x^{-1}, 0, \dots, 0)$ with $x \in \mathbb{K}^*$.

8.1.5 It is immediate to see that there is no polynomial of degree 1 in $\mathcal{I}_{n,m}$.

8.1.6 Use the universal property of the product.

8.2.9 Identify $\mathbb{P}^n \times \mathbb{A}^m$ with an open subset of $\mathbb{P}^n \times \mathbb{P}^m$ and note that closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$ are intersection with closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$.

8.2.10 To prove that the diagonal of V is closed it suffices to reduce to the case $V = \mathbb{P}^n$. A point $([x_0, \dots, x_n], [y_0, \dots, y_n]) \in \mathbb{P}^n \times \mathbb{P}^n$ sits in $\Delta_{\mathbb{P}^n}$ if and only if $x_i y_j = x_j y_i$, i.e., if and only if $w_{ij} = w_{ji}$, for $i, j = 0, \dots, n$. Hence

$$\Delta_{\mathbb{P}^n} = \mathrm{Seg}_{n,n} \cap Z_p(w_{ij} - w_{ji})_{i,j=0,\dots,n},$$

i.e., $\Delta_{\mathbb{P}^n}$ is the intersection of $\mathrm{Seg}_{n,n}$ with a linear subspace of $\mathbb{P}^{n^2+2n}$, thus it is a closed subset.

To prove that Δ_V is isomorphic to V, apply Theorem 8.2.4 with $V = W$, $Z = V$ and $f, g = \mathrm{id}_V$.

8.2.11 That $V \times_Z W$ is closed follows from $V \times_Z W = (f \times g)^{-1}(\Delta_Z)$, where

$$f \times g : (P, Q) \in V \times W \to (f(P), g(P)) \in Z \times Z$$

is clearly a morphism.

Next, suppose that V, W are subvarieties of Z and $f : V \to Z$ and $g : W \to Z$ the immersions. Then the projection onto V [or onto W] maps $V \times_Z W$ isomorphically to $V \cap W$, which in general is not irreducible.

8.2.12 The projection $p_1 : V \times W \to V$ induces an isomorphism of Γ_f to V. In fact the inverse of $p_{1|\Gamma_f}$ is the map $\mathrm{id}_V \times f$.

8.2.14 Let $P_j = [p_{j,0}, \dots, p_{j,n_j}]$. Set $N(n_1, \dots, n_r) = (n_1+1)\cdots(n_r+1) - 1$ and assume that the points of $\mathbb{P}^{N(n_1,\dots,n_{j-1},n_{j+1},\dots,n_r)}$ have homogeneous coordinates $[w_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_r}]$, with $i_h = 0, \dots, n_h$, for $h = 1, \dots, j-1, j+1, \dots, r$. Consider the map

$$\omega_{P_j} : [w_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_r}] \in \mathbb{P}^{N(n_1,\dots,n_{j-1},n_{j+1},\dots,n_r)} \to$$
$$\to [p_{j,i_j} w_{i_1,\dots,i_{j-1},i_{j+1},\dots,i_r}] \in \mathbb{P}^{N(n_1,\dots,n_r)}.$$

It is easy to verify that this is a projectivity which embeds $\mathbb{P}^{N(n_1,\dots,n_{j-1},n_{j+1},\dots,n_r)}$ into $\mathbb{P}^{N(n_1,\dots,n_r)}$ and sends $\mathrm{Seg}_{n_1,\dots,n_{j-1},n_{j+1},\dots,n_r}$ into $\sigma_{n_1,\dots,n_r}(p_j^{-1}(P_j))$.

8.2.17 If $Z = \mathrm{Seg}_{1,1}$, then $S_{(1,1)}/\mathcal{I}_s(Z) = S_{(1,1)} = S_3$, whereas $S(Z) = S_3/\mathcal{I}_p(Z)$, which is not isomorphic to S_3.

8.3.9 Recall that $\tilde{\mathbb{A}}^2$ has in $\mathbb{A}^2 \times \mathbb{P}^1$ equation $x_1 y_2 = x_2 y_1$, where $(x_1, x_2) \in \mathbb{A}^2$ and $[y_1, y_2] \in \mathbb{P}^1$. Consider the two open subsets $U_i = \{[y_1, y_2] \in \mathbb{P}^1 : y_i \neq 0\} \cong \mathbb{A}^1$, with $i = 1, 2$. Then $\mathbb{A}^2 \times \mathbb{P}^1$

can be covered with the two open subsets $V_i = \mathbb{A}^2 \times U_i \cong \mathbb{A}^3$, $i = 1, 2$. We can see $\tilde{\mathbb{A}}^2 \cap V_i$ as a closed subvariety of $V_i \cong \mathbb{A}^3$ for $i = 1, 2$. For example in V_1 we have coordinates (x_1, x_2, t), where $t = \frac{y_2}{y_1}$. Then $\tilde{\mathbb{A}}^2 \cap V_1$ has in V_1 the equation $x_2 = x_1 t$. This is a quadric Q_1 in $\mathbb{A}^3$, which is isomorphic to $\mathbb{A}^2$, via the isomorphism $(x_1, t) \in \mathbb{A}^2 \to (x_1, x_1 t, t) \in Q_1$. Similarly, in V_2 we have coordinates (x_1, x_2, s), where $s = \frac{y_1}{y_2}$. Then $\tilde{A}^2 \cap V_2$ has in V_2 the equation $x_1 = x_2 s$ and we can repeat the same argument as before. Note that both in V_1 and V_2 the exceptional locus has equations $x_1 = 0, x_2 = 0$.

8.3.10 First we look at the total transform of C_1 on $\tilde{\mathbb{A}}^2$. Let us look at what happens in the open subset V_1 (see solution of Exercise 8.3.9, from which we keep the notation). Here the total transform of C_1 has equations

$$x_1^2 = x_2^3, \quad x_2 = x_1 t.$$

This system is equivalent to

$$x_2 = x_1 t, \quad x_1^2(x_1 t^3 - 1) = 0$$

which in turn is equivalent to the union of the two systems

$$x_1^2 = 0, x_2 = 0 \quad \text{and} \quad x_2 = x_1 t, x_1 t^3 = 1.$$

The solutions to the first of the two systems are the set of point of the exceptional locus E. The presence of this locus in the total transform of C_1 is obvious. The solutions to the second system are the set of points of the strict transform $\tilde{C}_1$ of C_1 on V_1. In this open set $\tilde{C}_1$ does not intersect E, because there is no solution to the second system with $x_1 = 0$.

Let us look now to what happens in the open subset V_2. Here the total transform of C_1 has equations

$$x_1^2 = x_2^3, \quad x_1 = x_2 s.$$

This system is equivalent to

$$x_1 = x_2 s, \quad x_2^2(x_2 - s^2) = 0$$

which in turn is equivalent to the union of the two systems

$$x_1 = 0, x_2^2 = 0 \quad \text{and} \quad x_1 = x_2 s, x_2 = s^2.$$

Again the solutions to the first of the two systems fill up the exceptional locus E. The solutions to the second system are the strict transform $\tilde{C}_1$ of C_1 on V_2. In this open set $\tilde{C}_1$ intersects E (which has equation $x_1 = x_2 = 0$) at one point, namely the point $x_1 = x_2 = s = 0$.

The analysis in the case of C_2 is similar.

Chapter 9
More on Elimination Theory

9.1 The Fundamental Theorem of Elimination Theory

Let us consider the projective space $\mathcal{L}_{n,d}$ of dimension

$$N(n,d) = \binom{n+d}{n} - 1$$

whose points are in 1:1 correspondence with effective divisors of degree d of $\mathbb{P}^n$, or, equivalently, with proportionality equivalence classes of non–zero homogeneous polynomials of degree d in the variables $x_0, \dots, x_n$ (see Sect. 1.6.5). Given the positive integers $d_1, \dots, d_h$ we will set

$$\mathcal{L}_{n,d_1,\dots,d_h} = \mathcal{L}_{n,d_1} \times \cdots \times \mathcal{L}_{n,d_h}.$$

Then we will denote by $Z(n; d_1, \dots, d_h)$ the subset of $\mathcal{L}_{n,d_1,\dots,d_h}$ formed by all h–tuples $(F_1, \dots, F_h) \in \mathcal{L}_{n,d_1,\dots,d_h}$ such that $\text{supp}(F_1) \cap \cdots \cap \text{supp}(F_h) \neq \emptyset$. This is the same as looking at the set of h–tuples $(f_1, \dots, f_h)$ of non–zero homogeneous polynomials of degrees $d_1, \dots, d_h$ in the variables $x_0, \dots, x_n$ such that the system

$$f_1(x_0, \dots, x_n) = 0, \dots, f_h(x_0, \dots, x_n) = 0 \tag{9.1}$$

has some non–trivial solution.

Theorem 9.1.1 (Fundamental Theorem of Elimination Theory) *The set $Z(n; d_1, \dots, d_h)$ is closed in $\mathcal{L}_{n,d_1,\dots,d_h}$.*

Proof First of all recall (see Sect. 3.1) that the system (9.1) has no non–trivial solution if and only if there is a $d \in \mathbb{N} \setminus \{0\}$ such that $\mathcal{I} = (f_1, \dots, f_h) \supseteq S_{n,d}$. Let M_a, $a = 0, \dots, N(n,d)$ be the monomials of degree d in $x_0, \dots, x_n$. Then $S_{n,d} \subseteq \mathcal{I}$ is equivalent to say that we have relations of the sort

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$$M_a = \sum_{i=1}^{h} f_i(\mathbf{x}) F_{a,i}(\mathbf{x}), \quad \text{for} \quad a = 0, \ldots, N(n, d),$$

where $\mathbf{x} = (x_0, \ldots, x_n)$ and $F_{a,i}(\mathbf{x})$ are homogeneous polynomials of degree $d - d_i$ if $d - d_i \geqslant 0$, or zero if $d < d_i$. Let us denote by N_{i,a_i} the monomials of degree $d - d_i$ in $x_0, \ldots, x_n$, if $d - d_i > 0$, with $a_i = 0, \ldots, M(n, d - d_i)$. Then to say that $S_{n,d} \subseteq \mathcal{I}$ is equivalent to say that the polynomials $f_i N_{i,a_i}$ span $S_{n,d}$ as a $\mathbb{K}$–vector space, with $i = 1, \ldots, h$ and $a_i = 0, \ldots, M(n, d - d_i)$. Conversely, to say that $S_{n,d}$ is not contained in $\mathcal{I}$ is equivalent to say that the polynomials $f_i N_{i,a_i}$ do not span $S_{n,d}$ as a $\mathbb{K}$–vector space, with $i = 1, \ldots, h$ and $a_i = 0, \ldots, M(n, d - d_i)$, i.e., they form a system of rank $r \leqslant N(n, d)$. Let us order the coefficients of the polynomials $f_i N_{i,a_i}$ with $i = 1, \ldots, h$ and $a_i = 0, \ldots, M(n, d - d_i)$ in a matrix $\mathbf{A}$ of type $\sum_{i=1}^{h}(N(n, d - d_i) + 1) \times (N(n, d) + 1)$, where we set $N(n, d - d_i) = -1$ if $d < d_i$. Then the condition that $S_{n,d}$ is not contained in $\mathcal{I}$ is equivalent to say that all the minors of order $N(n, d) + 1$ of the matrix $\mathbf{A}$ are zero. On the other hand, each of these minors is in turn a polynomial function of the coefficients of $f_1, \ldots, f_h$. If we replace f_i with λf_i, each of these minors is multiplied by $\lambda^{N(n,d-d_i)+1}$. This shows that there are polynomials

$$g_{d,i}(\mathbf{a}_1, \ldots, \mathbf{a}_h), \quad i = 1, \ldots, m_d,$$

where $(\mathbf{a}_i)$ are the coefficients of the polynomial f_i (i.e., they are the homogeneous coordinates in $\mathcal{L}_{n,d_i}, i = 1, \ldots, h$), that are homogeneous of degree $N(n, d - d_i) + 1$ in each set of variables $\mathbf{a}_i$ for $i = 1, \ldots, h$, and the closed subset T_d of $\mathcal{L}_{n,d_1,\ldots,d_h}$ with equations

$$g_{d,i}(\mathbf{a}_1, \ldots, \mathbf{a}_h) = 0, \quad i = 1, \ldots, m_d,$$

is formed by all h–tuples $(f_1, \ldots, f_h)$ such that $S_{n,d}$ is not contained in the ideal $\mathcal{I} = (f_1, \ldots, f_h)$. Since $Z(n; d_1, \ldots, d_h) = \bigcap_{d \in \mathbb{N} \setminus \{0\}} T_d$, the assertion follows. □

From the proof of Theorem 9.1.1, it follows that the polynomials $g_{d,i}$ have coefficients in the fundamental field $\mathbb{F}$ of $\mathbb{K}$. Since $Z(n; d_1, \ldots, d_h)$ has equations

$$g_{d,i} = 0, \quad \text{for} \quad d \in \mathbb{N} \setminus \{0\}, \quad i = 1, \ldots, m_d,$$

it follows that there is a finite set $g_1, \ldots, g_t$ of plurihomogeneous polynomials in $\mathbf{a}_1, \ldots, \mathbf{a}_h$ such that

$$g_i(\mathbf{a}_1, \ldots, \mathbf{a}_h) = 0, \quad i = 1, \ldots, t \tag{9.2}$$

is a necessary and sufficient condition for the system (9.1) to have a non–trivial solution, where $f_1, \ldots, f_h$ have coefficients $\mathbf{a}_1, \ldots, \mathbf{a}_h$.

The plurihomogeneous ideal $\mathcal{I}(n; d_1, \ldots, d_h) = \mathcal{I}_s(Z(n; d_1, \ldots, d_h))$ is called the *resultant ideal* of h polynomials in $n + 1$ variables of degrees $d_1, \ldots, d_h$.

Exercise 9.1.2 Study $Z(1; n, m)$. One has to understand when the system

$$a_0x_0^n + a_1x_0^{n-1}x_1 + \cdots + a_nx_1^n = 0$$
$$b_0x_0^m + b_1x_0^{m-1}x_1 + \cdots + b_mx_1^m = 0$$

has a non–trivial solution. Here $[a_0, \ldots, a_n]$ have to be seen as homogeneous coordinates in $\mathcal{L}_{1,n}$ and $[b_0, \ldots, b_m]$ as homogeneous coordinates in $\mathcal{L}_{1,m}$. Prove that $Z(1; n, m)$ is defined by the vanishing of the Sylvester determinant, i.e., by the equation

$$\begin{vmatrix} a_0 & a_1 & \ldots & a_n & 0 & 0 & \ldots & 0 \\ 0 & a_0 & a_1 & \ldots & a_n & 0 & \ldots & 0 \\ & \ldots & & & & & & \\ 0 & \ldots & 0 & 0 & a_0 & a_1 & \ldots & a_n \\ b_0 & b_1 & \ldots & b_m & 0 & 0 & \ldots & 0 \\ 0 & b_0 & b_1 & \ldots & b_m & 0 & \ldots & 0 \\ & \ldots & & & & & & \\ 0 & \ldots & 0 & 0 & b_0 & b_1 & \ldots & b_m \end{vmatrix} = 0.$$

9.2 Morphisms on Projective Varieties Are Closed

An important application of the Fundamental Theorem of the Theory of Elimination, is the following:

Theorem 9.2.1 *Morphism are closed maps on projective varieties.*

Proof Let $V \subseteq \mathbb{P}^n$ be a projective variety, W a quasi–projective variety, $f : V \to W$ a morphism. Let us recall that the graph Γ_f is closed in $V \times W$ (see Exercises 8.2.11 and 8.2.12), and $f(V) = p_2(\Gamma_f)$, where $p_2 : V \times W \to W$ is the projection to the second factor. So, in order to prove the assertion it suffices to prove that p_2 is a closed map. To prove this it suffices to reduce to the case $V = \mathbb{P}^n$, because $V \times W$ is closed in $\mathbb{P}^n \times W$ and if Z is closed in $V \times W$ it is also closed in $\mathbb{P}^n \times W$. Moreover, since the concept of being closed is local (see Exercise 5.1.3), and since W is covered by affine open subsets, we can reduce to the case W is affine. Finally, arguing as we did before, we can actually assume that $W = \mathbb{A}^m$.

Let Z be a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$, so that it is defined by a system of equations of the form

$$f_i(x_0, \ldots, x_n; y_1, \ldots, y_m) = 0, \quad i = 1, \ldots, h \tag{9.3}$$

where the polynomials f_i are homogeneous of degrees d_i in the variables $x_0, \ldots, x_n$ for $i = 1, \ldots, h$ (see Exercise 8.2.9). Note that $(p_1, \ldots, p_m) \in \mathbb{A}^m$ belongs to $p_2(Z)$ if and only if the system

$$f_i(x_0, \ldots, x_n; p_1, \ldots, p_m) = 0, \quad i = 1, \ldots, h$$

has a non–trivial solution. Let

$$\mathbf{a}_i = (a_{i,0}(y_1, \ldots, y_m), \ldots, a_{i,N(n,d_i)}(y_1, \ldots, y_m))$$

be the vector of the coefficients of f_i as a homogeneous polynomial of degree d_i in $x_0, \dots, x_n$, for $i = 1, \dots, h$. Then the system (9.3) has non–trivial solution in $x_0, \dots, x_n$ if and only if one has

$$g_j(a_{i,0}(y_1, \dots, y_m), \dots, a_{i,N(n,d_i)}(y_1, \dots, y_m)) = 0, \quad j = 1, \dots, t \tag{9.4}$$

where the plurihomogeneous polynomials g_j are the ones in (9.2). Hence $p_2(Z)$ is defined by the system of equations (9.4) in $y_1, \dots, y_m$. The assertion follows. □

Exercise 9.2.2 Let $V \subseteq \mathbb{P}^n$ be a projective variety and consider the image of V via a projection to a subspace from a centre which does not intersect V. Prove that the image of V via this projection is a projective variety, which is called the *projection* of V to the given subspace from the given centre.

Exercise 9.2.3 Prove that a proper non–empty open subset of $\mathbb{P}^n$ is not isomorphic to a projective variety.

Exercise 9.2.4 Let V, W be varieties, V projective and let $f : V \to W$ be a dominant morphism. Prove that f is surjective.

Exercise 9.2.5 * Consider in $\mathcal{L}_{n,d}$ the set $R_{n,d}$ of reducible divisors. Prove that $R_{n,d}$ is a proper closed subset of $\mathcal{L}_{n,d}$. Prove also that the subset $\bar{R}_{n,d}$ of $\mathcal{L}_{n,d}$ of points corresponding to divisors having some multiple component is closed in $\mathcal{L}_{n,d}$.

9.3 Solutions of Some Exercises

9.2.5 For every positive integer $i < d$ consider the map

$$\phi_i : (F_1, F_2) \in \mathcal{L}_{n,i} \times \mathcal{L}_{n,d-i} \to F_1 + F_2 \in \mathcal{L}_{n,d}.$$

It is easy to see that this is a morphism. Then its image $R_{n,d;i}$ is a closed subset of $\mathcal{L}_{n,d}$. Then $R_{n,d} = \bigcup_{i=1}^{d-1} R_{n,d;i}$ is closed. The second assertion is proved in a similar way.

Chapter 10
Finite Morphisms

10.1 Definitions and Basic Results

Let V, W be affine varieties and let $f : V \to W$ be a dominant morphism, hence $f^* : A(W) \to A(V)$ is an injective homomorphism of $\mathbb{K}$-algebras. Via f^* we may identify $A(W)$ as a sub-ring of $A(V)$. We will say that f is a *finite morphism* if any element of $A(V)$ is integral over $A(W)$, in which case we will say that $A(V)$ is *integral* over $A(W)$.

Example 10.1.1 Let V be an irreducible hypersurface of $\mathbb{A}^n$ of degree d with equation

$$f(x_1, \dots, x_n) = x_n^d + x_n^{d-1} f_1(x_1, \dots, x_{n-1}) + \cdots + f_n(x_1, \dots, x_{n-1}) = 0$$

with f_i polynomial of degree at most i in $x_1, \dots, x_{n-1}$, for $i = 1, \dots, n$, so that the projective closure of V does not pass through the point at infinity of the x_n axis. Consider the projection p of V from the point at infinity of the x_n axis to the hyperplane $x_n = 0$, which can be identified with $\mathbb{A}^{n-1}$, i.e.,

$$p : (x_1, \dots, x_n) \in V \to (x_1, \dots, x_{n-1}) \in \mathbb{A}^{n-1}.$$

This is a morphism which is clearly surjective, hence dominant, and the counterimage of any point of $\mathbb{A}^{n-1}$ has at most order d. The morphism p corresponds to the injective homomorphism

$$p^* : A_{n-1} \to A(V) = \mathbb{K}[x_1, \dots, x_n]/(f)$$

such that $p^*(x_i) = x_i$ for all $i = 1, \dots, n-1$. Hence $A(V)$ is obtained by adding x_n to $A(W)$ and, since x_n is integral over $A(W)$, then $A(V)$ is integral over $A(W)$, hence p is a finite morphism.

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In order to prove some important properties of finite morphisms, we need an algebraic lemma:

Lemma 10.1.2 *Let B be a ring which is a finitely generated module over a subring A. Let $\mathcal{I}$ be a proper ideal of A. Then $\mathcal{I}B \neq B$.*

Proof Let $b_1, \ldots, b_n$ be a set of generators of B over A. If $\mathcal{I}B = B$, we have relations of the form

$$b_i = \sum_{j=1}^{n} a_{ij} b_j, \quad \text{with} \quad a_{ij} \in \mathcal{I} \quad \text{and} \quad i = 1, \ldots, n,$$

i.e.,

$$\sum_{j=1}^{n} (a_{ij} - \delta_{ij}) b_j = 0 \quad \text{for} \quad i = 1, \ldots, n, \tag{10.1.1}$$

where δ_{ij} is the *Kronecker symbol*. Let d be the determinant of the matrix $(a_{ij} - \delta_{ij})_{i,j=1,\ldots,n}$. From (10.1.1) one easily deduces that $db_i = 0$, for $i = 1, \ldots, n$ (see the solution of Exercise 5.4.4), hence we have $dB = \{0\}$ and therefore $d = 0$. But then, expanding the determinant d, we see that $1 \in \mathcal{I}$, so $\mathcal{I} = A$. □

We are now able to prove the following:

Theorem 10.1.3 *A finite morphism between affine varieties has the following properties:*

(a) it has finite fibres;
(b) it is surjective;
(c) it is a closed map.

Proof Let $f : V \to W$ be a finite morphism and take $Q \in W$. Suppose $V \subseteq \mathbb{A}^n$. To prove (a) it suffices to prove that every coordinate x_i, $i = 1, \ldots, n$, takes a finite number of values on the set $f^{-1}(Q)$. For every $i = 1, \ldots, n$, we have a relation of the type

$$x_i^{m_i} + b_{i1} x_i^{m_i - 1} + \cdots + b_{im_i} = 0,$$

with $b_{ij} \in A(W)$ and $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. If $P = (p_1, \ldots, p_n) \in f^{-1}(Q)$, we have

$$p_i^{m_i} + b_{i1}(Q) p_i^{m_i - 1} + \cdots + b_{im_i}(Q) = 0, \quad \text{with} \quad i = 1, \ldots, n$$

and this proves that p_i can take only finitely many values for all $i = 1, \ldots, n$, proving (a).

Let us prove (b). Let $Q \in W \subseteq \mathbb{A}^m$, and suppose $Q = (q_1, \ldots, q_m)$. Let $(y_1, \ldots, y_m)$ be the coordinates in $\mathbb{A}^m$. Let $f^*(y_i) = f_i(x_1, \ldots, x_n) \in A(V)$, for $i = 1, \ldots, m$. Then $P = (p_1, \ldots, p_n)$ belongs to $f^{-1}(Q)$ if and only if $f_i(p_1, \ldots, p_n) = q_i$, for

$i = 1, \ldots, m$. If $\mathfrak{m}_Q = (y_1 - q_1, \ldots, y_m - q_m)$ is the maximal ideal of $A(W)$ corresponding to the point Q, then $P \in f^{-1}(Q)$ if and only if $g(P) = 0$ for every $g \in \mathcal{J} = \mathfrak{m}_Q A(V)$. If $f^{-1}(Q) = \emptyset$, we would have $\mathcal{J} = A(V)$, since there would be no maximal ideal of $A(V)$ containing $\mathcal{J}$. Then by Exercise 5.4.7 and Lemma 10.1.2 we would have a contradiction.

Finally, let us prove (c). To prove this it suffices to prove that if $Z \subseteq V$ is an irreducible closed subset, then $f(Z)$ is closed in W. Consider the map $\bar{f} = f_{|Z} : Z \to \overline{f(Z)}$. We claim this map is finite. We have the following commutative diagram

$$\begin{array}{ccc} A(W) & \xrightarrow{\beta} & A(\overline{f(Z)}) \\ f^* \downarrow & & \downarrow \bar{f}^* \\ A(V) & \xrightarrow{\alpha} & A(Z) \end{array}$$

where α and β are clearly surjective. If $g \in A(Z)$ and $G \in A(V)$ is such that $g = \alpha(G)$, we have that G is integral over $A(W)$, i.e., we have a relation of the type

$$G^h + f^*(a_1)G^{h-1} + \cdots + f^*(a_h) = 0$$

with $a_1, \ldots, a_h \in A(W)$. Hence

$$g^h + \alpha(f^*(a_1))g^{h-1} + \cdots + \alpha(f^*(a_h)) = 0.$$

Because of the commutativity of the diagram, this reads

$$g^h + \bar{f}^*(\beta(a_1))g^{h-1} + \cdots + \bar{f}^*(\beta(a_h)) = 0$$

and this proves that g is integral over $A(\overline{f(Z)})$. But then by (b) the map $\bar{f}$ is surjective, hence $f(Z) = \overline{f(Z)}$. □

The following theorem proves that finiteness is a local property:

Theorem 10.1.4 *Let V, W be affine varieties and $f : V \to W$ a morphism. The following propositions are equivalent:*

(a) f is a finite morphism;
(b) for every point $P \in W$ there is an open affine neighborhood U of P in W such that $U' = f^{-1}(U)$ is affine and $f : U' \to U$ is a finite morphism.

Proof Clearly (a) implies (b). Let us prove the converse. First of all, by taking into account Exercises 5.1.6, 6.3.6 and 10.1.8 below, we may assume that for every point $P \in W$ there is a principal open affine neighborhood U of P in W such that $U' = f^{-1}(U)$ is principal affine and $f : U' \to U$ is a finite morphism. So, since W is compact, there are finitely many, non-zero elements of $A(W)$, $g_1, \ldots, g_n$, such that the morphisms induced by f

$$f_i : U'_i = U_V(f^*(g_i)) \to U_i = U_W(g_i), \quad i = 1, \ldots, n,$$

are finite and $\{U_i\}_{i=1,\dots,n}$ form a cover of W. Note that

$$\mathcal{O}(U_i) = A(U_W(g_i)) = A(W)\Big[\frac{1}{g_i}\Big], \quad \mathcal{O}(U_i') = A(U_V(f^*(g_i)) = A(V)\Big[\frac{1}{g_i}\Big],$$

for $i = 1, \dots, n$, (see Lemma 6.3.1), where, as usual, we may think to the injective map f^* as an identification, so that

$$f_i^* : A(W)\Big[\frac{1}{g_i}\Big] \to A(V)\Big[\frac{1}{g_i}\Big], \quad i = 1, \dots, n,$$

is the map naturally induced by f^*. By the hypotheses and by Exercise 5.4.7, $A(V)[\frac{1}{g_i}]$ has a finite basis ω_{ij} over $A(W)[\frac{1}{g_i}]$, for $i = 1, \dots, n$, $j = 1, \dots, n_i$, where we may suppose that $\omega_{ij} \in A(V)$. We now show that $\{\omega_{ij}\}_{i=1,\dots,n,j=1,\dots,n_i}$ is a basis of $A(V)$ over $A(W)$, which will imply the assertion by Exercise 5.4.6. Let $b \in A(V)$. For all $i = 1, \dots, n$ we have

$$b = \sum_{j=1}^{n_i} \frac{a_{ij}}{g_i^{m_i}} \omega_{ij},$$

with m_j suitable positive integers and $a_{ij} \in A(W)$. Since $\bigcap_{i=i}^n Z_V(g_i^{m_i}) = \emptyset$, because $\{U_i'\}_{i=1,\dots,n}$ is an open cover of V, there is no maximal ideal of $A(W)$ containing $\mathcal{I} = (g_1^{m_1}, \dots, g_n^{m_n})$, hence $\mathcal{I} = A(W)$. So there are $h_1, \dots, h_n \in A(W)$ such that

$$\sum_{i=1}^{n} h_i g_i^{m_i} = 1.$$

Then we have

$$b = b\Big(\sum_{i=1}^{n} h_i g_i^{m_i}\Big) = \sum_{i=1}^{n} h_i g_i^{m_i} b = \sum_{i=1}^{n}\sum_{j=1}^{m_i} a_{ij} h_i \omega_{ij},$$

which proves the assertion. □

Theorem 10.1.4 suggests the way of extending the notion of finite morphism to morphisms between quasi-projective varieties. Let V, W be quasi-projective varieties and $f : V \to W$ a morphism. We will say that f is *affine* if for every point $P \in W$ there is an open neighborhood U of P in W, such that $U' = f^{-1}(U)$ is affine. If, in addition $f_{|U'} : U' \to U$ is finite, one says that f is *finite*. Of course Theorem 10.1.3 still holds for finite morphisms as defined above. Finally, let $\phi : V \dashrightarrow W$ be a rational map (in particular a morphism). One says that ϕ is *generically finite* if there are non-empty open subsets U of W and U' of V such that ϕ induces a finite morphism $\phi_{|U'} : U' \to U$.

Theorem 10.1.5 *Let $\phi : V \dashrightarrow W$ be a dominant rational map between quasi-projective varieties. Then ϕ is generically finite if and only if the homomorphism $\phi^* : K(W) \to K(V)$ is an algebraic extension, i.e., if and only if* $\dim_{\mathrm{tr}}(V) = \dim_{\mathrm{tr}}(W)$.

Proof It suffices to prove the assertion in the case in which V, W are affine and ϕ is a morphism. If ϕ is generically finite, it is clear that $\phi^* : K(W) \to K(V)$ is an algebraic extension and then $\dim_{\mathrm{tr}}(V) = \dim_{\mathrm{tr}}(W)$. Let us prove the converse. Let $g \in K(V)$, so that g is algebraic over $K(W)$, hence over $A(W)$. In particular, every $g \in A(V)$ is algebraic over $A(W)$, so that we have a relation of the sort

$$a_0 g^n + a_1 g^{n-1} + \cdots + a_n = 0,$$

with $a_0, \ldots, a_n \in A(W)$ and $a_0 \neq 0$. Then we have

$$a_0^n g^n + a_1 a_0^{n-1} g^{n-1} + \cdots + a_n a_0^{n-1} = 0$$

hence $a_0 g$ is integral over $A(W)$. Let $b_1, \ldots, b_m$ be a system of generators of $A(V)$ as a $\mathbb{K}$-algebra and let $a_1, \ldots, a_m \in A(W)$ such that $a_i b_i$ is integral over $A(W)$, for $i = 1, \ldots, m$. Set $F = a_1 \cdots a_m$ and let $U = U_W(F)$ and $U' = U_V(\phi^*(F))$. In relation with the morphism $\phi_{|Y'} : U' \to U$ we have the inclusion which extends ϕ^*

$$\bar{\phi}^* : A(W)\Big[\frac{1}{F}\Big] \to A(V)\Big[\frac{1}{F}\Big].$$

An element of $A(V)[\frac{1}{F}]$ is of the form $c = \frac{b}{F^h}$, with $b \in A(V)$ and h a positive integer. Since $b_1, \ldots, b_m$ are integral over $A(W)[\frac{1}{F}]$, because $a_1, \ldots, a_m$ are invertible in $A(W)[\frac{1}{F}]$, and since b is a combination of products of $b_1, \ldots, b_m$ with coefficients in $\mathbb{K}$, it follows that $A(V)[\frac{1}{F}]$ is integral over $A(W)[\frac{1}{F}]$, which proves the assertion. □

We finish this section with the following:

Theorem 10.1.6 *Let $f : V \to W$ be a dominant morphism between quasi-projective varieties. Then $f(V)$ contains a non-empty open subset of W.*

Proof It suffices to reduce to the case in which both V, W are affine. Then $K(V)$ is an extension of $K(W)$. Let $u_1, \ldots, u_r$ be a transcendence basis of $K(V)$ over $K(W)$ and we may assume $u_1, \ldots, u_r \in A(V)$. Then we have the chain of inclusions

$$A(W) \xrightarrow{\alpha} A(W)[u_1, \ldots, u_r] \xrightarrow{\beta} A(V)$$

such that $f^* = \beta \circ \alpha$. Note that $A(W)[u_1, \ldots, u_r]$ is isomorphic to $A(W) \otimes_{\mathbb{K}} A_r$, hence $A(W)[u_1, \ldots, u_r] = A(W \times \mathbb{A}^r)$, and α corresponds to the projection p_1 of $W \times \mathbb{A}^r$ to the first factor. Then β corresponds to a generically finite morphism $g : V \to W \times \mathbb{A}^r$. By Theorem 10.1.5 there is a non-empty open subset U of $W \times \mathbb{A}^r$, such that $U \subseteq g(V)$. We may assume that U is a principal set, i.e., $U = U_{W \times \mathbb{A}^r}(F)$, with

$$F = \sum_{i_1 \dots i_r} g_{i_1 \dots i_r} u_1^{i_1} \cdots u_r^{i_r}, \quad \text{with} \quad g_{i_1 \dots i_r} \in A(W) \quad \text{not all zero.}$$

Let $P \in W$ which does not belong to $\bigcup_{i_1 \dots i_r} Z_W(g_{i_1 \dots i_r})$. Then there is some $Q \in W \times \mathbb{A}^r$ such that $F(Q) \neq 0$, so that $Q \in U$ and $P = p_1(Q) \in p_1(U)$. Therefore $p_1(U) \supseteq W \setminus \bigcup_{i_1 \dots i_r} Z_W(g_{i_1 \dots i_r})$ so that

$$f(V) = p_1(g(V)) \supseteq p_1(U) \supseteq W \setminus \bigcup_{i_1 \dots i_r} Z_W(g_{i_1 \dots i_r})$$

and $W \setminus \bigcup_{i_1 \dots i_r} Z_W(g_{i_1 \dots i_r})$ is a non-empty open subset of W because at least one of the functions $g_{i_1 \dots i_r}$ is non-zero. □

Exercise 10.1.7 * Let A, B be finitely generated $\mathbb{K}$-algebras, suppose that $A \subseteq B$ and that B is integral over A. Let $x \in \mathbb{Q}(B)$ be integral over B. Prove that x is also integral over A.

Exercise 10.1.8 * Let V, W be affine varieties and $f : V \to W$ a finite morphism. Let $g \in A(W) \setminus \{0\}$ and $U_W(g)$ be the principal (affine) open subset associated to g (see Sect. 5.1). Prove that $f^{-1}(U_W(g))$ is also a principal (affine) open subset of V and that the map $f_{|f^{-1}(U_W(g))} : f^{-1}(U_W(g)) \to U_W(g)$ is a finite morphism.

Exercise 10.1.9 * Let V be a hypersurface in $\mathbb{A}^n$ and consider the projection p of V to an hyperplane from a point at infinity which is not in the projective closure of V. Prove that p is a finite map.

Exercise 10.1.10 Consider an irreducible quadric of $\mathbb{A}^n$ with equation

$$f(x_1, \dots, x_n) = x_n f_1(x_1, \dots, x_{n-1}) + f_2(x_1, \dots, x_{n-1}) = 0$$

where f_i is a polynomial of degree at most i in $x_1, \dots, x_{n-1}$, for $i = 1, 2$, with f_1 non-zero and not dividing f_2. Consider the projection p of V from the point at infinity of the x_n axis to the hyperplane $x_n = 0$ identified with $\mathbb{A}^{n-1}$. Prove that p is a finite morphism if and only if f_1 is a non-zero constant.

Note that in this case p is finite if and only if p is surjective.

Exercise 10.1.11 * Prove that the composition of two finite morphisms between affine varieties is finite.

Exercise 10.1.12 * Let V, W be affine varieties and $f : V \to W$ a finite morphism. Prove that $\dim_{\mathrm{tr}}(V) = \dim_{\mathrm{tr}}(W)$.

Exercise 10.1.13 Give an example of a morphism which is surjective, closed and with finite fibres which is not finite.

Exercise 10.1.14 * Prove that the restriction of a finite morphism between affine varieties to a closed subvariety is still finite onto its image.

Exercise 10.1.15 Prove that any morphism between affine varieties is affine.

Exercise 10.1.16 Prove that the composition of finite morphisms between quasi-projective varieties is finite.

Exercise 10.1.17 Prove that if V, W are quasi-projective varieties and $f : V \to W$ is a finite morphism, then $\dim_{tr}(V) = \dim_{tr}(W)$.

Exercise 10.1.18 Prove that there are morphisms which are not affine.

Exercise 10.1.19 * Let $V \subset \mathbb{P}^n$ be an irreducible hypersurface of degree d, let P be a point which does not belong to V and H a hyperplane not containing P. Consider the restriction p to V of the projection of $\mathbb{P}^n$ to H from P. Prove that p is a finite morphism.

Exercise 10.1.20 Prove that an irreducible ipersurface of $\mathbb{A}^n$ or of $\mathbb{P}^n$ has transcendent dimension $n - 1$.

Exercise 10.1.21 Let V be an irreducible ipersurface of $\mathbb{P}^n$, let $P \in V$ and consider the projection p of V to a hyperplane H from P, that is a rational map. Prove that p is generically finite, unless V is a cone with vertex P.

10.2 Projections and Noether's Normalization Theorem

Let $V \subseteq \mathbb{P}^n$ be a projective variety and let $\mathbb{P}_1, \mathbb{P}_2$ be two subspaces of $\mathbb{P}^n$ of dimensions r and $n - r - 1$ respectively, which are skew. Suppose that $\mathbb{P}_1 \cap V = \emptyset$. We can then consider the projection

$$p : V \to \mathbb{P}_2$$

of V to $\mathbb{P}_2$ from $\mathbb{P}_1$ (see Exercise 7.1.12), which is a morphism and whose image $V' = p(V)$ is a projective subvariety of $\mathbb{P}_2$ by Theorem 9.2.1. The projection p is called an *external projection* of V.

Theorem 10.2.1 *An external projection $p : V \to V'$ is a finite morphism.*

Proof We use the above notation, and assume that p is the projection of $V \subseteq \mathbb{P}^n$ from $\mathbb{P}_1$, of dimension r, to $\mathbb{P}_2$ of dimension $n - r - 1$, with $\mathbb{P}_1 \cap V = \emptyset$. We fix homogeneous coordinates $[y_0, \dots, y_{n-r-1}]$ on $\mathbb{P}_2$ and, as usual, we denote by $U_i \cong \mathbb{A}^{n-r-1}$ the open subset of $\mathbb{P}_2$ where $y_i \neq 0$, and we set $V_i' = U_i \cap V'$ for all $i = 0, \dots, n - r - 1$ and $V_i = p^{-1}(V_i')$, for $i = 0, \dots, n - r - 1$. For all $i = 0, \dots, n - r - 1$, both V_i and V_i' are affine varieties. To prove the assertion we will prove that $p_{|V_i} : V_i \to V_i'$ is a finite morphism, for all $i = 0, \dots, n - r - 1$. We will consider the case $i = 0$, because the other cases are analogous.

We may suppose that the morphism p is given by relations of the type

$$y_i = f_i(x_0, \dots, x_n), \quad \text{for} \quad i = 0, \dots, n - r - 1$$

where $[x_0, \dots, x_n]$ are homogeneous coordinates in $\mathbb{P}^n$ and $f_1, \dots, f_{n-r-1}$ are linear forms. Since $\mathbb{P}_1 \cap V = \emptyset$, the system

$$f_0 = \cdots = f_{n-r-1} = 0 \tag{10.2.1}$$

has no non-trivial solution on V.

Let $g \in A(V_0)$. We want to prove that g is integral over $A(V_0')$. The function g is the restriction to V_0 of a function of the form

$$\frac{G(x_0, \dots, x_n)}{f_0(x_0, \dots, x_n)^m}$$

where m is a suitable positive integer and G is a homogeneous polynomial of degree m.

Consider the map $q : V \to \mathbb{P}^{n-r}$ given by the formulas

$$z_i = f_i^m, \quad \text{for} \quad i = 0, \dots, n-r-1, \quad \text{and} \quad z_{n-r} = G.$$

where $[z_0, \dots, z_{n-r}]$ are homogeneous coordinates in $\mathbb{P}^{n-r}$. Since the system (10.2.1) has no solution on V, the map q is a morphism, and we set $V'' = q(V)$, which is a closed subvariety of $\mathbb{P}^{n-r}$. We let $h_1, \dots, h_l$ be a basis of the ideal $\mathcal{I}_p(V'')$.

Since the system (10.2.1) has no solution on V, the point $P = [0, \dots, 0, 1] \in \mathbb{P}^{n-r}$ does not belong to V''. This means that the system

$$z_0 = \cdots = z_{n-r-1} = h_1 = \cdots = h_l = 0$$

has no non-trivial solutions. Then there is an integer $d > 0$ such that the ideal $(z_0, \dots, z_{n-r-1}, h_1, \dots, h_l)$ contains $S_{n-r,d}$ (see Sect. 3.1), in particular $z_{n-r}^d \in (z_0, \dots, z_{n-r-1}, h_1, \dots, h_l)$, i.e., we have a relation of the form

$$z_{n-r}^d = \sum_{i=0}^{n-r-1} z_i P_i + \sum_{i=1}^{l} h_i Q_i$$

with $P_1, \dots, P_{n-r-1}, Q_1, \dots, Q_l$ suitable homogeneous polynomials of the appropriate degrees, in particular $P_1, \dots, P_{n-r-1}$ have degree $d-1$. Then

$$H(z_0, \dots, z_{n-r}) = z_{n-r}^d - \sum_{i=0}^{n-r-1} z_i P_i$$

vanishes on V''. The homogeneous polynomial H of degree d is a monic polynomial in z_{n-r} and we can write

$$H = z_{n-r}^d - \sum_{i=0}^{d-1} a_i(z_0, \dots, z_{n-r-1}) z_{n-r}^i$$

with $a_i(z_0, \dots, z_{n-r-1})$ homogeneous polynomial of degree $d-i$ for $i = 0, \dots, d-1$. From the fact that H vanishes on V'' we deduce that

$$H(f_0^m, \dots, f_{n-r-1}^m, G) = 0 \quad \text{on} \quad V.$$

Dividing this relation by f_0^{md} we find the required integral dependence relation

$$g^d - \sum_{i=0}^{d-1} a_i(1, \xi_1 \dots, \xi_{n-r-1}) g^i = 0$$

where $\xi_i = \frac{y_i}{y_0} \in A(V_0')$, for $i = 1, \dots, n-r-1$. □

An interesting consequence of this result is the:

Corollary 10.2.2 *Let $V \subseteq \mathbb{P}^n$ be a projective variety. Let $f_0, \dots, f_h \in S_{n,d}$ be linearly independent polynomials such that $V \cap Z_p(f_0, \dots, f_h) = \emptyset$. Then the morphism*

$$\phi : P \in V \to [f_0(P), \dots, f_h(P)] \in \mathbb{P}^h$$

is a finite morphism onto its image.

Proof Consider the dual Veronese morphism

$$\check{v}_{n,d} : \mathbb{P}^n \to \check{\mathcal{L}}_{n,d}$$

(see Sect. 6.4), that is an isomorphism of $\mathbb{P}^n$ onto its image. Let us set $V' = \check{v}_{n,d}(V)$. The polynomials $f_0, \dots, f_h$ are the images via the homomorphism $\vartheta_{n,d}$ (see Sect. 6.3) of linearly independent linear forms $F_0, \dots, F_h$ in $S_{N(n,d),1}$, so that we have to prove that the morphism

$$\psi : P \in V' \to [F_0(P), \dots, F_h(P)] \in \mathbb{P}^h$$

is finite onto its image. So we are reduced to the case $d = 1$. In this case ϕ is the restriction to V of a degenerate projectivity τ (see Exercise 7.1.11) with centre $\mathbb{P}_1$ such that $\mathbb{P}_1 \cap V = \emptyset$. On the other hand τ is composed of a projection and of projectivities (see Exercise 7.1.13). The assertion follows from Theorem 10.2.1. □

Remark 10.2.3 By taking into account the proof of Theorem 10.2.1, we see that Corollary 10.2.2 says more than stated. The full result is that, if $[x_0, \dots, x_h]$ are homogeneous coordinates in $\mathbb{P}^h$ and if $U_i \cong \mathbb{A}^h$ is the open subset of $\mathbb{P}^h$ where $x_i \neq 0$, for $i = 0, \dots, h$, then the morphism $\phi_{|\phi^{-1}(U_i)} : \phi^{-1}(U_i) \to \phi(V) \cap U_i$ is a finite morphism of affine varieties.

Corollary 10.2.4 *Let $V \subseteq \mathbb{P}^n$ be a projective variety of transcendent dimension m. Then there is a finite morphism $p : V \to \mathbb{P}^m$.*

Proof It suffices to compose external projections. □

Remark 10.2.5 Let us keep the notation of Corollary 10.2.4 and set $m = \dim_{\text{tr}}(V)$. Then Corollary 10.2.4 asserts that there is a subspace $\mathbb{P}_1$ of $\mathbb{P}^n$ of dimension $n - m - 1$ such that $\mathbb{P}_1 \cap V = \emptyset$, such that the projection of V from $\mathbb{P}_1$ to a skew subspace $\mathbb{P}_2$ of dimension m is finite.

Corollary 10.2.6 *Let $V \subseteq \mathbb{P}^n$ be a projective variety of transcendent dimension m. Let $r \geqslant -1$ be the maximum integer such that there are subspaces of $\mathbb{P}^n$ of dimension r with empty intersection with V. Then $r = n - m - 1$.*

Proof By Remark 10.2.5, there are subspaces of dimension $n - m - 1$ with empty intersection with V. To finish we have to prove that if $r > n - m - 1$, any subspace with dimension r has non-empty intersection with V. Let us argue by contradiction. Suppose there is subspace $\mathbb{P}_1$ with dimension $r > n - m - 1$ with empty intersection with V. Let us fix a subspace $\mathbb{P}_2$ of dimension $n - r - 1 < m$ and let us consider the external projection of V to $\mathbb{P}_2$ from $\mathbb{P}_1$. Then this would be a finite morphism to its image, whose dimension would be at most $n - r - 1 < m = \dim_{\text{tr}}(V)$, a contradiction. □

Another consequence of Corollary 10.2.4 is the following result:

Corollary 10.2.7 *(*Emmy Noether's Normalization Theorem*) Let V be an affine variety with $\dim_{\text{tr}}(V) = m$. Then there is a finite morphism of V onto $\mathbb{A}^m$. In algebraic terms, given any finitely generated $\mathbb{K}$-algebra A with no zero divisors such that $\mathbb{Q}(A)$ has transcendence degree m on $\mathbb{K}$, there is an injective homomorphism $A_m \to A$ such that A is integral over A_m.*

Proof One proceeds as in the projective case, by projecting from points at infinity which do not belong to the projective closure of V. □

10.3 Normal Varieties and Normalization

Let V be a quasi-projective variety and let $P \in V$. One says that V is *normal* at P if the ring $\mathcal{O}_{V,P}$ is integrally closed. One says that V is *normal* if V is normal at any point $P \in V$. In order to explain the meaning of this definition, we need some results of algebra.

Lemma 10.3.1 *Let A, B be rings with A a subring of B and let C be the integral closure of A in B. Let S be a multiplicatively closed subset of A. Then C_S is the integral closure of A_S in B_S.*

Proof It is easy to see that C_S is integral over A_S. If $\frac{b}{s} \in B_S$ is integral over A_S, we have a relation of the type

$$\left(\frac{b}{s}\right)^n + \frac{a_1}{s_1}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_n}{s_n} = 0$$

with $a_i \in A$, $s_i \in S$, $i = 1, \ldots, n$. Set $t = s_1 \cdots s_n$, and multiply both members of the above relation by $(st)^n$. Then $bt \in C$ and $\frac{b}{s} = \frac{bt}{st} \in C_S$. □

Lemma 10.3.2 *Let A be a domain. The following are equivalent:*

(a) A is integrally closed;
(b) for every prime ideal $\mathcal{I} \subseteq A$, the ring $A_{\mathcal{I}}$ is integrally closed;
(c) for every maximal ideal $\mathfrak{m} \subseteq A$, the ring $A_{\mathfrak{m}}$ is integrally closed.

Proof By Lemma 10.3.1, (a) implies (b). Moreover (b) trivially implies (c). Let us prove that (c) implies (a). This follows from the fact that $A = \bigcap A_{\mathfrak{m}}$, where the intersection is made on all the maximal ideals of A (see Exercise 5.5.9). □

Proposition 10.3.3 *Let V be a quasi-projective variety:*

(a) if V is affine, then V is normal if and only if $A(V)$ is integrally closed;
(b) V is normal if and only if there is an open cover $\{U_i\}_{i \in \mathcal{I}}$ of normal affine subsets of V.

Proof Part (a) follows from Lemma 10.3.2. Part (b) is obvious. □

Let now V be a quasi-projective variety. We will say that the pair (V', ϕ) is a *normalization* of V, if V' is a normal variety and $\phi : V' \to V$ is a birational, finite morphism. Often one says that V' is a normalization of V, when ϕ is understood.

Let us state the following result of algebra, for which see [9, Theorem 9, p. 267]:

Theorem 10.3.4 *(*Finiteness of Integral Closure*) Let A be a finitely generated $\mathbb{K}$-algebra with no zero divisors and let K be a finite extension of $\mathbb{Q}(A)$. The integral closure of A in K is an A-module of finite type and it is also a finitely generated $\mathbb{K}$-algebra.*

This theorem implies the following:

Theorem 10.3.5 *Let V be an affine variety. Then:*

(a) there is a normalization (V', ϕ) of V with V' affine, which has the following properties:

  (a1) if W is an affine variety and $g : W \to V$ is a finite birational morphism, then there is a unique morphism $h : V' \to W$ such that $\phi = g \circ h$;
  (a2) if W is a normal affine variety and $g : W \to V$ is a dominant morphism, then there is a unique morphism $h : W \to V'$ such that $g = \phi \circ h$;

(b) if (V'', ψ) is still a normalization of V, there is an isomorphism $g : V' \to V''$ such that $\phi = \psi \circ g$.

Proof From Theorem 10.3.4 it follows that the integral closure B of $A(V)$ is a finitely generated $\mathbb{K}$-algebra with no zero divisors. Hence there is an affine variety V' such that $B = A(V')$, and accordingly there is a finite, birational morphism $\phi : V' \to V$. Hence (V', ϕ) is a normalization of V. If W verifies the hypotheses of (a1), there is an inclusion $A(V) \subseteq A(W) \subseteq K(V)$. Since $A(W)$ is integral on $A(V)$, we have an inclusion $A(W) \subseteq B = A(V')$ and accordingly we have the morphism $h : V' \to W$ such that $\phi = g \circ h$. Finally, if W is a variety verifying the hypotheses of (a2), we have $K(V) \subseteq K(W)$ and $A(V) \subseteq A(W)$. If $f \in B = A(V')$, then $f \in K(W)$ and it is integral over $A(V)$, so it is integral over $A(W)$, thus $f \in A(W)$. Then we have $A(V') \subseteq A(W)$ and accordingly we have a morphism $h : W \to V'$ such that $g = \phi \circ h$. Part (b) follows from (a1) and (a2). □

Exercise 10.3.6 Prove that $\mathbb{A}^n$, $\mathbb{P}^n$ and the Segre varieties are normal.

Exercise 10.3.7 Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Prove that any irreducible quadric in $\mathbb{P}^3$ is normal.

Exercise 10.3.8 Prove that the projective plane curves with equations $x_0x_1^2 = x_3$ and $x_2^2x_0 = x_1^2(x_1 + x_0)$ are not normal.

Exercise 10.3.9 A projective variety V is said to be *projectively normal* if $S(V)$ is integrally closed. Prove that $\mathbb{P}^n$, the Veronese varieties, the Segre varieties are projectively normal. Prove that a projectively normal variety is normal.

Exercise 10.3.10 * Prove that there are isomorphic projective varieties one of which is projectively normal the other is not, so that projective normality is not an intrinsic property, but depends on the immersion in projective space.

Exercise 10.3.11 Consider the affine plane curves V and V' with equations $x_1^2 = x_3$ and $x_2^2 = x_1^2(x_1 + 1)$, which are not normal (see Exercise 10.3.8). Consider the morphisms

$$\phi : t \in \mathbb{A}^1 \to (t^3, t^2) \in V, \quad \phi' : t \in \mathbb{A}^1 \to (t^2 - 1, t^3 - t) \in V'.$$

Prove that $(\mathbb{A}^1, \phi)$ and $(\mathbb{A}^1, \phi')$ are normalizations of V and V' respectively.

Exercise 10.3.12 * Let V be a quasi-projective variety. Prove that the set $N(V)$ of points P of V such that V is not normal in P is a proper closed subset of V.

10.4 Ramification

In this section we will look at the following question: given a finite morphism $\phi : V \to W$, what is the degree of the fibres of ϕ? We start with a definition. Let $\phi : V \dashrightarrow W$ be a generically finite rational map between quasi-projective varieties. Then $\phi^* : K(W) \to K(V)$ is an algebraic extension. We define *degree* of ϕ, denoted by $\deg(\phi)$, the degree of the field extension $\phi^* : K(W) \to K(V)$. Moreover we will say that ϕ is *separable* or *inseperable* if so is the field extension $\phi^* : K(W) \to K(V)$.

In order to prove our next result, we need some algebraic preliminaries.

Lemma 10.4.1 *Let A be a noetherian subring of a domain B and let C be the integral closure of A in B. Let $f, g \in B[x]$ be monic polynomials such that $fg \in C[x]$. Then $f, g \in C[x]$.*

Proof In the algebraic closure of $\mathbb{Q}(B)$ we have

$$f(x) = \prod_{i=1}^{n}(x - \xi_i), \quad g(x) = \prod_{i=1}^{m}(x - \eta_i).$$

Now $\xi_i, \eta_j, i = 1, \ldots, n, j = 1, \ldots, m$, are roots of $f(x)g(x)$, hence they are integral over C. So the coefficients of $f(x)$ and $g(x)$, which are polynomial expressions in the ξ_i and in the η_j respectively, are elements of B integral over C, so they belong to C (see Exercise 5.4.8). □

Proposition 10.4.2 *Let $A \subseteq B$ be noetherian domains with A integrally closed and B integral on A. Then the extension $\mathbb{Q}(A) \to \mathbb{Q}(B)$ is algebraic and for every element $b \in B$, its minimal polynomial over $\mathbb{Q}(A)$ has coefficients in A.*

Proof Let $b \in B$ and let $f(x) \in A[x]$ be a monic polynomial of minimal degree which has b as a root. If $f(x)$ is not the minimal polynomial of b on $\mathbb{Q}(A)$, there are monic polynomials $g, h \in \mathbb{Q}(A)[x]$ with $\deg(g), \deg(h)$ both positive, such that $f = gh$. By Lemma 10.4.1 with $B = \mathbb{Q}(A)$ and $C = A$, we have that $g, h \in A[x]$, and this is a contradiction. □

Theorem 10.4.3 *Let $f : V \to W$ be a finite morphism between quasi-projective varieties, with W normal. Then for all $P \in W$, the fibre $f^{-1}(P)$ consists of at most* $\deg(f)$ *distinct points.*

Proof We can reduce to the case in which both V, W are affine. Set $A = A(V)$, $B = A(W)$, $K = \mathbb{Q}(A) = K(V)$, $L = \mathbb{Q}(B) = K(W)$ and $n = \deg(f) = [K : L]$. If $a \in A \subseteq K$, since A is integral on B and B is integrally closed, then by Proposition 10.4.2 the minimal polynomial of a over L has coefficients in B. Let us set $f^{-1}(P) = \{Q_1, \ldots, Q_m\}$ and let us choose an element $a \in A$ such that $a(Q_1), \ldots, a(Q_m)$ are distinct elements of $\mathbb{K}$.

The existence of a is proved by showing that, if $V \subseteq \mathbb{A}^n$, there is a polynomial $F \in A_n$ which takes different values on $Q_1, \ldots, Q_m$. This is trivial if $m = 1$. If $m > 1$ one proceeds by induction on m. Let F_1 be such a polynomial for $Q_1, \ldots, Q_{m-1}$. If $F_1(Q_m)$ is different from $F_1(Q_1), \ldots, F_1(Q_{m-1})$, we take $F = F_1$. If $F_1(Q_m)$ is equal to one of the values $F_1(Q_1), \ldots, F_1(Q_{m-1})$, let G be a polynomial, which certainly exists, such that $G(Q_i) = 0$ for $i = 1, \ldots, m-1$ and $G(Q_m) \neq 0$. Then take $F = F_1 + hG$, with h suitable in $\mathbb{K}^*$. Indeed $F(Q_i) = F_1(Q_i)$, for $i = 1, \ldots, m-1$, are $m-1$ distinct values in $\mathbb{K}$, whereas $F(Q_m) = F_1(Q_m) + hG(Q_m)$ takes infinitely many values of $\mathbb{K}$, when h varies. So we can choose h so that $F(Q_m)$ is different from $F(Q_i)$ for $i = 1, \ldots, m-1$.

Let now $P(T) \in B[T]$ be the minimal polynomial of a. Of course $l = \deg(P(T)) \leqslant [K : L] = n$. If

$$P(T) = T^l + a_1 T^{l-1} + \cdots + a_l, \quad a_i \in B,$$

let us consider the polynomial on $\mathbb{K}$

$$\bar{P}(T) = T^l + a_1(P)T^{l-1} + \cdots + a_l(P), \quad a_i \in B,$$

which has the m roots $a(Q_1), \ldots, a(Q_m)$, so that $m \leqslant l \leqslant n$. □

A finite morphism $f : V \to W$ of quasi-projective varieties is said to be *not branched* at $P \in W$, if the fibre $f^{-1}(P)$ consists exactly of $\deg(f)$ distinct points. If this is not the case, one says that f is *branched* at P, and P is called a *branch point* for f.

Theorem 10.4.4 *Let $f : V \to W$ be a finite morphism between quasi-projective varieties, with W normal. Then:*

(a) if f is inseparable, then f is branched at all points of W;
(b) if f is separable, the branch points for f in W form a proper closed subset of W.

Proof We keep the notation of the proof of Theorem 10.4.3. Let Q be a point of W where f is not branched. Then $l = n$ and $\bar{P}(T)$ has n distinct roots. This means that the *discriminant* $D(\bar{P})$ of $\bar{P}(T)$, i.e., the resultant of $\bar{P}(T)$ and of the derivative $\bar{P}'(T)$, is not zero. Consider the discriminant $D(P) \in B$ of $P(T)$. Since $D(\bar{P}) = D(P)(Q)$, we have $D(P) \neq 0$. If f is inseparable, this is impossible, and this proves (a). If f is separable, since $D(P)(Q) \neq 0$, there is an open neighborhood U of Q in W, such that for all $Q' \in U$ one has $D(P)(Q') \neq 0$. The solutions of the equation

$$T^l + a_1(Q')T^{l-1} + \cdots a_l(Q') = 0$$

are exactly the values that a assumes at the points of $f^{-1}(Q')$, it follows that f is not branched in all points of U. It remains to be proved that if f is separable, there are points of W in which f is not branched. Note that the extension $L \to K$ is separable. So, for the Theorem of the Primitive Element 7.2.2, there is an $\alpha \in B$ such that $K = L(\alpha)$. If $P(T)$ is the minimal polynomial of α, one has $\deg(P(T)) = n$ and $D(P) \neq 0$ for the separability of the extension. Hence there is a $Q \in W$ such that $D(P)(Q) \neq 0$ and Q is not a branch point for f. □

Exercise 10.4.5 Prove that Theorem 10.4.3 no longer holds if W is not normal.

Exercise 10.4.6 * Prove that if $A \subseteq B$ are noetherian rings and C is the integral closure of A in B, then $C[x]$ is the integral closure of $A[x]$ in $B[x]$.

Exercise 10.4.7 * Prove that if (W, ϕ) is a normalization of the affine variety V, then $(W \times \mathbb{A}^n, \phi \times \mathrm{id}_{\mathbb{A}^n})$ is a normalization of $V \times \mathbb{A}^n$. In particular, if V is normal, so is $V \times \mathbb{A}^n$, hence also $V \times \mathbb{P}^n$, for all positive integers n.

10.5 Solutions of Some Exercises

10.1.7 Use Lemma 5.4.1.

10.1.8 Consider the inclusion $f^* : A(W) \to A(V)$, so via f^* we will consider $A(W)$ as a subring of $A(V)$. As for the first assertion note that $f^{-1}(U_W(g)) = U_V(g)$. Moreover

$$\mathcal{O}(U_W(g)) = A(W)\Big[\frac{1}{g}\Big], \quad \mathcal{O}(U_V(g)) = A(V)\Big[\frac{1}{g}\Big]$$

and $f_{|f^{-1}(U_W(g))}$ corresponds to the inclusion $A(W)\Big[\frac{1}{g}\Big] \to A(V)\Big[\frac{1}{g}\Big]$. Since f is finite, $A(V)$ is a finitely generated $A(W)$-module (see Exercise 5.4.7), then also $A(V)\Big[\frac{1}{g}\Big]$ is a finitely generated $A(W)\Big[\frac{1}{g}\Big]$-module, and we are done by Exercise 5.4.6.

10.1.10 If f_1 is non-constant, it suffices to prove that $x_n = \frac{f_2}{f_1}$ is not integral over A_{n-1}. Suppose by contradiction that $\frac{f_2}{f_1}$ is integral over A_{n-1}. We would have a relation of the type

$$\Big(\frac{f_2}{f_1}\Big)^n + a_1\Big(\frac{f_2}{f_1}\Big)^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n \in A_{n-1}$. This implies

$$f_2^n + a_1 f_2^{n-1} f_1 + \cdots + a_n f_1^n = 0,$$

which yields that f_1 divides f_2, a contradiction.

The final assertion is easy.

10.1.12 Every element of $A(V)$ is integral over $A(W)$, so it is algebraic over $A(W)$. Hence every element of $K(V)$ is algebraic over $K(W)$. Thus $K(V)$ and $K(W)$ have the same transcendence degree over $\mathbb{K}$.

10.1.13 Consider for instance the projection of the plane curve $x_1x_2^2 + x_1 - x_2 = 0$ on the x_1 axis from the point at infinity of the x_2 axis.

10.1.14 Suppose we have a finite morphism $f : V \to W$ of affine varieties. Let $V' \subseteq V$ be a closed subvariety and let $W' = f(V')$ be its image. Let $x \in A(V')$ be any element. Then x comes as the image of an element y via the surjective homomorphism $A(V) \to A(V')$. Since f is finite, we have a relation of the form

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0,$$

with $a_1, \ldots, a_n \in A(W)$. Then mapping to $A(W')$ via the homomorphism $A(W) \to A(W')$ we get

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0,$$

with $b_1, \ldots, b_n \in A(W')$ the images of $a_1, \ldots, a_n \in A(W)$.

10.1.15 The principal open subset form a basis for the Zariski topology of an affine variety. Moreover the counterimage of a principal open subset is an open principal subset, so it is affine.

10.1.18 For example the projection $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is not affine.

10.1.21 We can assume that $P = [0, \ldots, 0, 1]$ and $H = Z_p(x_n)$. If V has degree d it has equation of the sort

$$f = x_n^a f_{d-a}(x_0, \ldots, x_{n-1}) + \cdots + f_d(x_0, \ldots, x_{n-1}) = 0$$

with f_i homogenous polynomials of degree i in the variables $x_0, \ldots, x_{n-1}$, for $i = d - a, \ldots, d$, for some non-negative integer $a < d$, and $f_{d-a} \neq 0$. If $a = 0$, then the variable x_n does not appear in f and V is a cone with vertex $P = [0, \ldots, 0, 1]$ (see Exercise 3.2.9), and $p(V \setminus \{P\}) = Z_p(f_d) \subset H$, so that p is not dominant. If $a > 0$, then $p(V \setminus \{P\})$ contains $H \setminus Z_p(f_{d-a})$, and p is dominant.

10.3.6 To see that $\mathbb{A}^n$ is normal apply Proposition 10.3.3, (a) and Exercise 5.4.3. Since $\mathbb{P}^n$ and the Segre varieties can be covered by open sets isomorphic to affine spaces, their normality follows by Proposition 10.3.3, (b).

10.3.7 If the matrix of the quadric has rank 4, then the quadric is isomorphic to the Segre variety $\mathrm{Seg}_{1,1}$ (see Exercise 8.2.16) so it is normal by Exercise 10.3.6. Consider the case in which the matrix of the quadric has rank 3, in which case the quadric is a cone (see Exercise 3.2.10). It is well known that any such quadric is projectively equivalent to the one V with equation $x_1^2 + x_2^2 = x_3^2$. Let us prove that this quadric is normal. We claim that $S(V) = S_3/(x_1^2 + x_2^2 - x_3^2)$ is integrally closed. Then the normality of V follows by Theorem 5.5.3 and Lemma 10.3.1. To prove that $S(V)$ is integrally closed, note that $\mathbb{Q}(S(V))$ is obtained by adding to $\mathbb{K}(x_0, x_1, x_2)$ the roots of the polynomial $X^2 - (x_1^2 + x_2^2)$ in the variable X, which is irreducible over $\mathbb{K}(x_0, x_1, x_2)$ (see Lemma 2.2.4 and Gauss Lemma 2.2.5). Then any element of $\mathbb{Q}(S(V))$ can be written in a unique way as $u + vx_3$, with $u, v \in \mathbb{K}(x_0, x_1, x_2)$. Similarly, any element of $S(V)$ can be written in a unique way as $u + vx_3$, with $u, v \in \mathbb{K}[x_0, x_1, x_2]$, so that $S(V)$ is a finitely generated $\mathbb{K}[x_0, x_1, x_2]$-module, hence $S(V)$ is integral over $\mathbb{K}[x_0, x_1, x_2]$. Now let $a = u + vx_3 \in \mathbb{Q}(S(V))$ be integral over $S(V)$. Then a is integral over $\mathbb{K}[x_0, x_1, x_2]$ (see Exercise 10.1.14), and its minimal polynomial is

$$(X - u - vx_3)(X - u + vx_3) = X^2 - 2uX + \Big(u^2 - v^2(x_1^2 + x_2^2)\Big).$$

Therefore $u \in \mathbb{K}[x_0, x_1, x_2]$ and $v^2(x_1^2 + x_2^2) \in \mathbb{K}[x_0, x_1, x_2]$ and this implies that $v \in \mathbb{K}[x_0, x_1, x_2]$, so that $a \in S(V)$, as wanted.

10.3.8 For instance, consider the first curve V. Its affine part has equation $x_1^2 = x_2^3$. Now $A(V)$ is not integrally closed, because $\frac{x_1}{x_2}$ is solution of the equation $X^2 - x_2 = 0$ with coefficients in $A(V)$, but it does not lie in $A(V)$. Then V is not normal by Proposition 10.3.3. Similarly, the second curve V' has affine equation $x_2^2 = x_1^2(x_1 + 1)$ and $A(V')$ is not normal because $\frac{x_2}{x_1}$ verifies the equation $X^2 - (x_1 + 1) = 0$ but is does not lie in $A(V')$.

10.3.9 It is clear that $\mathbb{P}^n$ is projectively normal. As for Veronese varieties $V_{n,d}$, note that $S(V_{n,d}) = \bigoplus_{a\in\mathbb{N}} S_{n,ad}$ and this ring is easily seen to be integrally closed. Similar arguments for the Segre verities. The final assertion follows by Theorem 5.5.3 and Lemma 10.3.1.

10.3.10 Consider the rational normal curve $V = V_{1,4}$ which is the image of the morphism

$$v_{1,4} : [x_0, x_1] \in \mathbb{P}^1 \to [x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4] \in \mathbb{P}^4$$

and consider its projection to the hyperplane $x_2 = 0$ form the point $[0, 0, 1, 0, 0]$. This projection is an isomorphism onto its image V'. Indeed one can verify that the inverse morphism is given by assigning to a point $[a_0, a_1, a_3, a_4] \in V'$ the point $v_{1,4}(a_0, a_1) \in V$ if $a_0 \neq 0$, and the point $v_{1,4}(a_3, a_4) \in V$ if $a_4 \neq 0$. By Exercise 10.3.9 we know that V is projectively normal. By contrast V' is not projectively normal. Indeed, assuming the homogeneous coordinates in $\mathbb{P}^3$ to be $[y_0, y_1, y_2, y_3]$, in $S(V')$ we have the relation $(y_0y_2)^2 = y_0y_1^2y_3$, and this implies that $\frac{y_0y_2}{y_1}$ is integral over $S(V')$. However $\frac{y_0y_2}{y_1}$ does not lie in $S(V')$. In fact, if $\frac{y_0y_2}{y_1} \in S(V')$, since $S(V')$ is graded, we would have $y_0y_2 = y_1 f$ with f homogeneous of degree 1, i.e., $f = a_0y_0 + a_1y_1 + a_2y_2 + a_3y_3$. Then we would have a quadric with equation

$$y_0y_2 = y_1(a_0y_0 + a_1y_1 + a_2y_2 + a_3y_3)$$

containing V', which is not the case. We would have in fact that for any $[x_0, x_1] \in \mathbb{P}^1$, there is a relation

$$x_0^2x_1^2 = a_0x_0^4 + a_1x_0^3x_1 + a_2x_0x_1^3 + a_3x_0^4,$$

which is not possible.

10.3.11 It is not difficult to see that both ϕ, ϕ' are birational, in particular they are dominant, so they correspond to inclusions

$$\phi^* : A(V) \to \mathbb{K}[t], \quad (\phi')^* : A(V') \to \mathbb{K}[t],$$

which induce isomorphisms $K(V) \cong \mathbb{K}(t) \cong A(V')$. Now $\mathbb{K}[t]$ is integral over $A(V)$. Indeed, $t = \frac{x_1}{x_2}$ which, as we saw in the solution to Exercise 10.3.8, is integral over $A(V)$. Similarly, in the case of V', $t = \frac{x_2}{x_1}$ which is integral over $A(V')$. This implies that the maps ϕ and ϕ' are finite. Finally $\mathbb{K}[t]$ is integrally closed because it is a unique factorization domain (see Exercise 5.4.3). This implies that $(\mathbb{A}^1, \phi)$ and $(\mathbb{A}^1, \phi')$ are normalizations of V and V' respectively.

10.3.12 Let U be a non-empty open affine subset of V and let (U', ϕ) be a normalization of U. Then there are two non-empty open subsets of U and U' which are isomorphic, and this implies that there are points of U which are normal for U, hence there are normal points of V. Let $P \in V$ be a normal point. Let again U be an open neighborhood of P in V and let (U', ϕ) be a normalization of U. Let $P' \in U$ be any point such that $\phi(P') = P$. Then $\phi^* : \mathcal{O}_{U,P} \to \mathcal{O}_{U',P'}$ is injective and $\mathbb{Q}(\mathcal{O}_{U,P}) = \mathbb{Q}(\mathcal{O}_{U',P'}) = K(V)$ and $\mathcal{O}_{U',P'}$ is integrally closed. Then ϕ^* is an isomorphism. By Corollary 7.1.3, there is an open subset of U containing P and an open subset of U' containing P' such that ϕ induces an isomorphism between them. Hence there is a whole open neighborhood of P in V consisting of normal points. This proves the assertion.

10.4.5 Consider for example the affine plane curve W with equation $x_1^3 = x_2^2 - x_1^2$. We know that it is not normal (see Exercise 10.3.11) and its normalization is $\mathbb{A}^1$. So we have a finite birational morphism $\phi' : \mathbb{A}^1 \to W$ (see again Exercise 10.3.11), which has degree 1, but the counterimage of the point $(0, 0)$ consists of two distinct points.

10.4.6 Let $f \in B[x]$ be integral over $A[x]$. We have a relation of the sort

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0 \quad \text{with} \quad g_1, \ldots, g_m \in A[x].$$

Let r be an integer which is larger than the degrees of $g_1, \ldots, g_m$, and set $f_1 = f - x^r$. Hence we have

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_m = 0,$$

which implies a relation of the form

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0 \quad \text{with} \quad h_1, \ldots, h_m \in A[x],$$

hence

$$(-f_1)(f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}) = h_m.$$

If $r \gg 0$, then h_m is monic and so is also f_1, hence also $f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}$ is monic. By applying Lemma 10.4.1, we have $f_1 \in C[x]$ hence $f \in C[x]$. Conversely, if $f \in C[x]$, then f is integral over over $A[x]$. Indeed, if $a_0, \ldots, a_n$ are the coefficients of f, then $(A[x])[f] \subseteq (A[a_0, \ldots, a_n])[x]$ which is finitely generated over $A[x]$, since $A[a_0, \ldots, a_n]$ is finitely generated on A.

10.4.7 It suffices to prove the assertion for $n = 1$. Then we may apply Exercise 10.4.6 to the following situation: $A = A(V)$, $B = K(V)$, $C = A(W)$, with W a normalization of V. Then $A(V \times \mathbb{A}^1) = A[x]$, $K(V \times \mathbb{A}^1) = B(x)$, and $A(W \times \mathbb{A}^1) = C[x]$, and Exercise 10.4.6 implies that $A(W \times \mathbb{A}^1)$ is the integral closure of $A(V \times \mathbb{A}^1)$ in $K(V)[x]$. But, since $K(V)[x]$ is a unique factorization domain and $K(V)(x) = \mathbb{Q}(K(V)[x])$, then $K(V)[x]$ is integrally closed in $K(V)(x) = B(x)$. In conclusion $A(W \times \mathbb{A}^1)$ is the integral closure of $A(V \times \mathbb{A}^1)$ in $K(V \times \mathbb{A}^1)$.

Chapter 11
Dimension

11.1 Characterization of Hypersurfaces

As we know, any hypersurface of $\mathbb{A}^n$ or $\mathbb{P}^n$ has transcendent dimension $n-1$ (see Exercise 10.1.20). As a first result of this chapter, we invert this result. We start with the following:

Lemma 11.1.1 *Let V, W be quasi-projective varieties with $W \subseteq V$. Then* $\dim_{\text{tr}}(W) \leq \dim_{\text{tr}}(V)$. *If, in addition, W is closed in V and* $\dim_{\text{tr}}(W) = \dim_{\text{tr}}(V)$, *then $V = W$.*

Proof It suffices to reduce to the case in which V and W are affine. Then we may assume $W \subseteq V \subseteq \mathbb{A}^n$, so that $A(V)$ and $A(W)$ are generated, as $\mathbb{K}$-algebras, by $x_1, \dots, x_n$. Let $m = \dim_{\text{tr}}(V)$. Then any $(m+1)$-tuple $(x_{i_1}, \dots, x_{i_{m+1}})$ of elements of $\{x_1, \dots, x_n\}$ is algebraically dependent. This implies that there is a non-zero polynomial $F \in \mathbb{K}[T_{i_1}, \dots, T_{i_{m+1}}]$, such that $F(x_{i_1}, \dots, x_{i_{m+1}}) = 0$ in $A(V)$. Namely, $F(x_{i_1}, \dots, x_{i_{m+1}}) \in \mathcal{I}_a(V) \subseteq \mathcal{I}_a(W)$. Hence $F(x_{i_1}, \dots, x_{i_{m+1}}) = 0$ also in $A(W)$, i.e., $(x_{i_1}, \dots, x_{i_{m+1}})$ are algebraically dependent on $A(W)$. This implies that $\dim_{\text{tr}}(W) \leq m = \dim_{\text{tr}}(V)$.

Assume now also $\dim_{\text{tr}}(W) = m$, with W closed in V. We want to show that $V = W$, i.e., that $\mathcal{I}_a(V) = \mathcal{I}_a(W)$. We argue by contradiction and assume that $\mathcal{I}_a(V) \neq \mathcal{I}_a(W)$.

Since $\dim_{\text{tr}}(W) = m$, it is possible to choose m elements in $\{x_1, \dots, x_n\}$ which are algebraically independent over $\mathbb{K}$. We may assume these are $x_1, \dots, x_m$. By the same argument we made before, we see they are algebraically independent also as elements of $A(V)$. Take now $f \in \mathcal{I}_a(W) \setminus \mathcal{I}_a(V)$. Then $f(x_1, \dots, x_n)$ can be considered as a non-zero element of $A(V)$ and, as such, it algebraically depends on $x_1, \dots, x_m$, i.e., there is a relation of the form

$$a_0(x_1, \dots, x_m) f^l + \cdots + a_l(x_1, \dots, x_m) = 0 \tag{11.1}$$

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where the polynomial appearing in the left hand side of (11.1) can be supposed to be non-zero and irreducible, in particular we may assume that $a_l(x_1, \dots, x_m) \neq 0$ in $A(V)$ hence in A_n. The relation (11.1) holds also in $A(W)$, which is the quotient of $A(V)$ modulo $\mathcal{I}_V(W)$. Since in $A(W)$ one has $f = 0$, we get $a_l(x_1, \dots, x_m) = 0$ in $A(W)$. But in $A(W)$ the elements $x_1, \dots, x_m$ are algebraically independent, so we would have $a_l = 0$ in A_n, a contradiction. This implies that $\mathcal{I}_a(V) = \mathcal{I}_a(W)$, so that $V = W$. □

Theorem 11.1.2 *Any variety of transcendent dimension $n - 1$ in $\mathbb{A}^n$ or $\mathbb{P}^n$ is an irreducible hypersurface.*

Proof It suffices to consider the affine case. Take a subvariety V of dimension $n - 1$ in $\mathbb{A}^n$. There is an irreducible polynomial $f \in \mathcal{I}_a(V)$, where $\mathcal{I}_a(V)$ is a prime ideal of A_n. Then $V \subseteq Z_a(f)$, and $Z_a(f)$ is an irreducible variety of transcendent dimension $n - 1$, with V closed in $Z_a(f)$. Then apply Lemma 11.1.1. □

11.2 Intersection with Hypersurfaces

The next step is to consider the intersection of an affine or projective variety with a hypersurface. We start with the following algebraic lemma:

Lemma 11.2.1 *Let A be a domain containing A_n and integral over A_n. Let $x, y \in A_n$ be non-zero, coprime elements and let $z \in A$ be such that x divides yz in A. Then there is a positive integer m such that x divides z^m in A.*

Proof Suppose we have $xw = yz$ in A and let

$$F(T) = T^l + b_1 T^{l-1} + \cdots + b_l$$

be the minimal polynomial of w over $\mathbb{Q}(A_n)$. By Proposition 10.4.2 we have $b_1, \dots, b_l \in A_n$. Since $z = \frac{x}{y} w$ and $\frac{x}{y} \in \mathbb{Q}(A_n)$, the minimal polynomial of z on $\mathbb{Q}(A_n)$ has also degree l, and one has

$$0 = F(w) = F\left(\frac{y}{x} z\right) = \left(\frac{y}{x}\right)^l z^l + b_1 \left(\frac{y}{x}\right)^{l-1} z^{l-1} + \cdots + b_l,$$

so that the minimal polynomial of z is

$$G(T) = \left(\frac{x}{y}\right)^l F\left(\frac{y}{x} T\right) = T^l + \frac{x}{y} b_1 T^{l-1} + \cdots + \left(\frac{x}{y}\right)^l b_l.$$

Again by Proposition 10.4.2, we have $(\frac{x}{y})^i b_i \in A_n$ for $i = 1, \dots, l$. Since x, y are coprime, then y^i divides b_i for $i = 1, \dots, l$. From the relation $G(z) = 0$ we deduce that x divides z^l. □

Theorem 11.2.2 *Let $V \subseteq \mathbb{P}^m$ be a projective variety with $n = \dim_{\mathrm{tr}}(V) \geq 1$, and let H be a hypersurface not containing V. Then each irreducible component of $V \cap H$ has transcendent dimension $n - 1$.*

Proof Set $V_1 = V \cap H$. Then V_1 is a non-empty algebraic set (see Exercise 6.4.15). There is a hypersurface H_1 which does not contain any of the irreducible components of V_1. Then $V_2 = V_1 \cap H_1$ is an algebraic set which is either empty, or any of its irreducible components is strictly contained in some component of V_1. By repeating this argument, we obtain a sequence of algebraic sets

$$V := V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \ldots \supsetneq V_i \supsetneq V_{i+1} \supsetneq \ldots$$

such that $V_{i+1} = V_i \cap H_i$, where H_i is a hypersurface which does not contain any component of V_i. Let now n_i be the maximum dimension of the irreducible components of V_i. By Lemma 11.1.1 we have

$$n := n_0 > n_1 > n_2 > \ldots > n_i > n_{i+1} > \ldots$$

so that certainly $V_{n+1} = \emptyset$, i.e., $V \cap H \cap H_1 \cap \ldots \cap H_n = \emptyset$. Let $f_0 = 0$, $f_1 = 0, \ldots, f_n = 0$ be the equations of $H, H_1, \ldots, H_n$. We claim we may suppose that $f_0, f_1, \ldots, f_n$ have the same degree. More precisely, given any positive integer d we can assume $f_1, \ldots, f_n$ of the same degree d, because, for any $i = 1, \ldots, n$, H_i has simply to satisfy the hypothesis of not containing any component of V_i. Then we can choose $f_1, \ldots, f_n$ of the same degree of f_0. Since $Z_p(f_0, \ldots, f_n) \cap V = \emptyset$, the morphism

$$\phi : P \in V \to [f_0(P), \ldots, f_n(P)] \in \mathbb{P}^n$$

is finite onto its image (see Corollary 10.2.2). Then $\phi(V)$ is a closed subset of $\mathbb{P}^n$ and $\dim_{\mathrm{tr}}(\phi(V)) = \dim_{\mathrm{tr}}(V) = n$, so that $\phi(V) = \mathbb{P}^n$ by Lemma 11.1.1. Now, if we had $n_1 < n - 1$, we would have $V_n = \emptyset$, hence $V \cap H \cap H_1 \cap \ldots \cap H_{n-1} = \emptyset$, namely $Z_p(f_0, \ldots, f_{n-1}) \cap V = \emptyset$. This would imply that $[0, \ldots, 0, 1] \notin \phi(V)$, a contradiction. This proves that $n_1 = n - 1$, hence there is some component of V_1 of dimension $n - 1$.

