Dynamic portfolio strategies: quantitative methods and empirical rules for incomplete information

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DYNAMIC PORTFOLIO STRATEGIES: QUANTITATIVE METHODS AND EMPIRICAL RULES FOR INCOMPLETE INFORMATION

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Introduction

This book investigates optimal investment problems for stochastic financial market models. It is addressed to academics and students, who are interested in the mathematics of finance, stochastic processes, and optimal control, and also to practitioners in risk management and quantitative analysis who are interested in new strategies and methods of stochastic analysis.

There are many works devoted to the solution of optimal investment problems for different models. In fact, the "optimality" of any strategy is not something absolute but very much depends on a model (in particular, on prior distributions of parameters such as the appreciation rates and the volatility). In other words, any "optimal" strategy is optimal only for a given model of price evolution, for a given utility function, and for a given probability measure (prior distributions). On the other hand, strategies based on "technical analysis" are model-free: they require only historical data. This is why the technical analysis is even more popular among traders than the analysis based on stochastic models (see, e.g., survey and discussion in Lo *et al.* (2000)).

Our aim is to reduce the gap between model-free strategies and strategies that are "optimal" for stochastic models. We hope that specialists who prefer methods of "technical analysis" (which are rather empirical) will be interested in some of the new strategies suggested in this book and evaluated via stochastic market models.

We consider an optimal investment problem for strategies based on historical data with several new features. First, we introduce and investigate optimal investment problems for strategies that use *historical prices as well as trading volume* for underlying assets. It is shown in numerical experiments with real data that the joint distribution of prices and volume contains important information; in fact, we improve the performance of strategy by including volume in our consideration (note that "technical analysis" usually takes volume into account). Second, our model involves *additional constraints* of a very general type for the wealth process. In particular, these constraints cover the problem of replication of a given claim with a guaranteed error bound (gap). More precisely, our admissible strategies ensure *almost surely*, that the replication errors do not exceed a given level. Note that the strategy is uniquely determined by the claim in the classical problem (for a complete market) of an exact replication. In an incomplete market, where an exact replication is no longer generally possible, it is sensible to consider replications with some gap, which in turn allows choice among many possible strategies. Finally, the utility function under consideration in our model is a fairly general one, covering the mean-variance criterion, noncontinuous functions, and nonlinear concave functions as special cases. In particular, our general utility function and constraints incorporate the so-called *goal-achieving problem*, mean-variance hedging, problems with logical constraints, and many other problems. In addition, we consider some special problems such as optimization with maximin criterion, optimal portfolio compression, and superreplication under uncertainty; some new filters (estimators) for the appreciation rates of the stocks are given as an auxiliary tools.

The results are based on a stochastic diffusion market model, but the dependence on the model is partially lifted in the following sense: some empirical "model-free" strategies are presented, and they are shown to be apparently optimal for stochastic market models in the class of admissible strategies based on historical prices only. This conclusion is correct for a special but very important classes of prior distributions of market parameters. The corresponding class of admissible strategies is such that the appreciation rates of stocks are not supposed to be directly observable, but should be estimated from the historical data via some filters (estimators). The strategies based on filters presented in this book outperform classic strategies based on Kalman–Bucy filters for the appreciation rates, and they are surprisingly simple; thus, they may be interesting for practitioners.

There are two different types of market models: discrete-time models and continuous-time models. Of course, the real stock prices are presented as time series, so the discrete-time models are more realistic. However, it is commonly recognized that the continuous-time models give a good description of real (discrete-time) markets. Moreover, they lead to explicit and clear solutions of many analytical problems including investment problems. On the other hand, it appears that a formula for strategy derived for a continuous-time model can often be effectively used after a natural discretization. In this book, some "good" strategies in model-free discrete-time setting are presented, and then it is shown that these strategies can be interpreted as optimal for some continuous-time stochastic model.

For stochastic market models, the price of the stocks evolves as a random process with some standard deviations of the stock returns (volatility coefficient, or volatility) and some appreciation rate; they are the often mentioned parameters of a market. For a continuous-time model, it is assumed that the vector of stock prices $S(t)$ evolves according to an Itô stochastic differential equation, with the vector of appreciation rates $a(t)$ as a coefficient of the drift, and the volatility matrix $\sigma(t)$ as a diffusion coefficient.

The problem of optimal investment goes back to Merton (1969, 1973). He found strategies that solve an optimization problem in which $\mathbf{E}U(X(T))$ is to be maximized, where $X(T)$ represents the wealth at the final time T and $U(\cdot)$ is a utility function. If market parameters are allowed to be directly observed, then the optimal strategies (i.e. the current vector of stock portfolio) are functions of the current vector $(a(t), \sigma(t), S(t), X(t))$ for a general problem (see, e.g., the survey in Hakansson (1997) and Karatzas and Shreve (1998)).

Another problem of wide interest is mean-variance hedging, or the problem of minimizing $\mathbf{E}|X(T)-\xi|^2$, where ξ is a given random claim. For this problem, explicit solutions were obtained for the case of observable appreciation rates; see, e.g., Föllmer and Sondermann (1986), Duffie and Richardson (1991), Pham *et al.* (1998), Kohlmann and Zhou (1998), Pham *et al.* (1998), and Laurent and Pham (1999). The resulting optimal hedging strategies are combinations of the Merton strategy and the Black and Scholes strategy, which depends on the direct observation of the appreciation rates.

But in practice, $(a(t), \sigma(t))$ has to be replaced by estimates based on historical data. Many papers have been devoted to estimations of $(a(t), \sigma(t))$, mainly based on modifications of the Kalman–Bucy filter or the maximum likelihood principle (see, e.g., Lo (1988), Chen and Scott (1993), Pearson and Sun (1994)). In practice, the volatility coefficient can be currently estimated from stock prices. For an idealized continuous-time market model, the volatility is an explicit function of past historical prices (see (1.11) below). For the real market, the volatility can be calculated directly either from stock prices (historical volatility) or from option prices for the given asset (implied volatility). There is much of empirical research on the distribution of the real volatility (see, e.g., Black and Scholes (1972), Hauser and Lauterbach (1997), Mayhew (1995)). Unfortunately, the process $a(t)$ is usually hard to estimate in real-time markets, because the drift term defined by $a(t)$ is usually overshadowed by the diffusion term defined by $\sigma(t)$. Thus, there is a problem of optimal investment with a sufficient error in estimation of $a(t)$. A popular tool for this problem is the so-called "Separation Theorem" or "certainty equivalence principle": *Agents who know the solution to the optimal investment problem for the case of directly observable $a(t)$ can solve this problem by substituting $\mathbf{E}\{a(t)|S(\tau), \tau < t\}$, referred to as the real $a(t)$ (or "certainty equivalent estimate")* (see, e.g., Gennotte (1986)). Unfortunately, this principle does not hold in the general case of nonlog utilities (see Kuwana (1995)). Of course, one can hardly expect the conditional expectation alone (i.e. the L_2-estimate) to be the optimal estimate of $a(t)$ for all utilities in a wide class of possible estimates (which include, for example, Bayesian estimates and L_q-estimates with $q \neq 2$).

If $a(t)$ is Gaussian, then the pair $\mathbf{E}\{a(t)|S(\tau), \tau < t\}$ and the conditional variance perfectly describes the distribution $\mathcal{P}_{a(t)}(\cdot\,|S(\tau), \tau < t)$. Williams (1977), Detempte (1986), Dothan and Feldman (1986), Gennotte (1986), and Brennan (1998) obtained a solution of the investment problem for Gaussian nonobservable $a(\cdot)$. In particular, they showed that for a case of a power utility function, for which the certainty equivalence principle does not hold, a correction of myopic strategy can be calculated via solution of a Bellman parabolic equation (Brennan (1998)). However, this approach cannot be extended to the non-Gaussian case.

Karatzas (1997), and Karatzas and Zhao (1998) have obtained the first optimal strategies that are optimal in a class of strategies determined by all historical prices for a wide class of distribution of a, but under the crucial condition that appreciation rates and volatilities do not depend on time. This assumption ensures that optimal wealth has the form $X(t) = H(S(t), t)$, where $H(\cdot)$ satisfies a deterministic parabolic backward Kolmogorov equation of dimension n for the market with n stocks. Even if one accepts this restrictive condition, the solution of the problem is difficult to realize in practice for large n (say, $n > 4$), since it is usually difficult to solve the parabolic equation. Karatzas (1997) gave an explicit solution of a specific goal achieving problem for a case of one stock with conditionally normal distribution. Karatzas and Zhao (1998) solved a problem for $n > 1$ with a diagonal volatility matrix for a general utility function. It can be added that, in a similar setting, Dokuchaev and Zhou (2001) considered the problem as an extension of the goal-achieving problem into one with specific bounded risk constraints. Dokuchaev and Teo (2000) used a similar approach and considered a more general setting, in terms of both the utility function and the constraints.

In this book, we are study investment problems for the following different cases:

(i) There are no equations for stock prices or any other model of evolution; rather, there are only time series of historical prices. All market parameters are unknown and nonobservable, their prior distribution is unknown and they are not currently observed;

(ii) The evolution of stock prices is described by the Itô equation, where the appreciation rate $a(\cdot)$ and the volatility $\sigma(\cdot)$ are known or directly observable and their prior distributions are known;

(iii) The appreciation rate $a(\cdot)$, the volatility $\sigma(\cdot)$, and the risk-free interest rate $r(t)$ are directly observable, but their prior distributions are unknown;

(iv) $a(\cdot)$ and $\sigma(\cdot)$ are unknown and cannot be observed directly, but their prior distributions are known, and $a(\cdot)$ and $\sigma(\cdot)$ are currently estimated from historical prices;

(v) $r(\cdot)$, $a(\cdot)$, $\sigma(\cdot)$ and their prior distributions are unknown (but $r(t)$ is directly observable).

In case (i), we can consider the problem only for a discrete-time setting, because the real stock prices are given as time series; $a(\cdot)$ and $\sigma(\cdot)$ are completely excluded from the model (Chapter 2).

Let us describe our motivation for studying the "model-free" case (i). As was mentioned, if the market parameters can be directly observed, then the optimal strategies (i.e. the current vector of stock portfolio) are functions of the current vector of the volatilities and the appreciation rates. Moreover, their evolution law is supposed to be known (i.e., the volatilities and the appreciation rates evolve according to known equations). Such strategies are optimal for a given evolution law and for a given utility function. However, if this evolution law is changed, the optimality property of the strategy may disappear.

It is therefore tempting to look at strategies that do not employ any distribution assumptions on stock evolution or utility functions. Such a strategy was introduced first by Cover (1991) for the distribution of wealth between given number of stocks (the so-called *universal portfolio* strategy). The algorithm asymptotically outperforms the best stock in the market under some conditions on stationarity. But it is not a bounded risk algorithm, because the wealth may tend to zero for some "bad" samples of stock prices. Some statistical analysis of performance of this strategy for real data has been done in Blædel *et al.* (1999). It appears that the spectacular results of universal portfolios do not necessary materialize for a given historical market.

Chapter 2 considers a generic market model consisting of two assets only: a risky stock and a risk-free bond (or bank account). Following Cover (1991), we also reduce assumptions on the probability distribution of the price evolution. It is assumed that the price of the stock evolves arbitrarily with an interval uncertainty. The dynamics of the bond is exponentially increasing, also with interval uncertainty. Under such mild assumptions, the market is incomplete. Section 2.2 presents a bounded risk strategy such that

(a) the strategy uses only stock price observations and does not require any knowledge about the market appreciation rate, the volatility, or other parameters;

(b) the strategy bounds risk closely to the risk-free investment; and

(c) the strategy gives some additional gain from dealing with risky assets, and this gain is mainly positive.