To finish the proof we have to show that every irreducible component of V_1 has dimension at least $n - 1$. To do this, consider in $\mathbb{P}^n$ the open subset $U_j \cong \mathbb{A}^n$, where $x_j \neq 0$, for $j = 0, \ldots, n$ and set $V^j = \phi^{-1}(U_j)$. We will show that, for every $j = 0, \ldots, n$, every irreducible component of $V_1 \cap V^j$ has transcendent dimension at least $n - 1$, which will prove the assertion. It suffices to do this for $j = n$, the proof being analogous in the other cases. Let us set $a_{i+1} = \frac{f_i}{f_n}$, for $i = 0, \ldots, n - 1$ and consider the restriction of ϕ to V^n given by

$$\phi : P \in V^n \to (a_1(P), \ldots, a_n(P)) \in \mathbb{A}^n$$

which is a finite morphism (see Remark 10.2.5). Let us prove that the restriction of $a_2, \dots, a_n$ to each of the components of $V_1 \cap V^n = Z_{V^n}(a_1)$, are algebraically independent. To do this, consider $P(T_2, \dots, T_n) \in \mathbb{K}[T_2, \dots, T_n]$ a non-zero polynomial. We have to prove that $P(a_2, \dots, a_n)$ is non-zero on any component of $Z_{V^n}(a_1)$. To see this it suffices to prove that, if for $Q \in A(V^n)$ one has $Q \cdot P(a_2, \dots, a_n) = 0$ on $V_1 \cap V^n$, then $Q = 0$ on $V_1 \cap V^n$. Indeed, if $P(a_2, \dots, a_n)$ would be zero on some component X of $V_1 \cap V^n$, it would suffice to take Q zero on all other components of $V_1 \cap V^n$ but not on X, and then we would have $Q \cdot P(a_2, \dots, a_n) = 0$ on $V_1 \cap V^n$ but Q would not be zero on $V_1 \cap V^n$. So assume $Q \cdot P(a_2, \dots, a_n) = 0$ on $V_1 \cap V^n$. By applying the Nullstellensatz (see the version in Exercise 5.5.14), we have that a_1 divides $(Q \cdot P(a_2, \dots, a_n))^l$ in $A(V^n)$, for some positive integer $l > 0$. We claim that there is an $m > 0$ such that a_1 divides Q^m in $A(V^n)$, so that $Q = 0$ on $V_1 \cap V^n$, as needed. This follows by Lemma 11.2.1, by taking $A = A(V^n)$, $x = a_1$, $y = P(T_2, \dots, T_n)^l$ and $z = Q^l$. □

The previous theorem has some remarkable consequences:

Corollary 11.2.3 *Let $V \subseteq \mathbb{P}^m$ be a projective variety with* $\dim_{\mathrm{tr}}(V) = n$ *and let $f_1, \dots, f_r$ be homogeneous polynomials in S_m. Then every irreducible component of $V \cap Z_p(f_1, \dots, f_r)$ has transcendent dimension at least* $\max\{n - r, 0\}$. *In particular if $r \leq n$, then $V \cap Z_p(f_1, \dots, f_r) \neq \emptyset$. If V is quasi-projective, then every irreducible component of $V \cap Z_p(f_1, \dots, f_r)$ has transcendent dimension at least $n - r$ provided $V \cap Z_p(f_1, \dots, f_r) \neq \emptyset$.*

Proof If V is projective, the assertion follows by iterated applications of Theorem 11.2.2. If V is quasi-projective, then V is open in $\bar{V}$. One has $V \cap Z_p(f_1, \dots, f_r) = (\bar{V} \cap Z_p(f_1, \dots, f_r)) \cap V$. Then, either $V \cap Z_p(f_1, \dots, f_r) = \emptyset$ or every irreducible component of $V \cap Z_p(f_1, \dots, f_r)$ is a non-empty open subset of $\bar{V} \cap Z_p(f_1, \dots, f_r)$ and we may apply the result in the projective case. □

Corollary 11.2.4 *Let $V, W \subseteq \mathbb{P}^r$ be quasi-projective varieties of respective transcendent dimensions n, m, with $r \leq n + m$. Then, if $V \cap W \neq \emptyset$, for every irreducible component Z of $V \cap W$ one has* $\dim_{\mathrm{tr}}(Z) \geq n + m - r$.

Proof It suffices to reduce to the case in which V, W are affine. Then $V \cap W = (V \times W) \cap \Delta$, where Δ is the diagonal of $\mathbb{A}^r \times \mathbb{A}^r$ (see Exercise 6.2.23). On the other hand Δ is defined in $\mathbb{A}^r \times \mathbb{A}^r = \mathbb{A}^{2r}$ by r linear equations. Then it suffices to apply Corollary 11.2.3. □

Corollary 11.2.5 *Let V be a quasi-projective variety of transcendent dimension n. Then there are on V subvarieties of any transcendent dimension s with* $0 \leq s \leq n$.

Proof Obvious. □

Corollary 11.2.6 *Let V be a quasi-projective variety. Then* $\dim_{\mathrm{tr}}(V) = \dim_{\mathrm{top}}(V)$.

Proof From the proof of Theorem 11.2.2 it follows that if $n = \dim_{\mathrm{tr}}(V)$, then there is a chain of subvarieties

$$V := V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \ldots \supsetneq V_n$$

with V_n consisting of a point. Hence $\dim_{\text{top}}(V) \geq n = \dim_{\text{tr}}(V)$. On the other hand, Lemma 11.1.1 immediately implies that $\dim_{\text{top}}(V) \leq n = \dim_{\text{tr}}(V)$. □

From now on if V is a quasi-projective variety we will set $\dim(V) := \dim_{\text{tr}}(V) = \dim_{\text{top}}(V)$ and $\dim(V)$ will be simply called the *dimension* of V. If Z is a locally closed subset of V, we will say that the *dimension* of Z, denoted by $\dim(Z)$, is the maximum dimension of the irreducible components of Z. If W is a subvariety of V, then $\dim(V) - \dim(W)$ will be called the *codimension* of W in V, denoted by $\text{codim}_V(W)$.

Corollary 11.2.7 *Let $V \subseteq \mathbb{P}^m$ be a quasi-projective variety of dimension n and let $W \subseteq V$ be a subvariety of codimension r. Then there is a chain of subvarieties*

$$V := V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \ldots \supsetneq V_r = W.$$

Proof If $V = W$ there is nothing to prove. Otherwise, there is a hypersurface H of $\mathbb{P}^m$ such that $W \subseteq H$ but V is not contained in H. Let V_1 be one of the irreducible components of $V \cap H$ containing W. One has $\dim(V_1) = n - 1$. The assertion follows by iterating this argument. □

Corollary 11.2.8 *Let $V \subseteq \mathbb{P}^m$ be a quasi-projective variety of dimension n and let $W \subseteq V$ be a subvariety of codimension r. Then $\dim_K(\mathcal{O}_{V,W}) = r$.*

Proof The assertion follows from the definition of the Krull dimension of a ring (see Sect. 4.3), from Exercise 5.5.7 and by Corollary 11.2.7. □

Exercise 11.2.9 Let $V \subseteq \mathbb{P}^n$ be a quasi-projective variety and let $Z \subseteq V$ be an algebraic subset of V such that any irreducible component of Z has codimension r in V. One says that Z is a *set-theoretic complete intersection* in V if there are homogeneous polynomials $f_1, \ldots, f_r$ in S_n such that $Z = V \cap Z_p(f_1, \ldots, f_r)$. If $V \subseteq \mathbb{P}^n$ (resp. $V \subseteq \mathbb{A}^n$) is a projective (resp. affine) variety and $Z \subseteq V$ is like above, one says that Z is a *complete intersection* in V if $\mathcal{I}_{p,Z}(V)$ (resp. $\mathcal{I}_Z(V)$) is generated by r elements of $S(V)$ (resp. of $A(V)$).

Prove that any complete intersection is also a set-theoretic complete intersection.

Exercise 11.2.10 Let $Z \subset \mathbb{P}^2$ be the set consisting of three non-collinear points. Prove that Z is not a complete intersection in $\mathbb{P}^2$.

Exercise 11.2.11 Prove that the reducible curve in $\mathbb{A}^3$ consisting of the union of the three coordinate axes is not a complete intersection.

Exercise 11.2.12 *Let $Z \subset \mathbb{A}^2$ be a finite set. Prove that Z is a set-theoretic complete intersection in $\mathbb{A}^2$.

Exercise 11.2.13 *Let $Z \subset \mathbb{P}^2$ be a finite set. Prove that if $P \in \mathbb{P}^2$ is any point not in Z and not lying on any line joining a pair of distinct points of Z, then $Z \cup \{P\}$ is a set-theoretic complete intersection in $\mathbb{P}^2$.

Exercise 11.2.14 Let $Z \subset \mathbb{P}^2$ be a finite set of points for which there is a point $P \in Z$ such that there is no pair of distinct points of $Z \setminus \{P\}$ such that P lies on the line joining that pair of points. Deduce from Exercise 11.2.13 that Z is a set-theoretic complete intersection in $\mathbb{P}^2$. In particular if Z consists of three distinct non-collinear points, then Z is a set-theoretic complete intersection. Note that a set-theoretic complete intersection is not necessarily a complete intersection.

Exercise 11.2.15 Prove that the projective twisted cubic is not a complete intersection in $\mathbb{P}^3$.

Exercise 11.2.16 Prove that the affine twisted cubic is a complete intersection in $\mathbb{A}^3$.

Exercise 11.2.17 *Consider a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

over a field k. Set

$$\mathbf{A}_1 = \begin{pmatrix} a_{11} & \dots & a_{1,n-1} \\ & \dots & \\ a_{n-1,1} & \dots & a_{n-1,n-1} \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{n-1,1} & \dots & a_{n-1,n} \end{pmatrix}, \mathbf{A}_3 = \begin{pmatrix} a_{11} & \dots & a_{1,n-1} \\ & \dots & \\ a_{n1} & \dots & a_{n,n-1} \end{pmatrix}.$$

Prove that if $\det(\mathbf{A}) = \det(\mathbf{A}_1) = 0$, then either $\mathrm{rk}(\mathbf{A}_2) < n-1$ or $\mathrm{rk}(\mathbf{A}_3) < n-1$.

Exercise 11.2.18 *Prove that the twisted cubic in $\mathbb{P}^3$ is a set-theoretic complete intersection. Note: it is an open problem to see if any curve in $\mathbb{P}^3$ is a set-theoretic complete intersection.

Exercise 11.2.19 *Consider the image V of the morphism

$$\phi : t \in \mathbb{A}^1 \to (t^3, t^4, t^5) \in \mathbb{A}^3.$$

Prove that V is an irreducible affine curve in $\mathbb{A}^3$, which is not a complete intersection.

Exercise 11.2.20 Let $F \subset \mathbb{P}^n$ be an irreducible hypersurface and let $V \subset F$ be a subvariety of codimension 1 in F which is a complete intersection in $\mathbb{P}^n$. Prove that V is a complete intersection in F if and only if we can assume that one of the generators of $\mathcal{I}_p(V)$ coincides with a reduced equation of F.

Deduce that if Q is an irreducible quadric in $\mathbb{P}^3$ and L is a line on Q, then L is not a complete intersection on Q.

Exercise 11.2.21 Let Q be an irreducible quadric cone in $\mathbb{P}^3$ and let L be a line on Q. Prove that Q is a set theoretic intersection on Q.

Exercise 11.2.22 *Let Q be an irreducible quadric in $\mathbb{P}^3$ which is not a cone (so it is projectively equivalent to the quadric $x_0x_1 - x_2x_3 = 0$) and let L be a line on Q. Prove that Q is not a set theoretic intersection on Q.

Exercise 11.2.23 *Let H be a closed algebraic subset in the Veronese variety $V_{n,d}$, such that all irreducible components of H have codimension 1 in $V_{n,d}$. Prove that H is a set theoretic complete intersection on $V_{n,d}$.

Exercise 11.2.24 Let $V \subseteq \mathbb{P}^n$ be a quasi-projective variety of dimension n and let $f_1, \dots, f_r \in \mathcal{O}(V)$. Prove that, if $Z_V(f_1, \dots, f_r) \neq \emptyset$, then any of its irreducible components has dimension at least $\max\{n-r, 0\}$. In particular, if $f \in \mathcal{O}(V) \setminus \mathbb{K}$, then all irreducible components of $Z_V(f)$ have codimension 1 in V.

Exercise 11.2.25 Let A be a domain which is also a finitely $\mathbb{K}$-algebra. Prove that:

(a) $\dim_K(A)$ equals the transcendence degree of $\mathbb{Q}(A)$ over $\mathbb{K}$;
(b) for every prime ideal $\mathcal{I}$ of A one has

$$\mathrm{height}(\mathcal{I}) + \dim_K(A/\mathcal{I}) = \dim_K(A).$$

11.3 Morphisms and Dimension

In this section we investigate what is the behavior of the dimension with respect to morphisms between varieties. We have the following basic:

Theorem 11.3.1 *Let V, W be quasi-projective varieties of dimensions n, m respectively and let $f : V \to W$ be a dominant morphism. Then we have:*

(a) $n \geq m$ and for any point $P \in f(V)$ every irreducible component of the fibre *of f over P, i.e., of $V_P := f^{-1}(P)$, has dimension at least $n-m$;*
(b) there is a non-empty open subset U of $f(V)$ such that for every point $P \in U$ every irreducible component of V_P has dimension $n-m$.

Proof Let us prove (a). To prove that $n \geq m$, we may assume that V and W are affine. Then $f^* : A(W) \to A(V)$ is injective, and this extends to the injective homomorphism $f^* : K(W) \to K(V)$. Whence immediately follows that $\dim(V) = \dim_{\mathrm{tr}}(V) \geq \dim_{\mathrm{tr}}(W) = \dim(W)$.

Let now $P \in f(V)$. Since the problem is of a local nature, we may suppose that W is affine, i.e., $W \subseteq \mathbb{A}^r$. By the proof of Theorem 11.2.2 and by Corollary 11.2.7 and after may be shrinking W to an open affine principal neighborhhod of P, we can find polynomials $f_1, \ldots, f_m \in A_r$ such that $\{P\} = W \cap Z_a(f_1, \ldots, f_m)$. Then, if we set $g_i = f^*(f_i) = f_i \circ f \in \mathcal{O}(V)$, for $i = 1, \ldots, m$, one has $V_P = Z_V(g_1, \ldots, g_m)$. So the second assertion of (a) follows from Exercise 11.2.24.

Let us now prove (b). Let us start by assuming V, W to be affine varieties. From the injective homomorphism $f^* : K(W) \to K(V)$ we have that $K(V)$ has transcendence degree $n-m$ over $K(W)$. So if $t_1, \ldots, t_h$ are generators of $A(V)$ as a $\mathbb{K}$-algebra, among $t_1, \ldots, t_h$ there are $(n-m)$-tuples of elements which are algebraically independent over $K(W)$, hence on $A(W)$. Let $t_{i_1}, \ldots, t_{i_{n-m}}$ be one of these $(n-m)$-tuples. Then we have non-zero polynomials

$$F_{i_1,\ldots,i_{n-m};i}(T) \in A(W)[t_{i_1}, \ldots, t_{i_{n-m}}][T], \quad i = 1, \ldots, h,$$

with $i \neq i_j$ for $j = 1, \ldots, n-m$, such that $F_{i_1,\ldots,i_{n-m};i}(t_i) = 0$. Set

$$F_{i_1,\ldots,i_{n-m};i}(T) = a_{0,i}(t_{i_1}, \ldots, t_{i_{n-m}})T^{l_i} + \cdots + a_{l_i,i}(t_{i_1}, \ldots, t_{i_{n-m}}),$$

and we may assume that $a_{0,i}(t_{i_1}, \ldots, t_{i_{n-m}}) \in A(W)[t_{i_1}, \ldots, t_{i_{n-m}}]$ is not zero.

Consider the closed subset $X_{i_1,\ldots,i_{n-m};i}$ in W formed by the points $P \in W$ such that the polynomials $\bar{a}_{0,i}(t_{i_1}, \ldots, t_{i_{n-m}}) \in \mathbb{K}[t_{i_1}, \ldots, t_{i_{n-m}}]$ obtained from $a_{0,i}(t_{i_1}, \ldots, t_{i_{n-m}})$ by computing its coefficients in P, are identically zero. Then $X_{i_1,\ldots,i_{n-m};i}$ is a proper closed subset of W and the same happens for $X = \bigcup X_{i_1,\ldots,i_{n-m};i}$, where the union is taken over all indices i and over all $(n-m)$-tuples of generators of $A(V)$ which are algebraically independent on $A(W)$. Set $U = f(V) \cap (W \setminus X)$ which is a non-empty open subset of $f(V)$. Take $P \in U$ and let Z be an irreducible component of V_P. Let $\bar{t}_i$ be the image of t_i in $A(Z)$ for all $i = 1, \ldots, h$. Then $\bar{t}_1, \ldots, \bar{t}_h$ generate $A(Z)$ as a $\mathbb{K}$-algebra. By (a), there are at least $n-m$ elements among

$\bar{t}_1, \ldots, \bar{t}_h$ which are algebraically independent over $\mathbb{K}$, and we may suppose they are $\bar{t}_1, \ldots, \bar{t}_{n-m}$. Then $t_1, \ldots, t_{n-m}$ are algebraically independent in $A(V)$. Since the polynomials $\bar{a}_{0,i}(t_{i_1}, \ldots, t_{i_{n-m}})$ are non-zero in P, we have $\bar{a}_{0,i}(\bar{t}_1, \ldots, \bar{t}_{n-m}) \neq 0$, so that $F_{1,\ldots,n-m;i}(\bar{t}_i) = 0$, with $F_{1,\ldots,n-m;i} \neq 0$. This implies that $\dim(Z) \leq n - m$. Then (a) implies the assertion.

As for the general case, let U_1 be a non-empty affine open subset of W and let $\{U_{1,i}\}_{i=1,\ldots,s}$ be a covering with affine open subsets of $f^{-1}(U_1)$. For every $i = 1, \ldots, s$, there is a non-empty open subset $U_i' \subseteq U_1 \cap f(V)$, such that for every $P \in U_i'$, every irreducible component of $V_P \cap U_{1,i}$ has dimension $n - m$. If we set $U = \bigcap_{i=1}^{s} U_i'$, then $U \cap f(V)$ verifies the assertion. □

This theorem has a couple of important corollaries:

Corollary 11.3.2 *Let V, W be projective varieties of respective dimensions n, m and let $f : V \to W$ be a surjective morphism. Then for all integers $l = n - m, \ldots, n$, the subsets W_l of W formed by all points $P \in W$ such that V_P has a component of dimension at least l, are closed subsets of W.*

Proof By Theorem 11.3.1, we have $W_{n-m} = W$ and there is a proper closed subset X of W such that $W_l \subseteq X$ for $l > n - m$. Suppose $l > n - m$ and $W_l \neq \emptyset$. In order to show that W_l is closed it suffices to prove that its intersection with any irreducible component of X is closed. Then we may assume X to be irreducible and let $Y_1, \ldots, Y_h$ be the irreducible components of $f^{-1}(X)$ such that $f_i = f_{|Y_i} : Y_i \to X$ is dominant, hence surjective since V is projective. If $l \leq \dim(Y_i) - \dim(X)$ for some $i = 1, \ldots, h$, by Theorem 11.3.1 we have $W_l = X$. If for any $i = 1, \ldots, h$ one has $l > \dim(Y_i) - \dim(X)$, then W_l is contained in a proper closed subset of X. The assertion follows by iterating the above argument. □

Corollary 11.3.3 *Let V, W be projective varieties and $f : V \to W$ a morphism. Let Z be a closed subset of V such that $f(Z) = W$ and for any $P \in W$, $Z_P := V_P \cap Z$ is irreducible of constant dimension n as P varies. Then Z is irreducible of dimension* $\dim(Z) = \dim(W) + n$.

Proof Set $m = \dim(W)$. Let $Z = Z_1 \cup \ldots \cup Z_h$ be an irredundant decomposition of Z into irreducible components. Since $f(Z) = W$, there are irreducible components of Z, we may assume they are $Z_1, \ldots, Z_l$, such that $f(Z_i) = W$ for $i = 1, \ldots, l$, whereas $f(Z_i)$ is a proper closed subset of W if $i = l + 1, \ldots, h$. Let U be the non-empty open subset $U = W \setminus (\bigcup_{l+1}^{h} f(Z_i))$. For $i = 1, \ldots, l$ and any $P \in U$, we denote by $n_i(P)$ the maximal dimension of an irreducible component of $V_P \cap Z_i$, and let n_i be the minimum of $n_i(P)$ as P varies in U. By Theorem 11.3.1, there is a non-empty open subset U' of U such that for all $P \in U'$ one has $n_i = n_i(P)$, for $i = 1, \ldots, l$. Moreover, if $P \in U$, one has $Z_P = (V_P \cap Z_1) \cup \ldots \cup (V_P \cap Z_l)$, so that there is an $i = 1, \ldots, l$, such that $Z_P = V_P \cap Z_i$, and we may suppose that this happens for $i = 1$. Then $n_1 = n$, so that $\dim(Z_1) = n + m$. For every $P \in W$ the dimension of every component of $V_P \cap Z_1$ is at least n. On the other hand, since $V_P \cap Z_1 \subseteq Z_P$ and $n = \dim(Z_P)$, the dimension of every component of $V_P \cap Z_1$ is exactly n. This implies that $Z_1 = Z$, proving the assertion. □

Exercise 11.3.4 Let $V \subseteq \mathbb{P}^n$ be a projective variety and let p be the projection of V to a subspace $\mathbb{P}_2$ from a centre $\mathbb{P}_1$ such that $\mathbb{P}_1 \cap V = \emptyset$. We know that p is a finite morphism, so that $p(V)$ is a closed subvariety of $\mathbb{P}_2$ with $\dim(V) = \dim(p(V))$. The closed subset $W := \overline{p^{-1}(p(V))}$ is called *(projective) cone* on V with vertex $\mathbb{P}_1$. Prove that $W = \bigcup_{P \in V}(\mathbb{P}_1 \vee P) = \bigcup_{P \in p(V)}(\mathbb{P}_1 \vee P)$ and W is irreducible. Prove also that $\dim(W) = \dim(V) + \dim(\mathbb{P}_1) + 1$.

Exercise 11.3.5 If V is an affine variety, one has $\dim(V) = \dim_K(A(V))$ (see Sect. 4.3 and Corollary 11.2.6). Prove that instead, if V is a projective variety, one has $\dim(V) = \dim_K(S(V)) - 1$.

Exercise 11.3.6 *Let $f(\mathbf{x}^1, \ldots, \mathbf{x}^r)$ be a non-constant plurihomogeneous irreducible polynomial in the variables $\mathbf{x}^i = (x_{i0}, \ldots, x_{in_i})$, with $i = 1, \ldots, r$. Consider the zero set $V = Z_s(f)$ in $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$. Prove that V is irreducible of dimension $n_1 + \cdots + n_r - 1$.

Exercise 11.3.7 *Consider a codimension 1 irreducible closed subset V of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$. Prove that there is a non-constant plurihomogeneous irreducible polynomial in the variables $\mathbf{x}^i = (x_{i0}, \ldots, x_{in_i})$, with $i = 1, \ldots, r$, such that $V = Z_s(f)$.

Exercise 11.3.8 *Let $X \subseteq \mathbb{P}^r$ be a projective variety of dimension n. Consider the Zariski closure $\mathrm{Sec}(X)$ of the union of all lines in $\mathbb{P}^r$ joining distinct points of X. Prove that $\mathrm{Sec}(X)$ is a closed subset of $\mathbb{P}^r$ of dimension $m \le 2n + 1$, which is called the *secant variety* of X.

11.4 Elimination Theory Again

In this section we apply the results of the previous sections to give some interesting complement to elimination theory.

First of all recall the notation introduced in Sect. 9.1, and let us prove the following:

Proposition 11.4.1 *The closed subset $Z(n; d_1, \ldots, d_h)$ of $\mathcal{L}(n; d_1, \ldots, d_h)$ is irreducible and*

$$\operatorname{codim}_{\mathcal{L}_{n,d_1,\ldots,d_h}}(Z(n; d_1, \ldots, d_h)) \ge h - n \tag{11.2}$$

Proof Consider the subset $\Gamma(n; d_1, \ldots, d_h)$ of $\mathcal{L}_{n,d_1,\ldots,d_h} \times \mathbb{P}^n$ consisting of all pairs $((H_1, \ldots, H_h), P)$ such that $P \in H_1 \cap \ldots \cap H_h$. It is easy to check (we leave the details to the reader) that $\Gamma(n; d_1, \ldots, d_h)$ is closed in $\mathcal{L}_{n,d_1,\ldots,d_h} \times \mathbb{P}^n$. Moreover we have the second projection $p_2 : \mathcal{L}_{n,d_1,\ldots,d_h} \times \mathbb{P}^n \to \mathbb{P}^n$, whose restriction p to $\Gamma(n; d_1, \ldots, d_h)$, is clearly surjective. For every point $P \in \mathbb{P}^n$, consider the hyperplane $\mathcal{L}_{n,d_i}(P)$ of $\mathcal{L}_{n,d_i}$ consisting of all $H \in \mathcal{L}_{n,d_i}$ such that $P \in H$, for $i = 1, \ldots, h$. It is the clear that $p^{-1}(P) = \mathcal{L}_{n,d_1}(P) \times \cdots \times \mathcal{L}_{n,d_h}(P)$. So for all $P \in \mathbb{P}^n$, one has that $p^{-1}(P)$ is irreducible, of dimension $N(n, d_1) + \cdots + N(n, d_h) - h$. By Corollary 11.3.3, we deduce that $\Gamma(n; d_1, \ldots, d_h)$ is irreducible of dimension $N(n, d_1) + \cdots + N(n, d_h) + n - h$. Since $Z(n; d_1, \ldots, d_h) = p(\Gamma(n; d_1, \ldots, d_h))$ we have that $Z(n; d_1, \ldots, d_h)$ is irreducible and $\dim(Z(n; d_1, \ldots, d_h)) \le \dim(\Gamma(n; d_1, \ldots, d_h)) = N(n, d_1) + \cdots + N(n, d_h) + n - h$, as wanted. □

Corollary 11.4.2 *If $h \le n$ one has $Z(n; d_1, \ldots, d_h) = \mathcal{L}_{n,d_1,\ldots,d_h}$. If $h > n$ then the equality holds in* (11.2).

Proof If $h \leq n$, the assertion follows by Corollary 11.2.3. Suppose next that $h > n$. By Theorem 11.3.1, to prove the assertion it suffices to prove that there is a point $(H_1, \ldots, H_h) \in Z(n; d_1, \ldots, d_h)$ such that $q^{-1}(H_1, \ldots, H_h)$ is a finite subset of $\Gamma(n; d_1, \ldots, d_h)$, where q is the restriction to $\Gamma(n; d_1, \ldots, d_h)$ of the projection to $\mathcal{L}_{n,d_1,\ldots,d_h}$. In other words, it suffices to find h homogeneous polynomials of degrees $d_1, \ldots, d_h$ such that their system has finitely many non-trivial solutions. A set of such polynomials is $x_1^{d_1}, \ldots, x_n^{d_n}, x_n^{d_{n+1}}, \ldots, x_n^{d_h}$, whose only non-trivial solution is $(1, 0, \ldots, 0)$. □

We will now restrict our attention to the first non-trivial case $h = n + 1$, where $Z(n; d_1, \ldots, d_{n+1})$ has codimension 1 in $\mathcal{L}_{n,d_1,\ldots,d_{n+1}}$.

Let $\mathbf{x}_i$ be the natural homogeneous coordinates in $\mathcal{L}_{n,d_i}$, for $i = 1, \ldots, n+1$. By Exercise 11.3.7, there is an irreducible, non-constant, plurihomogeneous polynomial $R(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ in the variables $\mathbf{x}_i$, for $i = 1, \ldots, n+1$, defined up to a non-zero multiplicative constant, such that $Z(n; d_1, \ldots, d_{n+1}) = Z_s(R)$. The polynomial R is called the *resultant polynomial* of $n+1$ homogeneous polynomials of degrees $d_1, \ldots, d_{n+1}$ in $n+1$ variables, with indeterminate coefficients. Its vanishing is a necessary and sufficient condition in order that the system of such polynomials has a non-trivial solution.

Fix now an index $i \in \{1, \ldots, n+1\}$ and consider the projection

$$p_i : Z(n; d_1, \ldots, d_{n+1}) \to \mathcal{L}_{n,d_1,\ldots,d_{i-1},d_{i+1},\ldots,d_n}.$$

We will consider the case $i = n+1$, because the other cases are analogous, and we will set $p_{n+1} = \pi$. The morphism π is surjective. In fact, if we fix a point $(H_1, \ldots, H_n) \in \mathcal{L}_{n,d_1,\ldots,d_n}$, by Corollary 11.2.3 we have $H_1 \cap \ldots \cap H_n \neq \emptyset$. Then for every point $P \in H_1 \cap \ldots \cap H_n$, $\pi^{-1}(H_1, \ldots, H_n)$, as a subset of $\{(H_1, \ldots, H_n)\} \times \mathcal{L}_{n,d_{n+1}} \cong \mathcal{L}_{n,d_{n+1}}$ contains the hyperplane $\mathcal{L}_{n,d_{n+1}}(P)$, and so it is non-empty. Moreover we have the following two possibilities:

(a) $H_1 \cap \ldots \cap H_n = \{P_1, \ldots, P_l\}$ is a finite set, then $\pi^{-1}(H_1, \ldots, H_n)$ coincides with $\bigcup_{i=1}^{l} \mathcal{L}_{n,d_{n+1}}(P_i)$, which is a divisor in $\mathcal{L}_{n,d_{n+1}}$;
(b) $H_1 \cap \ldots \cap H_n$ has some component of positive dimension, and then $\pi^{-1}(H_1, \ldots, H_n) = \mathcal{L}_{n,d_{n+1}}$ by Exercise 6.4.15.

Since $\dim(Z(n; d_1, \ldots, d_{n+1})) - \dim(\mathcal{L}_{n,d_1,\ldots,d_n}) = \dim(\mathcal{L}_{n,d_{n+1}}) - 1$, by Theorem 11.3.1 we have that there is a non-empty open subset $U \subseteq \mathcal{L}_{n,d_1,\ldots,d_n}$ such that for all $(H_1, \ldots, H_n) \in U$ case (a) and not case (b) occurs.

Theorem 11.4.3 *Let r, n be a positive integers with $r \leq n$. Then there is a non-empty open subset U of $\mathcal{L}_{n,d_1,\ldots,d_r}$ such that for all $(H_1, \ldots, H_r) \in U$, every irreducible component of $H_1 \cap \ldots \cap H_r$ has dimension $n - r$.*

Proof The case $r = n$ follows from the above considerations. To prove the assertion for $r < n$ we proceed by descending induction. Suppose the assertion holds for $\mathcal{L}_{n,d_1,\ldots,d_{r+1}}$ and let U' be the non-empty open subset of $\mathcal{L}_{n,d_1,\ldots,d_{r+1}}$ such that for all

$(H_1, \ldots, H_{r+1}) \in U'$, every component of $H_1 \cap \ldots \cap H_{r+1}$ has dimension $n - r - 1$. Since the projection

$$q : \mathcal{L}_{n,d_1,\ldots,d_{r+1}} \to \mathcal{L}_{n,d_1,\ldots,d_r}$$

is surjective, we have that $q_{|U'}$ is dominant to $\mathcal{L}_{n,d_1,\ldots,d_r}$, hence there is a non-empty open subset $U \subseteq \mathcal{L}_{n,d_1,\ldots,d_r}$ such that $U \subseteq q(U')$ (see Theorem 10.1.6). For all $(H_1, \ldots, H_r) \in U$, there is an $H_{r+1} \in \mathcal{L}_{n,d_{r+1}}$ such that every component of $H_1 \cap \ldots \cap H_{r+1}$ has dimension $n - r - 1$. Then every component of $H_1 \cap \ldots \cap H_r$ has dimension $n - r$ by Corollary 11.2.3. □

Le us now go back to the study of $Z(n; d_1, \ldots, d_{n+1})$ and of $R(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$.

Proposition 11.4.4 *The polynomial $R(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ is not constant with respect to any set of variables $\mathbf{x}_i$, for $i = 1, \ldots, n + 1$.*

Proof We prove that R is not constant with respect to the variables $\mathbf{x}_{n+1}$, the proof for $i = 1, \ldots, n$ being analogous. We keep the notation introduced in Theorem 11.4.3 and before. Let $U \subset \mathcal{L}_{n,d_1,\ldots,d_n}$ be the non-empty open subset for which case (a) above occurs for all $(H_1, \ldots, H_n) \in U$. Suppose H_j has homogeneous coordinates $[\mathbf{a}_j]$ in $\mathcal{L}_{n,d_j}$ for $j = 1, \ldots, n$. Then the equation $R(\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{x}_{n+1}) = 0$ defines the divisor $\sum_{i=1}^{l} \mathcal{L}_{n,d_{n+1}}(P_h)$, where $H_1 \cap \ldots \cap H_n = \{P_1, \ldots, P_l\}$. Hence $R(\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{x}_{n+1})$ is not constant, proving the assertion. □

Let α_i be the degree of R with respect to the variables $\mathbf{x}_i$, for $i = 1, \ldots, n + 1$.

Corollary 11.4.5 *In the above setting one has that, for all $i = 1, \ldots, n + 1$, α_i is bounded below by the maximum of the order of a finite set of points which is the intersection of n hypersurfaces of degrees $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n+1}$. In particular one has*

$$\alpha_i \geq d_1 \cdots d_{i-1} \cdot d_{i+1} \cdots d_{n+1}. \tag{11.3}$$

Proof The first part of the assertion is an immediate consequence of the proof of Proposition 11.4.4. As for the final part, we note that if we take H_j reducible in d_j suitable distinct hyperplanes for $j = 1, \ldots, i - 1, i + 1, \ldots, n + 1$, then $H_1 \cap \ldots \cap H_{i-1} \cap H_{i+1} \cap \ldots \cap H_{n+1}$ consists of $d_1 \cdots d_{i-1} \cdot d_{i+1} \cdots d_{n+1}$ distinct points (we leave the details to the reader). □

We can be more precise:

Theorem 11.4.6 *Equality holds in* (11.3).

Proof We will prove the assertion for $i = n + 1$, the proof being analogous in the other cases.

We need some preliminaries. We fix a positive integer d and consider $\mathcal{L}_{n,d_0,\ldots,d_{N(n,d)}}$ with $d_i = d$ for $i = 0, \ldots, N(n, d)$. Let $\mathbf{x}$ be the obvious coordinates in $\mathcal{L}_{n,d}$, so that $\mathbf{x}$ is a non-zero vector of order $N(n, d) + 1$ on $\mathbb{K}$. If $(H_0, \ldots, H_{N(n,d)}) \in \mathcal{L}_{n,d_0,\ldots,d_{N(n,d)}}$, and if $H_i = [\mathbf{x}_i]$, the matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{x}_0 \\ \dots \\ \mathbf{x}_{N(n,d)} \end{pmatrix}$$

is a square matrix of order $N(n, d) + 1$, and its determinant $\Delta(n, d)$ is a polynomial in $\mathbf{x}_0, \dots, \mathbf{x}_{N(n,d)}$ which is linear in each of these sets of variables, i.e., $\Delta(n, d)$ is a plurihomogeneous polynomial of degree 1 in $\mathbf{x}_0, \dots, \mathbf{x}_{N(n,d)}$, which is clearly irreducible. We will denote by $D(n, d)$ the zero locus of $\Delta(n, d)$ in $\mathcal{L}_{n,d_0,\dots,d_{N(n,d)}}$, which is a subvariety of codimension 1 in $\mathcal{L}_{n,d_0,\dots,d_{N(n,d)}}$. Then $D(n, d)$ represents the $(N(n, d) + 1)$-tuples of divisors of degree d in $\mathbb{P}^n$ that, as points of $\mathcal{L}_{n,d}$, are not linearly independent, i.e., they are contained in some proper projective subspace of $\mathcal{L}_{n,d}$.

Let us now go back to the study of $R(n; d_1, \dots, d_{n+1})$. Set $d = d_1 + \cdots + d_{n+1} - n$ and let us remark that the distinct monomials with coefficient 1 of degree d in $x_0, \dots, x_n$ enjoy the property that in each of them at least one of the variables x_i appears with degree at least d_{i+1}. Then these monomials can be all obtained, only once, under the form $x_i^{d_{i+1}} \mu_i$, with $i = 0, \dots, n$, where μ_i is a monomial of degree $d - d_{i+1}$ such that:

(a) if $i = 0$ does not satisfy any other condition;
(b) if $i = 1$ contains x_0 at degree at most $d_1 - 1$;
(c) if $i = 2$ contains x_j at degree at most $d_{j+1} - 1$, for $j = 0, 1$, etc.

Note that the number of monomials of type μ_{n+1} is $d_1 \cdots d_n$. Let ν_i be the number of monomials μ_i, for $i = 0, \dots, n + 1$. One has

$$\nu_0 + \cdots + \nu_n = N(n, d) + 1.$$

Denote by $M_{i,j}$, for $j = 1, \dots, \nu_i$, the monomials of type μ_i, for $i = 0, \dots, n$. Let $H_{i,j}$ be the divisor of $\mathbb{P}^n$ of degree $d - d_{i+1}$ with equation $M_{i,j} = 0$. Finally consider the morphism

$$\psi : \mathcal{L}_{n,d_1,\dots,d_{n+1}} \to \mathcal{L}_{n,d}^{N(n,d)+1}$$

that maps $(H_1, \dots, H_{n+1}) \in \mathcal{L}_{n,d_1,\dots,d_{n+1}}$ to

$$(H_1 + H_{0,1}, \dots, H_1 + H_{0,\nu_0}, \dots, H_{n+1} + H_{n,1}, \dots, H_{n+1} + H_{n,\nu_n}).$$

If $H_i = [\mathbf{a}_i] \in \mathcal{L}_{n,d_i}$, for $i = 1, \dots, n + 1$, then

$$\psi([\mathbf{a}_1], \dots, [\mathbf{a}_{n+1}]) = ([\mathbf{x}_0], \dots, [\mathbf{x}_{N(n,d)}])$$

where the vectors $\mathbf{x}_i$ are linear functions of $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$, and we will write $\mathbf{x}_i = \mathbf{x}_i(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})$.

Now $\psi^{-1}(D(n, d))$ is a closed subset of $\mathcal{L}_{n,d_1,\dots,d_{n+1}}$, which is defined by the equation

$$\Delta_n := \Delta(n,d)(\mathbf{x}_0(\mathbf{x}_1,\ldots,\mathbf{x}_{n+1}),\ldots,\mathbf{x}_{N(n,d)}(\mathbf{x}_1,\ldots,\mathbf{x}_{n+1})=0.$$

This equation is not identically zero, because $\psi^{-1}(D(n,d))$ is not equal to $\mathcal{L}_{n,d_1,\ldots,d_{n+1}}$. Note in fact that if H_i has equation $x_{i-1}^{d_i}$, for $i=1,\ldots,n+1$, then clearly $(H_1,\ldots,H_{n+1}) \notin \psi^{-1}(D(n,d))$. On the other hand it is clear that $Z(n;d_1,\ldots,d_{n+1}) \subseteq \psi^{-1}(D(n,d))$, hence R divides Δ_n. This polynomial has degree ν_i in $\mathbf{x}_i$, in particular it has degree $\nu_{n+1}=d_1\cdots d_n$ in $\mathbf{x}_{n+1}$. This implies that $\alpha_n \le d_1\cdots d_n$, as wanted. □

As an immediate consequence of the previous arguments we have:

Corollary 11.4.7 *The maximum number of finitely many points in common to n divisors of degrees $d_1,\ldots,d_n$ in $\mathbb{P}^n$ is $d_1\cdots d_n$.*

The arguments in the proof of Theorem 11.4.6 imply the following:

Corollary 11.4.8 (Lasker's Theorem) *Suppose that $(H_1,\ldots,H_{n+1}) \notin Z(n;d_1,\ldots,d_{n+1})$ and that H_i has equation $f_i=0$ for $i=1,\ldots,n+1$, then the ideal $(f_1,\ldots,f_{n+1})$ contains $S_{n,d}$ for every $d \ge d_1+\cdots+d_{n+1}-n$.*

Proof We keep the notation of the proof of Theorem 11.4.6. Let $\Delta_0,\ldots,\Delta_{n-1}$ be the polynomials obtained as Δ_n by exchanging the roles of the variables $x_0,\ldots,x_n$. Then R coincides with the largest common divisor D of $\Delta_0,\ldots,\Delta_n$. Indeed, R divides D and it has degree not smaller than the degree of D in each set of variables. Then if $R \neq 0$ there is some $i=0,\ldots,n$ such that $\Delta_i \neq 0$ and the assertion follows by taking into account the geometric interpretation of the loci $Z_s(\Delta_i)$. □

Let now $H_1,\ldots,H_n$ be divisors in $\mathbb{P}^n$ of degrees $d_1,\ldots,d_n$ with finitely many common points $\{P_1,\ldots,P_h\}$, with H_i having coordinates $\mathbf{a}_i$ in $\mathcal{L}_{n,d_i}$ for $i=1,\ldots,n$ and $P_j=[p_{j0},\ldots,p_{jn}]$ for $j=1,\ldots,h$. Consider $Z(n;d_1,\ldots,d_n,1)$, with equation $R(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{u})=0$, with $\mathbf{u}=(u_0,\ldots,u_n)$. The zero set of the equation $R(\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{u})=0$ consists of the union of the hyperplanes $\mathcal{L}_{n,1}(P_i)$, for $i=1,\ldots,h$, so that there are positive integers $r_1,\ldots,r_h$ such that

$$R(\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{u}) = \alpha\prod_{j=1}^{h}(p_{j0}u_0+\cdots+p_{jn}u_n)^{r_j},$$

where $\alpha \in \mathbb{K}^*$. The integer r_i is called the *intersection multiplicity* of $H_1,\ldots,H_n$ at P_i and it is denoted by $i(P_i;H_1,\ldots,H_n)$, for $i=1,\ldots,h$. Since the degree of R in the variables $\mathbf{u}$ is $d_1\cdots d_n$, we have

$$\sum_{i=1}^{h} i(P_i;H_1,\ldots,H_n) = d_1\cdots d_n. \tag{11.4}$$

We will set $i(P;H_1,\ldots,H_n)=0$ if $P \notin H_1\cap\ldots\cap H_n$. We will say that $H_1,\ldots,H_n$ intersect *transversally* at P if $i(P;H_1,\ldots,H_n)=1$. In conclusion we have the:

Theorem 11.4.9 (Bézout Theorem) *Let $H_1, \ldots, H_n$ be divisors of $\mathbb{P}^n$ of degrees $d_1, \ldots, d_n$ respectively, having only finitely many common points $\{P_1, \ldots, P_h\}$. Then (11.4) holds. Accordingly $h \leq d_1 \cdots d_n$ and the equality holds if and only if $H_1, \ldots, H_n$ intersect transversally at any of their common points.*

In particular, if $H_1, \ldots, H_n$ have more than $d_1 \cdots d_n$ distinct points in common, their intersection has an irreducible component of positive dimension.

If $H_1, \ldots, H_n$ are divisors in $\mathbb{P}^n$ such that $P \in H_1 \cap \ldots \cap H_n$ and $H_1 \cap \ldots \cap H_n$ has a component of positive dimension containing P, then one says that the intersection multiplicity of $H_1, \ldots, H_n$ at P is *infinite* and one writes $i(P; H_1, \ldots, H_n) = \infty$.

Exercise 11.4.10 Prove that $D(n, 1)$ coincides with $Z(n; 1, \ldots, 1)$ and $\Delta(n, 1)$ coincides with $R(n; 1, \ldots, 1)$; where 1 is repeated $n + 1$ times.

Exercise 11.4.11 Prove that $R(1; n, m)$ coincides with the Sylvester determinant, which is therefore irreducible.

Exercise 11.4.12 *Assume $\mathbb{K}$ of characteristic 0. Consider the divisor H of $\mathbb{P}^n$ of degree $d > 1$ with equation $f(x_0, \ldots, x_n) = 0$. Consider the polynomials $f_i = \frac{\partial f}{\partial x_i}$, for $i = 0, \ldots, n$. Consider the set $\mathfrak{D}(n, d) \subseteq \mathcal{L}_{n,d}$ consisting of all divisors H with equation $f = 0$ such that the system $f_0 = \ldots = f_n = 0$ has some non-trivial solution. Prove that $\mathfrak{D}(n, d)$ is a proper subset of $\mathcal{L}_{n,d}$.

Exercise 11.4.13 *Continuing Exercise 11.4.12, prove that $\mathfrak{D}(n, d)$ is a closed codimension 1 subset of $\mathcal{L}_{n,d}$ whose equation is

$$\mathcal{D}(n, d)(\mathbf{x}) = R(\mathbf{x}'_0, \ldots, \mathbf{x}'_n)$$

where R is the resultant polynomial of $n + 1$ homogeneous polynomials of degree $d - 1$ and $[\mathbf{x}]$ are the obvious homogeneous coordinates in $\mathcal{L}_{n,d}$ and $\mathbf{x}'_i$ is the vector of the coefficients of the polynomial f_i, where f is a polynomial such that $f = 0$ is the equation of $H = [\mathbf{x}]$. The polynomial $\mathcal{D}(n, d)$ is called the *discriminant* of homogenous polynomials of degree d in $n + 1$ variables. The locus $\mathfrak{D}(n, d)$ is called the *discriminant hypersurface* in $\mathcal{L}_{n,d}$.

Exercise 11.4.14 Continuing Exercise 11.4.13, note that the coordinates in $\mathcal{L}_{n,2}$ are of the form $[\mathbf{A}]$ where $\mathbf{A}$ is a non-zero symmetric matrix of order $n + 1$ on $\mathbb{K}$. Prove that

$$\mathcal{D}(n, d)(\mathbf{A}) = \det(\mathbf{A}).$$

So the discriminant hypersurface in $\mathcal{L}_{n,2}$ consists of quadrics not of maximal rank.

Exercise 11.4.15 *Continuing Exercise 11.4.13, prove that $\mathfrak{D}(1, d)$ consist of the set of non-reduced effective divisors of degree d on $\mathbb{P}^1$.

Exercise 11.4.16 Continuing Exercise 11.4.13, prove that if $n > 1$ and $H \in \mathcal{L}_{n,d}$ is reducible, then $H \in \mathfrak{D}(n, d)$.

Exercise 11.4.17 *Let $H_1, \ldots, H_n$ be divisors of $\mathbb{P}^n$ such that $H_1 \cap \ldots \cap H_n$ consists of finitely many points. Let $(i_1, \ldots, i_n)$ be any permutation of the set $\{1, \ldots, n\}$. Prove that $i(P; H_1, \ldots, H_n) = i(P, H_{i_1}, \ldots, H_{i_n})$ for all points $P \in H_1 \cap \ldots \cap H_n$.

11.5 Solutions of Some Exercises

11.2.10 By locating the three points at $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$, one sees that there are three independent homogeneous quadratic polynomials in $\mathcal{I}_p(Z)$. Whence we deduce that $\mathcal{I}_p(Z)$ cannot be generated by two elements.

11.2.12 Suppose Z consists of n distinct points $P_1, \ldots, P_n$. Up to a change of coordinates, we may assume that $P_i = (a_i, b_i)$, for $i = 1, \ldots, n$, with $a_1, \ldots, a_n$ all distinct. Consider the polynomial $f(x_1) = \sum_{i=1}^{n} b_i q_i(x_1)$, where

$$q_i(x_1) = \frac{(x_1 - a_1)\cdots(x_1 - a_{i-1})(x_1 - a_{i+1})\cdots(x_1 - a_n)}{(a_i - a_1)\cdots(a_i - a_{i-1})(a_i - a_{i+1})\cdots(a_i - a_n)},$$

for $i = 1, \ldots, n$. Show that $Z = Z_a(x_2 - f(x_1), \prod_{i=1}^{n}(x_1 - a_i))$.
11.2.13 Look, as we can, at $Z \setminus \{P\}$ as a subset of $\mathbb{A}^2 \subset \mathbb{P}^2$. Then, as we saw in the solution of Exercise 11.2.12, up to a change of coordinates in $\mathbb{A}^2$ we have that $Z \setminus \{P\}$ is a set-theoretic complete intersection $Z \setminus \{P\} = Z_a(x_2 - f(x_1), \prod_{i=1}^{n}(x_1 - a_i))$ in $\mathbb{A}^2$. Note now that $Z_p(\beta(x_2 - f(x_1)), \prod_{i=1}^{n}(x_1 - a_i x_0)) = Z \setminus \{P\} \cup \{[0, 0, 1]\}$, where β is the homogenizing operator (see Sect. 1.5). Moreover notice that the point $[0, 0, 1]$ can be chosen arbitrarily, provided it does not lie on any line joining a pair of distinct points of $Z \setminus \{P\}$. In particular we can choose $P = [0, 0, 1]$. This proves the assertion.
11.2.15 Argue as in Exercise 11.2.10.
11.2.17 Suppose that $\mathrm{rk}(\mathbf{A}_2) = n - 1$. Then the homogeneous linear system

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ &\cdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= 0 \end{aligned}$$

is equivalent to the one formed by the first $n - 1$ equations, which has a unique solution, up to a factor, consisting of the maximal minors of $\mathbf{A}_2$ with alternate signs. Since $\det(\mathbf{A}_1) = 0$, this solution is of the form $(b_1, \ldots, b_{n-1}, 0)$, hence we have

$$\begin{aligned} a_{11}b_1 + \ldots + a_{1,n-1}b_{n-1} &= 0 \\ &\cdots \\ a_{n1}b_1 + \ldots + a_{n,n-1}b_{n-1} &= 0 \end{aligned}$$

with $b_1, \ldots, b_{n-1}$ not all zero. This implies that $\mathrm{rk}(\mathbf{A}_3) < n - 1$.
11.2.18 Consider the matrix

$$\mathbf{A} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & f \end{pmatrix}$$

where f is any homogeneous polynomial of degree 1. With the same notation as in Exercise 11.2.17, if in a point of $\mathbb{P}^3$ one has $\det(\mathbf{A}) = \det(\mathbf{A}_1) = 0$, at that point one has either $\mathrm{rk}(\mathbf{A}_2) < 2$ or $\mathrm{rk}(\mathbf{A}_3) < 2$. But $\mathbf{A}_2$ and $\mathbf{A}_3$ are equal, up to transposition and $\mathrm{rk}(\mathbf{A}_2) < 2$ is just the matrix equation of the projective twisted cubic (see Sect. 3.2.2). So the projective twisted cubic is the set-theoretic intersection of the two surfaces with equations $\det(\mathbf{A}) = 0$ and $\det(\mathbf{A}_1) = 0$.
11.2.19 V is the affine part of the image of the morphism

$$\psi : [\lambda, \mu] \in \mathbb{P}^1 \to [\lambda^3\mu^2, \lambda^4\mu, \lambda^5, \mu^5] \in \mathbb{P}^3,$$

and this proves it is an affine curve in $\mathbb{A}^3$. To prove that it it not a complete intersection, argue as follows. First, observe that V is non-degenerate, i.e., it does not lie on any hyperplane in $\mathbb{A}^3$. Indeed, if a hyperplane with equation

$$a_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$$

contains V, we would have

$$a_0 + a_1t^3 + a_2t^4 + a_3t^5 = 0$$

for all $t \in \mathbb{K}$, and this would imply $a_0 = a_1 = a_2 = a_3 = 0$. Similarly verify that the unique quadratic polynomial in $\mathcal{I}_a(V)$ is $F = x_2^2 - x_1x_3$. Verify in addition that every polynomial in $\mathcal{I}_a(V)$ has no constant and linear terms. Suppose now, by contradiction, that V is a complete intersection, so that $\mathcal{I}_a(V) = (f, g)$. We have the following decompositions in homogeneous components

$$f = f_2 + f_3 + \cdots, \quad g = g_2 + g_3 + \cdots.$$

Note that the polynomials

$$\tilde{f} = f_3 + \cdots, \quad \tilde{g} = g_3 + \cdots$$

must be proportional. Indeed, we have

$$F = Af + Bg \tag{11.5}$$

with A, B suitable polynomials. Since F is homogeneous, we claim that A and B are constant. In fact we have

$$F = A_0 f + B_0 g + (A - A_0)f + (B - B_0)g$$

with A_0, B_0 the constant terms in A, B, and $(A - A_0)f + (B - B_0)g$ has only homogeneous components of degree larger than 2, so that $A = A_0$ and $B = B_0$. On the other hand, if A, B verify (11.5), then

$$F = Af_2 + Bg_2 + A\tilde{f} + B\tilde{g}$$

and, A, B being constant, we have $A\tilde{f} + B\tilde{g} = 0$ for the same reasons as above, so that $\tilde{f}$ and $\tilde{g}$ are proportional. We can then assume that one of the two polynomials f, g is equal to f_2 or g_2, namely equal to F. So we may assume that $g = F$. Note now that in $\mathcal{I}_a(V)$ there are the degree 3 polynomials $G = x_1^2x_2 - x_3^2$, $H = x_2x_3 - x_1^3$. Since

$$G = Af + Bg = A_0 f_2 + B_0 F + \cdots$$

with A, B suitable polynomials, and the dots stay for higher order terms, we must have

$$A_0 f_2 = -x_3^2 - B_0(x_2^2 - x_1x_3).$$

We have in addition

$$H = A'f + B'g = A_0' f_2 + B_0' F + \cdots$$

so that x_2x_3 should be a linear combination of F and of f_2, which is clearly impossible.

11.2.22 Suppose L is a complete intersection on Q, so that $L = Q \cap Z_p(f)$. Let L' be another line on Q that does not intersect L (see Exercise 8.2.16). Prove that there is a projective transformation which fixes Q and maps L to L', so that also L' is a set theoretic complete intersection on Q. Then $L' = Q \cap Z_p(f')$. Then $Q \cap Z_p(f, f') = \emptyset$, contrary to Corollary 11.2.3.

11.2.23 It suffices to consider the case in which H is irreducible. Then $v_{n,d}^{-1}(H)$ is an irreducible hypersurface H' of a certain degree m in $\mathbb{P}^n$, and let $f = 0$ be an equation of H, with f a homogeneous polynomial of degree m. By the proof of Proposition 6.4.1, there is a homogeneous polynomial $F \in \mathbb{K}[v_{i_0 \dots i_n}]_{i_0 + \cdots + i_n = d}$ of degree m such that $\theta_{n,d}(F) = f^d$ and $H = V_{n,d} \cap Z_p(F)$.

11.2.24 This is an immediate consequence of Corollary 11.2.3. Indeed, locally, one has $f_i = \frac{P_i}{P_0}$, for $i = 1, \dots, r$, with $P_0, \dots, P_r$ homogeneous polynomials of the same degree, so that $Z_V(f_1, \dots, f_r)$ can be locally written as $V \cap Z_p(P_1, \dots, P_r)$.

11.3.4 It is clear that $W = \bigcup_{P \in V}(\mathbb{P}_1 \vee P) = \bigcup_{P \in p(V)}(\mathbb{P}_1 \vee P)$. Consider the blow-up $\pi : \tilde{\mathbb{P}} \to \mathbb{P}^n$ of $\mathbb{P}^n$ along $\mathbb{P}^1$ (see Exercise 8.3.13) and take $\tilde{W}$ the proper transform of W. Use Corollary 11.3.3

to show that $\tilde{W}$ is irreducible of dimension $\dim(V) + \dim(\mathbb{P}_1) + 1$. Deduce that W is irreducible of dimension $\dim(V) + \dim(\mathbb{P}_1) + 1$.

11.3.5 Suppose $V \subseteq \mathbb{P}^n$ is a projective variety. Let us think to $\mathbb{P}^n$ as embedded in $\mathbb{P}^{n+1}$ as the hyperplane at infinity of $\mathbb{A}^{n+1}$. Consider the affine cone V' with vertex the origin over V, whose closure in $\mathbb{P}^{n+1}$ is the projective cone. By Exercise 11.3.4 we have $\dim(V') = \dim(V) + 1$. But $A(V') = S(V)$ because $\mathcal{I}_a(V') = \mathcal{I}_p(V)$ (see Sect. 3.2.3), and the assertion follows.

11.3.6 The irreducibility of V is clear. Let us prove the dimensional statement. Since f is not constant, in f really appear the variables of at least one of the sets $\mathbf{x}^1, \ldots, \mathbf{x}^r$. We may assume that f is not constant in the variables $\mathbf{x}^r$. Consider the projection $p : V \to \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_{r-1}}$. If $P = ([\mathbf{a}^1], \ldots, [\mathbf{a}^{r-1}]) \in \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_{r-1}}$, then the fibre V_P is the subset of $\{P\} \times \mathbb{P}^{n_r} \cong \mathbb{P}^{n_r}$, with equation $f(\mathbf{a}^1, \ldots, \mathbf{a}^{r-1}, \mathbf{x}^r) = 0$ in $\mathbb{P}^{n_r}$, so that we have the three possibilities:

(i) $V_P = \{P\} \times \mathbb{P}^{n_r}$;
(ii) V_P is a divisor in $\{P\} \times \mathbb{P}^{n_r} \cong \mathbb{P}^{n_r}$;
(iii) $V_P = \emptyset$.

Case (i) happens if and only if $f(\mathbf{a}^1, \ldots, \mathbf{a}^{r-1}, \mathbf{x}^r)$ is identically 0, and, by the assumptions we made on f, this may occur only if P belongs to a proper closed subset of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_{r-1}}$. Case (iii) occurs only if $f(\mathbf{a}^1, \ldots, \mathbf{a}^{r-1}, \mathbf{x}^r)$ is a non-zero constant, which cannot happen under the given assumptions for f. So, if P varies in a non-empty open subset of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_{r-1}}$ only case (ii) can occur. Then by Theorem 11.3.1 the assertion follows.

11.3.7 This can be proved as Theorem 11.1.2.

11.3.8 Consider the set

$$I = \{(P, Q, R) \in (X \times X \setminus \Delta) \times \mathbb{P}^r : R \in \langle P, Q \rangle\}$$

where Δ is the diagonal in $X \times X$. One sees that this is a closed subset in $(X \times X \setminus \Delta) \times \mathbb{P}^r$, of dimension $2n + 1$ (use Theorem 11.3.1). Therefore its closure $\bar{I}$ in $X \times X \times \mathbb{P}^r$ also has dimension $2n + 1$. Then $\mathrm{Sec}(X)$ is the image of $\bar{I}$ via the projection of $X \times X \times \mathbb{P}^r$ on $\mathbb{P}^r$. The assertion follows.

11.4.12 For example the divisor with equation $x_0^d + \cdots + x_n^d = 0$ does not lie in $\mathfrak{D}(n, d)$.

Chapter 12
The Cayley Form

12.1 Definition of the Cayley Form

Let V be a projective variety of dimension m in $\mathbb{P}^n$. We can associate to V a variety of codimension 1 in $\mathcal{L}_{n,1}^{m+1}$ in the following way. Set $\tilde{V} = \mathcal{L}_{n,1}^{m+1} \times V$ and consider the two projections $p : \tilde{V} \to \mathcal{L}_{n,1}^{m+1}$ and $q : \tilde{V} \to V$. Consider the subset W of $\tilde{V}$ defined in the following way

$$W = \{(H_0, \dots, H_m, P) \in \tilde{V} : P \in H_0 \cap \dots \cap H_m\}.$$

Lemma 12.1.1 *W is a closed subset of $\tilde{V}$.*

Proof $\tilde{V}$ is a closed subset of $\mathcal{L}_{n,1}^{m+1} \times \mathbb{P}^n$, so it suffices to show that there is a closed subset W' of $\mathcal{L}_{n,1}^{m+1} \times \mathbb{P}^n$ such that $W = W' \cap \tilde{V}$. Let us set

$$W' = \{(H_0, \dots, H_m, P) \in \mathcal{L}_{n,1}^{m+1} \times \mathbb{P}^n : P \in H_0 \cap \dots \cap H_m\}.$$

One has $W = W' \cap \tilde{V}$. Moreover W' is closed in $\mathcal{L}_{n,1}^{m+1} \times \mathbb{P}^n$ since it is defined by the equations

$$\begin{gathered} u_{00}x_0 + \cdots + u_{0n}x_n = 0 \\ \dots \\ u_{m0}x_0 + \cdots + u_{mn}x_n = 0 \end{gathered}$$

where $[u_{i0}, \dots, u_{in}]$ are the homogeneous coordinates in the $(i + 1)$th factor of $\mathcal{L}_{n,1}^{m+1}$, for $i = 0, \dots, m$ and $[x_0, \dots, x_n]$ are the homogeneous coordinates in $\mathbb{P}^n$. □

Lemma 12.1.2 *W is irreducible of dimension $n(m + 1) - 1$.*

Proof For every point $P \in V$, the hyperplanes of $\mathbb{P}^n$ containing P form a hyperplane $\mathcal{L}_{n,1}(P)$ in $\mathcal{L}_{n,1}$. It is then clear that

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$$W \cap q^{-1}(P) = \mathcal{L}_{n,1}(P)^{m+1} \times \{P\},$$

hence $W \cap q^{-1}(P)$ is isomorphic to $\mathcal{L}_{n,1}(P)^{m+1} \cong (\mathbb{P}^{n-1})^{m+1}$, so that it is irreducible of dimension $(m+1)(n-1)$. By applying Corollary 11.3.3, we see that W is irreducible of dimension

$$\dim(V) + (m+1)(n-1) = m + (m+1)(n-1) = n(m+1) - 1.$$

□

Lemma 12.1.3 *In the above setting, $p(W)$ is an irreducible variety of codimension 1 in $\mathcal{L}_{n,1}(P)^{m+1}$.*

Proof By applying Theorem 11.3.1 it suffices to show that there are points $Q \in W$ such that $W \cap p^{-1}(p(Q))$ is a finite set. By iterated applications of Theorem 11.2.2 we see that for any point $P \in V$ we can find $(H_0, \ldots, H_m) \in \mathcal{L}_{n,1}^{m+1}$ such that $V \cap H_0 \cap \ldots \cap H_m = \{P\}$ and the assertion follows. □

We will set $C_V = p(W) \subset \mathcal{L}_{n,1}^{m+1} \cong (\mathbb{P}^n)^{m+1}$ and we will call C_V the *Cayley variety* of V. Then there is an irreducible plurihomogeneous polynomial $F_V(\mathbf{u}_0, \ldots, \mathbf{u}_m)$ in the variables $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, for $i = 0, \ldots, m$, such that $C_V = Z_s(F_V)$. Of course F_V is determined up to a non-zero multiplicative constant. We will say that F_V is the *Cayley form* of V. We will denote by d_i the degree of F_V in the variables $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, for $i = 0, \ldots, m$.

Lemma 12.1.4 *One has $d_0 = \cdots = d_m$.*

Proof Let $0 \le i < j \le m$ and consider the map

$$\phi_{ij} : (H_0, \ldots, H_i, \ldots, H_j \ldots, H_m) \in \mathcal{L}_{n,1}^{m+1} \to (H_0, \ldots, H_j, \ldots, H_i \ldots, H_m) \in \mathcal{L}_{n,1}^{m+1}$$

which is an isomorphism. It is clear that ϕ_{ij} maps C_V to itself. The assertion follows. □

Thus, given V, we have the polynomial F_V, determined up to a non-zero multiplicative constant, which is plurihomogeneous of degree $d = d_0 = \cdots = d_m$ in each set of variables $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, for $i = 0, \ldots, m$. The positive integer d is called the *degree* of the variety V, and it is denoted by $\deg(V)$.

Consider now $\mathcal{V}_{n,m,d}$ the set of all projective varieties of dimension m and degree d in $\mathbb{P}^n$. Moreover set $\mathbb{P}_{n,m,d} := \mathbb{P}(S_{\mathbf{n},\mathbf{d}})$ where $\mathbf{n}$ is the vector with $m+1$ entries $(n, \ldots, n)$ and $\mathbf{d}$ is the vector with $m+1$ entries $(d, \ldots, d)$. Then we have the *Cayley map*

$$\gamma_{n,m,d} : V \in \mathcal{V}_{n,m,d} \to [F_V] \in \mathbb{P}_{n,m,d}.$$

Theorem 12.1.5 *The Cayley map $\gamma_{n,m,d}$ is injective for all n, m, d.*

Proof Consider $V \in \mathcal{V}_{n,m,d}$. If $P \in \mathbb{P}^n \setminus V$, by Theorem 11.2.2 we can find a point $(H_0, \dots, H_m) \in \mathcal{L}_{n,1}^{m+1}$, such that $P \in H_0 \cap \dots \cap H_m$ with $H_0 \cap \dots \cap H_m \cap V = \emptyset$. Consider $V' \in \mathcal{V}_{n,m,d}$ with $V' \neq V$, so that there is a point $P \in V'$ with $P \notin V$. By the previous observation, we can find a point $(H_0, \dots, H_m) \in C_{V'}$ but $(H_0, \dots, H_m) \notin C_V$, proving the assertion. □

Theorem 12.1.5 says that any variety V can be reconstructed from its Cayley form. Actually, given F_V, hence C_V, V is the set of points $P \in \mathbb{P}^n$ such that for all $(H_0, \dots, H_m) \in \mathcal{L}_{n,1}(P)^{m+1}$ one has $(H_0, \dots, H_m) \in C_V$.

Exercise 12.1.6 Let $V \subseteq \mathbb{P}^n$ be a variety, let F_V be its Cayley form and let $\tau : \mathbb{P}^n \to \mathbb{P}^n$ be a projectivity, defined by a matrix $\mathbf{A} \in \mathrm{GL}(n+1, \mathbb{K})$. Prove that

$$F_{\tau(V)}(\mathbf{u}_0, \dots, \mathbf{u}_n) = F_V(\mathbf{u}_0 \cdot \mathbf{B}, \dots, \mathbf{u}_n \cdot \mathbf{B})$$

where $\mathbf{B} = (\mathbf{A}^t)^{-1}$. This proves that the theory related to the Cayley form is invariant by change of variables or projectivities.

Exercise 12.1.7 Determine the Chow form of a point $P \in \mathbb{P}^n$.

Exercise 12.1.8 More generally, determine the Chow form of a linear subspace of $\mathbb{P}^n$ and deduce that the linear subspaces have degree 1.

12.2 The Degree of a Variety

Next we interpret geometrically the notion of degree of a variety. By Corollary 10.2.6 and by Theorem 11.2.2, given a variety V of dimension m and degree d in $\mathbb{P}^n$, there are projective subspaces Π of $\mathbb{P}^n$ of dimension $n - m$ such that $\Pi \cap V$ is a finite set. Hence it is not empty the subset Σ_V of $\mathcal{L}_{n,1}^m$ of points $(H_1, \dots, H_m) \in \mathcal{L}_{n,1}^m$ such that $H_1 \cap \dots \cap H_m \cap V$ is a finite set of points $\{P_1, \dots, P_h\}$. We set $P_i = [p_{i0}, \dots, p_{in}]$, for $i = 1, \dots, h$, and $H_j = [\mathbf{a}_j] = [a_{j0}, \dots, a_{jn}]$, for $j = 1, \dots, m$. Let $F_V(\mathbf{u}_0, \dots, \mathbf{u}_m)$ be the Cayley form of V. The set of zeros in $\mathcal{L}_{n,1}$ of the equation

$$F_V(\mathbf{u}_0, \mathbf{a}_1 \dots, \mathbf{a}_m) = 0$$

is the union of the h hyperplanes $\mathcal{L}_{n,1}(P_i)$, for $i = 1, \dots, h$, which have equation

$$u_{00}p_{i0} + \dots + u_{0n}p_{in} = 0, \quad \text{for} \quad i = 1, \dots, n.$$

Hence we have

$$F_V(\mathbf{u}_0, \mathbf{a}_0, \dots, \mathbf{a}_m) = \alpha \prod_{i=1}^{h} (u_{00}p_{i0} + \dots + u_{0n}p_{in})^{r_i} \tag{12.1}$$

where $\alpha \in \mathbb{K}^*$ and $r_1, \dots, r_h$ are suitable positive integers such that $r_1 + \dots + r_h = d$, in particular $h \leq d$. If Π is the $(n - m)$-dimensional subspace $\Pi = H_1 \cap \dots \cap$

H_m, we will say that the positive integer r_i is the *intersection multiplicity* of Π and V at P_i and it is denoted by $i(P_i; V, \Pi)$, for $i = 1, \ldots, h$. This number only depends on Π and not on the choice of the hyperplanes $H_1, \ldots, H_m$ such that $\Pi = H_1 \cap \ldots \cap H_m$ (see Exercise 12.2.4). We will set $i(P; V, \Pi) = 0$ if $P \notin \Pi \cap H$ and we will say that Π and V intersect *transversally* at P if $i(P; V, \Pi) = 1$. If $P \in \Pi \cap V$ and $\Pi \cap V$ has some positive dimensional component containing P, one says that the intersection multiplicity of Π and V at P is *infinite* and one writes $i(P; V, \Pi) = \infty$.

One has:

Proposition 12.2.1 *If $H_1, \ldots, H_m$ are hyperplanes of $\mathbb{P}^n$ such that $V \cap H_1 \cap \ldots \cap H_m$ is a finite set of points of order h, then $h \le \deg(V)$. More precisely, if Π is a linear subspace of $\mathbb{P}^n$ of dimension $n - m$ such that $\Pi \cap V$ is a finite set $\{P_1, \ldots, P_h\}$ then $\deg(V)$ equals the sum of the intersection multiplicities of V and Π at the points $P_1, \ldots, P_h$.*

We can be more precise. For this we need an algebraic preliminary:

Lemma 12.2.2 *Let $F(\mathbf{x}, \mathbf{y}) \in \mathbb{K}[\mathbf{x}, \mathbf{y}]$ be an irreducible polynomial in the variables $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_m)$. Then either there is some point $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{A}^m$ such that $F(\mathbf{x}, \mathbf{a})$ has no multiple factor, or there is a polynomial $G(\mathbf{x}, \mathbf{y}) \in \mathbb{K}[\mathbf{x}, \mathbf{y}]$ such that*

$$F(x_1, \ldots, x_n, \mathbf{y}) = G(x_1^p, \ldots, x_n^p, \mathbf{y})$$

where $p > 0$ is the characteristic of $\mathbb{K}$. In particular, if $\mathbb{K}$ has characteristic 0, only the first alternative occurs.

Proof It clearly suffices to consider only the case $n = 1$. If $\frac{\partial F}{\partial x_1} = 0$, then it is immediate that the characteristic of $\mathbb{K}$ is $p > 0$ and there is a polynomial $G(x_1, \mathbf{y})$ such that $F(x_1, \mathbf{y}) = G(x_1^p, \mathbf{y})$. Suppose next that $\frac{\partial F}{\partial x_1} \ne 0$. Then $Z_a(F, \frac{\partial F}{\partial x_1})$ is a proper closed subset of $Z_a(F) \subset \mathbb{A}^{m+1}$, hence each component of $Z_a(F, \frac{\partial F}{\partial x_1})$ has dimension $h < m$. The projection of $Z_a(F, \frac{\partial F}{\partial x_1})$ from the point at infinity of the x_1 axis to the hyperplane $x_1 = 0$ is dominant on a closed subset Z any component of which has dimension $l < m$. By contrast, the projection of $Z_a(F)$ from the point at infinity of the x_1 axis to the hyperplane $x_1 = 0$ is dominant because F depends on the variable x_1. Hence there are points $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{A}^m$ such that for any point $b \in \mathbb{K}$ such that $F(b, \mathbf{a}) = 0$ one has $\frac{\partial F}{\partial x_1}(b, \mathbf{a}) \ne 0$. The assertion follows. □

We can now prove the:

Theorem 12.2.3 *Let $V \subseteq \mathbb{P}^n$ be a projective variety of dimension m and degree d. The two subsets $U_1 \subseteq U_2$ of $\mathcal{L}^m_{n,1}$ respectively formed by the m-tuples $(H_1, \ldots, H_m)$ such that $V \cap H_1 \cap \ldots \cap H_m$ consists of d distinct points and of finitely many points, are non-empty open subsets of $\mathcal{L}^m_{n,1}$.*

In particular there are subspaces Π of $\mathbb{P}^n$ of dimension $n - m$ intersecting V transversally in d distinct points.

Proof The set U_2 consists of all m-tuples $(H_1, \dots, H_m)$, with $H_i = [\mathbf{a}_i]$, $i = 1, \dots, m$, such that $F_V(\mathbf{u}_0, \mathbf{a}_1, \dots, \mathbf{a}_m) \neq 0$. It is the clear that U_2 is a non-empty open subset of $\mathcal{L}^m_{n,1}$. Moreover we have a morphism

$$\phi : U_2 \to \bar{\mathcal{L}}_{n,d},$$

where $\bar{\mathcal{L}}_{n,d}$ is the projective space whose points represent the divisors of degree d of $\mathcal{L}_{n,1}$, which maps $(H_1, \dots, H_m)$, with $H_i = [\mathbf{a}_i]$, $i = 1, \dots, m$, to the divisor with equation $F_V(\mathbf{u}_0, \mathbf{a}_1, \dots, \mathbf{a}_m) = 0$. Let $\mathcal{U}$ be the non-empty open subset of $\bar{\mathcal{L}}_{n,d}$ whose points represent the hypersurfaces with no multiple components (see Exercises 9.2.5 and 11.4.15). One has $U_1 = \phi^{-1}(\mathcal{U})$, hence U_1 is open.

Finally we have to prove that U_1 is non-empty. By Lemma 12.2.2, this is clear if $\mathrm{char}(\mathbb{K}) = 0$. If $\mathrm{char}(\mathbb{K}) = p > 0$, by Lemma 12.2.2 this is still clear, unless there is a polynomial $G(\mathbf{u}_0)$ such that

$$F_V(\mathbf{u}_0, \mathbf{a}_1, \dots, \mathbf{a}_m) = G(u_{00}^p, \dots, u_{0n}^p).$$

Since $F_V(\mathbf{u}_0, \dots, \mathbf{u}_m)$ is symmetric with respect to the variables $\mathbf{u}_0, \dots, \mathbf{u}_m$ (see the proof of Lemma 12.1.4), This would imply that there is a polynomial $G_V(\mathbf{u}_0, \dots, \mathbf{u}_m)$ such that

$$F_V(\mathbf{u}_0, \dots, \mathbf{u}_m) = G_V(u_{00}^p, \dots, u_{0n}^p, \dots, u_{m0}^p, \dots, u_{mn}^p)$$

But this implies that F_V is the p–power of a polynomial, hence it is reducible, a contradiction. □

Exercise 12.2.4 *Let V be a projective variety of dimension m in $\mathbb{P}^n$, let Π be a projective subspace of $\mathbb{P}^n$ of dimension $n - m$ such that $\Pi \cap V$ is a finite set. Let P be a point in $\Pi \cap V$. Prove that $i(P; V, \Pi)$ depends only on Π and not on the choice of the hyperplanes $H_1, \dots, H_m$ such that $\Pi = H_1 \cap \dots \cap H_m$.

Exercise 12.2.5 *Prove that if $V \subseteq \mathbb{P}^n$ is a subvariety of dimension m and degree 1, then V is a linear subspace.

Exercise 12.2.6 *Let $V \subseteq \mathbb{P}^n$ be a variety of dimension m which is the set–theoretic complete intersection of $n - m$ hypersurfaces $H_{m+1}, \dots, H_n$ of respective degrees $d_{m+1}, \dots, d_n$. Prove that $\deg(V) \leq d_{m+1} \cdots d_n$.

Exercise 12.2.7 *Let $H \subseteq \mathbb{P}^n$ be an irreducible hypersurface of degree d. Prove that $\deg(H) = d$.

Exercise 12.2.8 *Let $H \subseteq \mathbb{P}^n$ be an irreducible hypersurface of degree d. Find the Cayley form of H.

Exercise 12.2.9 *Let $H \subseteq \mathbb{P}^n$ be a hypersurface of degree d with equation $f(\mathbf{x}) = f(x_0, \dots, x_n) = 0$, where f is a homogeneous polynomial of degree d. Let L be a line not contained in H so that $L \cap H$ is a finite set $\{P_1, \dots, P_h\}$, with $P_i = [\mathbf{p}_i]$ for $i = 1, \dots, h$. Let $P = [\mathbf{p}]$, $Q = [\mathbf{q}]$ be distinct points of L, so that L is parametrically represented by

$$\mathbf{x} = \lambda\mathbf{p} + \mu\mathbf{q}, \quad \text{with} \quad [\lambda, \mu] \in \mathbb{P}^1.$$

Then for all $i = 1, \dots, h$, we have $\mathbf{p}_i = \lambda_i\mathbf{p} + \mu_i\mathbf{q}$ for suitable $[\lambda_i, \mu_i] \in \mathbb{P}^1$. Then the non-trivial solutions of the equation $f(\lambda\mathbf{p} + \mu\mathbf{q}) = 0$ in (λ, μ) are exactly (λ_i, μ_i) up to a multiplicative constant, for $i = 1, \ldots, h$. This means that there is a constant $\alpha \in \mathbb{K}^*$ and there are positive integers $s_1, \ldots, s_h$ such that

$$f(\lambda \mathbf{p} + \mu \mathbf{q}) = \alpha \prod_{i=1}^{h} (\lambda \mu_i - \mu \lambda_i)^{s_i}.$$

Prove that, for all $i = 1, \ldots, h$, one has $s_i = i(P_i; L, H)$.

Exercise 12.2.10 Prove that the Veronese variety $V_{n,d}$ in $\mathbb{P}^{N(n,d)}$ has degree d^n (see Corollary 11.4.7).

Exercise 12.2.11 *Prove that the Segre variety $\mathrm{Seg}_{n,1}$ has degree $n + 1$.

Exercise 12.2.12 *Let $V \subseteq \mathbb{P}^n$ be a projective subvariety of dimension m and let P be a point of V. Suppose that for any point $Q \in V \setminus \{P\}$ the line $P \vee Q$ is contained in V. Prove that V is a cone with vertex P over a variety W of dimension $m - 1$ contained in a hyperplane $H \cong \mathbb{P}^{n-1}$ of $\mathbb{P}^n$ non passing through P. Prove that $\deg(V) = \deg(W)$.

Exercise 12.2.13 *Let $V \subseteq \mathbb{P}^n$ be a projective subvariety of dimension m and let P be a point of V. Suppose that there is a point $Q \in V \setminus \{P\}$ such that the line $P \vee Q$ is not contained in V. Then the projection of $\mathbb{P}^n$ from P to a hyperplane $H \cong \mathbb{P}^{n-1}$ not containing P induces a rational map $\phi : V \dashrightarrow H$ which is dominant to a variety $W \subseteq H$. Prove that W has also dimension m and that $\deg(W) < \deg(V)$.

Exercise 12.2.14 Let $V \subseteq \mathbb{P}^n$ be a projective subvariety of dimension m and let P be a point not on V. Prove that the cone W with vertex P over V has degree $\deg(W) \leq \deg(V)$.

Exercise 12.2.15 Let $V \subsetneq \mathbb{P}^n$ be a projective subvariety of dimension $m < n$ and degree 2. Prove that V is contained in a subspace Π of dimension $m + 1$ and it is a quadric in Π.

Exercise 12.2.16 *Let $V \subseteq \mathbb{P}^n$ be a non-degenerate variety of degree d and dimension m. Prove that

$$d \geq n - m + 1. \tag{12.2}$$

Varieties for which the equality holds in (12.2) are called *varieties of minimal degree*. Examples of varieties of minimal degree are quadrics, the Veronese surface $V_{2,2}$, the rational normal curves, cones over rational normal curves.

Exercise 12.2.17 *Prove that any variety of minimal degree is rational.

Exercise 12.2.18 Prove that there is a non-empty open subset U of $\mathcal{L}_{n,d_1,\ldots,d_n}$ such that for all $(H_1, \ldots, H_n) \in U$ the intersection of $H_1 \ldots, H_n$ consists of $d_1 \cdots d_n$ distinct points.

12.3 The Cayley Form and Equations of a Variety

In this section we see that from the Cayley form of a variety $V \subseteq \mathbb{P}^n$ of dimension m we can reconstruct finitely many homogeneous polynomials $f_1, \ldots, f_h$ in $x_0, \ldots, x_n$, such that $V = Z_p(f_1, \ldots, f_h)$.

Let $F_V(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ be the Cayley form of V. Given $P = [p_0, \ldots, p_n] \in \mathbb{P}^n$, we have that $P \in V$ if and only if for any $(H_0, \ldots, H_m) \in \mathcal{L}_{n,1}(P)^{m+1}$, with $H_i = [\mathbf{v}_i]$, with $i = 0, \ldots, m$, one has $F_V(\mathbf{v}_0, \ldots, \mathbf{v}_m) = 0$.

Let us express the condition for a hyperplane $H \in \mathcal{L}_{n,1}$ with equation

$$u_0 x_0 + \cdots + u_n x_n = 0$$

to belong to $\mathcal{L}_{n,1}(P)$.

Lemma 12.3.1 *In the above setting H belongs to $\mathcal{L}_{n,1}(P)$ if and only if there is a non-zero antisymmetric matrix $\mathbf{S} = (s_{ij})_{i,j=0,\dots,n}$ of order $n+1$ on $\mathbb{K}$ such that*

$$u_i = \sum_{j=0}^{n} s_{ij} p_j, \quad \textit{for} \quad i = 0, \dots, n$$

Proof The condition is sufficient, because

$$\sum_{i=0}^{n} u_i p_i = \sum_{i=0}^{n} \sum_{j=0}^{n} s_{ij} p_j p_i = 0$$

since $\mathbf{S}$ is antisymmetric. Let us prove that the condition is also necessary. Let $P_1, \dots, P_{n-1}$ be points of H such that $P_1, \dots, P_{n-1}, P$ are linearly independent. Suppose $P_i = [p_{i0}, \dots, p_{in}]$, for $i = 1, \dots, n-1$. Then $(u_0, \dots, u_n)$ is a solution of the system

$$\begin{aligned}
&u_0 p_0 + \ldots + u_n p_n = 0 \\
&u_0 p_{10} + \ldots + u_n p_{1n} = 0 \\
&\qquad\ldots \\
&u_0 p_{n-1,0} + \ldots + u_n p_{n-1,n} = 0
\end{aligned}$$

So $(u_0, \dots, u_n)$ is proportional to the minors of maximal order of the matrix

$$\begin{pmatrix} p_0 & \cdots & p_n \\ p_{01} & \cdots & p_{0n} \\ & \cdots & \\ p_{n-1,1} & \cdots & p_{n-1,n} \end{pmatrix}$$

taken with alternate signs. So we can write an equation of H in the form

$$\begin{vmatrix} x_0 & \cdots & x_n \\ p_0 & \cdots & p_n \\ p_{01} & \cdots & p_{0n} \\ & \cdots & \\ p_{n-1,1} & \cdots & p_{n-1,n} \end{vmatrix} = 0. \tag{12.3}$$

If we expand the determinant in (12.3) with the Laplace rule applied to the first two rows, we have that (12.3) can be written as

$$\sum_{0\le i<j\le n}(x_i p_j - x_j p_i)p^{ij} = 0$$

where p^{ij} is the determinant of the matrix obtained from the last $n-1$ rows of the determinant in (12.3) by deleting the ith and jth columns, taken with the sign $(-1)^{i+j}$. We will set $p^{ji} = -p^{ij}$, so we give sense to p^{ji} even if $j > i$. Then we have

$$\sum_{0\le i<j\le n}(x_i p_j - x_j p_i)p^{ij} = \sum_{0\le i<j\le n} x_i p_j p^{ij} + \sum_{0\le i<j\le n} x_i p_j p^{ji} = \sum_{i,j=0,\dots,n} x_i p_j p^{ij}.$$

Then the assertion follows by setting $s_{ij} = \rho p^{ij}$ with ρ a suitable non-zero factor. □

Let us introduce now $m+1$ antisymmetric matrices $(s^h_{ij})_{i,j=0,\dots,n}$, $h = 0,\dots,m$, with entries indeterminates on $\mathbb{K}$. If $P = [x_0,\dots,x_n] \in \mathbb{P}^n$, we set

$$u^h_i(\mathbf{x}) = \sum_{j=0}^{n} s^h_{ij}x_j, \quad \text{for} \quad i = 0,\dots,n,\ h = 0,\dots,m,$$

and $\mathbf{u}^h = (u^h_0(\mathbf{x}),\dots,u^h_n(\mathbf{x}))$ for $h = 0,\dots,m$. Then P belong to V if and only if

$$F_V(\mathbf{u}^0,\dots,\mathbf{u}^m) \equiv 0. \tag{12.4}$$

The left hand side of (12.4) is a polynomial in the variables s^h_{ij} and x_j. This polynomial is identically zero if and only if are zero all the coefficients of the independent monomials in the variables s^h_{ij}. These coefficients are homogeneous polynomials in $\mathbf{x}$, so by equating to zero these coefficients one gets a set of equations for V.

12.4 Cycles and Their Cayley Forms

In this section we extend the notion of Cayley form to not necessarily irreducible algebraic closed subset of projective space.

Let $V \subseteq \mathbb{P}^n$ be a pure closed subset, so that all of its irreducible components $V_1,\dots,V_h$ have the same dimension m (see Sect. 5.5). The set $C_V \subset \mathcal{L}^{m+1}_{n,1}$ of points $(H_0,\dots,H_m)$ such that $H_0 \cap \dots \cap H_m \cap V \neq 0$ clearly coincides with $C_{V_1} \cup \dots \cup C_{V_h}$, and actually $C_{V_1},\dots,C_{V_h}$ are the irreducible components of C_V. We will set $F_V = F_{V_1}\cdots F_{V_h}$. The equation $F_V = 0$ defines C_V in $\mathcal{L}^{m+1}_{n,1}$ and it is the minimal degree equation defining C_V in $\mathcal{L}^{m+1}_{n,1}$. We call F_V the *Cayley form* of V, and we define *degree* of V the degree of F_V. With this definition the analogues of Theorems 12.1.5 and 12.2.3 still hold.

We can further extend this definition. Let $D_{n,m}$ be the free group generated by the varieties of dimension m in $\mathbb{P}^n$, whose elements will be called m-dimensional *cycles* of $\mathbb{P}^n$ (if $m = 0$, the cycle is called a 0-*cycle*). An element of $D_{n,m}$ is either zero or of the form $V = \sum_{i=1}^{h} m_i V_i$, where $V_1, \ldots, V_h$ are distinct varieties of dimension m in $\mathbb{P}^n$ which are called the *irreducible components* of V, and $m_1, \ldots, m_h$ are non-zero integers which are called the *multiplicities* of $V_1, \ldots, V_h$ for V. The integer $\sum_{i=1}^{h} m_d \deg(V_i)$ is called the *degree* of V and is denoted by $\deg(V)$. If $V = \sum_{i=1}^{h} m_i V_i$, then the closed subset $\bigcup_{i=1}^{h} V_i$ is called the *support* of V. If either $V = 0$ or $m_i > 0$ for all $i = 1, \ldots, h$, then V is called *effective*. We say that the effective cycle $V = \sum_{i=1}^{h} m_i V_i$ is *irreducible* if $h = 1$ and $m_1 = 1$, so that V can be identified with the variety V_1. The cycle 0 is assumed to be reducible. We will denote by $D^+_{n,m}$ the semigroup of effective cycles of $\mathbb{P}^n$. Note that for $m = n - 1$ these definitions coincide with the ones of divisors in projective space we gave in Sect. 1.6.4.

Given a non-zero effective cycle $V = \sum_{i=1}^{h} m_i V_i$ we define its *Cayley form* to be $F_V = F_{V_1}^{m_1} \cdots F_{V_h}^{m_h}$. Again, the analogues of Theorems 12.1.5 and 12.2.3 still hold.

This extension of the concept of Cayley form is useful in various circumstances, for instance in the intersection of varieties. For example, let $V \subseteq \mathbb{P}^n$ be a variety of dimension m and let H be a hyperplane not containing V, so that $V' = H \cap V$ is pure of dimension $m - 1$ with distinct components $V_1, \ldots, V_h$. The set $\{H\} \times C_{V'} \subset \mathcal{L}^{m+1}_{n,1}$ coincides with the set of points $(H, H_1, \ldots, H_m) \in \mathcal{L}^{m+1}_{n,1}$ such that $H \cap H_1 \cap \ldots \cap H_m \cap V \neq \emptyset$. If $H = [\mathbf{v}]$ in $\mathcal{L}_{n,1}$ and if $F_V(\mathbf{u}_0, \ldots, \mathbf{u}_m)$ is the Cayley form of V, it is clear that $F_V(\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_m)$ is an equation of $C_{V'}$ in $\mathcal{L}^m_{n,1}$. We have that

$$F_V(\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_m) = F_{V_1}^{m_1} \cdots F_{V_h}^{m_h}$$

where $m_1 \ldots, m_h$ are suitable positive integers. So $F_V(\mathbf{v}, \mathbf{u}_1, \ldots, \mathbf{u}_m)$ is the Cayley form of the cycle $\sum_{i=1}^{h} m_i V_i$. This cycle is denote by $H \cdot V$ and it is called the *intersection cycle* of H with V. The integer m_i is called the *intersection multiplicity* of H and V along V_i, for $i = 1, \ldots, h$. Note that $\deg(V) = \deg(H \cdot V)$.

Even more generally, if $V = \sum_{i=1}^{h} m_i V_i$ is an effective non-zero cycle of dimension m in $\mathbb{P}^n$, and H is a hyperplane which does not contain any of its components, let $V_{i,1}, \ldots, V_{i,k_i}$ be the distinct irreducible $(m-1)$-dimensional components of $H \cap V_i$, for $i = 1, \ldots, h$. We will define the *intersection cycle* of H with V and denote it still with $H \cdot V$, as the cycle $\sum_{i=1}^{h} m_i \sum_{j=1}^{k_j} m_{ij} V_{i,j}$, where m_{ij} is the intersection multiplicity of H with V_i along V_{ij}, for $i = 1, \ldots, h$ and $j = 1, \ldots, k_i$. Again we have $\deg(V) = \deg(H \cdot V)$.

Similarly, let $V \subseteq \mathbb{P}^n$ be an m-dimensional variety, and let $H_0, \ldots, H_i$ with $1 \leq i < m$ be independent hyperplanes such that $\Pi = H_0 \cap \ldots \cap H_i$ is a linear subspace of dimension $n - i - 1$. Assume that $\Pi \cap V$ is pure of dimension $m - i - 1$ with distinct irreducible components $V_1, \ldots, V_h$. Suppose $H_i = [\mathbf{v}_i]$ in $\mathcal{L}_{n,1}$. Then

$$F_V(\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_m) = 0$$

is an equation of $C_{\Pi \cap V}$ in $\mathcal{L}^{m-i}_{n,1}$. Then we have

$$F_V(\mathbf{v}_0, \ldots, \mathbf{v}_i, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_m) = F_{V_1}^{m_1} \cdots F_{V_h}^{m_h}.$$

It is easy to see that $m_1, \ldots, m_h$ only depend on Π. As above, we define $\Pi \cdot V = \sum_{i=1}^h m_i V_i$ and we have $\deg(\Pi \cdot V) = \deg(V)$.

12.5 Solutions of Some Exercises

12.1.7 If $P = [p_0, \ldots, p_n]$, then the Chow form F_P depends on a unique set of variables $u_0, \ldots, u_n$ and it is given by

$$F_P(u_0, \ldots, u_n) = p_0 u_0 + \cdots + p_n u_n$$

whose zero locus C_P in $\mathcal{L}_{n,1}$ is the hyperplane $\mathcal{L}_{n,1}(P)$.

12.1.8 Let $\Pi \subseteq \mathbb{P}^n$ be a linear subspace of dimension m defined by the independent system of equations

$$\begin{aligned} a_{m+1,0}x_0 + \ldots + a_{m+1,n}x_n &= 0 \\ &\cdots \\ a_{n,0}x_0 + \ldots + a_{n,n}x_n &= 0 \end{aligned}$$

Consider $(H_0, \ldots, H_m) \in \mathcal{L}_{1,n}^{m+1}$ with $H_i = [\mathbf{u}_i] = [u_{i,0}, \ldots, u_{in}]$, for $i = 0, \ldots, m$. Then $(H_0, \ldots, H_m) \in C_\Pi$ if and only if the linear system

$$\begin{aligned} u_{0,0}x_0 + \ldots + u_{0,n}x_n &= 0 \\ &\cdots \\ u_{m,0}x_0 + \ldots + u_{m,n}x_n &= 0 \\ a_{m+1,0}x_0 + \ldots + a_{m+1,n}x_n &= 0 \\ &\cdots \\ a_{n,0}x_0 + \ldots + a_{n,n}x_n &= 0 \end{aligned}$$

has non-trivial solutions, hence if and only if one has

$$\begin{vmatrix} u_{0,0} & \ldots & u_{0,n} \\ & \ldots & \\ u_{m,0} & \ldots & u_{m,n} \\ a_{m+1,0} & \ldots & a_{m+1,n} \\ & \ldots & \\ a_{n,0} & \ldots & a_{n,n} \end{vmatrix} = 0. \tag{12.5}$$

The determinant in (12.5) is the Cayley form of Π since it is irreducible, because it is linear in each set of variables $\mathbf{u}_i$, for $i = 0, \ldots, m$. Hence $\deg(\Pi) = 1$.

12.2.4 The assertion is trivial if $m = 1$. So assume $m > 1$. Keep the notation of Sect. 12.2. Consider the polynomial $F_V(\mathbf{u}_0, \mathbf{a}_1 \ldots, \mathbf{a}_m)$, in the variables $\mathbf{u}_0 = (u_{00}, \ldots, u_{0n})$, where $[\mathbf{a}_j] = [a_{j0}, \ldots, a_{jn}]$ are the homogeneous coordinates in $\mathcal{L}_{n,1}$ of m independent hyperplanes $H_1, \ldots, H_m$ intersecting along Π. We know that (12.1) holds, with $\{P_1, \ldots, P_h\} = \Pi \cap V$. As $H_1, \ldots, H_m$ vary among all the infinitely many m-tuples of independent hyperplanes intersecting along Π, the polynomial $F_V(\mathbf{u}_0, \mathbf{a}_1 \ldots, \mathbf{a}_m)$ varies in an algebraic way, i.e., its coefficents in the variables $\mathbf{u}_0 = (u_{00}, \ldots, u_{0n})$ vary as polynomials in the variables $\mathbf{a}_1 \ldots, \mathbf{a}_m$. From (12.1) we see that the only thing that can algebraically vary in $F_V(\mathbf{u}_0, \mathbf{a}_1 \ldots, \mathbf{a}_m)$ is $\alpha \in \mathbb{K}^*$, whereas the integers $r_1, \ldots, r_h$ have to stay constant. The assertion follows.

12.2.5 The assertion is trivial if $m = 0, n$, so assume $0 < m < n$. Let us prove that for all pairs of distinct points $P, Q \in V$ the line $P \vee Q$ is contained in V. Whence the assertion follows.

By the definition of the degree, the assertion is trivial if $m = 1, n-1$, so we may assume $1 < m < n-1$. Suppose that for P, Q distinct points in V, the line $P \vee Q$ is not contained in

V. Then $(P \vee Q) \cap V = F$ is a finite set. Set $r = n - m - 1 \geq 1$ and let us prove that for all $i = 1, \dots, r$, there is a linear subspace Σ_i of dimension i such that $P \vee Q \subseteq \Sigma_i$ and $\Sigma_i \cap V = F$. The case $m = n - 2$ implies $i = 1$ and the assertion is trivially true. So let us assume $m \leq n - 3$ and let us start with the case $i = 2$. Let us fix a linear subspace Σ of dimension $n - 2$ skew with $P \vee Q$ and consider the morphism

$$\phi : P' \in V \setminus F \to (P' \vee P \vee Q) \cap \Sigma \in \Sigma.$$

Since $m \leq n - 3$, then ϕ is not surjective, and this implies that there is some plane Σ_2 containing $P \vee Q$ and such that $\Sigma_2 \cap V = F$. If we iterate this argument, we prove the existence of Σ_i for $i > 2$. Let us now fix a subspace Σ' of dimension $n - r - 1 = m$ skew with Σ_r and consider the morphism

$$\psi : P' \in V \setminus F \to (P' \vee \Sigma_r) \cap \Sigma' \in \Sigma'.$$

Since for every point $Q' \in \Sigma'$, the subspace $Q' \vee \Sigma_r$ has at least two points in common with V, then it has infinitely many points in common with V, and so infinitely many points in common with $V \setminus F$. This implies that ψ is surjective, and moreover that any fibre of ψ has some component of positive dimension, which leads to a contradiction.

12.2.6 Suppose that $H_i = [\mathbf{a}_i]$ in $\mathcal{L}_{n,d_i}$ for $i = 1, \dots, n - m$. Consider the variety $Z(n; 1, \dots, 1, d_{m+1}, \dots, d_n)$, where 1 is repeated $m + 1$ times, and consider its equation $R(\mathbf{u}_0, \dots, \mathbf{u}_m, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n) = 0$, which defines C_V, hence $F_V(\mathbf{u}_0, \dots, \mathbf{u}_m)$ divides $R(\mathbf{u}_0, \dots, \mathbf{u}_m, \mathbf{a}_{m+1}, \dots, \mathbf{a}_n)$. Since the degree of $R(\mathbf{u}_0, \dots, \mathbf{u}_m, \mathbf{a}_{m+1}, \dots, \mathbf{a}_n)$ with respect to the variables $\mathbf{u}_0, \dots, \mathbf{u}_m$ is $d_{m+1} \cdots d_n$, the assertion follows.

12.2.7 Suppose that H has equation $f(x_0, \dots, x_n) = 0$, where f is an irreducible homogeneous polynomial of degree d. By Exercise 12.2.6 we know that $\deg(H) \leq d$. So it suffices to find some line intersecting H in exactly d distinct points. Up to a change of coordinates we may assume that H does not contain any of the vertices of the fundamental pyramid. Passing to affine coordinates, the equation of H becomes $F(x_1, \dots, x_n) = f(1, x_1, \dots, x_n) = 0$, and by the hypothesis in $F(x_1, \dots, x_n)$ all the variables appear with degree d. We claim that there is an $i \in \{1, \dots, n\}$ such that

$$\frac{\partial F}{\partial x_i} \neq 0. \tag{12.6}$$

In fact, if this is not the case, then there is a polynomial $G(x_1, \dots, x_n)$ such that

$$f(1, x_1, \dots, x_n) = F(x_1, \dots, x_n) = G(x_1^p, \dots, x_n^p) = g(x_1, \dots, x_n)^p,$$

with p the characteristic of $\mathbb{K}$ that divides d and g a suitable polynomial of degree $\frac{d}{p}$. But then we have

$$f(x_0, x_1, \dots, x_n) = x_0^d F\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) =$$
$$= x_0^d G\left(\left(\frac{x_1}{x_0}\right)^p, \dots, \left(\frac{x_n}{x_0}\right)^p\right) = x_0^d g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)^p,$$

and this implies that f is reducible, against the hypothesis. So we can assume that (12.6) holds for $i = 1$. By the proof of Lemma 12.2.2 we can find a point $(a_2, \dots, a_n) \in \mathbb{A}^{n-1}$ such that $F(x_1, a_2, \dots, a_n) = 0$ has d distinct roots. This implies that the line of $\mathbb{P}^n$ with equations $x_i = a_i x_0$, with $i = 2, \dots, n$ has exactly d distinct points in common with H, as wanted.

12.2.8 Suppose that H has equation $f(x_0, \dots, x_n) = 0$, where f is an irreducible homogeneous polynomial of degree d. Consider the matrix of type $n \times (n + 1)$

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_0 \\ \dots \\ \mathbf{u}_{n-1} \end{pmatrix}$$

where $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, for $i = 0, \ldots, n-1$. We denote by $u_0, \ldots, u_n$ the maximal minors of $\mathbf{U}$ taken with alternate signs. We claim that $F_H = f(u_0, \ldots, u_n)$. First of all note that $f(u_0, \ldots, u_n)$ has degree d in each set of variables $\mathbf{u}_i$, for $i = 0, \ldots, n-1$. So it suffices to prove that F_H and $f(u_0, \ldots, u_n)$ have the same zero set. To see this, take a point $(H_0, \ldots, H_{n-1}) \in \mathcal{L}^n_{n,1}$, with $H_j = [\mathbf{v}_j]$ for $j = 0, \ldots, n-1$. We consider the matrix

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_0 \\ \ldots \\ \mathbf{v}_{n-1} \end{pmatrix}$$

and we denote by $v_0, \ldots, v_n$ the maximal minors of $\mathbf{V}$ taken with alternate signs. Then we have the following possibilities:

(a) one has $v_0 = \ldots = v_n = 0$, and then $f(v_0, \ldots, v_n) = 0$; on the other side $H_0, \ldots, H_{n-1}$ are linearly dependent, so that $H_0 \cap \ldots \cap H_{n-1}$ contains a line, so $H \cap H_0 \cap \ldots \cap H_{n-1} \neq \emptyset$ and $F_V(v_0, \ldots, v_n) = 0$;
(b) one has $(v_0, \ldots, v_n) \neq \mathbf{0}$, then $H_0 \cap \ldots \cap H_{n-1} = \{P\}$, with $P = [v_0, \ldots, v_n]$, hence again $f(v_0, \ldots, v_n) = 0$ implies $F_V(v_0, \ldots, v_n) = 0$.

This shows that $f(u_0, \ldots, u_n)$ and F_V have the same zero locus, hence they are equal up to a multiplicative constant.
12.2.9 First we note that the definition of the integers s_i, for $i = 1, \ldots, h$, is invariant by change of coordinates and also by the choice of the point P, Q on L. This is easy to check and can be left to the reader. Then we may change coordinates and we may assume that the line L has equations $x_i = 0$, for $i = 2, \ldots, n$. By the definition of intersection multiplicity and by Exercise 12.2.8, to compute the intersection multiplicities of L with H at their intersection points, we have to solve the equation $f(u_0, \ldots, u_n) = 0$, where $u_0, \ldots, u_n$ are the maximal minors with alternate signs of the matrix

$$\begin{pmatrix} u_{0,0} & u_{0,1} & u_{0,2} & \ldots & u_{0,n} \\ 0 & 0 & 1 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

namely $u_0 = u_{0,1}, u_1 = -u_{0,0}, u_i = 0$, for $i = 2, \ldots, n$. So $f(u_0, \ldots, u_n) = 0$ is equivalent to $f(u_{0,1}, -u_{0,0}, 0, \ldots, 0) = 0$, where $u_{0,1}, u_{0,0}$ are variables. On the other hand the line L is parametrically represented by $x_0 = \lambda, x_1 = \mu, x_i = 0$, for $i = 2, \ldots, n$. So the integers s_i are obtained by solving the equation $f(\lambda, \mu, 0, \ldots, 0) = 0$. So we see that the two equations are the same (up to the name of variables) and we are done.
12.2.10 The degree of $V_{n,d}$ equals the order of the maximum number of points in common to n hypersurfaces of degree d in $\mathbb{P}^n$, which is d^n
12.2.11 Consider first in general the case of $\mathrm{Seg}_{n,m}$, whose degree is the maximum number of finitely many solutions of a system of equations of the form

$$\begin{aligned} f_1(\mathbf{x}, \mathbf{y}) &= 0 \\ &\ldots \\ f_{n+m}(\mathbf{x}, \mathbf{y}) &= 0 \end{aligned} \tag{12.7}$$

where $\mathbf{x} = (x_0, \ldots, x_n)$, $\mathbf{y} = (y_0, \ldots, y_m)$ and $f_1, \ldots, f_{n+m}$ are bihomogeneous of degree 1 in the variables $\mathbf{x}$ and $\mathbf{y}$. Suppose $n \geq m$, and set

$$f_i(\mathbf{x}, \mathbf{y}) = f_{i,0}(\mathbf{y})x_0 + \cdots + f_{i,n}(\mathbf{y})x_n, \quad \text{for} \quad i = 1, \ldots, n+m.$$

The system (12.7) has non-trivial solutions if and only if

$$\operatorname{rank}\begin{pmatrix} f_{1,0}(\mathbf{y}) & \dots & f_{1,n}(\mathbf{y}) \\ & \dots & \\ f_{n+m,0}(\mathbf{y}) & \dots & f_{n+m,n}(\mathbf{y}) \end{pmatrix} \leq n. \tag{12.8}$$

So we are led to compute the maximum number of finitely many solutions of a determinantal equation of the form (12.8). One can prove that this number is $\binom{n+m}{n}$, which is therefore the degree of $\mathrm{Seg}_{n,m}$. In particular, in the case $m = 1$ the equation (12.8) reduces to

$$\begin{vmatrix} f_{1,0}(y_0, y_1) & \dots & f_{1,n}(y_0, y_1) \\ & \dots & \\ f_{n+1,0}(y_0, y_1) & \dots & f_{n+1\,n}(y_0, y_1) \end{vmatrix} = 0 \tag{12.9}$$

and the determinant appearing in left hand side of (12.9) is a homogeneous polynomial of degree $n + 1$ in y_0, y_1 and any such polynomial can clearly be obtained in this way. So the maximum number of finitely many solutions of the equation (12.9) is $n + 1$ which is also the degree of $\mathrm{Seg}_{n,1}$.
12.2.12 The assertion that V is a cone is trivial. It is also obvious that $\deg(W) \leq \deg(V)$. Let us prove the opposite inequality. There is a linear subspace Π of $\mathbb{P}^n$ of dimension $n - m$ such that $\Pi \cap V$ consists of $d := \deg(V)$ distinct points. By Theorem 12.2.3 we may assume that $P \notin \Pi$. This implies that two distinct points in $\Pi \cap V$ are not aligned with P. Then the projection of Π from P to H is a linear subspace of H of dimension $n - m$ which cuts W in d distinct points, namely the projections of the points of $\Pi \cap V$. This proves that $d = \deg(V) \leq \deg(W)$ as wanted.
12.2.13 The map $\phi : V \dashrightarrow W$ is generically finite, hence $\dim(W) = \dim(V)$. Let now U be a non-empty open subset of W (existing because ϕ is generically finite) such that for all points $Q \in U$ the line $P \vee Q$ is not contained in V so it intersects V in a finite set. Let Π be a linear space of dimension $n - m - 1$ contained in H which intersects W in $\delta = \deg(W)$ distinct points, which we may assume to be all contained in U. Then $P \vee \Pi$ is a linear space of dimension $n - m$ in $\mathbb{P}^n$ which cuts V in more than δ distinct points, hence $\deg(V) \geq \delta + 1 > \deg(W)$.
12.2.15 Proceed by induction on m. Let $m = 1$. The assertion is clear if $n = 2$. If $n > 2$ consider three independent points of V. Then all hyperplanes containing these three points contain V, hence V is contained in the plane spanned by the three points, concluding the proof. Suppose next $m > 1$ and the assertion true for $m - 1$. Fix $P \in V$. If for any point $Q \in V \setminus \{P\}$ the line $P \vee Q$ is contained in V, then V is a cone over a variety W of dimension $m - 1$ contained in a hyperplane H of $\mathbb{P}^n$, and $\deg(W) = \deg(V) = 2$ (see Exercise 12.2.12). Then by induction W is a quadric in a linear subspace Π of dimension m of H. Thus V sits in the $(m + 1)$-dimensional subspace $P \vee \Pi$ and it is a quadric there. If there is a point $Q \in V \setminus \{P\}$ such that the line $P \vee Q$ is not contained in V, then the image W of the projection of V from P is contained in a hyperplane H, and W has dimension m and degree $\deg(W) < 2$ (see Exercise 12.2.13), hence $\deg(W) = 1$. Then W is a linear subspace of dimension m (see Exercise 12.2.5), thus V sits in the linear subspace $P \vee W$ of dimension m, as wanted.
12.2.16 To prove (12.2) one proceeds by induction on the codimension $n - m$ of V in $\mathbb{P}^n$. If V is a hypersurface, it is clear that $d \geq 2$. If $n - m > 1$, note that there is some point $P \in V$ such that the projection ϕ of V from P to a hyperplane $H \cong \mathbb{P}^{n-1}$ not containing P is generically finite onto its image W. Indeed, if this is not the case, then for any pair of distinct points $P, Q \in V$ the line $P \vee Q$ is contained in V, and V would be a linear subspace of $\mathbb{P}^n$, contradicting the non-degeneracy of V. If ϕ is generically finite, by Exercise 12.2.13 we have $\deg(W) < \deg(V) = d$. On the other hand, by induction, we have $\deg(W) \geq (n - 1) - m + 1 = n - m$, hence $d \geq \deg(W) + 1 \geq n - m + 1$, as wanted.
12.2.17 This can be proved by induction on the codimension $n - m$. If $m = n - 1$, then V is a quadric in $\mathbb{P}^n$ and therefore it is rational (see Exercise 7.3.10). Assume $n - m \geq 2$. Arguing as in the solution of Exercise 12.2.16 one proves that there is a point P of V such that the projection of V from P to a hyperplane $H \cong \mathbb{P}^{n-1}$ not containing P is birational onto its image W, which is again a variety of minimal degree, with lower codimension. Then the assertion follows by applying induction.

Chapter 13
Grassmannians

In all this chapter we will assume that $\mathbb{K}$ has characteristic $p \neq 2$.

13.1 Plücker Coordinates

Let $\Pi \subseteq \mathbb{P}^n$ be a subspace of dimension m, defined by the system of independent equations

$$\begin{aligned} a_{m+1,0}x_0 + \ldots + a_{m+1,n}x_n &= 0 \\ \ldots \\ a_{n,0}x_0 + \ldots + a_{n,n}x_n &= 0 \end{aligned} \tag{13.1}$$

As we saw in the solution to Exercise 12.1.8, given $(H_0, \ldots, H_m) \in \mathcal{L}_{1,n}^{m+1}$ with $H_i = [\mathbf{u}_i] = [u_{i,0}, \ldots, u_{in}]$, for $i = 0, \ldots, m$, $(H_0, \ldots, H_m) \in C_\Pi$ if and only if the linear system

$$\begin{aligned} u_{0,0}x_0 + \ldots + u_{0,n}x_n &= 0 \\ \ldots \\ u_{m,0}x_0 + \ldots + u_{m,n}x_n &= 0 \\ a_{m+1,0}x_0 + \ldots + a_{m+1,n}x_n &= 0 \\ \ldots \\ a_{n,0}x_0 + \ldots + a_{n,n}x_n &= 0 \end{aligned}$$

has non-trivial solutions, hence if and only if one has

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$$\begin{vmatrix} u_{0,0} & \dots & u_{0,n} \\ & \dots & \\ u_{m,0} & \dots & u_{m,n} \\ a_{m+1,0} & \dots & a_{m+1,n} \\ & \dots & \\ a_{n,0} & \dots & a_{n,n} \end{vmatrix} = 0. \tag{13.2}$$

The determinant in (13.2) is the Cayley form of Π.

We want to write the Cayley form F_Π more explicitly. Consider the matrix of type $(n-m) \times (n+1)$

$$\mathbf{A} = \begin{pmatrix} a_{m+1,0} & \dots & a_{m+1,n} \\ & \dots & \\ a_{n,0} & \dots & a_{n,n} \end{pmatrix}.$$

If $(i_0, \dots, i_{n-m-1})$ is a $(n-m)$-tuple of elements of $\{0, \dots, n\}$, we will set

$$p^{i_0,\dots,i_{n-m-1}} = \begin{vmatrix} a_{m+1,i_0} & \dots & a_{m+1,i_{n-m-1}} \\ & \dots & \\ a_{n,i_0} & \dots & a_{n,i_{n-m-1}} \end{vmatrix}.$$

Of course $p^{i_0,\dots,i_{n-m-1}} = 0$ if two indices are equal. So we will assume that the indices are all distinct. Moreover we will assume that $i_0 < i_1 < \dots < i_{n-m-1}$. With this assumption we may consider the non-zero vector

$$\check{\mathbf{p}} = (p^{i_0,\dots,i_{n-m-1}})_{0 \le i_0 < i_1 < \dots < i_{n-m-1} \le n}$$

with lexicographically ordered entries. The length of this vector is $\binom{n+1}{n-m} = \binom{n+1}{m+1}$. The vector $\check{\mathbf{p}}$ is called the vector of *dual Plücker coordinates* of Π. It has an interpretation in terms of multilinear algebra. Consider $\mathbb{A}^{n+1}$ as a vector space $\check{\mathbf{V}}$ of dimension $n+1$ on $\mathbb{K}$, generated by the independent vectors $\mathbf{e}_0 = (1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, \dots, 0, 1)$. Then $\wedge^{n-m}\check{\mathbf{V}}$ is a vector space of dimension $\binom{n+1}{n-m} = \binom{n+1}{m+1}$ over $\mathbb{K}$, which has as a basis the set of vectors

$$\mathbf{e}_{i_0,\dots,i_{n-m-1}} = \mathbf{e}_{i_0} \wedge \dots \wedge e_{i_{n-m-1}}, \quad \text{with} \quad 0 \le i_0 < i_1 < \dots < i_{n-m-1} \le n. \tag{13.3}$$

If we interpret the rows of the matrix $\mathbf{A}$ as vectors $\mathbf{a}_{m+1}, \dots, \mathbf{a}_n$ in $\check{\mathbf{V}}$, then $\check{\mathbf{p}}$ is just the vector of the components of $\mathbf{a}_{m+1} \wedge \dots \wedge \mathbf{a}_n$ with respect to the basis (13.3), namely

$$\mathbf{a}_{m+1} \wedge \dots \wedge \mathbf{a}_n = \sum p^{i_0,\dots,i_{n-m-1}} \mathbf{e}_{i_0,\dots,i_{n-m-1}}$$

where the sum is made over all sets of indices $i_0, \dots, i_{n-m-1}$ such that $0 \le i_0 < i_1 < \dots < i_{n-m-1} \le n$.

Next we consider the point $[\check{\mathbf{p}}]$ of $\mathbb{P}^{M(m,n)}$, where $M(m,n) = \binom{n+1}{m+1} - 1$ and we point out two facts:

(a) $[\check{\mathbf{p}}]$ does not depend on the particular set of equations (13.1) defining Π;
(b) $[\check{\mathbf{p}}]$ determines Π, i.e., if $\Pi' \neq \Pi$ is another subspace of dimension m of $\mathbb{P}^n$ with dual Plücker coordinates $\check{\mathbf{p}}'$, then $[\check{\mathbf{p}}] \neq [\check{\mathbf{p}}']$.

Both facts follow from the following circumstance. If we expand the determinant in (13.2) with the Laplace rule applied to the first $m+1$ rows, we have

$$F_\Pi = \sum p^{i_0,\dots,i_{n-m-1}} u^{j_0,\dots,j_m} \varepsilon^{i_0,\dots,i_{n-m-1}} \tag{13.4}$$

where the sum is made over all sets of indices $i_0, \dots, i_{n-m-1}$ such that $0 \le i_0 < i_1 < \cdots < i_{n-m-1} \le n$, one has $\{j_0, \dots, j_m\} = \{0, \dots, n\} \setminus \{i_0, \dots, i_{n-m-1}\}$ with $j_0 < \cdots < j_m$, $\varepsilon^{i_0,\dots,i_{n-m-1}}$ equals 1 or -1 according to the fact that $i_0 + \cdots + i_{n-m-1}$ is even or odd, and $u^{j_0,\dots,j_m}$ is the maximal minor of the matrix

$$\mathbf{U} = \begin{pmatrix} u_{0,0} & \dots & u_{0,n} \\ & \dots & \\ u_{m,0} & \dots & u_{m,n} \end{pmatrix}$$

determined by the columns of order $j_0, \dots, j_m$. In conclusion F_Π is determined by $[\check{\mathbf{p}}]$, and this proves (a) and (b) above by Theorem 12.1.5.

To get the same conclusions as above, we may argue in a slightly different way. Given the subspace $\Pi \subseteq \mathbb{P}^n$ of dimension m, we can choose $m+1$ independent points $P_i = [\mathbf{p}_i]$, $i = 0, \dots, m$ of Π. Let $(H_0, \dots, H_m) \in \mathcal{L}_{n,1}^{m+1}$ with $H_i = \mathbf{u}_i$, for $i = 0, \dots, m$. Set

$$a_{ij} = \mathbf{u}_i \times \mathbf{p}_j, \quad \text{for} \quad i, j = 0, \dots, m.$$

Then $(H_0, \dots, H_m) \in C_\Pi$ if and only if $\det(a_{ij})_{i,j=0,\dots,m} = 0$, hence

$$F_\Pi = \det(a_{ij})_{i,j=0,\dots,m}.$$

On the other hand, the square matrix $(a_{ij})_{i,j=0,\dots,m}$ of order $m+1$ is the product rows by columns of the two matrices $\mathbf{U}$ and $\mathbf{P}^t$, where

$$\mathbf{P} = \begin{pmatrix} p_{0,0} & \cdots & p_{0,n} \\ & \cdots & \\ p_{m,0} & \cdots & p_{m,n} \end{pmatrix} \tag{13.5}$$

If $0 \le j_0 < \cdots < j_m \le n$ we set

$$p_{j_0,\cdots,j_m} = \begin{vmatrix} p_{0,j_0} & \cdots & p_{0,j_m} \\ & \cdots & \\ p_{m,j_0} & \cdots & p_{m,j_m} \end{vmatrix} \tag{13.6}$$

and

$$\mathbf{p} = (p_{j_0,\cdots,j_m})$$

with lexicographically ordered entries, a non-zero vector of order $\binom{n+1}{m+1}$.

The vector $\mathbf{p}$ has again an interpretation in terms of multilinear algebra. Namely $\mathbf{p}$ is the vector of the components of $\mathbf{p}_0 \wedge \ldots \wedge \mathbf{p}_m$ in the $\binom{n+1}{m+1}$-dimensional vector space $\wedge^{m+1}\mathbf{V}$ (with $\mathbf{V}$ dual of $\check{\mathbf{V}}$) with respect to the basis

$$\mathbf{e}^{j_0,\ldots,j_m} = \mathbf{e}^{j_0} \wedge \ldots \wedge \mathbf{e}^{j_m}$$

with $(\mathbf{e}^1, \ldots, \mathbf{e}^n)$ dual basis of $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

The vector $\mathbf{p}$ is called the vector of *Plücker coordinates* of Π. For $[\mathbf{p}]$ the same properties (a) and (b) as above hold. Indeed by expanding $F_\Pi = \mathbf{U} \cdot \mathbf{P}^t$, we have

$$F_\Pi(\mathbf{u}_0, \ldots, \mathbf{u}_m) = \sum p_{j_0,\cdots,j_m} u^{j_0,\ldots,j_m} \tag{13.7}$$

where the sum is made over all indices such that $0 \le j_0 < \cdots < j_m \le n$. From this and from (13.4) we get the following:

Proposition 13.1.1 *With the above notation consider the square matrix of order* $n+1$

$$\mathbf{M} = \begin{pmatrix} \mathbf{P} \\ \mathbf{A} \end{pmatrix}.$$

Then the maximal minors of $\mathbf{A}$ *[resp. of* $\mathbf{P}$*] are proportional to their cofactors in* $\mathbf{M}$.

13.2 Grassmann Varieties

Let us fix the non-negative integers n, m with $m < n$ and consider the set $\mathcal{G}(m,n)$ of all subspaces of dimension m of $\mathbb{P}^n$. Given an element $\Pi \in \mathcal{G}(m,n)$, we have the non-zero vectors of order $\binom{n+1}{m+1}$ of its Plücker coordinates $\mathbf{p} = (p_{j_0,\ldots,j_m})$ and of its dual Plücker coordinates $\check{\mathbf{p}} = (p^{i_0,\ldots,i_{n-m-1}})$, which are defined up to a non-zero factor and are proportional. We have the two coinciding injective maps

$$g_{m,n} : \Pi \in \mathcal{G}(m,n) \to [\mathbf{p}] \in \mathbb{P}^{M(m,n)}$$

and

$$\check{g}_{m,n} : \Pi \in \mathcal{G}(m,n) \to [\check{\mathbf{p}}] \in \mathbb{P}^{M(m,n)}.$$

The image set $\mathbb{G}(m,n)$ of $g_{m,n}$ (or of $\check{g}_{m,n}$) is bijective with $\mathcal{G}(m,n)$, and it is called, for reasons which will be soon clear, the *Grassmann variety* or the *Grassmannian* of type (m,n). The same name will have any transformation of $\mathbb{G}(m,n)$ via a projectivity of $\mathbb{P}^{M(m,n)}$. By abuse of notation, we will identify a point $\pi \in \mathbb{G}(m,n)$ with the subspace $\Pi \in \mathcal{G}(m,n)$ such that $\pi = g_{m,n}(\Pi)$.

Given $\Pi \in \mathbb{G}(m,n)$, we have the subspace $\Pi^\perp$ of dimension $m - m - 1$ of $\mathcal{L}_{n,1} = \check{\mathbb{P}}^n$ which consists of all the hyperplanes containing Π. The dual Plücker coordinated of Π coincide with the Plücker coordinated of $\Pi^\perp$. This implies that $\mathbb{G}(m,n) = \mathbb{G}(n-m-1,n)$.

It is also useful to have in mind the following algebraic interpretation of $\mathbb{G}(m,n)$. As we did in Sect. 13.1, consider $\mathbb{A}^{n+1}$ has a vector space $\mathbf{V}$ of dimension $n+1$ on $\mathbb{K}$. Then $\wedge^{m+1}\mathbf{V}$ is a vector space of dimension $M(m,n)+1$ over $\mathbb{K}$. A non-zero tensor $T \in \wedge^{m+1}\mathbf{V}$ is said to be *indecomposable* if there are $\mathbf{p}_0,\ldots,\mathbf{p}_m \in \mathbf{V}$ such that $T = \mathbf{p}_0 \wedge \ldots \wedge \mathbf{p}_m$. The indecomposable tensors generate $\wedge^{m+1}\mathbf{V}$ as a $\mathbb{K}$-vector space. Note that $\mathcal{G}(m,n)$ can be interpreted as the set of all vector subspaces of dimension $m+1$ of $\mathbf{V}$. So, given $\Pi \in \mathcal{G}(m,n)$, and given a basis $\mathbf{p}_0,\ldots,\mathbf{p}_m$ of Π, we have the indecomposable tensor $\mathbf{p}_0 \wedge \ldots \wedge \mathbf{p}_m \in \wedge^{m+1}\mathbf{V}$. If we change basis in Π, this indecomposable tensor varies, but it is easy to check that it changes only by the product of a non-zero element of $\mathbb{K}$ (namely, the determinant of the matrix of the basis change). Hence, if we set $\mathbb{P}^{M(m,n)} = \mathbb{P}(\wedge^{m+1}\mathbf{V})$, we have the map

$$\gamma_{m,n} : \Pi \in \mathcal{G}(m,n) \to [\mathbf{p}_0 \wedge \ldots, \wedge \mathbf{p}_m] \in \mathbb{P}^{M(m,n)}$$

which clearly coincides with $g_{m,n}$. Hence $\mathbb{G}(m,n)$ can be interpreted as the set of points of $\mathbb{P}(\wedge^{m+1}\mathbf{V})$ corresponding to proportionality equivalence classes of indecomposable tensors. From this description it follows that $\mathbb{G}(m,n)$ is non-degenerate in $\mathbb{P}^{M(m,n)}$.

Next we want to prove that $\mathbb{G}(m,n)$ is a subvariety of $\mathbb{P}^{M(m,n)}$. This is trivial if $m=0$ or $m=n-1$. Indeed $\mathbb{G}(0,n) = \mathbb{P}^n$ and $\mathbb{G}(n-1,n) = \mathcal{L}_{n,1} = \check{\mathbb{P}}^n$. To see this in general we need a number of preliminaries.

Let $\Pi \in \mathbb{G}(m,n)$ with Plücker coordinates $\mathbf{p} = [p_{j_0,\ldots,j_m}]$. If we chose $m+1$ independent points $P_i = [\mathbf{p}_i]$, $i = 0,\ldots,m$ of Π, the Plücker coordinates of Π are given by the minors of maximal order of the matrix $\mathbf{P}$ in (13.5).Note that, in the above notation we have $0 \le j_0 < \cdots < j_m \le n$. However we want to make sense of the symbol $p_{j_0,\ldots,j_m}$ in the more general case in which $\{j_0,\ldots,j_m\}$ is any disposition (with perhaps repetitions) of $m+1$ elements in the set $\{0,\ldots,n\}$. Precisely we define $p_{j_0,\ldots,j_m}$ as in (13.6). Note then that $\{p_{j_0,\ldots,j_m}\}$ is an *alternating set*, namely

$$p_{j_0,\ldots,j_m} = \varepsilon p_{j'_0,\ldots,j'_m}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ whenever the $(m+1)$-tuple $(j'_0,\ldots,j'_m)$ is obtained from $(j_0,\ldots,j_m)$ with an even or odd number of transpositions. In particular $p_{j_0,\ldots,j_m} = 0$ if two of the indices in $(j_0,\ldots,j_m)$ are equal.

Fix now $(i_1,\ldots,i_m)$ a disposition with repetitions of m elements in the set $\{0,\ldots,n\}$. If the columns of the matrix $\mathbf{P}$ of order $i_1,\ldots,i_m$ are linearly dependent, we have $p_{i,i_1,\ldots,i_m} = 0$, for all $i = 0,\ldots,n$. Suppose by contrast that the columns of the matrix $\mathbf{P}$ of order $i_1,\ldots,i_m$ are linearly independent. Then, since the matrix $\mathbf{P}$ has rank $m+1$, the vector $(p_{0,i_1,\ldots,i_m},\ldots,p_{n,i_1,\ldots,i_m})$ is non-zero and we can consider the point $P_{i_1,\ldots,i_m} = [p_{0,i_1,\ldots,i_m},\ldots,p_{n,i_1,\ldots,i_m}] \in \mathbb{P}^n$. Of course $P_{i_1,\ldots,i_m}$ does not depend on the ordering $(i_1,\ldots,i_m)$, so that we may suppose $0 \le i_1 < \cdots < i_m \le n$.

Lemma 13.2.1 *In the above setting the point $P_{i_1,\dots,i_m}$ is the unique intersection point of Π with the subspace $\Pi_{i_0,\dots,i_m}$ with equations*

$$x_{i_1} = \cdots = x_{i_m} = 0.$$

Proof A point $P = [\mathbf{p}]$ belongs to Π if and only if there are $\lambda_0, \dots, \lambda_m$ in $\mathbb{K}$ such that

$$\mathbf{p} = \lambda_0 \mathbf{p}_0 + \cdots + \lambda_m \mathbf{p}_m,$$

hence P belongs to $\Pi_{i_0,\dots,i_m}$ if and only if

$$\begin{aligned} \lambda_0 p_{0,i_1} + \ldots + \lambda_m p_{m,i_1} &= 0 \\ \ldots& \\ \lambda_0 p_{0,i_m} + \ldots + \lambda_m p_{m,i_m} &= 0 \end{aligned} \tag{13.8}$$

Since the columns of $\mathbf{P}$ of order $i_1, \dots, i_m$ are linearly independent, the relations (13.8) uniquely determine $[\lambda_0, \dots, \lambda_m]$, i.e., $\lambda_0, \dots, \lambda_m$ are proportional to the maximal minors with alternate signs of the matrix

$$\begin{pmatrix} p_{0,i_1} & \cdots & p_{m,i_1} \\ & \cdots & \\ p_{0,i_m} & \cdots & p_{m,i_m} \end{pmatrix}.$$

Then we have

$$p_i = \lambda_0 p_{0i} + \cdots + \lambda_m p_{mi} =$$

$$= \begin{vmatrix} p_{0,i} & \cdots & p_{m,i} \\ p_{0,i_1} & \cdots & p_{m,i_1} \\ & \cdots & \\ p_{0,i_m} & \cdots & p_{m,i_m} \end{vmatrix} = p_{i,i_1,\dots,i_m}$$

as wanted. □

Consider now the matrix

$$\mathbf{P}_{i_1,\dots,i_m} = \begin{pmatrix} p_{0,i_1,\dots,i_m} & \cdots & p_{n,i_1,\dots,i_m} \\ p_{0,0} & \cdots & p_{0,n} \\ & \cdots & \\ p_{m,0} & \cdots & p_{m,n} \end{pmatrix}.$$

By Lemma 13.2.1, we have that $\mathbf{P}_{i_1,\dots,i_m}$ has rank $m+1$, thus all its maximal minors vanish. So if $(i_0, j_0, \dots, j_m)$ is a disposition (with perhaps repetitions) of $m+2$ elements in the set $\{0, \dots, n\}$, we have

$$\begin{vmatrix} p_{i_0,i_1,\ldots,i_m} & p_{j_0,i_1,\ldots,i_m} & \cdots & p_{j_m,i_1,\ldots,i_m} \\ p_{0,i_0} & p_{0,j_0} & \cdots & p_{0,j_m} \\ & \cdots & & \\ p_{m,i_0} & p_{m,j_0} & \cdots & p_{m,j_m} \end{vmatrix} = 0$$

which reads

$$p_{i_0,i_1,\ldots,i_m} p_{j_0,j_1,\ldots,j_m} - p_{j_0,i_1,\ldots,i_m} p_{i_0,j_1,\ldots,j_m} + \cdots + (-1)^m p_{j_m,i_1,\ldots,i_m} p_{i_0,j_0,\ldots,j_{m-1}} = 0$$

or equivalently

$$p_{i_0,i_1,\ldots,i_m} p_{j_0,j_1,\ldots,j_m} = p_{j_0,i_1,\ldots,i_m} p_{i_0,j_1,\ldots,j_m} + \\ + p_{j_1,i_1,\ldots,i_m} p_{j_0,i_0,\ldots,j_m} + \cdots + p_{j_m,i_1,\ldots,i_m} p_{j_0,\ldots,j_{m-1},i_0}.$$

These relations, which hold for every $(i_0, \ldots, i_m)$ and $(j_0, \ldots, j_m)$, are called *Plücker relations*.

Now we introduce the $M(m, n) + 1$ indeterminates $x_{i_0,\ldots,i_m}$, with $0 \leq i_0 < \cdots < i_m \leq n$. We also introduce symbols $x_{j_0,\ldots,j_m}$ in the more general case in which $\{j_0, \ldots, j_m\}$ is any disposition (with perhaps repetitions) of $m + 1$ elements in the set $\{0, \ldots, n\}$. As usual we define

$$x_{j_0,\ldots,j_m} = \varepsilon x_{i_0,\ldots,i_m}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ whenever the $(m + 1)$-tuple $(j_0, \ldots, j_m)$ is obtained from $(i_0, \ldots, i_m)$ with $0 \leq i_0 < \cdots < i_m \leq n$ with an even or odd number of transpositions. Moreover we define $x_{j_0,\ldots,j_m} = 0$, if two indices among $j_0, \ldots, j_m$ are equal. With this notation, we see that $\mathbb{G}(m, n)$ is contained in the closed subset of $\mathbb{P}^{M(m,n)}$ defined by the set of equations

$$x_{i_0,i_1,\ldots,i_m} x_{j_0,j_1,\ldots,j_m} = x_{j_0,i_1,\ldots,i_m} x_{i_0,j_1,\ldots,j_m} + \\ + x_{j_1,i_1,\ldots,i_m} x_{j_0,i_0,\ldots,j_m} + \cdots + x_{j_m,i_1,\ldots,i_m} x_{j_0,\ldots,j_{m-1},i_0}. \tag{13.9}$$

for every $(i_0, \ldots, i_m)$ and $(j_0, \ldots, j_m)$. These equations are identically zero if $m = 0, n - 1$ because in these cases $\mathbb{G}(0, n) = \mathbb{P}^n = \mathbb{P}^{M(0,n)}$ and $\mathbb{G}(n - 1, n) = \check{\mathbb{P}}^n = \mathbb{P}^{M(n-1,n)}$. So we will assume from now on that $1 \leq m \leq m - 2$. In this case we can choose $i_2 = j_2, \ldots, i_m = j_m$ and we set $i_0 = i, i_1 = j, j_0 = h, j_1 = k$. Then (13.9) become

$$x_{i,j,i_2,\ldots,i_m} x_{h,k,i_2,\ldots,i_m} = x_{h,i,i_2,\ldots,i_m} x_{i,k,i_2,\ldots i_m} + x_{h,j,i_2,\ldots,i_m} x_{h,i,i_2,\ldots,i_m} \tag{13.10}$$

which are not identically zero if i, j, h, k are all distinct and different from $i_2, \ldots, i_m$ as it is possible if $1 \leq m \leq m - 2$. The equations (13.10) are called the *three terms Plücker relations*.

Now we can prove the:

Theorem 13.2.2 *$\mathbb{G}(m, n)$ is a closed subset of $\mathbb{P}^{M(m,n)}$, which is proper if and only if $1 \leq m \leq m-2$.*

Proof We already saw that $\mathbb{G}(0, n)=\mathbb{P}^n=\mathbb{P}^{M(0,n)}$ and $\mathbb{G}(n-1, n)=\check{\mathbb{P}}^n = \mathbb{P}^{M(n-1,n)}$. So we may assume that $1 \leq m \leq m-2$. Since $\mathbb{G}(m, n)$ verifies the Plücker relations (13.9) and among these there are the three terms relations (13.10) which are not identically zero, then $\mathbb{G}(m, n)$ is a proper subset of $\mathbb{P}^{M(m,n)}$ if $1 \leq m \leq m-2$. In order to show that $\mathbb{G}(m, n)$ is a closed subset, we will prove that it is defined by the Plücker relations (13.9).

Suppose that $(p_{i_0,\ldots,i_m})$ is non-zero and verifies the Plücker relations. To fix the ideas, let us suppose that $p_{0,\ldots,m} \neq 0$. Then it makes sense to consider the following points of $\mathbb{P}^n$

$$P_0 = [p_{i,1,\ldots,m}]_{i=0,\ldots,n},\ P_1 = [p_{0,i,2,\ldots,m}]_{i=0,\ldots,n},\ \ldots,\ P_m = [p_{0,\ldots,m-1,i}]_{i=0,\ldots,n}.$$

They are linearly independent. Indeed a matrix that has as rows the homogeneous coordinates of these points is given by

$$\begin{pmatrix} p_{0,1,\ldots,m} & 0 & \cdots & 0 & p_{m+1,1,\ldots,m} \cdots & p_{n,1,\ldots,m} \\ 0 & p_{0,1,\ldots,m} & \cdots & 0 & p_{0,m+1,\ldots,m} \cdots & p_{0,n,\ldots,m} \\ & \cdots & & & & \\ 0 & 0 & \cdots & p_{0,1,\ldots,m} & p_{0,\ldots,m-1,m} \cdots & p_{0,\ldots,m-1,n} \end{pmatrix}$$

whose minor determined by the first $m+1$ columns is $p_{0,1,\ldots,m}^{m+1} \neq 0$. Hence $P_0, \ldots, P_m$ span a linear space $\Pi \in \mathbb{G}(m, n)$. Let now $(\pi_{i_0,\ldots,i_m})$ be the vector of Plücker coordinates of Π. We will conclude the proof by showing that the vector $(\pi_{i_0,\ldots,i_m})$ is proportional to $(p_{i_0,\ldots,i_m})$, more precisely, we will show that for all $(i_0, \ldots, i_m)$ one has

$$\pi_{i_0,\ldots,i_m} = p_{0,1,\ldots,m}^{m} p_{i_0,\ldots,i_m}. \tag{13.11}$$

First we show that (13.11) holds if only one of the indices $i_0, \ldots, i_m$ is larger than m. Indeed, if $m < i \leq n$, one has

$$\pi_{0,\ldots,l-1,i,l+1,\ldots,m} = \begin{vmatrix} p_{0,\ldots,m} & 0 & \cdots\ p_{i,1,\ldots,m} \cdots & 0 \\ 0 & p_{0,\ldots,m} & \cdots\ p_{0,i,\ldots,m} \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots\ p_{0,\ldots,m-1,i}\ \cdots & p_{0,\ldots,m} \end{vmatrix}$$

where the column $(p_{i,1,\ldots,m}, p_{0,i,\ldots,m}, \ldots, p_{0,\ldots,m-1,i})^t$ appears at the lth place, and from this we see that $\pi_{0,\ldots,l-1,i,l+1,\ldots,m} = p_{0,1,\ldots,m}^{m} p_{0,\ldots,l-1,i,l+1,\ldots,m}$.

Then we proceed by induction, supposing that (13.11) holds if $i_0, \ldots, i_m$ contain ν numbers greater that m, and we show that (13.11) holds if $i_0, \ldots, i_m$ contain $\nu + 1$ numbers greater that m. Since Plücker relations hold for $(p_{i_0,\ldots,i_m})$, we have

$$
\begin{aligned}
p_{0,\ldots,m} p_{i_0,\ldots,i_m} = &\; p_{i_0,1,\ldots,m} p_{0,i_1,\ldots,i_m} + \\
&+ p_{i_1,1,\ldots,m} p_{i_0,0,i_2 m \ldots,i_m} + \cdots + p_{i_m,1,\ldots,m} p_{i_0,\ldots,i_{m-1},0}.
\end{aligned} \tag{13.12}
$$

On the right hand side of (13.12) the only non-zero terms may be the ones of the form $p_{i_h,1,\ldots,m} p_{i_0,\ldots,i_{h-1},0,i_{h+1},\ldots,i_m}$ with $i_h > m$, where we assume, as it is possible, that $0 \notin \{i_0, \ldots, i_h\}$. By multiplying both members of (13.12) by $p^{2m}_{0,\ldots,m}$ and applying induction, we have

$$
\pi_{0,\ldots,m} p^{m}_{0,\ldots,m} p_{i_0,\ldots,i_m} = \pi_{i_0,1,\ldots,m} \pi_{0,i_1,\ldots,i_m} + \cdots + \pi_{i_m,1,\ldots,m} \pi_{i_0,\ldots,i_{m-1},0}.
$$

On the other hand, Plücker relations hold for $(\pi_{i_0,\ldots,i_m})$, hence we have

$$
\pi_{0,\ldots,m} \pi_{i_1,\ldots,i_m} = \pi_{i_0,1,\ldots,m} \pi_{0,i_1,\ldots,i_m} + \cdots + \pi_{i_m,1,\ldots,m} \pi_{i_0,\ldots,i_{m-1},0}
$$

whence (13.11) holds because $\pi_{0,\ldots,m} \neq 0$. This end the proof of the Theorem. □

Consider now $(\mathbb{P}^n)^{m+1}$ and let $[\mathbf{x}_i] = [x_{i0}, \ldots, x_{in}]$, for $i = 0, \ldots, m$, the homogeneous coordinates in the $(i+1)$th factor of the product $(\mathbb{P}^n)^{m+1}$. Consider the subset $\mathcal{Z}(m,n)$ of $(\mathbb{P}^n)^{m+1}$ consisting of all $(m+1)$-tuples $(P_0, \ldots, P_m)$, with $P_i = [\mathbf{p}_i] = [p_{i0}, \ldots, p_{in}]$, for $i = 0, \ldots, m$, such that $P_0, \ldots, P_m$ are linearly dependent in $\mathbb{P}^n$. Then $\mathcal{Z}(m,n)$ is a proper closed subset of $(\mathbb{P}^n)^{m+1}$ which is defined by the matrix equation

$$
\operatorname{rank} \begin{pmatrix} \mathbf{x}_0 \\ \ldots \\ \mathbf{x}_m \end{pmatrix} < m + 1.
$$

We set $\mathcal{D}(m,n) = (\mathbb{P}^n)^{m+1} \setminus \mathcal{Z}(m,n)$. We have the map

$$
\phi_{m,n} : (P_0, \ldots, P_m) \in \mathcal{D}(m,n) \to g_{m,n}(P_0 \vee \ldots \vee P_m) \in \mathbb{G}(m,n) \subseteq \mathbb{P}^{M(m,n)}.
$$

Lemma 13.2.3 *The map $\phi_{m,n} : \mathcal{D}(m,n) \to \mathbb{P}^{M(m,n)}$ is a morphism and its image is $\mathbb{G}(m,n)$.*

Proof The fact that the image of $\phi_{m,n}$ is $\mathbb{G}(m,n)$ is obvious. To show that $\phi_{m,n}$ is a morphism, note that $\phi_{m,n}$ sends the point $([\mathbf{p}_0], \ldots, [\mathbf{p}_m]) \in \mathcal{D}(m,n)$ to the point $[p_{i_0,\ldots,i_m}]$ where $(p_{i_0,\ldots,i_m})$ are the Plücker coordinates of $P_0 \vee \ldots \vee P_m$, hence are the minors of maximal order of the matrix

$$
\begin{pmatrix} \mathbf{p}_0 \\ \ldots \\ \mathbf{p}_m \end{pmatrix}
$$

and these minors are polynomials in the coordinates of $P_0, \ldots, P_m$. □

Theorem 13.2.4 $\mathbb{G}(m,n)$ *is irreducible of dimension* $(n-m)(m+1)$.

Proof The irreducibility follows from the surjectivity of $\phi_{m,n}$ and the fact that $\mathcal{D}(m,n)$ is irreducible, being an open set of $(\mathbb{P}^n)^{m+1}$. Let now $\Pi \in \mathcal{G}(m,n)$. Then

$$\phi_{m,n}^{-1}(g_{m,n}(\Pi)) = \Pi^{m+1} \cap \mathcal{D}(m,n)$$

which has dimension $m(m+1)$. Then, by Theorem 11.3.1, we have

$$\dim(\mathbb{G}(m,n)) = n(m+1) - m(m+1) = (n-m)(m+1).$$

□

Theorem 13.2.5 $\mathbb{G}(m,n)$ *is rational.*

Proof Consider the set

$$\mathcal{I}(m,n) = \{(\Pi, \Pi') \in \mathbb{G}(m,n) \times \mathbb{G}(n-m,n), \Pi \vee \Pi' \neq \mathbb{P}^n\},$$

which is a proper closed subset of $\mathbb{G}(m,n) \times \mathbb{G}(n-m,n)$. In fact, if $\Pi = P_0 \vee \ldots \vee P_m$ and $\Pi' = P_{m+1} \vee \ldots \vee P_{n+1}$, with $P_i = [\mathbf{p}_i]$, for $i = 0, \ldots, n+1$, then $(\Pi, \Pi') \in \mathcal{I}(m,n)$ if and only if

$$\operatorname{rank} \begin{pmatrix} \mathbf{p}_0 \\ \ldots \\ \mathbf{p}_{n+1} \end{pmatrix} < n+1, \tag{13.13}$$

i.e., if and only if all maximal minors of the matrix in (13.13) vanish. Expanding these minors with Laplace rule applied to the first $m+1$ rows, one obtains algebraic relations (of degree 1) between the Plücker coordinates of Π and Π', which are necessary and sufficient conditions in order that $(\Pi, \Pi') \in \mathcal{I}(m,n)$.

Let $p_1 : \mathcal{I}(m,n) \to \mathbb{G}(m,n)$ and $p_2 : \mathcal{I}(m,n) \to \mathbb{G}(n-m,n)$ be the projections to the two factors. If we fix $\Pi'_0 \in \mathbb{G}(n-m,n)$, we set

$$\mathbb{G}(m,n,\Pi'_0) = p_1(p_2^{-1}(\Pi'_0)).$$

It is clear that $\mathbb{G}(m,n,\Pi'_0)$ is a proper closed subset of $\mathbb{G}(m,n)$. Fix now $\Pi \in \mathbb{G}(m,n)$, fix $P_0, \ldots, P_m$ independent points in Π, and take $\Pi'_0, \ldots, \Pi'_m \in \mathbb{G}(n-m,n)$ such that $\{P_i\} = \Pi \cap \Pi'_i$ for all $i = 0, \ldots, m$. Set

$$U = \mathbb{G}(m,n) \setminus \bigcup_{i=0}^{m} \mathbb{G}(m,n,\Pi'_i)$$

which is an open dense subset of $\mathbb{G}(m,n)$, and let U' be the dense open subset $\mathcal{D}(m,n) \cap (\Pi'_0 \times \ldots \times \Pi'_m)$ of $\Pi'_0 \times \ldots \times \Pi'_m$. Consider the map

$$\psi : P \in U \to (P \cap \Pi'_0, \ldots, P \cap \Pi'_m) \in \Pi'_0 \times \ldots \times \Pi'_m$$

which is a morphism. In fact, if $P \in U$, suppose that P has equations

$$\begin{aligned} u_{00}x_0 + \ldots + u_{0n}x_n &= 0 \\ &\ldots \\ u_{n-m-1,0}x_0 + \ldots + u_{n-m-1,n}x_n &= 0 \end{aligned}$$

Suppose now Π_0' has equations

$$\begin{aligned} u_{n-m,0}x_0 + \ldots + u_{n-m,n}x_n &= 0 \\ &\ldots \\ u_{n-1,0}x_0 + \ldots + u_{n-1,n}x_n &= 0 \end{aligned}$$

From the definition of **U** it follows that the matrix

$$\mathbf{M} = \begin{pmatrix} u_{0,0} & \ldots & u_{0,n} \\ & \ldots & \\ u_{n-1,0} & \ldots & u_{n-1,n} \end{pmatrix}$$

has maximal rank n and the coordinates of the point $P \cap \Pi_0'$ are given by the maximal minors of **M** with alternate signs. These minors are polynomials of degree 1 in the dual Plücker coordinates of P. The same argument holds for the coordinates if the points $P \cap \Pi_i'$, for $i = 1, \ldots, m$. This proves that ψ is a morphism, which determines a rational map

$$\psi : \mathbb{G}(m, n) \dashrightarrow \Pi_0' \times \ldots \times \Pi_m'.$$

We have also the map

$$\phi : (Q_0, \ldots, Q_m) \in U' \to Q_0 \vee \ldots \vee Q_m \in \mathbb{G}(m, n)$$

which is also a morphism because it is the restriction to U' of $\phi_{m,n}$. Since clearly $\phi^{-1}(\phi(P_0, \ldots, P_m)) = (P_0, \ldots, P_m)$, and since $\dim(\Pi_0' \times \ldots \times \Pi_m') = (m+1)(n-m) = \dim(\mathbb{G}(m, n))$, the map ϕ is dominant because of Theorem 11.3.1, so it determines a dominant rational map

$$\phi : \Pi_0' \times \cdots \times \Pi_m' \dashrightarrow \mathbb{G}(m, n).$$

Take now $P \in U$ such that $\psi(P) \in U'$. Then one has $\phi(\psi(P)) = P$. Similarly, if $(Q_0, \ldots, Q_m) \in U'$ and $\phi(Q_0, \ldots, Q_m) \in U$, then $\psi(\phi(Q_0, \ldots, Q_m)) = (Q_0, \ldots, Q_m)$. Hence ϕ and ψ are birational transformations one inverse to the other. Since $\Pi_0' \times \ldots \times \Pi_m'$ is rational, the assertion follows. □

Exercise 13.2.6 *Let $\tau : \mathbb{P}^n \to \mathbb{P}^n$ be a projectivity. Prove that there is a projectivity ω_τ of $\mathbb{P}^{M(m,n)}$, fixing $\mathbb{G}(m, n)$, such that for any point $\Pi \in \mathbb{G}(m, n)$ one has $\omega_\tau(\Pi) = \tau(\Pi)$.

Exercise 13.2.7 Prove that $\mathbb{G}(1,3)$ is a quadric in $\mathbb{P}^5$, having Plücker equation

$$x_{01}x_{23} + x_{12}x_{03} - x_{13}x_{02} = 0.$$

Prove that the determinant of the matrix of this quadric is non-zero. This is called the *Klein quadric*.

Exercise 13.2.8 Fix a point $P \in \mathbb{P}^n$ and $H \subset \mathbb{P}^n$ a hyperplane not containing P. Consider the map

$$g_P : Q \in H \cong \mathbb{P}^{n-1} \to P \vee Q \in \mathbb{G}(1,n).$$

Prove that g_P is a projectivity of H onto a subspace of dimension $n-1$ contained in $\mathbb{G}(1,n)$.

Exercise 13.2.9 Exercise 13.2.8 can be generalised in the following way. Let Π_0 be a subspace of $\mathbb{P}^n$ of dimension h, and fix $m \geq h+1$. Consider the subset of $\mathbb{G}(m,n)$

$$\mathbb{G}(\Pi_0, m) = \{\Pi \in \mathbb{G}(m,n) : \Pi_0 \subset \Pi\}.$$

Prove that $\mathbb{G}(\Pi_0, m)$ is a closed subset of $\mathbb{G}(m,n)$ isomorphic to $\mathbb{G}(n-m-1, n-h-1)$.

Exercise 13.2.10 This is similar to Exercise 13.2.9. Let Π_0 be a subspace of $\mathbb{P}^n$ of dimension h, and fix $m < h$. Consider the subset of $\mathbb{G}(m,n)$

$$\mathbb{G}(m, \Pi_0) = \{\Pi \in \mathbb{G}(m,n) : \Pi \subset \Pi_0\}.$$

Prove that $\mathbb{G}(m, \Pi_0)$ is a closed subset of $\mathbb{G}(m,n)$ isomorphic to $\mathbb{G}(m,h)$.

Exercise 13.2.11 Prove that on the Klein quadric $\mathbb{G}(1,3)$ there are two families Σ_1, Σ_2 of planes, the former containing the planes corresponding to the lines passing through a given point of $\mathbb{P}^3$ (these are called *stars*), the latter containing the planes corresponding to the lines contained in a fixed plane of $\mathbb{P}^3$ (these are called *ruled planes*). Prove that two planes of the same family intersect each other in a point, whereas two planes of two different families intersect each other either in the empty set or along a line.

Exercise 13.2.12 Let $r = [p_{ij}]$ and $s = [q_{ij}]$ be two distinct points of $\mathbb{G}(1,3)$. Prove that the line $r \vee s$ is contained in $\mathbb{G}(1,3)$ if and only if

$$p_{01}q_{23} + q_{01}p_{23} + p_{12}q_{03} + q_{12}p_{03} - p_{13}q_{02} - q_{13}p_{02} = 0.$$

Prove that this happens if and only if the lines r and s intersect at a point, i.e, if and only if they are coplanar. Conclude that r, s are coplanar if and only if the line $r \vee s$ is contained in $\mathbb{G}(1,3)$.

Exercise 13.2.13 Let r be a line in $\mathbb{P}^3$. Let Q_r be the subset of $\mathbb{G}(1,3)$ consisting of r and of all lines distinct from r which are coplanar with r. Prove that Q_r is the section of $\mathbb{G}(1,3)$ with a hyperplane $H_r \cong \mathbb{P}^4$ and it is a quadric cone with vertex r in H_r.

Exercise 13.2.14 Prove that any line contained in $\mathbb{G}(1,3)$ is the intersection of a plane of Σ_1 with a plane in Σ_2.

Exercise 13.2.15 Prove that any plane contained in $\mathbb{G}(1,3)$ is either in Σ_1 or in Σ_2.

Exercise 13.2.16 *Fix positive integers n, m, d with $m \leq n$ and consider the set

$$\mathcal{I}(m,n,d) = \{(\Pi, Z) \in \mathbb{G}(m,n) \times \mathcal{L}_{n,d} : \Pi \subseteq Z\}.$$

Prove that $\mathcal{I}(m,n,d)$ is an irreducible closed subset of $\mathbb{G}(m,n) \times \mathcal{L}_{n,d}$ of codimension $N(m,d) + 1 = \binom{m+d}{d}$.

Exercise 13.2.17 Prove that if $N(m,d)+1>(m+1)(n-m)$, then there is a non-empty open subset $U \subset \mathcal{L}_{n,d}$ such that for any hypersurface $Z \in U$, Z contains no subspace $\Pi \in \mathbb{G}(m,n)$.

Exercise 13.2.18 Prove that if $Z \in \mathcal{L}_{n,d}$, the set

$$\mathbb{G}(m,Z)=\{\Pi \in \mathbb{G}(m,n) : \Pi \subseteq Z\}$$

is a closed subset of $\mathbb{G}(m,n)$. This is called the *family of m-dimensional subspaces* of Z.

Exercise 13.2.19 *From Exercises 13.2.16 and 13.2.17 it follows that if $d \geq 4$, the surfaces of degree d in $\mathbb{P}^3$ containing some line form a proper irreducible closed subset $\mathcal{Z}(1,3,d)$ of $\mathcal{L}_{3,d}$, which is the image of $\mathcal{I}(1,3,d)$ via the projection q to the second factor. Prove that $\dim(\mathcal{Z}(1,3,d)) = \dim(\mathcal{I}(1,3,d)) = N(3,d)-(d-3)$.

Exercise 13.2.20 Prove that any cubic surface in $\mathbb{P}^3$ contains at least a line and there is a dense open subset U in $\mathcal{L}_{3,3}$ such that for any $Z \in U$, Z has finitely many lines.

Exercise 13.2.21 An irreducible surface Z in $\mathbb{P}^3$ is said to be a *scroll* if it contains infinitely many lines, i.e., if $\mathbb{G}(1,Z) \subset \mathbb{G}(1,3)$ has a component of dimension at least 1. For example, according to this definition, a plane is a scroll. Prove that if Z is a cone of degree $d>1$, it is a scroll and $\mathbb{G}(1,Z)$ is a curve of degree d contained in the plane of Σ_1 corresponding to the star of lines containing the vertex of Z. Moreover $\mathbb{G}(1,Z)$ is isomorphic to a plane section of Z with a plane not containing the vertex of Z.

Prove that, conversely, if C is an irreducible curve in a plane of Σ_1 corresponding to the star of lines through the point $P \in \mathbb{P}^3$, then there is a cone Z with vertex P such that $C=\mathbb{G}(1,Z)$.

Exercise 13.2.22 *Prove that any irreducible quadric Q in $\mathbb{P}^3$ is a scroll. If Q is a cone we have the same situation as in Exercise 13.2.21, and $\mathbb{G}(1,Q)$ is a conic sitting the plane of Σ_1 corresponding to the star of lines containing the vertex of Q. Prove that if Q is not a cone, then $\mathbb{G}(1,Q)$ consists of two irreducible, disjoint conics Γ_1, Γ_2. The lines corresponding to points in Γ_i are pairwise skew, for $i=1,2$, whereas the lines corresponding to points of Γ_1 intersect in one point each line corresponding to a point of Γ_2.

Exercise 13.2.23 *Continuing Exercise 13.2.22, prove that if we fix three distinct lines r_1, r_2, r_3 of Γ_1, the plane in which Γ_2 sits is the intersection of the three hyperplanes $H_{r_1}, H_{r_2}, H_{r_3}$ introduced in Exercise 13.2.13. The same if we exchange Γ_1 with Γ_2. Deduce from this the well known fact that a quadric in $\mathbb{P}^3$ which is not a cone is the locus of all lines which are coplanar with three pairwise skew lines.

Exercise 13.2.24 *Suppose that the surface $Z \subset \mathbb{P}^3$ is a scroll. Prove that Z is a plane if and only if $\mathbb{G}(1,Z)$ has a component of dimension $n \geq 2$.

Exercise 13.2.25 Let $V \subseteq \mathbb{P}^n$ be a variety of dimension m. Consider the set $\mathbb{G}_V=\{\Pi \in \mathbb{G}(n-m-1,n) : \Pi \cap V \neq \emptyset\}$. Prove that $\mathbb{G}_V$ is an irreducible closed subset of $\mathbb{G}(n-m-1,n)$ of codimension 1.

Exercise 13.2.26 Let $V \subseteq \mathbb{P}^n$ be a variety of degree d and dimension m. Prove that the two sets $U_V=\{\Pi \in \mathbb{G}(n-m,n) : V \cap \Pi$ consists of $d$distinct points$\}$ and $U'_V=\{\Pi \in \mathbb{G}(n-m,n) : V \cap \Pi$consists of finitely many points$\}$, are open dense subsets of $\mathbb{G}(n-m,n)$.