This strategy bounds risk closely to the risk-free investment, but it also uses the risky asset. The additional gain is positive on average for *any* non-risk-neutral probability measure, under some additional assumptions for probability distributions such that the market is still incomplete, though the strategy itself does not use probability assumptions. Thus, this is a strategy for someone who basically prefers risk-free investments but accepts some bounded risk for the sake of an additional gain.

Section 2.3 presents an empirical strategy that

(a) uses only stock price observations and does not require any knowledge about the market appreciation rate, the volatility or other parameters;

(b) gives some systematic additional gain in comparison with the "buy-and-hold" strategy for a given risky asset; and

(c) has a risk similar to the "buy-and-hold" strategy, i.e., is a bounded risk strategy if the risky asset is taken as a numéraire.

Again, the additional gain is positive on average for *any* non-risk-neutral probability measure, under some additional assumptions for the probability distributions, though the strategy itself does not use probability assumptions.

In other words, a strategy is presented for someone who has made the principal decision to keep the given risky asset, but it admits some dynamic adjusting of the total amount of shares to improve performance. In particular, it means that this investor accepts a risk of losses in case of the stock falls. This model of preferences can be realistic, for example, for a holder of the controlling share of a company.

In Chapter 3, we consider "model-free" strategies of investments in options. It is shown that there exists a correct proportion between "put" and "call" options with the same expiration time on the same underlying security (so-called *long strangle* combination) so that the average gain is almost always positive for a generic Black and Scholes stochastic model. This gain is zero if and only if the market price of risk is zero. A paradox related to the corresponding loss of option's seller is discussed.

In Chapter 4, we consider diffusion and multistock analogue of the model-free winning empirical strategy described in Theorem 2.2 from Chapter 2. This strategy is extended to the case of a continuous diffusion market model, when the trader does transactions at any time when the price variation exceeds a given level. A number of transactions are known and finite, and the stopping time is random (but the expectation of the stopping time is finite). Another continuous-time variant of the strategy has an infinite number of transactions and a fixed and given horizon. In both cases, the strategy is expressed as an explicit function of historical prices. Again, the strategies ensure a positive average gain for any non-risk-neutral probability measure. The strategies bound risk, do not require forecasting the volatility coefficient and appreciation rate estimation, and depend on the historical volatility (Sections 4.4–4.5). Moreover, the strategies ensure a positive average gain for all volatilities and appreciation rates from a wide class that includes random bounded volatilities. As the number of the traded stocks increases, the strategies converge to arbitrage with a given positive gain that is ensured with probability arbitrarily close to 1.

We do not investigate the optimality of model-free strategies in Chapters 2–4. The optimality depends on a model: any "optimal" strategy is optimal only for a given utility function and a given probability measure (i.e., prior probability distributions of parameters). However, it will be shown later in Chapter 9 that the good performance of these strategies is based on some optimal properties: these strategies are optimal for the investment problem with $U(x) \equiv \log x$ under a special but important hypothesis concerning the prior distribution of parameters for the stochastic diffusion market model.

Chapter 5 considers a model where the process $(r(\cdot), a(\cdot), \sigma(\cdot))$, which describes the market parameters is currently observable (case (ii)). We give a survey of the dynamical programming approach and derive Merton-type strategies. Further, a simpler method than dynamic programming is proposed: a nonlinear parabolic Bellman equation is replaced for a finite-dimensional optimization problem and a linear parabolic equation. Under assumptions that an only one scalar parameter of distribution of $r(\cdot), a(\cdot), \sigma(\cdot)$ is known, we derive an optimal strategy explicitly for a very general utility function.

In Chapter 6, we study the portfolio compression problem. By this we mean that admissible strategies may include no more than m different stocks concurrently, where m may be less than the total number n of available stocks. Although this problem has not been treated extensively in the literature, it is of interest to the investor. It is obviously not realistic to include *all* available stocks in the portfolio; the total number of assets in the market is too large. In fact, the number of stocks in the portfolio should be limited by the equity in the account (say, several hundred stocks for a large fund, and less for an individual investor) because of the need to have a large enough position in each stock so that management fees and commissions are only a small proportion of the value of the portfolio. There is no point in having too many stocks in a small portfolio. Even in a large portfolio, it makes sense to limit the number of stocks to those that can be watched closely. On the other hand, there should be a certain minimum number of stocks so that a sufficient degree of diversification can be achieved. For example, the need to limit the portfolio diversification was mentioned by Murray (2000), who estimated that 256 stocks ought to be enough even for some large mutual funds and it should certainly be enough for the typical institutional or individual portfolio.

In Chapter 7, we do not assume to know the distributions of $(r(\cdot), a(\cdot), \sigma(\cdot))$ (case (iii)). Following Cvitanić and Karatzas (1999) and Cvitanić (2000), we consider instead the problem as a *maximin* problem: Find a strategy that maximizes the infimum of $\mathbf{E}U(X(T))$ over all admissible $(r(\cdot), a(\cdot), \sigma(\cdot))$ from a given class; the process $(r(\cdot), a(\cdot), \sigma(\cdot))$ is supposed to be currently observable. For this problem, it is shown that the duality theorem holds under some nonrestrictive conditions. Thus, the *maximin* problem, which as far as we know, cannot be solved directly, is effectively reduced to a *minimax* problem. Moreover, it is proved that the *minimax* problem requires minimization only *over a single scalar parameter* R even for a multistock market, where $R = \int_0^T |\sigma(t)^{-1}(a(t) - r(t)\mathbf{1})|^2 dt$. This interesting effect follows from the result of Chapter 6 for the optimal compression problem. Using this effect, the original *maximin* problem is solved explicitly; the optimal strategy is derived explicitly via solution of a linear parabolic equation.

Cvitanić and Karatzas (1999), and Cvitanić (2000) consider related *minimax* and *maximin* problems of minimizing $\mathbf{E}(\xi_1 - X(T))^+$ subject to $X(T) \geq \xi_2$, where ξ_1 and ξ_2 are given claims, for similar admissible strategies which allow direct observations of appreciation rates (adapted to the driving Brownian motion); however, the maximization over parameters in the dual *minimax* problem was not reduced to the scalar minimization, and the explicit solution was not given for the general case.

In Chapters 8–12 we consider the investment problem for a class of strategies of the form $\pi(t) = f(S(\tau), \eta(\tau), \tau < t)$, where $f(\cdot)$ is a deterministic function and $\eta(\cdot)$ is a directly observable process correlated with the random and nonobservable $a(t)$ (for example, $\eta(t)$ can describe trading volume). The condition that coefficients do not depend on time is dropped in Chapters 8-10 and 12. In this case, the optimal wealth $X(t)$ is not a function of current stock prices. We emphasize that the optimal investment problem has not been solved before under these assumptions (i.e., for this class of strategies, random $a(t)$, and time dependent coefficients). In Chapter 9, we give a solution that does not require the solution of a parabolic equation of high dimension. The solution in Chapters 10 and 11 is based on linear parabolic equations. We assume that the volatility and the appreciation rates are random and that they are not adapted to the driving Wiener process, so the market is incomplete. In fact, we prove the existence of an optimal strategy and find the optimal strategy in the following two cases: (i) the utility function is the log, or (ii) the volatility is nonrandom and $\eta(t) \equiv 0$. (However, some results such as the duality theorem in Chapter 12, are proved for the general case).

In Chapter 8, we show that there exists an optimal claim that is attainable under some (mild) conditions, and we find this optimal claim. A general model that allows us to take into account volume of trade or other observable market data is introduced here. In Chapter 9 we obtain optimal strategies constructively for some special cases, including isoelastic utilities, without solving a parabolic equation. As a consequence, it is shown that the "certainty equivalence principle" for power utilities can be reformulated with the following correction: the "U-optimal filter" of $a(t)$ must be derived (in place of $\mathbf{E}\{a(t)|S(\tau), \tau < t\}$). In general, this filter is neither $\mathbf{E}\{a(t)|S(\tau), \tau < t\}$ nor any other function of $\mathcal{P}_{a(t)}(\cdot\,|S(\tau), \tau < t)$. The only exception is the case of the logarithmic utility function; for the case of Gaussian $a(t)$, which has been studied by Gennotte (1986), $\mathbf{E}\{a(t)|S(\tau), \tau < t\}$ can be found by the Kalman–Bucy filter.

In this terminology, "U" refers to the fact that the filter depends on the utility function in the problem. For the case of power utility, the U-optimal filter is not a function of $\mathcal{P}_{a(t)}(\cdot\,|S(\tau),\tau < t)$ even under the Gaussian assumption. However, this estimate can be written as a conditional expectation of $a(t)$ *under a new measure*, and this measure is obtained explicitly; i.e., a new, convenient way of calculating the correction of the myopic strategy is given. Thus, under a Gaussian assumption on $a(t)$, the certainty equivalent estimate can be obtained by a Kalman–Bucy filter, *but* with some corrections to the parameters.

One might think that a Gaussian prior is the most natural assumption; however, our experiments with historical data show that the optimal strategy based on this assumption (and using the Kalman–Bucy filter) is outperformed by a strategy based on the assumption that $a(t)$ has a distribution with a two-point supporter. It is shown here that the joint distribution of prices and volume contains important information; we improve the performance of a strategy by including volume in the hypothesis on the prior distribution. The simple strategies introduced here appear to be rather interesting and deserve further statistical evaluation for large data sets.

In Chapter 10, we present the solution of the optimal investment problem with additional constraints and utility functions of a very general type, including discontinuous functions. Optimal portfolios are obtained for the class of strategies based on historical prices under some additional restrictions on the prior distributions of market parameters. More precisely, it is assumed here that $\sigma(t)$ is deterministic and $a(t) - r(t)\mathbf{1} = \sum_{i=1}^{L}\theta_i e_i(t)$, where θ_i are random variables and $e_i(t)$ are deterministic and known vectors, $L < +\infty$. This assumption allows us to express optimal investment strategies via the solution of a linear deterministic parabolic backward equation.

In Chapter 11, an optimal portfolio is obtained for the class of strategies based on historical prices under some additional conditions that ensure that the optimal normalized wealth $\tilde{X}(t) \triangleq \exp\{-\int_0^t r(s)ds\}X(t)$ and the optimal strategy are functions of the current vector $\tilde{S}(t) \triangleq \exp\{-\int_0^t r(s)ds\}S(t)$ of normalized stock prices. In particular, these conditions are satisfied if $\sigma(t)$ is deterministic and if σ, $\tilde{a}$ are time independent. The solution is obtained for optimal investment problems with very general utilities and additional constraints. A solution of a goal-achieving problem and a problem of European put option replicating with a possible gap are given among others. Explicit formulas for optimal claims and numerical examples are provided.

In Chapter 12, we do not assume to know the distributions of $(r(\cdot), a(\cdot), \sigma(\cdot))$, and we do not use direct observation of $a(\cdot), \sigma(\cdot)$. As in Chapter 7, we consider the problem as a *maximin* problem: Find a strategy based on historical data that maximizes the infimum of $\mathbf{E}U(\tilde{X}(T))$ over all admissible distributions of $(r(\cdot), a(\cdot), \sigma(\cdot))$, where additional constraints are required to be satisfied with probability 1 for all such admissible $(r(\cdot), a(\cdot), \sigma(\cdot))$. It is shown that the duality theorem holds under some nonrestrictive conditions. Thus, the *maximin* problem which, as far as we know, cannot be solved directly, is effectively reduced to a *minimax* problem. In fact, the original problem is solved for any case in which the optimal investment problem can be solved for strategies based on historical prices with nonobservable market parameters, but with known distributions. Some of these cases are described in Chapters 9-11. For the special case in which the distributions of parameters have support on a finite set, the *minimax* problem is further reduced to a finite-dimensional optimization problem.