13.3 Solutions of Some Exercises

13.2.6 Suppose that τ has matrix equation $\mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$, where $\mathbf{A}$ is a non-degenerate square matrix of order $n+1$. Take $\Pi \in \mathbb{G}(m, n)$. Choose $m+1$ independent points $P_i = [\mathbf{p}_i]$, $i = 0, \ldots, m$ of Π. The Plücker coordinates of Π are given by the minors of maximal order of the matrix $\mathbf{P}$ in (13.5). Then the Plücker coordinates of $\tau(\Pi)$ are given by the minors of maximal order of the matrix $\mathbf{A} \cdot \mathbf{P}^t$ in (13.5). By expanding the minors of maximal order of $\mathbf{A} \cdot \mathbf{P}^t$, we see that these are linear combinations of the minors of maximal order of $\mathbf{P}$ with coefficients depending on the entries of $\mathbf{A}$. These linear combinations define a projective transformation ω_τ of $\mathbb{P}^{M(m,n)}$ enjoying the required property. The projective transformation is a projectivity because it is bijective on $\mathbb{G}(m, n)$ and $\mathbb{G}(m, n)$ is non-degenerate in $\mathbb{P}^{M(m,n)}$.

13.2.8 We may assume $P = [1, 0, \ldots, 0]$ and H with equation $x_0 = 0$. Then given $Q = [0, x_1, \ldots, x_n]$, the Plücker coordinates of $P \vee Q$ are all zero, except

$$p_{1i} = x_i, \quad \text{for} \quad i = 1, \ldots, n.$$

This proves the assertion.

13.2.9 The proof is analogous to the one of Exercise 13.2.8 and can be left to the reader.

13.2.16 Take $(\Pi_0, Z_0) \in \mathcal{I}(m, n, d)$ and let us suppose that Π_0 has Plücker coordinates $[p^0_{i_0,\ldots,i_m}]$. We may assume, with no loss of generality, that $p^0_{0,\ldots,m} \neq 0$. Then there is an open neighborhood U of (Π_0, Z_0) in $\mathbb{G}(m, n) \times \mathcal{L}_{n,d}$ such that same happens for every $(\Pi, Z) \in U$. So for every $(\Pi, Z) \in U$ a set of $m+1$ linearly independent points of Π is given by the points $P_j = [p_{0,\ldots,j-1,i,j+1,\ldots,m}]_{i=0,\ldots,n}$, for $j = 0, \ldots, m$. The points of Π have homogeneous coordinates $[x_0, \ldots, x_n]$ given by

$$x_i = \sum_{j=0}^{m} \lambda_j p_{0,\ldots,j-1,i,j+1,\ldots,m}, \quad \text{, for} \quad i = 0, \ldots, n$$

with $[\lambda_0, \ldots, \lambda_m] \in \mathbb{P}^m$. If Z has equation $f(x_0, \ldots, x_n) = 0$ in $\mathbb{P}^n$, then (Π, Z) is in $\mathcal{I}(m, n, d)$ if and only if the polynomial

$$f(\ldots, \sum_{j=0}^{m} \lambda_j p_{0,\ldots,j-1,i,j+1,\ldots,m}, \ldots,)$$

in $(\lambda_0, \ldots, \lambda_m)$ is identically zero. The coefficients of this polynomial are of the form

$$\phi_h(\ldots a_{j_0,\ldots,j_n} \ldots, \ldots p_{i_0,\ldots,i_m} \ldots), \quad \text{for} \quad h = 0, \ldots, N(m, d)$$

where $a_{j_0,\ldots,j_n}$ are the coefficients of f and $(p_{i_0,\ldots,i_m})$ are the Plücker coordinates of Π. So $U \cap \mathcal{I}(m, n, d)$ is defined by the equations $\phi_h = 0$, and it is therefore a closed subset. This proves that $\mathcal{I}(m, n, d)$ is closed, because the notion of being closed is local.

Consider the projection

$$p : \mathcal{I}(m, n, d) \to \mathbb{G}(m, n)$$

to the first factor. To prove the rest of the assertion we apply Corollary 11.3.3 and prove that for any $\Pi \in \mathbb{G}(m, n)$, $p^{-1}(V)$ identifies with the set of all hypersurfaces in $\mathcal{L}_{n,d}$ containing Π, which is irreducible of codimension $N(m, d) + 1$ in $\mathcal{L}_{n,d}$. Let in fact $\mathcal{I}(n, d, \Pi)$ be such a subset of $\mathcal{L}_{n,d}$, which is of course closed. It is clear that $\mathcal{I}(n, d, \Pi)$ is a subspace of $\mathcal{L}_{n,d}$. In order to determine the dimension of $\mathcal{I}(n, d, \Pi)$ we can reduce ourselves to the case in which Π has equations $x_{m+1} = \ldots, x_n = 0$. In this case the hypersurface with equation $f(x_0, \ldots, x_n) = 0$ contains Π if and only if the polynomial $f(x_0, \ldots, x_m, 0, \ldots, 0)$ in $x_0, \ldots, x_m$ is identically zero, i.e., if and only if in f do not appear the $N(m, d) + 1$ monomials in $x_0, \ldots, x_m$. This proves the assertion.

13.2.17 Consider the projection

$$q : \mathcal{I}(m, n, d) \to \mathcal{L}_{n,d}$$

to the second factor, and let $\mathcal{R}(m, n, d)$ be its image, which is a closed subset of $\mathcal{L}_{n,d}$ of dimension $\dim(\mathcal{R}(m, n, d)) \leq \dim(\mathcal{I}(m, n, d)) = (m+1)(n-m) + N(n, d) - N(m, d) - 1$. If $(m+1)(n-m) + N(n, d) - N(m, d) - 1 < \dim(\mathcal{L}_{n,d}) = N(n, d)$, i.e., if $N(m, d) + 1 > (m+1)(n-m)$, then $\dim(\mathcal{R}(m, n, d)) < \dim(\mathcal{L}_{n,d})$, hence $\mathcal{R}(m, n, d)$ is a proper closed subset of $\mathcal{L}_{n,d}$. Then $U = \mathcal{L}_{n,d} \setminus \mathcal{R}(m, n, d)$ is the required open subset.

13.2.19 To prove the assertion it suffices to verify that there are surfaces $Z \in \mathcal{L}_{3,d}$ such that $q^{-1}(Z)$ is finite, i.e., such that Z contains a finite number of lines. Consider the irreducible surface of degree d of $\mathbb{A}^3$ with equation $x_1^{d-2}x_2x_3 = 1$. This surface does not contain any affine line. Indeed, such a line has parametric equations of the form

$$x_i = a_i + tb_i, \quad \text{with} \quad i = 1, 2, 3 \quad \text{and} \quad t \in \mathbb{K}$$

with $(b_1, b_2, b_3) \neq \mathbf{0}$. For this line to be contained in the surface, the polynomial

$$(a_1 + tb_1)^{d-2}(a_2 + tb_2)(a_3 + tb_3) - 1$$

in t should be identically zero, which is easily seen to be impossible. On the other hand the projective closure of this surface has exactly three lines on the plane at infinity.

13.2.20 Same argument as in the solution of Exercise 13.2.19.

13.2.22 It is well known that all quadrics Q in $\mathbb{P}^3$ which are not cones are projectively equivalent, so we can argue on one specific of them, e.g., the quadric Q with equation $x_0x_1 = x_2x_3$. We know (see Exercise 8.2.16) that Q has two families of lines $\mathcal{L}_1 = \{L_P\}_{P\in\mathbb{P}^1}$, $\mathcal{L}_2 = \{M_P\}_{P\in\mathbb{P}^1}$, with $P = [\lambda, \mu] \in \mathbb{P}^1$, where L_P has equations

$$\lambda x_0 = \mu x_2, \quad \mu x_1 = \lambda x_3$$

and M_P has equation

$$\lambda x_0 = \mu x_3, \quad \mu x_1 = \lambda x_2.$$

The lines in $\mathcal{L}_i$ are pairwise skew, for $i = 1, 2$, whereas the lines in $\mathcal{L}_1$ intersect in one point each line in $\mathcal{L}_2$. One can see, with a direct computation, that the two maps

$$\omega_1 : P \in \mathbb{P}^1 \to L_P \in \mathcal{L}_1 \subset \mathbb{G}(1, Q), \quad \omega_2 : P \in \mathbb{P}^1 \to M_P \in \mathcal{L}_2 \subset \mathbb{G}(1, Q),$$

are morphisms whose respective images Γ_1 and Γ_2 are two disjoint irreducible conics.

13.2.24 If Z is a plane, one has $\mathbb{G}(1, Z) \cong \mathbb{P}^2$ so it has dimension 2. Suppose conversely that Z is an irreducible surface that $\mathbb{G}(1, Z)$ has an irreducible component V of dimension $n \geq 2$. Consider the set $\tilde{V} = \{(P, r) \in Z \times V : P \in r\}$, which is easily proved to be a closed subset of $Z \times V$. The projection to the second factor V is surjective and the fibres are lines, so $\tilde{V}$ is irreducible of dimension $n + 1$. Consider the projection to the first factor Z. This is also surjective, and therefore the fibre of any point $P \in Z$ has dimension $n + 1 - 2 = n - 1 \geq 1$. This means that there are infinitely many lines of V containing any point of Z. This implies that Z is a cone with vertex any point $P \in Z$. As a consequence one has that given any two distinct points $P, Q \in Z$, the line $P \vee Q$ is contained in Z, and this implies that Z is a plane.

Chapter 14
Smooth and Singular Points

14.1 Basic Definitions

Let $V \subseteq \mathbb{A}^n$ be an affine variety, with $\mathcal{I}_a(V) = (f_1, \dots, f_m)$ and let $P = (p_1, \dots, p_n)$ be a point of V. Let r be a line passing through P, so that r has parametric equations of the form

$$x_i = p_i + \lambda_i t, \quad \text{with} \quad t \in \mathbb{K} \quad \text{for} \quad i = 1, \dots, n, \quad \text{and} \quad (\lambda_1, \dots, \lambda_n) \neq \mathbf{0}.$$

The polynomial system in t

$$f_i(t) := f_i(p_1 + \lambda_1 t, \dots, p_n + \lambda_n t) = 0, \quad i = 1, \dots, m$$

has the solution $t = 0$. If the polynomials $f_1(t), \dots, f_m(t)$ are all identically 0, this means that r is contained in V. Otherwise, the greatest common divisor of the polynomials $f_1(t), \dots, f_m(t)$ is a non-zero polynomial of the form

$$f(t) = \alpha t^c \prod_{i=1}^{h} (t - \alpha_i)^{c_i} \tag{14.1}$$

where $\alpha, \alpha_1, \dots, \alpha_h \in \mathbb{K}^*$ with $\alpha_1, \dots, \alpha_h$ all distinct, and $c, c_1, \dots, c_h$ are positive integers, with $c \geqslant 1$. The integer c is said to be the *intersection multiplicity* of r and V at P, and it is denoted by $i(P; r, V)$. One sets $i(P; r, V) = \infty$ if $r \subseteq V$. One says that the line r *touches* V in P if $i(P; r, V) > 1$, and in that case one says that r is *tangent* to V at P. The definition of tangency of a line to the variety V is independent on the basis $f_1, \dots, f_m$ of $\mathcal{I}_a(V)$ (see Exercise 14.1.6).

Lemma 14.1.1 *The set of all lines tangent to V at P is an affine subspace $T_P(V)$ of $\mathbb{A}^n$.*

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
https://doi.org/10.1007/978-3-030-71021-7_14

Proof We keep the notation introduced above. Let $f(x_1, \dots, x_n) \in A_n$ be a polynomial which vanishes at P. By expanding f in Taylor series with initial point P, we have

$$f(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x_i - p_i) + o(2)$$

where $o(2) \in \mathfrak{m}_P^2$, with $\mathfrak{m}_P = (x_1 - p_1, \dots, x_n - p_n)$ the maximal ideal corresponding to P. We set

$$d_P f(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x_i - p_i)$$

hence $d_P f$ is a linear polynomial which vanishes at P, i.e., it belong to $\mathfrak{m}_P$. We have

$$f_i(t) = d_P f_i(\lambda_1 t, \dots, \lambda_n t) + o(2) = t d_P f_i(\lambda_1, \dots, \lambda_n) + o(2), \quad i = 1, \dots, m,$$

where, in $o(2)$, t appears at least with exponent 2. It is clear that r is tangent to V at P if and only if $d_P f_i(\lambda_1, \dots, \lambda_n) = 0$ for $i = 1, \dots, m$. Therefore the union of the tangent lines to V at P is defined in $\mathbb{A}^n$ by the equations

$$d_P f_i = 0, \quad i = 1, \dots, m,$$

which are linear and therefore define an affine subspace of $\mathbb{A}^n$. □

The affine subspace $T_P(V)$, considered as a vector space with zero at P, is called the *(Zariski) tangent space* of V at P. Looking at the proof of Lemma 14.1.1, we see that

$$\dim(T_P(V)) = n - \rho_P,$$

where

$$\rho_P = \operatorname{rank}\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i=1,\dots,m;\, j=1,\dots,n}.$$

Let Ω_P be the vector space of dimension n of all linear polynomials which vanish at P. Consider the map

$$d_P : f \in A_n \to d_P f \in \Omega_P$$

which is linear and verifies the *Leibnitz rule*

$$d_P(fg) = d_P f \cdot g(P) + d_P g \cdot f(P).$$

Taking this into account, we see that d_P induces a homomorphism

$$d_P : f \in A(V) \to d_P f \in T_P(V)^\vee$$

(where $T_P(V)^\vee$ is the dual of $T_P(V)$), which is clearly surjective. Since d_P takes the value 0 on the constants, the map

$$d_p : \mathfrak{m}_{V,P} \to T_P(V)^\vee$$

(where $\mathfrak{m}_{V,P}$ is the maximal ideal of P in $A(V)$) is still surjective. Moreover, because of the Leibniz rule, $\mathfrak{m}^2_{V,P}$ is contained in the kernel of d_P, so that we have the map

$$d_P : \mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P} \to T_P(V)^\vee. \tag{14.2}$$

Lemma 14.1.2 *The map d_P in (14.2) is an isomorphism.*

Proof It suffices to prove that the map d_P in (14.2) is injective. Let $g \in \mathfrak{m}_{V,P}$ be such that $d_P g = 0$. Suppose that g is induced by a polynomial $G \in A_n$. Then $d_P G$ vanishes on $T_P(V)$ and then we have a relation of the form

$$d_P G = \lambda_1 d_P f_1 + \cdots + \lambda_m d_P f_m \quad \text{with} \quad \lambda_1 \ldots, \lambda_m \in \mathbb{K}.$$

Set $G' = G - \lambda_1 f_1 - \cdots - \lambda_m f_m$. Then G' vanishes at P and it has no terms of degree 1 in $x_1 - p_1, \ldots, x_n - p_n$, hence $G' \in (x_1 - p_1, \ldots, x_n - p_n)^2$. Furthermore $G'_{|V} = G_{|V} = g$, hence $g \in \mathfrak{m}^2_{V,P}$ as wanted. □

In conclusion d_P induces an identification

$$T_P(V) = (\mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P})^\vee. \tag{14.3}$$

This identity suggests that we can extend the notion of Zariski tangent space to any quasi-projective variety. Indeed, if V is such a variety and $P \in V$ is a point, we define the Zariski tangent space $T_P(V)$ to V at P as the vector space on $\mathbb{K} = \mathcal{O}_{V,P}/\mathfrak{m}_P$ given by (14.3).

In case V is projective, this vector space can be identified with an affine subspace of the projective space in which V sits, and its projective closure $T_{V,P}$ is called the *(projective) tangent space* to V at P.

Next we want to understand what is the dimension of the Zariski tangent space. Consider again an affine variety $V \subseteq \mathbb{A}^n$ and consider in $V \times \mathbb{A}^n$ the closed subset $T(V)$ consisting of all pairs $(P, Q) \in V \subseteq \mathbb{A}^n$, with $Q = (x_1, \ldots, x_n)$ such that

$$d_P f_1(x_1, \ldots, x_n) = \cdots = d_P f_m(x_1, \ldots, x_n) = 0,$$

i.e., this is the set of all pairs $(P, Q) \in V \subseteq \mathbb{A}^n$ such that $Q \in T_P(V)$. This closed set is called the *tangent fibration* to V: the fibres of the projection of $T(V)$ to the first factor V are just the Zariski tangent spaces to V at its points. Consider the rank ρ of the matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}\right)_{i=1,\ldots,m;\, j=1,\ldots,m}.$$

where its elements are considered modulo $\mathcal{I}_a(V)$, i.e., as elements of $A(V)$. Let $F_1(x_1, \dots, x_n), \dots, F_h(x_1, \dots, x_n)$ be its minors of order ρ, that are not identically zero. Hence

$$\mathrm{Sing}(V) := Z_V(F_1, \dots, F_h) = V \cap Z_a(F_1, \dots, F_h)$$

is a proper closed subset of V. If $P \in V \setminus \mathrm{Sing}(V)$, the rank of J, computed at P is ρ, and therefore $\dim(T_P(V)) = n - \rho$. If $P \in \mathrm{Sing}(V)$ the rank of J computed at P is strictly smaller that ρ and therefore $\dim(T_P(V)) > n - \rho$. The points in $\mathrm{Sing}(V)$ are called *singular* of *multiple* points for V, whereas the points in $V \setminus \mathrm{Sing}(V)$ are called *smooth points* or also *simple points* of V. Note that the smooth points of V fill up a dense open subset of V, whereas the singular points fill up a proper closed subset of V. A variety with no singular points is said to be *smooth*.

As for the determination of ρ, we have the following:

Theorem 14.1.3 *In the above setting one has $\rho = n - \dim(V)$, i.e., in any smooth point $P \in V$ one has $\dim(T_P(V)) = \dim(V)$, whereas in a singular point $\dim(T_P(V)) > \dim(V)$.*

Proof We start by remarking that the assertion holds for affine hypersurfaces (see Exercise 14.1.8). The assertion follows from the fact that any variety is birational to an affine hypersurface (see Theorem 7.2.3). □

Exercise 14.1.4 Prove that $\mathbb{A}^n$ and $\mathbb{P}^n$ are smooth.

Exercise 14.1.5 Prove that the blow-up of $\mathbb{P}^n$ along a subspace is smooth.

Exercise 14.1.6 Prove that the polynomial $f(t)$ in (14.1.1) is the greatest common divisor of all polynomials of the form

$$g(p_1 + \lambda_1 t, \dots, p_n + \lambda_n t)$$

with $g \in \mathcal{I}_a(V)$.

Exercise 14.1.7 Prove that if Π is an affine subspace of $\mathbb{A}^n$, then it coincides with its tangent space at any of its points.

Exercise 14.1.8 Let V be an irreducible hypersurface of $\mathbb{A}^n$ with reduced equation $f(x_1, \dots, x_n) = 0$ and let $P = (p_1, \dots, p_n)$ be a point of V. Prove that $T_P(V)$ is:

(a) the whole space $\mathbb{A}^n$ if and only if $\frac{\partial f}{\partial x_i}(P) = 0$ for all $i = 1, \dots, n$;
(b) the hyperplane of $\mathbb{A}^n$ with equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x_i - p_i) = 0$$

if the *gradient* $\mathrm{grad} f(P) = (\frac{\partial f}{\partial x_1}(P), \dots, \frac{\partial f}{\partial x_n}(P))$ of f at P is non-zero.

Prove that there is a dense open subset of V such that case (b) occurs.

Exercise 14.1.9 *Let H be an irreducible hypersurface in $\mathbb{P}^n$ and let P be a point of H. Prove that P is a smooth point for H if and only if there is some line passing through P having with H at P intersection multiplicity 1. Prove that in this case the tangent hyperplane to H at P is the locus of all lines passing through P having with H at P intersection multiplicity at least 2.

Exercise 14.1.10 *Let H be an irreducible hypersurface in $\mathbb{P}^n$ and let P be a point of H. The point P is said to have *multiplicity* m (or P is said to be a m*-tuple point* for H), and one writes $m = m_P(H)$, if every line passing through P has with H at P intersection multiplicity at least m and there is some line through P having with H at P intersection multiplicity exactly m. In particular a point P is smooth for H if and only if it has multiplicity 1. Prove that if P is a m-tuple point for H, the union of lines having with H at P intersection multiplicity larger than m is a divisor of degree m, which is a cone with vertex P, called the *tangent cone* to H at P. It is denoted by $TC_{H,P}$.

Exercise 14.1.11 *Let H be a divisors in $\mathbb{P}^n$ with equation $f(x_0, \ldots, x_n) = 0$. One can extend to H the notions of simple and multiple point in an obvious way: P is said to have *multiplicity* m (or P is said to be a m*-tuple point* for H), and one writes $m = m_P(H)$, if every line passing through P has with H at P intersection multiplicity at least m and there is some line through P having with H at P intersection multiplicity exactly m. A simple point is a point with multiplicity $m = 1$.

Prove that P is a point of multiplicity m for H if and only if all derivatives of f of order $i \leqslant m - 1$ vanish at P whereas not all derivatives of order m of f vanish at P. Prove that if P is a point of multiplicity m for H, then the union of all lines having with H at P intersection multiplicity larger than m form a divisor (which is a cone with vertex P), with equation

$$\sum_{i_1+\ldots+i_n=m} \frac{\partial^m f}{\partial x_0^{i_0} \cdots \partial x_n^{i_n}}(P) x_0^{i_0} \cdots x_n^{i_n} = 0. \tag{14.4}$$

This is called again the *tangent cone* to H at P and denoted by $TC_{H,P}$. If $m = 1$ the tangent cone is a hyperplane, called the *tangent hyperplane* and denoted by $T_{H,P}$.

Exercise 14.1.12 *Let H be a divisor of degree d with equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{A}^n$ and let $P = (p_1, \ldots, p_n)$ be a point of H. Expand the polynomial f in Taylor series with initial point P. Then we have

$$f = f_1 + f_2 + \cdots + f_d$$

where

$$f_i = \sum_{j_1+\cdots+j_n=i} \frac{\partial^i f}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}(P)(x_1 - p_1)^{j_1} \cdots (x_n - p_n)^{j_n}$$

is a homogeneous polynomial of degree i in $x_1 - p_1, \ldots .x_n - p_n$, for $i = 1, \ldots, d$. One defines P to be a point of multiplicity m for H, and one writes $m = m_P(H)$, if and only if $f_1, \ldots, f_{m-1}$ are identically zero, whereas f_m is not identically zero. Prove that P is a point of multiplicity m for H if and only if all derivatives of f of order $i \leqslant m - 1$ vanish at P, whereas there is some derivative of f of order m at P which is non-zero. Prove that P has multiplicity m for H if and only if every line passing through P has with H at P intersection multiplicity at least m and there is some line through P having with H at P intersection multiplicity exactly m. Prove that P is smooth for H if and only if $m = 1$. Prove that P has multiplicity m for H if and only if it has multiplicity m for the projective closure of H in $\mathbb{P}^n$. Prove that if P is a point of multiplicity m for H then the projective closure of the affine hypersurface with equation $f_m = 0$ is the tangent cone to H at P. The affine hypersurface with equation $f_m = 0$ is also called the *tangent cone* to H at P. Prove that this is the union of all lines having with H at P intersection multiplicity larger than m.

Exercise 14.1.13 *Let $H \subset \mathbb{A}^n$ be a divisor with equation $f(x_1, \ldots, x_n) = 0$ and let P be any point of H. For all $i = 1, \ldots, n$ consider the hypersurface H_i with equation $\frac{\partial f}{\partial x_i} = 0$. Prove that $m_P(H_i) \geqslant m_P(H) - 1$. Prove an analogous result for projective hypersurfaces.

Exercise 14.1.14 *Prove that an irreducible affine or projective curve has finitely many singular points.

Exercise 14.1.15 Prove that if the hypersurface H in $\mathbb{P}^n$ has components of multiplicities $m_1, \ldots, m_h$, then any point in the intersection of these components has multiplicity at least $m = m_1 + \cdots + m_h$ for H.

Exercise 14.1.16 Prove that the multiplicity of a point of a hypersurface of degree d cannot be higher than d. Prove that a hypersurface H in $\mathbb{P}^n$ of degree d is a cone with vertex P if and only if P is a point of multiplicity d for H. In that case $H = TC_{H,P}$.

Exercise 14.1.17 *Prove that an affine or projective hypersurface V of degree d is a monoid of vertex P (see Exercise 7.3.5) if and only if P has multiplicity $d - 1$ for V.

Exercise 14.1.18 Suppose that $\mathrm{char}(\mathbb{K}) \neq 2$. Consider an irreducible quadric Q in $\mathbb{P}^n$, having equation

$$\sum_{0 \leqslant i \leqslant j \leqslant n} a_{ij} x_i x_j = 0,$$

with symmetric matrix $\mathbf{A} = (a_{ij})_{0 \leqslant i \leqslant j \leqslant n}$. Prove that Q is smooth if and only if $\det(\mathbf{A}) \neq 0$. More generally, prove that $\mathrm{Sing}(Q)$ is the linear space with equations

$$\sum_{j=0}^{n} a_{ij} x_j = 0, \quad \text{for} \quad i = 0, \ldots, n.$$

Exercise 14.1.19 Prove that there is a non-empty open subset U of $\mathcal{L}_{n,d}$ such that for all $H \in U$, H is a smooth, irreducible hypersurface.

Exercise 14.1.20 Prove that Segre varieties are smooth.

Exercise 14.1.21 Prove that Grassmann varieties are smooth.

Exercise 14.1.22 *Let $f : X \to Y$ be a morphism of varieties, let $P \in X$ and $Q = f(P)$. Prove that there is a natural linear map

$$df_P : T_P(X) \to T_Q(Y)$$

induced by the map $f^* : \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$. The map df_P is called the *differential of f at P*.

Exercise 14.1.23 *Let $X \subseteq \mathbb{P}^r$ be a smooth projective variety of dimension n. Let $\mathrm{Tan}(X)$ be the union of all projective tangent spaces to X at its points. Prove that $\mathrm{Tan}(X)$ is a variety of dimension $m \leqslant 2n$. It is called the *tangential variety* of X.

14.2 Some Properties of Smooth Points

14.2.1 Regular Rings

Let V be a quasi-projective variety of dimension n and let P be a point of V. Recall that $\dim_K(\mathcal{O}_{V,P}) = n$. Hence P is a smooth point if and only if

$$\dim(\mathfrak{m}_{V,P}/\mathfrak{m}_{V,P}^2) = \dim_K(\mathcal{O}_{V,P}).$$

This suggests to give the following algebraic definition. The local ring $(A, \mathfrak{m})$ is said to be *regular* if and only if

$$\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = \dim_K(A).$$

So $P \in V$ is smooth if and only if $(\mathcal{O}_{V,P}, \mathfrak{m}_{V,P})$ is regular.

14.2.2 System of Parameters

Let V be a quasi-projective variety of dimension n and let $P \in V$ be a smooth point. Given $u_1, \dots, u_n \in \mathfrak{m}_{V,P}$, one says that $u_1, \dots, u_n$ is a *system of parameters* of V at P, if the classes of $u_1, \dots, u_n$ generate $\mathfrak{m}_{V,P}/\mathfrak{m}^2_{V,P}$.

We need the following result of algebra:

Lemma 14.2.1 (Nakayama's Lemma) *Let $(A, \mathfrak{m})$ be a local domain and let M be a finitely generated A-module. Let $u_1, \dots, u_m \in M$. Then $u_1, \dots, u_m$ generate M if and only if their classes generate $M/\mathfrak{m}M$.*

Proof One implication is obvious. We prove only the other.

Let us start proving that if $M = \mathfrak{m}M$ then $M = 0$. Indeed, if M is not zero, let $u_1, \dots, u_n$ be a minimal set of generators of M. Then we have $u_n = a_1u_1 + \cdots + a_nu_n$, with $a_1, \dots, a_n \in \mathfrak{m}$. Hence

$$(1 - a_n)u_n = a_1u_1 + \cdots a_{n-1}u_{n-1}$$

and $1 - a_n \notin \mathfrak{m}$, so that it is invertible. Hence we have

$$u_n = b_1u_1 + \cdots b_{n-1}u_{n-1}$$

with $b_i = \frac{a_i}{1-a_n}$, for $i = 1, \dots, n-1$, a contradiction.

It follows that if $N \subseteq M$ is a finitely generated submodule of M, then $\mathfrak{m}M + N = M$ implies that $M = N$. In fact it suffices to apply what we proved above to M/N and remark that $\mathfrak{m}(M/N) = (\mathfrak{m}M + N)/N$.

Finally, if N is the submodule generated by $u_1, \dots, u_n$, one has that $\mathfrak{m}M + N = M$, and we conclude by what we proved above. □

As an immediate consequence we have:

Theorem 14.2.2 *Let V be a quasi-projective variety of dimension n and let $P \in V$ be a smooth point. Given $u_1, \dots, u_n \in \mathfrak{m}_{V,P}$, then $u_1, \dots, u_n$ is a system of parameters of V in P if and only if they generate $\mathfrak{m}_{V,P}$.*

14.2.3 Auslander–Buchsbaum Theorem

We state the following famous result:

Theorem 14.2.3 (Theorem of Auslander–Buchsbaum) *Any regular local ring is UFD.*

As an immediate consequence we have:

Corollary 14.2.4 *Let V be a quasi-projective variety and $P \in V$ a smooth point. Then $\mathcal{O}_{V,P}$ is a UFD. Hence V is also normal.*

Proof Immediate consequence of Theorem 14.2.3, and the fact that any UFD is integrally closed (see Exercise 5.4.3). □

For the proof of Theorem 14.2.3, we defer the reader to [5, Chapt. 7], or to [7, p. 101 and foll.]. Here we will content ourselves to prove Theorem 14.2.3 in the case the ring has Krull dimension 1, hence Corollary 14.2.4 in the case of curves.

We start with the following:

Theorem 14.2.5 (Krull's Theorem) *If A is a noetherian domain and $\mathcal{I}$ is a proper ideal of A, then $\bigcap_{n\in\mathbb{N}} \mathcal{I}^n = (0)$.*

Proof We argue by contradiction and assume we have $a \in \bigcap_{n\in\mathbb{N}} \mathcal{I}^n$ non-zero. Let $a_1, \ldots, a_r$ be a set of generators of $\mathcal{I}$. For every $n \in \mathbb{N}$ there is a homogeneous polynomial $F_n \in A[x_1, \ldots, x_r]$, such that $a = F_n(a_1, \ldots, a_r)$. The ideal generated by $\{F_n\}_{n\in\mathbb{N}}$ is finitely generated, hence there is an $m \in \mathbb{N}$ such that $(F_n)_{n\in\mathbb{N}} = (F_1, \ldots, F_m)$. Thus we have

$$F_{m+1} = G_1 F_1 + \cdots + G_m F_m$$

with $G_1, \ldots, G_m \in A[x_1, \ldots, x_r]$ homogeneous of the appropriate degrees. Then

$$a = G_1(a_1, \ldots, a_r)a + \cdots + G_m(a_1, \ldots, a_r)a$$

hence

$$1 = G_1(a_1, \ldots, a_r) + \cdots + G_m(a_1, \ldots, a_r)$$

therefore $1 \in \mathcal{I}$, a contradiction. □

Now we have the following result which proves Theorem 14.2.3 in the 1-dimensional case:

Theorem 14.2.6 *Let $(A, \mathfrak{m})$ be a local noetherian, regular domain of dimension 1. Then A has principal ideals, hence it is a UFD. More precisely one has $\mathfrak{m} = (u)$ with the class of u a generator of $\mathfrak{m}/\mathfrak{m}^2$, and all non-zero proper ideals of A are of the form $\mathfrak{m}^n = (u^n)$, with $n \in \mathbb{N}$.*

Proof By Nakayama's Lemma (see also Theorem 14.2.2) we have that $\mathfrak{m} = (u)$ with the class of u a generator of $\mathfrak{m}/\mathfrak{m}^2$. Let $\mathcal{I}$ be a non-zero proper ideal of A. Then there is a positive integer n such that $\mathcal{I} \subseteq \mathfrak{m}^n = (u^n)$ and $\mathcal{I} \subsetneq \mathfrak{m}^{n+1}$. Then $u^{-n}\mathcal{I}$ is an ideal which is not contained in $\mathfrak{m}$, hence $u^{-n}\mathcal{I} = A$, thus $\mathcal{I} = (u^n) = \mathfrak{m}^n$. □

It is useful to go deeper into the study of local noetherian, regular domains of dimension 1. Let us first make a definition. Let k be a field. A *discrete valuation* on k is a surjective map

$$v : k^* = k \setminus \{0\} \to \mathbb{Z}$$

such that:

(a) for all $x, y \in k^*$ one has $v(xy) = v(x) + v(y)$, namely $v : k^*(\cdot) \to \mathbb{Z}(+)$ is a homomorphism;
(b) $v(x + y) \geqslant \min\{v(x), v(y)\}$.

Sometimes one sets $v(0) = \infty$.

If v is a discrete valuation on k, then $A = \{0\} \cup \{x \in k^* : v(x) \geqslant 0\}$ is a ring, which is called the *ring of the valuation* v. A domain A is called a *discrete valuation ring* (briefly DVR) if there is a discrete valuation v on $k = \mathbb{Q}(A)$ such that A is the ring of the valuation v.

Lemma 14.2.7 *If A is a DVR with respect to the valuation v on $\mathbb{Q}(A)$, then A is a local ring with maximal ideal* $\mathfrak{m} = \{x \in A : v(x) > 0\}$.

Proof It is clear that $\mathfrak{m}$ is an ideal. It is immediate that $v(1) = 0$ so that $1 \notin \mathfrak{m}$, so $\mathfrak{m}$ is a proper ideal. Moreover it is also obvious that for all $x \in \mathbb{Q}(A)^*$ one has $v(x) = -v(x^{-1})$, hence if $x \in A \setminus \mathfrak{m}$, one has $v(x) = 0$, then also $v(x^{-1}) = 0$, which implies that $x^{-1} \in A$, so x is invertible. This proves the assertion (see Proposition 5.3.1). □

Lemma 14.2.8 *If A is a DVR with respect to the valuation v on $\mathbb{Q}(A)$, then given $x, y \in A$ such that $v(x) = v(y)$, one has $(x) = (y)$.*

Proof If $v(x) = v(y)$, then $v(xy^{-1}) = 0$, so that $a = xy^{-1} \in A$ is invertible (by Lemma 14.2.7), hence $x = ay$ and therefore $(x) = (y)$. □

Lemma 14.2.9 *If A is a DVR with respect to the valuation v on $\mathbb{Q}(A)$, there is an element $u \in \mathfrak{m}$ such that $v(u) = 1$ and any such element is such that $(u) = \mathfrak{m}$.*

Proof Since the valuation map is surjective, it is clear that there is some $u \in A$ such that $v(u) = 1$ and then $u \in \mathfrak{m}$. Let $x \in \mathfrak{m}$. Then $v(x) = n > 0$. We have $v(u^n) = n$, so $v(u^n) = v(x)$ and therefore $(u^n) = (x)$. This implies that $(u) = \mathfrak{m}$. □

Lemma 14.2.10 *If A is a DVR with respect to the valuation v on $\mathbb{Q}(A)$, then the only non-trivial ideals of A are powers of* $\mathfrak{m}$. *As consequences, A is noetherian with* $\dim_K(A) = 1$ *and moreover A has principal ideals, hence it is a UFD.*

Proof Given a non-trivial ideal $\mathcal{I}$, there is a minimal positive integer n such that $v(x) = n$ for $x \in \mathcal{I}$. Let u be such that $v(u) = 1$. Then with the same argument as in the proof of Lemma 14.2.9, we have $(x) = (u^n)$. Moreover, if y is such that $v(y) = m \geqslant n$, then the usual argument implies that $(y) = (xu^{m-n})$. This yields that

$$\mathcal{I} = \{y \in A : v(y) \geqslant n\} = \mathfrak{m}^n$$

as wanted. □

Next we prove the following basic:

Theorem 14.2.11 *Let $(A, \mathfrak{m})$ be a local noetherian domain of Krull dimension 1, with $k = A/\mathfrak{m}$. The following propositions are equivalent:*

(a) A is a DVR;
(b) A is integrally closed;
(c) $\mathfrak{m}$ is principal;
(d) $(A, \mathfrak{m})$ is regular, i.e., $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$*;*
(e) every non-trivial ideal of A is a power of $\mathfrak{m}$;
(f) there is $u \in A$ such that every non-zero ideal of A is of the form (u^n), with $n \in \mathbb{N}$.

Proof (a) implies (b) because if A is a DVR, then it is UFD and this implies that it is integrally closed (see Exercise 5.4.3).
(b) implies (c). First we need a couple of preliminaries.

Claim A: Let $\mathcal{I}$ be an ideal of A. Then $\mathrm{rad}(\mathcal{I}) = \mathfrak{m}$.

In fact, since $\dim_K(A) = 1$, then $\mathfrak{m}$ is the only non-zero prime ideal of A. Since $\mathrm{rad}(\mathcal{I})$ is the intersection of all prime ideals containing $\mathcal{I}$ (see Exercise 2.5.6), the Claim follows.
Claim B: If A is a noetherian ring, then any ideal $\mathcal{I}$ of A contains a power of its radical.

Let $x_1, \ldots, x_h$ be a set of generators of $\mathrm{rad}(\mathcal{I})$, so that we have relations of the sort $x_i^{n_i} \in \mathcal{I}$, with n_i suitable positive integers, for $i = 1, \ldots, h$. Set $m = \sum_{i=i}^h (n_i - 1) + 1$. Then $\mathrm{rad}(\mathcal{I})^m$ is generated by products of the form $x_1^{r_1} \cdots x_h^{r_h}$, with $\sum_{i=1}^h r_i = m$. From the definition of m, we have that there is at least an index $i = 1, \ldots, h$ such that $r_i \geqslant n_i$, so that each of the above products belongs to $\mathcal{I}$, hence $\mathrm{rad}(\mathcal{I})^m \subseteq \mathcal{I}$.

To prove the implication, take $u \in \mathfrak{m}$ with $u \neq 0$. By Claims A and B there is a positive integer n such that $\mathfrak{m}^n \subseteq (u)$ but $\mathfrak{m}^{n-1} \subsetneq (u)$. Take $v \in \mathfrak{m}^{n-1}$ with $v \notin (u)$, in particular $v \neq 0$. Set $x = \frac{u}{v} \in \mathbb{Q}(A)$. One has $x^{-1} \notin A$, because $v \notin (u)$. Hence x^{-1} is not integral on A. Then we claim that $x^{-1}\mathfrak{m} \subsetneq \mathfrak{m}$. Indeed, if $x^{-1}\mathfrak{m} \subseteq \mathfrak{m}$, then $\mathfrak{m}$ would be an $A[x^{-1}]$-module which is finitely generated as an A-module. If $x_1, \ldots, x_h$ is a system of generators of $x^{-1}\mathfrak{m}$, we would have relations of the sort

$$x_i x^{-1} = a_{i1}x_1 + \cdots + a_{ih}x_h, \quad \text{for} \quad i = 1, \ldots, h,$$

with $a_{ij} \in A$, for $i, j = 1, \ldots, h$. With an argument we already made in the proof of Lemma 5.4.1, this would imply that x^{-1} is integral over A, a contradiction. On the other hand $x^{-1}\mathfrak{m} = \frac{v}{u}\mathfrak{m} \subseteq A$, because $v \in \mathfrak{m}^{n-1}$, hence $v\mathfrak{m} \subseteq \mathfrak{m}^n$ and $\mathfrak{m}^n \subseteq (u)$. So in conclusion $x^{-1}\mathfrak{m} = A$, hence $\mathfrak{m} = xA = (x)$, as wanted.
(c) implies (d): suppose that $\mathfrak{m} = (u)$, then u generates $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{m}/\mathfrak{m}^2$ is non-zero because of Krull's Theorem 14.2.5. Thus $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$.
(d) implies (e): Let $\mathcal{I} \neq (0)$ be a proper ideal of A. By Krull's Theorem, there is a positive integer n such that $\mathcal{I} \subseteq \mathfrak{m}^n$ but $\mathcal{I} \subsetneq \mathfrak{m}^{n+1}$. By Nakayama's Lemma $\mathfrak{m}$ is principal, i.e., $\mathfrak{m} = (u)$ for a suitable u. So there a $y \in \mathcal{I}$ such that $y = au^n$, with $a \notin (u) = \mathfrak{m}$, hence a is invertible. So $u^n \in \mathcal{I}$. Then $\mathfrak{m}^n = (u^n) \subseteq \mathcal{I} \subseteq \mathfrak{m}^n$, hence $\mathcal{I} = \mathfrak{m}^n$, as wanted.
(e) implies (f): by Krull's Theorem there is $u \in \mathfrak{m}$ such $u \notin \mathfrak{m}^2$. By the hypothesis (e), one has $(u) = \mathfrak{m}^r$, for some $r \geqslant 1$ and therefore $r = 1$. Then $\mathfrak{m} = (u)$ and every non-zero ideal is of the form $\mathfrak{m}^n = (u^n)$, with $n \in \mathbb{N}$.
(f) implies (a): By Krull's Theorem we have $(u^n) \neq (u^{n+1})$ for all non-negative integers n. So if $a \in A \setminus \{0\}$, there is a unique non-negative integer n such that $(a) = (u^n)$. We set $v(a) = n$. Then we extend the definition of v to $\mathbb{Q}(A)$ by setting $v(\frac{a}{b}) = v(a) - v(b)$, with $b \neq 0$. This defines a discrete valuation and A is a DVR. □

As a consequence we have:

Corollary 14.2.12 *A curve C is smooth if and only if it is normal.*

Remark 14.2.13 Theorem 14.2.11 and its proof have an important consequence concerning the local behaviour of rational maps on smooth curves. Let V be a curve and $P \in V$ a smooth point. Since the question we want to treat is local, we may assume that V is affine. Consider a non-zero rational function f on V. Theorem 14.2.11 tells us that there is a discrete valuation v_P defined on $K(V)$, such that $(\mathcal{O}_{V,P}, \mathfrak{m}_P)$ is the DVR of v_P. Then f is defined at P if and only if $v_P(f) \geqslant 0$, and $f(P) = 0$ if and only if $v_P(f) > 0$. In this case we will say that P is a *zero* of f and $v_P(f)$ will be called the *order of zero* of f at P. If $v_P(f) = 0$, then f is defined at P, it is not zero at P, and it is invertible at P. Finally, suppose that f is not defined at P, hence $v_P(f) < 0$. We will then say that f has a *pole* at P. In this case $v_P(f^{-1}) = -v_P(f) > 0$ and f^{-1} is defined at P and it has order of zero equal to $-v_P(f)$ at P. This will be called the *order of pole* of f at P.

If f has a zero of order m at P, then f^{-1} has a pole of order m at P. If f has a pole of order m at P, and g is a non-zero rational function with a zero at P, then $v_P(fg^m) = v_P(f) + mv_P(g) \geqslant 0$, so that fg is defined at P.

14.2.4 Local Equations of a Subvariety

Let V be a quasi-projective variety, let P be a point of V and let W be a closed subset of V containing P. Take $f_1, \ldots, f_m \in \mathcal{O}_{V,P}$. We say that $f_1 = \cdots = f_m = 0$ is a *system of equations* for W in P, if there is an affine open neighborhood U of P in V in which $f_1, \dots, f_m$ are all defined, and such that $f_1, \dots, f_m$ generate in $\mathcal{O}(U)$ the ideal of $W' = W \cap U$. It is also well defined the ideal $\mathcal{I}_{V,W,P}$ of $\mathcal{O}_{V,P}$ consisting of all functions $f \in \mathcal{O}_{V,P}$ which are zero on W in a neighborhood of P. Note that if V is an affine variety, then

$$\mathcal{I}_{V,W,P} = \{\frac{u}{v} : u, v \in A(V), u \in \mathcal{I}_a(W)/\mathcal{I}_a(V), v(P) \neq 0\}.$$

Lemma 14.2.14 *In the above setting, given $f_1, \dots, f_m \in \mathcal{O}_{V,P}$, $f_1 = \cdots = f_m = 0$ is a system of equations of W in P if and only if $\mathcal{I}_{V,W,P} = (f_1, \dots, f_m)$.*

Proof Since the question is local, we may and will assume that V is affine.

Suppose $f_1 = \cdots = f_m = 0$ is a system of equations of W in P. Up to shrinking V, we may assume that $f_1, \dots, f_m$ are all defined in V, and they generate the ideal of W in $A(V)$. Then it is clear that $\mathcal{I}_{V,W,P} = (f_1, \dots, f_m)$.

Suppose conversely that $\mathcal{I}_{V,W,P} = (f_1, \dots, f_m)$, with $f_1, \dots, f_m \in \mathcal{O}_{V,P}$, and let $\mathcal{I}_a(W)/\mathcal{I}_a(V) = (g_1, \dots, g_h)$, with $g_1, \dots, g_h$ the classes of elements $G_1, \dots, G_h \in A(V)$. Since $g_i \in \mathcal{I}_{V,W,P}$ for $i = 1, \dots, h$, we have relations of the sort

$$g_i = \sum_{j=1}^{m} a_{ij} f_j, \quad \text{for} \quad i = 1, \dots, h. \tag{14.5}$$

The functions f_j, a_{ij} are all regular in some principal open neighborhood U of P in V, for $i = 1, \dots, h$, $j = 1, \dots, m$. Then (14.5) implies that

$$(g_1, \dots, g_h) = (\mathcal{I}_a(W)/\mathcal{I}_a(V))A(U) \subseteq (f_1, \dots, f_m).$$

Next we will prove that $(\mathcal{I}_a(W)/\mathcal{I}_a(V))A(U)$ coincides with the ideal $\mathcal{I}$ of $W' = W \cap U$ in $A(U)$. Whence it follows that $\mathcal{I} \subseteq (f_1, \dots, f_m)$, but we may assume that $f_i \in \mathcal{I}$, for $i = 1, \dots, m$, and this implies the assertion of the lemma.

We are left to prove that

$$\mathcal{I} = (\mathcal{I}_a(W)/\mathcal{I}_a(V))A(U).$$

The inclusion

$$(\mathcal{I}_a(W)/\mathcal{I}_a(V))A(U) \subseteq \mathcal{I}$$

is clear. Let us prove the opposite inclusion. Suppose $U = U_V(v)$ for some $v \in A(V) \setminus \{0\}$. Then $A(U) = A(V)_v$ (see Exercise 6.3.6) consists of elements of the form $\frac{u}{v^l}$, with $u \in A(V)$ and $l \in \mathbb{N}$. Take $x \in \mathcal{I}$. Then $x = \frac{u}{v^l}$, with $u \in A(V)$, hence $u = xv^l$ and $u \in \mathcal{I}_a(W)/\mathcal{I}_a(V)$. Since $\frac{1}{v^l} \in A(U)$, then $x = \frac{u}{v^l} \in (\mathcal{I}_a(W)/\mathcal{I}_a(V))A(U)$ as wanted. □

We can now prove the:

Theorem 14.2.15 *If V is a quasi-projective variety, $P \in V$ is a simple point for V and W is a subvariety of codimension 1 of V containing P, then W has a unique local equation at P, i.e., $\mathcal{I}_{V,W,P}$ is principal.*

Proof The proof is quite similar to the one of Theorem 11.1.2.

Since the question is local, we may assume that V is affine. Let $f \in \mathcal{O}_{V,P}$ that vanishes on W in a neighborhood of P. Since $\mathcal{O}_{V,P}$ is UFD by Corollary 14.2.4, we can factor f into prime factors. Since W is irreducible, one prime factor g of f has to vanish on W. Let us prove that $g = 0$ is a local equation of W. Replacing V with a smaller open affine neighborhood of P we may assume that g is regular on V.

Since $W \subseteq Z_V(g)$, and since both W and $Z_V(g)$ have codimension 1 in V, we may write $Z_V(g) = W \cup W'$, where either W' is empty or it is different from W and also of codimension 1. Let us assume the latter thing happens. If $P \in W'$, then we can find functions h, h', with h vanishing on W and not on W', h' vanishing on W' and not on W, both vanishing at P, such that hh' vanishes on $Z_V(g)$ but neither one of h and h' vanishes on $Z_V(g)$. Therefore g divides $(hh')^l$ in $A(V)$ for some positive integer l, hence g divides $(hh')^l$ also in $\mathcal{O}_{V,P}$. Since $\mathcal{O}_{V,P}$ is UFD, then either g divides h or it divides h'. Thus either h or h' vanishes on $Z_V(g)$ in a neighborhood of P and, by passing to a smaller neighborhood of P we may assume it vanishes on the whole of $Z_V(g)$. This is a contradiction. So we conclude that either W' is empty or $P \notin W'$. In this latter case, by further reducing the neighborhood of P we may assume that $W' = \emptyset$, so $Z_V(g) = W$. If now h is a function vanishing on W, we have that g divides h^l in $A(V)$ for some positive integer l, and then g divides h^l in $\mathcal{O}_{V,P}$. It follows that g divides h in $\mathcal{O}_{V,P}$. This implies that $\mathcal{I}_{V,W,P} = (g)$, as wanted. □

As a consequence of Theorem 14.2.11 we have the next basic:

Theorem 14.2.16 *Let V be a smooth quasi-projective variety and let $\phi : V \dashrightarrow \mathbb{P}^n$ be a rational map. Then the closed set of points of V where ϕ is not defined has codimension at least 2.*

In particular, if V is a smooth curve, then ϕ is a morphism.

Proof Recall that the set of points Z where ϕ is not defined is a closed subset of V. If Z is empty, there is nothing to prove. Otherwise, fix a point $P \in Z$. Since the question is local, we may assume that V is affine. By applying Proposition 6.2.5, we know that there are rational functions $f_0, \ldots, f_n$ on V such that $\phi(Q) = [f_0(Q), \ldots, f_n(Q)]$ and $f_0, \ldots, f_n$ are all defined and not all zero in all points $Q \in V \setminus Z$. Now $f_0, \ldots, f_n \in \mathbb{Q}(\mathcal{O}_{V,P})$ and, without changing ϕ, we can multiply $f_0, \ldots, f_n$ by a common factor so that $f_0, \ldots, f_n \in \mathcal{O}_{V,P}$. Moreover, again without changing ϕ we may assume that $f_0, \ldots, f_n$ are coprime in $\mathcal{O}_{V,P}$ (remember that $\mathcal{O}_{V,P}$ is UFD since V is smooth). Since $P \in Z$, then $f_0, \ldots, f_n$ all vanish at P. So the local equations of Z in a neighborhood of P are $f_0 = \ldots = f_n = 0$. Now we claim that no subvariety W of V of codimension 1 can be contained in the locus where $f_0 = \ldots = f_n = 0$. Indeed, if W has codimension 1, then by Theorem 14.2.11 we have that W is defined by an equation $g = 0$ in a neighborhood of P, and therefore $f_0, \ldots, f_n$ would all be divisible by g, a contradiction. □

Corollary 14.2.17 *If two smooth projective curves are birationally equivalent they are isomorphic.*

Proof Let C, C' be two smooth projective curves and let $\phi : C \dashrightarrow C'$ be a birational map. By Theorem 14.2.15, both ϕ and ϕ^{-1} are morphisms, and clearly one is inverse of the other, so they are isomorphisms. □

14.3 Smooth Curves and Finite Maps

In this section we prove an important result which relates smoothness and finiteness of maps in the case of curves. We start with the following:

Theorem 14.3.1 *Let V, W be varieties and $f : V \to W$ a birational morphism. Given a point $P \in V$ assume that $Q = f(P)$ is smooth for W and that the inverse map $g = f^{-1}$ is not defined at Q. Then there is a subvariety $Z \subset V$ of codimension 1 in V, containing P, such that $f(Z)$ has codimension at least 2 at Q on W.*

Proof We may replace V with an affine neighborhood of P, so we may suppose that V is affine. So assume that $V \subseteq \mathbb{A}^n$, with coordinates $(x_1, \ldots, x_n)$. There are rational functions $g_1, \ldots, g_n \in K(W)$, such that $g = f^{-1}$ is given by the equations $x_i = g_i$, for $i = 1, \ldots, n$. Since g is not defined at Q, at least one of the functions $g_1, \ldots, g_n$ is not defined at Q. We may suppose that g_1 is not defined at Q. So we may write $g_1 = \frac{u}{v}$ with $u, v \in \mathcal{O}_{W,Q}$ and $v(Q) = 0$. Since W is smooth at Q, the ring $\mathcal{O}_{W,Q}$ is a UFD, so we may assume that u, v have no common factor.

The map f induces an isomorphism $f^* : K(W) \to K(V)$, and we have

$$x_1 = f^*(g_1) = f^*\left(\frac{u}{v}\right) = \frac{f^*(u)}{f^*(v)}$$

therefore we have

$$f^*(v)x_1 = f^*(u). \tag{14.6}$$

Moreover $f^*(v)(P) = v(Q) = 0$. Set $Z = Z_V(f^*(v))$. Then $P \in Z$, so Z is non-empty. By Exercise 11.2.24, Z has codimension 1 in V. From (14.6) we see that $f^*(u)$ is zero on Z. So we have that u, v are both zero on $f(Z)$.

We are left to prove that $f(Z)$ has codimension at least 2 at Q. Suppose by contradiction this is not the case. Then $f(Z)$ has a component Z' of codimension 1 at Q. Then by Theorem 14.2.16, Z' has a local equation $h = 0$ at Q. But then, since u and v vanish on Z', then h divides both u and v, contrary to the assumption that u and v have no common factor. □

Corollary 14.3.2 *Let V, W be two curves with W smooth and let $f : V \to W$ be a birational morphism. Then $f(V)$ is open in W and f induces an isomorphism from V to $f(V)$.*

Proof Since $f : V \to W$ is a birational morphism, there are two non-empty open subsets $U \subseteq V$, $U' \subseteq W$, such that f induces an isomorphism between U and U' (see Corollary 7.1.3). Since $U' = f(U)$ is obtained by subtracting finitely many points from W, the same is true for $f(V)$ hence $f(V)$ is open in W. Now look at the morphism $f : V \to f(V)$. If this were not an isomorphism, we would get a contradiction by Theorem 14.3.1, since there would be some point in V mapped by f to the empty set. □

Theorem 14.3.3 *Let V be a smooth, irreducible, projective curve and let $f : V \to W$ be a surjective morphism with W a curve. Then f is finite.*

Proof Take any non-empty affine subset U in W. Set $B = A(U)$ and note that $\mathbb{Q}(B) = K(W)$. The morphism f induces an inclusion $f^* : K(W) \to K(V)$. So we may view B as a subring of $K(V)$. Let A be the integral closure of B in $K(V)$. We claim that $\mathbb{Q}(A) = K(V)$. The inclusion $\mathbb{Q}(A) \subseteq K(V)$ is obvious. Let us prove the opposite inclusion. Take $f \in K(V)$. Since the field extension $f^* : K(W) \to K(V)$ is algebraic, we have a relation of the form

$$f^n + g_1 f^{n-1} + \cdots + g_n = 0, \quad \text{with} \quad g_1, \ldots, g_n \in K(W) = \mathbb{Q}(B). \tag{14.7}$$

We can write $g_i = \frac{b_i}{a_i}$, with $a_i, b_i \in B$, and $a_i \neq 0$, for $i = 1, \ldots, n$. Set $a = a_1 \cdots a_n \neq 0$. If we multiply both sides of (14.7) by a^n, we have a relation of the sort

$$(af)^n + c_1(af)^{n-1} + \cdots + c_n = 0, \quad \text{with} \quad c_1, \ldots, c_n \in B,$$

hence $af \in A$, and therefore $f \in \mathbb{Q}(A)$, as wanted.

By Theorem 10.3.4, A is a finitely generated B-module. So we may set $A = A(Z)$, with Z a suitable affine normal (hence smooth) curve. Since $K(Z) = \mathbb{Q}(A) = K(V)$, then Z is birational to V, we have a birational map $g : Z \dashrightarrow V$, which is a morphism because Z is smooth. Then by Corollary 14.3.2, we may assume that g is an isomorphism of Z onto $g(Z)$ which is an open subset of V. We will identify Z with $g(Z)$ via g. The inclusion $B \subseteq A$ tells us that f induces a morphism $f_{|V} : Z \to U$, which is finite. We claim that $Z = f^{-1}(U)$. This will prove the finiteness assertion.

To prove this last claim, we argue by contradiction and assume there is a point $P \in U$ for which there is a point $Q \notin Z$ such that $f(Q) = P$. Take a rational function $g \notin \mathcal{O}_{V,Q}$, but g is regular and vanishes at all (finitely many) points $Q' \in Z$ such that $g(Q') = P$ (the existence of this function is easily proved and we leave it as an exercise for the reader). Suppose g has poles at $Q_1, \ldots Q_m \in Z$. Then, by the above assumptions on g, $f(Q_i) := P_i \neq P$, for $i = 1, \ldots, m$. We can find a function $h \in B = A(U)$ such that $h(P) \neq 0$ and $gh \in \mathcal{O}_{Z,Q_i}$, for $i = 1, \ldots, m$. For this, it suffices to take a sufficiently high power of a function in B that does not vanish at P

but vanishes at $P_1, \dots, P_m$ (see Remark 14.2.13). Then $0 \neq y := gh \in A = A(Z)$, so it is integral over B, namely we have a relation of the sort

$$y^n + b_1 y^{n-1} + \cdots + b_n = 0, \quad \text{with} \quad b_1, \dots, b_n \in B$$

whence

$$y = -b_1 - \frac{b_2}{y} - \cdots - \frac{b_n}{y^{n-1}}. \tag{14.8}$$

Remember that $y \notin \mathcal{O}_{V,Q}$ and therefore $y^{-1} \in \mathfrak{m}_{V,Q}$. But then from (14.8) we get a contradiction, because the right hand side member belongs to $\mathcal{O}_{V,Q}$, but the left hand side does not. □

14.4 A Criterion for a Map to Be an Isomorphism

In this section we will prove a useful criterion for a finite morphism to be an isomorphism.

Theorem 14.4.1 *Let $f : V \to W$ be a finite morphism between two varieties. Then f is an isomorphism if an only if f is bijective between V and W and the differential of f (see Exercise 14.1.22) is injective at each point of V.*

Proof Set $g = f^{-1}$. The assertion will be proved if we prove that g is a morphism. The problem is local in the following sense. Fix a point $Q \in W$ and take the unique point $P \in V$ such that $f(P) = Q$. Choose affine neighborhoods U of P and U' of Q such that $f(U) = U'$ and $A(U)$ is integral over $A(U')$. It suffices to prove that, for suitable choices of U and U', $f : U \to U'$ is an isomorphism, because g is then a morphism at Q.

We have the injective map $f^* : \mathcal{O}_{W,Q} \to \mathcal{O}_{V,P}$. The hypothesis on the differential of f is equivalent to say that the induced map

$$f^* : \mathfrak{m}_Q/\mathfrak{m}_Q^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$$

is surjective. Suppose $\mathfrak{m}_Q = (u_1, \dots, u_k)$. Then $f^*(u_i) + \mathfrak{m}_P^2$, for $i = 1, \dots, k$, generate $\mathfrak{m}_P/\mathfrak{m}_P^2$. By applying Nakayama's Lemma 14.2.1 to $\mathfrak{m}_P$ as an $\mathcal{O}_{V,P}$-module, we have that $\mathfrak{m}_P = (f^*(u_1), \dots, f^*(u_k))$, namely

$$\mathfrak{m}_P = f^*(\mathfrak{m}_Q)\mathcal{O}_{V,P}. \tag{14.9}$$

Next we claim that $\mathcal{O}_{V,P}$ is a finite module over $\mathcal{O}_{W,Q}$. Since $A(U)$ is a finite module over $A(U')$, and since every element of $\mathcal{O}_{V,P}$ is of the form $\frac{u}{v}$ with $u, v \in A(U)$ and $v \notin \mathfrak{m}_P$, to prove the claim it suffices to show that every element of $\mathcal{O}_{V,P}$ is of the form $\frac{h}{f^*(z)}$, with $h \in A(U)$ and $z \in A(U')$, with $z \notin \mathfrak{m}_Q$. To prove this it suffices to prove the following other claim: for every $v \in A(U)$, with $v \notin \mathfrak{m}_P$, there is a $z \in A(U')$, with $z \notin \mathfrak{m}_Q$ and a $w \in A(U)$, such that $f^*(z) = vw$. In fact, if this is the case, then

$$\frac{u}{v} = \frac{uw}{f^*(z)} = \frac{h}{f^*(z)}, \quad \text{with} \quad h = uw.$$

To prove this final claim, note that, since f is finite, hence closed by Theorem 10.1.3, we have that $Z = f(Z_U(v))$ is closed in U' and since f is bijective, then $Q \notin Z$. So there is $t \in A(U')$, such that $t = 0$ on Z and $t(Q) \neq 0$. Then $f^*(t) = 0$ on $Z_U(v)$ and $f^*(t)(P) \neq 0$. By the Nullstellensatz, there is a positive integer n such that $f^*(t)^n = vw$ for some $w \in A(U)$. We can set $z = t^n$ and we are done with the claim.

Now we can apply Nakayama's Lemma to $\mathcal{O}_{V,P}$ as an $\mathcal{O}_{W,Q}$-module. The (14.9) shows that

$$\mathcal{O}_{V,P}/f^*(\mathfrak{m}_Q)\mathcal{O}_{V,P} \cong \mathcal{O}_{V,P}/\mathfrak{m}_P = \mathbb{K}$$

so it is generated by 1. Then Nakayama's Lemma implies that $\mathcal{O}_{V,P} = f^*(\mathcal{O}_{W,Q})$.

Finally, let $a_1, \ldots, a_l$ be a basis of $A(U)$ as a $A(U')$-module. We have $a_i \in \mathcal{O}_{V,P} = f^*(\mathcal{O}_{W,Q})$, for $i = 1, \ldots, l$. Then we can find a function $b \in A(U')$ such that $U'' = U' \setminus Z_V(b)$ is an open affine neighborhood of Q, $a_1, \ldots, a_l$ are in $f^*(A(U''))$ and $(f^*)^{-1}(a_i)$ are regular in $U^* = U \setminus Z_U(f^*(b))$, for $i = 1, \ldots, l$. Then it is clear that $A(U^*)$ is generated by $a_1, \ldots, a_l$ over $f^*(A(U''))$. On the other hand, $a_1, \ldots, a_l$ are in $f^*(A(U''))$, so $A(U^*) \cong A(U'')$, so $f : U^* \to U''$ is an isomorphism, as wanted. □

Exercise 14.4.2 *Let $V \subseteq \mathbb{P}^r$ be a projective variety and let P be a point of $\mathbb{P}^r$ such that any line through P intersects V in at most one point and that P does not sit on any (projective) tangent space to V. Prove that the projection of V from P to $\mathbb{P}^{r-1}$ is an isomorphism of V onto its image.

Exercise 14.4.3 *Let V be a smooth projective variety of dimension n. Prove that there is a morphism $f : V \to \mathbb{P}^{2n+1}$ which is an isomorphism of V onto its image.

Exercise 14.4.4 Find zero and poles, with their orders, of the rational functions

$$\frac{x^2 - 2x + 1}{x - 3}, \quad x^3 - x$$

on $\mathbb{P}^1$ on the field of complex numbers.

14.5 Solutions of Some Exercises

14.1.5 We prove the assertion for the blow-up of $\mathbb{P}^2$ at $P = [1, 0, 0]$, the proof in general being similar. Fix coordinates $[x_0, x_1, x_2]$ in $\mathbb{P}^2$ and $[y_1, y_2]$ in $\mathbb{P}^1$, and recall that the blow-up $\tilde{\mathbb{P}}^2$ of $\mathbb{P}^2$ at $P = [1, 0, 0]$ sits in $\mathbb{P}^2 \times \mathbb{P}^1$ and it is defined there by the equation

$$x_1 y_2 = x_2 y_1.$$

The projection to the first factor $\sigma : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ is an isomorphism between $\tilde{\mathbb{P}}^2 \setminus E$ and $\mathbb{P}^2 \setminus \{P\}$, where $E = \sigma^{-1}(P) \cong \mathbb{P}^1$ is the exceptional locus of the blow-up. By the smoothness of $\mathbb{P}^2$, we have that $\tilde{\mathbb{P}}^2$ is smooth at all points not on E. It remains to prove that the points of E are also smooth for $\tilde{\mathbb{P}}^2$. Let us consider the affine open subset $U_1 \cong \mathbb{A}^1$ of $\mathbb{P}^1$ where $y_1 \neq 0$. So we may assume that $y_1 = 1$. Since we are on $E = \sigma^{-1}(P)$ we may also work in the open subset $U \cong \mathbb{A}^2$ of $\mathbb{P}^2$ where $x_0 = 1$. So in the open subset $U \times U_1 \cong \mathbb{A}^3$ the blow-up has equation

$$f(x_1, x_2, y_2) = x_2 - x_1 y_2 = 0.$$

Then we have to compute the rank of the matrix

$$\Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial y_2}\Big) = (-y_2, 1, -x_1)$$

which is always 1. This implies that all points of E in this open set are smooth for $\tilde{\mathbb{P}}^2$. A similar computation works in the open subset $U_2 \cong \mathbb{A}^1$ of $\mathbb{P}^1$ where $y_2 \neq 0$.

14.1.10 Suppose that H has degree d and equation $f(x_0, \dots, x_n) = 0$ (with f irreducible) and $P = [p_0, \dots, p_n]$. Take a point $Q = [q_0, \dots, q_n]$ different from P and consider the line $r = \langle P, Q\rangle$, which has parametric equations

$$x_i = \lambda p_i + \mu q_i, \quad i = 0, \dots, n,$$

with $[\lambda, \mu] \in \mathbb{P}^1$. The the intersection of r with H is governed by the equation

$$f(\lambda p_0 + \mu q_0, \dots, \lambda p_n + \mu q_n) = 0 \tag{14.10}$$

in λ, μ. We can expand the polynomial in (14.10) in Taylor series with initial point $(\lambda p_0, \dots, \lambda p_n)$ and increments $(\mu q_0, \dots, \mu q_n)$. One gets

$$f(\lambda p_0 + \mu q_0, \dots, \lambda p_n + \mu q_n) = \lambda^d f(P) + \dots + \frac{\lambda^{d-i}\mu^i}{i!}\Delta^i_Q f(P) + \dots + \mu^d f(Q)$$

where

$$\Delta^i_Q f(P) := \Big(q_0\frac{\partial}{\partial x_0} + \dots + q_n\frac{\partial}{\partial x_n}\Big)^{(i)} f(p_0, \dots, p_n)$$

where (i) stays, as usual, for the *symbolic power*.

The intersection multiplicity of r with H at P is at least k if and only if

$$f(P) = \Delta^1_Q f(P) = \ldots = \Delta^{k-1}_Q f(P) = 0.$$

So P has multiplicity m if and only if $\Delta^1_Q f(P) = \ldots = \Delta^{m-1}_Q f(P) = 0$ for all Q, whereas there is some point Q such that $\Delta^m_Q f(P) \neq 0$, and the union of the lines having intersection multiplicity larger than m with H at P is the set of all points Q such that $\Delta^m_Q f(P) = 0$, and this is an equation in the variables $q_0, \dots, q_n$. Note that $\Delta^m_Q f(P)$ has degree m in $q_0, \dots, q_n$. Moreover the divisor with equation $\Delta^m_Q f(P) = 0$ is a cone with vertex P because it is union of lines passing through P.

14.1.11 With the same notation as in the solution of Exercise 14.1.10, note that P is a point of multiplicity m for H if and only if $\Delta^1_Q f(P) = \ldots = \Delta^{m-1}_Q f(P) = 0$ identically in Q and $\Delta^m_Q f(P)$ is not identically zero in Q. The assertion immediately follows, by taking in account that (14.4) is the same as $\Delta^m_Q f(P) = 0$, by setting $q_i = x_i$, for $i = 0, \dots, n$.

14.1.12 This is proved by passing to affine coordinates in the Exercises 14.1.10 and 14.1.11. The details may be left to the reader.

14.1.15 Suppose that the components of H of multiplicities $m_1, \dots, m_h$ have reduced equations $f_1, \dots, f_h$, and P verifies $f_1(P) = \ldots = f_h(P) = 0$. The equation of H is then of the form $f =$

$f_1^{m_1} \cdots f_h^{m_h} g$, with g not divisible by $f_1, \ldots, f_h$. If we compute the derivatives of order i of f at P, we see that they vanish if $i \leqslant m - 1$, which proves the assertion.

14.1.16 The first assertion is trivial. As for the second assertion, if H is a cone of degree d with vertex P, then it is clear that any line through P either belongs to H or it intersects H only at P, so it has intersection multiplicity d with H at P, so that P has multiplicity d for H and $H = TC_{H,P}$.

Conversely, if H has degree d and it has a point P of multiplicity d, then any line containing P and a point $Q \in H \setminus \{P\}$ is contained in H, so H is a cone with vertex in P. It is then clear that $H = TC_{H,P}$.

14.1.19 The open U in question is the complement of the discriminant $\mathcal{D}(n, d)$, see Exercises 11.4.12 and 11.4.13.

14.1.20 Consider a Segre variety $\mathrm{Seg}_{n,m}$ (the argument is similar for the Segre varieties with more indices). By Exercise 8.1.6, there is a group of projectivities of $\mathbb{P}^{nm+n+m}$ which acts transitively on $\mathrm{Seg}_{n,m}$. This proves that all points of $\mathrm{Seg}_{n,m}$ have isomorphic local ring. This proves the assertion.

14.1.21 The proof is similar to the one of Exercise 14.1.20, by taking into account Exercise 13.2.6.

14.1.23 Use a strategy analogous to the one in the solution of Exercise 11.3.8.

14.4.2 Apply Theorem 14.4.1.

14.4.3 We may suppose that $X \subseteq \mathbb{P}^r$. If $r \leqslant 2n + 1$ the assertion is trivially true. Assume $r > 2n + 1$. Since $\mathrm{Sec}(X)$ has dimension at most $2n + 1$ and $\mathrm{Tan}(X)$ has dimension al most $2n$, we can find a point $P \in \mathbb{P}^r$ off $\mathrm{Sec}(X)$ and $\mathrm{Tan}(X)$. Then any line through P intersects X in at most one point and P does not sit on any tangent space to X. So by Exercise 14.4.2 we may project X to $\mathbb{P}^{r-1}$ and the projection is an isomorphism of X onto its image. Then repeat this argument till we prove the assertion.

Chapter 15
Power Series

In this chapter we introduce some algebraic tools which will be essential in what follows for the local study of curves.

15.1 Formal Power Series

A *formal power series* on the field k in the indeterminate x is an expression of the form $\sum_{i=0}^{\infty} a_n x^n$ with $a_n \in k$, for any $n \in \mathbb{N}$. The elements $a_n \in k$, for $n \in \mathbb{N}$, are called the *coefficients* of the series $\sum_{i=0}^{\infty} a_n x^n$. The set of all power series on k is denoted by $k[[x]]$. Of course $k[x] \subset k[[x]]$. Addition and multiplication of power series can be defined so that $k[[x]]$ is a domain. To be precise, one sets

$$\sum_{i=0}^{\infty} a_n x^n + \sum_{i=0}^{\infty} b_n x^n = \sum_{i=0}^{\infty} (a_n + b_n) x^n$$

and

$$\sum_{i=0}^{\infty} a_n x^n \cdot \sum_{i=0}^{\infty} b_n x^n = \sum_{i=0}^{\infty} c_n x^n$$

where

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ &\dots \\ c_n &= a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 \\ &\dots \end{aligned}$$

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
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We use the symbol $f(x)$ (or simply f) to denote a power series $\sum_{i=0}^{\infty} a_n x^n$. We can make sense of $f(0)$ by setting $f(0) = a_0$. In addition, if $f(x) \in k[x]$ and $g(x) \in k[[x]]$, it makes sense to consider $f(g(x)) \in k[[x]]$.

Lemma 15.1.1 *The power series $f(x) = \sum_{i=0}^{\infty} a_n x^n$ is invertible in $k[[x]]$ if and only if $f(0) = a_0 \neq 0$.*

Proof If $a_0 \neq 0$ we inductively define the series $g(x) = \sum_{i=0}^{\infty} b_n x^n$ so that the following relations hold

$$\begin{aligned} &a_0 b_0 = 1 \\ &a_0 b_1 + a_1 b_0 = 0 \\ &a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ &\dots \\ &a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = 0 \\ &\dots \end{aligned}$$

Then one has $fg = 1$. Conversely, if there is a series $g(x) = \sum_{i=0}^{\infty} b_n x^n$ such that $fg = 1$, then $a_0 b_0 = 1$, and $a_0 \neq 0$. □

Lemma 15.1.1 implies that $k[[x]]$ is a local domain with maximal ideal $\mathfrak{m} = (x)$, which contains all series of the form $\sum_{i=1}^{\infty} a_n x^n$.

We will denote by $k((x))$ the quotient field $\mathbb{Q}(k[[x]])$.

Lemma 15.1.2 *Any element of the quotient field $k((x))$ can be written in the form*

$$\frac{f(x)}{x^h},$$

where f is a suitable power series and h a suitable positive integer.

Proof Suppose we have an element of $k((x))$ which is of the form

$$\frac{\sum_{i=0}^{\infty} b_n x^n}{\sum_{i=0}^{\infty} c_n x^n}$$

with $\sum_{i=0}^{\infty} c_n x^n \neq 0$. Let h be the minimum integer such that $c_h \neq 0$. Consider the series $\sum_{i=h}^{\infty} c_n x^{n-h}$. Since $c_h \neq 0$ this series is invertible, with inverse $\sum_{i=0}^{\infty} d_n x^n$. So we have

$$\frac{\sum_{i=0}^{\infty} b_n x^n}{\sum_{i=0}^{\infty} c_n x^n} = \frac{(\sum_{i=0}^{\infty} b_n x^n)(\sum_{i=0}^{\infty} d_n x^n)}{(\sum_{i=0}^{\infty} c_n x^n)(\sum_{i=0}^{\infty} d_n x^n)} = \frac{(\sum_{i=0}^{\infty} b_n x^n)(\sum_{i=0}^{\infty} d_n x^n)}{x^h}$$

as wanted. □

By Lemma 15.1.2, every non–zero element of $k((x))$ can be uniquely written as

$$f = x^h \cdot \sum_{i=0}^{\infty} a_n x^n$$

with $a_0 \neq 0$ and h a suitable integer, which is called the *order* of f and is denoted by $o(f)$. One sets $o(0) = \infty$. The order function

$$o : k((x)) \setminus \{0\} \to \mathbb{Z}$$

is a valuation, and it is clear that $k[[x]]$ is the DVR related to this valuation. As a consequence $k[[x]]$ is a UFD, and factorization in $k[[x]]$ is very easy: a non zero f divides g if and only if $o(f) \leqslant o(g)$.

15.2 Congruences, Substitution and Derivatives

15.2.1 Conguences

In the ring $k[[x]]$ we can consider the equivalence relation determined by the ideal $\mathfrak{m}^n$, with n a positive integer. This is called *congruence* modulo x^n. One has that f and g are congruent modulo x^n, and one writes $f \equiv g \mod x^n$, if $f - g \in \mathfrak{m}^n$, or, which is the same, if $f - g$ is divisible by x^n, or if $o(f - g) = n$, or also if the first n coefficients of f and g coincide.

Lemma 15.2.1 *One has:*

(a) let f_1, f_2 and g_1, g_2 be power series such that $f_1 \equiv f_2, g_1 \equiv g_2 \mod x^n$, then

$$f_1 \pm f_2 \equiv g_1 \pm g_2 \mod x^n, \quad f_1 f_2 \equiv g_1 g_2 \mod x^n;$$

(b) let f, g be power series such that $f \equiv g \mod x^n$ for $n \gg 0$, then $f = g$;
(c) let $f_1(x), f_2(x)$ be two polynomials and $g_1(x), g_2(x)$ two power series such that $o(g_1), o(g_2) > 0$. If $f_1 \equiv f_2, g_1 \equiv g_2 \mod x^n$, then $f_1(g_1(x)) \equiv f_2(g_2(x)) \mod x^n$;
(d) if we have a sequence $(f_n)_{n \in \mathbb{N}\setminus\{0\}}$ of power series such that

$$f_{n+1} \equiv f_n \mod x^n \quad \textit{for all} \quad n \in \mathbb{N} \setminus \{0\},$$

then there is a unique power series f such that

$$f \equiv f_n \mod x^n \quad \textit{for all} \quad n \in \mathbb{N} \setminus \{0\}.$$

Proof Parts (a) and (b) are easy and left to the reader.

Proof of (c). Write $f_1 = f_2 + x^n f_3$, with f_3 a polynomial. By applying (a), we have

$$f_1(g_1(x)) \equiv f_2(g_2(x)) + g_2(x)^n f_3(g_2(x)), \quad \mod x^n.$$

Since

$$o(g_2(x)^n f_3(g_2(x))) = no(g_2) + o(f_3(g_2(x))) \geqslant n$$

the assertion follows.

Proof of (d). One has

$$\begin{aligned}
f_1 &= a_{10} + a_{11}x + a_{12}x^2 + \cdots \\
f_2 &= a_{10} + a_{21}x + a_{22}x^2 + \cdots \\
f_3 &= a_{10} + a_{21}x + a_{32}x^2 + \cdots \\
&\cdots \\
f_n &= a_{10} + a_{21}x + a_{32}x^2 + \cdots + a_{nn}x^n + \cdots \\
&\cdots
\end{aligned}$$

So we set

$$f = a_{10} + a_{21}x + a_{32}x^2 + \cdots + a_{n+1,n}x^n + \cdots$$

Then $f \equiv f_n \mod x^n$ for all $n \in \mathbb{N}$. If g is another series such that $g \equiv f_n \mod x^n$ for all $n \in \mathbb{N}$, then $g \equiv f \mod x^n$ for all $n \in \mathbb{N}$ and then $f = g$ by part (b). □

15.2.2 Substitution

Let f, g be formal power series in $k[[x]]$, with $o(g) > 0$. For all positive integers n, we will denote by f_n and g_n the *truncations* of f and g at the n–th term, i.e., f_n and g_n are the polynomials of degree at most $n - 1$ obtained by taking the sum of the first n terms of f and g. Then, for all positive integers n we have

$$f_{n+1} \equiv f_n, \quad g_{n+1} \equiv g_n \mod x^n$$

hence, by Lemma 15.2.1, (c), we have

$$f_{n+1}(g_{n+1}) \equiv f_n(g_n) \mod x^n.$$

By Lemma 15.2.1, (d), there is a unique power series $h \in k[[x]]$ such that

$$h \equiv f_n(g_n) \mod x^n$$

for all $n \in \mathbb{N}$. The series h is said to be obtained by *substitution* of g in f, and it is denoted by $h = f(g)$.