Cvitanić and Karatzas (1999) and Cvitanić (2000) consider related *minimax* and *maximin* problems for another class of admissible strategies that allow direct observations of appreciation rates. In Chapter 12 we obtain the duality theorem for the class strategies based on historical prices. Furthermore, we consider more general utility functions and constraints.

In Chapter 13, some aspects of the replication of given claims are discussed. This problem is connected with the solution of the optimal problems proposed in the previous chapters, where the investment problem was decomposed into two different problems: calculation of the optimal claim and calculation of a strategy that replicates the optimal claim. In Chapter 13, some possibilities are considered for replicating the desired claim by purchasing options are considered. In addition, an example is considered of an incomplete market with transactions costs and with nonpredictable volatility, when replication is replaced by rational superreplication.

The results presented in Section 2.2 and in Chapters 4 and 13 were obtained by the author together with A.V. Savkin (Dokuchaev and Savkin (1997), (1998a) and (1998b)). The results presented in Chapters 5 and 6, as well as in Chapters 8 and 9 for the case when $\eta \equiv 0$, were obtained by the author together with U.G. Haussmann (Dokuchaev and Haussmann (2001a), (2001b)). The duality theorem from Chapter 12 was obtained by the author together with K.L. Teo for a slightly less general case when $\eta \equiv 0$ (Dokuchaev and Teo (1998)). The results presented in Chapter 11 (Sections 11.2.1-11.2.3, and 11.3) were obtained by the author together with X.Y. Zhou; furthermore, the proof of the results of Chapter 10 is based on the proofs from Dokuchaev and Zhou (2001), and Dokuchaev and Teo (2000). The results presented in Section 2.3 and in Chapters 3, 7, and 10 were obtained by the author, as well as the results presented in Chapters 8,9, and 12 for the case when $\eta(\cdot) \neq 0$.

I

BACKGROUND

Chapter 1

STOCHASTIC MARKET MODEL

Abstract In this chapter we briefly describe the basic concepts of stochastic market models. Further, we introduce a multystock stochastic continuous-time market model that will be used in the following chapters, and we give some necessary definitions.

1.1. Brief introduction to stochastic market models

Consider a risky asset (stock, bond, foreign currency unit, etc.) with time series prices $S_1, S_2, S_3, \ldots$, for example, daily prices. The premier model of price evolution is such that $S_k = S(t_k)$, where

$$S(t) = S(0)e^{at+\xi(t)}, \tag{1.1}$$

where $\xi(t)$ is a *martingale*, i.e., $\mathbf{E}\{\xi(T)|\xi(\cdot)|_{[0,t]}\} = \xi(t)$ for any t and $T > t$. For the simplest model, $\xi(t)$ is a Gaussian process,

$$\mathbf{E}\{\xi(t+\Delta t)|\xi(\cdot)|_{[0,t]}\} = \xi(t),$$
$$\mathrm{Var}\,[\xi(t+\Delta t) - \xi(t)] \sim \sigma^2 \cdot \Delta t \quad \forall t > 0,\ \Delta t > 0,$$

such that $\xi(t+\Delta) - \xi(t)$ does not depend on $\xi(\cdot)|_{[0,t]}$ for any $t \geq 0$. Here $a \in \mathbf{R}$, $\sigma \in \mathbf{R}$ are parameters.

It is convenient to rewrite equation (1.1) as the following *Itô's equation* :

$$dS(t) = S(t)[a(t)dt + \sigma(t)dw(t)], \tag{1.2}$$

where $w(t)$ is a Brownian motion, $a(t)$ is the appreciation rate, $\sigma(t)$ is the volatility; and a and σ are market parameters.

Let us discuss some basic properties of Itô's equation (1.2). The solution $S(t)$ of this equation has the following properties:

- sample paths maintain continuity;

N. Dokuchaev, *Dynamic Portfolio Strategies: Quantitative Methods and Empirical Rules for Incomplete Information*

- paths are nondifferentiable;
- paths are not absolutely continuous;
- if a, σ are deterministic, then

$$\mathrm{Var}\,\frac{S(t+\Delta t)}{S(t)} = \sigma^2 \cdot \Delta t \quad \forall t > 0,\ \Delta t > 0;$$

- if a, σ are deterministic, then the law of $S(t)$ is log-normal (i.e., its logarithm follows a normal law);
- if a, σ are deterministic, then the relative-increments $[S(t)-S(\tau)]/S(\tau)$ are independent of the σ-algebra $\sigma(S(\cdot)|_{[0,\tau]})$, $0 \le \tau < t$.
- if a, σ are deterministic and constant, then the relative increments law of $[S(t) - S(\tau)]/S(\tau)$ is identical to the law of $[S(t-\tau) - S(0)]/S(0)$, $0 \le \tau < t$.

For a multistock market model, $S(t) = \{S_i(t)\}$, $a = \{a_i\}$, $w = \{w_i\}$ are vectors, and $\sigma = \{\sigma_{ij}\}$ is a matrix.

We assume that there is a riskless asset (bond) with price

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right),$$

where $r(t)$ is a process of risk-free interest rates.

The portfolio is a process $(\gamma(\cdot), \beta(\cdot))$ with values in $\mathbf{R}^n \times \mathbf{R}$, $\gamma(\cdot) = (\gamma_1(t), \dots, \gamma_n(t))$, where $\gamma_i(t)$ is the quantity of the ith stock; and $\beta(t)$ is the quantity of the bond.

A portfolio $(\gamma(\cdot), \beta(\cdot))$ is said to be *self-financing* if there is no income from or outflow to external sources. In that case,

$$dX(t) = \sum_{i=1}^{n} \gamma_i(t)dS_i(t) + \beta(t)dB(t).$$

It can be seen that

$$\beta(t) = \frac{X(t) - \sum_{i=1}^{n} \gamma_i(t)S_i(t)}{B(t)}.$$

Let

$$\pi_0(t) \triangleq \beta(t)B(t),$$
$$\pi_i(t) \triangleq \gamma_i(t)S_i(t), \quad \pi(t) = (\pi_1(t), \dots, \pi_n(t))^\top.$$

By the definitions, the process $\pi_0(t)$ is the investment in the bond, and $\pi_i(t)$ is the investment in the ith stock. It can be seen that the vector π alone suffices to specify the self-financing portfolio. We shall use the term *self-financing strategy* for a vector process $\pi(\cdot) = (\pi_1(t), \dots, \pi_n(t))$, where the pair $(\pi_0(t), \pi(t))$ describes the self-financing portfolio at time t:

$$X(t) = \sum_{i=1}^{n} \pi_i(t) + \pi_0(t).$$

There are the following key problems:

- *Optimal investment problem: To find a strategy of buying and selling stocks*
- *Pricing problem: To find a "fair" price for derivatives (i.e. options, futures, etc.)*

There is an auxiliary problem:

- *To estimate the parameters* $(a(t), \sigma(t))$ *from market statistics.*

In fact, the estimation of $\sigma(\cdot)$ is easy, since $\sigma(t)\sigma(t)^\top$ is an explicit function of $S(\cdot)$ (see (1.11) below). The estimation of $a(\cdot)$ is much more difficult.

In this book, we shall study the optimal investment problem only, and we are leaving the very important pricing problem out of consideration.

If a and σ are constant and deterministic, then the process $S(t)$ is log-normal (i.e., the process $\{\log S_i(t)\}$ is Gaussian). Empirical research has shown that the real distribution of stock prices is not exactly log-normal. The imperfection of the log-normal hypothesis on the prior distribution of stock prices can be taken into account by assuming that a and σ are random processes. This more sophisticated model is much more challenging: for example, the market is *incomplete* (i.e., an arbitrary random claim cannot be replicated by an adapted self-financing strategy).

Generic investment problem

We can state a generic *optimal investment problem:*

$$\begin{aligned}&\text{Maximize} \quad \mathbf{E}U(X(T))\\ &\text{over self-financing strategies} \quad \pi(\cdot).\end{aligned}$$

Here T is the terminal time, and $U(\cdot)$ is a given *utility function* that describes risk preferences. The most common utilities are log and power, i.e., $U(x) = \log x$ and $U(x) = \delta x^\delta$, $\delta < 1$.

There are many modifications of the generic optimal investment problem:

- optimal investment-consumption problems
- optimal hedging of nonreplicable claims

- problem with constraints
- $T = +\infty$
- etc.

Some examples

For simplicity, let $r(t) \equiv r$ be constant.

EXAMPLE 1.1 Consider the trivial "buy-and-hold" strategy for the stock, such that $\beta(t) \equiv 0$ and $\gamma(t) \equiv X(0)S(0)^{-1}$. Then $X(t) \equiv S(t)$. Let $\theta = \min\{t : S(t) = Ke^{rt}\}$, where $K > 0$ is a given number. Then $X(\theta) = Ke^{rt}$ a.s., where K may be large enough, and $\theta < +\infty$ a.s. But $\mathbf{E}\theta = +\infty$ for every $K \neq S_0$; hence this strategy cannot ensure the gain K in practice.

EXAMPLE 1.2 Consider the trivial "keep-only-bonds" strategy for the diffusion market model such that the portfolio contains only the bonds, $\gamma(t) \equiv 0$. In that case the corresponding total wealth is $X(t) \equiv \beta(0)B(t) = e^{rt}X(0)$.

Merton's strategy

We describe now strategies that are optimal for the generic model with $U(x) = \log x$ or $U(x) = \delta x^{\delta}$:

$$\pi(t)^{\top} = \nu(a(t) - r(t)\mathbf{1})^{\top}[Q(t)X(t) + f(t)],$$

where $\nu = \nu(\delta)$ is a coefficient, $Q(t) \triangleq (\sigma(t)\sigma(t)^{\top})^{-1}$, $r(t)$ is the interest rate for a risk-free investment, and $\mathbf{1}^{\top} \triangleq (1, 1, \dots, 1)^{\top}$.

The term $f(t)$ describes the correlation between (σ, a) and $w(\cdot)$; if they are independent, then $f \equiv 0$.

Note that these strategies require direct observation of (σ, a). But, in practice, the parameters $a(\cdot), \sigma(\cdot)$ need to be estimated from historical market data. Thus, the investment problem can be reformulated as follows:

$$\text{Maximize} \quad \mathbf{E}U(X(T))$$

over strategies that use historical data only.

This problem is studied in Chapters 8–11 below.

1.2. Options market

Rather than trade directly in stocks, investors can purchase securities representing a claim – an option – on a particularly stock. This option gives the holder the right to receive or deliver shares of stock under specified conditions. The option need not be exercised: an investor can simply trade these derivative securities. Gains or losses will depend on the difference between the purchase price and the sale price.

A *call* is an option to buy a stated amount of a particular stock at a specified price. A *put* is an option to sell a stated amount of a particular stock at a specified price. An *American* put (call) option gives the owner the right to purchase (sell) stated amount of a particular stock at a specified price at any time before the specified expiration date. A *European* put (call) option gives the owner the right to purchase (sell) stated amount of a particular stock at a specified price only at the specified expiration date.

In the case of the standard call option of European type, the option writer (seller) obligation is $F(S(T))$, where $F(x) = (x-K)^+ = \max(0, x-K)$. In the case of the standard put option of European type, the option writer (seller) obligation is $F(S(T))$, where $F(x) = (K - x)^+ = \max(0, x - K)$. Here, K is the option striking price, T is the expiration time, $S(T)$ is the underlying stock price at the time T.

Profit and loss diagrams

Each type of option has its own *profit and loss diagrams*, which offer a convenient way to see what happens with option strategies as the value of the underlying security. The vertical axis of the diagrams reflects profits or losses X on option expiration day resulting from a particular strategy, where the horizontal axis reflects the stock prices S. Figures 1.1 and 1.2 present profit/loss diagrams for generic European put and call options, i.e., they show the wealth of European call and put option holder as a function of the stock price at the terminal time.