Lemma 15.2.2 *Let $g, h \in k[[x]]$ be such that $o(g) > 0$ and $o(h) > 0$. Then for any $f \in k[[x]]$, substituting h in $f(g)$ is the same as substituting $g(h)$ in f.*

Proof Set $m = f(g)$ and $l = g(h)$. Let f_n, g_n, h_n be the truncations of f, g, h at the n–th term, and set $m_n = f_n(g_n)$ and $l_n = g_n(h_n)$. For all $n \in \mathbb{N} \setminus \{0\}$, one has

$$m_n(h_n) = f_n(l_n).$$

Moreover

$$m \equiv m_n, \quad h \equiv h_n, \quad f \equiv f_n, \quad l \equiv l_n \mod x^n$$

for all $n \in \mathbb{N} \setminus \{0\}$, and therefore

$$m_n(h_n) \equiv m(h), \quad f_n(l_n) \equiv f(l) \mod x^n,$$

whence

$$m(h) \equiv f(l) \mod x^n.$$

Since this is true for all $n \in \mathbb{N} \setminus \{0\}$, one has $m(h) = f(l)$ (Lemma 15.2.1, (b)), as wanted. □

Lemma 15.2.3 *Let $g, f \in k[[x]]$ be such that $o(g) = 1$ and set $h = f(g)$. There exists an $l \in k[[x]]$ with $o(l) = 1$ such that $f = h(l)$.*

Proof Suppose

$$g = a_1 x + a_2 x^2 + \cdots, \quad \text{with} \quad a_1 \neq 0.$$

We search for a

$$l = b_1 x + b_2 x^2 + \cdots, \quad \text{with} \quad b_1 \neq 0$$

such that $g(l) = 1$. We have

$$\begin{aligned} g(l) = {} & a_1 b_1 x + (a_1 b_2 + a_2 b_1^2)x + (a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3)x^3 + \cdots \\ & + (a_1 b_n + F_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}))x^n + \cdots \end{aligned}$$

where F_n is a suitable polynomial in $a_2, \ldots, a_n, b_1, \ldots, b_{n-1}$. The required series l is obtained by defining $b_1 = a_1^{-1}$ and b_n recursively by the formula

$$b_n = -a_1^{-1} F_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}).$$

The conclusion follows by Lemma 15.2.2. □

15.2.3 Derivatives

Given a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we define the *derivative* of f as

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Lemma 15.2.4 *Let f, g be formal power series. One has*

$$(f+g)' = f' + g', \quad (fg)' = fg' + f'g.$$

Proof The linearity of derivative with respect to the sum is obvious. Let us prove the assertion for the product (i.e., the derivative of the product verifies *Leibnitz rule*). Note first that if $f \equiv g \mod x^n$, then $f' \equiv g' \mod x^{n-1}$.

With the usual notation, we have

$$f \equiv f_n, \quad g \equiv g_n \mod x^n$$

for all $n \in \mathbb{N} \setminus \{0\}$. Then $fg \equiv f_n g_n \mod x^n$ for all $n \in \mathbb{N} \setminus \{0, 1\}$ and therefore

$$\begin{aligned} &(fg)' \equiv (f_n g_n)' \mod x^{n-1} = \\ &= f_n g_n' + f_n' g_n \equiv fg' + f'g \mod x^{n-1} \end{aligned}$$

and since this is true for all $n \in \mathbb{N} \setminus \{0, 1\}$, the assertion follows. □

Exercise 15.2.5 Suppose that

$$f = a_0 + a_1 x + a_2 x^2 + \cdots, \quad g = b_1 x + b_2 x^2 + \cdots.$$

Prove that

$$\begin{aligned} f(g) &= a_0 + a_1 g + a_2 g^2 + \cdots = \\ &= a_0 + a_1 b_1 x + (a_1 b_2 + a_2 b_1^2) x_2 + \\ &+ (a_1 b_3 + 2 a_1 b_1 b_2 + a_3 b_1^3) x^3 + \cdots \end{aligned}$$

Exercise 15.2.6 Let $g \in k[[x]]$ be such that $o(g) > 0$ and let $f \in k[[x]]$ be non–zero. Prove that $o(f(g)) = o(f) \cdot o(g)$. In particular, if $o(g) = 1$, then $o(f(g)) = o(f)$.

Exercise 15.2.7 Let $g \in k[[x]]$ be such that $o(g) > 0$. Prove that the map

$$f \in k[[x]] \to f(g) \in k[[x]]$$

is a homomorphism. Prove in addition that if $o(g) = 1$ the above map is an automorphism of $k[[x]]$ which is order preserving. Prove that conversely any order preserving automorphism of $k[[x]]$ is of this kind.

15.3 Fractional Power Series

Let us introduce the symbol $x^{\frac{1}{n}}$, with x an indeterminate over the field k, and $n \in \mathbb{N}$ non–zero. We use the following conventions

$$x^0 = 1, \quad x^{\frac{1}{1}} = x, \quad (x^{\frac{1}{mn}})^m = x^{\frac{1}{n}}, \quad x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m$$

for every $n, m \in \mathbb{N} \setminus \{0\}$. From this it follows that

$$x^{\frac{rm}{rn}} = x^{\frac{m}{n}}$$

for every $r, n, m \in \mathbb{N}$, with $r, n \neq 0$.

From $(x^{\frac{1}{rn}})^r = x^{\frac{1}{n}}$, it follows that $k((x^{\frac{1}{n}})) \subseteq k((x^{\frac{1}{rn}}))$. So we can consider the union $k\{x\}$ of all the fields $k((x^{\frac{1}{n}}))$, for $n \in \mathbb{N}$. If $\xi, \eta \in k\{x\}$, there are $n, m \in \mathbb{N}$ such that $\xi \in k((x^{\frac{1}{n}}))$ and $\eta \in k((x^{\frac{1}{m}}))$. Hence $\xi, \eta \in k((x^{\frac{1}{nm}}))$, and so their sum, product and quotient (if $\eta \neq 0$), also belong to $k((x^{\frac{1}{nm}}))$ and therefore to $k\{x\}$. Thus $k\{x\}$ is in a natural way a field.

If

$$\xi = a_1 x^{\frac{m_1}{n_1}} + a_2 x^{\frac{m_2}{n_2}} + \cdots \in k\{x\}$$

with

$$\frac{m_1}{n_1} < \frac{m_2}{n_2} < \cdots, \quad \text{and} \quad a_1 \neq 0,$$

we define the *order* $o(\xi)$ of ξ to be $o(\xi) = \frac{m_1}{n_1}$. The set of elements with non–negative order of $k\{x\}$ is a domain, which we denote by $k\langle x\rangle$. If $\xi \in k\langle x\rangle$, the coefficient of $x^0 = 1$ in ξ is denoted by $\xi(0)$.

In order to prove or main result on the field $\mathbb{K}\{x\}$ with $\mathbb{K}$ algebraically closed, which is Theorem 15.3.2 below, we need some preliminaries concerning solutions of equations with coefficients in $k\{x\}$. Start with a polynomial $f(x, y) \in k\{x\}[y]$ which is not constant, i.e., $f(x, y) \notin k\{x\}$. We set

$$f(x, y) = \mathfrak{a}_0 + \mathfrak{a}_1 y + \cdots + \mathfrak{a}_n y^n \tag{15.1}$$

with $\mathfrak{a}_0, \ldots, \mathfrak{a}_n \in k\{x\}$, $n > 0$, $\mathfrak{a}_n \neq 0$. We will set $\alpha_i = o(\mathfrak{a}_i)$, for $i = 0, \ldots, n$. If $\mathfrak{a}_i \neq 0$, we have

$$\mathfrak{a}_i = a_i x^{\alpha_i} + \cdots, \quad \text{and} \quad a_i \neq 0, \tag{15.2}$$

for $i = 0, \ldots, n$.

Suppose that there is $\eta \in k\{x\} \setminus \{0\}$ such that $f(x, \eta) = 0$. Then we can write

$$\eta = b_1 x^{\gamma_1} + b_2 x^{\gamma_1+\gamma_2} + b_3 x^{\gamma_1+\gamma_2+\gamma_3} + \cdots$$

with $b_1, b_2, b_3, \ldots$ (which can be finitely many or not) all non–zero and $\gamma_2, \gamma_3, \ldots$ all positive rational numbers. We set

$$\eta = x^{\gamma}(b + \eta_1)$$

where we put $\gamma = \gamma_1$, $b = b_1$, and

$$\eta_1 = b_2 x^{\gamma_2} + b_3 x^{\gamma_2+\gamma_3} + \cdots .$$

Then we have

$$\begin{aligned}0 = f(x, \eta) &= \mathfrak{a}_0 + \mathfrak{a}_1 x^{\gamma}(b + \eta_1) + \cdots + \mathfrak{a}_n x^{n\gamma}(b + \eta_1)^n = \\ &= \mathfrak{a}_0 + b\mathfrak{a}_1 x^{\gamma} + \cdots + b^n \mathfrak{a}_n x^{n\gamma} + g(x, \eta_1),\end{aligned}$$

where g contains all the terms in which η_1 appears. Since $o(\eta_1) = \gamma_2 > 0$, all terms appearing in g have order greater than the order of some of the terms $b^i \mathfrak{a}_i x^{i\gamma}$, for $i = 0, \ldots, n$. We can summarize the contents of these remarks in the following:

Lemma 15.3.1 *In the above setting we have:*

(a) at least two of the elements $b^i \mathfrak{a}_i x^{i\gamma}$, with $i = 0, \ldots, n$, have the same order, which is less than or equal to the order of any other $b^i \mathfrak{a}_i x^{i\gamma}$. In other terms there are two distinct integers $j, k \in \{0, \ldots, n\}$ such that

$$o(b^j \mathfrak{a}_j x^{j\gamma}) = o(b^k \mathfrak{a}_k x^{k\gamma}) \leqslant o(b^i \mathfrak{a}_i x^{i\gamma}), \quad \textit{for} \quad i = 0, \ldots, n,$$

namely

$$\alpha_j + j\gamma = \alpha_k + k\gamma \leqslant \alpha_i + i\gamma \quad \textit{for} \quad i = 0, \ldots, n; \tag{15.3}$$

(b) the coefficients of all terms of lowest order must cancel out, i.e.,

$$\sum a_h b^h = 0 \tag{15.4}$$

where the sum is made over all indices h such that $\alpha_h + h\gamma = \alpha_j + j\gamma$.

In relation with (15.3), we introduce the so–called *Newton polygonal* of $f(x, y)$. To do so, we fix a system of orthogonal Cartesian coordinates (u, v) in the Euclidean plane and we mark the points P_i with coordinates $u = i$, $v = \alpha_i$, for $i = 0, \ldots, n$, provided $\alpha_i < \infty$. Condition (15.3) says that there is a β such that all points P_i, $i = 0, \ldots, n$, lie above or on the line $r_{\gamma,\beta}$ with equation $v + \gamma u = \beta$ and at least two of the points P_i do lie on $r_{\gamma,\beta}$. Now we join the point P_0 to the point P_n with a convex polygonal, with vertices points in the set $\{P_0, \ldots, P_n\}$ in such a way that no point P_i lies below the polygonal. This is the Newton polygonal of $f(x, \eta)$. Then the only possible lines $r_{\gamma,\beta}$, hence the only values of γ, are the ones determined by the segments of the polygonal and by their slopes. Once one of the above segments has been chosen, so that γ has been determined by its slope, then b has to satisfy (15.4), where the sum is extended to the indices h of points P_h lying on the segment in question.

In this way we have given necessary conditions for b_1 and γ_1 appearing in η. To proceed to finding analogous conditions for b_2 and γ_2, one defines

$$f_1(x, y_1) = x^{-\beta} f(x, x^{\gamma_1}(b_1 + y_1))$$

and consider the root η_1 of $f_1(x, y_1) = 0$ in the variable y_1. Note that since $\gamma_2 > 0$, only segments of the Newton polygonal can be considered. Then one repeats the above considerations for $f_1(x, \eta_1) = 0$, and finds necessary conditions for b_2 and γ_2. This can be continued to give necessary conditions for b_n and γ_n for all $n \in \mathbb{N}$.

From now on we assume $k = \mathbb{K}$ to be algebraically closed and we now prove the main result about $\mathbb{K}\{x\}$:

Theorem 15.3.2 $\mathbb{K}\{x\}$ *is an algebraically closed field.*

Proof We keep all notation we introduced above. The proof consists in showing that the process we indicated above can be carried out on any polynomial of the form (15.1), to construct a root of the equation $f(x, y) = 0$, where $f(x, y)$ is given by (15.1). In order to see this, we have three basic facts to verify:

(a) at each step the Eq. (15.4) has a non–zero solution;
(b) after the first step, the Newton polygonal has some segment with negative slope;
(c) after a certain step all the γ_is have a common denominator.

Property (a) is easy to verify. Indeed, since the segments of the Newton polygonal contain at least two points P_i, the left hand side of (15.4) has at least two non–zero terms, hence (15.4) has certainly a non–zero solution (remember that $\mathbb{K}$ is an algebraically closed field).

As for property (b), we need to make a deeper analysis of the Newton polygonal. First of all for all $i = 0, \ldots, n$, there is a positive integer n_i such that $\mathfrak{a}_i \in \mathbb{K}\{x^{\frac{1}{n_i}}\}$. Then we can find a positive integer m such that $\mathfrak{a}_i \in \mathbb{K}\{x^{\frac{1}{m}}\}$, for all $i = 0, \ldots, n$. So we have

$$\alpha_i = \frac{m_i}{m}, \quad \text{for all} \quad i = 0, \ldots, n.$$

Let now P_j and P_k, with $j < k$, be the left and right end of the segment Σ of slope γ_1 lying on the line r_{γ_1, β_1} with equation $v + \gamma_1 u = \beta_1$, of the Newton polygonal we have chosen. Then we have

$$\alpha_j + j\gamma_1 = \alpha_k + k\gamma_1,$$

which implies

$$\gamma_1 = \frac{\alpha_j - \alpha_k}{k - j} = \frac{m_j - m_k}{m(k - j)}$$

and so we can write

$$\gamma_1 = \frac{p}{mq}$$

with p, q coprime and $q > 0$. If P_h is on the segment Σ, one has

$$\frac{p}{mq} = \gamma_1 = \frac{\alpha_j - \alpha_h}{h - j} = \frac{m_j - m_h}{m(h - j)}$$

and therefore

$$q(m_j - m_h) = p(h - j).$$

Since p, q are coprime, we have that q divides $h - j$ and therefore $h = j + tq$, where t is a non–negative integer. Thus (15.4), which is an equation in b, has the form

$$b^j \phi(b^q) = 0$$

where $\phi(z)$ is a polynomial such that $\phi(0) \neq 0$. Since the degree of the polynomial appearing in (15.4) is k, then the polynomial $\phi(z)$ has degree $\frac{k-j}{q} \leqslant k - j$.

Suppose now $b_1 \neq 0$ is a root of $\phi(z^q) = 0$, with multiplicity $r \geqslant 1$, so that

$$\phi(z^q) = (z - b_1)^r \psi(z), \quad \text{with} \quad \psi(b_1) \neq 0.$$

Note that

$$r \leqslant \frac{k - j}{q} \leqslant k - j. \tag{15.5}$$

Now, as we indicated above, we consider the polynomial

$$\begin{aligned} f_1(x, y_1) &= x^{-\beta_1} f(x, x^{\gamma_1}(b_1 + y_1)) = \\ &= x^{-\beta_1}[\mathfrak{a}_0 + \mathfrak{a}_1 x^{\gamma_1}(b_1 + y_1) + \cdots + \mathfrak{a}_n x n \gamma_1 (b_1 + y_1)^n] = \\ &= x^{-\beta_1} \sum \mathfrak{a}_h x^{h\gamma_1}(b_1 + y_1)^h + x^{-\beta_1} \sum \mathfrak{a}_l x^{l\gamma_1}(b_1 + y_1)^l \end{aligned}$$

where the first sum runs over the indices h of points P_h lying on the segment Σ, the second sum on the indices l of the remaining points. Recalling the expressions (15.2) of the $\mathfrak{a}_i$, for $i = 0, \ldots, n$, we have

$$\begin{aligned} f_1(x, y_1) = x^{-\beta_1} &\sum a_h x^{\alpha_h + h\gamma_1}(b_1 + y_1)^h + \\ &+ x^{-\beta_1} \sum (\mathfrak{a}_h - a_h) x^{h\gamma_1}(b_1 + y_1)^h + x^{-\beta_1} \sum \mathfrak{a}_l x^{l\gamma_1}(b_1 + y_1)^l. \end{aligned}$$

Since for the indices h of points lying on Σ we have $\alpha_h + h\gamma_1 = \beta_1$, the first summand in the above expression of $f_1(x, y_1)$ coincides with

$$\sum a_h (b_1 + y_1)^h = (b_1 + y_1)^j \phi((b_1 + y_1)^q) = y_1^r (b_1 + y_1)^j \psi(b_1 + y_1).$$

Now notice that $o(\mathfrak{a}_h - a_h) > \alpha_h$ and $o(\mathfrak{a}_l x^{l\gamma_1}) = \alpha_l + l\gamma_1 > \beta_1$ for the indices l in the second summation. So we can write

$$f_1(x, y_1) = c_1 y_1^r + c_2 y_1^{r+1} + \cdots + g(x, y_1)$$

where $c_1, c_2, \ldots$ are constants, with $c_1 = b_1^j \psi(b_1) \neq 0$, and we collected in $g(x, y_1)$ all terms in which each power of y_1 has a coefficient of positive order. In conclusion we can write

$$f_1(x, y_1) = \mathfrak{c}_0 + \mathfrak{c}_1 y_1 + \cdots + \mathfrak{c}_n y_1^n,$$

with $\mathfrak{c}_0, \ldots, \mathfrak{c}_n \in \mathbb{K}\{x\}$, and

$$\begin{aligned} &o(\mathfrak{c}_i) \geqslant 0, \quad \text{for} \quad i = 0, \ldots, n \\ &o(\mathfrak{c}_i) > 0, \quad \text{for} \quad i = 0, \ldots, r-1 \\ &o(\mathfrak{c}_r) = 0. \end{aligned}$$

Now suppose first that $\mathfrak{c}_0 = 0$. In this case the equation $f_1(x, y_1) = 0$ has the root $\eta_1 = 0$ and this implies that $\eta = b_1 x_1^\gamma$ is a solution of $f(x, y) = 0$. If instead $\mathfrak{c}_0 \neq 0$, then for the Newton polygonal of $f_1(x, y_1)$ the point P_0 has u coordinate equal to zero and positive v coordinate, whereas P_r has positive u coordinate and the v coordinate is 0. Then since in the next step we have to take positive values of γ_2, this is actually obtained from a segment in the arch which goes from P_0 to P_r in the Newton polygonal of $f_1(x, y_1)$. This proves (b) in making the second step. The proof for the following steps is the same.

Finally we have to prove (c). This will be done if we prove that, with the above notation, after a certain number of steps the value of q is constantly equal to 1. To see this proceed as follows. Recall first (15.5). Moreover, as we saw, the horizontal length of the segment of the Newton polygonal to be taken in the next step is at most r. Hence r', the value of r to be taken in the subsequent step, is such that $0 \leqslant r' \leqslant r$. Since r is an integer, after a finite number of steps the value of r stabilizes, so that $r = r_0$ for large number of steps. By taking into account (15.5) this implies that q also stabilizes to the value 1, as needed. ◻

The following corollary is now immediate:

Corollary 15.3.3 *Given the equation $f(x, y) = 0$ where $f(x, y)$ is as in (15.1) with $\mathfrak{a}_0, \ldots, \mathfrak{a}_n \in \mathbb{K}\{x\}$, $n > 0$, $\mathfrak{a}_n \neq 0$, there are distinct elements $\eta_1, \ldots \eta_h \in \mathbb{K}\{x\}$ and positive integers $n_1, \ldots, n_h$ such that*

$$f(x, y) = \mathfrak{a}_n \prod_{i=1}^{h} (y - \eta_i)^{n_i} \tag{15.6}$$

with $n_1 + \cdots + n_h = n$. Such an expression of $f(x, y)$ is uniquely determined.

If $o(\mathfrak{a}_i) \geqslant 0$ for $i = 0, \ldots, n$ and $\mathfrak{a}_n = 1$, then $o(\eta_j) \geqslant 0$ for all $j = 1, \ldots, h$.

Proof The only non–trivial assertion is the last one. Let us prove it. Let η be one of the η_j, with $j = 1, \ldots, h$, and assume that $o(\eta) < 0$. One has

$$f(x, \eta) = \mathfrak{a}_0 + \mathfrak{a}_1 \eta + \cdots + \mathfrak{a}_{n-1}\eta^{n-1} + \eta^n = 0$$

Then $o(f(x, \eta)) = no(\eta) < 0$ which contradicts $f(x, \eta) = 0$. □

Exercise 15.3.4 Prove that if $\xi \in k\langle x \rangle$, then $\xi(0) \neq 0$ if and only if $o(\xi) = 0$.

Exercise 15.3.5 Find the first terms of a solution of the equation

$$f(x, y) = (-x^3 + x^4) - 2x^2 y - xy^2 + 2xy^4 + y^5 = 0$$

in y over the complex field. This exercise is taken form [8, p. 102].

Exercise 15.3.6 Find the first terms of a solution of the equation

$$f(x, y) = x + (1 - x)y + (3x + x^2)y^2.$$

in y over the complex field.

Exercise 15.3.7 * Looking at Corollary 15.3.3, prove that if $o(\mathfrak{a}_n) \leqslant o(\mathfrak{a}_i)$, for $i = 1, \ldots, n-1$, then $\eta_1, \ldots, \eta_h$ belong to $\mathbb{K}\langle x \rangle$.

Exercise 15.3.8 * Continuing Exercise 15.3.7, prove that if $o(\mathfrak{a}_n) = 0$, $o(\mathfrak{a}_0) > 0$ and $o(\mathfrak{a}_i) \geqslant 0$, for $i = 1, \ldots, n-1$, then there is a $j = 1, \ldots, h$ such that $o(\eta_j) > 0$.

Exercise 15.3.9 * Let $f(x, y) \in \mathbb{K}[x, y]$ be such that it has no factor not involving y. Prove that f has a multiple factor in $\mathbb{K}[x, y]$ if and only if $f(x, y) = 0$ has a multiple root in $\mathbb{K}\{x\}$. Prove that if f has a multiple factor of multiplicity n in $\mathbb{K}[x, y]$ depending on y then $f(x, y) = 0$ has a multiple root of multiplicity at least n in $\mathbb{K}\{x\}$.

15.4 Solutions of Some Exercises

15.3.5 To draw the Newton polygonal of $f(x, y)$ we have to plot the points $P_0, \ldots, P_5$ (the degree of f in y is 5). We have $P_0 = (0, 3)$, $P_1 = (1, 2)$, $P_2 = (2, 1)$, P_3 is indeterminate, $P_4 = (4, 1)$, $P_5 = (5, 0)$. So the Newton polygonal consists of two segments, the one joining P_0 and P_2, then contains also P_1, and the one joining P_2 and P_5 which contains no other point. The point P_4 is above the polygonal. For the first segment we have

$$\gamma_1 = 1, \quad \beta_1 = 3, \quad p = q = 1.$$

Equation (15.4) becomes $-1 - 2b - b^2 = 0$, so that it has the only solution $b_1 = -1$ with multiplicity $r = 2$.

Having determined b_1 and γ_1, we have to go to the second step. So we define

$$f_1(x, y_1) = x^{-3} f(x, x(-1 + y_1)) = (x + x^2) - 3x^2 y_1 + (-1 + 2x^2)y_1^2 + 2x^2 y_1^3 - 3x^2 y_1^4 + x^2 y_1^5.$$

The Newton polygonal of $f_1(x, y_1)$ consists of only one segment, the one joining $P_0 = (0, 1)$ with $P_2 = (2, 0)$. So here we have

$$\gamma_2 = \frac{1}{2}, \quad \beta_2 = 1, \quad p = 1, \quad q = 2.$$

Equation (15.4) becomes $1 - b_2^2 = 0$, with solutions $b_2 = \pm 1$, with multiplicity 1 and we may choose $b_2 = 1$.

Then we have to make the next step. We set

$$\begin{aligned} f_2(x, y_2) = x^{-1} f_1(x, x^{\frac{1}{2}}(1+y_2)) = \\ &= (x - 3x^{\frac{3}{2}} + \cdots) + (-2 - 3x^{\frac{3}{2}} + 4x^2 + \cdots)y_2 + \\ &+ (-1 + 2x^2 + \cdots)y_2^2 + \cdots . \end{aligned}$$

Now we are at the point where $r = p = q = 1$, so from now on r stabilizes to the value 1 so that η_2 becomes a power series in $x^{\frac{1}{2}}$. Rather than continuing with the Newton polygonal algorithm, we now set

$$\eta_2 = b_3 x^{\frac{1}{2}} + b_4 x + b_5 x^{\frac{3}{2}} + \cdots$$

and $f_2(x, \eta_2) = 0$, and from this equation we iteratively compute $b_3, b_4, b_5, \ldots$. We find

$$\begin{aligned} f_2(x, \eta_2) &= (x - 3x^{\frac{1}{2}} + \cdots) + (-2 - 3x^{\frac{3}{2}} + 4x^2 + \cdots)(b_3 x^{\frac{1}{2}} + b_4 x + b_5 x^{\frac{3}{2}} + \cdots) + \\ &+ (-1 + 2x^2 + \cdots)(b_3 x^{\frac{1}{2}} + b_4 x + b_5 x^{\frac{3}{2}} + \cdots)^2 + \cdots = \\ &= -2b_3 x^{\frac{1}{2}} + (-2b_4 + 1 - b_3^2)x + (-2b_5 - 3 - 2b_3 b_4)x^{\frac{3}{2}} + \cdots = 0 \end{aligned}$$

whence

$$b_3 = 0, \quad b_4 = \frac{1}{2}, \quad b_5 = -\frac{3}{2}, \quad \ldots$$

Taking into account that we found $b_1 = -1, b_2 = 1$, we have for a root η of $f(x, y) = 0$ the expression

$$\eta = -x + x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{5}{2}} - \frac{1}{3}x^3 + \cdots .$$

15.3.7 Suppose the equation $f(x, y) = 0$ has n distinct roots $\eta_1, \ldots, \eta_n$, the argument is similar otherwise. Then by expanding (15.6), we see that $\mathfrak{a}_{n-1} = -\mathfrak{a}_n(\eta_1 + \cdots + \eta_n)$. If $\mathfrak{a}_{n-1} \neq 0$, this implies that $o(\eta_i) \geqslant 0$ for all $i = 0, \ldots, n$ as wanted. If $\mathfrak{a}_{n-1} = 0$, we have $\eta_1 + \cdots + \eta_n = 0$. On the other hand $\mathfrak{a}_{n-2} = \mathfrak{a}_n \sum_{1 \leqslant i < j \leqslant n} \eta_i \eta_j$. Suppose that $\mathfrak{a}_{n-2} \neq 0$, This implies that $o(\eta_i \eta_j) \geqslant 0$ for all $1 \leqslant i < j \leqslant n$. Then one has $0 \leqslant o(\eta_1(\eta_2 + \cdots + \eta_n)) = o(\eta_1^2)$, which yields $o(\eta_1) \geqslant 0$. Similarly one finds $o(\eta_i) \geqslant 0$ for all $i = 1, \ldots, n$ as wanted. If $\mathfrak{a}_{n-2} = 0$, one repeats this argument.

15.3.8 Again we pretend the equation $f(x, y) = 0$ has n distinct roots $\eta_1, \ldots, \eta_n$, the argument is similar otherwise. By Exercise 15.3.7, $\eta_1, \ldots, \eta_n$ are such that $o(\eta_i) \geqslant 0$ for all $i = 1, \ldots, n$. We have $\mathfrak{a}_0 = \mathfrak{a}_n \eta_1 \cdots \eta_n$. If $o(\eta_i) = 0$ for all $i = 1, \ldots, n$, we would have $o(\mathfrak{a}_0) = 0$, a contradiction.

15.3.9 One implication is trivial. Let us prove the other one. Suppose that $f(x, y) = 0$ has a multiple root in $\mathbb{K}\{x\}$. Then the resultant R of $f(x, y)$ and $\frac{\partial f}{\partial y}$ after elimination of y is non–zero. But the resultant of $f(x, y)$ and $\frac{\partial f}{\partial y}$ is the same independently of the fact that we consider the polynomials in $\mathbb{K}\{x\}[y]$ or in $\mathbb{K}[x, y]$. So this means that $f(x, y)$ and $\frac{\partial f}{\partial y}$ have a non–trivial common factor g in $\mathbb{K}[x, y]$, and g depends on y. We claim this is a multiple factor for f. In fact, suppose the factor in question g is not multiple. Let us assume also that g is irreducible. Then we may write $f = gh$, with h not divisible by g. Then g has to divide also $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} h + \frac{\partial h}{\partial y} g$, hence it has to divide $\frac{\partial g}{\partial y} h$. Since g does not divide h it has to divide $\frac{\partial g}{\partial y}$. Since the degree of $\frac{\partial g}{\partial y}$ in y is smaller than the degree of g in y, this is possible only if $\frac{\partial g}{\partial y} = 0$, i.e., if and only if g does not depend on y, which is a contradiction.

The final assertion is obvious.

Chapter 16
Affine Plane Curves

16.1 Multiple Points and Principal Tangent Lines

Let $X \subset \mathbb{A}^2$ be a effective divisor of degree d, which we will call from now on an *affine plane curve* of degree d. We will use coordinates (x, y) in $\mathbb{A}^2$ hence X has equation of the form $f(x, y) = 0$. Let $P = (a, b)$ be a point of X, consider the expansion of f in Taylor series with initial point P, which is

$$f = f_1 + \cdots + f_d$$

with f_i homogeneous of degree i in $x - a, x - b$, for $i = 1, \ldots, d$. Recall that P is a point of multiplicity m for X if and only if $f_i = 0$ for $i = 1, \ldots, m - 1$, whereas $f_m = 0$ defines the tangent cone to X at P (see Exercise 14.1.12). We will write $m_P(X) = m_P(f) = m$. The homogeneous polynomial $f_m(\xi, \eta)$ of degree m can be written as

$$f_m(\xi, \eta) = \prod_{i=1}^{h} (\xi\eta_i - \eta\xi_i)^{m_i}$$

where (ξ_i, η_i) are the non–zero distinct solutions of the equation $f_m = 0$, up to a proportionality factor, and m_i are positive integers such that

$$m_1 + \cdots + m_h = m.$$

Then the equation $f_m = 0$ of the tangent cone can be written as

$$\prod_{i=1}^{h} ((x - a)\eta_i - (y - b)\xi_i)^{m_i} = 0$$

which defines the h distinct lines $r_1, \ldots, r_h$ containing P with equations

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
https://doi.org/10.1007/978-3-030-71021-7_16

$$(x-a)\eta_i = (y-b)\xi_i, \quad i = 1, \ldots, h.$$

The lines $r_1, \ldots, r_h$ are called the *principal tangent lines* to X at P, and the integers $m_1, \ldots, m_h$ are called the *multiplicities* of these lines. So X has in P just $m_P(X)$ principal tangent lines, provided each tangent line is counted with its multiplicity. Note that if $m = 1$, then P is smooth for X and the unique principal tangent line to X at P is just the tangent line to X at P. The point P is called an *ordinary point* of multiplicity m if X has in P exactly m distinct principal tangent lines, each with multiplicity 1. A *double point* P for X, i.e., $m_P(X) = 2$, is called a *node*, if it is an ordinary point of multiplicity 2. A curve X of degree d with a point P of multiplicity d coincides with its tangent cone at P and therefore it is a union of d lines through P, each line to be counted with its multiplicity.

All the above definitions are invariant by affinities and by change of coordinates.

Recall that $P \in X$ is smooth for X if and only if $\mathcal{O}_{X,P}$ is a DVR.

Theorem 16.1.1 *Let X be an affine plane curve with equation $f = 0$ and let P be a smooth point of X. Let r be a line passing through P, with equation $g = 0$. If r is not tangent to X at P, then the class of g in $\mathcal{O}_{X,P}$ generates the maximal ideal $\mathfrak{m}_{X,P}$ hence it is a local parameter at P.*

Proof Since r contains P, then g vanishes at P and therefore the class of g in $\mathcal{O}_{X,P}$ sits in $\mathfrak{m}_{X,P}$. We need to prove that it generates $\mathfrak{m}_{X,P}$. After a change of coordinates we may assume that $P = (0, 0)$ and that the tangent line to X at P is the x axis, with equation $y = 0$. Hence the equation of X is of the form

$$f = y + o(1) = 0$$

where $o(1)$ stays for terms of degree larger than 1. Moreover we may assume that r is the y axis with equation $x = 0$. Now we can write

$$f(x, y) = yG - x^2H$$

where $H \in \mathbb{K}[x]$ and $G = 1 + o(1)$. Then in $\mathcal{O}_{X,P}$ we have the relation

$$yG = x^2H$$

where we abuse notation and denote the classes in $\mathcal{O}_{X,P}$ of the polynomials with the same symbols denoting the polynomials. Moreover $G \notin \mathcal{O}_{X,P}$, because G is not zero in P, so G is invertible in $\mathcal{O}_{X,P}$ and we have

$$y = x^2HG^{-1}$$

in $\mathcal{O}_{X,P}$. So $y \in (x)$ in $\mathcal{O}_{X,P}$. Since $\mathcal{O}_{X,P} = (x, y)$, the assertion follows. □

Exercise 16.1.2 Prove that the affine plane curve X with equation $f = 0$ has a node at P if and only if $\text{grad} f(P) = 0$ and

$$\frac{\partial^2 f}{\partial x \partial y}(P) \neq \frac{\partial^2 f}{\partial x^2}(P) \cdot \frac{\partial^2 f}{\partial y^2}(P).$$

Exercise 16.1.3 Let X be an affine plane curve and let P be a point of X which lies on an irreducible component Y of X of multiplicity h. Prove that $m_P(X) \geqslant h$ and the equality holds if and only if P is smooth for Y and P does not belong to any other irreducible component of X.

16.2 Parametrizations and Branches of a Curve

Let X be a reduced affine plane curve with equation $f(x, y) = 0$. Let $\xi(t), \eta(t) \in \mathbb{K}[[t]]$ be formal power series, not both belonging to $\mathbb{K}$. We will say that

$$x = \xi(t), \quad y = \eta(t)$$

(or simply $(\xi(t), \eta(t))$) is a *parametrization* of X if $f(\xi(t), \eta(t)) = 0$. The point $P = (p, q)$ with $p = \xi(0)$ and $q = \eta(0)$, is called the *centre* of the parametrization and it belongs to X. Note that the notion of parametrization and of centre of a parametrization is invariant by change of coordinates. Note also that if $g \in \mathbb{K}[[\tau]]$ is such that $o(g) > 0$, we can substitute $\xi(\tau) = \xi(g(\tau))$ and $\eta(\tau) = \eta(g(\tau))$ and $(\xi(\tau), \eta(\tau))$ is still a parametrization of X with the same centre. It is said to be obtained from $(\xi(t), \eta(t))$ with a *change of parameter*. If $o(g) = 1$ one says that the change of parameter is *regular*. If two parametrizations differ by a regular change of parameters they are said to be *equivalent*. By taking into account the properties substitutions (see Sect. 15.2.2) one sees that this is in fact an equivalence relation between parametrizations. If $(\xi(t), \eta(t))$ is a parametrization of the curve X, and $\xi(t), \eta(t) \in \mathbb{K}[[t^h]]$, with $h > 1$, the parametrization, or one equivalent to it, is said to be *reducible*. In this case, if h is maximum so that $\xi(t), \eta(t) \in \mathbb{K}[[t^h]]$, we may replace t^h with a new variable τ, so that $(\xi(t), \eta(t))$ becomes a new parametrization $(\xi(\tau), \eta(\tau))$, which is now *irreducible* and with the same centre as $(\xi(t), \eta(t))$. We will mainly deal with parametrizations that are irreducible and we will soon give an irreducibility criterion for parametrizations.

Lemma 16.2.1 *Any parametrization of a curve is equivalent, up to a suitable choice of coordinates, to one of the form*

$$x = t^n, \quad y = a_1 t^{n_1} + a_2 t^{n_2} + \cdots \tag{16.1}$$

with $a_1, a_2, \ldots$ *non zero,* $n > 0$, $0 < n_1 < n_2 < \cdots$.

Proof First of all we can put the centre of the parametrization at the origin of the coordinate system. Then the parametrization takes the form

$$x = t^n(b_0 + b_1 t + \cdots), \quad x = t^m(c_0 + c_1 t + \cdots)$$

where $n, m > 0$ and $b_0, c_0 \neq 0$. Now we make a substitution

$$t = d_1\tau + d_2\tau^2 + \cdots$$

with $d_1 \neq 0$ and try to determine $d_1, d_2, \dots$ in such a way that after the substitution the parametrization takes the form (16.1). After substituting we have

$$\begin{aligned} x &= \tau^n(d_1 + d_2\tau + \cdots)^n[b_0 + b_1(d_1 + d_2\tau + \cdots) + \cdots] = \\ &= \tau^n[d_1^n b_0 + (nd_1^{n-1}d_2b_0 + d_1^{n+1}b_1)\tau + \cdots + \\ &+ (nd_1^{n-1}d_ib_0 + P_i(b_1, \dots, b_i, d_1, \dots, d_{i-1}))\tau^i + \cdots], \end{aligned}$$

where P_i is a suitable polynomial in its arguments. We define recursively $d_1, d_2, \dots$ in the following way

$$\begin{aligned} d_1 &= b_0^{-1}, \\ d_2 &= -(nd_1^{n-1}b_0)^{-1}d_1^{n+1}b_1, \\ d_i &= -(nd_1^{n-1}b_0)^{-1}P_i(b_1, \dots, b_i, d_1, \dots, d_{i-1}), \quad i \geqslant 3. \end{aligned}$$

With these definitions we have $x = \tau^n$, as wanted. □

A parametrization of the type (16.1) is said to be in *standard form.*

Next we give the announced irreducibility criterion:

Lemma 16.2.2 *A parametrization of a curve in standard form* (16.1) *is reducible if and only if the integers $n, n_1, n_2, \dots$ have a common factor larger than 1.*

Proof One implication is obvious. We prove the other.

Suppose that there is a substitution $t = f(\tau)$, with $o(f) = 1$ such that after the substitution we have $x = \xi(\tau)$, $y = \eta(\tau)$, with $\xi(\tau), \eta(\tau) \in \mathbb{K}[[\tau^h]]$, with $h > 1$.

We first claim that $\frac{f(\tau)}{\tau} \in \mathbb{K}[[\tau^h]]$. In fact, if this is not the case, since $o(f) = 1$ we have

$$f(\tau) = \tau(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh} + c\tau^k + \cdots),$$

with $b_0 \neq 0$, $c \neq 0$, and h does not divide k. Then

$$\begin{aligned} x = \xi(\tau) &= \tau^n[(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh}) + c\tau^k + \cdots]^n = \\ &= \tau^n(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh})^n + nc\tau^{n+k}(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh})^{n-1} + \cdots. \end{aligned}$$

Since $\xi(\tau) \in \mathbb{K}[[\tau^h]]$, we have that h divides n because $\xi(\tau)$ starts with $b_0\tau^n$. Then $\xi(\tau) - \tau^n(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh})^n \in \mathbb{K}[[\tau^h]]$, but

$$\xi(\tau) - \tau^n(b_0 + b_1\tau^h + \cdots + b_l\tau^{lh})^n = ncb_0^{n-1}\tau^{n+k} + \cdots$$

and h does not divide $n + k$ because it does not divide k. So we have a contradiction which proves our claim. Then we can write $f(\tau) = \tau g(\tau)$, with $g(\tau) \in \mathbb{K}[[\tau^h]]$.

Next we prove the assertion of the lemma. Note that the above argument proves that h divides n. We want to prove that also $n_1, n_2, \ldots$ are all divisible by h. We argue by contradiction and assume this is not the case. Suppose that $n_1, \ldots, n_s$ are divisible by h but n_{s+1} is not divisible by h. Then

$$\begin{aligned}&\eta(\tau) - (a_1\tau^{n_1}g(\tau)^{n_1} + \cdots + a_s\tau^{n_s}g(\tau)^{n_s}) = \\ &= a_{s+1}\tau^{n_{s+1}}(b_0 + b_1\tau^h + \cdots)^{n_{s+1}} + \cdots = a_{s+1}b_0^{n_{s+1}}\tau^{n_{s+1}} + \cdots .\end{aligned}$$

The left hand side belongs to $\mathbb{K}[[\tau^h]]$ because $\eta(\tau), g(\tau) \in \mathbb{K}[[\tau^h]]$ and $n_1, \ldots, n_s$ are all divisible by h, whereas the right hand side does not belong to $\mathbb{K}[[\tau^h]]$ because it starts with $a_{s+1}b_0^{n_{s+1}}\tau^{n_{s+1}}$, and n_{s+1} is not divisible by h. This is a contradiction, which proves that $n_1, n_2, \ldots$ are all divisible by h. □

An equivalence class of irreducible equivalent parametrizations of X is called a *branch* of X. A branch is determined by any of the equivalent parametrizations which represent it. All parametrizations of the same branch have the same centre belonging to X, which is called the *centre* of the branch.

Proposition 16.2.3 *Consider $f(x, y) \in \mathbb{K}[x, y]$ an irreducible polynomial. To each root $\eta(x) \in \mathbb{K}\{x\}$ of $f(x, y) = 0$ such that $o(\eta) > 0$ corresponds a unique branch of the curve X with equation $f(x, y) = 0$ with centre the origin. Conversely, to any branch (ξ, η) of X with centre the origin correspond $o(\xi)$ roots $\eta(x) \in \mathbb{K}\{x\}$ of $f(x, y) = 0$, with $o(\eta) > 0$.*

Proof Let $\eta(x) \in \mathbb{K}\{x\}$ be a root of $f(x, y) = 0$ with $o(\eta) > 0$. Let n be the minimum such that $\eta \in \mathbb{K}(x^{\frac{1}{n}})$, so that we can write $\eta = \eta(x^{\frac{1}{n}})$. Then if we set $\xi = t^n, \eta = \eta(t)$, then (ξ, η) is parametrization with centre the origin of X. Moreover by the definition of n and by Lemma 16.2.2, this parametrization is irreducible, so that it defines a branch of the curve X with centre the origin.

Conversely, consider an irreducible parametrization (ξ, η) of X with centre the origin, and assume that $o(\xi) = n > 0$ and $o(\eta) > 0$. By Lemma 16.2.1 this parametrization is equivalent to a standard one like (16.1). Clearly two parametrizations like this can only differ by substitution of t with εt, where $\varepsilon^n = 1$, so that there are n distinct such parametrizations. We will prove that each of them determines a different root of $f(x, y) = 0$. Indeed, consider two distinct values ε_1 and ε_2 of the n–th root of unity ε. The values of η corresponding to ε_1 and ε_2 are

$$\eta_1 = a_1\varepsilon_1^{n_1}t^{n_1} + a_2\varepsilon_1^{n_2}t^{n_2} + \cdots \quad \text{and} \quad \eta_2 = a_1\varepsilon_2^{n_1}t^{n_1} + a_2\varepsilon_2^{n_2}t^{n_2} + \cdots .$$

Suppose that $\eta_1 = \eta_2$, which implies

$$\varepsilon_1^{n_i} = \varepsilon_2^{n_i} \quad \text{for} \quad i = 1, 2, \ldots . \tag{16.2}$$

Since the integers $n, n_1, n_2, \ldots$ have no common factor larger than 1 (see Lemma 16.2.2), we can find a positive integer m such that the greatest common divisor of $n, n_1, n_2, \ldots, n_m$ is 1, so that there are integers $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$1 = \lambda n + \lambda_1 n_1 + \lambda_2 n_2 + \cdots + \lambda_m n_m.$$

Taking into account (16.2) and $\varepsilon_1^n = \varepsilon_2^n = 1$, we find the relation

$$\varepsilon_1 = \varepsilon_1^{\lambda n+\lambda_1 n_1+\lambda_2 n_2+\cdots+\lambda_m n_m} = \varepsilon_2^{\lambda n+\lambda_1 n_1+\lambda_2 n_2+\cdots+\lambda_m n_m} = \varepsilon_2$$

which is a contradiction. Hence for each of the distinct roots ε_i of $\varepsilon^n = 1$, we have different series η_i, for $i = 1, \ldots, n$. Each of them gives rise to a root of $f(x, y) = 0$ of the form

$$y = a_1 x^{\frac{n_1}{n}} + a_2 x^{\frac{n_2}{n}} + \cdots$$

as desired. □

Now we are in a position to prove the fundamental:

Theorem 16.2.4 *Given an irreducible affine plane curve X, every point of X is the centre of at least one branch of X.*

Proof Let P be any point of X and assume that X has degree d. Up to appropriately choosing the coordinates, we may assume that P is the origin and that the projective closure of X does not pass through the point at infinity of the y axis. Then the equation of X is of the form

$$f(x, y) = a_0(x) + a_1(x)y + \cdots + y^d$$

with $a_0(0) = 0$. By Exercise 15.3.8, there is at least one solution η of $f(x, y) = 0$ as an equation in y, such that $o(\eta) > 0$. By Proposition 16.2.3 it determines a branch of X with centre P. □

Remark 16.2.5 The notions of parametrization and of branch can be extended in a natural way to the case in which X has multiple components. Suppose X has equation $f(x, y) = 0$ with f with no factor not depending on y, namely X has no line component parallel to the y axis (we may achieve this by an appropriate choice of coordinates). By Exercise 15.3.9 we have that $f(x, y) = 0$ has a multiple root in $\mathbb{K}\{x\}$ if and only if f has a multiple factor. We extend the correspondence between roots of $f(x, y)$ and branches of X by saying that a root of $f(x, y) = 0$ with multiplicity n corresponds to a branch of X of multiplicity n. Then every branch of an irreducible component of X of multiplicity m counts with multiplicity at least m.

16.3 Intersections of Affine Curves

Let $X, Y \subset \mathbb{A}^2$ be two affine plane curves of degrees n, m. We can look at their projective closures $\bar{X}, \bar{Y}$. If X and Y have no common component, then also $\bar{X}$ and $\bar{Y}$ have no common components and, according to Bezout Theorem, they have only finitely many points in common, and the number of these points is nm, provided we count each point $P \in \bar{X} \cap \bar{Y}$ with its intersection multiplicity $i(P; \bar{X}, \bar{Y})$.

For any point $P \in \mathbb{A}^2$ we define the *intersection multiplicity* of X and Y at P as $i(P; X, Y) := i(P; \bar{X}, \bar{Y})$.

If X and Y have equations $f(x, y) = 0$ and $g(x, y) = 0$ respectively, one sets $i(P; f, g) := i(P; X, Y)$.

In this section we will interpret the intersection multiplicity of two affine curves at a common point in various ways which will be useful for our further purposes.

16.3.1 Intersection Multiplicity and Resultants

First of all we want to go back to the general definition of intersection multiplicity given in Sect. 11.4, and provide a more flexible and computable interpretation of it in the curve case. Interpreting $\mathbb{A}^2$ as usual as the open subset U_0 of $\mathbb{P}^2$ (see Sect. 1.5), in $\mathbb{P}^2$ we have homogeneous coordinates $[x_0, x_1, x_2]$ such the point (x, y) of $\mathbb{A}^2$ has homogeneous coordinates $[1, x, y]$.

Consider the two affine curves X, Y with equations $f(x, y) = 0$ and $g(x, y)=0$ respectively, with no common components. Their projective closures $\bar{X}, \bar{Y}$ have equations $\beta(f) = 0$ and $\beta(g) = 0$ where β is the homogenizing operator defined in Sect. 1.5. To ease notation we set from now on $\beta(f) = f_h$ and similarly $\beta(g) = g_h$. Suppose we have an intersection point $P = [p_0, p_1, p_2]$ of $\bar{X}$ and $\bar{Y}$. Then recall from Sect. 11.4 that $i(P; \bar{X}, \bar{Y})$, is the multiplicity which the factor $u_0 p_0 + u_1 p_1 + u_2 p_2$ has in the resultant of the system of polynomals

$$f_h(x_0, x_1, x_2), \quad g_h(x_0, x_1, x_2), \quad u_0x_0 + u_1x_1 + u_2x_2. \tag{16.3}$$

Note that the equation

$$u_0x_0 + u_1x_1 + u_2x_2 = 0 \tag{16.4}$$

where u_0, u_1, u_2 are indeterminate, represents an indeterminate line ℓ of $\mathbb{P}^2$. This can be also represented in a different way. Take two points $A = [\boldsymbol{\alpha}] = [\alpha_0, \alpha_1, \alpha_2]$ and $B = [\boldsymbol{\beta}] = [\beta_0, \beta_1, \beta_2]$ of $\mathbb{P}^2$, with indeterminate coordinates, and consider the line $\ell = \langle A, B\rangle$, which can be parametrically represented by

$$x_i = \lambda\alpha_i + \mu\beta_i, \quad i = 0, 1, 2. \tag{16.5}$$

The line ℓ can be represented as well by the equation (16.4) where u_0, u_1, u_0 are proportional to the maximal minors with alternate signs of the matrix

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix}. \tag{16.6}$$

Let us plug (16.5) in $f_h(x_0, x_1, x_2)$ and $g_h(x_0, x_1, x_2)$. In this way we obtain two homogeneous polynomials in λ, μ and we can consider their resultant $R(\boldsymbol{\alpha}, \boldsymbol{\beta})$ which is a polynomial in $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This resultant vanishes if and only if the line $\ell = \langle A, B\rangle$ contains one of the intersection points $P = [\mathbf{p}] = [p_0, p_1, p_2]$ of $\bar{X}$ and $\bar{Y}$, i.e., it vanishes if and only if

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p}) := \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ p_0 & p_1 & p_2 \end{vmatrix} = 0$$

which by the way is equivalent to $u_0 p_0 + u_1 p_1 + u_2 p_2 = 0$, if u_0, u_1, u_2 are the minors of maximal order of the matrix (16.6) with alternate signs. This implies that we have a decomposition

$$R(\boldsymbol{\alpha}, \boldsymbol{\beta}) = c \prod (\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p})^{r_P} \tag{16.7}$$

where $c \neq 0$ and the product runs over all points $P \in \bar{X} \cap \bar{Y}$. Taking into account that each factor of the form $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p})$ appearing in R is also of the form $u_0 p_0 + u_1 p_1 + u_2 p_2$, it is clear that $R(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is nothing else than the resultant of the polynomials in (16.3), and therefore the exponent r_P is just the intersection multiplicity of $\bar{X}$ and $\bar{Y}$ at P.

In order to make it easier to compute the intersection multiplicities, it is useful to make some specializations. For instance, we take $A = [1, u, 0]$, $B = [0, v, 1]$, $\lambda = 1$, $\mu = t$, so that (16.5) becomes

$$x_0 = 1, \quad x_1 = u + tv, \quad x_2 = t$$

and $R(\boldsymbol{\alpha}, \boldsymbol{\beta})$ becomes a polynomial $N(u, v)$ which is called the *Netto's resolvent* of f and g. This is nothing but the resultant of $f_h(1, u + tv, t) = f(u + tv, t)$ and $g_h(1, u + tv, t) = g(u + tv, t)$ with respect to t. By taking into account (16.7), we have

$$N(u, v) = c \prod (up_0 - p_1 + vp_2)^{r_P}$$

Again the product runs over all the intersection points of $\bar{X}$ and $\bar{Y}$.

Next we specialize further setting $v = 0$, provided $\bar{X}$ and $\bar{Y}$ do not both pass through the point at infinity of the y axis which has coordinates $[0, 0, 1]$. In this case we have the resultant $R(u)$ of the polynomials $f(u, t)$, $g(u, t)$ with respect to t and

$$R(u) := N(u, 0) = c \prod (up_0 - p_1)^{r_P}$$

This form applies well to the intersection points $P = (p, q)$ of X and Y in $\mathbb{A}^2$, for which $p_0 = 1$, $p_1 = p$. Then we have

$$R(u) = c \prod (u - p)^{r_P}$$

and the product runs over all the intersection points of X and Y. So the factor $u - p$ appears in $R(u)$ in correspondence to all intersection points of X and Y with the same x–coordinate p, and the exponent with which $u - p$ appears in $R(u)$ is the sum of the intersection multiplicities of X and Y at their common points with first coordinate p.

In conclusion we can summarize what we have proved in this:

Theorem 16.3.1 *Let X and Y be affine plane curves, with no common component, with equations $f(x, y) = 0$ and $g(x, y) = 0$ respectively and such that their projective closures do not both pass through the point at infinity of the y axis. Let $R(x)$ be the resultant of f and g with respect to y. Then the solutions of the equation $R(x) = 0$ are the x–coordinates of the intersection points of X and Y and their multiplicities are the sums of the intersection multiplicities of X and Y at the common points with the same x–coordinate.*

In particular, if no common points of X and Y are aligned on lines with equations $x = \text{const.}$, then the solutions of the equation $R(x) = 0$ are the x–coordinates of the intersection points of X and Y and their multiplicities are the intersection multiplicities of X and Y at the unique common point with the corresponding first coordinate.

We can add a useful remark given by the following:

Proposition 16.3.2 *Let X and Y be affine plane curves, with no common component, with equations $f(x, y) = 0$ and $g(x, y) = 0$ respectively and such that their projective closures do not both pass through the point at infinity of the y axis. Let $R(x)$ be the resultant of f and g with respect to y. Then $R(x)$ is not identically 0. Moreover there is no common branch to X and Y.*

Proof The fact that $R(x)$ is not identically 0 follows from Theorem 16.3.1, since the equation $R(x) = 0$ has as solutions the x–coordinates of the finitely many intersection points of X and Y.

As for the other assertion, suppose that X and Y have a common branch. We may assume that the centre of the branch is the origin, and that the branch has a standard parametrization of the type

$$x = t^n, \quad y = \sum_{i=1}^{\infty} a_i t^i$$

so that

$$f\left(t^n, \sum_{i=1}^{\infty} a_i t^i\right) = 0, \quad g\left(t^n, \sum_{i=1}^{\infty} a_i t^i\right) = 0.$$

Then we have

$$f\left(x, \sum_{i=1}^{\infty} a_i x^{\frac{i}{n}}\right) = 0, \quad g\left(x, \sum_{i=1}^{\infty} a_i x^{\frac{i}{n}}\right) = 0.$$

so that $\xi = \sum_{i=1}^{\infty} a_i x^{\frac{i}{n}}$ is a common root of the equations $f(x, y) = 0$ and $g(x, y) = 0$ in the variable y. This implies that $R(x) = 0$, a contradiction. □

16.3.2 Order of a Curve at a Branch and Intersection Multiplicities

Consider a branch γ of an affine plane curve determined by a parametrization $(\xi(t), \eta(t))$. Let X be any affine plane curve curve, with equation $f(x, y) = 0$. We define the *order* of X (or of f) at γ, denoted by $o_\gamma(X) = o_\gamma(f)$, to be ∞ if $f(\xi(t), \eta(t)) = 0$ (i.e., if γ is a branch of Y), or otherwise the order of the power series $f(\xi(t), \eta(t))$ in t. It is a simple verification, that we leave to the reader, that this definition does not depend on the single parametrization that determines the branch, moreover it is independent of affinities or change of coordinates. It is also easy to verify that, for any pair of polynomials $f(x, y)$, $g(x, y)$ and any branch γ one has

$$o_\gamma(fg) = o_\gamma(f) + o_\gamma(g), \quad o_\gamma(f \pm g) \geqslant \min\{o_\gamma(f), o_\gamma(g)\}.$$

One has $o_\gamma(X) > 0$ if and only if X contains the centre of γ.

Theorem 16.3.3 *Let X, Y be affine plane curves with equations $f(x, y) = 0$, $g(x, y) = 0$ respectively, with no common component. Suppose that the point P is contained in $X \cap Y$. Then the sum of the orders of X at the branches of Y with centre P (counted with their multiplicities, see Remark 16.2.5) equals the sum of the orders of Y at the branches of X with centre P (counted with their multiplicities), and this number equals $i(P; X, Y)$.*

Proof We can choose the coordinates in such a way that P is the origin, the projective closures of X and Y do not contain the point at infinity of the y axis, and no other intersection point of X and Y sits on the y axis. Then the coefficient of the highest power of y appearing in f and g can be assumed to be 1, and by Corollary 15.3.3 we have

$$f(x, y) = \prod_{i=1}^{n}(y - \xi_i), \quad g(x, y) = \prod_{j=1}^{m}(y - \eta_j)$$

for some $\xi_i, \eta_j \in \mathbb{K}[[x^{\frac{1}{h}}]]$, for a suitable h, and $i = 1, \ldots, n$, $j = 1, \ldots, m$. Consider an $i = 1, \ldots, n$ such that $o(\xi_i) > 0$, so that ξ_i corresponds to a branch γ of X with centre the origin (see Proposition 16.2.3), that is represented by a parametrization of the type

$$x = t^r, \quad y = \sum_{i=1}^{\infty} a_i t^i.$$

Then we have

$$g\left(t^r, \sum_{i=1}^{\infty} a_i t^i\right) = bt^k + \cdots, \quad \text{with} \quad b \neq 0, \tag{16.8}$$

where $k = o_\gamma(g)$ is not ∞ by Proposition 16.3.2.

Now γ has equivalent parametrizations of the form

$$x = t^r, \quad y = \sum_{i=1}^{\infty} \varepsilon_j^i a_i t^i.$$

where ε_j, for $j = 1, \ldots, r$, is any r–th root of the unity. Each of these parametrizations gives rise to a different root

$$\xi_j = \sum_{i=1}^{\infty} \varepsilon_j^i a_i x^{\frac{i}{r}}, \quad j = 1, \ldots, r$$

of the equation $f(x, y) = 0$ in y. By (16.8) we have

$$g(x, \xi_j) = b\varepsilon^k x^{\frac{k}{r}} + \cdots$$

hence

$$\prod_{j=1}^{r} g(x, \xi_j) = cx^k + \cdots, \quad \text{with} \quad c \neq 0.$$

Applying the same argument to any branch γ of X with centre the origin, we find

$$o\left(\prod_{l=1}^{u} g(x, \xi_l)\right) = \sum_{\gamma} o_\gamma(g)$$

where the product on the left hand side is on all roots $\xi_1, \ldots, \xi_u$ of $f(x, y)$ such that $o(\xi_l) > 0$ for $i = 1, \ldots, u$ (see Proposition 16.2.3), and the sum on the right hand side is made on all branches of X with centre the origin.

Let now $\eta_1, \ldots, \eta_v$ be the roots of $g(x, y)$ with positive order, and $\eta'_1, \ldots, \eta'_{v'}$ the roots of $g(x, y)$ with zero order. Then we have

$$\sum_{\gamma} o_\gamma(g) = \prod_{l=1}^{u} g(x, \xi_l) = o\left(\prod_{l=1,\ldots,u, r=1,\ldots,v} (\xi_l - \eta_r) \cdot \prod_{l=1,\ldots,u, r=1,\ldots,v'} (\xi_l - \eta'_r)\right) =$$
$$= o\left(\prod_{l=1,\ldots,u, r=1,\ldots,v} (\xi_l - \eta_r) \cdot h(x)\right)$$

where $o(h) = 0$. Therefore

$$\sum_{\gamma} o_\gamma(g) = o\Big(\prod_{l=1}^{u} g(x,\xi_l)\Big) = o\Big(\prod_{l=1,\dots,u,r=1,\dots,v} (\xi_l - \eta_r)\Big). \tag{16.9}$$

On the other hand, by exchanging the roles of f and g in the above argument, we have

$$\sum_{\gamma'} o_\gamma(f) = o\Big(\prod_{l=1,\dots,u,r=1,\dots,v} (\eta_r - \xi_l)\Big) \tag{16.10}$$

where the sum on the left hand side is made over all branches γ' of g with centre the origin. From (16.9) and (16.10), we have

$$\sum_{\gamma'} o_\gamma(f) = \sum_{\gamma} o_\gamma(g).$$

Let us now denote by $\xi'_1, \dots, \xi'_{u'}$ the roots of $f(x, y)$ such that $o(\xi'_t) = 0$ for $t = 1, \dots, u'$. Then, with the above notation, we have

$$f(x,y) = \prod_{l=1,\dots,u} (y - \xi_l) \prod_{t=1,\dots,u'} (y - \xi'_t),\ \ g(x,y) = \prod_{r=1,\dots,v} (y - \eta_r) \prod_{s=1,\dots,v'} (y - \eta'_s).$$

If $R(x)$ is the resultant of $f(x, y)$ and $g(x, y)$ by the elimination of y, we have (see Exercise 2.1.6)

$$\begin{aligned} R(x) = & \prod_{l=1,\dots,u,r=1,\dots,v} (\xi_l - \eta_r) \prod_{l=1,\dots,u,s=1,\dots,v'} (\xi_l - \eta'_r)\cdot \\ & \cdot \prod_{t=1,\dots,u',r=1,\dots,v} (\xi'_t - \eta_r) \prod_{t=1,\dots,u',s=1,\dots,v'} (\xi'_t - \eta'_r). \end{aligned}$$

It is clear that

$$o\Big(\prod_{l=1,\dots,u,s=1,\dots,v'} (\xi_l - \eta'_r)\Big) = o\Big(\prod_{t=1,\dots,u',r=1,\dots,v} (\xi'_t - \eta_r)\Big) = 0.$$

We claim that also

$$o\Big(\prod_{t=1,\dots,u',s=1,\dots,v'} (\xi'_t - \eta'_r)\Big) = 0.$$

Indeed, if $o((\xi'_l - \eta'_r))) > 0$ for some $t = 1, \dots, u', s = 1, \dots, v'$, then ξ'_l and η'_r would start with the same constant term $d \neq 0$. The corresponding branches would then have the same centre at the point with coordinates $(0, d)$. But this is not possible because we assumed that X and Y have only the origin as a common point on the y axis. In conclusion we have

$$o(R(x)) = o\Big(\prod_{l=1,\dots,u,r=1,\dots,v} (\xi_l - \eta_r)\Big)$$

which, as we saw, equals $\sum_{\gamma'} o_\gamma(f) = \sum_\gamma o_\gamma(g)$. On the other hand $o(R(x))$ is the multiplicity of the root 0 for $R(x)$, which is equal to the intersection multiplicity of X and Y at the origin (see Theorem 16.3.1). This completes the proof of the theorem. □

16.3.3 More Properties of Branches and of Intersection Multiplicity

Lemma 16.3.4 *Let γ be a branch of a curve with centre the point P. Consider the integer*

$$o(\gamma) = \min\{o_\gamma(L) : L \quad \text{is a line containing} \quad P\}.$$

Then for all lines L containing P, one has $o_\gamma(L) = o(\gamma)$, except for only one line L_0 containing P such that $o_\gamma(L_0) > o(\gamma)$.

Proof Consider a parametrization

$$x = \xi(t) = \sum_{i=0}^{\infty} a_i t^i, \quad y = \eta(t) = \sum_{i=0}^{\infty} b_i t^i \tag{16.11}$$

of γ. Take any line L containing the centre $P = (a_0, b_0)$ of γ. The equation of L is of the form

$$\ell(x, y) = a(x - a_0) + b(y - b_0) = 0$$

with $(a, b) \neq (0, 0)$. We have

$$\ell(\xi(t), \eta(t)) = \sum_{i=1}^{\infty} (aa_i + bb_i)t^i.$$

If r is the minimum positive integer such that a_n, b_n are not both zero, then $o_\gamma(L) = r$ unless $aa_r + bb_r = 0$. Thus the condition $aa_r + bb_r = 0$ identifies the unique line L_0 through P such that $o_\gamma(L_0) > r$. □

The positive integer $o(\gamma)$ defined in the statement of Lemma 16.3.4, is called the *order* of the branch γ. The branch is called *linear* if $o(\gamma) = 1$. The proof of Lemma 16.3.4 shows that a branch with parametrization (16.11) is linear if and only if either a_1 or b_1 is non–zero. The unique line L_0 in the statement of Lemma 16.3.4 is called the *tangent* of the branch. The positive quantity $c(\gamma) = o_\gamma(L_0) - o(\gamma)$ is called the *class* of the branch.

Proposition 16.3.5 *Let X be a reduced affine plane curve. One has:*

(a) if a point $P \in X$ has multiplicity m then the sum of the order of the branches of X with centre at P is m;
(b) a point $P \in X$ is smooth for X if and only if it is the centre of a unique linear branch;
(c) the principal tangents to X at a point $P \in X$ coincide with the tangents to the branches of X with centre P.

Proof (a) A line L through P, which is not tangent to any branch of X with centre P, has intersection multiplicity $i(P; L, X) = \sum_\gamma o(\gamma)$, where the sum is taken over all branches γ with the centre at P. On the other hand $i(P; L, X) = m$, except for the finitely many lines L which are principal tangent lines to X at P. This implies $m = \sum_\gamma o(\gamma)$.

Part (b) is an immediate consequence of (a).

(c) Let m be the multiplicity of P for X. By (a), we have $m = \sum_\gamma o(\gamma)$, where the sum is taken over all branches γ with the centre at P. A line L is a principal tangent line to X at P if and only if $i(P; L, X) > m$. As we saw in the proof of (a), a line L is tangent to one of the branches of X with centre at P if and only if $i(P; L, X) > \sum_\gamma o(\gamma) = m$. This proves the assertion. □

Proposition 16.3.6 *Let γ be a branch of a reduced affine plane curve with centre at P and let X be a reduced affine plane curve which has multiplicity n at P. Then $o_\gamma(X) \geqslant no(\gamma)$ and the equality holds if and only if the tangent to γ at P is not one of the principal tangents to X at P.*

Proof The assertion is trivial if $o_\gamma(X) = \infty$. So we will assume this is not the case.

Fix coordinates in such a way that P is the origin and the x and y axes are neither tangent to γ nor among the principal tangents to X at P. Set $m = o(\gamma)$. Then γ and the branches $\gamma_1, \dots, \gamma_h$ of X at P may be assumed to have parametrizations given by

$$x = t^m, \quad y = at^m + \cdots, \quad \text{and} \quad x = t^{m_i}, \quad y = a_i t^{m_i} + \cdots, \quad i = 1, \dots, h$$

with $a, a_i \neq 0$, for $i = 1, \dots, h$ and

$$m_1 + \cdots + m_l = n.$$

So the tangent line to γ has equation $y = ax$ and the tangent line to γ_i has equation $y = a_i x$, for $i = 1, \dots, h$. The corresponding fractional power series are

$$\eta_j = ax + \cdots, \quad j = 1, \dots, m$$

and

$$\eta_{i,l} = a_i x + \cdots, \quad i = 1, \dots, h, l = 1, \dots, m_i$$

where the indices j and l correspond to the m–th and m_i–th roots of the unity. Let $f(x, y) = 0$ be the equation of X. As we saw in the proof of Theorem 16.3.3, we have

$$\prod_{j=1}^{m} f(x, \eta_j) = \prod_{j=1,\dots,m, i=1,\dots,h, l=1,\dots,m_i} (\eta_j - \eta_{i,l})h(x) =$$
$$= \prod_{j=1,\dots,m, i=1,\dots,h, l=1,\dots,m_i} \Big((a - a_i)x + \cdots\Big)$$
$$= \prod_{i=1,\dots,h,} \Big((a - a_i)^{mm_i} x^{mm_i} + \cdots\Big),$$

where $o(h) = 0$. Thus $o_\gamma(f) \geqslant m(m_1 + \cdots + m_l) = mn$. The equality holds if and only if a is different from all of $a_1, \dots, a_h$, i.e., if and only if the tangent to γ at P is not one of the principal tangents to X at P. □

As an immediate consequence we have:

Corollary 16.3.7 *Let X and Y be two reduced affine plane curves with no common components. Suppose P is a point of multiplicity n for X and m for Y. Then $i(P; X, Y) \geqslant mn$ and the equality holds if and only if X and Y have no common principal tangent line at P.*

16.3.4 Further Interpretation of the Intersection Multiplicity

In this section we give another important interpretation of the intersection multiplicity of two curves. First of all we list the main properties that the intersection multiplicity has:

(a) if X, Y are two affine plane curves and $P \in \mathbb{A}^2$ is a point, then the intersection multiplicity $i(P; X, Y) \in \mathbb{N} \cup \{\infty\}$ and precisely $i(P; X, Y) = 0$ if $P \notin X \cap Y$, $i(P; X, Y) \in \mathbb{N} \setminus \{0\}$ if $P \in X \cap Y$ and X and Y have no common component passing through P, $i(P; X, Y) = \infty$ if P sits in a common component of X and Y;
(b) $i(P; X, Y)$ depends only on the components of X and Y containing P;
(c) $i(P; X, Y)$ is invariant under affinities, i.e., if $\tau : \mathbb{A}^2 \to \mathbb{A}^2$ is an affinity then $i(P; X, Y) = i(\tau(P); \tau(X), \tau(Y))$;
(d) $i(P; X, Y) = i(P; Y, X)$;
(e) $i(P; X, Y) \geqslant m_P(X)m_P(Y)$ and the equality holds if and only if X and Y have no principal tangents in common at P;
(f) if $f = \prod_{i=1,\dots,h} f_i^{n_i}$ and $g = \prod_{j=1,\dots,k} g_j^{m_j}$, then

$$i(P; f, g) = \sum_{i=1,\dots,h,\, j=1,\dots,k} n_i m_j i(P; f_i, g_j);$$

(g) for every triple of polynomials $f(x, y), g(x, y), h(x, y)$ one has

$$i(P; f, g) = i(P; f, g + hf).$$

Most of these properties have been proved above. For property (f) see Exercise 16.3.16. Property (g) follows from Theorem 16.3.3.

To prove the main theorem of this section we need some preliminary results.

Lemma 16.3.8 *Let $\mathcal{I} \subseteq A_n$ be an ideal. Then $Z_a(\mathcal{I})$ is a finite set if and only if $A_n/\mathcal{I}$ is a finitely generated $\mathbb{K}$–vector space. If this is the case, then the number of points in $Z_a(\mathcal{I})$ is at most* $\dim(A_n/\mathcal{I})$.

Proof Let $P_1, \dots, P_m$ be distinct points in $Z_a(\mathcal{I})$. We can choose polynomials $f_1, \dots, f_m \in A_n$ such that $f_i(P_j) = \delta_{ij}$ for $i, j = 1, \dots, m$, where δ_{ij} is the Kronecker symbol (we leave to the reader the easy task to prove that there are such polynomials). Denote by $\bar{f_i}$ the class of f_i in $A_n/\mathcal{I}$, for $i = 1, \dots, m$. If $\sum_{i=1}^m t_i \bar{f_i} = 0$, with $t_1, \dots, t_m \in \mathbb{K}$, then $\sum_{i=1}^m t_i f_i \in \mathcal{I}$, so $t_j = \sum_{i=1}^m t_i f_i(P_j) = 0$, for all $j = 1, \dots, m$. This yields that $\bar{f}_1, \dots, \bar{f}_m$ are linearly independent over $\mathbb{K}$. Thus $\dim(A_n/\mathcal{I}) \geqslant m$, and therefore $\dim(A_n/\mathcal{I})$ is infinite dimensional if $Z_a(\mathcal{I})$ is not finite.

Suppose now $Z_a(\mathcal{I})$ to be finite, consisting of the distinct points $P_1, \dots, P_m$. Set $P_i = (p_{i1}, \dots, p_{in})$ for $i = 1, \dots, m$. Define

$$g_j = \prod_{i=1}^m (x_j - p_{ij}), \quad \text{for} \quad j = 1, \dots, n.$$

Then $g_j \in \mathcal{I}_a(Z_a(\mathcal{I}))$, for $j = 1, \dots, n$. So by the Hilbert Nullstellensatz, there is a positive integer h such that $g_j^h \in \mathcal{I}$, for $j = 1, \dots, n$. Denote with an upper bar the images of polynomials in $A_n/\mathcal{I}$. We have that $\bar{g}_j^h = 0$, hence $\bar{x}_j^{mh}$ is a $\mathbb{K}$–linear combination of $1, \bar{x}_j, \dots, \bar{x}_j^{mh-1}$, for $j = 1, \dots, n$. It follows that for any $s \geqslant mh$, $\bar{x}_j^s$ is a $\mathbb{K}$–linear combination of $1, \bar{x}_j, \dots, \bar{x}_j^{mh-1}$, for $j = 1, \dots, n$. This proves that the set

$$\{\bar{x}_1^{l_1}, \dots, \bar{x}_n^{l_n} : l_1, \dots, l_n < mh\}$$

generates $A_n/\mathcal{I}$. □

Next we make a definition. Let A be a ring and let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of A. In general one has

$$\mathcal{I}\mathcal{J} \subseteq \mathcal{I} \cap \mathcal{J}. \tag{16.12}$$

The ideals $\mathcal{I}$ e $\mathcal{J}$ are said to be *comaximal* if $\mathcal{I} + \mathcal{J} = A$.

Lemma 16.3.9 *Let A be a ring and let $\mathcal{I}$ and $\mathcal{J}$ be two comaximal ideals of A. One has:*

(a) $\mathcal{I}\mathcal{J} = \mathcal{I} \cap \mathcal{J}$;
(b) $\mathcal{I}^n$ *and* $\mathcal{J}^m$ *are comaximal for any pairs of positive integers* n, m.

Moreover, if $\mathcal{I}_1, \ldots, \mathcal{I}_h$ *are ideals in A such that for any* $i \in \{1, \ldots, h\}$ *the ideals* $\mathcal{I}_i$ *and* $\mathcal{J}_i = \bigcap_{j \neq i} \mathcal{I}_j$ *are comaximal, then for any positive integer n one has*

$$\mathcal{I}_1^n \cap \ldots \cap \mathcal{I}_h^n = (\mathcal{I}_1 \cdots \mathcal{I}_h)^n = (\mathcal{I}_1 \cap \ldots \cap \mathcal{I}_h)^n.$$

Proof (a) Since $A = \mathcal{I} + \mathcal{J}$, we have

$$\begin{aligned}\mathcal{I} \cap \mathcal{J} &= (\mathcal{I} \cap \mathcal{J})A = (\mathcal{I} \cap \mathcal{J})(\mathcal{I} + \mathcal{J}) = \\ &= (\mathcal{I} \cap \mathcal{J})\mathcal{I} + (\mathcal{I} \cap \mathcal{J})\mathcal{J} \subseteq \mathcal{J}\mathcal{I} + \mathcal{I}\mathcal{J} = \mathcal{I}\mathcal{J},\end{aligned}$$

which, with (16.12), proves the assertion.

(b) Since $A = \mathcal{I} + \mathcal{J}$ we have a relation of the form $1 = u + v$, with $u \in \mathcal{I}$ and $v \in \mathcal{J}$. Then $1 = 1^n = (u + v)^m$ and expanding the power we see that $1 \in \mathcal{I} + \mathcal{J}^m$, i.e., $\mathcal{I}$ and $\mathcal{J}^m$ are comaximal. Hence we have $1 = u + v$, with $u \in \mathcal{I}$ and $v \in \mathcal{J}^m$. Again $1 = 1^n = (u + v)^n$ and expanding we see that $1 \in \mathcal{I}^n + \mathcal{J}^m$, hence $\mathcal{I}^n$ and $\mathcal{J}^m$ are comaximal.

To prove the last assertion, we proceed by induction on h. For $h = 2$ and for all n, by (a) we have $\mathcal{I}_1^n \cap \mathcal{I}_2^n = \mathcal{I}_1^n\mathcal{I}_2^n$ because $\mathcal{I}_1^n$ and $\mathcal{I}_2^n$ are comaximal by (b). On the other hand we have the trivial identity $\mathcal{I}_1^n\mathcal{I}_2^n = (\mathcal{I}_1\mathcal{I}_2)^n$ and finally $\mathcal{I}_1^n \cap \mathcal{I}_2^n = (\mathcal{I}_1\mathcal{I}_2)^n = (\mathcal{I}_1 \cap \mathcal{I}_2)^n$.

Now we assume the assertion is true for a number of ideals smaller than h. We have $\mathcal{I}_1^n\mathcal{J}_1^n = \mathcal{I}_1^n \cap \mathcal{J}_1^n$ because $\mathcal{I}_1^n$ and $\mathcal{J}_1^n$ are comaximal by the hypothesis and by (b). For all $i \in \{2, \ldots, h\}$ we still have that $\mathcal{I}_i$ and $\bigcap_{2 \leqslant j \neq i} \mathcal{I}_j$ are comaximal. So we can apply induction and we have

$$\mathcal{I}_1^n\mathcal{J}_1^n = \mathcal{I}_1^n(\mathcal{I}_2 \cap \cdots \cap \mathcal{I}_h)^n = \mathcal{I}_1^n(\mathcal{I}_2 \cdots \mathcal{I}_h)^n = (\mathcal{I}_1 \cdots \mathcal{I}_h)^n.$$

On the other hand, again by induction, we have

$$\mathcal{I}_1^n\mathcal{J}_1^n = \mathcal{I}_1^n \cap \mathcal{J}_1^n = \mathcal{I}_1^n \cap (\mathcal{I}_2 \cap \ldots \cap \mathcal{I}_h)^n = \mathcal{I}_1^n \cap (\mathcal{I}_2^n \cap \ldots \cap \mathcal{I}_h^n) = \mathcal{I}_1^n \cap \ldots \cap \mathcal{I}_h^n.$$

Finally, by (a) we have

$$\mathcal{I}_1^n\mathcal{J}_1^n = (\mathcal{I}_1\mathcal{J}_1)^n = (\mathcal{I}_1 \cap \mathcal{J}_1)^n = (\mathcal{I}_1 \cap \ldots \cap \mathcal{I}_h)^n.$$

The assertion follows. □

Lemma 16.3.10 *Let $\mathcal{I}$ be an ideal of A_n such that* $Z_a(\mathcal{I}) = \{P_1, \ldots, P_m\}$ *is finite. Set* $\mathfrak{O}_i = \mathcal{O}_{\mathbb{A}^n, P_i}$, *for* $i = 1, \ldots, m$. *Then there is a natural isomorphism*

$$\phi : A_n/\mathcal{I} \to \prod_{i=1}^{m} \mathfrak{O}_i/\mathcal{I}\mathfrak{O}_i.$$

In particular

$$\dim_{\mathbb{K}}(A_n/\mathcal{I}) = \sum_{i=1}^{m} \dim_{\mathbb{K}}(\mathfrak{O}_i/\mathcal{I}\mathfrak{O}_i),$$

and, if $Z_a(\mathcal{I}) = \{P\}$*, then* $A_n/\mathcal{I}$ *is isomorphic to* $\mathcal{O}_{\mathbb{A}^n,P}/\mathcal{I}\mathcal{O}_{\mathbb{A}^n,P}$.

Proof Denote by $\mathfrak{m}_i$ the maximal ideal in A_n corresponding to the point P_i, for $i = 1, \dots, m$. Set $R = A_n/\mathcal{I}$ and $R_i = \mathfrak{O}_i/\mathcal{I}\mathfrak{O}_i$, for $i = 1, \dots, m$. There are natural homomorphisms $\phi_i : R \to R_i$, for $i = 1, \dots, m$, and these induce a homomorphism $\phi : R \to \prod_{i=1}^{m} R_i$.

By the Nullstellensatz, we have $\mathrm{rad}(\mathcal{I}) = \bigcap_{i=1}^{m} \mathfrak{m}_i$. It is immediate that there is a positive integer h such that $(\bigcap_{i=1}^{m} \mathfrak{m}_i)^h = \mathrm{rad}(\mathcal{I})^h \subseteq \mathcal{I}$. Moreover, let $i \in \{1, \dots, m\}$ and consider the ideals $\mathfrak{m}_i$ and $\bigcap_{j \neq i} \mathfrak{m}_j$ which are clearly comaximal. Then, by Lemma 16.3.9, we have

$$\bigcap_{i=1}^{m} \mathfrak{m}_i^h = (\mathfrak{m}_1 \cdots \mathfrak{m}_m)^h = \left(\bigcap_{i=1}^{m} \mathfrak{m}_i\right)^h \subseteq \mathcal{I}.$$

Now we proceed with the proof of the assertion. For each $i = 1, \dots, m$, fix a polynomial $f_i \in A_n$ such that, for all $j = 1, \dots, m$, one has $f_i(P_j) = \delta_{ij}$ (see the proof of Lemma 16.3.8). We set $g_i = 1 - (1 - f_i^h)^h$, for $i = 1, \dots, m$. We have $g_i = f_i^h p_i$, for some suitable polynomial p_i, for $i = 1, \dots, m$. So we have $g_i \in \mathfrak{m}_j^h$ if $1 \leqslant i, j \leqslant m$ and $i \neq j$. Thus, if $1 \leqslant i, j \leqslant m$ and $i \neq j$ we have

$$g_i g_j \in \bigcap_{l=1}^{m} \mathfrak{m}_l^h \subseteq \mathcal{I}.$$

Moreover, for every $j = 1, \dots, m$, we have

$$1 - \sum_{i=1}^{m} g_i = (1 - g_j) - \sum_{i \neq j} g_i \in \bigcap_{l=1}^{m} \mathfrak{m}_l^h \subseteq \mathcal{I}.$$

Furthermore

$$g_i - g_i^2 = g_i(1 - f_i^h)^h \in \left(\bigcap_{i \neq j} \mathfrak{m}_j^h\right) \cdot \mathfrak{m}_i^h \subseteq \mathcal{I}.$$

So if we denote by $\mathfrak{g}_i$ the class of g_i in R, for $i = 1, \dots, m$, we have $\mathfrak{g}_i \mathfrak{g}_j = \delta_{ij} \mathfrak{g}_i$ and $\sum_{i=1}^{m} \mathfrak{g}_i = 1$.