We shall use profit/loss diagrams to demonstrate claims for different optimal strategies.

Figure 1.1. Profit/loss diagram for long call: c is the price of the call option, S is the stock price at the terminal time, and K is the strike price.

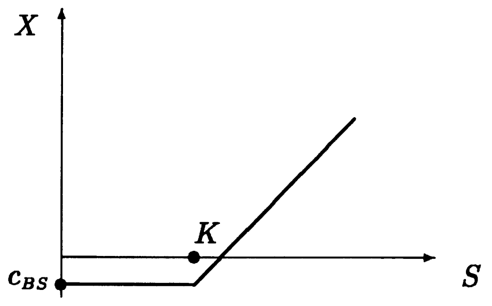
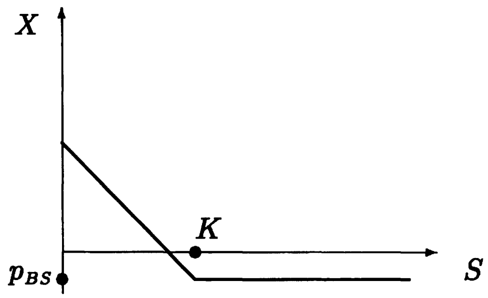


Figure 1.2. Profit/loss diagram for a long put: p is the price of the put option, S is the stock price at the terminal time, and K is the strike price.



Black and Scholes model

Consider the generic model of a financial market consisting of a risk-free asset (bond, or bank account) with price $B(t)$ and a risky asset (stock) with price $S(t)$. We are given a standard probability space with a probability measure $\mathbf{P}$ and a standard Wiener process (Brownian motion) $w(t)$. The bond and stock prices evolve as

$$B(t) = e^{rt}B_0, \tag{1.3}$$

$$dS(t) = aS(t)dt + \sigma S(t)dw(t). \tag{1.4}$$

Here, $r \geq 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, and $a \in \mathbf{R}$ is the appreciation rate. We assume that $t \in [0, T]$, where $T > 0$ is a given terminal time. Equation (1.4) is Itô's equation.

Further, we assume that $\sigma, r, B_0 = B(0) > 0$ and $S_0 \triangleq S(0) > 0$ are given, but a is unknown.

In the approach of Black and Scholes, the rational (fair) price of an option with the option writer obligation ξ is the initial wealth that may be raised to ξ by some investment transactions (see Black and Scholes (1973)).

DEFINITION 1.1 *Let Π be the set of all values of the initial wealth X_0 such that there exists an admissible strategy such that*

$$X(T) \geq \xi \quad a.s.$$

Then the fair (rational) price $\hat{C}$ for the option with the claim ξ in this class of admissible strategies is defined as

$$\hat{C} = \inf_{X_0 \in \Pi} X_0.$$

THEOREM 1.1 *The fair price of an option does not depend on a, and it is* $\hat{C} = \mathbf{E}\{\xi \mid a = r\}$.

Let

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

$$d \triangleq \frac{\log(S_0/K) + T\left(r + \sigma^2/2\right)}{\sigma\sqrt{T}}, \quad d^- = d - \sigma\sqrt{T}. \tag{1.5}$$

The premier result of the pricing theory in mathematical finance is the following *Black–Scholes formula*:

$$c_{BS}(S_0, K, r, T, \sigma) = S_0\Phi(d) - Ke^{-rT}\Phi(d^-), \tag{1.6}$$

$$p_{BS}(S_0, K, r, T, \sigma) = c_{BS}(S_0, K, r, T, \sigma) - S_0 + Ke^{-rT}, \tag{1.7}$$

where $p_{BS}(S_0, K, r, T, \sigma)$ denote the "fair" price for "put" option, and $c_{BS}(S_0, K, r, T, \sigma)$ denote the "fair" price for "call" option (Black-Scholes price). Here, $S_0 = S(0)$ is the initial stock price, K is the strike price, r is the risk-free interest rate, σ is the volatility, and T is the expiration time (see, e.g., Strong (1994), Duffie (1988)).

1.3. Continuous-time multistock stochastic market model

In this section, we describe the continuous-time diffusion stochastic market model, which will be the main model for this book. Consider a diffusion model of a market consisting of a risk-free bond or bank account with the price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \dots, n$, where $n < +\infty$ is given. The prices of the stocks evolve according to

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right), \quad t > 0, \tag{1.8}$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given nonrandom constant. The price of the bond evolves according to

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right), \tag{1.9}$$

where $B(0)$ is a given constant that we take to be 1 without loss of generality and $r(t)$ is a random process of risk-free interest rate.

We are given a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the set of all events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure. Introduce the vector processes ($^\top$ denoted transpose)

$$\begin{aligned} w(t) &= (w_1(t), \ldots, w_n(t))^\top, \\ a(t) &= (a_1(t), \ldots, a_n(t))^\top, \\ S(t) &= (S_1(t), \ldots, S_n(t))^\top \end{aligned}$$

and the matrix process $\sigma(t) = \{\sigma_{ij}(t)\}_{i,j=1}^n$.

We assume that $\{w(t)\}_{0 \le t \le T}$ is a standard Wiener process, and that $a(t)$, $r(t)$, and $\sigma(t)$ are measurable random processes, independent of future increments of w, such that

$$\sigma(t)\sigma(t)^\top \geq c_1 I_n,$$

where $c_1 > 0$ is a constant and I_n is the identity matrix in $\mathbf{R}^{n \times n}$. Under these assumptions, the solution of (1.8) is well defined, if a and σ are uniformly bounded.

Set $\mathbf{1} \triangleq (1, \ldots, 1)^\top \in \mathbf{R}^n$,

$$\begin{aligned} V(t) &\triangleq \sigma(t)\sigma(t)^\top, \\ Q(t) &\triangleq V(t)^{-1}, \\ \tilde{a}(t) &\triangleq a(t) - r(t)\mathbf{1}, \\ \theta(t) &\triangleq \sigma(t)^{-1}\tilde{a}(t). \end{aligned} \tag{1.10}$$

Let $\mathbf{S}(t) \triangleq \operatorname{diag}(S_1(t), \ldots, S_n(t))$ be the diagonal matrix with the corresponding diagonal elements.

Let $\mu(t) \triangleq (r(t), \tilde{a}(t), \sigma(t))$. Let $\{\mathcal{F}_t^a\}_{0 \le t \le T}$ be the filtration generated by the process $(S(t), \mu(t))$ completed with the null sets of $\mathcal{F}$. Let $\{\mathcal{F}_t^S\}_{0 \le t \le T}$ be the filtration generated by the process $(S(t), r(t))$ completed with the null sets of $\mathcal{F}$.

REMARK 1.1 *The volatility coefficients can be effectively estimated from $S_i(t)$. In fact, if $V(t) = \{V_{ij}(t)\}_{i,j=1}^n$, then direct calculations show*

$$\begin{aligned} \int_0^t V_{ii}(\tau)d\tau &= 2\int_0^t \frac{dS_i(\tau)}{S_i(\tau)} - 2\log\frac{S_i(t)}{S_i(0)}, \\ \int_0^t V_{ij}(\tau)d\tau &= \int_0^t \frac{d(S_i(\tau)S_j(\tau))}{S_i(\tau)S_j(\tau)} \\ &\quad - \log\frac{S_i(t)S_j(t)}{S_i(0)S_j(0)} - \frac{1}{2}\int_0^t V_{ii}(\tau)d\tau - \frac{1}{2}\int_0^t V_{jj}(\tau)d\tau. \end{aligned} \tag{1.11}$$

By (1.8),

$$dw(t) = \sigma(t)^{-1}\mathbf{S}(t)^{-1}\left[dS(t) - \mathbf{S}(t)a(t)dt\right]. \tag{1.12}$$

By Remark 1.1, it follows that $\{\mathcal{F}_t^S\}$ coincides with the filtration generated by the processes $(S(t), r(t), V(t))$. By (1.12), it follows that $\{\mathcal{F}_t^a\}$ coincides with the filtration generated by the processes $(w(t), \mu(t))$.

Set

$$\begin{aligned} p(t) &\triangleq \exp\left(-\textstyle\int_0^t r(s)ds\right) = B(t)^{-1}, \\ \tilde{S}(t) &\triangleq p(t)S(t). \end{aligned} \tag{1.13}$$

It is easy to see that $\mathcal{F}_t^a$ coincides with the filtration generated by the processes $(\tilde{S}(t), \mu(t))$.

Let

$$w_*(t) \triangleq w(t) + \int_0^t \theta(s)ds.$$

Let

$$\mathcal{Z}(t) \triangleq \exp\left(\int_0^t \theta(s)^\top dw(s) + \frac{1}{2}\int_0^t |\theta(s)|^2 ds\right). \tag{1.14}$$

Clearly,

$$\mathcal{Z}(t) = \exp\left(\int_0^t \theta(s)^\top dw_*(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds\right). \tag{1.15}$$

Our standing assumptions imply that $\mathbf{E}\mathcal{Z}(T)^{-1} = 1$. Define the (equivalent martingale) probability measure $\mathbf{P}_*$ by

$$\frac{d\mathbf{P}_*}{d\mathbf{P}} = \mathcal{Z}(T)^{-1}.$$

Let $\mathbf{E}_*$ be the corresponding expectation. Girsanov's Theorem implies that w_* is a standard Wiener process under $\mathbf{P}_*$. Then $(w_*(t), \mu(t))$ also generate $\{\mathcal{F}_t\}$ and $\mathbf{P}$, $\mathbf{P}_*$ have the same null sets.

It follows from (1.14) that

$$d\mathcal{Z}(t) = \mathcal{Z}(t)\theta(t)^\top dw_*(t) = \mathcal{Z}(t)\tilde{a}(t)^\top Q(t)\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \tag{1.16}$$

where $\tilde{\mathbf{S}}(t) \triangleq \operatorname{diag}(\tilde{S}_1(t), \dots, \tilde{S}_n(t))$. Note that $\mathbf{E}_*\mathcal{Z}(T) = 1$.

Portfolio and strategies

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$, $X(0) = X_0$. We assume that

$$X(t) = \pi_0(t) + \sum_{i=1}^n \pi_i(t), \tag{1.17}$$

where the pair $(\pi_0(t), \pi(t))$ describes the portfolio at time t. The process $\pi_0(t)$ is the investment in the bond, and $\pi_i(t)$ is the investment in the ith stock, $\pi(t) = (\pi_1(t), \dots, \pi_n(t))^\top$, $t \geq 0$.

The portfolio is said to be self-financing if

$$dX(t) = \pi(t)^\top \mathbf{S}(t)^{-1} dS(t) + \pi_0(t) B(t)^{-1} dB(t). \tag{1.18}$$

It follows that for such portfolios,

$$dX(t) = r(t)X(t)\,dt + \pi(t)^\top \left(\tilde{a}(t)\,dt + \sigma(t)\,dw(t)\right), \tag{1.19}$$

$$\pi_0(t) = X(t) - \sum_{i=1}^{n} \pi_i(t),$$

so π alone suffices to specify the portfolio; it is called a self-financing strategy.

DEFINITION 1.2 *The process* $\tilde{X}(t) \triangleq p(t)X(t)$ *is called the normalized wealth.*

It satisfies

$$\begin{aligned} \tilde{X}(t) &= X(0) + \int_0^t p(s)\pi(s)^\top \sigma(s)\,dw_*(s) \\ &= X(0) + \int_0^t p(s)\pi(s)^\top \tilde{\mathbf{S}}(s)^{-1} d\tilde{S}(s). \end{aligned} \tag{1.20}$$

Special classes of admissible strategies are described below (see Definitions 5.1 and 8.2).