Now we claim that, if $g \in A_n$ is such that $g(P_i) \neq 0$ for some $i = 1, \dots, m$, then there is a $t \in R$ such that $\mathfrak{g}_i = t\mathfrak{g}$, where $\mathfrak{g}$ is the class of g in R.

To prove the claim, assume that $g(P_i) = 1$. Set $q = 1 - g$ so that $q \in \mathfrak{m}_i$. We have

$$\begin{aligned} g_i - g_i q^h &= g_i(1 - q^h) = g_i(1 - q)(1 + q + \cdots + q^{h-1}) = \\ &= (1 - q)(g_i + g_i q + \cdots + g_i q^{h-1}) = g(g_i + g_i q + \cdots + g_i q^{h-1}) \end{aligned}$$

and $q^h g_i \in \bigcap_{l=1}^m \mathfrak{m}_l^h \subseteq \mathcal{I}$. So in R we have the relation

$$\mathfrak{g}_i = \mathfrak{g}(\mathfrak{g}_i + \mathfrak{g}_i \mathfrak{q} + \cdots + \mathfrak{g}_i \mathfrak{q}^{h-1})$$

with $\mathfrak{q}$ the class in R of q. The claim follows by setting $t = \mathfrak{g}_i + \mathfrak{g}_i \mathfrak{q} + \cdots + \mathfrak{g}_i \mathfrak{q}^{h-1}$.

Finally we have to prove that ϕ is injective and surjective. First we prove that ϕ is injective.

Suppose we have a polynomial $f \in A_n$ such that its class $\mathfrak{f}$ in R is such that $\phi(\mathfrak{f}) = 0$. By the definition of localization this means that for any $i = 1, \ldots, m$, there is a polynomial $u_i \in A_n$ such that $u_i(P_i) \neq 0$ and that $u_i f \in \mathcal{I}$, so that $\mathfrak{u}_i \mathfrak{f} = 0$ in R (again, $\mathfrak{u}_i$ is the class of u_i in R). By the above claim, for every $i = 1, \ldots, m$, we can find a $t_i \in R$ such that $\mathfrak{g}_i = t_i \mathfrak{u}_i$. Then we have

$$\mathfrak{f} = \sum_{i=1}^m \mathfrak{g}_i \mathfrak{f} = \sum_{i=1}^m t_i \mathfrak{u}_i \mathfrak{f} = 0$$

as wanted.

Finally, let us prove the ϕ is surjective. Since $g_i(P_i) = 1$ for all $i = 1, \ldots, m$, then $\phi_i(\mathfrak{g}_i)$ is invertible in R_i. As $\phi_i(\mathfrak{g}_i)\phi(\mathfrak{g}_j) = \phi_i(\mathfrak{g}_i \mathfrak{g}_j) = 0$, we have $\phi_i(\mathfrak{g}_j) = 0$, for $i, j = 1, \ldots, m$ with $i \neq j$. Thus

$$\phi_i(\mathfrak{g}_i) = \phi_i\left(\sum_{j=1}^m \mathfrak{g}_j\right) = \phi_i(1) = 1.$$

Now take an element $x = (x_1, \ldots, x_m) \in \prod_{i=1}^m R_i$, so that for all $i = 1, \ldots, m$ we may write $x_i = \frac{\xi_i}{\eta_i}$, where η_i is the class of a polynomial e_i such that $e_i(P_i) \neq 0$. We will denote by $\mathfrak{x}_i$ an element in R such that $\phi_i(\mathfrak{x}_i) = \xi_i$ (this is possible because ϕ_i is clearly surjective for all $i = 1, \ldots, m$). By the claim, for all $i = 1, \ldots, m$, we can find a $t_i \in R$ such that $\mathfrak{g}_i = t_i \mathfrak{e}_i$, where $\mathfrak{e}_i$ is the class of e_i in R. We have

$$\phi_i(t_i)\eta_i = \phi_i(t_i \mathfrak{e}_i) = \phi_i(\mathfrak{g}_i) = 1, \quad \text{for} \quad i = 1, \ldots, m.$$

Then we have

$$x_i = \frac{\xi_i}{\eta_i} = \xi_i \phi_i(t_i) = \phi_i(t_i \mathfrak{x}_i), \quad \text{for} \quad i = 1, \ldots, m.$$

and therefore

$$\phi_i\left(\sum_{j=1}^{m} t_j \mathfrak{x}_j \mathfrak{g}_j\right) = \phi_i(t_i \mathfrak{x}_i) = \frac{\xi_i}{\eta_i} = x_i, \quad \text{for} \quad i = 1, \ldots, m.$$

Hence $\phi(\sum_{j=1}^{m} t_j \mathfrak{x}_j \mathfrak{g}_j) = x$, as desired. □

Lemma 16.3.11 *Let $V \subseteq \mathbb{A}^n$ be an affine variety, let P be a point of V and let $\mathcal{J} \subseteq A_n$ be an ideal containing $\mathcal{I}_a(V)$. Let $\mathcal{J}' \subseteq A(V)$ be the image of $\mathcal{J}$. There is a natural homomorphism*

$$\phi : \mathcal{O}_{\mathbb{A}^n,P}/\mathcal{J}\mathcal{O}_{\mathbb{A}^n,P} \to \mathcal{O}_{V,P}/\mathcal{J}'\mathcal{O}_{V,P}$$

which is an isomorphism.

In particular $\mathcal{O}_{\mathbb{A}^n,P}/\mathcal{I}_a(V)\mathcal{O}_{\mathbb{A}^n,P}$ is isomorphic to $\mathcal{O}_{V,P}$.

Proof The map ϕ sends the class $\bar{f}$ of a function f in $\mathcal{O}_{\mathbb{A}^n,P}/\mathcal{J}\mathcal{O}_{\mathbb{A}^n,P}$ in the class of the same function in $\mathcal{O}_{X,P}/\mathcal{J}'\mathcal{O}_{X,P}$. The map is easily seen to be well defined, and to be a surjective homomorphism. Let us prove that it is injective. Indeed, if $\phi(\bar{f}) = 0$, this means that the class of f belongs to $\mathcal{J}'\mathcal{O}_{X,P}$, and this implies that the class of f belongs to $\mathcal{J}\mathcal{O}_{\mathbb{A}^n,P}$, hence $\bar{f} = 0$. □

We are now ready to prove the:

Theorem 16.3.12 *Let X, Y be two affine plane curves with respective equations $f(x, y) = 0$ and $g(x, y) = 0$ and let $P \in \mathbb{A}^2$ be a point. Then*

$$i(P; X, Y) = \dim_{\mathbb{K}}(\mathcal{O}_{\mathbb{A}^2,P}/(f, g)) \tag{16.13}$$

where we abuse notation and denote by f and g their classes in $\mathcal{O}_{\mathbb{A}^2,P}$.

Proof The proof consists of two main steps. In the first step we prove uniqueness of intersection multiplicity, i.e., that however given plane curves X, Y and a point $P \in \mathbb{A}^2$, there is a unique way to define $i(P; X, Y)$ so that properties (a)–(g) listed above are verified. The second step consists in proving that if one defines $i(P; X, Y)$ as in (16.13), then properties (a)–(g) are verified.

We proceed with step 1. Suppose that, for every plane curves X, Y and a point $P \in \mathbb{A}^2$, we have the definition of a number $i(P; X, Y)$ so that properties (a)–(g) are verified. We will see that $i(P; X, Y)$ is uniquely determined. First of all by property (c) we may assume that P is the origin. If X and Y contain a component through P, then $i(P; X, Y) = \infty$ by (a). So we may assume that P sits in no common component of X and Y. Then, still by (a), we have $i(P; X, Y) = 0$ if and only if $P \notin X \cap Y$. We argue by induction and suppose we uniquely determine the case in which $i(P; X, Y) < n$, for $n \in \mathbb{N}$, and we prove that we can uniquely determine when $i(P; X, Y) = n$. Consider the polynomials $f(x, 0), g(x, 0) \in \mathbb{K}[x]$ and assume their respective degrees are r and s, where r or s are assumed to be 0 if the polynomial vanishes. We may suppose that $r \leqslant s$ by (d).

Case 1: $r = 0$. Then y divides f, so we may write $f = yh$. By (f) we have

$$n = i(P; f, g) = i(P; y, g) + i(P; h, g).$$

Note that $g(x, 0)$ cannot be identically 0, otherwise g is also divisible by y, and X and Y have a common component, i.e., the x axis, passing through P. So we can write

$$g(x, 0) = x^l(a_0 + a_1x + \cdots), \quad \text{with} \quad a_0 \neq 0 \quad \text{and} \quad l > 0.$$

Then

$$i(P; y, g) = i(P; y, g(x, 0)) = l$$

by (b), (e), (f) and (g). Then $i(P; h, g) = i(P; f, g) - i(P; y, g) = n - l < n$, and by induction we can uniquely define $i(P; h, g)$, so we can uniquely define $i(P; f, g)$.
Case 2: $r > 0$. Multiply f and g by constants so to make $f(x, 0)$ and $g(x, 0)$ monic. Let $h = g - x^{s-r}f$. Then

$$i(P; f, g) = i(P; f, h)$$

by (g). Moreover $\deg(h(x, 0)) := t < s$. We can repeat this process finitely many times, perhaps interchanging the role of f and g if $t < r$, so that we end up with two polynomials $v(x, y)$, $w(x, y)$ such that $i(P; v, w) = i(P; f, g)$ and v, w fall in Case 1. This ends step 1.

Next we go to step 2. It is clear that (b), (c), (d) and (g) are satisfied. We may again assume that P is the origin and that all components of X and Y pass through P. To ease notation, we set $\mathfrak{O} = \mathcal{O}_{\mathbb{A}^2,P}$.

If X and Y have no common irreducible component Z, then by Lemma 16.3.10, $\mathcal{O}_{\mathbb{A}^2,P}/(f, g)$ is a finitely generated $\mathbb{K}$–vector space. If X and Y have a common component, then f and g have a non–constant irreducible factor h such that $Z = Z_a(h)$, so $(f, g) \subset (h)$. Hence there is a surjective homomorphism $\mathfrak{O}/(f, g) \to \mathfrak{O}/(h)$. We show that $\mathfrak{O}/(h)$ is infinite dimensional over $\mathbb{K}$, which implies that also $\mathfrak{O}/(f, g)$ is infinite dimensional over $\mathbb{K}$. By Lemma 16.3.11, $\mathfrak{O}/(h)$ is isomorphic to $\mathcal{O}_{Z,P}$ and $A(Z) \subseteq \mathcal{O}_{Z,P}$. By Lemma 16.3.8, $A(Z)$ is infinite dimensional over $\mathbb{K}$, as wanted. This proves (a).

To prove (f) it suffices to prove that, given polynomials f, g, h, we have

$$\dim_{\mathbb{K}}(\mathfrak{O}/(f, gh)) = \dim_{\mathbb{K}}(\mathfrak{O}/(f, g)) + \dim_{\mathbb{K}}(\mathfrak{O}/(f, h)). \tag{16.14}$$

We may assume that f and gh have no common non–constant factor, because otherwise (16.14) is trivially true. We have a natural surjective homomorphism

$$\phi : \mathfrak{O}/(f, gh) \to \mathfrak{O}/(f, g).$$

Then we define the $\mathbb{K}$–linear map

$$\psi : \mathfrak{O}/(f, h) \to \mathfrak{O}/(f, gh)$$

in the following way: given $t \in \mathfrak{O}$, we set $\psi(\bar{t}) = \overline{tg}$, where the bar denotes the class modulo the appropriate ideals. We claim that ψ is injective and that $\mathrm{im}(\psi) = \ker(\phi)$. This will imply (16.14).

Let us prove the claim. The proof that $\mathrm{im}(\psi) = \ker(\phi)$ is trivial. So we focus on proving the injectivity of ψ. We keep the above notation and suppose that $\psi(\bar{t}) = 0$, i.e., $\overline{tg} = 0$. This means that $tg = uf + vgh$, where $u, v \in \mathfrak{O}$. Fix a polynomial $w \in A_2$ such that $w(P) \neq 0$, and set $a = wu$, $b = wv$, $c = wt$, which can be considered as polynomials in A_2. Then we have $g(c - bh) = af$ in A_2. Since f and g have no common factor, f divides $c - bh$, hence we have a relation of the sort $c - bh = df$. Since w is invertible in $\mathfrak{O}$, we have

$$t = \frac{c}{w} = h\frac{b}{w} + \frac{d}{w}f$$

so that $\bar{t} = 0$ as wanted.

Finally we prove that property (e) holds. We set $m = m_P(X)$, $n = m_P(Y)$. Let $\mathfrak{m} = (x, y)$ be the maximal ideal in A_2 corresponding to P which is the origin. Consider the following linear maps of $\mathbb{K}$–vector spaces

$$A_2/\mathfrak{m}^n \times A_2/\mathfrak{m}^m \xrightarrow{\lambda} A_2/\mathfrak{m}^{m+n} \xrightarrow{\mu} A_2/(\mathfrak{m}^{m+n}, f, g)$$

$$\mathfrak{O}/(f, g) \xrightarrow{\pi} \mathfrak{O}/(\mathfrak{m}^{m+n}, f, g)$$

and

$$A_2/(\mathfrak{m}^{m+n}, f, g) \xrightarrow{\alpha} \mathfrak{O}/(\mathfrak{m}^{m+n}, f, g)$$

where μ, π and α are the natural ring homomorphisms, and λ is defined by setting $\lambda(\bar{a}, \bar{b}) = \overline{af + bg}$, where a, b are polynomials in A_2 and the bar denotes as usual the class modulo the appropriate ideal. Note that μ and π are clearly surjective, and α is an isomorphism by Lemma 16.3.10. It is moreover clear that $\mathrm{im}(\lambda) = \ker(\mu)$. Then we have

$$\dim_{\mathbb{K}}(A_2/\mathfrak{m}^m) + \dim_{\mathbb{K}}(A_2/\mathfrak{m}^n) \geqslant \dim(\ker(\mu))$$

with equality holding if and only if λ is injective. Moreover

$$\dim_{\mathbb{K}}(A_2/(\mathfrak{m}^{m+n}, f, g)) = \dim_{\mathbb{K}}(A_2/\mathfrak{m}^{m+n}) - \dim(\ker(\mu)).$$

Hence we get

$$\begin{aligned}\dim_{\mathbb{K}}(\mathfrak{O}/(f,g)) &\geqslant \dim_{\mathbb{K}}(\mathfrak{O}/(\mathfrak{m}^{m+n},f,g)) = \\ &= \dim_{\mathbb{K}}(A_2/(\mathfrak{m}^{m+n},f,g)) \geqslant \\ &\geqslant \dim_{\mathbb{K}}(A_2/\mathfrak{m}^{m+n}) - \dim_{\mathbb{K}}(A_2/\mathfrak{m}^{m}) - \dim_{\mathbb{K}}(A_2/\mathfrak{m}^{n}) = nm.\end{aligned} \tag{16.15}$$

Indeed, for all positive integers h one has $\dim_{\mathbb{K}}(A_2/\mathfrak{m}^h) = \frac{h(h+1)}{2}$. In fact for all positive integers h we have a surjective homomorphism

$$r_h : f \in A_2 \to f_0 + \cdots + f_{h-1} \in A_{2,h-1}$$

where $A_{2,h-1}$ is the vector space of polynomials in A_2 of degree at most $h-1$, and $f = f_0 + f_1 + \cdots$ is the decomposition in homogeneous components. One has $\ker(r_h) = \mathfrak{m}^h$, hence $A_2/\mathfrak{m}^h$ is isomorphic to $A_{2,h-1}$ whose dimension is $\frac{h(h+1)}{2}$.

This proves the first part of property (e). One has $\dim_{\mathbb{K}}(\mathfrak{O}/(f,g)) = nm$ if and only if both inequalities in (16.15) are equalities. The first inequality is an equality if and only if π is an isomorphism, i.e., if and only if $\mathfrak{m}^{n+m}\mathfrak{O} \subseteq (f,g)\mathfrak{O}$. The second is an equality if and only if λ is injective. We finish by proving the following:

Claim:

(i) if X and Y have no common principal tangent lines, then $\mathfrak{m}^s\mathfrak{O} \subseteq (f,g)\mathfrak{O}$ for all $s \geqslant n+m-1$;
(ii) λ is injective if and only if X and Y have distinct principal tangent lines.

Proof of (i). First of all we prove that if $t \gg 0$, then $\mathfrak{m}^t\mathfrak{O} \subseteq (f,g)\mathfrak{O}$. This is a consequence of Hilbert's Nullstellensatz. In fact set $Z_a(f,g) = \{P, Q_1, \ldots, Q_l\}$. Let us choose a polynomial h such that $h(P) \neq 0$ and $h(Q_i) = 0$ for $i = 1, \ldots, l$. Then xh and yh are in $\mathcal{I}_a(Z_a(f,g))$, so there is a positive integer r such that $(xh)^r, (yh)^r \in (f,g) \subset A_2$. Since h is invertible in $\mathfrak{O}$, then x^r, y^r are in $(f,g)\mathfrak{O}$, and this implies that $\mathfrak{m}^{2r}\mathfrak{O} \subseteq (f,g)\mathfrak{O}$.

Next we let $r_1, \ldots, r_m$ be equations of the principal tangents to X at P and $\ell_1, \ldots, \ell_n$ be equations of the principal tangents to Y at P (in $r_1, \ldots, r_m$ and $\ell_1, \ldots, \ell_n$ there could be repetitions). We set $r_0 = \ell_0 = 1$. Then we define r_i for all $i > m$ by setting $r_i = r_m$ and similarly $\ell_j = \ell_n$ for all $j > n$. Then for all $i, j \geqslant 0$ we set $s_{ij} = r_1 \cdots r_i \ell_1 \cdots \ell_j$.

We claim that the set $\Sigma_d = \{s_{ij} : i + j = d\}$ is a basis for the vector space $S_{1,d}$ of dimension $d+1$ of all homogeneous polynomials of degree d in x, y. Since Σ_d consists of $d+1$ elements it suffices to show that the elements of Σ_d are independent. Suppose we have a relation of the form

$$a_0 s_{0,d} + a_1 s_{1,d-1} + \cdots + a_d s_{d,0} = 0 \quad \text{with} \quad a_0, \ldots, a_d \in \mathbb{K}.$$

Since r_1 appears as a factor of $s_{1,d-1}, \ldots, s_{d,0}$, it has to divide $a_0 s_{0,d}$. But since it does not appear as a factor in $s_{0,d}$, then $a_0 = 0$. So the above relation reduces to

$$a_1 s_{1,d-1} + \cdots + a_d s_{d,0} = 0$$

and since r_1 appears a factor of $s_{1,d-1}, \dots, s_{d,0}$, we have

$$a_1 \frac{s_{1,d-1}}{r_1} + \cdots + a_d \frac{s_{d,0}}{r_1} = 0.$$

Now we have that r_2 appears as a factor in $\frac{s_{2,d-1}}{r_1}, \dots, \frac{s_{d,0}}{r_1}$, but not in $\frac{s_{1,d-1}}{r_1}$. By arguing as above, this implies that $a_1 = 0$. Going on in this way we have $a_0 = \cdots = a_d = 0$, which proves the independence of the elements of Σ_d.

Going back to the proof of (i), it suffices to prove that $s_{ij} \in (f, g)\mathfrak{O}$ (we abuse notation here and denote by s_{ij} also its class in $\mathfrak{O}$), as soon as $t = i + j \geqslant n + m - 1$. We do this by descending induction on t, given the fact that, as we saw, the assertion is true for $t \gg 0$. So we assume that the assertion is true for $t + \epsilon$, for all $\epsilon \in \mathbb{N} \setminus \{0\}$ and prove it for t.

Note that $t = i + j \geqslant n + m - 1$ implies that either $i \geqslant m$ or $j \geqslant n$. Suppose that $i \geqslant m$ (the argument is the same otherwise). Then $s_{ij} = s_{m0}a$, where a is a homogeneous polynomial of degree $i + j - m$. Note that s_{m0} can be assumed to be equal to the homogeneous component of minimal degree of f, hence $f = s_{m0} + f^*$, where all terms of f^* have degree at least $m + 1$. Then $s_{ij} = a(f - f^*) = af - af^*$, where each term of af^* has degree at least $(m + 1) + (i + j - m) = i + j + 1 = t + 1$. By induction we have that the class of af^* is in $(f, g)\mathfrak{O}$, hence $s_{ij} \in (f, g)\mathfrak{O}$ as wanted for the proof of (i).

Finally we prove (ii). Suppose that $\lambda(\bar{a}, \bar{b}) = \overline{af + bg} = 0$. This means that $af + bg$ has only terms of degree at least $n + m$. Write a and b as the sum of their homogeneous components

$$a = a_r + a_{r+1} + \cdots, \quad b = b_s + b_{r+1} + \cdots, \quad \text{with} \quad a_r, b_s \neq 0.$$

We want to prove that $r \geqslant n$ and $s \geqslant m$, because this implies that $(\bar{a}, \bar{b}) = (0, 0)$ as wanted for the injectivity of λ. Suppose, to fix the ideas, that $r < n$ (otherwise the argument is similar). We have

$$af + bg = a_r f_m + b_s g_n + \cdots$$

where $\cdots$ stay as usual for higher order terms. Since $af + bg$ has only terms of degree at least $n + m$, and $a_r f_m$ has degree $r + m < n + m$, we have $r + m = s + n$ and $a_r f_m = -b_s g_n$. But f_m and g_n have no common factor, so g_n has to divide a_r, which is impossible because $r < n$. So $r \geqslant n$ and $s \geqslant m$ as wanted.

Conversely, suppose there is a common principal tangent $r = 0$ to X and Y at P. Then we may write $f_m = rf', g_n = rg'$. Then $(\bar{g}', -\bar{f}')$ is non–zero and $\lambda(\bar{g}', -\bar{f}') = 0$, so λ is not injective. This ends the proof of (ii) and the proof of the theorem. □

To state the next result, we first give a definition. Let $(A, \mathfrak{m})$ be a DVR, so that there is a discrete valuation v defined on $\mathbb{Q}(A)$ and A is the valuation ring of v. Given $g \in A$, one has $v(g) = n \in \mathbb{N}$. We define n to be the *order* of g in A, and we write

$n = o_A(g)$, or simply $n = o(g)$ if there is no danger of confusion. Remember that $\mathfrak{m} = (u)$. By looking at the proof of Theorem 14.2.11, we see that $n = o(g)$ is the unique positive integer such that $g = wu^n$ with $w \notin \mathfrak{m}$.

If X is an affine plane curve and $P \in X$ is a smooth point for X, then $\mathcal{O}_{X,P}$ is a DVR (see again Theorem 14.2.11). If $g \in \mathcal{O}_{X,P}$, we set

$$o_{X,P}(g) := o_{\mathcal{O}_{X,P}}(g).$$

If $g \in \mathbb{K}[x, y]$, then we will abuse notation and denote by g its class in $\mathcal{O}_{X,P}$.

Proposition 16.3.13 *Let X be an irreducible affine plane curve with equation $f(x, y) = 0$ and P a smooth point of X. Let Y be a curve with equation $g(x, y) = 0$. Then*

$$i(P; X, Y) = o_{X,P}(g).$$

Proof By Exercise 16.3.29 one has $o_{X,P}(g) = \dim_{\mathbb{K}}(\mathcal{O}_{X,P}/(g))$. On the other hand we claim that

$$\mathcal{O}_{X,P}/(g) = \mathcal{O}_{\mathbb{A}^2,P}/(f, g) \tag{16.16}$$

whence the assertion follows by Theorem 16.3.12. To prove (16.16), we first may assume that P is the origin, then we note that

$$\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}$$

where $\mathfrak{m}_P$ is the maximal ideal of $A(X)$ corresponding to P. One has $A(X) = \mathbb{K}[x, y]/(f)$ and $\mathfrak{m}_P = (x, y)$. So

$$\mathcal{O}_{X,P} = (\mathbb{K}[x, y]/(f))_{(x,y)}$$

and it is easy to check that

$$(\mathbb{K}[x, y]/(f))_{(x,y)} = (\mathbb{K}[x, y]_{(x,y)})/(f) = \mathcal{O}_{\mathbb{A}^2,P}/(f).$$

The assertion follows. □

Exercise 16.3.14 Consider the two affine plane curves X and Y with equations

$$f(x, y) = x^3 + y^2 - 2xy = 0, \quad g(x, y) = x^2 - x^2 y + y^3 = 0.$$

Find the intersection multiplicity of X and Y at the origin and verify that it is equal to $\sum_{\gamma'} o_\gamma(f) = \sum_\gamma o_\gamma(g)$, where γ' [resp. γ] runs through all branches of Y [resp. X] at the origin.

Exercise 16.3.15 Consider the two affine plane curves X and Y with equations

$$f(x, y) = x^3 + y^3 - 2xy = 0, \quad g(x, y) = 2x^3 - 4x^2 y + 3xy^2 + y^3 - 2y^2 = 0.$$

Find the intersections of X and Y and the corresponding intersection multiplicities. Find the branches at the origin of X and Y.

Exercise 16.3.16 * Let X and Y be two affine plane curves and $P \in \mathbb{A}^2$ a point. Let $X_1, \ldots, X_h$ [resp. $Y_1, \ldots, Y_k$] be the components of X of respective multiplicities $n_1, \ldots, n_h$ [resp. of Y of respective multiplicities $m_1, \ldots, m_k$] passing through P. Prove that

$$i(P; X, Y) = \sum_{i=1,\ldots,h, j=1,\ldots,k} n_i m_j i(P; X_i, Y_j).$$

Exercise 16.3.17 Consider two affine plane curves X and Y of degrees n and m respectively, with no common factor. Prove that the sum of the orders of X at all branches of Y is at most nm.

Exercise 16.3.18 Consider the affine plane curve X with equation

$$x^n + ax - y = 0$$

with $a \in \mathbb{K}$ and $n \geqslant 2$. Prove that X is irreducible and smooth, passing through the origin and compute the intersection multiplicity of X with its tangent line at the origin.

Exercise 16.3.19 Consider the affine plane curve X with equation

$$y = a_0 + a_1 x + \cdots + a_n x^n$$

with $a_0, \ldots, a_n \in \mathbb{K}$, $a_n \neq 0$ and $n \geqslant 2$. Prove that X is irreducible and smooth, passing through the point $P = (0, a_0)$, and compute the intersection multiplicity of X with its tangent line at P.

Exercise 16.3.20 Consider the affine plane curve X with equation

$$(x^2 + y^2)^2 - 4(x^2 + y^2)(15x^2 + 11y^2) + 36(25x^2 + 13y^2) = 0.$$

Determine its tangent lines at its intersection points with the y axis.

Exercise 16.3.21 A smooth point P of a curve X is said to be a *flex* if the linear branch of which it is the centre has class $n \geqslant 2$, in which case it is called, more precisely, an $(n-1)$*–flex* (if $n = 2$ it is called a *simple flex*).

Determine the flexes of the curve with equation

$$x^3 - a(x^2 - y^2) = 0$$

with $a \in \mathbb{K}^*$.

Exercise 16.3.22 Consider the rational map

$$\phi : t \in \mathbb{A}^1 \dashrightarrow \left(\frac{t}{1+t^3}, \frac{t^2}{1+t^3}\right) \in \mathbb{A}^2.$$

Prove that ϕ is dominant on a curve X of degree 3. Determine the singular points of X, the principal tangents there, and the flexes of X.

Exercise 16.3.23 Prove that the class of a branch γ is ∞ if and only if γ is a branch of a line.

Exercise 16.3.24 The branches of order 2 are called *quadratic cusps*. A quadratic cusp with class 1 is called an *ordinary cusp*, a quadratic cusp with class 2 is called a *ramphoid cusp*, a quadratic cusp of class $n \geqslant 1$ is called an *n–cusp*. Write down standard parametrizations of an n–cusp with centre the origin.

Exercise 16.3.25 * Let P be an ordinary n–tuple point for the curve X. Prove that P is the centre of exactly n linear branches for X.

Exercise 16.3.26 * Let X and Y be two reduced affine plane curves with no common components. Suppose P is an ordinary point of multiplicity n for X, centre of the branches $\gamma_1, \ldots, \gamma_n$ and there is some positive integer $m \leqslant n$ such that $o_{\gamma_i}(Y) \geqslant m$, for $i = 1, \ldots, n$. Prove that P is a point of multiplicity at least m for Y.

Exercise 16.3.27 * Let γ, γ' be two branches. We define the *intersection multiplicity* of γ and γ', which will be denoted by $i(\gamma, \gamma')$, in the following way.

If γ, γ' have distinct centres, we define $i(\gamma, \gamma') = 0$, If $\gamma = \gamma'$, we define $i(\gamma, \gamma') = \infty$. If γ, γ' have the same centre but they are different, we proceed as follows. First we may assume that their common centre is the origin. Then we may assume that they are given by parametrizations

$$x = t^n, \quad y = a_1 t + a_2 t^2 + \cdots \quad \text{and} \quad x = t^m, \quad y = b_1 t + b_2 t^2 + \cdots .$$

Consider the corresponding fractional power series

$$\xi_i = \sum_{j=1}^{\infty} a_j \varepsilon_i^j x^{\frac{j}{n}}, \quad \text{with} \quad \varepsilon_i^n = 1, \quad i = 1, \ldots, n$$

and

$$\eta_l = \sum_{j=1}^{\infty} b_j \eta_l^j x^{\frac{j}{m}}, \quad \text{with} \quad \eta_l^m = 1, \quad l = 1, \ldots, m.$$

Then we define $i(\gamma, \gamma') = o(\prod_{i=1,\ldots,n, l=1,\ldots,m}(\xi_i - \eta_l))$.

Prove the following facts:

(a) if γ, γ' have the same centre, then $i(\gamma, \gamma') \geqslant o(\gamma)o(\gamma')$ and the equality holds if and only if the two branches have distinct tangent lines;
(b) if X is an affine curve and γ is a branch, then $o_\gamma(X)$ equals the sum of $i(\gamma, \gamma')$, with γ' varying among all branches of X;
(c) if X and Y are two reduced affine curves with no common components, and if $P \in X \cap Y$, then $i(P; X, Y)$ is the sum of $i(\gamma, \gamma')$ with γ [resp. γ'] varying among all branches of X [resp. of Y] with centre P.

Exercise 16.3.28 Let P be a double point for a curve X. Prove that either P is the centre of a unique quadratic cuspidal branch of order $n \geqslant 1$, or P is the centre of exactly two linear branches which have intersection multiplicity $n \geqslant 1$. In the latter case, when $n = 1$, we have a node. If $n > 1$ we say that the point is a *n–tacnode*. If $n = 2$ one simply says it is a *tacnode*.

Write down the equation of an irreducible curve with a tacnode at the origin.

Exercise 16.3.29 * Let $(A, \mathfrak{m})$ be a DVR and take $g \in A$. Prove that $o_A(g) = \dim_{A/\mathfrak{m}}(A/(g))$.

Exercise 16.3.30 * Let X, Y be two affine plane curves with equations $f(x, y) = 0$ and $g(x, y) = 0$ respectively, that have no common components. Prove that

$$\sum_{P \in X \cap Y} i(P; X, Y) = \dim(A_2/(f, g)).$$

16.4 Solutions of Some Exercises

16.3.16 Let $f(x, y) = 0$ be the equation of X and let $f_i(x, y) = 0$ be the equation of X_i for $i = 1, \ldots, h$. For each $j = 1, \ldots, k$ consider the branches $\gamma_{j,1}, \ldots, \gamma_{j,k_j}$ of Y_j with centre P.

Taking into account Exercise 15.3.9, we see that the branches of Y with centre P are γ_{jl}, with $j = 1, \dots, k$ and $l = 1, \dots, k_j$, each with multiplicity m_j. By Theorem 16.3.3 we have

$$
\begin{aligned}
i(P; X, Y) &= \sum_{j=1,\dots,k, l=1,\dots,k_i} m_j o_{\gamma_{jl}}(f) = \\
&= \sum_{j=1,\dots,k, l=1,\dots,k_i} m_j o_{\gamma_{jl}}(f_1^{n_1} \cdots f_h^{n_h}) = \\
&= \sum_{j=1,\dots,k, l=1,\dots,k_i, i=1,\dots,h} n_i m_j o_{\gamma_{jl}}(f_i) = \\
&= \sum_{i=1,\dots,h, j=1,\dots,k} n_i m_j \Big(\sum_{l=1,\dots,k_i} o_{\gamma_{jl}}(f_i) \Big) = \\
&= \sum_{i=1,\dots,h, j=1,\dots,k} n_i m_j i(P; X_i, Y_j)
\end{aligned}
$$

as wanted.

16.3.26 Suppose by contradiction that P has multiplicity $\mu < m$ for Y. By Proposition 16.3.6, the distinct tangent lines to $\gamma_1, \dots, \gamma_n$ at P must be also among the principal tangent lines to Y at P. This is a contradiction because Y, having multiplicity $\mu < m$, can have at most μ principal tangent lines at P.

16.3.30 Apply Lemma 16.3.10.

Chapter 17
Projective Plane Curves

17.1 Some Generalities

17.1.1 Recalling Some Basic Definitions

Let $X \subset \mathbb{P}^2$ be an effective divisor of degree d, with equation $f(x_0, x_1, x_2) = 0$, where f is a homogeneous polynomial of degree d, which we will call a *projective plane curve* or simply a *curve*. Recall from Sect. 1.6.5 that if we have the decomposition in distinct irreducible components

$$f = f_1^{h_1} \cdots f_n^{h_n}$$

then the curves $X_i = Z_p(f_i)$, $i = 1, \ldots, n$, are called the *irreducible components* of X and one writes $X = \sum_{i=1}^n h_i X_i$, where h_i is called the *multiplicity* of X_i in X, for $i = 1, \ldots, n$. Recall form Exercises 14.1.10 and 14.1.11 that if P is a point of X we defined the *multiplicity* $m_P(X)$ of P for X. We defined also the tangent cone $TC_{X,P}$ of X at P. This is the union of $m = m_P(X)$ lines through P, each counted with a certain multiplicity, that are called the *principal tangent lines* to X at P.

If $P \in U_0 \cong \mathbb{A}^2$, then the affine equation of $X_0 = X \cap U_0$ in U_0 is

$$\phi(x, y) = f(1, x, y) = 0,$$

and $m_P(X_0) = m_P(X)$. Moreover the projective closure of the tangent cone to X_0 at P (see Exercise 14.1.12) coincides with the tangent cone $TC_{X,P}$ of X at P.

17.1.2 The Bézout Theorem

Recall the Bezout Theorem, that says that if X, Y are curves in $\mathbb{P}^2$ with no common components, of degrees n and m, then

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https://doi.org/10.1007/978-3-030-71021-7_17

$$nm = \sum_{P \in X \cap Y} i(P; X, Y).$$

As for the notion of intersection multiplicity of two curves X, Y at a point P, we may always assume that $P \in U_0 \cong \mathbb{A}^2$, consider the affine curves $X_0 = X \cap U_0$, $Y_0 = Y \cap U_0$, and then

$$i(P; X, Y) = i(P; X_0, Y_0)$$

and we can carry over to $i(P; X, Y)$ all the considerations we made in the affine case in Sect. 16.3. As immediate consequences of the properties of intersection multiplicity, we have the following:

Lemma 17.1.1 *Let X, Y be projective plane curves of degrees n, m. Then:*

(a) if X, Y have no common components, one has

$$nm \geqslant \sum_{P \in X \cap Y} m_P(X) m_P(Y);$$

(b) if X and Y intersect exactly in nm points, then these points are smooth for both X and Y and X and Y intersect transversely *there, i.e., they have no common tangent line at those points;*

(c) if X and Y have more than nm points in common, then they have a common component.

17.1.3 Linear Systems

Recall from Sect. 1.6.5 the general notion of linear system, which applies to linear systems of projective plane curves of degree d. Such a linear system of dimension r is a projective subspace of dimension r of $\mathcal{L}_{2,d}$. Note that

$$\dim(\mathcal{L}_{2,d}) = \frac{d(d+3)}{2}.$$

Projectivities send linear systems of plane curves of degree d to linear systems of the same dimension of plane curves of the same degree d. Recall from Exercise 1.6.31 that, if P is any point of $\mathbb{P}^2$, the set $\mathcal{L}_{2,d}(-P)$ of all plane curves of degree d containing P is a linear system of codimension 1 in $\mathcal{L}_{2,d}$ (the notation adopted here for $\mathcal{L}_{2,d}(-P)$ is slightly different from the one introduced on Sect. 11.4). We also recall from Exercise 1.6.32, that if $P_1, \ldots, P_h \in \mathbb{P}^2$ are distinct points and $\mathcal{L}_{2,d}(-P_1 - \cdots - P_h)$ is the set of all plane curves of degree d containing $P_1, \ldots, P_h$, then $\mathcal{L}_{2,d}(-P_1 - \cdots - P_h)$ is a linear system of codimension at most h, and that for any

$h \in \mathbb{N}$ there are points $P_1, \dots, P_h$ such that

$$\dim(\mathcal{L}_{2,d}(-P_1 - \dots - P_h)) = \max\left\{-1, \frac{d(d+3)}{2} - h\right\}$$

(recall that dimension -1 means empty system).

Now we want to extend these results. Fix a point P of $\mathbb{P}^2$, fix a positive integer m and consider the set $\mathcal{L}_{2,d}(-mP)$ consisting of all curves of degree d in $\mathbb{P}^2$ having in P multiplicity at least m. Of course $\mathcal{L}_{2,d}(-mP)$ is empty if $m > d$.

Lemma 17.1.2 *In the above setting, if $m \leqslant d$ then $\mathcal{L}_{2,d}(-mP)$ is a linear system of dimension*

$$\dim(\mathcal{L}_{2,d}(-mP)) = \frac{d(d+3)}{2} - \frac{m(m+1)}{2}.$$

Proof Acting with a projectivity we may assume that $P = [1, 0, 0]$. Then passing to non-homogeneous coordinates the equation of an element of $\mathcal{L}_{2,d}(-mP)$ is of the form $\phi(x, y) = 0$, where ϕ has homogeneous decomposition of the form

$$\phi(x, y) = \phi_m(x, y) + \dots + \phi_d(x, y)$$

with $\phi_i(x, y)$ homogeneous of degree i, for $i = m, \dots, d$. Namely, all the homogeneous components of ϕ of degree $i < m$ vanish. The linear combination of two such polynomials is of the same type, so this proves that $\mathcal{L}_{2,d}(-mP)$ is a linear system. To prove the dimension statement, notice that for any $i = 0, \dots, m-1$, the polynomial ϕ_i has $i + 1$ coefficients. Therefore to be in $\mathcal{L}_{2,d}(-mP)$ is equivalent to the vanishing of

$$1 + 2 + \dots + m = \frac{m(m+1)}{2}$$

distinct coefficients of ϕ. These coefficients can be interpreted as distinct homogeneous coordinates in $\mathcal{L}_{2,d}$, and this proves the assertion. □

Let now $P_1, \dots, P_h \in \mathbb{P}^2$ be distinct points, and let $m_1, \dots, m_h$ be positive integers. We denote by $\mathcal{L}_{2,d}(-m_1P_1 - \dots - m_hP_h)$ the set of all curves of degree d having at $P_1, \dots, P_h$ points of multiplicity at least $m_1, \dots, m_h$ respectively. With the same argument as in the proof of Lemma 17.1.2 one proves that $\mathcal{L}_{2,d}(-m_1P_1 - \dots - m_hP_h)$ is a linear system and it is easy to see (we leave it as a simple exercise for the reader) that

$$\dim(\mathcal{L}_{2,d}(-m_1P_1 - \dots - m_hP_h)) \geqslant \frac{d(d+3)}{2} - \sum_{i=1}^{h} \frac{m_i(m_i+1)}{2}. \tag{17.1}$$

We will need the following:

Lemma 17.1.3 *Let X be an irreducible curve of degree n in $\mathbb{P}^2$. One has*

$$\frac{(n-1)(n-2)}{2} \geqslant \sum_{P\in X} \frac{m_P(X)(m_P(X)-1)}{2}.$$

Proof By Exercise 17.3.8 we have

$$r := \frac{(n-1)(n+2)}{2} - \sum_{P\in X} \frac{m_P(X)(m_P(X)-1)}{2} > \\ > \frac{n(n-1)}{2} - \sum_{P\in X} \frac{m_P(X)(m_P(X)-1)}{2} \geqslant 0.$$

Then there is a curve Y of degree $n-1$ such that in each singular point P of X has multiplicity at least $m_P(X)-1$ and moreover it passes through $Q_1, \ldots, Q_r$ further distinct points of X. Since X is irreducible and Y has degree one less than the degree of X, then X and Y have no common component, hence by the Bézout Theorem applied to X and Y we have

$$n(n-1) \geqslant \sum_{P\in X} m_P(X)(m_P(X)-1) + r \geqslant \\ \geqslant \sum_{P\in X} \frac{m_P(X)(m_P(X)-1)}{2} + \frac{(n-1)(n+2)}{2}$$

whence the assertion follows. □

Exercise 17.1.4 * Let $P_1, \ldots, P_h \in \mathbb{P}^2$ be distinct points, and let $m_1, \ldots, m_h$ be positive integers. Assume that

$$d \geqslant \left(\sum_{i=1}^{h} m_i\right) - 1.$$

Then equality holds in (17.1).

17.2 M. Noether's $Af + Bg$ Theorem

In Sect. 12.4 we introduced the notion of cycle of a variety and in particular of 0-cycle. In the case of $\mathbb{P}^2$ a 0-cycle is an element of the free abelian group $D_{2,0}$ generated by the points of $\mathbb{P}^2$, i.e., it is an object of the form

$$\mathfrak{C} = \sum_{P\in\mathbb{P}^2} n_P P$$

with $n_P \in \mathbb{Z}$ and $n_P \neq 0$ for only finitely many points $P \in \mathbb{P}^2$. The integer

$$\deg(\mathfrak{C}) = \sum_{P\in\mathbb{P}^2} n_P$$

is called the *degree* of $\mathfrak{C}$. In $D_{2,0}$ there is a partial ordering given by

$$\sum_{P\in\mathbb{P}^2} n_P P \geqslant \sum_{P\in\mathbb{P}^2} m_P P \quad \text{if and only if} \quad n_P \geqslant m_P \quad \text{for all} \quad P \in \mathbb{P}^2.$$

Given two curves X, Y in $\mathbb{P}^2$ with no common components of degrees n and m, we can define the *intersection cycle* of X and Y as

$$X \cdot Y = \sum_{P\in\mathbb{P}^2} i(P; X, Y)P.$$

The Bézout theorem says that $\deg(X \cdot Y) = nm$.

The following properties are easy to verify:

(a) if the curves X and Y have no common components, then $X \cdot Y = Y \cdot X$;
(b) if the curves X and $Y + Z$ have no common components, $X \cdot (Y + Z) = X \cdot Y + X \cdot Z$;
(c) If the curves X, Y, Z have equations $f = 0, g = 0, g + af = 0$ respectively, and they have no common components, then $X \cdot Y = X \cdot Z$.

We want to treat now the following problem. Suppose we have three curves X, Y, Z, with respective equations $f = 0$, $g = 0$, $h = 0$ in $\mathbb{P}^2$, such that X has no common component with Y or Z, and suppose that we have $X \cdot Z \geqslant X \cdot Y$. The question is whether it is possible to find a curve V, with equation $B = 0$, with no common component with X such that

$$X \cdot Z = X \cdot Y + X \cdot V.$$

If this is the case, by the Bézout Theorem we have the relation

$$\deg(Z) = \deg(Y) + \deg(V).$$

We find such a curve V if we are able to find homogeneous polynomials A, B such that $h = Af + Bg$. Indeed in this case we have

$$\begin{aligned} X \cdot Z = Z_p(f) \cdot Z_p(h) = Z_p(f) \cdot Z_p(Af + Bg) = \\ = Z_p(f) \cdot Z_p(g) + Z_p(f) \cdot Z_p(B) = X \cdot Y + X \cdot V. \end{aligned}$$

To attack the above problem, we first give a definition. Fix a point $P \in \mathbb{P}^2$. We may assume that $P \in U_0 \cong \mathbb{A}^2$. Let X and Y be two curves in $\mathbb{P}^2$, with respective equations $f = 0$ and $g = 0$, with no common components, containing P. Let Z be a third curve in $\mathbb{P}^2$ with equation $h = 0$. We will say that Z satisfies *Noether's conditions* at P with respect to X and Y, if $h_* \in (f_*, g_*)$ in $\mathcal{O}_{\mathbb{A}^2,P}$, where the lower asterisk denotes the dehomogenization of forms and we abuse notation identifying f_*, g_*, h_* with their classes in $\mathcal{O}_{\mathbb{A}^2,P}$. Note that $\mathcal{O}_{\mathbb{A}^2,P} = \mathcal{O}_{\mathbb{P}^2,P}$. We can now state the:

Theorem 17.2.1 (M. Noether's $Af + Bg$ Theorem) *Let X, Y, Z be projective plane curves of degrees n, m, q, with respective equations $f = 0$, $g = 0$, $h = 0$, with X and Y with no common components. There is a relation of the form $h = Af + Bg$, with A, B homogeneous polynomials of degrees $q - n$ and $q - m$ respectively, if and only if Z satisfies Noether's conditions at every point $P \in X \cap Y$, with respect to X and Y.*

Proof If $h = Af + Bg$, then dehomogenizing we have $h_* = A_* f_* + B_* g_* \in (f_*, g_*)$, hence $h_* \in (f_*, g_*)$ in $\mathcal{O}_{\mathbb{A}^2,P}$ for all points $P \in X \cap Y$.

Let us prove the other implication. First we may assume that no point in $X \cap Y$ sits on the line at infinity $x_0 = 0$. If $P_1, \dots, P_h$ are the distinct intersection points of X and Y, we set $\mathfrak{O}_i = \mathcal{O}_{\mathbb{A}^2,P_i}$. By dehomogenizing we get the three polynomials f_*, g_*, h_*. We know that $h_* \in (f_*, g_*)$ in $\mathfrak{O}_i$ for all $i = 1, \dots, h$. We set $\mathcal{I} = (f_*, g_*)$. We apply Lemma 16.3.10, which tells us that we have an isomorphism

$$\phi : A_2/\mathcal{I} \to \prod_{i=1}^{h} \mathfrak{O}_i/\mathcal{I}\mathfrak{O}_i.$$

By the hypothesis, h_* has zero class in $\mathfrak{O}_i/\mathcal{I}\mathfrak{O}_i$, for all $i = 1, \dots, h$. Hence h_* has zero class in $A_2/\mathcal{I}$, so we have a relation of the form

$$h_* = af_* + bg_*$$

with $a, b \in A_2$ suitable polynomials. By homogenizing we get a relation of the form

$$x_0^r h = A' f + B' g$$

with A', B' suitable homogeneous polynomials, and r a suitable positive integer. Indeed by homogenizing f_* and g_* we obtain f and g because, by the hypothesis, neither X nor Y contains the line at infinity. By Exercise 17.3.5, multiplication by x_0 is injective on $\Gamma := S_2/(f, g)$. Therefore, from the fact that the class of $x_0^r h$ is zero in Γ, we deduce that the class of h is zero in Γ, and this implies that $h = Af + Bg$. By taking the suitable homogeneous component of this relation, we get the assertion. □

Theorem 17.2.1 becomes useful only if we give conditions under which Noether's conditions are verified. This is the purpose of the next:

Proposition 17.2.2 *Let X, Y, Z be curves in $\mathbb{P}^2$, with X and Y with no common components. Let $P \in X \cap Y$. Then Z verifies Noether's conditions at P with respect to X and Y if one of the following facts happens:*

(a) $i(P; X, Y) = 1$ *and* $P \in Z$*;*
(b) P *is a smooth point for* X *and*

$$i(P; X, Z) \geqslant i(P; X, Y); \tag{17.2}$$

(c) X *and* Y *have distinct principal tangents at* P *and*

$$m_P(Z) \geqslant m_P(X) + m_P(Y) - 1.$$

Proof Case (a) is a consequence of both (b) and (c), and it is also easy to be verified directly, we leave it to the reader as an exercise.

Let us prove (b). Relation (17.2) means that

$$o_{X,P}(h) \geqslant o_{X,P}(g)$$

hence

$$\bar{h}_* \in (\bar{g}_*) \subset \mathcal{O}_{X,P},$$

where, as usual, the asterisk denotes dehomogenization and the bar denotes the class. Now we claim that

$$\mathcal{O}_{X,P}/(\bar{g}_*) \cong \mathcal{O}_{\mathbb{A}^2,P}/(\bar{f}_*, \bar{g}_*). \tag{17.3}$$

In fact we have an obvious surjective homomorphism

$$\rho : \mathcal{O}_{\mathbb{A}^2,P} \to \mathcal{O}_{X,P}$$

whose kernel is clearly $(\bar{f}_*)$, hence

$$\mathcal{O}_{X,P} \cong \mathcal{O}_{\mathbb{A}^2,P}/(\bar{f}_*)$$

whence (17.3) follows. Then, since $\bar{h}_*$ is zero in $\mathcal{O}_{X,P}/(\bar{g}_*)$, it is also zero in $\mathcal{O}_{\mathbb{A}^2,P}/(\bar{f}_*, \bar{g}_*)$, as wanted.

Finally we prove (c). With the usual notation we have

$$m_P(h_*) \geqslant m_P(f_*) + m_P(g_*) - 1.$$

This implies that

$$h_* \in \mathfrak{m}^{m_P(f_*)+m_P(g_*)-1}.$$

On the other hand part (i) of the Claim in the proof of Theorem 16.3.12 tells us that $\mathfrak{m}^{m_P(f_*)+m_P(g_*)-1}\mathcal{O}_{\mathbb{A}^2,P} \subseteq (\bar{f}_*, \bar{g}_*)\mathcal{O}_{\mathbb{A}^2,P}$, hence $\bar{h}_* \in (\bar{f}_*, \bar{g}_*)\mathcal{O}_{\mathbb{A}^2,P}$, as wanted. □

Corollary 17.2.3 *If X, Y, Z are curves in $\mathbb{P}^2$ with equations $f = 0$, $g = 0$, $h = 0$ respectively, X and Y with no common component, and if all points in $X \cap Y$ are smooth for X, then $X \cdot Z \geqslant X \cdot Y$ implies that there is a curve V such that*

$$X \cdot V = X \cdot Z - X \cdot Y.$$

Proof By (b) of Proposition 17.2.2, we have that Z verifies Noether's conditions at all points of $X \cap Y$. Then by Theorem 17.2.1 there is a relation of the form $h = Af + Bg$, with A, B homogeneous polynomials of degrees $p - n$ and $p - m$ respectively. This implies that $X \cdot Z = X \cdot Z_p(B) + X \cdot Y$, and the assertion follows with $V = Z_p(B)$. □

17.3 Applications of the $Af + Bg$ Theorem

17.3.1 Pascal's and Pappo's Theorems

An *exagon* is an ordered sixtuple of distinct lines $(\ell_1, \dots, \ell_6)$ in $\mathbb{P}^2$ such that no three of them pass through the same point. The lines $\ell_1, \dots, \ell_6$ are called the *sides* of the exagon. Two sides ℓ_i, ℓ_j are said to be *opposite* if $(i, j) = (1, 4), (2, 6), (3, 6)$. We call *vertices* of the exagon the points $P_i = \ell_i \cap \ell_{i+1}$, for $i = 1, \dots, 6$, where we set $\ell_1 = \ell_7$. We will say that the exagon is *inscribed* in a curve X if the vertices $P_1, \dots, P_6$ of the exagon lie on X and no side of the exagon is contained in X.

Theorem 17.3.1 (Pascals's Theorem) *If an exagon is inscribed in a conic, then the opposite sides intersect in three collinear points.*

Proof Let C be the conic in which the exagon $(\ell_1, \dots, \ell_6)$ is inscribed. Let $X = \ell_1 + \ell_3 + \ell_5$, $Y = \ell_2 + \ell_4 + \ell_6$. Then apply Exercise 17.3.9 to the two cubics X and Y. □

Corollary 17.3.2 (Pappo's Theorem) *Let r_1, r_2 be two distinct lines in $\mathbb{P}^2$. Let $P_1, P_2, P_3 \in r_1$ and $Q_1, Q_2, Q_3 \in r_2$ be distinct points and distinct also from $r_1 \cap r_2$. Let $\ell_{ij} = \langle P_i, Q_j \rangle$, for $i, j = 1, 2, 3$. For every triple (i, j, k) such that $\{i, j, k\} = \{1, 2, 3\}$ set $R_k = \ell_{ij} \cap \ell_{ji}$. Then R_1, R_2, R_3 are aligned.*

Proof The sextuple of lines $(\ell_{12}, \ell_{32}, \ell_{31}, \ell_{21}, \ell_{23}, \ell_{13})$ form an exagon, which is inscribed in the conic $r_1 + r_2$. Then apply Pascal's Theorem. □

17.3.2 The Group Law on a Smooth Cubic

Let X be a smooth cubic curve in $\mathbb{P}^2$. Assume $\mathbb{K}$ has characteristic zero. By Exercise 17.3.12, X certainly has some flex O. If t is the tangent line to X at O, we have $t \cdot X = 3O$. For every point $P \in X$ distinct from O, let us denote by P' the point such that the line r containing O and P is such that $r \cdot X = O + P + P'$. If $P = O$ we set $O' = O$. Note that it could be the case that $P' = P$. This happens if $P = O$, and for those points $P \neq O$ such that the tangent line to X at P contains O. Of course $P'' = (P')' = P$. Fix now two (not necessarily distinct) points $P, Q \in X$. Consider the line $r = \langle P, Q \rangle$ if $P \neq Q$, whereas r is the tangent line to X at P if $P = Q$. Then $r \cdot X = P + Q + R$. We define a sum operation $\oplus$ on X by setting

$$P \oplus Q = R'.$$

Theorem 17.3.3 *The operation $\oplus$ is a commutative group operation.*

Proof It is easy to see that O is the zero for $\oplus$, that $\oplus$ is commutative and the opposite of P is P'. It is more complicated to check the associativity of $\oplus$. To prove it, we argue as follows. Consider three (not necessarily distinct) points P, Q, R of X. There are lines $\ell_1, \ell_2, m_1, m_1'$ such that

$$\begin{aligned} &\ell_1 \cdot X = P + Q + A', \quad m_1 \cdot X = O + A + A', \quad \text{so} \quad P \oplus Q = A \\ &\ell_2 \cdot X = A + R + T', \quad m_1' \cdot X = O + T + T', \quad \text{so} \quad (P \oplus Q) \oplus R = T. \end{aligned}$$

Then there are lines ℓ_3, ℓ_3' and m_2, m_3 such that

$$\begin{aligned} &m_2 \cdot X = Q + R + U', \quad \ell_3 \cdot X = O + U' + U, \quad \text{so} \quad Q \oplus R = U \\ &m_3 \cdot X = P + U + V, \quad \ell_3' \cdot X = O + V + V', \quad \text{so} \quad P \oplus (Q \oplus R) = V'. \end{aligned}$$

To prove associativity we have to prove that $T = V'$. For this set $Y = \ell_1 + \ell_2 + \ell_3$ and $Z = m_1 + m_2 + m_3$ and apply Exercise 17.3.11. □

Exercise 17.3.4 * In this exercise and in the next two we will indicate a new proof of Bézout Theorem in the case of projective plane curves. We will assume here that the intersection multiplicity is given by formula (16.13).

Let X, Y be curves in $\mathbb{P}^2$ with no common components of degrees n and m respectively, so that $X \cap Y$ is a finite set. Suppose that X has equation $f = 0$, Y has equation $g = 0$. Moreover we may assume that the line at infinity $x_0 = 0$ does not contain any intersection point of X and Y. As usual we will denote by h_* the dehomogenization of a homogeneous polynomial h. For every $P \in X \cap Y$ we have

$$i(P; f, g) = i(P; f_*, g_*)$$

hence

$$\sum_{P \in X \cap Y} i(P; f, g) = \sum_{P \in X \cap Y} i(P; f_*, g_*).$$

Moreover, by Exercise 16.3.30, we have

$$\sum_{P\in X\cap Y} i(P; f_*, g_*) = \dim_{\mathbb{K}}(A_2/(f_*, g_*)).$$

Recall that $S_2 = \mathbb{K}[x_0, x_1, x_2]$ and set

$$R = S_2, \quad \Gamma = R/(f, g), \quad \Gamma_* = A_2/(f_*, g_*).$$

The rings R and Γ are graded and, as usual, we will denote by R_d and Γ_d their homogeneous parts of degree $d \in \mathbb{N}$. The Bézout Theorem will be proved by proving that

$$\dim(\Gamma_d) = nm \quad \text{and} \quad \dim(\Gamma_*) = \dim(\Gamma_d) \quad \text{for} \quad d \geqslant n + m. \tag{17.4}$$

In this exercise prove that $\dim(\Gamma_d) = nm$ for $d \geqslant n + m$.

Exercise 17.3.5 * (*Continue Exercise* 17.3.4) Consider the map

$$\alpha : \bar{h} \in \Gamma \to \overline{x_0 h} \in \Gamma$$

where $h \in R$ is a polynomial and $\bar{h}$ is its class in Γ. Prove that α is injective.

Exercise 17.3.6 * (*Continue Exercise* 17.3.4) Assume $d \geqslant n + m$ and choose $A^1, \ldots, A^{nm} \in R_d$ such that their classes $a_1, \ldots, a_{nm} \in \Gamma_d$ are a basis of Γ_d. Consider the classes α_i of A^i_* in Γ_*, for $i = 1, \ldots, nm$. Prove that $\alpha_1, \ldots, \alpha_{nm}$ form a basis of Γ_*.

This ends the proof of (17.4) and therefore of the Bézout Theorem.

Exercise 17.3.7 Prove that any projective smooth curve in $\mathbb{P}^2$ is irreducible. Prove that this not always the case in $\mathbb{A}^2$.

Exercise 17.3.8 Let X be an irreducible curve of degree n in $\mathbb{P}^2$. Prove that

$$n(n-1) \geqslant \sum_{P\in X} m_P(X)(m_P(X) - 1).$$

Exercise 17.3.9 Let X, Y be cubics in $\mathbb{P}^2$ with no common components, and let $X \cdot Y = \sum_{i=1}^9 P_i$, with $P_1, \ldots, P_9$ smooth points for X. Let Z be a conic such that $X \cdot Z = \sum_{i=1}^6 P_i$. prove that there is a line L such that $X \cdot L = P_7 + P_8 + P_9$.

Exercise 17.3.10 Prove the inverse of Pascal's Theorem: if the opposite sides of an exagon intersect in three collinear points, then the vertices of the exagon lie on a conic.

Exercise 17.3.11 * Let X be an irreducible cubic in $\mathbb{P}^2$. Let Y, Z two more cubics. Suppose that $X \cdot Y = \sum_{i=1}^9 P_i$, where $P_1, \ldots, P_9$ are (not necessarily distinct) smooth points of X. Suppose that $X \cdot Z = \sum_{i=1}^8 P_i + Q$. Prove that $Q = P_9$.

Exercise 17.3.12 * Let X be a curve in $\mathbb{P}^2$ of degree n with equation $f = 0$. According to Exercise 16.3.21, we say that a smooth point $P \in X$ is a *flex* if the intersection multiplicity of the tangent line to X at P with X is $m \geqslant 3$. The flex is *simple* if $m = 3$. If $m > 3$ we say we have a $(m-2)$-*flex*. Note that lines are characterized by the condition that all points on them are ∞-flexes.

Consider the *Hessian polynomial* of f, defined as

$$\text{hess}(f) = \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=0,1,2}.$$

Note that hess(f) could be identically zero. If this is not the case, the curve of degree $3(n-2)$ with equation hess$(f) = 0$ is called the *Hessian* of X, denoted by Hess(X).

Assume that $\mathbb{K}$ has characteristic zero. Prove that given a point $P \in X$, one has hess$(f)(P) = 0$ if and only if either P is singular for X or P is a flex of X.

Exercise 17.3.13 * Assume that $\mathbb{K}$ has characteristic zero. Let X be a curve of degree n in $\mathbb{P}^2$ with equation $f = 0$ and P a m-flex of X. Prove that $o_P(\text{hess}(f)) = m$ and conversely.

Exercise 17.3.14 * Assume that $\mathbb{K}$ has characteristic zero. Let X be a reduced curve in $\mathbb{P}^2$ with equation $f = 0$. Prove that if there is an irreducible component of X such hess(f) vanishes on it, then this component is a line.

Exercise 17.3.15 Prove that a smooth cubic in $\mathbb{P}^2$ has exactly 9 (simple) flexes.

Exercise 17.3.16 Prove that a line containing two flexes of a cubic contains a third flex.

Exercise 17.3.17 Prove that the flexes of a smooth cubic X are the points of order 3 of the group law on the cubic and they form a group isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Exercise 17.3.18 Assume $\mathbb{K}$ of characteristic different form 2 and 3. Let X be a smooth cubic curve in $\mathbb{P}^2$. Prove that we can change coordinates so that X has affine equation of the type

$$y^2 = x^3 + ax + b, \quad \text{with} \quad a, b \in \mathbb{K} \tag{17.5}$$

with $x^3 + ax + b$ with no multiple roots. This is called *Weierstrass normal form* of the equation of the cubic.

Exercise 17.3.19 Assume $\mathbb{K}$ of characteristic 2 or 3. Let X be a smooth cubic curve in $\mathbb{P}^2$. Prove that we can change coordinates so that X has affine equation of the type

$$y^2 = x^3 + ax^2 + bx + c, \quad \text{with} \quad a, b, c \in \mathbb{K}$$

if $\mathbb{K}$ has characteristic 3, and of the types

$$y^2 + cy = x^3 + ax + b, \quad \text{with} \quad a, b, c \in \mathbb{K}$$

or

$$y^2 + xy = x^3 + ax + b, \quad \text{with} \quad a, b \in \mathbb{K}$$

if $\mathbb{K}$ has characteristic 2.

Exercise 17.3.20 Prove that a point P of a smooth cubic X is a point of order 2 of the group law on the cubic if and only if the tangent line to X at P pass through the zero given by the flex O.

Exercise 17.3.21 Assume $\mathbb{K}$ has characteristic zero. Let X in $\mathbb{P}^2$ be a smooth cubic curve. Prove that there are exactly four points of order 2 (including the zero) with respect to the group law with zero at the flex O and that they form a group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 17.3.22 Assume $\mathbb{K}$ has characteristic zero. Let X in $\mathbb{P}^2$ be a smooth cubic curve endowed with its group law with neutral element a flex of X. Prove that the maps

$$(P, Q) \in X \times X \to P \oplus Q \in X, \quad P \in X \to \ominus P \in X$$

are morphisms.

Exercise 17.3.23 Let X be a projective variety which has an additive group law $(X, \oplus)$ such that the maps

$$(P, Q) \in X \times X \to P \oplus Q \in X, \quad P \in X \to \ominus P \in X$$

are morphisms. Prove that X is smooth. Such an X is called an *abelian variety*.

Exercise 17.3.24 Assume $\mathbb{K}$ has characteristic zero. Prove that there are abelian varieties of any dimension.

Exercise 17.3.25 * Let X in $\mathbb{P}^2$ be an irreducible curve which is not a line. Consider the map $\mu : X \dashrightarrow \check{\mathbb{P}}^2$, which sends any smooth point of X to its tangent line. Prove that μ is a rational map. Prove that the closure $\check{X}$ of the image of the set of smooth points of X via μ is an irreducible curve called the *dual curve* of X. Let $m(X)$ be the degree of $\check{X}$, also shortly denoted by m and called the *class* of X. Prove that m is the maximum number of tangents in smooth points of X passing through a given point P of $\mathbb{P}^2$.

Exercise 17.3.26 * Let X in $\mathbb{P}^2$ be an irreducible curve with equation $f = 0$. Given two points $P = [p_0, p_1, p_2]$ and $Q = [q_0, q_1, q_2]$ of the plane, we set

$$f_Q(P) = \sum_{i=0}^{2} \frac{\partial f}{\partial x_i}(P) q_i.$$

Prove that $f_Q(P) = 0$ if either P is a singular point of X or if P is a smooth point of X and the tangent line to X at P contains Q.

Given $Q \in \mathbb{P}^2$, consider the polynomial

$$f_Q = \sum_{i=0}^{2} \frac{\partial f}{\partial x_i} q_i$$

and consider the curve X'_Q with equation $f_Q = 0$. This is called the *polar* of X with respect to Q. Prove that the tangent at a smooth point P of X contains Q if and only if $P \in X'_Q$. Prove also that all polars pass through the singular points of X.

Exercise 17.3.27 * Assume $\mathbb{K}$ has characteristic zero. Consider the irreducible plane curve X of degree n and suppose that X has only nodes and ordinary cusps as singularities. Prove that

$$m(X) = n(n-1) - 2\delta - 3\kappa \tag{17.6}$$

where δ is the number of nodes of X and κ the number of cusps. Relation (17.6) is called the *first Plücker formula*.

Exercise 17.3.28 * Assume $\mathbb{K}$ has characteristic zero. Consider the irreducible plane curve X of degree $n > 1$ and suppose that X has only nodes and ordinary cusps as singularities and it has only simple flexes. Moreover let us assume that the nodes are *simple*: a node is simple if each branch at the node is of class 1, i.e., the principal tangent lines at the node have intersection multiplicity 2 with the corresponding branch, and therefore 3 with the curve at the node.

Let i be the number of flexes of X, δ the number of (simple) nodes and κ the number of cusps. Prove that

$$i = 3n(n-2) - 6\delta - 8\kappa \tag{17.7}$$

Relation (17.7) is called the *second Plücker formula*.

17.4 Solutions of Some Exercises

17.1.4 First of all treat the case in which $m_1 = \cdots = m_h = 1$, and make induction on h. The case $h = 1$ is trivial. Next assume $h > 1$ and $d \geqslant h - 1$. Set $\mathcal{L}_i = \mathcal{L}_{2,d}(-P_1 - \cdots - P_i)$ for all $i = 1, \ldots, h$. By induction we may assume that

$$\dim(\mathcal{L}_{h-1}) = \frac{d(d+3)}{2} - h + 1.$$

Since $\mathcal{L}_h$ has codimension at most 1 in $\mathcal{L}_{h-1}$, it suffices to prove that $\mathcal{L}_h \subsetneq \mathcal{L}_{h-1}$. Choose lines r_i such that $P_i \in r_i$ and $P_i \notin r_j$ for $j \neq i$, for $i = 1, \dots, h$ and choose a line ℓ not containing any of the points $P_1, \dots, P_h$. Then

$$X = r_1 + \cdots + r_{h-1} + (d-h+1)\ell$$

is a curve of degree d containing $P_1, \dots, P_{h-1}$ but not P_h, hence $X \in \mathcal{L}_{h-1} \setminus \mathcal{L}_h$ as required.

Next treat the general case and make induction on $m := (\sum_{i=1}^{h} m_i) - 1$. If $m = 0$, the assertion is trivial. So assume $m > 0$. We may assume there is an $i = 1, \dots, h$ such that $m_i > 1$, and we may actually assume that $m_1 > 1$ and, by changing coordinates, that $P_1 = [1, 0, 0]$. Set $\mathcal{L} = \mathcal{L}_{2,d}(-(m_1 - 1)P_1 - m_2P_2 - \cdots - m_hP_h)$. Let $f = 0$ be the equation of a curve in $\mathcal{L}$. Then, passing to affine coordinates, the equation of this curve becomes $\phi(x, y) = f(1, x, y) = 0$, and the decomposition in homogeneous components is of the form

$$\phi(x, y) = \phi_{m_1-1}(x, y) + \cdots + \phi_d(x, y)$$

and

$$\phi_{m_1-1}(x, y) = \sum_{i=0}^{m_1-1} a_i x^i y^{m_1-1-i}.$$

Denote by $\mathfrak{L}_i$ the subspace of the curves in $\mathcal{L}$ such that, with the above notation, have the property that $a_j = 0$ for all j such that $0 \leqslant j \leqslant i \leqslant m_1 - 1$. Then we have

$$\mathcal{L} \supseteq \mathfrak{L}_0 \supseteq \cdots \supseteq \mathfrak{L}_{m_1-1} = \mathcal{L}_{2,d}(m_1P_1 - m_2P_2 - \cdots - m_hP_h).$$

By induction we have

$$\dim(\mathcal{L}) = \frac{d(d+3)}{2} - \frac{m_1(m_1-1)}{2} - \sum_{i=2}^{h} \frac{m_i(m_i+1)}{2}.$$

We claim that

$$\mathcal{L} \supsetneq \mathfrak{L}_0 \supsetneq \cdots \supsetneq \mathfrak{L}_{m_1-1}. \tag{17.8}$$

If this is the case then

$$\dim(\mathcal{L}_{2,d}(m_1P_1 - m_2P_2 - \cdots - m_hP_h)) = \dim(\mathfrak{L}_{m_1-1}) \leqslant$$
$$\leqslant \dim(\mathcal{L}) - m_1 = \frac{d(d+3)}{2} - \sum_{i=1}^{h} \frac{m_i(m_i+1)}{2}.$$

Since also the opposite inequality holds by (17.1), we have the equality as desired.

So let us prove (17.8). Set

$$\Lambda = \mathcal{L}_{2,d}(-(m_1-2)P_1 - m_2P_2 - \cdots - m_hP_h).$$

Similarly as before, if $\psi(x, y) = 0$ is the affine equation of a curve in Λ we have the decomposition in homogeneous components

$$\psi(x, y) = \psi_{m_1-2}(x, y) + \cdots + \psi_d(x, y)$$

with

$$\psi_{m_1-2}(x, y) = \sum_{i=0}^{m_1-2} b_i x^i y^{m_1-2-j}.$$

As before we define Λ_i to be the subspace of Λ of the curves for which $b_j = 0$ for all j such that $0 \leqslant j \leqslant i \leqslant m_1 - 2$. We have

$$\Lambda := \Lambda_{-1} \supseteq \Lambda_0 \supseteq \cdots \supseteq \Lambda_{m_1-2} = \mathcal{L}_{2,d-1}(-(m_1-1)P_1 - m_2P_2 - \cdots - m_hP_h).$$

By induction we have

$$\dim(\Lambda) = \frac{(d-1)(d+2)}{2} - \frac{(m_1-2)(m_1-1)}{2} - \sum_{i=2}^{h} \frac{m_i(m_i+1)}{2}$$

$$\dim(\Lambda_{m_1-2}) = \frac{(d-1)(d+2)}{2} - \frac{m_1(m_1-1)}{2} - \sum_{i=2}^{h} \frac{m_i(m_i+1)}{2}$$

so that

$$\dim(\Lambda) - \dim(\Lambda_{m_1-2}) = m_1 - 1.$$

Since we have

$$\dim(\Lambda_i) \geqslant \dim(\Lambda_{i-1}) - 1, \quad \text{for} \quad i = 0, \ldots, m_1 - 2,$$

we deduce that

$$\Lambda \supsetneq \Lambda_0 \supsetneq \cdots \supsetneq \Lambda_{m_1-2}.$$

Finally, if $\phi_i = 0$ is the affine equation of a curve sitting in Λ_i but not in Λ_{i+1}, for $i = -1, \ldots, m_1 - 3$, then $y\phi_i = 0$ is the equation of a curve sitting in $\mathfrak{L}_i$ but not in $\mathfrak{L}_{i+1}$ (again we set $\mathcal{L} := \mathfrak{L}_{-1}$) and, if $\phi_{m_1-2} = 0$ is the equation of a curve in Λ_{m_1-2}, then $x\phi_{m_1-2} = 0$ is the equation of a curve in $\mathfrak{L}_{m_1-2}$ but not in $\mathfrak{L}_{m_1-1}$. This proves (17.8) as wanted.

17.3.4 Let $\pi : R \to \Gamma$ be the natural surjective homomorphism, whose kernel is $\mathcal{I} := (f, g)$. Consider the homomorphisms

$$\phi : (A, B) \in R \times R \to Af + Bg \in \mathcal{I}, \quad \psi : C \in R \to (gC, -fC) \in R \times R.$$

One easily proves that:

(a) ψ is injective;

(b) ϕ is surjective onto $\mathcal{I}$;

(c) $\mathrm{im}(\psi) = \ker(\phi)$ (here one uses the fact that f and g have no common factor).

By restricting to homogeneous parts, one has the linear maps of vector spaces

$$\phi_{d,n,m} : R_{d-m} \times R_{d-n} \to \mathcal{I}_d, \quad \psi_{d,n,m} : R_{d-n-m} \to R_{d-m} \times R_{d-n}.$$

and again one has

(a) $\psi_{d,n,m}$ is injective;

(b) $\phi_{d,n,m}$ is surjective onto $\mathcal{I}_d$;

(c) $\mathrm{im}(\psi_{d,n,m}) = \ker(\phi_{d,n,m})$.

This implies that

$$\dim(\Gamma_d) = \dim(R_d) - \dim(R_{d-m} \times R_{d-n}) - \dim(R_{d-n-m}). \tag{17.9}$$

Since

$$\dim(R_h) = \frac{(h+1)(h+2)}{2}$$

as soon as $h \geqslant 0$, the assertion follows from (17.9) with easy calculations.

17.3.5 The map α is a group homomorphism for the additive structure of Γ. So to prove injectivity we have to prove that $\ker(\alpha) = 0$. Suppose we have an element $\bar{h}$ such that $\overline{x_0 h} = \alpha(\bar{h}) = 0$. We want to prove that $\bar{h} = 0$, i.e., that $h \in (f, g)$. Since $\overline{x_0 h} = 0$, we have that $x_0 h \in (f, g)$. Given any polynomial $p(x_0, x_1, x_2) \in R$, we set $p_0 = (0, x, y) \in A_2$. Since

$$x_0 h = Af + Bg$$

we have

$$A_0 f_0 = -B_0 g_0.$$

The system $f_0 = g_0 = 0$ has no solutions since there is no point of $X \cap Y$ on the line at infinity $x_0 = 0$. Then f_0 divides B_0 and g_0 divides A_0, i.e., there is a polynomial $C \in A_2$ such that

$$B_0 = f_0 C, \quad A_0 = -g_0 C.$$

Set

$$A' = A + Cg, \quad B' = B - Cf$$

and notice that

$$A'f + B'g = Af + Bg = x_0 h$$

and

$$A'_0 = B'_0 = 0.$$

Then A' and B' are divisible by x_0, i.e.,

$$A' = x_0 A'', \quad B' = x_0 B''$$

for suitable polynomials A'', B''. So we have

$$x_0 h = A'f + B'g = x_0 A'' f + x_0 B'' g$$

and therefore

$$h = A'' f + B'' g \in (f, g)$$

as wanted.

17.3.6 First we notice that the map $\alpha : \Gamma \to \Gamma$ restricts to an isomorphism between Γ_d and Γ_{d+1}, because it induces a linear injective map from Γ_d to Γ_{d+1} and these are vector spaces of the same dimension nm. Therefore, by iterating the application of α, we have that the classes of $x_0^r A^1, \ldots, x_0^r A^{nm}$ give a basis of Γ_{d+r} for all $d \geqslant n + m$ and $r \geqslant 0$.

Next we prove that $\alpha_1, \ldots, \alpha_{nm}$ generate Γ_*. Fix $\bar{p} \in \Gamma_*$ with $p \in A_2$ and $\bar{p}$ the class of p in Γ_*. Recall that we denote by p_h the homogenization of p. Then consider $N \in \mathbb{N}$ large enough so that $x_0^N p_h$ is a homogeneous polynomial of degree $d + r$ with $r \gg 0$. Then, since $x_0^r A^1, \ldots, x_0^r A^{nm}$ are a basis of Γ_{d+r}, we have a relation of the form

$$x_0^N p_h = \sum_{i=1}^{nm} \lambda_i x_0^r A^i + Bf + Cg$$

with $\lambda_1, \ldots, \lambda_{nm} \in \mathbb{K}$ and $B, C \in R$. If we dehomogenize this relation, we get

$$p = \sum_{i=1}^{nm} \lambda_i A_*^i + B_* f_* + C_* g_*$$

whence

$$\bar{p} = \sum_{i=1}^{nm} \lambda_i \alpha_i$$

as wanted.

Finally we prove that $\alpha_1, \dots, \alpha_{nm}$ are linearly independent. Suppose we have a relation of the form

$$\sum_{i=1}^{nm} \lambda_i \alpha_i = 0$$

with $\lambda_1, \dots, \lambda_{nm} \in \mathbb{K}$. This implies a relation of the form

$$\sum_{i=1}^{nm} \lambda_i A_*^i = B f_* + C g_*$$

with $B, C \in A_2$ suitable polynomials. We homogenize the above relation getting

$$x_0^r \sum_{i=1}^{nm} \lambda_i A^i = x_0^s B_h f + x_0^t C_h g$$

for suitable positive integers r, s, t. Therefore we have

$$\sum_{i=1}^{nm} \lambda_i \overline{x_0^r A^i} = 0$$

in Γ_{d+r} (with the upper bar we denote, as usual, the class in Γ_{d+r}). But $x_0^r A^i$, for $i = 1, \dots, nm$ form a basis of Γ_{d+r}, hence $\lambda_i = 0$ for $i = 1, \dots, nm$, as desired.

17.3.9 This is an immediate application of Corollary 17.2.3.

17.3.11 Suppose by contradiction $Q \neq P_9$. Let r be a line through P_9 not passing through Q. Let $r \cdot X = P_9 + R + S$. Then

$$(r + Z) \cdot X = \sum_{i=1}^{8} P_i + Q + P_9 + R + S.$$

Since $X \cdot Y = \sum_{i=1}^{9} P_i$, there is a line r' such that $r' \cdot X = R + S + Q$. As r' contains R and S, then $r = r'$, and so $Q = P_9$, a contradiction.

17.3.12 Suppose that $P = [\mathbf{p}] = [p_0, p_1, p_2]$ is a smooth point for the curve X of degree n with equation $f = 0$. Let $Q = [\mathbf{q}] = [q_0, q_1, q_2]$ be any other point of the plane. By applying Taylor formula we have

$$f(\lambda \mathbf{p} + \mu \mathbf{q}) = f(\mathbf{p})\lambda^n + \sum_{i=0}^{2} \frac{\partial f}{\partial x_i}(\mathbf{p}) q_i \lambda^{n-1} \mu + \frac{1}{2} \sum_{i,j=0}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) q_i q_j \lambda^{n-2} \mu^2 + \cdots . \tag{17.10}$$

We have $f(\mathbf{p}) = 0$. Moreover the tangent line t to X at P has equation

$$\sum_{i=0}^{2} \frac{\partial f}{\partial x_i}(\mathbf{p}) x_i = 0. \tag{17.11}$$

Consider the polynomial

$$\sum_{i,j=0}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) x_i x_j. \tag{17.12}$$

From (17.10) it is clear that P is a flex for X if and only of the polynomial appearing in (17.11) divides the polynomial in (17.12). Hence if P is a flex, then the polynomial in (17.12) is either zero or reducible, and this implies that $\mathrm{hess}(f)(P) = 0$.