The following definition is standard.

DEFINITION 1.3 *Let* ξ *be a given random variable. An admissible strategy* $\pi(\cdot)$ *is said to replicate the claim* ξ *if* $X(T, \pi(\cdot)) = \xi$ *a.s.*

Some notations

Throughout the book, a vector (strict) inequality will mean component-wise (strict) inequalities; $\chi\{\cdot\}$ denotes the indicator function.

We shall denote by $B([0,T]; E)$ the set of bounded measurable functions $f(t) : [0,T] \to E$ for an Euclidean space E.

We shall use notations

$$\phi^+(x) \triangleq \max(0, \phi(x)),$$

$$\phi^-(x) \triangleq \max(0, -\phi(x)),$$

$$\mathbf{R}^n_+ \triangleq \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_i \geq 0, i = 1, 2, \ldots, n\},$$

$$\overset{\circ}{\mathbf{R}}{}^n_+ \triangleq \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : x_i > 0, i = 1, 2, \ldots, n\}.$$

II

MODEL-FREE EMPIRICAL STRATEGIES AND THEIR EVALUATION

Chapter 2

TWO EMPIRICAL MODEL-FREE "WINNING" STRATEGIES AND THEIR STATISTICAL EVALUATION

Abstract In this chapter, we consider a generic market model that consists of two assets only: a risky stock and a locally risk-free bond (or bank account). We reduce assumptions on the probability distribution of the price evolution and assume that the price of the stock evolves arbitrarily with interval uncertainty. The dynamics of the bond is exponentially increasing along with interval uncertainty. Under such mild assumptions, the market is incomplete. We further assume that only historical prices are available. Thus, admissible strategies for this model are similar to strategies from "technical analysis" and they are almost model free. We present two original empirical strategies that bound risk closely to a risk-free numéraire and risky numéraire respectively. The important feature of the strategies is that they guarantee a positive average gain for any non-risk-neutral probability measure. Some statistical tests of profitability of these strategies as applied to historical data are provided.

2.1. A generic discrete-time market model

We introduce a simplest model of a market, consisting of the risk-free bond or bank account with price B_k and the risky stock with tprice S_k, $k = 0, 1, 2, \ldots$. The initial prices $S_0 > 0$ and $B_0 > 0$ are given nonrandom variables.

Set

$$\rho_k \triangleq \frac{B_k}{B_{k-1}}, \quad \xi_k \triangleq \frac{S_k}{\rho_k S_{k-1}} - 1.$$

In other words,

$$S_k = \rho_k S_{k-1}(1 + \xi_k), \quad k = 1, 2, \ldots.$$

We assume that

$$|\xi_k| \le 1, \quad \rho_k \ge 1 \quad \forall k. \tag{2.1}$$

Note that these conditions are not restrictive, since the usual change in real stock market prices is about 1% and no more than 5% per day; in other words,

$|\xi_k|$ is about 0.01–0.05 in the case of everyday transactions; in this case,

$$\rho_k = 1 + \text{interest rate}/365.$$

Let $X_0 > 0$ be the initial wealth at time $k = 0$.

Let X_k be the wealth at time $k > 0$. We assume that the wealth X_k at time $k \geq 0$ is

$$X_k = \beta_k B_k + \gamma_k S_k, \tag{2.2}$$

where β_k is the quantity of the bond portfolio, and γ_k is the quantity of the stock portfolio. The pair (β_k, γ_k) describes the state of the bond–stocks securities portfolio at time k. We call sequences of these pairs strategies.

We consider the problem of trading or choosing a strategy in a class of strategies that does not use any information about the probability distribution of the market dynamics or about the future variables of S_k. Some constraints will be imposed on current operations in the market, or in other words, on strategies.

DEFINITION 2.1 *A sequence $\{(\beta_k, \gamma_k)\}$ is said to be an admissible strategy if there exist measurable functions $F_k : \mathbf{R}^{2k+2} \to \mathbf{R}^2$ such that*

$$(\beta_k, \gamma_k)^\top = F_k(S_0, B_0, S_1, B_1, \dots, S_k, B_k).$$

(In other words, β_k and γ_k do not depend on the "future", or on S_{k+m}, B_{k+m} for $m > 0$).

The main constraint in choosing a strategy is the so-called condition of self-financing.

DEFINITION 2.2 *A strategy $\{(\beta_k, \gamma_k)\}$ is said to be self-financing, if*

$$X_{k+1} - X_k = \beta_k \left(B_{k+1} - B_k\right) + \gamma_k \left(S_{k+1} - S_k\right). \tag{2.3}$$

For the trivial, risk-free, "keep-only-bonds" strategy, the portfolio contains only the bonds, $\gamma_k \equiv 0$, and the corresponding total wealth is $X_k \equiv \beta_0 B_k \equiv \prod_{m=1}^k \rho_m X_0$. Strategies that bound risk are said to be bounded risk strategies. Some loss is possible for a strategy that deals with risky assets. It is natural to estimate the loss and compare it with the "keep-only-bonds" strategy.

DEFINITION 2.3 *The process $\tilde{X}_k \triangleq \left(X_0 \prod_{m=1}^k \rho_m\right)^{-1} X_k$ is called the normalized wealth.*

REMARK 2.1 *This definition is slightly different from Definition 1.2. To reduce the difference, it suffices to assume that $X_0 = 1$; thiscan be done without loss of generality.*

Notice that $\tilde{X}_0 = 1$.

DEFINITION 2.4 *Let $\{h_k\}$ be a sequence such that $0 < h_k \leq 1$, $k = 1, 2, \ldots, n$. An admissible strategy (β_k, γ_k) is said to be a bounded risk strategy with the bounds $\{h_k\}$, if $\tilde{X}_k \geq h_k$ $(\forall k = 1, 2, \ldots)$.*

Set

$$\tilde{S}_k \triangleq \left(\prod_{m=1}^{k} \rho_m\right)^{-1} S_k, \quad k > 1,$$
$$\tilde{S}_0 \triangleq S_0.$$

We have that

$$\tilde{S}_k = \tilde{S}_{k-1}(1 + \xi_k).$$

PROPOSITION 2.1 *Let $\{(\tilde{X}_k, \gamma_k)\}_{k=1}^{n}$ be a sequence such that*

$$\tilde{X}_{k+1} - \tilde{X}_k = \frac{\gamma_k}{X_0}(\tilde{S}_{k+1} - \tilde{S}_k), \quad k = 0, 1, \ldots$$

Then $X_k \triangleq X_0 \prod_{m=1}^{k} \rho_m \tilde{X}_k$ is the wealth corresponding to the strategy (β_k, γ_k), where $\beta_k = (X_k - \gamma_k X_k)B_k^{-1}$, which is self-financing.

2.2. A bounded risk strategy

We present below a strategy that bounds risk closely to a risk-free investment and guarantees at the same time a positive average gain for any non-risk-neutral probability measure. The strategy uses only stock price observations and does not require any knowledge about the market appreciation rate, the volatility, or other parameters; it bounds risk closely to the risk-free investment, and it gives some additional gain from trading of the risky asset, and this gain is mostly positive. In fact, the additional gain is positive on average for *any* non-risk-neutral probability measure, under some additional assumptions about probability distributions such that the market is still incomplete, though the strategy itself does not use probability assumptions. Thus, we present a strategy for someone who basically prefers risk-free investments but accepts some bounded risk for the sake of an additional gain.

2.2.1 The strategy

Set

$$c_k \triangleq 1 - \xi_k^2, \quad v_0 \triangleq 1, \quad v_k \triangleq \prod_{m=1}^{k} c_m. \tag{2.4}$$

Then $0 < c_k \leq 1$ and $0 < v_k \leq 1$.

THEOREM 2.1 *Let*

$$\begin{cases} \tilde{X}_k \triangleq \frac{1}{2}\left(\frac{\tilde{S}_k}{S_0} + \frac{v_k S_0}{\tilde{S}_k}\right), \quad X_k \triangleq X_0 \tilde{X}_k \prod_{m=1}^k \rho_m, \\ \gamma_k \triangleq \frac{X_0}{2}\left(\frac{1}{S_0} - \frac{v_k S_0}{\tilde{S}_k^2}\right), \\ \beta_k \triangleq \frac{X_k - \gamma_k S_k}{B_k}, \quad k = 0, 1, 2, \dots, n. \end{cases} \tag{2.5}$$

Then the pair (β_k, γ_k) is an admissible and self-financing strategy with the corresponding wealth X_k and the normalized wealth $\tilde{X}_k$, and

$$\tilde{X}_k = \frac{1}{2}\left(\prod_{m=1}^k (1+\xi_m) + \prod_{m=1}^k (1-\xi_m)\right), \tag{2.6}$$

$$\tilde{X}_k \ge \sqrt{v_k}, \tag{2.7}$$

for all admissible sequences $S_1, \dots, S_k$, $k = 1, 2, \dots$.

Notice that the strategy (2.5) at time k uses only $\{B_m, S_m, \ m \le k\}$.

COROLLARY 2.1 *Assume that $|\xi_k| \le \varepsilon$, where $\varepsilon \in (0,1)$ is a given number. Then the pair (β_k, γ_k) is a bounded risk strategy with the bounds $h_k = (1 - \varepsilon^2)^{k/2}$.*

Let $n \ge 1$ be a given integer. Denote by n_+ the random number of positive ξ_k in the set $\{\xi_k\}_{k=1}^n$.

THEOREM 2.2 *Let $h > 0$ be a constant. Assume that $n \to \infty$ and $|\xi_k| = n^{-1}h$. Then*

$$\tilde{X}_n \to \frac{1}{2}(e^{h(2\nu-1)} + e^{h(1-2\nu)}),$$

where $\nu \triangleq n_+/n$.

Notice that $(e^y + e^{-y})/2 > 1$ ($\forall y \in \mathbf{R}, \ y \ne 0$).

COROLLARY 2.2 *The strategy (2.5) ensures a positive gain for large n and $\varepsilon = hn^{-1}$ in the case of a "good" value of ν ($\nu \ne 1/2$).*

For a real market, changes of stock prices are usually no more than 1%–5% per day, hence $|\xi_k|$ is about 0.01–0.05 for everyday transactions.

Consider examples of a possible gain and the maximum loss in comparison with the risk-free "keep-only-bonds" strategy. Let $n = 100$, and let $\rho_k \equiv \rho$ be a constant.

EXAMPLE 2.1 Let $|\xi_k| \le 0.05$; then $v_k = 0.9975^k$ and $X_{100} \ge 0.9975^{50}\rho^{100}X_0 = 0.8824\rho^{100}X_0$ for all admissible $S_1, \dots, S_{100}$.

If $\xi_k = \pm 0.05$ and either $\nu = 0.6$ or $\nu = 0.4$, then $X_{100} = 1.3624\rho^{100}X_0$. In other words, if $\nu = 0.6$, then there are 60% positive ξ_k and 40% negative ξ_k.

If $\xi_k = \pm 0.05$ and either $\nu = 0.65$ or $\nu = 0.35$, then $X_{100} = 2.0780\rho^{100}X_0$.

EXAMPLE 2.2 If $|\xi_k| \leq 0.02$, then $v_k = 0.9996^k$ and $X_{100} \geq 0.9996^{50}\rho^{100}X_0 = 0.9802\rho^{100}X_0$

If $\xi_k = \pm 0.02$ and either $\nu = 0.6$ or $\nu = 0.4$, then $X_{100} = 1.0597\rho^{100}X_0$.

If $\xi_k = \pm 0.02$ and either $\nu = 0.65$ or $\nu = 0.35$, then $X_{100} = 1.1620\rho^{100}$.