To see the converse, let us assume that $\mathrm{hess} f(P) = 0$. If the polynomial in (17.12) is identically zero, then it is divisible by the polynomial in (17.11) and by (17.10) P is a flex. If the polynomial in (17.12) is not identically zero, it defines a conic Q. By Euler's formula (1.6), we have

$$\sum_{i,j=0}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) p_i p_j = n(n-1) f(\mathbf{p}) = 0$$

so Q contains P. Moreover t is tangent to Q at P, because this tangent has equation

$$\sum_{i,j=0}^{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) p_i x_j = 0$$

and by Euler's formula this is

$$(n-1) \sum_{j=0}^{2} \frac{\partial f}{\partial x_j}(\mathbf{p}) x_j = 0.$$

So if $\mathrm{hess}(f)(P) = 0$, the conic Q is reducible, hence it must contain t, so the polynomial in (17.12) is divisible by the polynomial in (17.11) and by (17.10) the point P is a flex.

Finally it is clear that singular points of X lie on the Hessian curve of f.

17.3.13 We can assume that $P \in U_0 \cong \mathbb{A}^2$ and in fact that P is the origin and that X has affine equation $\phi(x, y) = f(1, x, y) = 0$. By adjusting coordinates we may assume that the tangent line to X at P is the x axis and that P is the centre of a branch with equations

$$x = \xi(t) = t, \quad y = \eta(t) = t^{m+2} + \cdots. \tag{17.13}$$

It is easy to see that this implies that

$$\phi(x, y) = y - x^{m+2} + g(x, y) \tag{17.14}$$

where g contains no term in the only variable y. By Euler's formula we have

$$x_0 \frac{\partial^2 f}{\partial x_0 \partial x_i} = (n-1)\frac{\partial f}{\partial x_i} - x_1 \frac{\partial^2 f}{\partial x_1 \partial x_i} - x_2 \frac{\partial^2 f}{\partial x_2 \partial x_i}, \quad i = 0, 1, 2$$
$$x_0 \frac{\partial f}{\partial x_0} = nf - x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}.$$

Substituting in the Hessian determinant, and with easy computations, we have that

$$\mathrm{hess}(f) = \frac{(n-1)^2}{x_0^2} \cdot \begin{vmatrix} \frac{n}{n-1} f & \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix}.$$

Passing to affine coordinates we have, up to a constant term

$$h(x,y) := \mathrm{hess}(f)(1,x,y) = \begin{vmatrix} \frac{n}{n-1}\phi & \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x\partial y} \\ \frac{\partial f}{\partial x_2} & \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial y^2} \end{vmatrix}.$$

Using (17.13) and (17.14), we find

$$h(\xi(t),\eta(t)) =$$

$$= \begin{vmatrix} 0 & -(m+2)t^{m+1} + \frac{\partial g}{\partial x}(\xi(t),\eta(t)) & 1 + \frac{\partial g}{\partial y}(\xi(t),\eta(t)) \\ -(m+2)t^{m+1} + \frac{\partial g}{\partial x}(\xi(t),\eta(t)) & -(m+2)(m+1)t^m + \frac{\partial^2 g}{\partial x^2}(\xi(t),\eta(t)) & \frac{\partial^2 g}{\partial x\partial y}(\xi(t),\eta(t)) \\ 1 + \frac{\partial g}{\partial y}(\xi(t),\eta(t)) & \frac{\partial^2 g}{\partial x\partial y}(\xi(t),\eta(t)) & \frac{\partial^2 g}{\partial y^2}(\xi(t),\eta(t)) \end{vmatrix}.$$

Moreover we have

$$0 = t^{m+2} + \cdots - t^{m+2} + g(\xi(t),\eta(t))$$

so that $o(g(\xi(t),\eta(t))) \geqslant m+3$. Differentiating twice with respect to t we find

$$o\Big(\frac{\partial g}{\partial x}(\xi(t),\eta(t)) + \frac{\partial g}{\partial y}(\xi(t),\eta(t))((m+2)t^{m+1} + \cdots\Big) \geqslant m+2,$$

$$\begin{aligned} &o\Big(\frac{\partial^2 g}{\partial x^2}(\xi(t),\eta(t)) + 2\frac{\partial^2 g}{\partial x\partial y}(\xi(t),\eta(t))((m+2)t^{m+1} + \cdots) + \\ &+ \frac{\partial^2 g}{\partial y^2}(\xi(t),\eta(t))((m+2)t^{m+1} + \cdots) + \\ &+ \frac{\partial g}{\partial y}(\xi(t),\eta(t))((m+1)(m+2)t^m + \cdots)\Big) \geqslant m+1. \end{aligned}$$

From these relations we deduce that $o(\frac{\partial g}{\partial y}(\xi(t),\eta(t)) \geqslant 1$, $o(\frac{\partial g}{\partial x}(\xi(t),\eta(t)) \geqslant m+2$ and $o(\frac{\partial^2 g}{\partial x^2}(\xi(t),\eta(t))) \geqslant m+1$. With an easy computation it follows that $h(\xi(t),\eta(t)) = (m+1)(m+2)t^m + \cdots$ so that $o(h(\xi(t),\eta(t))) = m$ as wanted.

Conversely, if P is a smooth point of X and $o_P(\mathrm{hess}(f)) = m \geqslant 1$, then P is a flex by Exercise 17.3.12, and it is an m-flex just because $o_P(\mathrm{hess}(f)) = m$.

17.3.14 Let Y be a component of X such that $\mathrm{hess}(f)$ vanishes on Y. Let P be a smooth point of Y and let γ be the branch of Y with centre P. We can assume that $P \in U_0 \cong \mathbb{A}^2$ and in fact that P is the origin. We may also suppose that γ is determined by a parametrization like (17.13). If $m = \infty$, this means that $y = 0$, hence Y is a line, i.e., the x axis. If m is not ∞, then, with the notation of the solution of Exercise 17.3.13, we have that $o(h(\xi(t),\eta(t))) = m$. On the other hand $o_\gamma(h(x,y)) = \infty$ by assumption. Since $o_\gamma(h(x,y)) = o(h(\xi(t),\eta(t))) = m$ we have a contradiction.

17.3.18 The curve X has a flex O. We can change coordinates assuming that O is the point at infinity of the y axis and the tangent line at O is the line at infinity $x_0 = 0$. With this choice of coordinates the affine equation of X becomes of the form

$$\phi(x,y) = y^2 + y(\alpha + \beta x) + f(x) = 0$$

where $f(x)$ is a polynomial of degree 3 in the variable x. We have

$$\frac{\partial\phi}{\partial y} = 2y + \alpha + \beta x$$

and we can change coordinates so that this is equal to y, hence $\alpha = \beta = 0$. Then the affine equation of X becomes of the type

$$y^2 = Ax^3 + Bx^2 + Cx + D$$

with $A \neq 0$ because the curve has degree 3. We can then change coordinates by sending x to $\frac{x}{A^{\frac{1}{3}}}$. In this way we may assume that $A = 1$. Next we change again coordinates by sending x to $x - \frac{B}{3}$. It is easy to check that the equation then becomes of the form (17.5). The fact that $x^3 + ax + b$ must have no multiple roots is equivalent to the curve being smooth, since we require that there is no solution to the system of the equation of the curve and of the two derivatives

$$y = 0, \quad 3x^2 + a = 0.$$

17.3.19 Similar to the solution of Exercise 17.3.18. The details can be left to the reader.

17.3.25 Consider the homogeneous coordinate ring $S(X) = \mathbb{K}[x, y, z]/(f)$ of X. Consider the classes ξ, η, ζ in $S(X)$ of the polynomials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$. Consider the homomorphism $\phi : \mathbb{K}[u, v, w] \to S(X)$ obtained by sending u, v, w to ξ, η, ζ respectively. One easily checks that ϕ is a homogeneous homomorphism and its kernel $\mathcal{I}$ is a homogeneous prime ideal. Hence $Z := Z_p(\mathcal{I})$ is a closed irreducible subset of $\check{\mathbb{P}}^2$. By the very definition of ϕ and by the equation of the tangent line at a smooth point of X it follows that Z contains $\mu(S)$. Next one proves that Z is curve. Otherwise Z would be a point $[a, b, c]$, and therefore the fixed line with equation $ax + by + cz = 0$ would be tangent to X at any of its simple points, so X would coincide with that line, a contradiction. We set $Z = \check{X}$. We have $\mu(S) \subseteq \check{X}$, and $\mu(S)$ is infinite, so the Zariski closure of $\mu(S)$ is a curve contained in $\check{X}$, so it coincides with $\check{X}$ because $\check{X}$ is irreducible. The rest of the assertion follows from Theorem 12.2.3.

17.3.27 By Exercise 17.3.26 and Theorem 12.2.3 there is an open subset U of $\mathbb{P}^2$ such that for all points $Q \in U$, the curve X'_Q intersects X in exactly m points off the singular points of X. Since X'_Q has degree $n - 1$, we have that $m = n(n - 1) + h$, where h is the sum of the intersection multiplicities of X'_Q and X at the singular points of X. Let P be a node of X. We can change coordinates and can put P at the origin and the principal tangent lines to X at P coinciding with the coordinate axes. In this coordinate system the affine equation of X is of the form

$$\phi(x, y) = xy + \cdots = 0$$

where the dots stay for higher order terms. Among the polars of X we have all the curves with equation

$$a\frac{\partial \phi}{\partial x} + a\frac{\partial \phi}{\partial y} = ay + bx + \cdots = 0.$$

This implies that there is a non-empty open subset U_P of $\mathbb{P}^2$ such that for $Q \in U_P$ the polar X'_Q passes through P simply, with tangent different from the principal tangents to X at P. For these polars the intersection multiplicity with X at P is 2. Since we have finitely many nodes $P_1, \ldots, P_\delta$, for Q in the open subset $U \cap U_{P_1} \cap \cdots \cap U_{P_\delta}$, the intersection multiplicity of X'_Q with X at each of the points $P_1, \ldots, P_\delta$ is 2, hence the contribution of these singularities to the number h is 2δ.

Let P' be a cusp of X. We can change coordinates in such a way that P' is the origin of the coordinate system and the centre of a unique branch γ determined by an equation of the form

$$x = t^2, \quad y = at^3 + \cdots, \quad \text{with} \quad a \neq 0.$$

Then the affine equation of X is of the form

$$\phi(x, y) = y^2 + \cdots$$

where the dots stay for higher order terms. Among the polars of X we have the curve

$$\frac{\partial \phi}{\partial y} = 2y + \cdots = 0$$

which has order 3 on γ. It is easy to check that this is the minimum order that a polar has with γ, and therefore there is an open set $U_{P'}$ such that for all $Q \in U_{P'}$ the intersection multiplicity of X'_Q with X at P' is 3. Then with the same argument we made in the case of nodes we see that the contribution of the κ cusps to the number h is 3κ, and this proves (17.6).

17.3.28 By Exercise 17.3.13, the number of (simple) flexes is the number of intersections of the Hessian curve $\mathrm{Hess}(X)$ of X, which has degree $3(n-2)$ with X, off the singular points. Now, with a direct calculation similar to the ones we performed in the solutions of Exercises 17.3.13 and 17.3.27, one checks that this intersection multiplicity is exactly 6 at the nodes and 8 at the cusps. Then (17.7) follows.

Chapter 18
Resolution of Singularities of Curves

In this chapter we will prove that in any birational equivalence class of irreducible quasi-projective curves there is a smooth projective model. The process of passing from a curve to a smooth model is called *resolution of singularities*.

18.1 The Case of Ordinary Singularities

In this section we prove a preliminary important result. Let X be a projective, reduced, irreducible curve in $\mathbb{P}^2$ of degree d, with equation $f = 0$, and we suppose that X has only ordinary singularities at the points $P_1, \dots, P_h$, with multiplicities $m_1, \dots, m_h$. We want to prove the following:

Theorem 18.1.1 *In the above setting, consider the blow-up $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ at the points $P_1, \dots, P_h$. Then the proper transform $\tilde{X}$ of X on $\tilde{\mathbb{P}}^2$ is a smooth projective curve and $\pi_{|\tilde{X}} : \tilde{X} \to X$ is a birational morphism.*

Proof Since $\tilde{\mathbb{P}}^2$ is a projective variety and $\tilde{X}$ is closed in $\tilde{\mathbb{P}}^2$, then $\tilde{X}$ is a projective curve. Moreover, since $\pi : \tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ is an isomorphism between $\pi^{-1}(\mathbb{P}^2 \setminus \{P_1, \dots, P_h\})$ and $\mathbb{P}^2 \setminus \{P_1, \dots, P_h\}$, it is clear that $\pi_{|\tilde{X}} : \tilde{X} \to X$ is a birational morphism. In addition, since X is smooth at the points of $X \setminus \{P_1, \dots, P_h\}$ then $\tilde{X}$ is smooth at the points of $\tilde{X} \cap \pi^{-1}(\mathbb{P}^2 \setminus \{P_1, \dots, P_h\})$. So the only thing to be proved is that $\tilde{X}$ is smooth at every point whose image via π is one of the points $P_1, \dots, P_h$. To prove this we proceed in the following way.

First, since the question is local at the points $P_1, \dots, P_h$, we may assume that there is only one singular point P for X, which is an ordinary point of multiplicity $m > 1$. Moreover we may assume that $P \in U_0 \cong \mathbb{A}^2$ and that actually P is the origin of $\mathbb{A}^2$. We suppose that X has affine equation $f(x, y) = 0$ with the decomposition in homogeneous components given by

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$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots + f_d(x, y)$$

where d is the degree of X. Moreover the homogeneous polynomial $f_m(x, y)$ has m distinct roots, up to a proportionality factor, so that

$$f_m(x, y) = \prod_{i=1}^{m} (b_i x - a_i y)$$

with $(a_i, b_i) \neq (0, 0)$ non-proportional, for $i = 1, \ldots, m$. The principal tangents to X at P have affine equations

$$b_i x = a_i y, \quad i = 1, \ldots, m.$$

Next we blow-up $\mathbb{A}^2$ at the origin, getting $p : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$, with $\tilde{\mathbb{A}}^2$ contained in $\mathbb{A}^2 \times \mathbb{P}^1$, defined there by the equation $xv = yu$, where (x, y) are the coordinates in $\mathbb{A}^2$ and $[u, v]$ are the coordinates in $\mathbb{P}^1$. In the open set $A_0 \cong \mathbb{A}^1$ of $\mathbb{P}^1$ where $u \neq 0$, we may assume that $u = 1$. Then in the open set $B_0 = \mathbb{A}^2 \times A_0 \cong \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$ with coordinates (x, y, v), $\tilde{\mathbb{A}}_0 := \tilde{\mathbb{A}}^2 \cap B_0$ has equation $y = xv$. This is a surface isomorphic to $\mathbb{A}^2$, an isomorphism being given by

$$(x, v) \in \mathbb{A}^2 \to (x, xv, v) \in \tilde{\mathbb{A}}_0.$$

We may identify $\tilde{\mathbb{A}}_0$ to $\mathbb{A}^2$ via this map. The map $p : \tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ restricts to $\tilde{\mathbb{A}}_0$, to the map $p_0 : (x, y, v) \in \tilde{\mathbb{A}}_0 \to (x, y) = (x, xv) \in \mathbb{A}^2$. If $E = p^{-1}(P)$ is the exceptional locus of the bow up, its intersection $E_0 = E \cap \tilde{\mathbb{A}}_0$ has equation $x = 0$.

Similarly, in the open set $A_1 \cong \mathbb{A}^1$ of $\mathbb{P}^1$ where $v \neq 0$, we may assume that $v = 1$. Then in $B_1 = \mathbb{A}^2 \times A_1 \cong \mathbb{A}^2 \times \mathbb{A}^1 = \mathbb{A}^3$ with coordinates (x, y, u), $\tilde{\mathbb{A}}_1 := \tilde{\mathbb{A}}^2 \cap B_1$ has equation $x = yu$. This is isomorphic to $\mathbb{A}^2$, an isomorphism being given by

$$(y, u) \in \mathbb{A}^2 \to (yu, y, u) \in \tilde{\mathbb{A}}_1.$$

Again we may identify $\tilde{\mathbb{A}}_0$ to $\mathbb{A}^2$ via this map. The map p restricts to $\tilde{\mathbb{A}}_1$, to the map $p_1 : (x, y, u) \in \tilde{\mathbb{A}}_1 \to (x, y) = (yu, y) \in \mathbb{A}^2$. Finally $E_1 = E \cap \tilde{\mathbb{A}}_1$ has equation $y = 0$.

Let us now study $Y := \tilde{X} \cap \tilde{\mathbb{A}}^2$, and more precisely, we study $Y_i = Y \cap \tilde{\mathbb{A}}_i$, for $i = 0, 1$. First let us look at Y_0. We consider the total transform $p^{-1}(X_0)$ of $X_0 = X \cap U_0$. In the open set $\tilde{\mathbb{A}}_0$ it is given by the equation $f(x, vx) = 0$. We have

$$\begin{aligned} f(x, xv) &= f_m(x, xv) + \cdots + f_d(x, xv) = \\ &= x^m f_m(1, v) + \cdots + x^d f_d(1, v) = \\ &= x^m (f_m(1, v) + \cdots + x^{d-m} f_d(1, v)). \end{aligned}$$

So $f(x, xv)$ is reducible in the factor x^m and in the factor $f_m(1, v) + \cdots + x^{d-m} f_d(1, v)$. This tells us that $p^{-1}(X_0) \cap \tilde{\mathbb{A}}_0$ is also reducible: it contains the exceptional locus E_0 with multiplicity m, plus the other component with equation

$$f_m(1, v) + \cdots + x^{d-m} f_d(1, v) = 0 \tag{18.1}$$

which is nothing else than Y_0. We are interested in the intersection points of Y_0 with E_0, that are obtained from the system of (18.1) plus the equation $x = 0$, and this system is equivalent to

$$x = 0, \quad f_m(1, v) = \prod_{i=1}^{m} (b_i - a_i v) = 0.$$

So the distinct solutions of this system are the m points $Q_1, \ldots, Q_m$ of E_0 with coordinate $x = 0$ and

$$v = \frac{b_i}{a_i}, \quad i = 1, \ldots, m$$

provided $a_i \neq 0$. If there is an $i \in \{1, \ldots, m\}$ such that $a_i = 0$, hence we may assume $b_i = 1$, the degree of the polynomial $f_m(1, v)$ drops to $m - 1$ and we have only $m - 1$ solutions as above. Note however that we may have made a change of variables so that none of the principal tangents to X at P is the y axis $x = 0$. In this case $a_i \neq 0$ for all $i = 1, \ldots, m$ and $f_m(1, v) = 0$ has m distinct solutions as indicated above. Now we want to prove that the intersection points $Q_1, \ldots, Q_m$ of E_0 with Y_0 are smooth points of Y_0. Take one of these points Q_i, which has coordinates $x = 0$, $v = \frac{b_i}{a_i}$, for $i = 1, \ldots, m$. To make things easier, we may suppose to have chosen coordinates so that the principal tangent line to X at P with equation $b_i x = a_i y$ is the x axis with equation $y = 0$, so that $b_i = 0$ and we may assume $a_i = 1$. So Q_i in the $\mathbb{A}^2$ with coordinates (x, v) is the origin Q, and the equation of Y_0 is of the form

$$\psi(x, v) = v \prod_{i=1}^{m-1} (b_i - a_i v) + x f_{m-1}(1, v) + \cdots + x^{d-m} f_d(1, v) = 0$$

where $b_i \neq 0$ for all $i = 1, \ldots, m - 1$. Now we have

$$\frac{\partial \psi}{\partial v}(0, 0) = \prod_{i=1}^{m-1} b_i \neq 0$$

and this shows that Q is a smooth point for Y_0. The analysis is identical in the other open set $\tilde{\mathbb{A}}_1$. The conclusion is that in correspondence with the m principal tangent lines to X at P, there are exactly m distinct intersection points of $\tilde{X}$ with the exceptional locus E, and they are smooth for $\tilde{X}$. This proves the assertion. □

18.2 Reduction to Ordinary Singularities

18.2.1 Statement of the Main Theorem

In this section we prove the following fundamental result, that, together with Theorem 18.1.1, completes the reduction of singularities of curves:

Theorem 18.2.1 *Let X be an irreducible curve in $\mathbb{P}^2$. There is a birational transformation $\omega : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that, restricted to X, induces a birational transformation of X to a curve Y with only ordinary singularities.*

As an immediate consequence we have the:

Corollary 18.2.2 *In any birational equivalence class of irreducible quasi-projective curves there is a unique smooth projective model. More precisely, let X be any projective irreducible curve. There is a projective smooth curve C, uniquely determined up to isomorphism, with a birational morphism $\phi : C \to X$.*

Proof The first assertion follows from the second, since any irreducible quasi-projective curve is birational to a projective curve. Now, let X be any projective irreducible curve. By Theorem 7.2.3, X is birational to a projective curve Z in $\mathbb{P}^2$. By Theorem 18.2.1, Z is birational to a projective curve Y with ordinary singularities. By Theorem 18.1.1, there is a smooth curve C and a birational morphism $f : C \to Y$. According to Corollary 14.2.12, such a curve is unique up to isomorphism. Moreover there is a birational map $f : C \dashrightarrow X$. By Theorem 14.2.16, f is a morphism, proving the assertion.

For the proof of Theorem 18.2.1 we need some preliminaries which we will make now.

18.2.2 Standard Quadratic Transformations

First of all we briefly recall the contents of Exercises 7.1.19–7.1.24. Consider $\mathbb{P}^2$ with homogeneous coordinates $[x, y, z]$ and the *fundamental triangle* formed by the three lines

$$(a) \quad x = 0, \quad (b) \quad y = 0, \quad (c) \quad z = 0$$

which pairwise intersect at the *vertices of the triangle*

$$A = [1, 0, 0] = b \cap c, \quad B = [0, 1, 0] = a \cap c, \quad C = [0, 0, 1] = a \cap b.$$

Set $U = \mathbb{P}^2 \setminus (a \cup b \cup c)$.Consider the rational map

$$\omega : [x, y, z] \in \mathbb{P}^2 \dashrightarrow [yz, xz, xy] \in \mathbb{P}^2.$$

This is called the *standard quadratic transformation based* at the points A, B, C, which are also called the *fundamental points* of ω and the lines a, b, c are called the *exceptional lines* of ω. We recall the main properties of ω:

(a) the definition set of ω is $\mathbb{P}^2 \setminus \{A, B, C\}$;
(b) the restriction of ω to the line a [resp. to b and c] is the constant map whose image is the point A [resp. the points B and C]. One says that ω *contracts* a [resp. b and c] to the point A [resp. to the points B and C] or that it *blows-up* the point A [resp. to the points B and C] to the line a [resp. b and c];
(c) $\omega^2 = \mathrm{id}$, and ω induces an isomorphism of U to itself, so that ω is a birational map;
(d) consider a line r containing A, with equation $\lambda y + \mu z = 0$. Then the restriction of ω to r maps r to the line with equation $\mu y + \lambda z = 0$, which cuts the line a in the point $R = [0, -\lambda, \mu]$. The map which sends the line r through A to the point R in a is a projectivity from the pencil (A) of lines through A to the line a. The point R of a can be considered as the correspondent of the *direction* of the line r in A. Similar considerations can be made for the point B [resp. C] in relation with the line b [resp. c].

If $\tau : \mathbb{P}^2 \to \mathbb{P}^2$ is a projectivity, the composite map $\omega \circ \tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is still called a *standard quadratic transformation*. It is *based* at the points $\tau^{-1}(A), \tau^{-1}B), \tau^{-1}(C)$ and its has the exceptional lines $\tau^{-1}(a), \tau^{-1}(b), \tau^{-1}(c)$. All what holds for ω holds also for an application of the form $\omega \circ \tau$.

18.2.3 Transformation of a Curve via a Standard Quadratic Transformation

Let X be a reduced curve in $\mathbb{P}^2$ which does not contain any of the exceptional lines of ω. We will assume that X is irreducible, though all what we will say applies to reducible and reduced curves, not containing any of the exceptional lines of ω, with essentially no change. Of course $U \cap X$ is an open subset of X. Then $\omega(X \cap U)$ is a curve in U isomorphic to $X \cap U$. Its closure X' in $\mathbb{P}^2$ is called the *proper transform* of X via ω. Clearly X' is irreducible and birational to X and $(X')' = X$.

We want to determine the equation of X' starting from the equation of X. Suppose that X has degree n with equation

$$f(x, y, z) = 0.$$

Then we can consider the curve with equation

$$f_\omega(x, y, z) = f(yz, xz, xy) = 0.$$

This curve, denoted by X'_t is called the *total transform* of X via ω. Note that X'_t has degree $2n$.

Lemma 18.2.3 *In the above setting, suppose that* $m_A(X) = r$*,* $m_B(X) = s$ *and* $m_C(X) = t$*. Then* $\deg(X') = 2n - r - s - t$*.*

Proof Since $m_A(X) = r$, the equation of X is of the form

$$f_r(y, z)x^{n-r} + \cdots + f_n(y, z) = 0$$

where we wrote f as a polynomial in x with coefficients homogeneous polynomials in y, z of degree equal to the index, and $f_r(y, z)$ is not zero. We have

$$\begin{aligned} f_\omega(x, y, z) &= f_r(xz, xy)(yz)^{n-r} + \cdots + f_n(xz, xy) = \\ &= x^r f_r(z, y)(yz)^{n-r} + \cdots + x^n f_n(z, y) \end{aligned} \tag{18.2}$$

and therefore X'_t contains the line a exactly with multiplicity r. Similarly, if $m_B(X) = s$ and $m_C(X) = t$, we have that X'_t contains the lines b and c with multiplicities exactly s and t respectively. Then we have

$$f_\omega(x, y, z) = x^r y^s z^t f'(x, y, z) \tag{18.3}$$

and X' has equation $f' = 0$, and the assertion follows. □

Lemma 18.2.4 *In the above setting we have*

$$m_A(X') = n - s - t, \quad m_B(X') = n - r - t, \quad m_C(X') = n - r - s.$$

Proof It suffices to prove only the first equality, since the others are proved in a similar way. Taking into account (18.2), the equation of X' is of the form

$$\sum_{i=0}^{n-r} f_{r+i}(z, y)y^{n-r-s-i}z^{n-r-t-i}x^i = 0$$

hence the term with highest degree in x is

$$x^{n-r} f_n(z, y)y^{-s}z^{-t}.$$

Now

$$n - r = (2n - r - s - t) - (n - s - t) = \deg(X') - (n - s - t)$$

and $f_n(z, y)y^{-s}z^{-t}$ has to be a polynomial. This proves that $m_A(X') = n - s - t$. □

Note that the previous two lemmas imply another proof of the fact, contained in Exercises 7.1.23 and 7.1.24, that ω maps the lines of the plane to conics through the points A, B, C and viceversa.

We will say now that X is in *good position* with respect to ω if none of the principal tangents to X in A, B, C coincides with one of the exceptional lines.

Lemma 18.2.5 *In the above setting, if X is in good position with respect to ω, then also X' is in good position with respect to ω.*

Proof Suppose X is in good position with respect to ω and assume, by contradiction, that the line a is among the principal tangents to X' at B. Then we have

$$i(B : X', a) > m_B(X') = n - r - t. \tag{18.4}$$

By taking into account (18.2) and (18.3), we see that the equation of X' is of the form

$$f_r(z, y)y^{n-r-s}z^{n-r-t} + \cdots + x^{n-r} f_n(z, y)y^{-s}z^{-t} = 0.$$

Intersecting with $x = 0$, we get the equation

$$f_r(z, y)y^{n-r-s}z^{n-r-t} = 0. \tag{18.5}$$

In this equation z appears with an exponent larger that $n - r - t$ (and therefore (18.4) holds), if and only if z divides $f_r(z, y)$, i.e., $f_r(1, 0) = 0$, and this is equivalent to say that the line b is one of the principal tangent lines to X at A, a contradiction. □

We notice that if X is in good position with respect to ω, and if $P_1, \ldots, P_h$ are the non-fundamental points of X' on a, then $P_1, \ldots, P_h$ correspond to the directions of the principal tangent lines to X at A. In particular, if X has in A an ordinary multiple point of multiplicity r then $h = r$ and $P_1, \ldots, P_r$ correspond to the r tangent lines to X at P.

Corollary 18.2.6 *In the above setting, if X is in good position with respect to ω, then*

$$\sum_{i=1}^{h} i(P_i; X', a) = r \tag{18.6}$$

and similarly for the lines b and c. In particular:

(a) if $h = r$, i.e., if X has in A an ordinary r-tuple point, then $P_1, \ldots, P_r$ are all smooth for X';
(b) if $h > 1$, one has

$$m_A(X) > m_{P_i}(X'), \quad \textit{for all} \quad i = 1, \ldots, h.$$

Proof The intersections of X' with a are obtained solving the Eq. (18.5). By the good position hypothesis, $f_r(z, y)$ is neither divisible by y nor for z, so the intersection multiplicity of a with X' in B and C is $n - r - t$ and $n - r - s$ respectively, whereas the sum of the remaining intersection multiplicities is the degree of f_r, i.e., it is r.

As for the proof of (a), note that if $h = r$ from (18.6) we have $i(P_i; X', a) = 1$ for all $i = 1, \dots, r$, and the assertion follows.

For (b), since $h > 1$, we have

$$m_A(X) = r > i(P_i; X', a) \geqslant m_{P_i}(X'), \quad \text{for all} \quad i = 1, \dots, h$$

whence the assertion. □

Next we say that C is in *very good position* with respect to ω and to A, if it is in good position and moreover the line a intersects X in n distinct points not lying on b and c, whereas b and c intersect X off A in $n - r$ distinct points each.

Lemma 18.2.7 *In the above setting, if X is in very good position with respect to ω and A, then X' has the following singularities:*

(a) the singular points in $X' \cap U$ correspond to the singular points of $X \cup U$ and for them it is preserved the multiplicity and the fact that they are ordinary or not;
(b) A, B, C are ordinary of multiplicities n, $n - r$ and $n - r$ respectively;
(c) the intersection of b [resp. of c] with X' are only at the fundamental points A and C [resp. A and B]. If $P_1, \dots, P_h$ are the non-fundamental points on a, one has

$$\sum_{i=1}^{h} i(P_i; X', a) = r.$$

Proof Part (b) follows from (a) of Corollary 18.2.6. Part (c) follows from Corollary 18.2.6 and from the fact that X' has degree $2n - r$, because $s = t = 0$, and b and c intersect X' in two fundamental points of multiplicities n and $n - r$.

Let us prove part (a). Since ω induces an isomorphism between $X \cup U$ and $X' \cap U$, it is clear that ω maps smooth points of $X \cap U$ to smooth points of $X \cap U$ and singular points of $X' \cap U$ to singular points of $X' \cap U$. It remains to show that $\omega_{|U}$ preserves the multiplicity of the singular points and the fact that they are ordinary o not.

To prove this, we make the following argument. First we pass to affine coordinates by setting $z = 1$, so that ω becomes the following birational map of $\mathbb{A}^2$

$$(x, y) \in \mathbb{A}^2 \to \left(\frac{1}{x}, \frac{1}{y}\right) \in \mathbb{A}^2.$$

The open set U coincides with the open set of $\mathbb{A}^2$ which is the complement of the coordinate axes. Let us take a point $P = (a, b) \in U$, with a, b both non-zero. Set

$P' = \omega(P) = (\frac{1}{a}, \frac{1}{b})$. Consider a branch γ of curve with centre P. We may suppose γ is determined by a parametrization

$$x = a + t^n + \cdots, \quad y = b + ct^m + \cdots$$

with n, m positive integers, with $n \leqslant m$ and $c \neq 0$. Note that the tangent to γ is the line with equation $y = b$ if $m > n$ and is the line with equation $c(x - a) = y - b$ if $n = m$.

Then ω maps γ to the branch γ' determined by the parametrization

$$x = \frac{1}{a + t^n + \cdots} = \frac{1}{a} - \frac{1}{a^2}t^n + \cdots, \quad y = \frac{1}{b + ct^m + \cdots} = \frac{1}{b} - \frac{c}{b^2}t^m + \cdots.$$

The tangent to γ' is the line with equation $y = \frac{1}{b}$ if $m > n$ and is the line with equation $\frac{c}{b^2}(x - \frac{1}{a}) = \frac{1}{a^2}(y - \frac{1}{b})$.

This shows that ω maps branches of a curve to branches of a curve, preserving the order of the branch, and mapping branches with different tangents to branches with different branches. The assertion follows by taking into account Proposition 16.3.5.

□

18.2.4 Proof of the Main Theorem

In what follows we will apply standard quadratic transformations based at suitable singular points of X. We will abuse notation and we will still denote these points by A, B, C as if the transformation were ω. This makes no difference because we can change coordinates and put any three non-collinear points in the points A, B, C.

Keeping the above notation, we say that X' has *milder singularities* than X if:

(a) either the maximum multiplicity of a non-ordinary singular point of X' is smaller than the maximum multiplicity of a non-ordinary singular point of X,
(b) or the number of non-ordinary singular points of X' with maximum multiplicity is smaller than the number of non-ordinary singular points of X with maximum multiplicity.

Suppose A is a non-ordinary singular point of X with maximum multiplicity. Suppose that we are able to put X in very good position with respect to ω and A. Then, by Lemma 18.2.7 from which we keep the notation, if $h > 1$ the curve X' will have milder singularities with respect to X. We will say that a singular point A of X is a *bad point*, if $h = 1$ for it and for all the points deduced from it by iterated applications of standard quadratic transformations as above. If we are able:

(i) to put the curves in very good position and
(ii) to exclude the existence of bad points,

then iterated application of the procedure described above eventually leads to a curve with only ordinary singularities and this will prove Theorem 18.2.1. Therefore we have to prove (i) and (ii) above.

First we deal with (ii). For this we introduce for any curve X of degree n the following number

$$p(X) = \frac{(n-1)(n-2)}{2} - \sum_{P \in X} \frac{m_P(X)(m_P(X)-1)}{2}$$

which is non-negative by Lemma 17.1.3.

Lemma 18.2.8 *In the above setting, suppose that X is in very good position with respect to ω and A and that $P_1, \ldots, P_h$ are the non-fundamental points of X' on the line a. Then one has*

$$p(X') = p(X) - \sum_{i=1}^{h} \frac{m_{P_i}(X')(m_{P_i}(X')-1)}{2}. \tag{18.7}$$

Proof Suppose that the multiple points of X in U are $Q_1, \ldots, Q_l$ with multiplicities $m_1, \ldots, m_l$. Let r be the multiplicity of X in A. By the very goodness assumption there is no other singular point of X but A on $a \cup b \cup c$. Then we have

$$p(X) = \frac{(n-1)(n-2)}{2} - \frac{r(r-1)}{2} - \sum_{i=1}^{l} \frac{m_i(m_i-1)}{2}$$

whereas

$$p(X') = \frac{(2n-r-1)(2n-r-2)}{2} - \sum_{i=1}^{l} \frac{m_i(m_i-1)}{2} -$$
$$- \sum_{i=1}^{h} \frac{m_{P_i}(X')(m_{P_i}(X')-1)}{2} - \frac{n(n-1)}{2} - 2\frac{(n-r)(n-r-1)}{2}.$$

One finds

$$\frac{(2n-r-1)(2n-r-2)}{2} - \frac{n(n-1)}{2} - 2\frac{(n-r)(n-r-1)}{2} =$$
$$= \frac{(n-1)(n-2)}{2} - \frac{r(r-1)}{2},$$

whence (18.7) immediately follows. □

Corollary 18.2.9 *Suppose we can put X and its transformed curves in very good position, then there are no bad points for X.*

Proof This is an immediate consequence of Lemma 18.2.8 and of the non-negativity of $p(X)$. □

Finally to conclude the proof of Theorem 18.2.1, we need to deal with the very good position issue (i). This is done in the following:

Lemma 18.2.10 *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let X be an irreducible plane curve, P a point of X. Then we can choose a projectivity τ such that X is in very good position with respect to $\omega \circ \tau$ and P.*

Proof Let n be the degree of X and r the multiplicity at P. It suffices to find three lines a, b, c such that b, c contain P and intersect X off P in $n - r$ distinct points, and a line a which intersects X in n distinct points which are neither on b nor on c.

First of all we choose a. In fact by Theorem 12.2.3 there is a non-empty Zariski open subset U of $\check{\mathbb{P}}^2$ such that for each $a \in U$, a intersects X in n distinct points. So we can choose a in infinitely many ways.

To choose b and c we change coordinates and put P at the point at infinity of the y axis and a in the x axis, so that X has equation of the form

$$y^{n-r} f_r(x) + \cdots + f_n(x) = 0 \tag{18.8}$$

where $f_i(x)$ is a polynomial of degree at most i in x, for $i = r, \ldots, n$, $f_r(x)$ is not identically 0 and $f_n(x) = 0$ has n distinct solutions $h_1, \ldots, h_n$. The affine lines through P have equation $x = h$, with $h \in \mathbb{K}$, and we need to find two such lines which intersect X in $n - r$ distinct points in $\mathbb{A}^2 \setminus \{(h_1, 0), \ldots, (h_n, 0)\}$. The line $x = h$ intersects X in $\mathbb{A}^2$ in the points having the y-coordinate solution of the equation

$$y^{n-r} f_r(h) + \cdots + f_n(h) = 0. \tag{18.9}$$

Since $f_r(x)$ is not identically zero, the equation $f_r(x) = 0$ has only finitely many solutions, so we can choose h in the dense open Zariski subset U' of $\mathbb{K} = \mathbb{A}^1$ such that $f_r(h) \neq 0$ and h is distinct from $h_1, \ldots, h_n$. For these values of h, (18.9) is an equation of degree $n - r$ which does not have the solution $y = 0$ because $f_n(h) \neq 0$. We want this equation to have $n - r$ distinct solutions. This is not the case if and only if the polynomial in (18.9) has some common solution with its derivative with respect to y. Consider the system

$$\begin{aligned} &y^{n-r} f_r(x) + \cdots + f_n(x) = 0 \\ &(n-r)y^{n-r-1} f_r(x) + \cdots + f_{n-1}(x) = 0 \end{aligned} \tag{18.10}$$

formed by (18.8) and by its derivative with respect to y, which, since the characteristic of $\mathbb{K}$ is zero, is a non-zero polynomial. Since X is irreducible, the first polynomial in (18.10) is irreducible. Hence the two polynomials in (18.10) have no common factor, and therefore they have a finite set S of common solutions. If we take h in the dense open subset $U'' = U' \setminus (U' \cap S)$, then the Eq. (18.9) has exactly $n - r$

distinct solutions, none of them equal to 0. This implies that we can choose a, b, c in infinitely many ways so that b, c contain P and intersect X off P in $n - r$ distinct points, and a intersects X in n distinct points which are neither on b nor on c. This proves the assertion in this case. □

We can finally give the:

Proof (*of Theorem* 18.2.1) The proof is complete if the characteristic of $\mathbb{K}$ is 0. To conclude, we need to deal with the case in which $\mathrm{char}(\mathbb{K}) = p > 0$.

The argument in Lemma 18.2.10 goes through, except that the derivative of (18.8) with respect to y could be identically 0. In this case all the lines passing through P do not intersect X in $n - r$ distinct points off P and therefore a non-empty open subset of the set $(P) \subset \check{\mathbb{P}}^2$ of lines through P consists of tangent lines to X at smooth points. Then we say that P is a *nasty point* of X. If this is the case p divides $n - r$. Moreover, since the dual curve of X is irreducible (see Exercise 17.3.25) and it has infinitely many points in common with the line (P) of $\mathbb{P}^2$ then $\check{X} = (P)$ and this implies that P can be the only nasty point of X.

We now show that we can eliminate the nasty point P by making a suitable standard quadratic transformation. In fact make a standard quadratic transformation based at a point A of multiplicity $m = 0, 1$ of X, which is not nasty for X, so we can choose the line a intersecting X in n distinct points, and the lines b and c intersecting X in $n - m$ distinct points not on $a \cap X$. The transformed curve X' has acquired three more ordinary points of multiplicities $n, n - m$ and $n - m$, and has the same singularities as X besides them. So the image P' of P is still a point of maximum multiplicity among the non-ordinary singularities of X'. Note that X' has degree $n' = 2n - m$ and P' has still multiplicity r. Then $n' - r = 2n - m - r$ and this is congruent to $n - m$ modulo p. So appropriately choosing $m = 0, 1$ we can avoid that p divides $n' - r$ so P' is not nasty for X' and we can proceed. Repeating this process, and applying all the above considerations, we get rid of all non ordinary points, concluding the proof of the theorem. □

Exercise 18.2.11 * Let C be a smooth, projective, irreducible curve and let P be a point of C. Prove that there is a plane, irreducible, projective curve $X \subset \mathbb{P}^2$ with ordinary singularities such that $\pi : C \to X$ is the smooth model of X, and $\pi(P)$ is a smooth point of X.

Exercise 18.2.12 Analyze the singularities of the following projective curves over $\mathbb{C}$ and make a transformation of each of them in a curve with ordinary singularities

$$zy^2 = x^3$$
$$(x^2 + y^2)^2 - 4x^2z^2 - y^2z^2 = 0$$
$$y^3(y - 2z) + x^2z^2 = 0$$
$$4x^4 + x^3y + xy^2z - y^2z^2 = 0$$
$$(x^2 + y^2 - 2xz)^2 - xy^3 = 0$$
$$(x^2 + y^2 - 2xz)^2 - x^3z = 0$$

Chapter 19
Divisors, Linear Equivalence, Linear Series

In this chapter $X \subset \mathbb{P}^2$ will be an irreducible projective plane curve and we will denote by $\pi : C \to X$ its smooth birational model. A point P of C will be sometimes called a *place* of X *centered* at the point $Q = f(P)$.

19.1 Divisors

In this section we extend in the case of curves the concept of divisors already introduced in another situation in Sect. 1.6.5.

Let C be a smooth projective curve. A *divisor* on C is a formal sum

$$D = \sum_{P \in C} n_P P, \quad \text{with} \quad n_P \in \mathbb{Z}$$

and $n_P = 0$ except for a finite number of points of C. The integer n_P is called the *multiplicity* of D in P. The set of points P such that $n_P \neq 0$ is called the *support* of D and denoted by $\mathrm{Supp}(D)$.

The divisors of C form an abelian group $\mathrm{Div}(C)$, which is the free abelian group generated by the points of C. The *degree* of $D = \sum_{P \in C} n_P P$ is the integer $\deg(D) = \sum_{P \in C} n_P$ and the map

$$\deg : D \in \mathrm{div}(C) \to \deg(D) \in \mathbb{Z}$$

is a group homomorphism. A divisor $D = \sum_{P \in C} n_P P$ is said to be *effective* if $n_P \geqslant 0$ for all $P \in C$, and in this case one writes $D \geqslant 0$. An effective divisors $D = \sum_{P \in C} n_P P$ is said to be *reduced* if $n_P \leqslant 1$ for all $P \in C$. In this case one also says that D consists of $\deg(D)$ distinct points.

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Given two divisors D, D' one sets $D \geqslant D'$ if and only if $D - D' \geqslant 0$. This is a partial order relation in the group $\mathrm{Div}(C)$. If $D \geqslant D' \geqslant 0$, one says that D *contains* D'. If $P \in C$ is a point and $D \geqslant P$, one writes $P \in D$ and says that P *belongs* to D.

Consider $X \subset \mathbb{P}^2$ an irreducible projective plane curve of degree n and we will denote by $\pi : C \to X$ its smooth birational model. Since we will be interested in questions that are invariant under birational transformations, we may assume that X has only ordinary singularities. The process described in Sect. 18.1 shows that the places of X are in 1:1 correspondence with the branches of X, which are all linear.

Let now Y be a plane curve of degree m with equation $g = 0$, which does not contain X as a component. We define the effective divisor $\mathrm{div}_X(Y) := \mathrm{div}_X(g)$ (also denoted $\mathrm{div}(Y)$ or $\mathrm{div}(g)$ if there is no danger of confusion) on C in the following way. Let P be a point of C and let γ_P be the linear branch of X corresponding to P. Then we set

$$\mathrm{div}_X(Y) = \sum_{P \in C} o_{\gamma_P}(Y).$$

This divisor is called the *divisor cut out by Y on X*. By Theorem 16.3.3, for any point $Q \in X$ we have

$$i(Q; X, Y) = \sum_{P \in f^{-1}(Q)} o_{\gamma_P}(Y)$$

and by Bézout Theorem

$$\deg(\mathrm{div}_X(Y)) = nm.$$

Next consider a non-zero element $\phi \in K(C) = K(X)$. Let $P \in C$ be a zero of ϕ. We will denote by $\mathrm{ord}_P^0(\phi)$ the order of zero of ϕ at P (see Remark 14.2.13). We set

$$(\phi)_0 = \sum_{P \in C} \mathrm{ord}_P^0(\phi)$$

and this is called the *divisor of zeros* of ϕ. Similarly, if $P \in C$ is a pole of ϕ, we denote by $\mathrm{ord}_P^\infty(\phi)$ the order of pole of ϕ at P, and we set

$$(\phi)_\infty = \sum_{P \in C} \mathrm{ord}_P^\infty(\phi)$$

and call it the *divisor of poles* of ϕ.

Finally we set

$$\mathrm{div}(\phi) = (\phi) = (\phi)_0 - (\phi)_\infty$$

and call it the *divisor of* ϕ. Divisors of this type are called *principal divisors*. The map

$$\mathrm{div} : \phi \in K(C) \setminus \{0\} \to \mathrm{div}(\phi) \in \mathrm{Div}(C)$$

is a homomorphism for the multiplicative structure of $K(C)$.

Proposition 19.1.1 *For any $\phi \in K(C) \setminus \{0\}$ we have*

$$\deg(\mathrm{div}(\phi)) = 0.$$

Proof Let $X \subset \mathbb{P}^2$ be an irreducible curve with only ordinary singularities such that $\pi : C \to X$ is the smooth model. Any rational function $\phi \in K(C) \setminus \{0\} = K(X) \setminus \{0\}$ can be written as $\phi = \frac{g}{h}$, with $g, h \in S(X)$, $g, h \neq 0$, and g, h of the same degree (see Theorem 5.5.3). Let G, H be homogeneous polynomials in S_2 of the same degree whose classes in $S(X)$ are g, h. Then (see Exercise 19.1.2) one has

$$\mathrm{div}(\phi) = \mathrm{div}(G) - \mathrm{div}(H)$$

whose degree is 0 because $\mathrm{div}(G)$ and $\mathrm{div}(H)$ have the same degree. □

Exercise 19.1.2 *Consider $X \subset \mathbb{P}^2$ an irreducible projective plane curve and let $\pi : C \to X$ be its smooth birational model. Assume that X has only ordinary singularities. Let Z be a plane curve with equation $g = 0$ which does not contain X as a component. Fix $P \in C$ and $Q = f(P)$ and let γ_P be the linear branch of X with centre Q determined by P. The map f determines an injective homomorphism

$$f^* : \mathcal{O}_{X,Q} \to \mathcal{O}_{C,P}.$$

We may assume that $Q \in U_0 \cong \mathbb{A}^2$. Let g_* be, as usual, the dehomogenization of g. Interpret g_* as an element of $\mathcal{O}_{X,Q}$. Prove that $o_{\gamma_P}(Z)$ equals the order of $f^*(g_*)$ as an element of the DVR $\mathcal{O}_{C,P}$.

Exercise 19.1.3 *Let ϕ be a non-zero rational function on the smooth curve C. Prove that $\mathrm{div}(\phi) = 0$ if and only if $\phi \in \mathbb{K}^*$.

Exercise 19.1.4 *Let ϕ, ϕ' be non-zero rational functions on the smooth curve C. Prove that $\mathrm{div}(\phi) = \mathrm{div}(\phi')$ if and only if there is $\lambda \in \mathbb{K}^*$ such that $\phi' = \lambda\phi$.

19.2 Linear Equivalence

Two divisors on the smooth projective curve C are said to be *linearly equivalent*, and one writes $D \equiv D'$, if and only if there is a function $\phi \in K(C) \setminus \{0\}$ such that $D' = D + \mathrm{div}(\phi)$.

Proposition 19.2.1 *The linear equivalence has the following properties:*

(a) it is an equivalence relation;
(b) if $D \equiv D'$, then $\deg(D) = \deg(D')$;

(c) *if* $D \equiv D'$ *and* $D_1 \equiv D_1'$, *then* $D + D_1 \equiv D' + D_1'$;
(d) $D \equiv 0$ *if and only if there is a non-zero rational function* ϕ *such that* $D = \mathrm{div}(\phi)$;
(e) *if* $X \subset \mathbb{P}^2$ *is an irreducible curve with only ordinary singularities such that* $\pi : C \to X$ *is the smooth model, then two divisors* D, D' *on* C *are linearly equivalent if and only if there are two plane curves* Y, Y' *not containing* X *such that*

$$D + \mathrm{div}(Y) = D' + \mathrm{div}(Y').$$

Proof Parts (a)–(d) are easy and can be left to the reader as an exercise. As for part (e), note that $D \equiv D'$ is equivalent to say that there is $\phi \in K(X) \setminus \{0\}$ such that $D = D' + \mathrm{div}(\phi)$. Then $\phi = \frac{g}{g'}$ with $g, g' \in S(X)$ homogeneous of the same degree and $g, g' \neq 0$. Let $G, G' \in S_2$ be homogeneous polynomials of the same degree whose classes in $S(X)$ are g, g'. We have

$$\mathrm{div}(\phi) = \mathrm{div}(G) - \mathrm{div}(G')$$

whence the assertion immediately follows. □

Exercise 19.2.2 *Verify parts (a)–(d) of Proposition 19.2.1.

Exercise 19.2.3 *Prove that two divisors on $\mathbb{P}^1$ are linearly equivalent if and only if they have the same degree.

Exercise 19.2.4 *Let C be a smooth projective curve, let D be a divisor on C and S be a finite subset of C. Prove that there is a divisor $D' \equiv D$ such that $S \cap \mathrm{Supp}(D') = \emptyset$.

19.3 Fibres of a Morphism

Let $f : C \to C'$ be a surjective morphism between smooth, irreducible, projective curves. Let Q be a point on C' and suppose $f^{-1}(Q) = \{P_1, \ldots, P_h\}$. Fix a uniformizing parameter u in $\mathcal{O}_{C',Q}$. For all $i = 1, \ldots, h$, we have an inclusion $f^* : \mathcal{O}_{C',Q} \to \mathcal{O}_{C,P_i}$. We set $e_{f,P_i} = o_{P_i}(f^*(u))$, for $i = 1, \ldots, h$. It is easy to see that e_{f,P_i} does not depend on the uniformizing parameter u. We define the *fibre divisor* of f at Q as the divisor on C' given by

$$f^*(Q) = \sum_{i=1}^{h} e_{f,P_i} P_i.$$

More generally, let $D = \sum_{Q \in C'} n_Q Q$ be a divisor on C'. We define $f^*(D) = \sum_{Q \in C'} n_Q f^*(Q)$, which is called the *pull-back* of D via f. One has $\mathrm{Supp}(f^*(D)) = f^{-1}(\mathrm{Supp}(D))$.

Recall that the degree of the morphism $f : C \to C'$ is the degree of the field extension $f^* : K(C') \to K(C)$ (see Sect. 10.4). We will prove in Theorem 19.3.5 below that the degree of all fibre divisors of $f : C \to C'$ equals the degree of f. In order to do so, we need some algebraic preliminaries.

Let C be a smooth, projective curve. If S is a finite subset of C, we will set

$$\mathcal{O}_{C,S} = \bigcap_{P \in S} \mathcal{O}_{C,P},$$

so that $\mathcal{O}_{C,S}$ is the ring of rational functions on C which are defined at all points of S.

Fix the surjective morphism $f : C \to C'$ of smooth, projective curves. If $Q \in C'$ and $S = f^{-1}(Q)$, then $\mathcal{O}_{C',Q}$, identified with its image via the injection $f^* : K(C') \to K(C)$, is a subring of $\mathcal{O}_{C,S}$.

Lemma 19.3.1 *Let C be a smooth, projective curve. If S is a finite subset of C then:*

(a) $\mathcal{O}_{C,S}$ is a domain with principal ideals, therefore it is a UFD;
(b) if $S = \{P_1, \dots, P_h\}$, there are elements $t_1, \dots, t_h \in \mathcal{O}_{C,S}$ such that

$$o_{P_i}(t_j) = \delta_{ij}, \quad \textit{for} \quad i, j = 1, \dots, h. \tag{19.1}$$

Moreover $t_1, \dots, t_h$ are pairwise relatively prime in $\mathcal{O}_{C,S}$;
(c) if $u \in \mathcal{O}_{C,S}$ and $u \neq 0$, then

$$u = vt_1^{r_1} \cdots t_h^{r_h}$$

where $r_i = o_{P_i}(u)$, for $i = 1, \dots, h$, and v is invertible in $\mathcal{O}_{C,S}$.

Proof For each $i = 1, \dots, h$, fix a uniformizing parameter $u_i \in \mathcal{O}_{C,P_i}$. Then $\operatorname{div}(u_i) = P_i + D_i$, with P_i not appearing in D_i, for $i = 1, \dots, h$. By Exercise 19.2.4, we can find a rational function f_i such that the support of the divisor $D_i + \operatorname{div}(f_i)$ has empty intersection with S, for $i = 1, \dots, h$. Then (19.1) is verified for $t_i = f_i u_i$, that sits in $\mathcal{O}_{C,S}$, with $i = 1, \dots, h$. This proves the first part of (b).

Let us prove (a). Let $\mathcal{I}$ be an ideal of $\mathcal{O}_{C,S}$. Set

$$r_i = \min\{o_{P_i}(g), g \in \mathcal{I}\}, \quad i = 1, \dots, h$$

and

$$f = t_1^{r_1} \cdots t_h^{r_h}.$$

Then one has $uf^{-1} \in \mathcal{O}_{C,S}$ for all $u \in \mathcal{I}$, hence $\mathcal{I} \subseteq (f)$. The us prove the opposite inclusion. Consider the set $\mathcal{J}$ of all elements of $\mathcal{O}_{C,S}$ of the form $uf^{-1} \in \mathcal{O}_{C,S}$ with $u \in \mathcal{I}$. It is clear that $\mathcal{J}$ is an ideal of $\mathcal{O}_{C,S}$. Moreover

$$\min\{o_{P_i}(g), g \in \mathcal{J}\} = 0, \quad \text{for} \quad i = 1, \ldots, h.$$

Hence, for all $i = 1, \ldots, h$, we can find $u_i \in \mathcal{J}$ such that $o_{P_i}(u_i) = 0$, i.e., $u_i(P_i) \neq 0$. Consider the element

$$z = \sum_{i=1}^{h} u_i t_1 \cdots \hat{t_i} \cdots t_h \in \mathcal{J}.$$

It is clear that $o_{P_i}(z) = 0$ for all $i = 1, \ldots, h$, and therefore $z^{-1} \in \mathcal{O}_{C,S}$. Then $1 = z^{-1}z \in \mathcal{J}$, so $\mathcal{J} = \mathcal{O}_{C,S}$. This immediately implies that $(f) \subseteq \mathcal{I}$, as desired.

Next we finish the proof of (b), showing that $t_1, \ldots, t_h$ are pairwise relatively prime in $\mathcal{O}_{C,S}$. Suppose, for instance, that $t_1 = ws_1$, $t_2 = ws_2$, with $w \in \mathcal{O}_{C,S}$ a common factor. Since $t_1(P_i) \neq 0$ for $i = 2, \ldots, h$, then also, $w(P_i) \neq 0$ for $i = 2, \ldots, h$. Similarly, since $t_2(P_i) \neq 0$ for $i = 1, 3, \ldots, h$, then also $w(P_i) \neq 0$, for $i = 1, 3, \ldots, h$. So $w(P_i) \neq 0$ for $i = 1, \ldots, h$, and therefore w is invertible in $\mathcal{O}_{C,S}$.

Finally we prove (c). Fix $u \in \mathcal{O}_{C,S} \setminus \{0\}$, set $r_i = o_{P_i}(u)$ and note that $r_i \geqslant 0$ for $i = 1, \ldots, h$. Set $v = ut_1^{-r_1} \cdots t_h^{-r_h}$. Then $o_{P_i}(v) = 0$ for $i = 1, \ldots, h$, hence $v \in \mathcal{O}_{C,S}$ and also $v^{-1} \in \mathcal{O}_{C,S}$. This proves (c). □

Lemma 19.3.2 *In the same setting as in Lemma 19.3.1, one has*

$$\dim_{\mathbb{K}} \left(\mathcal{O}_{C,S}/(t_i^{r_i})\right) = r_i, \quad \textit{for all} \quad i = 1, \ldots, h.$$

Proof It suffices to prove the assertion for $i = 1$, the proof being analogous for $i > 1$. Let us set $P_1 = P$, $t_1 = t$ and $r_1 = r$. Fix $v \in \mathcal{O}_{C,S}$. The assertion will be proved if we prove that v can be uniquely written as

$$v = a_0 + a_1 t + \cdots + a_{r-1}t^{r-1} \quad \text{modulo} \quad t^r$$

and $a_0, \ldots, a_{r-1} \in \mathbb{K}$. We prove this by induction. Suppose we have a unique expression of the sort

$$v = a_0 + a_1 t + \cdots + a_{s-1}t^{s-1} \quad \text{modulo} \quad t^s$$

for a given $s < r$ (certainly there is one for $s = 1$, the first step of the induction). Then

$$w = t^{-s}(v - a_0 - a_1 t - \cdots - a_{s-1}t^{s-1}) \in \mathcal{O}_{C,S} \subset \mathcal{O}_{C,P}.$$

We set $w(P) = a_s \in \mathbb{K}$. Then $o_P(w - a_s) > 0$ and from Lemma 19.3.1 it follows that

$$w = a_0 + a_1 t + \cdots + a_s t^s \quad \text{modulo} \quad t^{s+1}$$

in a unique way. □

Lemma 19.3.3 *Let $f : C \to C'$ be a surjective morphism of projective curves, with C and C' smooth. Fix $Q \in C'$ and set $S = f^{-1}(Q) = \{P_1, \dots, P_h\}$. Then $\mathcal{O}_{C,S}$ is a finite $\mathcal{O}_{C',Q}$-module.*

Proof Recall that f is finite by Theorem 14.3.3. Then note that the problem is local. So we may assume that C and C' are affine and we set $A = A(C)$, $B = A(C')$, with $B \subseteq A$ and A a finitely generated B-module. We claim that $\mathcal{O}_{C,S} = A\mathcal{O}_{C',Q}$. The inclusion $A\mathcal{O}_{C',Q} \subseteq \mathcal{O}_{C,S}$ is obvious, so we prove the opposite inclusion.

Take any non-zero $\phi \in \mathcal{O}_{C,S}$, and let $\bar{P}_1, \dots, \bar{P}_l$ be the poles of ϕ. Then $f(\bar{P}_i) = Q_i \neq Q$, for $i = 1, \dots, l$. We can find a function $g \in B$ such that $g(Q) \neq 0, g(Q_i) = 0$ and $g\phi \in \mathcal{O}_{C,\bar{P}_i}$ for $i = 1, \dots, l$. Hence $g\phi \in A$. Moreover $g^{-1} \in \mathcal{O}_{C',Q}$, and we get $\phi = (g\phi)g^{-1} \in A\mathcal{O}_{C',Q}$. This proves that $\mathcal{O}_{C,S} \subseteq A\mathcal{O}_{C',Q}$, as wanted for the claim.

Finally, a set of finitely many generators of A over B, gives a set of generators of $\mathcal{O}_{C,S} = A\mathcal{O}_{C',Q}$ over $\mathcal{O}_{C',Q}$, and the assertion is proved. □

Lemma 19.3.4 *Let $f : C \to C'$ be a surjective morphism of smooth, projective curves. Fix $Q \in C'$ and set $S = f^{-1}(Q) = \{P_1, \dots, P_h\}$. Then $\mathcal{O}_{C,S}$ is a free $\mathcal{O}_{C',Q}$-module of rank $n = \deg(f)$, i.e.,*

$$\mathcal{O}_{C,S} \cong \mathcal{O}_{C',Q}^{\oplus n}.$$

Proof Since $\mathcal{O}_{C',Q}$ is a DVR, it has principal ideals (see Theorem 14.2.6). By the structure theorem of finitely generated modules over principal ideals domains, we have that $\mathcal{O}_{C,S}$ is the direct sum of a free module and a torsion module. However, $\mathcal{O}_{C,S}$ is contained in the field $K(C)$, so it has no torsion. Therefore $\mathcal{O}_{C,S}$ is a free $\mathcal{O}_{C',Q}$-module. We have to compute its rank, which is the maximum number m of elements of $\mathcal{O}_{C,S}$ that are linearly independent over $\mathcal{O}_{C',Q}$. This is the same as the maximum number of elements of $\mathcal{O}_{C,S}$ that are linearly independent over $\mathbb{Q}(\mathcal{O}_{C',Q}) = K(C')$.

Since the degree of $K(C)$ on $K(C')$ is n, we clearly have $m \leqslant n$. To finish we prove that $\mathcal{O}_{C,S}$ contains n elements that are linearly independent over $K(C')$. To see this, take $f_1, \dots, f_n \in K(C)$ that are a basis of $K(C)$ as a $K(C')$-vector space. Let r be the maximum order of poles of $f_1, \dots, f_n$ at the points $P_1, \dots, P_h$. Let u be a uniformizing parameter in $\mathcal{O}_{C',Q}$. The $f_1u^r, \dots, f_nu^r \in \mathcal{O}_{C,S}$ and still are linearly independent over $K(C')$, proving the assertion. □

This is the promised result about the degree of fibre divisors:

Theorem 19.3.5 *Let $f : C \to C'$ be a surjective morphism of smooth, projective curves. For every point $Q \in C'$, one has*

$$\deg(f^*(Q)) = \deg(f).$$

Proof We set $f^{-1}(Q) = \{P_1, \dots, P_h\}$ and $n = \deg(f)$. Let u be a uniformizing parameter at Q. By Lemma 19.3.1 we have $u = vt_1^{r_1} \cdots t_h^{r_h}$, with $r_i = o_{P_i}(u)$ for $i = 1, \dots, h$ and v is invertible in $\mathcal{O}_{C,S}$. By the definition of $f^*(Q)$ we have

$$f^*(Q) = \sum_{i=1}^{h} r_i P_i, \quad \text{hence} \quad \deg(f^*(Q)) = \sum_{i=1}^{h} r_i.$$

Consider now the obvious surjective linear map

$$\phi : \mathcal{O}_{C,S}/(u) \to \oplus_{i=1}^{h} \mathcal{O}_{C,S}/(t_i^{r_i}).$$

Since $t_1, \ldots, t_h$ are pairwise relatively prime in $\mathcal{O}_{C,S}$ by Lemma 19.3.1, one immediately sees that ϕ is also injective, so it is an isomorphism. By Lemma 19.3.2, we have $\dim(\mathcal{O}_{C,S}/(u)) = \sum_{i=1}^{h} r_i$. On the other hand, by Lemma 19.3.4, we have

$$\mathcal{O}_{C,S}/(u) \cong (\mathcal{O}_{C',Q}/(u))^{\oplus n}.$$

But $(u) = \mathfrak{m}_Q$, therefore $\mathcal{O}_{C',Q}/(u) \cong \mathbb{K}$, and so $\dim(\mathcal{O}_{C,S}/(u)) = n$. In conclusion we have

$$\deg(f) = n = \dim(\mathcal{O}_{C,S}/(u)) = \sum_{i=1}^{h} r_i = \deg(f^*(Q))$$

as wanted. □

Given a surjective morphism $f : C \to C'$, we can consider the family of effective divisors of degree $d = \deg(f)$ given by $\{f^*(P)\}_{P \in C'}$. This is called an *involution* of divisors *parametrized* by C', and usually it is denoted by the symbol γ_d^1. If $Q \in C$ is any point, there is a unique divisor $D \in \gamma_d^1$ such that P is contained in D. Moreover there is no point P contained in all divisors of the γ_d^1.

Exercise 19.3.6 *Let C, C' be smooth, irreducible, projective curves and $f : C \to C'$ a surjective morphism, with $\nu = \deg(f)$. Prove that if f is separable (in particular if $\mathrm{char}(\mathbb{K}) = 0$), then there is a non-empty open subset U of C' such that for all points $P \in U$ the fibre divisor $f^*(P)$ consists of d distinct points.

Exercise 19.3.7 Using Theorem 19.3.5 give another proof of Proposition 19.1.1.

Exercise 19.3.8 Every divisor is *locally principal* in the following sense. Let C be a smooth, irreducible, projective curve and $D = \sum_{P \in C} n_P P$ a divisor on C. Prove that there is an open cover $\{U_i\}_{i \in \mathcal{I}}$ of C such that for any $i \in \mathcal{I}$ there is a non-zero rational function ϕ_i in U_i such that the principal divisor of ϕ_i in U_i coincides with the restriction of D to U_i. Prove that for all pairs $(i, j) \in \mathcal{I} \times \mathcal{I}$ such that $U_i \cap U_j \neq \emptyset$, the function $\phi_{ij} = \frac{\phi_i}{\phi_j}$ is regular and never zero in $U_i \cap U_j$. The family $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ is said to be *compatible* and *determined* by D.

Exercise 19.3.9 Suppose we have a compatible family $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ on C. Prove that there is a unique divisor D on C, such that $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ is determined by D. In this case one says that D is *determined* by $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$.

Exercise 19.3.10 Prove that if there are two compatible families $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$ and $\{(U'_j, \phi'_j)\}_{j \in \mathcal{J}}$ on C, the two families determine the same divisor if and only if for all pairs $(i, j) \in \mathcal{I} \times \mathcal{J}$ such that $U_i \cap U'_j \neq \emptyset$, the function $\frac{\phi_i}{\phi'_j}$ is regular and never zero in $U_i \cap U'_j$.

Exercise 19.3.11 Suppose we have a surjective morphism $f : C \to C'$ between two smooth, irreducible projective curves. Let D be a divisor on C', and let $\{(U_i, \phi_i)\}_{i\in\mathcal{I}}$ be a compatible family determined by D. Consider the family $\{(f^{-1}(U_i), \phi_i \circ f_{|f^{-1}(U_i)})\}_{i\in\mathcal{I}}$. Prove that it is compatible and that it determines the divisor $f^*(D)$.

Exercise 19.3.12 Suppose we have a surjective morphism $f : C \to C'$ of degree n between two smooth, irreducible projective curves. Let D be a divisor on C'. Prove that $\deg(f^*(D)) = n\deg(D)$.

19.4 Linear Series

Let $D = \sum_{P\in C} n_P P$ be a divisor on a smooth, projective curve C. We make the following definition

$$L(D) = \{\phi \in K(C) : \quad \text{either} \quad \phi = 0 \quad \text{or} \quad D + \mathrm{div}(\phi) \geqslant 0\}.$$

We note that $L(D)$ is a vector space over $\mathbb{K}$. Indeed, if $a \in \mathbb{K}$ and $\phi \in L(D)$ it is clear that $a\phi \in L(D)$. Moreover, if $\phi, \psi \in L(D)$, we have

$$D + \mathrm{div}(\phi) \geqslant 0, \quad D + \mathrm{div}(\psi) \geqslant 0$$

which means that for all $P \in C$ we have

$$o_{C,P}(\phi) \geqslant -n_P, \quad o_{C,P}(\psi) \geqslant -n_P.$$

But

$$o_{C,P}(\phi + \psi) \geqslant \min\{o_{C,P}(\phi), o_{C,P}(\psi)\} \geqslant -n_P,$$

so hat $\phi + \psi \in L(D)$.

We will denote by $\ell(D)$ the dimension of $L(D)$, that, as we will soon see, is finite.

Lemma 19.4.1 *If $D \leqslant D'$ then $L(D) \subseteq L(D')$ and*

$$\dim(L(D')/L(D)) \leqslant \deg(D' - D). \tag{19.2}$$

Proof The first assertion is trivial: if $\phi \in L(D)$, one has $D + \mathrm{div}(\phi) \geqslant 0$, then $D' + \mathrm{div}(\phi) \geqslant 0$ and $\phi \in L(D')$.

To prove (19.2), it suffices to prove that

$$\dim(L(D + P)/L(D)) \leqslant 1.$$

To see this, consider a uniformizing parameter u in $\mathcal{O}_{C,P}$. Let m be the multiplicity of P in D. Consider the linear map

$$\mu : L(D + P) \to \mathbb{K}$$

which sends $\phi \in L(D+P)$ to $\mu(\phi) := (u^{m+1}\phi)(P)$. Since $o_{C,P}(\phi) \geqslant -m-1$, the function $u^{m+1}\phi$ is defined in P and so it makes sense to consider its value in P, and the map μ is therefore well defined. We claim that $\ker(\mu) = L(D)$. In fact $\mu(\phi) = 0$ means that $(u^{m+1}\phi)(P) = 0$. Since $o_{C,P}(u) = 1$, this means that $o_{C,P}(\phi) \geqslant -m$, as wanted. Therefore

$$\dim(L(D+P)/L(D)) \leqslant \dim(\mathbb{K}) = 1.$$

□

Lemma 19.4.2 *One has* $L(0) = \mathbb{K}$ *and* $L(D) = \{0\}$ *if* $\deg(D) < 0$.

Proof We have $\phi \in L(0)$ if and only if either $\phi = 0$ or $\operatorname{div}(\phi) \geqslant 0$. This means that ϕ has no poles, so it has also no zeros by Proposition 19.1.1, so $\operatorname{div}(\phi) = 0$ and ϕ is constant (see Exercise 19.1.3).

If $\phi \in L(D)$ and $\phi \neq 0$, then $D' = D + \operatorname{div}(\phi) \geqslant 0$ is linearly equivalent to D, hence $\deg(D) = \deg(D') \geqslant 0$. Thus if $\deg(D) < 0$ then $L(D) = \{0\}$. □

Lemma 19.4.3 $L(D)$ *has finite dimension and precisely* $L(D) = \{0\}$ *if* $\deg(D) < 0$ *whereas if* $\deg(D) \geqslant 0$ *then*

$$\ell(D) \leqslant \deg(D) + 1.$$

Proof If $\deg(D) < 0$ the assertion follows from Lemma 19.4.2. If $\deg(D) = n \geqslant 0$, we fix a point $P \in C$ and we set $D' = D - (n+1)P$. Then $L(D') = \{0\}$. By Lemma 19.4.1 we have

$$\ell(D) = \dim(L(D)/L(D')) \leqslant \deg(D - D') = n + 1$$

as wanted. □

Lemma 19.4.4 *If* $D \equiv D'$ *then* $L(D) \cong L(D')$.

Proof There is a $\psi \in K(X) \setminus \{0\}$ such that $D' = D + \operatorname{div}(\psi)$. We consider the map

$$\tau : \phi \in L(D') \to \phi\psi \in L(D).$$

This map is well defined. Indeed

$$D + \operatorname{div}(\phi\psi) = D + \operatorname{div}(\phi) + \operatorname{div}(\psi) = D' + \operatorname{div}(\phi) \geqslant 0.$$

It is moreover clear that τ is linear and bijective. □

Given the divisor D on C, we denote by $|D|$ the set of all effective divisors that are linearly equivalent to D. This is called the *complete linear series* determined by D. If $D' \equiv D$, then $|D'| = |D|$.

Proposition 19.4.5 *One has*

$$|D| = \mathbb{P}(L(D)).$$

Proof For every $\phi \in L(D) \setminus \{0\}$, the divisor $D' = D + \mathrm{div}(\phi)$ belongs to $|D|$. The same D' is obtained from a $\psi \in L(D) \setminus \{0\}$ if and only if there is a $\lambda \in \mathbb{K} \setminus \{0\}$ such that $\psi = \lambda\phi$. Indeed, one has

$$D' = D + \mathrm{div}(\phi) = D + \mathrm{div}(\psi)$$

if and only if $\mathrm{div}(\phi) = \mathrm{div}(\psi)$ and this is the case if and only if there is a $\lambda \in \mathbb{K}^*$ such that $\psi = \lambda\phi$ (see Exercise 19.1.4). Thus we have an application

$$\rho : \mathbb{P}(L(D))) \to |D|$$

which maps the proportionality class of a non-zero function $\phi \in L(D)$ to the divisor $D' = D + \mathrm{div}(\phi)$. As we said, this map is injective. Moreover it is also surjective. In fact, if $D' \geqslant 0$ and $D' \equiv D$, there is a non-zero rational function ϕ such that $D' = D + \mathrm{div}(\phi)$ and $\phi \in L(D)$. □

So $|D|$ is in a natural way a projective space of dimension $r(D) := \ell(D) - 1$. Any linear subspace of dimension r of $|D|$ is called a *linear series* on C of dimension r. A linear series of dimension -1 is empty. Two effective divisors are linearly equivalent if and only if they belong to a linear series. Two divisors belonging to the same linear series, being linearly equivalent, have the same degree. This is also called the *degree* of the linear series. A linear series of dimension r and degree d is usually denoted by the symbol g^r_d.

Given a linear series g^r_d on C, suppose there is an effective divisor D of degree δ that is contained in all divisors of the g^r_d. Then one says that D is a *base divisor* of the g^r_d. If we remove D from all divisors of the g^r_d we still obtain a linear series $g^r_{d-\delta}$ which is denoted by the symbol $g^r_d(-D)$.

Exercise 19.4.6 Consider a g^r_d on the curve C, and let $P \in C$ be a point which is not a base point for the g^r_d . Consider the set of all divisors of the g^r_d containing P. Prove that this set is a linear series g^{r-1}_d which has P has a base point. By removing P from all divisors of this g^{r-1}_d we obtain a g^{r-1}_{d-1} which is again denoted by $g^r_d(-P)$.

Exercise 19.4.7 Consider a g^r_d on the curve C, and let M be an effective divisor of degree $m \leqslant r$. Prove that the set of divisors $D \in g^r_d$ containing M is non-empty and they form a linear series of dimension at least $r - m$, having M in the base divisor. By removing M from the divisors of this series, one obtains a g^s_{d-m}, with $s \geqslant r - m$, which is denoted by $g^r_d(-M)$.

Exercise 19.4.8 Given a surjective morphism $f : C \to \mathbb{P}^1$ of degree d, with C a smooth, irreducible projective curve, prove that the corresponding involution γ^1_d parametrized by $\mathbb{P}^1$ is in fact a g^1_d.

19.5 Linear Series and Projective Morphisms

Let $\xi = g^r_d$ be a linear series on a smooth, projective curve C. It corresponds to an $(r + 1)$-dimensional vector subspace $V \subseteq L(D)$, with $D \in \xi$. Suppose that ξ has no base points: we will then say that ξ is *base point free*. Then for every point $P \in C$, the series $g^r_d(-P)$ has dimension $r - 1$ (see Exercise 19.4.6). Hence $g^r_d(-P)$ can be considered as a hyperplane in g^r_d. Denote by $\check{g}^r_d$ the dual projective space of g^r_d. We have a map

$$\omega_\xi : P \in C \to g^r_d(-P) \in \check{g}^r_d \cong \mathbb{P}^r.$$

Proposition 19.5.1 *In the above setting, the map ω_ξ is a morphism, which is said to be* determined *by ξ.*

Proof With the above notation, $g^r_d = \mathbb{P}(V)$ and therefore $\check{g}^r_d = \mathbb{P}(\check{V})$. Fix a point $P \in C$ and a divisor $D \in g^r_d$ such that P is not contained in D: this is possible because g^r_d is base point free. For every $g \in V \setminus \{0\}$, we have

$$D + \mathrm{div}(g) = D' \in g^r_d,$$

therefore the divisor of poles of g is such that

$$D \geqslant (g)_\infty.$$

Fix a basis $g_0, \dots, g_r$ of V, so that

$$D \geqslant (g_i)_\infty, \quad \text{for all} \quad i = 0, \dots, r. \tag{19.3}$$

On $\check{V}$ we have coordinates $(\lambda_0, \dots, \lambda_r)$ corresponding to the linear form mapping g_i in λ_i, for $i = 0, \dots, r$. Now $g^r_d(-P)$ corresponds to the subspace $V(-P)$ of V formed by 0 and by all non-zero functions $g = \mu_0 g_0 + \cdots + \mu_r g_r$ such that

$$D + \mathrm{div}(g) \geqslant P.$$

This, by (19.3) and by the fact that P is not contained in D, happens if and only if $g(P) = 0$, i.e., if and only if

$$\mu_0 g_0(P) + \cdots + \mu_r g_r(P) = 0. \tag{19.4}$$

Hence $V(-P)$ is the linear subspace of V of codimension 1 having Eq. (19.4) in the coordinates $(\mu_0, \dots, \mu_r)$ of V. Thus in $\check{V}$ it has coordinates

$$(g_0(P), \dots, g_r(P))$$

which are not all zero, because P is not a base point for the g^r_d. In conclusion, in the open neighborhood of P which is the complement of the support of D (containing all the poles of the functions $g_0, \dots, g_r$), ω writes as

$$\omega : P \in U = C \setminus \mathrm{Supp}(D) \to [g_0(P), \dots, g_r(P)] \in \mathbb{P}^r \tag{19.5}$$

where $g_0, \dots, g_r$ are regular functions in U, which proves the assertion. □

We go on keeping the notation of the proof of Proposition 19.5.1. We observe that, by the linear independence of the functions $g_0, \dots, g_r$, there is no hyperplane of $\mathbb{P}^r$ containing the curve $\omega(C)$, which is therefore non-degenerate.

Conversely, suppose we are given a morphism

$$\omega : C \to \mathbb{P}^r$$

so that $\omega(C)$ is non-degenerate. Given a hyperplane π of $\mathbb{P}^r$, with equation

$$a_0x_0 + \cdots + a_rx_r = 0 \tag{19.6}$$

we define the *divisor* $\mathrm{div}(\pi)$ *cut out by* π *on* C in the following way. Set

$$\mathrm{div}(\pi) = \sum_{P \in C} n_P P$$

and we define n_P for all $P \in C$. Fix $P \in C$. Suppose $C \subset \mathbb{P}^s$. In a neighborhood U of P on C there are regular functions $g_0, \dots, g_r$ on U, not all zero at any point $Q \in U$, such that

$$\omega(Q) = [g_0(Q), \dots, g_r(Q)], \quad \text{for all} \quad Q \in U$$

(see Exercise 6.2.13). Then we define n_P to be the order of the function $a_0g_0 + \cdots + a_rg_r$ at P. Divisors cut out by different hyperplanes on C are linearly equivalent. In fact, if the hyperplane π has Eq. (19.6) and the hyperplane π' has equation

$$a'_0x_0 + \cdots + a'_rx_r = 0$$

then, with the above notation, one has

$$\mathrm{div}(\pi) - \mathrm{div}(\pi') = \mathrm{div}\Big(\frac{a_0g_0 + \cdots + a_rg_r}{a'_0g_0 + \cdots + a'_rg_r}\Big).$$

Actually the divisors cut out by all hyperplanes vary in a linear series $\xi = g^r_d$ with no base point, and $\omega = \omega_\xi$.

In conclusion there is a 1:1 correspondence between base point free linear series g^r_d on C and morphisms $\omega : C \to \mathbb{P}^r$ up to projectivities of $\mathbb{P}^r$, such that $\omega(C)$ is non-degenerate.