2.2.2 Estimates of transaction costs

The problem of transactions costs for stochastic market models has been widely studied (see, e.g., Black and Scholes (1972), Edirisinghe *et al.* (1993), Jouini and Kallal (1995)). We show below that the transaction costs are not crucially large for our strategy. Consider the strategy defined in Theorem 2.1. In this section, we assume for the sake of simplicity that $\rho_k \equiv 1$ and that there exists a constant $\varepsilon > 0$ such that $|\xi_k| \leq \varepsilon$ for all k. We assume that transaction costs are

$$C^{(n)} = \mu \sum_{k=0}^{n-1} S_{k+1} \left|\gamma_{k+1} - \gamma_k\right|, \tag{2.8}$$

where $\mu > 0$ is a given constant that presents the brokerage fee percentage.

Set

$$\bar{S}_k \triangleq S_0 \prod_{m=1}^{k} (1 - \xi_m).$$

By (2.5),

$$\gamma_k = \frac{X_0}{2S_0}\left(1 - \frac{\bar{S}_k}{S_k}\right), \quad k = 0, 1, 2, \ldots, n,$$

and

$$\begin{aligned} S_{k+1}(\gamma_{k+1} - \gamma_k) &= \tfrac{S_{k+1}X_0}{2S_0}\tfrac{\bar{S}_k}{S_k}\left(\tfrac{1-\xi_{k+1}}{1+\xi_{k+1}} - 1\right) \\ &= -\tfrac{X_0}{S_0}\bar{S}_k\xi_{k+1}, \quad k \leq n-1. \end{aligned}$$

Hence

$$C^{(n)} \leq \mu \frac{X_0 \varepsilon}{S_0} \sum_{k=0}^{n-1} \bar{S}_k.$$

Let n_+, n_- and ν be as defined in Section 2.2.1: n_+ is the random number of positive ξ_k in the set $\{\xi_k\}_{k=1}^n$, $n_- \triangleq n - n_+$. It can easily be seen that $\sum_{i=k}^n \bar{S}_k$ takes the maximum value if $\xi_k = \varepsilon$ $(\forall k \leq n_-)$, $\xi_k = -\varepsilon$ $(\forall k > n_-)$. In this case, we have that

$$\sum_{k=0}^{n_-} \bar{S}_k = \sum_{k=0}^{n_-} (1+\varepsilon)^k, \qquad \sum_{k=n_-+1}^{n} \bar{S}_k = (1+\varepsilon)^{n_-} \sum_{k=n_-+1}^{n} (1-\varepsilon)^k.$$

Hence

$$\begin{aligned} C^{(n)} &\le \mu \tfrac{X_0\varepsilon}{S_0} \textstyle\sum_{k=0}^{n-1} \bar{S}_k \\ &\le \mu X_0 \varepsilon \left[\textstyle\sum_{k=0}^{n_-} (1+\varepsilon)^k + (1+\varepsilon)^{n_-} \sum_{k=1}^{n_+} (1-\varepsilon)^k\right] \\ &= \mu X_0 \varepsilon \left[\tfrac{(1+\varepsilon)^{n_-+1}-1}{(1+\varepsilon)-1} + (1+\varepsilon)^{n_-}(1-\varepsilon)\tfrac{1-(1-\varepsilon)^{n_+}}{1-(1-\varepsilon)}\right]. \end{aligned}$$

Hence we obtain the estimate of transaction costs

$$C^{(n)} \le \mu X_0 \left[(1+\varepsilon)^{n_-+1} - 1 + (1+\varepsilon)^{n_-}(1-\varepsilon)(1-(1-\varepsilon)^{n_+})\right].$$

Let n be large, or $n \to +\infty$. Assume that $\varepsilon \le hn^{-1}$, where $h > 0$ is a given constant. With this assumption, we have that

$$C^{(n)} \le \mu X_0 \Bigg[\left(1+\tfrac{h}{n}\right)^{n_-+1} -1 + \left(1+\tfrac{h}{n}\right)^{n_-}\left(1-\tfrac{h}{n}\right)\left[1-\left(1-\tfrac{h}{n}\right)^{n_+}\right]\Bigg].$$

Hence

$$\limsup_{n\to\infty} C^{(n)} \le \mu X_0 \left[\left(e^{h(1-\nu)}-1\right) + e^{h(1-\nu)}\left(1-e^{-h\nu}\right)\right]$$

and transaction costs are limited over $n \to \infty$.

Let $\varepsilon = hn^{-1}$. From Theorem 2.2, we have that

$$\tilde{X}_n \to \frac{X_0}{2}\left(e^{h(2\nu-1)} + e^{h(1-2\nu)}\right) \quad \text{as} \quad n \to +\infty.$$

The value of μ is about 0.01 in a real market; hence transaction costs are reasonably small, in comparison with a possible gain for "good" values of ν ($\nu \ne 1/2$).

2.2.3 Average performance under probability assumptions

We suppose that there is a probability space such that ξ_k and ρ_k are random variables, $k = 1, \ldots, n$, where $n > 0$ is given. We repeat that

$$B_k = \rho_k B_{k-1}, \quad S_k = \rho_k S_{k-1}(1+\xi_k), \quad k = 1, 2, \ldots, n. \tag{2.9}$$

Consider a probability measure $\mathbf{P}_\mu$ on the set of sequences $\{(\xi_k, \rho_k)\}_{k=1}^n$. Let $\mathbf{E}_\mu$ denote the corresponding expectation and $\mathbf{E}_\mu\{\cdot\,|\,\cdot\}$ denote a conditional expectation.

DEFINITION 2.5 *A measure $\mathbf{P}_\mu$ is said to be risk-neutral if*

$$\mathbf{E}_\mu\left\{\tilde{S}_k | \tilde{S}_{k-1}, \tilde{S}_{k-2}, \ldots, \tilde{S}_0\right\} = \tilde{S}_{k-1} \quad (\forall k = 1, \ldots, n).$$

REMARK 2.2 *By (2.9),*

$$\mathbf{E}_\mu \left\{ \tilde{S}_k | \tilde{S}_{k-1}, \tilde{S}_{k-2}, \dots, \tilde{S}_0 \right\} = \tilde{S}_k \left(1 + \mathbf{E}_\mu \{\xi_k | \xi_{k-1}, \dots, \xi_1\}\right).$$

Hence, if ξ_k does not depend on $\xi_{k-1}, \dots, \xi_1$, then the measure $\mathbf{P}_\mu$ is risk-neutral if and only if $\mathbf{E}_\mu \xi_k = 0$ ($\forall k = 1, \dots, n$).

Let c_k and v_k be such as defined by (2.4).

Consider the classic stochastic Cox–Ross–Rubinstein model (see Cox *et al.* (1979)). For this model, ρ_k are nonrandom, and ξ_k are independent random variables that have equal distribution and can have only two values, δ_1 and δ_2, where δ_1, δ_2 are given numbers, $-1 < \delta_1 < 0 < \delta_2 < 1$. It is easy to see that $v_k \geq \left(1 - \max(\delta_1^2, \delta_2^2)\right)^k$, $k \geq 0$. Denote by $\mathbf{P}_p$ the corresponding probability measure, which is uniquely defined by the sequence $\{\rho_k\}$ and a real number $p \in (0, 1)$ such that

$$\mathbf{P}(\xi_k = \delta_1) = p, \quad \mathbf{P}(\xi_k = \delta_2) = 1 - p \quad (\forall k = 1, 2, \dots, n).$$

Let $\mathbf{E}_p$ denote the corresponding expectation. It is easy to see that a measure $\mathbf{P}_p$ is risk-neutral if and only if $\mathbf{E}_p \xi_k = 0$ ($\forall k$), i.e., $\delta_1 p + \delta_2(1 - p) = 0$.

THEOREM 2.3 *Let (β_k, γ_k) be the strategy defined in Theorem 2.1. For the Cox–Ross–Rubinstein model, $\mathbf{E}_p \tilde{X}_k > 1$ ($\forall k = 1, 2, \dots, n$) for any non-risk-neutral measure and $\mathbf{E}_p \tilde{X}_k = 1$ ($\forall k = 1, 2, \dots, n$) for any risk-neutral measure.*

Consider now a more general model. We assume that the random variables ξ_k take values in the interval $[\delta_1, \delta_2]$, where $-1 < \delta_1 < 0 < \delta_2 < 1$, and ρ_k are not necessary nonrandom. Under these assumptions, the market is incomplete.

THEOREM 2.4 *Let $\mathbf{P}_\mu$ be a probability measure on the set of sequences $\{(\xi_k, \rho_k)\}_{k=1}^n$ such that ξ_k does not depend on $\xi_{k-1}, \dots, \xi_1$ for all k. Let (β_k, γ_k) be the strategy defined in Theorem 2.1. Then*

$$\mathbf{E}_\mu \tilde{X}_k = \frac{1}{2} \left(\prod_{m=1}^{k} (1 + \mathbf{E}_\mu \xi_m) + \prod_{m=1}^{k} (1 - \mathbf{E}_\mu \xi_m) \right), \quad k = 1, 2, \dots, n, \tag{2.10}$$

and

$$\tilde{X}_k \geq \left(1 - \max(\delta_1^2, \delta_2^2)\right)^{k/2} \quad (\forall k = 1, 2, \dots, n). \tag{2.11}$$

Furthermore, if

$$\mathbf{E}_\mu \xi_k \, \mathbf{E}_\mu \xi_m \geq 0 \quad \forall k, m = 1, 2, \dots, n, \tag{2.12}$$

then $\mathbf{E}_\mu \tilde{X}_n > 1$ for any non-risk-neutral measure $\mathbf{P}_\mu$, and $\mathbf{E}_\mu \tilde{X}_k = 1$ ($\forall k = 1, 2, \dots, n$) for any risk-neutral measure $\mathbf{P}_\mu$.

REMARK 2.3 *The condition (2.12) holds for the Cox–Ross–Rubinstein model.*

Notice that the strategy defined in Theorem 2.1 does not depend on the probability distributions of the stock appreciation rate or on the assumptions about a stochastic model. It should be pointed out that this strategy is both profitable in mean and risk bounded.

EXAMPLE 2.3 Consider a stochastic market model as in Theorem 2.3 and a self-financing admissible strategy $\gamma_k = \xi_k \tilde{S}_k^{-1}$. Suppose that (2.12) holds. Then

$$\begin{aligned} \mathbf{E}_\mu \tilde{X}_k &= 1 + \tfrac{1}{X_0} \textstyle\sum_{m=0}^{k-1} \mathbf{E}_\mu \gamma_m \left(\tilde{S}_{m+1} - \tilde{S}_m \right) \\ &= 1 + \tfrac{1}{X_0} \textstyle\sum_{m=0}^{k-1} \mathbf{E}_\mu \gamma_m \xi_{m+1} \tilde{S}_m = 1 + \tfrac{1}{X_0} \textstyle\sum_{m=0}^{k-1} \mathbf{E}_\mu \xi_{m+1} \mathbf{E}_\mu \xi_m. \end{aligned}$$

Hence $\mathbf{E}_\mu \tilde{X}_k > 1$, and this strategy also ensures a positive average gain for any non-risk-neutral measure $\mathbf{P}_\mu$. But we have that the wealth will be negative for large k for any sample sequence such that $\xi_{m+1}\xi_m < 0$ $(\forall m)$.

In Theorems 2.3 and 2.4, ξ_k are assumed to be independent. We now reduce this assumption.

Denote by n_+ the random number of positive ξ_k in the set $\{\xi_k\}_{k=1}^n$, $n_- = n - n_+$, $\nu = n_+/n$. Let $\varepsilon > 0$ be a given number. Introduce the following function:

$$f(\nu, \varepsilon) = \frac{1}{2} \left[(1+\varepsilon)^{n\nu} (1-\varepsilon)^{n(1-\nu)} + (1-\varepsilon)^{n\nu} (1+\varepsilon)^{n(1-\nu)} \right]. \quad (2.13)$$

Notice that $f(\nu, \varepsilon) = f(1-\nu, \varepsilon)$ and $f(0, \varepsilon) = f(1, \varepsilon) > 1$.

PROPOSITION 2.2 *Let ξ_k take only two values, $-\varepsilon$ and $+\varepsilon$, where $\varepsilon \in (0,1)$ is a given number, $k = 1, \dots, n$. Then $\tilde{X}_n = f(\nu, \varepsilon)$ for the strategy defined in Theorem 2.1.*

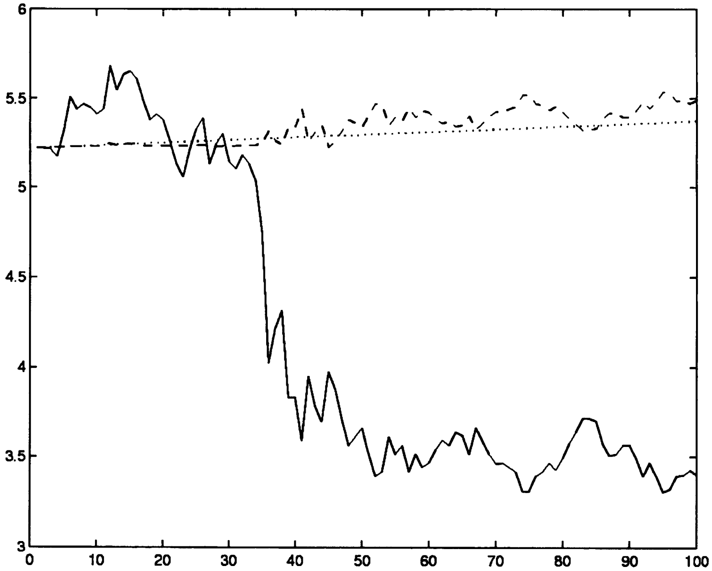
The following example demonstrates that the strategy ensures a positive average gain sometimes when $\mathbf{E}_\mu \xi_k \equiv 0$, but ξ_k are not independent.

EXAMPLE 2.4 Consider a stochastic market model as in Proposition 2.2. Let $\mathbf{P}_\mu$ be a given probability measure such that $\mathbf{E}\xi_m \equiv 0$ and $\mathbf{P}_\mu(\xi_1 = \xi_2 = \ldots = \xi_k) = 1$, where $k \le n$ is a given number such that $f(\delta, \varepsilon) > 1$, where $\delta = k/n$. It can be easily seen that if $\xi_1 = \varepsilon$, then $\nu > \delta$, and if $\xi_1 = -\varepsilon$, then $\nu < 1 - \delta$. Hence $\mathbf{E}_\mu f(\nu, \varepsilon) \ge f(\delta, \varepsilon) = f(1-\delta) > 1$. Consider the strategy defined in Theorem 2.1. By Proposition 2.2, $\mathbf{E}_\mu \tilde{X}_n \ge f(\delta, \varepsilon) > 1$.

2.2.4 Experiments with historical data

We have carried out the following experiments. We have applied our strategy to given sequences $\{S_1, \dots, S_{100}\}$ of 100 daily prices of 10 leading Australian stocks (AMC, ANZ, LLC, MAY, MWB, MIM, NAB, NBH, NCP, and NFM), taking all possible initial days from 1984 to 1996. For the sake of simplicity, we assume that ρ is constant and $\rho = 1 + 0.07/250$, where 250 is the approximate number of trading days, and we have not taken into account the dividends income of shareholders. We take the average over all such trials (the total number was 19,024; in fact full 13 years of data were not available for all the stocks). We have obtained the following result: in the case of the risk-free investment, $\rho^{100} = 1.0284$ and $\log \rho^{100} = 0.0280$. For our strategy, the average of X_{100}/X_0 is 1.0304, the minimal X_k/X_0 over $k = 1, \dots, 100$ is 0.8212, and the average of $\log(X_{100}/X_0)$ is 0.0292. It appears that our strategy reduces risk in comparison with the "buy-and-hold" strategy: for this experiment, the minimal S_k/S_1 over $k = 1, \dots, 100$ and over all trials is 0.2625. Figure 2.1 shows an example of the performance of our strategy when applied to the ANZ (Australia New Zealand) Bank stocks with daily transaction from September 1, 1987, to January 21, 1988 (i.e., for 100 trading days, including the October 1987 market crash).

Figure 2.1. The resulting wealth X_k and the stock price S_k for the strategy (2.5) applied for ANZ Bank stocks during 100 days from 1 September 1987 to 21 January 1988. ——: values of S_k; - - - -: values of X_k when $X_0 = S_0$; · · ·: values of B_k when $B_0 = S_0$.



2.3. A strategy with a risky numéraire

Now we present a strategy that has approximately the same performance as the "buy-and-hold" investment to the risky assets but that also bounds risk and gives a positive average gain in comparison with the "buy-and-hold" strategy for any non-risk-neutral probability measure.

More precisely, we present a strategy with the following properties:

(a) The strategy uses only observations of stock prices and does not require any knowledge about the market appreciation rate, the volatility, or other parameters;

(b) It gives some systematic additional gain in comparison with the "buy-and-hold" strategy for a given risky asset;

(c) The risk for the strategy is similar to the risk for the "buy-and-hold" strategy; i.e. it is a bounded risk strategy if the risky asset is taken as a numéraire.

We consider again a market model from Section 2.1 that consists of two assets: a risky stock and a risk-free bond (or bank account). As in Section 2.1, we assume that the price of the stock evolves arbitrarily with the interval uncertainty. The dynamics of the bond is exponentially increasing along with the interval uncertainty. Under these assumptions, the market is incomplete. A multi-period strategy will be presented that differs from the strategy of Cover (1991) and has the properties (a)–(c). The additional gain is positive on average for *any* non-risk-neutral probability measure, under some additional assumptions about the probability distributions such that the market is still incomplete, though the strategy itself does not use probability assumptions.

In other words, we present a strategy for an investor who wishes to keep the given risky asset as a core portfolio but who also admits some dynamic adjusting of the total amount of shares in order to improve performance. In particular, this investor accepts a risk of huge losses if the stock falls. This model of preferences can be realistic, as in the case of a holder of the controlling share of a company.

2.3.1 The strategy

Let $\varepsilon > 0$ be a parameter. Set

$$\begin{aligned}
&c_1 \triangleq 1+\varepsilon, \quad c_2 \triangleq 1-\varepsilon,\\
&\tilde{z}_{j,0} \triangleq 1, \quad z_{j,0} \triangleq 1, \quad v_0 \triangleq 1,\\
&\tilde{z}_{j,k} \triangleq \prod_{m=1}^{k}(1+c_j\xi_k), \quad z_{j,k} \triangleq \tilde{z}_{j,k}B_k,\\
&v_k \triangleq \prod_{m=1}^{k}\left(1-\frac{\varepsilon^2\xi_m^2}{(1+\varepsilon\xi)^2}\right)^{1/2}, \quad k>1.
\end{aligned}$$

THEOREM 2.5 *Let*

$$\begin{cases} \tilde{X}_k \triangleq \frac{1}{2}\left(\tilde{z}_{1,k} + \tilde{z}_{2,k}\right), \quad X_k \triangleq \tilde{X}_k B_k, \\ \gamma_k \triangleq \frac{1}{2\tilde{S}_k}\left(c_1\tilde{z}_{1,k} + c_2\tilde{z}_{1,k}\right), \\ \beta_k \triangleq \frac{X_k - \gamma_k S_k}{B_k}, \quad k = 0, 1, 2, \dots, n. \end{cases} \tag{2.14}$$

Then the pair (β_k, γ_k) *is an admissible and self-financing strategy with the corresponding wealth* X_k *and the normalized wealth* $\tilde{X}_k$*, and*

$$\tilde{X}_k \geq v_k \tilde{S}_k, \tag{2.15}$$

for all admissible sequences $S_1, \dots, S_k$.

Notice that the strategy (2.14) uses at time k only $B_k, S_k, \tilde{z}_{j,k}$, $j = 1, 2$, so we need to store only two numbers $\tilde{z}_{j,k}$ in addition to the current observation of B_k, S_k. And these two numbers depend on $\{B_m, S_m, \ m \leq k\}$ only.

COROLLARY 2.3 *Assume that* $|\xi_k| \leq \varepsilon_1$ $(\forall k)$*, where* $\varepsilon_1 \in (0,1)$*. Then*

$$\tilde{X}_k \geq (1 - \varepsilon^2\varepsilon_1^2(1 - \varepsilon\varepsilon_1)^{-2})^{k/2}\tilde{S}_k.$$

If $\varepsilon = 1$*, then* $\tilde{X}_k \geq 1/2$ *for all possible* S_k.

For a real market, changes in stock prices are usually no more than 1%– 5% per day, hence $|\xi_k|$ is about 0.01–0.05 for daily transactions.

2.3.2 Average performance on a probability space

We suppose that a probability space is given such that ξ_k and ρ_k are random variables. We repeat that

$$B_k = \rho_k B_{k-1}, \quad S_k = \rho_k S_{k-1}\left(1 + \xi_k\right), \quad k = 1, 2, \dots, n. \tag{2.16}$$

Consider a probability measure $\mathbf{P}_\mu$ on the set of sequences $\{(\xi_k, \rho_k)\}_{k=1}^n$. Let $\mathbf{E}_\mu$ and $\mathbf{E}_\mu\{\cdot\,|\,\cdot\}$ denote the corresponding expectation and a conditional expectation.

DEFINITION 2.6 *A measure* $\mathbf{P}_\mu$ *is said to be risk-neutral if*

$$\mathbf{E}_\mu\left\{\tilde{S}_k | \tilde{S}_{k-1}, \dots, \tilde{S}_0\right\} = \tilde{S}_{k-1} \quad \forall k = 1, \dots, n.$$

REMARK 2.4 *By (2.16),*

$$\mathbf{E}_\mu\left\{\tilde{S}_k | \tilde{S}_{k-1}, \dots, \tilde{S}_0\right\} = \tilde{S}_k\left(1 + \mathbf{E}_\mu\left\{\xi_k | \xi_{k-1}, \dots, \xi_1\right\}\right).$$

Hence, if ξ_k *does not depend on* $\xi_{k-1}, \dots, \xi_1$*, then the measure* $\mathbf{P}_\mu$ *is risk-neutral if and only if* $\mathbf{E}_\mu\xi_k = 0$ $(\forall k = 1, \dots, n)$.

THEOREM 2.6 *Let $\mathbf{P}_\mu$ be a probability measure on the set of sequences $\{(\xi_k, \rho_k)\}_{k=1}^n$ such that ξ_k does not depend on $\xi_{k-1}, \dots, \xi_1$ for all k, and $\mathbf{E}_\mu \xi_k \, \mathbf{E}_\mu \xi_m \geq 0 \quad (\forall k, m = 1, 2, \dots, n)$. Let (β_k, γ_k) be the strategy defined in Theorem 2.5. Then $\mathbf{E}_\mu \tilde{X}_n > 1$ for any non-risk-neutral measure $\mathbf{P}_\mu$, and $\mathbf{E}_\mu \tilde{X}_k = 1$ $(\forall k = 1, 2, \dots, n)$ for any risk-neutral measure $\mathbf{P}_\mu$.*

REMARK 2.5 *The market is incomplete under assumptions of Theorem 2.4. The conditions of this theorem are satisfied for the Cox–Ross–Rubinstein model of complete market (Cox et al. (1979)).*

Notice that the strategy defined in Theorem 2.5 does not require the probability distributions of the market parameters.