Given a base point free $\xi = g^r_d$ on C, it is called *simple* if ω_ξ is birational onto its image X. This is the same as saying that, except for finitely many pairs of points

$(P, Q) \in C \times C$, one has

$$\dim(g^r_d(-P - Q)) = r - 2. \tag{19.7}$$

Suppose that the base point free linear series $\xi = g^r_d$ on C is non-simple, i.e., $\omega_\xi : C \to X$ is not birational, where $X = \omega_\xi(C)$ is a non-degenerate curve in $\mathbb{P}^r$. Let $\pi : C' \to X$ be a smooth model of X. Then there is a rational map $\psi = \pi^{-1} \circ \omega_\xi : C \dashrightarrow C'$, which is a morphism because C is smooth, and $\omega_\xi = \pi \circ \psi$. Then one says that ω_ξ *factors* through ψ. Let $\nu = \deg(f)$. One says that ξ is *composed* with the involution γ^1_ν parametrized by C'. In this case given any point $P \in C$, if D is the unique divisor of the γ^1_ν containing P, then D is contained in any divisor of ξ containing P. This implies that d is divisible by ν and the birational morphism $\pi : C' \to X \subseteq \mathbb{P}^r$ is determined by a $g^r_{\frac{d}{\nu}}$. So if ξ is base point free but non-simple, then for any point $P \in C$ there is some point $Q \in C$ such that $\dim(g^r_d(-P - Q)) = r - 1$.

Given a linear series ξ on the smooth, irreducible, projective curve C, ξ is said to be *very simple* if it is base point free and moreover for all pairs of points $(P, Q) \in C \times C$, (19.7) holds. The following is a basic result:

Theorem 19.5.2 *Let ξ be a very simple linear series on the smooth, irreducible, projective curve C. Then the morphism $\omega_\xi : C \to \mathbb{P}^r$ is an isomorphism of C onto its image. In particular the image of C is smooth.*

Proof Let C' be the image of f. Then $f : C \to C'$ is a finite map by Theorem 14.3.3. The map $f : C \to C'$ is bijective, because it is of course surjective and injective by the very simplicity of ξ. Then we apply Theorem 14.4.1 and in order to do so we have to prove that for any point $P \in C$, the differential of f is injective at P. If we set $Q = f(P)$, this is equivalent to say that

$$f^* : \mathfrak{m}_Q/\mathfrak{m}_Q^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$$

is surjective. Suppose by contradiction that this is not the case. Then this means that

$$f^*(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P^2,$$

because, since C is smooth at P, one has $\dim(\mathfrak{m}_P/\mathfrak{m}_P^2) = 1$. Then for any function $u \in \mathfrak{m}_Q$, we have $o_P(f^*(u)) \geqslant 2$. This implies that for any divisor $D \in \xi$ such that $P \in D$, then $2P \leqslant D$. This contradicts the very simplicity of ξ. □

Exercise 19.5.3 *Prove that if $\xi = g^r_d$ on the smooth, projective curve C is base points free and simple then the image of ω_ξ is a curve of degree d and there is a dense open subset U of the g^r_d such that every divisor $D \in U$ consists of d distinct points.

Exercise 19.5.4 *Assume $\mathrm{char}(\mathbb{K}) = 0$. Suppose that $\xi = g^r_d$ on the curve C is base point free but not simple, so that $\omega_\xi : C \to \omega_\xi(C) = X \subset \mathbb{P}^r$ is not birational. Let $\pi : C' \to X$ be a smooth model of X and consider the morphism $\psi = \pi^{-1} \circ \omega_\xi : C \to C'$. Then ξ is composed with the γ^1_ν parametrized by C' determined by $\psi : C \to C'$. Prove that X has degree $\frac{d}{\nu}$ and that $\omega_\xi : C \to X$ has degree ν. Prove that again there is a dense open subset U of the g^r_d such that every divisor $D \in U$ consists of d distinct points.

19.6 Adjoint Curves

Let $X \subset \mathbb{P}^2$ be an irreducible curve with only ordinary singularities and $\pi : C \to X$ its smooth model. For any point $P \in C$ with $Q = f(P)$, we set

$$m_P = m_Q(X)$$

and we define the divisor

$$\Delta_X = \sum_{P \in C} (m_P - 1)P$$

that may be simply denoted by Δ if no confusion arises. We will call it the *multiple points divisor* of X. Note that $\Delta = 0$ if and only if X is smooth.

A plane curve Y is said to be *adjoint* to X if for any point $Q \in X$ one has $m_Q(Y) \geqslant m_Q(X) - 1$.

Proposition 19.6.1 *Let $X \subset \mathbb{P}^2$ be an irreducible curve with only ordinary singularities. Then Y is adjoint to X if and only if either Y contains X or*

$$\mathrm{div}_X(Y) \geqslant \Delta_X.$$

Proof One implication is trivial, so we focus on the non-trivial implication. We will argue as in Sect. 18.1. Suppose that P is a singular point of X with multiplicity m, that $P \in U_0 \cong \mathbb{A}^2$ and that actually P is the origin of $\mathbb{A}^2$. Then the affine equation of X is of the form

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots + f_d(x, y) = 0$$

where d is the degree of X and $f_i(x, y)$ is a form of degree i, for $i = m, \ldots, d$. The form $f_m(x, y)$ has m distinct roots, up to a proportionality factor, so that

$$f_m(x, y) = \prod_{i=1}^{m} (b_i x - a_i y)$$

with $(a_i, b_i) \neq (0, 0)$ non-proportional, for $i = 1, \ldots, m$. The principal tangents to X at P have affine equations

$$b_i x = a_i y, \quad i = 1, \ldots, m.$$

By a change of coordinates we may assume that $b_1 = 0$, so that one of the principal tangents to X at P is the x axis. Let us blow-up the point P. We restrict our attention to the open subset $\tilde{\mathbb{A}}_0$ of the blow-up, which is isomorphic to $\mathbb{A}^2$ with coordinates (x, v) and the blow-up map identifies with the map $(x, v) \to (x, xv)$. Let Z be the intersection of the strict transform of X with $\tilde{\mathbb{A}}_0$, which is an open subset of the smooth model C of X. The equation of Z in $\tilde{\mathbb{A}}_0 \cong \mathbb{A}^2$ is

$$f_m(1, v) + \cdots + x^{d-m} f_d(1, v) = 0, \quad \text{and} \quad f_m(1, v) = \prod_{i=1}^{m}(b_i - a_i v) = 0.$$

Now let us consider a curve Y in $\mathbb{P}^2$ which has affine equation

$$g(x, y) = g_r(x, y) + g_{r+1}(x, y) + \cdots$$

where, as usual $g_r(x, y), g_{r+1}(x, y), \ldots$ are homogeneous polynomials of degree equal to the index. The intersection of the total transform of Y with $\tilde{\mathbb{A}}_0 \cong \mathbb{A}^2$ has equation

$$x^r(g_r(1, v) + x g_{r+1}(1, v) + \cdots) = 0$$

It contains with multiplicity r the exceptional locus of the blow-up, which has equation $x = 0$, and the proper transform has equation

$$g_r(1, v) + x g_{r+1}(1, v) + \cdots = 0.$$

This tells us that for any point $Q_1, \ldots, Q_m$ of C in $f^{-1}(P)$, the multiplicity of $\mathrm{div}(Y)$ is at least r. Moreover we see that if $Q_1 = (0, 0)$ is the point corresponding to the principal tangent $y = 0$ to X at P, the multiplicity of $\mathrm{div}(Y)$ at Q_1 is larger than r if and only if the equation $g_r(1, v) = 0$ in v has the solution $v = 0$, i.e. if and only if one of the principal tangents to Y at P coincides with the x axis. The same happens for every other point $Q_2, \ldots, Q_m$. Hence, if $r < m - 1$, since Y has in P at most r principal tangents, then there is an $i \in \{1, \ldots, m\}$ such that the multiplicity of $\mathrm{div}(Y)$ in Q_i is $r < m - 1$, so it does not happen that $\mathrm{div}(Y) \geqslant \Delta_X$. This proves the assertion. □

The set $\mathrm{Adj}_d(X)$ of all adjoint curves of degree d to X is a linear system. If $d < n$, no curve in $\mathrm{Adj}_d(X)$ contains X. If $d \geqslant n$, then $\mathrm{Adj}_d(X)$ contains the subset $\mathrm{Adj}^0_d(X)$ of all the adjoints containing X. They consist of X plus any curve of degree $d - n$. So $\mathrm{Adj}^0_d(X)$ is a linear system and

$$\dim(\mathrm{Adj}^0_d(X)) = \frac{(d-n)(d-n+3)}{2}.$$

Now we are able to prove an important result:

Theorem 19.6.2 (M. Noether Restsatz) *Let $X \subset \mathbb{P}^2$ be an irreducible curve with only ordinary singularities and $\pi : C \to X$ its smooth model. Let D, D' be effective divisors on C such that $D \equiv D'$. Suppose there is an adjoint curve Y of degree m not containing X such that*

$$\mathrm{div}(Y) = \Delta + D + A$$

where A is an effective divisor. Then there is an adjoint curve Y' of degree m not containing X such that

$$\operatorname{div}(Y') = \Delta + D' + A.$$

Proof Since D and D' are linearly equivalent, there are curves Z, Z' of the same degree such that

$$D + \operatorname{div}(Z) = D' + \operatorname{div}(Z').$$

Then

$$\begin{aligned}\operatorname{div}(Y + Z) &= \operatorname{div}(Y) + \operatorname{div}(Z) = \\ &= \Delta + D + A + D' - D + \operatorname{div}(Z') = \\ &= \Delta + A + D' + \operatorname{div}(Z') \geqslant \operatorname{div}(Z') + \Delta.\end{aligned} \tag{19.8}$$

Now we make the following:

Claim (+): Let $X \subset \mathbb{P}^2$ be an irreducible curve with an ordinary m-tuple point P. Let Y, Z be curves with equations $g = 0$ and $h = 0$ respectively. Let $\gamma_1, \ldots, \gamma_m$ be the distinct linear branches of X with centre P. Then Z verifies at P the Noether conditions with respect to X and Y if

$$o_{\gamma_i}(h_*) \geqslant o_{\gamma_i}(g_*) + m - 1, \quad \text{for all} \quad i = 1, \ldots, m$$

where, as usual, g_* and h_* denote the dehomogenizations of g and h.

Take this claim for granted for the moment, and let us finish the proof of the theorem. By (19.8) it follows that $Y + Z$ verifies Noether's conditions with respect to X and Z' in any multiple point P of X, hence, assuming that X and Z' have equations $f = 0$ and $h' = 0$ respectively, we have a relation of the sort

$$gh = ah' + bf$$

with a, b suitable homogeneous polynomials. Then on C we have

$$\operatorname{div}(gh) = \operatorname{div}(ah') = \operatorname{div}(a) + \operatorname{div}(h')$$

hence

$$\operatorname{div}(a) = \operatorname{div}(Y + Z) - \operatorname{div}(Z') = \Delta + A + D'$$

so that, by Proposition 19.6.1, we may consider the adjoint curve Y' with equation $a = 0$, as desired.

We are left to prove Claim (+). As usual we may put P in the origin of $\mathbb{A}^2$ and we denote with a lower asterisk the dehomogenization of polynomials. Now Z verifies at P the Noether conditions with respect to X and Y if and only if $h_* \in (f_*, g_*) \subset \mathcal{O}_{\mathbb{A}^2,P}$, where we abuse notation identifying f_*, g_*, h_* with their classes in $\mathcal{O}_{\mathbb{A}^2,P}$. This is equivalent to say that $\bar{h}_* \in (\bar{g}_*) \subset \mathcal{O}_{X,P}$, where the upper bar denotes the class. But to say that $\bar{h}_* \in (\bar{g}_*) \subset \mathcal{O}_{X,P}$, is in turn equivalent to say that $\frac{\bar{h}_*}{\bar{g}_*} \in \mathcal{O}_{X,P}$.

Consider the smooth model $C \to X$, let $P_1, \ldots, P_m$ be the counterimages on C of P and consider a rational function ϕ on C which is defined at $P_1, \ldots, P_m$ and $o_{C,P_i}(\phi) \geqslant m-1$ for all $i = 1, \ldots, m$. Then we claim that $\phi \in \mathcal{O}_{X,P} \subseteq K(C) = K(X)$. This clearly implies Claim (+).

To prove this final claim, consider an open neighborhood U of P in X such that ϕ is defined for every $P' \in U$ except perhaps in P. Let U' be the counterimage of U on C, where C as usual is obtained by blowing-up at P. If $x = 0$ is the equation of the exceptional locus in the usual open set $\tilde{\mathbb{A}}_0$ of the blow-up, then $o_{C,P_i}(x) = 1$ for all $i = 1, \ldots, m$. Therefore $\frac{\phi}{x^{m-1}} \in \mathcal{O}(U')$. The assertion follows from the fact that $x^{m-1}\mathcal{O}(U') \subseteq \mathcal{O}(U)$.

In fact the affine equation of X is

$$f_*(x, y) = \sum_{i+j \geqslant m}^{d} a_{ij} x^i y^j = 0$$

with d the degree of X. We may suppose that, after making a change of coordinates, we have $a_{0m} \neq 0$. With the usual notation, the equation of C in $\tilde{\mathbb{A}}_0$ is of the form

$$\sum_{i+j \geqslant m}^{d} a_{ij} x^{i+j-m} v^j = 0.$$

Recalling that $y = xv$, we may write this equation in the form

$$\sum_{i+j \geqslant m}^{d} a_{ij} y^{i+j-m} v^{m-i} = 0. \tag{19.9}$$

The left hand side of (19.9) is a polynomial of degree m in v. Indeed, when $i > m$, so that $m - i < 0$, we may write

$$y^{i+j-m} v^{m-i} = x^{i-m} y^j.$$

The coefficient of v^m (19.9) is

$$\sum_{j \geqslant m}^{d} a_{0j} y^{j-m} \tag{19.10}$$

which has constant term $a_{0m} \neq 0$. So (19.10) is non-zero in U, thus, up to shrinking U, we may divide by (19.10), and the Eq. (19.9) becomes of the form

$$v^m + b_1 v^{m-1} + \cdots + b_m = 0$$

where $b_1, \ldots, b_m \in \mathcal{O}(U)$, so that $\mathcal{O}(U') = \mathcal{O}(U)[1, v, \ldots, v^{m-1}]$. But then

$$x^{m-1}v^j = x^{m-1-j}(xv)^j = x^{m-1-j}y^j \in \mathcal{O}(U), \quad \text{for all} \quad j = 1, \dots, m-1.$$

Hence $x^{m-1}\mathcal{O}(U') \subseteq \mathcal{O}(U)$ as desired.

In conclusion

$$\phi = x^{m-1} \cdot \frac{\phi}{x^{m-1}} \in x^{m-1}\mathcal{O}(U') \subseteq \mathcal{O}(U) \subseteq \mathcal{O}_{X,P}$$

proving Claim (+) and the theorem. □

19.7 Linear Systems of Plane Curves and Linear Series

Let $X \subset \mathbb{P}^2$ be an irreducible curve with only ordinary singularities and let $\pi : C \to X$ be its smooth model. Let $\mathcal{L}$ be a linear system of dimension r of curves of degree d in $\mathbb{P}^2$, such than no curve in $\mathcal{L}$ contains X. Then we can consider the set of divisors

$$\mathcal{L}_C = \{\mathrm{div}(Y) : Y \in \mathcal{L}\}.$$

The divisors in $\mathcal{L}_C$ are all linearly equivalent (see Proposition 19.2.1, (e)). Actually $\mathcal{L}_C$ is a linear series of dimension r that, as a projective space, is projectively equivalent to $\mathcal{L}$. In fact, let $V \subset S_{2,d}$ be the vector space such that $\dim(V) = r + 1$ and $\mathcal{L} = \mathbb{P}(V)$. Fix $f \in V \setminus \{0\}$ and consider the divisor $D = \mathrm{div}(f)$ on C. Consider the vector space $W \subset K(C)$ of all rational functions of the form $\frac{g}{f}$ with $g \in V$. Clearly $V \cong W$, hence $\dim(W) = r + 1$. We claim that $W \subseteq L(D)$. In fact, for all $\frac{g}{f} \in W$, we have

$$D + \mathrm{div}\Big(\frac{g}{f}\Big) = D + \mathrm{div}(g) - \mathrm{div}(f) = \mathrm{div}(g) \geqslant 0.$$

Therefore W corresponds to the linear series of dimension r given by $\mathbb{P}(W)$, and this clearly coincides with $\mathcal{L}_C$. Since $\mathbb{P}(W) \cong \mathbb{P}(V) = \mathcal{L}$, this series is projectively equivalent to $\mathcal{L}$. The linear series $\mathcal{L}_C$ is called the *series cut out* by $\mathcal{L}$ on C (or on X).

Suppose next that $\mathcal{L}$ contains curves containing X. Let $\mathcal{L}^0$ be the subset of $\mathcal{L}$ formed by such curves. This is a sublinear system of $\mathcal{L}$ whose dimension we denote by s. In $\mathcal{L}$, which is a projective space of dimension r, we can consider a subspace $\mathcal{L}'$ of maximal dimension $r - s - 1$ such that $\mathcal{L}' \cap \mathcal{L}^0 = \emptyset$. We can the consider the linear series $\mathcal{L}'_C$ cut out by $\mathcal{L}'$ on C, because no curve in $\mathcal{L}'$ contains X.

Lemma 19.7.1 *In the above setting, if $Y \in \mathcal{L}$ is any curve not containing X, then there is a unique curve $Y' \in \mathcal{L}'$ such that*

$$\mathrm{div}(Y) = \mathrm{div}(Y'). \tag{19.11}$$

Proof Consider the subspace $\langle \mathcal{L}^0, Y\rangle$ of $\mathcal{L}$. Since $Y \notin \mathcal{L}^0$, then $\langle \mathcal{L}^0, Y\rangle$ has dimension $s+1$. If X and Y have equations $f=0$ and $g=0$ respectively, all curves in $\langle \mathcal{L}^0, Y\rangle$ have equation of the form

$$\lambda g + \mu h = 0, \quad \text{with} \quad (\lambda, \mu) \in \mathbb{K}^2 \setminus \{(0,0)\},$$

with $h=0$ the equation of a curve in $\mathcal{L}^0$ and therefore h is divisible by f. If $\lambda \neq 0$, we have

$$\operatorname{div}(\lambda g + \mu h) = \operatorname{div}(g) = \operatorname{div}(Y)$$

so all curves in $\langle \mathcal{L}^0, Y\rangle$ not containing X cut out on C the same divisor. On the other hand, by Grassmann formula, $\langle \mathcal{L}^0, Y\rangle$ intersect $\mathcal{L}'$ in a unique point Y' for which (19.11) holds. □

In conclusion, the linear series cut out by $\mathcal{L}'$ on C, which has dimension $r-s-1$, coincides with the set of all divisors cut out on C by curves in $\mathcal{L}$ not containing X. So this series is independent of $\mathcal{L}'$, it is still denoted by $\mathcal{L}_C$ and it is called the *series cut out* by $\mathcal{L}$ on C (or on X).

If D is an effective divisor contained in all divisors of $\mathcal{L}_C$, we can subtract D for all divisors of $\mathcal{L}_C$ and obtain a new linear series which is said to be the *series cut out by $\mathcal{L}$ on C (or on X) off D*.

In particular, we can consider the linear system $\mathrm{Adj}_d(X)$. This linear system cuts out on C a linear series all of whose divisors contain the multiple point divisor Δ. Thus $\mathrm{Adj}_d(X)$ cuts out on C off Δ a linear series that we denote by $\mathrm{adj}_d(X)$. The Restsatz Theorem 19.6.2 implies that:

Theorem 19.7.2 *For every positive integer d, the linear series $\mathrm{adj}_d(X)$ is a complete linear series. Moreover all complete linear series are cut out on C by sublinear systems of adjoint curves of a sufficiently high degree d which cut out on C a fixed divisor A (off Δ).*

Exercise 19.7.3 Given any effective divisor D with $\deg(D) \leqslant 3$ on a smooth plane cubic or on a plane quartic with two nodes, determine the dimension of the complete linear series $|D|$.

Exercise 19.7.4 Given any effective divisor D with $\deg(D) = 3$ on a smooth plane quartic, determine the dimension of the complete linear series $|D|$.

Exercise 19.7.5 Given any effective divisor D with $\deg(D) = 3$ on a plane quartic, with a single node, determine the dimension of the complete linear series $|D|$.

19.8 Solutions of Some Exercises

19.1.2 We may assume that $Q \in U_0 \cong \mathbb{A}^2$ and that Q is the origin of $\mathbb{A}^2$. We may assume in addition that γ_P is determined by a parametrization of the form

$$x = t, \quad y = at^n + \cdots,$$

with $n \geqslant 2$ and $a \neq 0$. We may assume also that C is obtained, as in Sect. 18.1, by blowing-up at the singular points of X. So in particular we blow-up Q. As in Sect. 18.1, from which we keep the notation, we can restrict our attention to the open subset $\tilde{\mathbb{A}}_0$ of the blow-up, which is isomorphic to $\mathbb{A}^2$ with coordinates (x, v) and the blow-up map identifies with the map $(x, v) \to (x, xv)$. Let Y be the intersection of the strict transform of X with $\tilde{\mathbb{A}}_0$, which is an open subset of the smooth model C of X. The point P coincides with the point with coordinates (0, 0) in $\tilde{\mathbb{A}}_0 \cong \mathbb{A}^2$. The linear branch $\tilde{\gamma}$ of Y with centre P is determined by the parametrization obtained by the equations

$$x = t, \quad xv = y = at^n + \cdots, \quad \text{hence} \quad v = at^{n-1} + \cdots .$$

Now $o_{\gamma_P}(Z)$ is the order of the power series $g_*(t, at^m + \cdots)$ (as usual g_* denotes dehomogenization). We can consider the decomposition in homogeneous components

$$g_*(x, y) = g_m(x, y) + g_{m+1}(x, y) + \cdots$$

so

$$g_*(t, at^n + \cdots) = t^m \Big(g_m(1, at^{n-1} + \cdots) + t g_{m+1}(1, at^{n-1} + \cdots) + \cdots\Big)$$

and therefore the order of $g_*(t, at^m + \cdots)$ is m plus the order of the power series

$$g_m(1, at^{n-1} + \cdots) + t g_{m+1}(1, at^{n-1} + \cdots) + \cdots . \qquad (19.12)$$

By Propositions 16.3.2 and 16.3.13, to compute the order of $f^*(g_*)$ in $\mathcal{O}_{C,P}$, it suffices to compute the order of $f^*(g_*)$ on $\tilde{\gamma}$. Now $f^*(g_*)$ is the function

$$g_*(x, xv) = x^m (g_m(1, v) + x g_{m+1}(1, v) + \cdots)$$

and the order of $f^*(g_*)$ on $\tilde{\gamma}$ is the order of the power series

$$t^m (g_m(1, at^{n-1} + \cdots) + t g_{m+1}(1, at^{n-1} + \cdots) + \cdots)$$

which is m plus the order of the power series (19.12). This computation shows that $o_{\gamma_P}(Z)$ equals $o_{\mathcal{O}_{C,P}}(f^*(g_*))$, as desired.

19.2.3 Suppose we have two divisors D_1, D_2 of the same degree n on $\mathbb{P}^1$. Write $D_i = D_{i1} - D_{i2}$, with D_{i1}, D_{i2} effective and with no common points, for $i = 1, 2$. The divisor $D_1 - D_2 = (D_{11} + D_{22}) - (D_{12} + D_{21})$ has degree 0, hence $\deg(D_{11} + D_{22}) = \deg(D_{12} + D_{21})$. Write $D_{ij} = m_{ij,1} P_{ij,1} + \cdots m_{ij,h_{ij}} P_{ij,h_{ij}}$ and set $P_{ij,l} = [a_{ij,l}, b_{ij,l}]$, for $(i, j) \in \{1, 2\}^2$ and $l = 1, \dots, , h_{ij}$. Consider the rational function

$$\phi = \frac{\prod_{l=1}^{h_{11}} (x_0 b_{11,l} - x_1 a_{11,1})^{m_{11,l}} \cdot \prod_{l=1}^{h_{22}} (x_0 b_{22,l} - x_1 a_{22,1})^{m_{22,l}}}{\prod_{l=1}^{h_{12}} (x_0 b_{12,l} - x_1 a_{12,1})^{m_{12,l}} \cdot \prod_{l=1}^{h_{21}} (x_0 b_{21,l} - x_1 a_{21,1})^{m_{21,l}}}.$$

One has $\operatorname{div}(\phi) = D_1 - D_2$.

19.2.4 It suffices to prove the assertion if $D = \pm P$ with $P \in S$. We treat the case $D = P$, the case $D = -P$ being similar. By Exercise 18.2.11 there is a curve $X \subset \mathbb{P}^2$ with ordinary singularities such that $\pi : C \to X$ is the smooth model of X, and $Q = \pi(P)$ is a smooth point of X. Consider a line r in $\mathbb{P}^2$ which avoids any point in $\pi(S)$ and a line s passing through Q, not tangent to X at Q, and avoiding any other point of $\pi(S)$. Let $f = 0$ be the linear equation of r and $g = 0$ the linear equation of s and consider the rational function $\phi = \frac{f}{g}$. Then $P + \operatorname{div}(\phi)$ is linearly equivalent to P and misses all points of S.

19.3.6 It follows from Theorem 10.4.4.

19.5.3 Let X be the image of ω_ξ. Since ξ is simple, then $\omega_\xi : C \to X$ is birational. Let $U \subset X$ be a non-empty open subset such that ω_ξ induces an isomorphism between $\omega_\xi^{-1}(U)$ and U. We can assume that U consists only of smooth points for X. Then $S = X \setminus U$ is a finite set of X. Consider the following set

$$I = \{(P, \pi) \in U \times \check{\mathbb{P}}^r : T_{X,P} \subseteq \pi\}.$$

The first projection $p_1 : I \to U$ is surjective and for each point $P \in U$, $p_1^{-1}(P)$ is a projective space of dimension $r - 2$. This implies that I has dimension $r - 1$ (see Theorem 11.3.1). Then also the closure of I in $X \times \check{\mathbb{P}}^r$ has dimension $r - 1$ and therefore the image of the projection $p_2 : \bar{I} \to \check{\mathbb{P}}^r$ to the second factor is a proper closed subset Z of $\check{\mathbb{P}}^r$. Consider the open subset A of $\check{\mathbb{P}}^r$ consisting of the complement of the union of Z plus the set of hyperplanes passing through one of the finitely many points of S. Let π be a hyperplane in A. We claim that π intersects X in d distinct points. This will prove both assertions in the Exercise. First we notice that π intersects X only in points of the open subset U. Let P be one of these points. We may assume that $P \in U_0 \cong \mathbb{A}^r$. Let $f(x_1, \ldots, x_r) = 0$ be the linear affine equation of π. In a neighborhood of P we may identify X with C. The hyperplane π contains P but it does not contain the tangent line to X at P. This means that f can be interpreted as a non-zero linear map on the Zariski tangent space $(\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee$ of X at P (where $\mathfrak{m}_P$ denotes the maximal ideal of $\mathcal{O}_{X,P}$). This implies that $f \in \mathfrak{m}_P$ but $f \notin \mathfrak{m}_P^2$, so $o_{X,P}(f) = 1$. Since P is any point of $\pi \cap U = \pi \cap X$, we have that $\mathrm{div}(\pi)$ on C is reduced, so it consists of d distinct point, as wanted.

19.5.4 It follows from Exercises 19.5.3 and 19.3.6.

Chapter 20
The Riemann–Roch Theorem

20.1 The Riemann–Roch Theorem

In this section we will prove the *Riemann–Roch Theorem*, which computes the dimension of a complete linear series in terms of the degree of the series and of an invariant of the curve called the *genus*.

Let us consider an irreducible curve $X \subset \mathbb{P}^2$ of degree n with only ordinary multiple points $P_1, \dots, P_h$ with multiplicities $m_1, \dots, m_h$ respectively. We will consider, as usual, its smooth model $\pi : C \to X$.

Let us set

$$g(X) = \frac{(n-1)(n-2)}{2} - \sum_{i=1}^{h} \frac{m_i(m_i - 1)}{2},$$

which is often simply denoted by g if there is no danger of confusion. This number a priori depends on X, namely, if Y is again a projective, irreducible curve $Y \subset \mathbb{P}^2$ with only ordinary multiple points and if its smooth model is again $\pi' : C \to Y$, it is not a priori clear that $g(X) = g(Y)$. Actually we will see later that $g(X) = g(Y)$, so that $g(X)$ depends only on C, it will be denoted by $g(C)$ and called the *genus* of C.

For the moment we note that $g(X) \geqslant 0$ by Lemma 17.1.3. For instance, if X is a line or a conic, we have $g = 0$, if X is a smooth cubic we have $g = 1$, if C is a cubic with a node we have $g = 0$, etc.

If $g \geqslant 1$, the complete linear series $\text{adj}_{n-3}(X)$ is called the *canonical series* on C determined by X. Again, a priori this series depends on X but, as we will see later, it does not. The divisors linearly equivalent to the divisors of this series are called *canonical divisors*. Usually a canonical divisor on C is denoted by the symbol K_C or simply by K, so that the canonical series is denoted by $|K_C|$. Every effective divisor contained in some canonical divisor is said to be a *special divisor*, and the complete linear series determined by such a divisor will be called a *special linear series*. In particular, the canonical linear series is special.

C. Ciliberto, *An Undergraduate Primer in Algebraic Geometry*, UNITEXT 129,
https://doi.org/10.1007/978-3-030-71021-7_20

Lemma 20.1.1 *If* $g = g(X) \geqslant 1$ *the canonical series has degree* $2g - 2$ *and dimension* $r \geqslant g - 1$.

Proof The degree of the canonical series is

$$n(n-3) - \deg(\Delta) = n(n-3) - \sum_{i=1}^{h} m_i(m_i - 1) = 2g - 2.$$

The dimension of the canonical series equals the dimension r of $\mathrm{Adj}_{n-3}(X)$ and we have

$$r \geqslant \frac{n(n-3)}{2} - \sum_{i=1}^{h} \frac{m_i(m_i-1)}{2} = g - 1.$$

□

Example 20.1.2 In this example X is again an irreducible curve in $\mathbb{P}^2$ with at most ordinary singularities.

(a) If X is a smooth cubic one has $g = 1$, then $\mathrm{Adj}_{n-3}(X)$ is the linear system of curves of degree 0 and the canonical series consists of the only 0 divisor, so it is a g^0_0.
(b) If X is a quartic with $g = 1$, then X has two nodes P_1, P_2. There is a unique adjoint curve of degree $n - 3 = 1$, i.e., the line $\langle P_1, P_2\rangle$, which cuts, off Δ, the 0 divisor. Again the canonical series is a g^0_0.
(c) If X is a quartic with $g = 2$, then X has only one node P. The system $\mathrm{Adj}_{n-3}(X)$ is the pencil of lines through P, and it cuts, off Δ the canonical g^1_2.
(d) If X is a quartic with $g = 3$, then X is smooth and the canonical series is cut out on X by the lines of the plane, so it is a g^2_4.

Proposition 20.1.3 *In the above setting, let* D *be an effective divisor on* C *with* $\deg(D) = g + 1$. *Then* $\dim(|D|) \geqslant 1$.

Proof Let $n = \deg(X) = 1$, so that n is a line and $g = 0$. Then D consists of a unique point P, and the complete linear series $|P|$ is cut out on X by the lines of the plane, so it is a g^1_1, and the assertion holds.

Next assume $n \geqslant 2$. Set

$$m = n(n-1) - \sum_{i=1}^{h} m_i(m_i - 1)$$

which is the degree of $\mathrm{adj}_{n-1}(X)$. We have

$$m - 2g = (n(n-1) - \sum_{i=1}^{h} m_i(m_i - 1)) -$$
$$- ((n-1)(n-2) - \sum_{i=1}^{h} m_i(m_i - 1)) = 2(n-1) > 0,$$

hence

$$m - g \geqslant g + 1.$$

The linear system $\mathrm{Adj}_{n-1}(X)$ has dimension

$$r \geqslant \frac{(n-1)(n+2)}{2} - \sum_{i=1}^{h} \frac{m_i(m_i - 1)}{2} = m - g \geqslant g + 1$$

and therefore there is some adjoint curve of degree $n-1$ which cuts on C, off $\varDelta$, a divisor containing D (see Exercise 19.4.7). If Y is such an adjoint we have

$$\mathrm{div}(Y) = \varDelta + D + A$$

with $\deg(A) = m - g - 1$. From the Restsatz Theorem 19.6.2, the adjoint curves of degree $n-1$ that cut, off $\varDelta$, a divisor containing A, form a linear system of dimension

$$r' \geqslant r - (m - g - 1) \geqslant 1,$$

and cut out, off $\varDelta$ and A, the complete linear series $|D|$. The assertion follows. □

Corollary 20.1.4 *If $g = 0$, then C is isomorphic to $\mathbb{P}^1$, i.e., C and X are rational.*

Proof Let $P \in C$ be a point and consider the divisor $D = P$. By applying Proposition 20.1.3, we have that $|P| = g_1^1$. This linear series ξ determines a morphism $\omega_\xi : C \to \mathbb{P}^1$, which is surjective (because $\omega(C)$ is non-degenerate). Moreover ξ is clearly simple. So ω_ξ is birational and the assertion follows. □

For reasons that we will later understand, if $g = 0$, so that $C \cong \mathbb{P}^1$, all (linearly equivalent, see Exercise 19.2.3) divisors of degree -2 on C are called canonical divisors.

Theorem 20.1.5 (M. Noether Reduktionsatz) *Let X be an irreducible curve with only ordinary singularities and let $\pi : C \to X$ be its smooth model. Let D be a special divisor on C and let $P \in C$ be a point such that $\pi(P) = Q$ is a smooth point which is not a nasty point of X and P does not belong to all canonical divisors containing D. Then*

$$\dim(|D|) = \dim(|D + P|),$$

i.e., P is a base point for the series $(|D + P|$.

Proof Let $h = 0$ be the linear equation of a line r passing through Q, such that

$$\mathrm{div}(h) = P + P_2 + \cdots + P_n, \quad \text{with} \quad n = \deg(X), \tag{20.1}$$

with $P, P_2, \ldots P_n$ distinct (which exists because Q is not nasty for X). Let moreover $\phi = 0$ be the equation of an adjoint curve of degree $n - 3$ such that

$$\mathrm{div}(\phi) = \Delta + D + D'$$

with $P \notin D'$, which exists by the hypotheses. Then we have

$$\mathrm{div}(h\phi) = \Delta + D + P + P_2 + \cdots + P_n + D'.$$

By the Restsatz Theorem 19.6.2, the linear series $D + P$ is cut out on C by the linear system of the adjoint curves of degree $n - 2$ that cut out on C the divisor $\Delta + P_2 + \cdots + P_n + D'$, off this divisor. Since the curves of this system have degree $n - 2$ and contain $P_2 + \cdots + P_n$ which are distinct points of the line r, all these curves contain r. Because of (20.1), P belongs to all these curves, hence it is a base point of $|D + P|$. □

Let D be an effective divisor o C. We can consider the subseries of the canonical series $|K_C|$ consisting of all divisors of $|K_C|$ containing D. One sets $i(D)$ to be the dimension of this series plus 1, i.e., $i(D) = \dim(|K_C - D|) + 1 = \ell(K_C - D)$ and $i(D)$ is called the *index of speciality* of D. Hence $i(D) \geqslant 0$ and $i(D) > 0$ if and only if D is special. Note that if $\deg(D) > 2g - 2$ certainly D is non-special, hence $i(D) = 0$. By contrast, if $\deg(D) \leqslant g - 1$, by Lemma 20.1.1 the divisor D is certainly special and

$$i(D) \geqslant g - \deg(D). \tag{20.2}$$

Theorem 20.1.6 (Riemann–Roch Theorem) *Let D be an effective divisor on a smooth projective curve C. Then*

$$\dim(|D|) = \deg(D) - g + i(D). \tag{20.3}$$

Proof As usual we assume $\pi : C \to X$ to be a birational morphism, with $X \subset \mathbb{P}^2$ a curve of degree n with only ordinary singularities.

Let us start by proving the theorem for non-special divisors. We proceed by induction on $\dim(|D|)$. So we first assume $\dim(|D|) = i(D) = 0$. By (20.2) we have $\deg(D) \geqslant g$. If we have $\deg(D) > g$, we can take an effective divisor $D' \leqslant D$ with $\deg(D') = g + 1$. By Proposition 20.1.3 we have $\dim(|D|) \geqslant \dim(|D'|) \geqslant 1$, a contradiction. Hence we have $\deg(D) = g$, proving (20.3) in this case.

Now suppose that (20.3) holds for all effective divisors D' such that $i(D') = 0$ and $\dim(|D'|) = r \geqslant 0$. Let D be an effective divisor such that $i(D) = 0$ and $\dim(|D|) = r + 1$. Let $P \in C$ be not a base point for $|D|$, such that $\pi(P) = Q$ is a smooth point for X and not a nasty point for X (recall that there is at most one nasty point for X, see the proof of Theorem 18.2.1). Certainly there are divisors in $|D|$ that contain P. So we can find some effective divisor D' such that $D' + P \in |D|$, and

$$\dim(|D'|) = \dim(|D|) - 1 = r.$$

We have also $i(D') = 0$. Suppose to the contrary $i(D') > 0$. Since $i(D) = 0$, no canonical divisor containing D' contains also P. Then by the Reduktionsatz 20.1.5, P would be a base point for $|D|$ a contradiction. So we can apply the induction hypothesis to D' and we get

$$\begin{aligned} r + 1 = \dim(|D|) = \dim(|D'|) + 1 = \\ = \deg(D') - g + 1 = \deg(D) - g \end{aligned}$$

proving (20.3) also in this case.

Next we proceed by induction on $i(D)$. Suppose the theorem proved for any effective divisor D' such that $i(D') = i \geqslant 0$. Let us take an effective divisor D such that $i(D) = i + 1$. The adjoint curves of degree $n - 3$ that cut out on C a divisor containing $\Delta + D$, cut out off this divisor a linear series ξ of dimension $i(D) - 1 = i$. Let P be a point of C such that $Q = \pi(P)$ is smooth for X, not nasty and not a base point for ξ. Then we have $i(D + P) = i$. By induction we have

$$\begin{aligned} \dim(|D + P|) = \deg(D + P) - g + i(D + P) = \\ = \deg(D) + 1 - g + i = \deg(D) - g + i(D). \end{aligned}$$

On the other hand, by the Reduktionsatz we have

$$\dim(|D|) = \dim(|D + P|),$$

and this proves (20.3) also in this case. □

Corollary 20.1.7 *The canonical series has dimension $g - 1$.*

Proof If $g = 0$, the canonical series is empty, because it has degree $2g - 2 = -2 < 0$. So the assertion is true in this case. If $g \geqslant 1$, the canonical series has dimension at least $g - 1 \geqslant 0$ (see Lemma 20.1.1). Moreover $i(K) = 1$ and therefore

$$\dim(|K|) = \deg(K) - g + i(K) = 2g - 2 - g + 1 = g - 1$$

as wanted. □

Theorem 20.1.8 *Let X, X' be irreducible, projective, plane curves with ordinary singularities. If X and X' are birationally equivalent then $g(X) = g(X')$.*

Proof Let $\pi : C \to X$ and $\pi' : C' \to X'$ be smooth birational models. Then C is birational to C' so it is isomorphic to C'. For every effective divisor D on C we have $\dim(|D|) \leqslant \deg(D)$, so it makes sense to consider the non-negative integer $\mathfrak{p}(D) = \deg(D) - \dim(|D|)$, and we set

$$\mathfrak{p}(C) = \sup_{D \geqslant 0}\{\mathfrak{p}(D)\}.$$

Of course $\mathfrak{p}(C)$ is invariant under isomorphisms. On the other hand Riemann–Roch Theorem implies that $\mathfrak{p}(C) = g(X)$, and therefore $g(X') = \mathfrak{p}(C') = \mathfrak{p}(C) = g(X)$ and the assertion follows. □

This theorem proves that, as announced, if $\pi : C \to X$ is a smooth model of the plane curve X with ordinary singularities, then $g(X)$ depends only on C. If X is any curve, we will define its *genus* to be the genus of a smooth projective model of X.

Corollary 20.1.9 *Given a smooth, projective curve C of genus $g \geqslant 1$, the canonical series is the only g^{g-1}_{2g-2} on C, therefore it is a birational invariant.*

Proof Let D be an effective divisor on C such that $\deg(D) = 2g - 2$. Then either D is special, in which case it is a canonical divisor and then $|D| = |K| = g^{g-1}_{2g-2}$, or D is non-special, in which case $|D| = g^{g-2}_{2g-2}$. □

Exercise 20.1.10 Compute $g(X)$ for each of the following curves $X \subset \mathbb{P}^2$ over $\mathbb{C}$:

$$x^2y^2 - z^2(x^2 + y^2) = 0$$
$$(x^3 - y^3)(x^2 + z^2) + x^3y^2 + x^2y^3 = 0$$
$$(x + y)^4 + z^4 - 2x^2(x + y - z)^2 = 0$$

and for the curves in Exercise 18.2.12.

Exercise 20.1.11 *Compute the index of speciality of any effective divisor if X is a smooth quartic or a quartic with a node.

Exercise 20.1.12 *Prove that if D is any divisor on a smooth, projective, irreducible curve C of genus g such that $\deg(D) \geqslant g$, then $\ell(D) \geqslant 1$.

Exercise 20.1.13 *Prove the following general form of Riemann–Roch Theorem: let D be any divisor on C of genus g, then

$$\ell(D) = \deg(D) - g + \ell(K_C - D) + 1. \tag{20.4}$$

Exercise 20.1.14 Prove that if D is any divisor on C of genus g such that $\deg(D) \geqslant 2g - 1$, then $\dim(|D|) = \deg(D) - g$.

Exercise 20.1.15 Prove that a curve X is rational if and only if its genus is 0.

Exercise 20.1.16 Prove that a smooth projective curve C is isomorphic to $\mathbb{P}^1$ if and only if it carries a complete g^n_n, for some $n \geqslant 1$.

Exercise 20.1.17 Prove that for any $n \geqslant 1$, $\mathbb{P}^1$ carries a unique complete g^n_n. Prove that this series is base point free and very simple and that the image of $\mathbb{P}^1$ via such a series is a rational normal curve of degree n in in $\mathbb{P}^n$.

Exercise 20.1.18 Let C be a smooth, projective, irreducible curve of genus g. Let D be a divisor of degree $d \geqslant 2g + 1$. Prove that the complete series $|D| = g_g^{d-g}$ is base point free and very simple.

Exercise 20.1.19 Let C be a smooth, projective, irreducible curve of genus g. Prove that $g = 1$ if and only if for any effective divisor D with $\deg(D) = n \geqslant 1$ one has $\dim(|D|) = n - 1$.

Exercise 20.1.20 Let C be a smooth, projective, irreducible curve of genus 1. Prove that it is isomorphic to a smooth plane cubic.

20.2 Consequences of the Riemann–Roch Theorem

In this section we list some consequences of the Riemann–Roch Theorem.

Theorem 20.2.1 (Reciprocity Theorem) *Let D, D' be effective divisors such that $D + D'$ is a canonical divisor. Then*

$$\deg(D) - \deg(D') = 2\Big(\dim(|D|) - \dim(|D'|)\Big).$$

Proof We have $\dim(|D'|) = i(D) - 1$, hence

$$\dim(|D|) - \dim(|D'|) = \deg(D) - g + 1.$$

Moreover

$$\deg(D) + \deg(D') = 2g - 2$$

hence

$$\dim(|D|) - \dim(|D'|) = \deg(D) - \frac{1}{2}(\deg(D) + \deg(D'))$$

whence the assertion follows. □

Theorem 20.2.2 *If C is a smooth curve of genus $g \geqslant 1$, the canonical series is base point free.*

Proof If P is a base point of the canonical series, we have $i(P) = g$, and then

$$\dim(|P|) = \deg(P) - p + i(P) = 1$$

so that $|P| = g_1^1$. Then by the argument we made in the proof of Corollary 20.1.4, C would be isomorphic to $\mathbb{P}^1$, hence we would have $g = 0$, a contradiction. □

Theorem 20.2.2 tells us that the canonical series $|K_C|$ determines a morphism

$$\kappa_C : C \to \mathbb{P}^{g-1}$$

also simply denoted by κ if there is no danger of confusion. This is called the *canonical map* of C. The curve $\kappa(C)$ is called the *canonical image* or, if $g \geqslant 2$, the *canonical curve* of C.

If $g = 1$, the canonical map is constant and does not give any interesting information on C. For $g \geqslant 2$ we have the following:

Theorem 20.2.3 *Let C be a smooth, projective curve of genus $g \geqslant 2$. Let $D = P + Q$ be an effective divisor of degree 2 on C. Then either*

$$\dim(|K - D|) = g - 3$$

or

$$\dim(|K - D|) = g - 2$$

in which case $|D| = g^1_2$, and the canonical series is composed with this involution.

Proof We have

$$\dim(|D|) = \deg(D) - g + i(D) = 2 - g + i(D).$$

We have $\dim(|D|) < 2$ otherwise we would have a g^2_2 on C and C would be isomorphic to $\mathbb{P}^1$ (see Exercise 20.1.16) thus $g = 0$, a contradiction. Hence we either have $\dim(|D|) = 0$ or $\dim(|D|) = 1$. In the former case we have $i(D) = g - 2$, hence $\dim(|K - D|) = g - 3$. In the latter we have $|D| = g^1_2$, and $i(D) = g - 1$, hence $\dim(|K - D|) = g - 2$. In this case for any divisor $D' = P' + Q' \in |D|$, we have

$$\dim(|K - D'|) = g - 2$$

which means that any canonical divisor containing P' also contains Q'. On the other hand, for any point $P' \in C$ there is a point $Q' \in C$ such that $D' = P' + Q' \in |D|$. This proves the assertion. □

A curve of genus $g \geqslant 2$ with a g^1_2 is called *hyperelliptic*. If $g = 2$ certainly the curve is hyperelliptic because the canonical series is a g^1_2.

Theorem 20.2.3 tells us that there is the following dichotomy. If C is a smooth, projective, irreducible curve of genus $g \geqslant 2$, then:

(a) either the canonical series is very simple, and therefore the canonical image of C is a smooth curve of degree $2g - 2$ in $\mathbb{P}^{g-1}$ isomorphic to C (see Theorem 19.5.2 and Exercise 19.5.3);

(b) or C is hyperelliptic, in which case the canonical series is composed with a uniquely determined g_2^1, and each effective canonical divisor is the sum of $g-1$ divisors of the g_2^1. The g_2^1 determines a morphism $\omega : C \to \mathbb{P}^1$ of degree 2. Then $|K|$ consists of the pull-backs to C via ω of the divisors of the g_{g-1}^{g-1} on $\mathbb{P}^1$. This shows that $X = \kappa(C)$ is a rational normal curve of degree $g-1$ in $\mathbb{P}^{g-1}$ (see Exercise 20.1.17), and that $\kappa : C \to X \cong \mathbb{P}^1$ is the g_2^1.

Theorem 20.2.4 (Clifford's Theorem) *Let D be a special divisor on C. Then*

$$\deg(D) \geqslant 2\dim(|C|)$$

Proof By definition of $i(D)$, the codimension of $|K-D|$ in $|K|$ is $g-i(D)$. This codimensions is also called the *number of conditions* that D imposes to $|K|$. The same holds for any $D' \in |D|$. Let $r = \dim(|D|)$, and let $P_1, \ldots, P_r$ be arbitrary points on C. Then there is a divisor of $|D|$ containing $P_1, \ldots, P_r$. Since $P_1, \ldots, P_r$ are arbitrary, we may chose $P_1, \ldots, P_r$ so that

$$\dim(|K-(P_1+\ldots+P_r)|) = g-1-r.$$

On the other hand

$$\begin{aligned} g-i(D) = \dim(|K|) - \dim(|K-D|) \geqslant \\ \geqslant \dim(|K|) - \dim(|K-(P_1+\ldots+P_r)|) = r \end{aligned}$$

because

$$\dim(|K-D|) \leqslant \dim(|K-(P_1+\ldots+P_r)|).$$

But

$$\dim(|D|) = \deg(D) - g + i(D)$$

and therefore

$$\deg(D) - \dim(|D|) = g - i(D) \geqslant r = \dim(|D|)$$

as wanted. □

Exercise 20.2.5 Consider a non-hyperelliptic canonical curve C of genus g in $\mathbb{P}^{g-1}$. Given an effective divisor D on C, let $\langle D\rangle$ be the *span* of D, i.e., the subspace of $\mathbb{P}^{g-1}$ that is the intersection of all hyperplanes π of $\mathbb{P}^{g-1}$ such that $\mathrm{div}(\pi) \geqslant D$. Prove the following *geometric form of Riemann–Roch Theorem*: let D be an effective divisor on C, then $\dim(|D|) = \deg(D) - 1 - \dim(\langle D\rangle)$.

Exercise 20.2.6 Let C be a smooth, projective, irreducible curve of genus $g \geqslant 1$ and let D, D' be effective divisors of degree $g-1$ such that $D + D'$ is a canonical divisor. Prove that $|D|$ and $|D'|$ have the same dimension.

Exercise 20.2.7 *Let C be a smooth, projective, irreducible curve of genus $g \geqslant 4$. Prove that if C has a g_4^2, then C is hyperelliptic, and the g_4^2 is composed with the g_2^1.

Exercise 20.2.8 Let C be a smooth, projective, irreducible curve of genus $g \geqslant 1$ and P any point of C. Prove that $|K_C + P|$ has P as a base point.

Exercise 20.2.9 Let C be a smooth, projective, irreducible, hyperelliptic curve and P any point of C. Let $P, Q \in C$ be such that $P + Q \in g^1_2$. Prove that $|K_C + P + Q|$ is composed with the g^1_2.

Exercise 20.2.10 Let C be a smooth, projective, irreducible, non-hyperelliptic curve. Let $P, Q \in C$ be any two points. Prove that $|K_C + P + Q|$ is simple but not very simple.

Exercise 20.2.11 Prove that a non-hyperelliptic curve of genus 3 is isomorphic to a smooth plane quartic and viceversa. What is the canonical image of a hyperelliptic curve of genus 3?

Exercise 20.2.12 Prove that a non-hyperelliptic canonical curve of genus 4 is the complete intersection of a quadric and a cubic hypersurface.

Exercise 20.2.13 Prove that any curve of genus 4 has a g^1_3. Prove that a non-hyperelliptic curve of genus 4 has one or two series g^1_3.

Exercise 20.2.14 Prove that a non-hyperelliptic canonical curve of genus 5 is contained in three independent quadrics.

Exercise 20.2.15 *Prove that any curve of genus 5 has a g^1_4.

Exercise 20.2.16 *Prove that there are hyperelliptic curves of any genus $g \geqslant 2$.

Exercise 20.2.17 *Prove that for any $g \geqslant 3$ there are non-hyperelliptic curves of genus g.

Exercise 20.2.18 Prove that not every projective irreducible curve is isomorphic to a smooth plane curve.

Exercise 20.2.19 Prove that a hyperelliptic curve has a unique g^1_2.

20.3 Differentials

20.3.1 Algebraic Background

Let A be a ring containing a field k and let M be an A-module. We define a *derivation* of A to M over k to be a k-linear map $D : A \to M$ such that the *Leibnitz rule* $D(xy) = xD(y) + yD(x)$ holds for all $(x, y) \in A \times A$.

Lemma 20.3.1 *If D is a derivation of A to M over k then:*

(a) $D(x) = 0$ *for all* $x \in k$;

(b) *if* $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, *then for all* $a_1, \dots, a_n \in A$ *we have*

$$D(f(a_1, \dots, a_n)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, \dots, a_n) D(a_i).$$

Proof Let $x \in A$ be any element and let n be a positive integer. One has

$$D(x^n) = nx^{n-1}D(x).$$

Indeed, this trivially holds for $n = 1$. Then proceed by induction on n. Assume $n > 1$. We have

$$\begin{aligned} D(x^n) &= D(x \cdot x^{n-1}) = x^{n-1}D(x) + xD(x^{n-1}) = \\ &= x^{n-1}D(x) + x \cdot (n-1)x^{n-2}D(x) = nx^{n-1}D(x). \end{aligned}$$

Then (b) follows right away. To prove (a), it suffices to prove that $D(1) = 0$. To prove this note that for any positive integer n one has

$$D(1) = D(1^n) = nD(1)$$

which implies $D(1) = 0$. □

Lemma 20.3.2 *Let A be a domain containing a field k and M an A-module that is a vector space over $\mathbb{Q}(A)$. Then any derivation $D : A \to M$ over k extends uniquely to a derivation $\bar{D} : \mathbb{Q}(A) \to M$.*

Proof Suppose we have the derivation $\bar{D}$. Let $x \in \mathbb{Q}(A)$ so that $x = \frac{a}{b}$, with $a, b \in A$ and $b \neq 0$. Then $a = xb$ and therefore $D(a) = b\bar{D}(x) + xD(b)$. Hence

$$\bar{D}(x) = \frac{1}{b} \cdot (D(a) - xD(b)). \tag{20.5}$$

This shows that $\bar{D}$ is uniquely determined. If we define $\bar{D}$ with the formula (20.5), it is easy to see that $\bar{D}$ is a derivation from $\mathbb{Q}(A)$ to M. □

Let A be a domain containing a field k. For $a \in A$ we define the symbol $[a]$ and we consider the free A-module P on the set $\{[a], a \in A\}$. Consider the submodule N of P generated by the elements of the form:

(a) $[a + b] - [a] - [b]$, for all $a, b \in A$;

(b) $[ka] - k[a]$, for all $a \in A$ and $k \in k$;
(c) $[ab] - a[b] - b[a]$, for all $a, b \in A$.

We set $\Omega_k(A) = P/N$ and we denote by da the image of $[a]$ in $\Omega_k(A)$ for $a \in A$, and call it the *differential* of a. Then we define $d : A \to \Omega_k(A)$ the k-linear map that takes a to da, for all $a \in A$. The module $\Omega_k(A)$ is called the *module of differentials* of A over k with derivation d.

Lemma 20.3.3 *For any domain A containing a field k, any A-module M and any derivation $D : A \to M$ over k, there is a unique homomorphism of A-modules $\mu : \Omega_k(A) \to M$ such that $D(a) = \mu(da)$, for all $a \in A$.*

Proof Keeping the notation introduced above, we define the homomorphism

$$\nu : \sum_{i=1}^{h} a_i[b_i] \in P \to \sum_{i=1}^{h} a_i D(b_i) \in M.$$

It is immediate to see that N is contained in $\ker(\nu)$, hence ν determines a homomorphism $\mu : \Omega_k(A) = P/N \to M$, which verifies the assertion. □

In the above setting, if $f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$, and $a_1, \ldots, a_n \in A$ we have

$$d(f(a_1, \ldots, a_n)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) da_i.$$

If A is generated by $a_1, \ldots, a_n$, then $\Omega_k(A)$ is generated by $da_1, \ldots, da_n$. If $x \in \mathbb{Q}(A)$ so that $x = \frac{a}{b}$, with $a, b \in A$ and $b \neq 0$, then we have

$$dx = d(ab^{-1}) = b^{-1}da + ad(b^{-1}) = b^{-1}da - ab^{-2}db = b^{-1}da - b^{-1}xdb.$$

So, if $\mathbb{Q}(A) = k(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \in A$ then $\Omega_k(\mathbb{Q}(A))$ is a vector space on $\mathbb{Q}(A)$ generated by $da_1, \ldots, da_n$.

Proposition 20.3.4 *Suppose X is an irreducible curve (over the algebraically closed field $\mathbb{K}$). Then $\Omega_{\mathbb{K}}(K(X))$ is a 1-dimensional vector space over $K(X)$. Moreover, if $\mathrm{char}(\mathbb{K}) = 0$ and if $\xi \in K(X) \setminus \mathbb{K}$, then $d\xi$ is non-zero, hence it is a basis of $\Omega_{\mathbb{K}}(K(X))$ over $K(X)$.*

Proof We may assume that X is an affine plane curve with equation $f(x, y) = 0$, with f and irreducible polynomial. Then $A := A(X) = \mathbb{K}[x, y]/(f)$. We denote by ξ and η the classes of x and y in A. Then $K := K(X) = \mathbb{Q}(A) = \mathbb{K}(\xi, \eta)$. By an argument we already made in the proof of Theorem 7.2.3, we have that one of the derivatives of f with respect to x and y is non-zero. We may suppose that $\frac{\partial f}{\partial y} \neq 0$. So f does not divide $\frac{\partial f}{\partial y}$ and therefore $\frac{\partial f}{\partial y}(\xi, \eta) \neq 0$ in K. We know that $\Omega_{\mathbb{K}}(K)$ is generated by $d\xi$ and $d\eta$ over K. However, since $f(\xi, \eta) = 0$ in K, we have

$$0 = d(f(\xi, \eta)) = \frac{\partial f}{\partial x}(\xi, \eta)d\xi + \frac{\partial f}{\partial y}(\xi, \eta)d\eta$$

so

$$d\eta = -\frac{\frac{\partial f}{\partial x}(\xi, \eta)}{\frac{\partial f}{\partial y}(\xi, \eta)}d\xi$$

thus $d\xi$ generates $\Omega_{\mathbb{K}}(K)$ over K.

To prove that $\Omega_{\mathbb{K}}(K)$ has dimension 1, we must show that $\Omega_{\mathbb{K}}(K)$ is non-zero. By Lemmas 20.3.2 and 20.3.3 it suffices to prove that there is a non-zero derivation $D : A \to K$. Let $g(x, y) \in \mathbb{K}[x, y]$ and let $\bar{g}$ be its class in A. Then we set

$$D(\bar{g}) = \frac{\partial g}{\partial x}(\xi, \eta) - \frac{\frac{\partial f}{\partial x}(\xi, \eta)}{\frac{\partial f}{\partial y}(\xi, \eta)} \frac{\partial g}{\partial y}(\xi, \eta).$$

It is an easy verification that this is a well defined derivation. Since we have $D(\xi) = 1$, we are done. We want to stress that the above argument actually proves that $d\xi$ is non-zero, hence it is a basis of $\Omega_{\mathbb{K}}(K)$ over K.

We prove now the final assertion. Since $\xi \notin \mathbb{K}$, then $\mathbb{K}(\xi)$ has transcendence degree 1 over $\mathbb{K}$, and therefore the extension $\mathbb{K}(\xi) \subseteq K$ is algebraic. By the Primitive Element Theorem 7.2.2, there is a $\eta \in K$ such that $K = \mathbb{K}(\xi, \eta)$. So K is the quotient field of the domain $\mathbb{K}[x, y]/(f)$, where $f(x, y)$ is an irreducible polynomial and $\frac{\partial f}{\partial y} \neq 0$. The same argument we made to prove the first assertion shows that $d\xi$ is non-zero, as desired. □

If we assume $\mathrm{char}(\mathbb{K}) = 0$, and we fix $f, t \in K = K(X)$, with $t \notin \mathbb{K}$, there is a unique $g \in K$ such that $df = g dt$. One writes $g = \frac{df}{dt}$, and we call it the *derivative* of f with respect to t.

Lemma 20.3.5 *Let $(A, \mathfrak{m})$ be a DVR with quotient field K , and suppose k is a subfield of A such that $A/\mathfrak{m} \cong k$ and the composite map $k \to A \to A/\mathfrak{m}$ is an isomorphism. Let u be a uniformizing parameter in A and take $g \in A$. For any positive integer n there are $\lambda_0, \ldots, \lambda_{n-1} \in k$ and $h \in A$ such that*

$$g = \sum_{i=0}^{n-1} \lambda_i u^i + h u^n. \tag{20.6}$$

Proof We proceed by induction on n. For $n = 1$, consider the image λ_0 of g in $A/\mathfrak{m} = k$. Then $g - \lambda_0 \in \mathfrak{m}$, so there is an $h \in A$ such that $g - \lambda_0 = hu$, as wanted.

Next assume the result is true for $n - 1 \geqslant 1$. Then we have a relation of the sort

$$g = \sum_{i=0}^{n-2} \lambda_i u^i + z u^{n-1} \quad \text{with} \quad z \in A \quad \text{and} \quad \lambda_0, \ldots, \lambda_{n-2} \in k.$$

On the other hand we have

$$z = \lambda_n + hu, \quad \text{with} \quad h \in A \quad \text{and} \quad \lambda_n \in k.$$

Putting together the last two relations one gets (20.6). □

Lemma 20.3.6 *Let C be a smooth, irreducible, projective curve and let $P \in C$. Let u be a uniformizing parameter for $\mathcal{O}_{C,P}$ at P. Then for all $g \in \mathcal{O}_{C,P}$, one has $\frac{dg}{du} \in \mathcal{O}_{C,P}$.*

Proof We may assume that there is an irreducible curve $Y \subset \mathbb{P}^2$ such that $\pi : C \to Y$ is the smooth model of Y and $\pi(P) = Q$ is a smooth point of Y (see Exercise 18.2.11).

We may suppose that $Q \in U_0 \cong \mathbb{A}^2$ and that actually Q is the origin of $\mathbb{A}^2$. We will denote by X the affine curve $Y \cap U_0$. Then $\mathcal{O}_{C,P} = \mathcal{O}_{X,Q}$.

We use now the same notation we used in the proof of Proposition 20.3.1. Namely X has equation $f(x, y) = 0$, $A(X) = \mathbb{K}[x, y]/(f)$, ξ and η are the classes of x and y in $A(X)$ so that $A(X) = \mathbb{K}[\xi, \eta]$, $K = \mathbb{K}(\xi, \eta)$. Choose an integer n large enough so that

$$o_{X,Q}\Big(\frac{dx}{du}\Big) \geqslant -n, \quad o_{X,Q}\Big(\frac{dy}{du}\Big) \geqslant -n.$$

If $h \in A(X)$ we have

$$\frac{dh}{du} = \frac{\partial h}{\partial x}(\xi, \eta)\frac{dx}{du} + \frac{\partial h}{\partial y}(\xi, \eta)\frac{dy}{du}$$

so that $o_{X,Q}(\frac{dh}{du}) \geqslant -n$.

Take $g \in \mathcal{O}_{X,Q}$. By Lemma 20.3.5 we can write g as in (20.6). Then

$$\frac{dg}{du} = \sum_{i=0}^{n-1} i\lambda_i u^{i-1} + nhu^{n-1} + u^n \frac{dh}{du}. \tag{20.7}$$

Since $o_{X,Q}(\frac{dh}{du}) \geqslant -n$, each term in the sum (20.7) is in $\mathcal{O}_{X,Q}$, so $\frac{dg}{du} \in \mathcal{O}_{X,Q}$, as wanted. $\square$

20.3.2 Differentials and Canonical Divisors

Throughout this section we assume $\mathrm{char}(\mathbb{K}) = 0$.

Let C be a smooth, irreducible projective curve and let $K = K(C)$. We set $\Omega_C = \Omega_{\mathbb{K}}(K)$ (and also $\Omega_C = \Omega_X$ for any birational model X of C). Its elements are called the *differentials* on C (or on any birational model of C).

Let $\omega \in \Omega_C \setminus \{0\}$ and let P be a point of C. We define the *order* of ω at P, and denote it by $o_P(\omega)$, in the following way. We fix a uniformizing parameter $u \in \mathcal{O}_{C,P}$ and write $\omega = gdu$, with $g \in K$. Then we set $o_P(\omega) = o_P(g)$. This is well defined. Indeed, if $v \in \mathcal{O}_{C,P}$ is another uniformizing parameter, and if $gdu = hdv$, then by Lemma 20.3.6 we have

$$\frac{g}{h} = \frac{dv}{du} \in \mathcal{O}_{C,P}$$

and, by the same token, we have also $\frac{h}{g} \in \mathcal{O}_{C,P}$. This means that $\frac{g}{h}$ is invertible in $\mathcal{O}_{C,P}$, hence $o_P(g) - o_P(h) = o_P(\frac{g}{h}) = 0$, as wanted.

If $\omega \in \Omega_C \setminus \{0\}$, we define the *divisor of* ω as

$$\mathrm{div}(\omega) = \sum_{P \in C} o_P(\omega)P.$$

We will see in a moment that this definition is well posed, i.e., $o_P(\omega) \neq 0$ only for finitely many points $P \in C$. We note that two divisors of differentials are linearly equivalent. Indeed, if ω and ω' are two non-zero differentials, there is a $g \in K$ such that $\omega' = g\omega$. Hence $\mathrm{div}(\omega') = \mathrm{div}(g) + \mathrm{div}(\omega)$, so that $\mathrm{div}(\omega') \equiv \mathrm{div}(\omega)$. Conversely, for any divisor D that is linearly equivalent to $\mathrm{div}(\omega)$ for some $\omega \in \Omega_C \setminus \{0\}$, there is an $\omega' \in \Omega_C \setminus \{0\}$ such that $D = \mathrm{div}(\omega')$. Indeed, if $D = \mathrm{div}(g) + \mathrm{div}(\omega)$ for some $g \in K$, then $D = \mathrm{div}(g\omega)$. In conclusion the divisors of differentials form a complete linear equivalence class of divisors. If $\omega \in \Omega_C \setminus \{0\}$ is such that $\mathrm{div}(\omega)$ is effective, we will say that ω is a *differential of the first kind* on C.

The following result identifies the linear equivalence class of the divisors of differentials.

Theorem 20.3.7 *The divisors of differentials on C are linearly equivalent to canonical divisors on C.*

Proof We first examine the genus 0 case, i.e., $C = \mathbb{P}^1$. Consider homogeneous coordinates $[x_0, x_1]$ on $\mathbb{P}^1$ and on $U_0 \cong \mathbb{A}^1$ we pass to the affine coordinate $x = \frac{x_1}{x_0}$. Consider the differential dx. For all points $P \in U_0$, we have $o_P(dx) = 0$. So we have to understand what is the order of dx at the *point at infinity* $P_\infty = [0, 1]$. Now we pass to affine coordinates on $U_1 \cong \mathbb{A}^1$, where we have $x_1 \neq 0$. Here we have the affine coordinate $y = \frac{x_0}{x_1} = \frac{1}{x}$ and P_∞ becomes the origin in this coordinate. Moreover

$$dx = d\left(\frac{1}{y}\right) = -y^{-2}dy$$

so that $o_{P_\infty}(dx) = -2$. Thus $\mathrm{div}(dx) = -2P_\infty$, and this proves the assertion, because on $\mathbb{P}^1$ the canonical divisors have degree -2.

Next we suppose $g \geqslant 1$. We may assume to have a projective, irreducible, plane curve $X \subset \mathbb{P}^2$ of degree d with ordinary singularities such that $\pi : C \to X$ is the smooth model of X. We can fix homogeneous coordinates $[x_0, x_1, x_2]$ in $\mathbb{P}^2$ so that:

(a) the *line at infinity* ℓ_∞ with equation $x_0 = 0$ intersects X in d distinct points $P_1, \dots, P_d$, which implies that no multiple point of X lies on ℓ_∞;
(b) the point $P_x = [0, 1, 0]$ at infinity of the x axis does not sit on X;
(c) no principal tangent line in a multiple point of X contains P_x.

We let $f(x_0, x_1, x_2) = 0$ be the equation of X, so that f is homogeneous of degree d. We set $D_\infty = P_1 + \cdots + P_d$ the divisor cut out on C by ℓ_∞. By the definition of the canonical series we need to prove that, if ω is a non-zero differential on C, one has

$$\mathrm{div}(\omega) \equiv (d-3)D_\infty - \Delta \tag{20.8}$$

where Δ, as usual, denoted the multiple points divisor of X.

We can pass to affine coordinates $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$ in $U_0 \cong \mathbb{A}^2$ and we consider the affine equation $g(x, y) = f(1, x, y) = 0$ of X. We have

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x_1}(1, x, y), \quad \frac{\partial g}{\partial y} = \frac{\partial f}{\partial x_2}(1, x, y)$$

hence

$$\frac{\partial f}{\partial x_1}(x_0, x_1, x_2) = x_0^{d-1}\frac{\partial g}{\partial x}\Big(\frac{x_1}{x_0}, \frac{x_2}{x_0}\Big), \quad \frac{\partial f}{\partial x_2}(x_0, x_1, x_2) = x_0^{d-1}\frac{\partial g}{\partial y}\Big(\frac{x_1}{x_0}, \frac{x_2}{x_0}\Big).$$

Consider the differential $\omega = dx$ and the rational function

$$\phi = \frac{\frac{\partial f}{\partial x_2}}{x_0^{d-1}} = \frac{\partial g}{\partial y}\Big(\frac{x_1}{x_0}, \frac{x_2}{x_0}\Big).$$

To prove (20.8), we will prove that

$$\mathrm{div}(\omega) = (d-3)D_\infty - \Delta + \mathrm{div}(\phi). \tag{20.9}$$

As

$$\mathrm{div}(\phi) = \mathrm{div}\Big(\frac{\partial f}{\partial x_2}\Big) - (d-1)D_\infty$$

then (20.9) is equivalent to

$$\mathrm{div}(\omega) - \mathrm{div}\Big(\frac{\partial f}{\partial x_2}\Big) = -2D_\infty - \Delta. \tag{20.10}$$

Since

$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$

we have

$$\omega = dx = -\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial x}}dy = -\frac{\frac{\partial f}{\partial x_2}}{\frac{\partial f}{\partial x_1}}dy$$

hence for all points $P \in C$ we have

$$o_P(\omega) - o_P\Big(\frac{\partial f}{\partial x_2}\Big) = o_P(dy) - o_P\Big(\frac{\partial f}{\partial x_1}\Big). \tag{20.11}$$

Suppose $P \in C$ is such that $\pi(P) = P_i$, for $i = 1, \ldots, d$. Then $y^{-1} = \frac{x_0}{x_2}$ is a uniformizing parameter in $\mathcal{O}_{X,P_i} = \mathcal{O}_{C,P}$ and

$$dy = -y^{-2}d(y^{-1})$$

so $o_P(dy) = -2$. We claim that $\frac{\partial f}{\partial x_1}(P_i) \neq 0$, for all $i = 1, \ldots, d$. In fact, for $i = 1, \ldots, d$, we have $P_i = [0, p_{i1}, p_{i2}]$ and $p_{i2} \neq 0$ because P_x does not sit on X. By

Euler's relation we have

$$0 = d \cdot f(P_i) = \frac{\partial f}{\partial x_1}(P_i)p_{i1} + \frac{\partial f}{\partial x_2}(P_i)p_{i2}$$

and if $\frac{\partial f}{\partial x_1}(P_i) = 0$, we would have also $\frac{\partial f}{\partial x_2}(P_i) = 0$. Then ℓ_∞ would be tangent to X at P_i, a contradiction because ℓ_∞ intersects X in d distinct points. So from (20.11), we see that both sides of (20.10) have order -2 at P.

Now we come at points in U_0. So assume that $P \in C$ is a point such that $\pi(P) = Q \in U_0 \cong \mathbb{A}^2$. We can change coordinates and assume that Q is the origin of $\mathbb{A}^2$.

Suppose first that the x axis is tangent to X at Q. By the hypothesis (c), Q is a smooth point for X. Then x is a uniformizing parameter at Q and $\frac{\partial f}{\partial x_2}(Q) \neq 0$. Therefore by (20.11), we see that both sides of (20.10) have order 0 at P.

Suppose finally that the x axis is not tangent to X at Q. Then y is a uniformizing parameter at P, so that $o_P(dy) = 0$. Moreover we claim that $o_P(\frac{\partial f}{\partial x_1}) = m_Q(X) - 1$. If so, again by (20.11), we see that both sides of (20.11) have order $m_Q(X) - 1$ at P and we are done.

So, to finish we have to prove the above claim. Set $m = m_Q(X)$. The equation $g(x, y) = 0$ of X is of the form

$$g(x, y) = \prod_{i=1}^{m}(x - a_i y) + g_{m+1}(x, y) + \cdot + g_d(x, y) = 0,$$

with $a_1, \ldots, a_m$ distinct, where we indicated the homogeneous components of g. With an appropriate choice of coordinates we may assume that $a_1, \ldots, a_m$ are all distinct from 0.

The point P corresponds to a linear branch γ which we may suppose to be determined by a parametrization of the type

$$y = t, \quad x = a_1 t + \cdots . \tag{20.12}$$

To compute $o_P(\frac{\partial f}{\partial x_1})$ we have to compute $o_\gamma(\frac{\partial g}{\partial x})$. Now

$$\frac{\partial g}{\partial x} = \sum_{i=1}^{m}\prod_{j \neq i}(x - a_i y) + \cdots \tag{20.13}$$

where the dots stay for higher order terms. Then $o_\gamma(\frac{\partial g}{\partial x})$ is the order of the series obtained by substituting (20.12) in (20.13). The lower order term of this series is

$$\sum_{i=2}^{m}\prod_{j \neq i}(a_1 - a_i)t$$

which has order $m - 1$, as desired. □

20.3.3 The Riemann–Hurwitz Theorem

We still assume that $\mathrm{char}(\mathbb{K}) = 0$ in this section. Let C, C' be two smooth, projective, irreducible curves and $f : C \to C'$ a surjective morphism of degree n, which determines a γ^1_n. Then there is a dense open subset U of C' such that for all $Q \in C'$ the fibre divisor $f^*(Q)$ is reduced of order n (see Exercise 19.3.6).

Take $P \in C$ with $f(P) = Q$, and let $e_{f,P}$ be the multiplicity of P in $f^*(Q)$, also denoted by e_P is no confusion arises. Then we have $f^*(Q) = \sum_{f(P)=Q} e_{f,P} P$ and $n = \sum_{f(P)=Q} e_{f,P}$. We set $r_{f,P} = e_{f,P} - 1$, and call it the *ramification index* of f at P. One has $r_P = 0$ for all but finitely many points of C. We set $R_f = \sum_{P\in C} r_{f,P} P$. This is called the *ramification divisor* of f on C, whose degree we denote by r_f. We set $B_f = \sum_{P\in C} r_{f,P} f(P)$, which is a divisor on C', that has the same degree $\sum_{P\in C} r_{f,P}$ as R_f and is called the *branch divisor* of f. The points in the support of R_f [resp. of B_f] are called *ramification points* [resp. *branch points*] of f. If $C' = \mathbb{P}^1$, then $f : C \to \mathbb{P}^1$ is a base point free g^1_n and we talk about branch, ramification points and divisors of the g^1_n.

We want to compute the degree of the ramification (or of the branch) divisor, in terms of the genera of C and C', and of the degree of f.

Theorem 20.3.8 (Riemann–Hurwitz Theorem) *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let* C, C' *be two smooth, projective, irreducible curves of genera* g *and* g' *respectively and* $f : C \to C'$ *a surjective morphism of degree* n. *Then*

$$K_C \equiv f^*(K_{C'}) + R_f$$

and accordingly

$$2g - 2 = (2g' - 2)n + r_f.$$

Proof We have an inclusion $f^* : K(C') \to K(C)$. This yields on obvious inclusion $f^* : \Omega_{C'} \to \Omega_C$. Fix a non zero $\omega \in \Omega_{C'}$. We want to compare the canonical divisors $\mathrm{div}(\omega)$ on C' and $\mathrm{div}(f^*(\omega))$ on C.

Let P be a point of C and $Q = f(P) \in C'$. We have the inclusion $f^* : \mathcal{O}_{C',Q} \to \mathcal{O}_{C,P}$. Let u be a uniformizing parameter in $\mathcal{O}_{C',Q}$. Then there is a positive integer m such that $f^*(u) \in \mathfrak{m}_P^m$ but $f^*(u) \in \mathfrak{m}_P^{m+1}$. Hence there is a uniformizing parameter v in $\mathcal{O}_{C,P}$ such that $f^*(u) = av^m$, with a invertible in $\mathcal{O}_{C,P}$. Since $u = 0$ is a local equation of Q, it is clear from the definition of the pull back divisor $f^*(Q)$ that $m = e_{f,P}$.

Given a non-zero $\omega \in \Omega_{C'}$, there is a rational function $g \in K(C')$ such that $\omega = g du$. Then

$$f^*(\omega) = f^*(g)d(av^m) = f^*(g)(mv^{m-1}adv + v^m da).$$

Comparing the order of the two members and taking into account that $m = e_{f,P}$, we have

$$K_C \equiv \mathrm{div}(f^*(\omega)) = R_f + f^*(\mathrm{div}(\omega)) \equiv R_f + f^*(K_{C'}),$$

as wanted. □

Exercise 20.3.9 Let C be a smooth, projective, irreducible curve of genus g. Prove that the differentials of the first kind on C form a $\mathbb{K}$-vector space of dimension g.

Exercise 20.3.10 Prove that any base point free g^1_n on a curve of genus g, with $n \geqslant 2$ has $2n + 2g - 2 > 0$ branch points (to be counted with their multiplicities).

Exercise 20.3.11 Prove that the ramification and branch divisors of a γ^1_2 are reduced.

Exercise 20.3.12 Let C, C' be two smooth, projective, irreducible curves of genera g and g' respectively and $f : C \to C'$ a surjective morphism of degree n. Prove that $g \geqslant n(g' - 1) + 1$ and the equality holds if and only if there are no ramification points for f.

Exercise 20.3.13 Fix a line r in the projective plane $\mathbb{P}^2$ and consider in $\mathcal{L}_{2,2}$ the set X of all conics that have intersection multiplicity at least 2 with r at some point. Prove that X is a quadric in $\mathcal{L}_{2,2} \cong \mathbb{P}^5$.

Exercise 20.3.14 Prove that, given two different linear series g^1_2 on $\mathbb{P}^1$, there is a unique divisor that belongs to both of them.

Prove that, given two different linear series g^1_2 on a smooth curve C of genus 1, there is no divisor that belongs to both of them. Are there different linear series g^1_2 on a smooth curve C of genus 1?

Exercise 20.3.15 Identify the g^2_2 on $\mathbb{P}^1$ with $\mathbb{P}^2$. Prove that the set Γ of non-reduced divisors in the g^2_2 is an irreducible conic. Prove that the series g^1_2 with a base point correspond exactly to the lines in g^2_2 which are tangent to Γ.

Exercise 20.3.16 Find another proof of Lüroth Theorem 7.3.3 based on Riemann–Hurwitz Theorem 20.3.8.

20.4 Solutions of Some Exercises

20.1.12 Write $D = D_1 - D_2$ with D_1, D_2 effective divisors with no common support. Then $\deg(D_1) \geqslant g + \deg(D_2)$. By Riemann–Roch Theorem we have

$$\ell(D_1) \geqslant \deg(D_1) - g + 1 \geqslant \deg(D_2) + 1$$

so we can certainly find divisors in $|D_1|$ containing D_2, which proves the assertion.

20.1.13 The assertion is proved by Riemann–Roch Theorem (20.1.6) if $\ell(D) > 0$. So it suffices to prove (20.4) when $\ell(D) = 0$. First we do the case $\ell(D) = \ell(K_C - D) = 0$. Then by Exercise 20.1.12 we have that $\deg(D) \leqslant g - 1$ and $\deg(K_C - D) \leqslant g - 1$, whence $\deg(D) = g - 1$, which proves (20.4) in this case. Assume next $\ell(D) = 0$ and $\ell(K_C - D) > 0$. Applying Theorem 20.1.6 to $K_C - D$, we get

$$\ell(K_C - D) = \deg(K_C - D) - g + \ell(D) + 1 = g - 1 - \deg(D)$$

which again proves (20.4) in this case.

20.2.16 Suggestion: take irreducible plane curves of degree $g + 2$ with an ordinary multiple point of multiplicity g and no other singularity.

20.2.17 Suggestion: take irreducible plane curves of degree $n \geqslant 4$ with an ordinary multiple point of multiplicity $n - 3$ and no other singularity or with at most a further node.

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