2.3.3 Experiments

We have carried out the following experiments. We have applied the strategy with different parameter ε to sequences $\{S_0, S_1, \dots, S_n\}$, where $S_k \triangleq P_k/P_0$, $k = 1, \dots, n$, and sequences $\{P_0, P_1, \dots, P_n\}$ are samples of daily price data for 16 leading Australian stocks (AMC, ANZ, LEI, LLC, LLN, MAY, MLG, MMF, MWB, MIM, NAB, NBH, NCM, NCP, NFM and NPC), taking all possible initial days from 1984 to 1997. Thus, n is the number of periods, i.e., the number of portfolio adjustments, or transactions. The length of the real-time interval that corresponds to each single period is m days, where m is given. The case of $m = 1$ corresponds to daily portfolio adjusting. For the sake of simplicity, we assume that $\rho_k = (1 + 0.07/250)^m$, where 250 is the approximate number of trading days in one year. Thus, $T = n \times m/250$ is the length of the real-time interval of running the strategy in each trial; $T = 1$ corresponds to one year. Also, we have not taken into account the dividends income of shareholders. We calculate the average of $\tilde{X}_n$ and $\log \tilde{X}_n$ over all such trials (the total number was 35,430 for $n = 100$ and $m = 1$; the full 13 years of data were not available for all the stocks).

We compare the performance of the strategy with different ε with the performance of the "buy-and-hold" strategy (i.e., when $\gamma_k \equiv 1$, $X_k \equiv S_k$, and $\tilde{X}_k \equiv \tilde{S}_k$). Also, we compare our results with the performance of the classic Merton-type "myopic" strategy for the log utility function, which has the following closed-loop form:

$$\gamma_k = (a - r) \frac{\tilde{X}_k}{\sigma^2 \tilde{S}_k}, \tag{2.17}$$

i.e., the normalized wealth evolves as

$$\tilde{X}_{k+1} = \tilde{X}_k + \tilde{X}_k (a - r) \frac{\tilde{S}_{k+1} - \tilde{S}_k}{\sigma^2 \tilde{S}_k},$$

where a is the appreciation rate of the stocks, r is the risk-free interest rate and σ is the volatility. Assume that we have allowed the use of posterior historical data; the average of the appreciation rate for a single stock for this market falls within the interval $[0.14, 0.16]$ and the volatility within $[0.28, 0.3]$. Thus, we applied the strategy (2.17) with $a = 0.15$, $r = 0.07$ and $\sigma = 0.29$.

The results of the experiments with different ε, m, n are shown in Tables 2.1-2.3. It can be seen that:

- the average performance is robust with respect to changing in m with given T, and the performance is better for a more frequent portfolio adjusting; and

- the average gain is proportional to the length of the time interval.

These findings show that the results are stable.

Further, it can be seen that $\varepsilon = 0.5$ and $\varepsilon = 1.0$ ensure systematically better performance for both criteria $\tilde{X}_n$ and $\log \tilde{X}_n$.

Note that there is a difference between the average of $\tilde{X}_n = \tilde{S}_n$ for the "buy-and-hold" strategy in Tables 2.2 and 2.3 for a given n, because $\tilde{S}_n$ are calculated as prices on different days for different m (for example, for $m = 5$ and $m = 1$, the difference is 4 days).

Figures 2.2 and 2.3 show examples of the performance of our strategy applied with $\varepsilon = 1.5$ to the ANZ (Australia New Zealand) Bank stocks with daily transactions from September 1, 1987, to January 21, 1988 (including the October 1987 market crash) and from January 3, 1995, to May 30, 1995.

It will be shown below in Chapter 10 that the strategies introduced in Sections 2.2-2.3 are optimal for a generic investment problem under special assumptions about the distribution of market parameters (see Remark 9.1).

Table 2.1. Performance of strategy (2.14) for the Australian market with $n = 100$, $m = 1$, and $T = 0.4$ (average over 38,266 trials).

Type of strategy	$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$	$\varepsilon = 2.5$	Merton strategy	Buy-and-hold strategy
$\tilde{X}_n$	1.040	1.0499	1.0661	1.0907	1.1256	1.0269	1.0376
$\log \tilde{X}_n$	0.0106	0.0128	0.0136	-0.0072	-0.0293	0.0088	0.0091

Table 2.2. Performance of strategy (2.14) for the Australian market with $n = 20$, $m = 5$, and $T = 0.4$ (average over 38,202 trials).

Type of strategy	$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$	$\varepsilon = 2.5$	Merton strategy	Buy-and-hold strategy
$\tilde{X}_n$	1.0388	1.0458	1.0576	1.0745	1.0965	1.0248	1.0376
$\log \tilde{X}_n$	0.0092	0.0106	0.0083	-0.0178	-0.0880	0.0088	0.0091

Table 2.3. Performance of strategy (2.14) for the Australian market with $n = 50$, $m = 1$, and $T = 0.2$ (average over 36,180 trials).

Type of strategy	$\varepsilon = 0.5$	$\varepsilon = 1.0$	$\varepsilon = 1.5$	$\varepsilon = 2.0$	$\varepsilon = 2.5$	Merton strategy	Buy-and-hold strategy
$\tilde{X}_n$	1.0185	1.0213	1.0259	1.0326	1.0416	1.0161	1.0176
$\log \tilde{X}_n$	0.0052	0.0065	0.0077	0.0079	0.0053	0.0053	0.0047

2.4. Proofs

Proof of Proposition 2.1. We have that

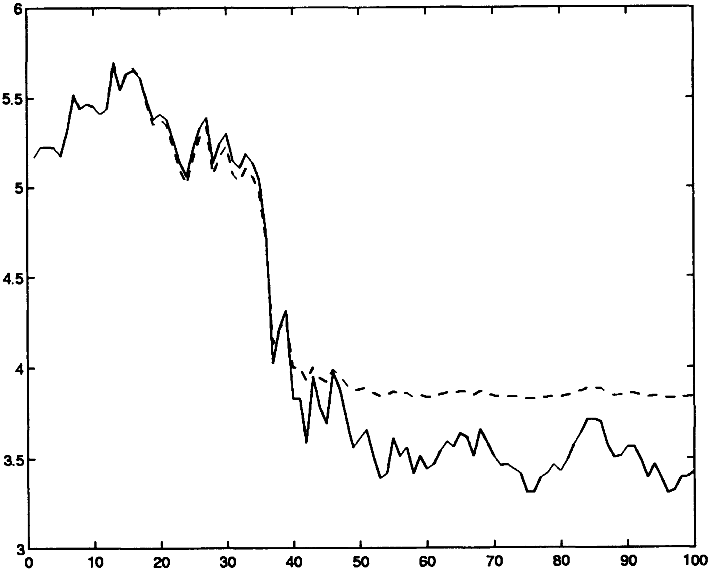
$$\begin{aligned} X_{k+1} - X_k &= X_0 \prod_{m=1}^{k+1} \rho_m \left(\tilde{X}_{k+1} - \tilde{X}_k\right) + X_0(\rho_{k+1} - 1) \prod_{m=1}^{k} \rho_m \tilde{X}_k \\ &= \prod_{m=1}^{k+1} \rho_m \gamma_k \left(\tilde{S}_{k+1} - \tilde{S}_k\right) + (\rho_{k+1} - 1) X_k \\ &= \gamma_k \left(S_{k+1} - \rho_{k+1} S_k\right) + (\rho_{k+1} - 1) X_k \\ &= \gamma_k \left(S_{k+1} - S_k\right) - (\rho_{k+1} - 1) S_k \gamma_k + (\rho_{k+1} - 1) X_k \\ &= \gamma_k \left(S_{k+1} - S_k\right) + (\rho_{k+1} - 1)\left(X_k - S_k \gamma_k\right) \\ &= \gamma_k \left(S_{k+1} - S_k\right) + (\rho_{k+1} - 1) \beta_k B_k \\ &= \gamma_k \left(S_{k+1} - S_k\right) + (B_{k+1} - B_k) \beta_k. \end{aligned}$$

This completes the proof of the proposition. □

Proof of Theorem 2.1. We have

$$\frac{v_k S_0}{2\tilde{S}_k} \left(\frac{\tilde{S}_k - \tilde{S}_{k+1}}{\tilde{S}_k} \right) = \frac{v_k S_0}{2\tilde{S}_k} \left(\frac{c_{k+1} \tilde{S}_k}{\tilde{S}_{k+1}} - 1 \right).$$

Figure 2.2. The resulting wealth X_k and the stock price S_k for the strategy (2.14) applied with $\varepsilon = 1.5$ for ANZ Bank stocks during 100 days from 1 September 1987 to 21 January 1988. ——: values of $X_k \equiv S_k$ for the "buy-and-hold" strategy; - - - - -: values of X_k when $X_0 = S_0$.



Then

$$\begin{aligned}\tilde{X}_{k+1} - \tilde{X}_k &= \frac{1}{2}\left(\frac{\tilde{S}_{k+1}}{S_0} - \frac{\tilde{S}_k}{S_0} + \frac{v_{k+1}S_0}{\tilde{S}_{k+1}} - \frac{v_k S_0}{\tilde{S}_k}\right) \\ &= \frac{1}{2}\left(\frac{\tilde{S}_{k+1}-\tilde{S}_k}{S_0} + \frac{v_k S_0}{\tilde{S}_k}\left[\frac{c_{k+1}\tilde{S}_k}{\tilde{S}_{k+1}} - 1\right]\right) = \gamma_k\left(S_{k+1} - S_k\right).\end{aligned}$$

Furthermore,

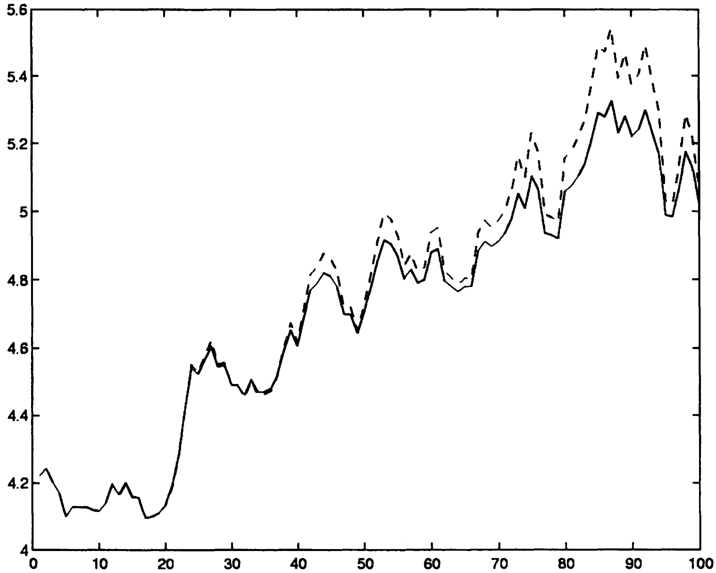
$$\frac{c_m}{1+\xi_m} = \frac{1-\xi_m^2}{1+\xi_m} = 1 - \xi_m.$$

Hence

$$\begin{aligned}2\tilde{X}_k &= \frac{\tilde{S}_k}{S_0} + \frac{v_k S_0}{\tilde{S}_k} \\ &= \prod_{m=1}^k (1+\xi_m) + \prod_{m=1}^k \frac{c_m}{1+\xi_m} = \prod_{m=1}^k (1+\xi_m) + \prod_{m=1}^k \frac{1-\xi_m^2}{1+\xi_m}.\end{aligned}$$

Therefore, we obtain equation (2.6). Furthermore, (2.7) follows from (2.6) and the elementary estimate $y^{-1} + y \geq 2$ $(\forall y > 0)$. From Proposition 2.1, it follows that X_k is the total wealth for the self-financing strategy (2.5) which is defined from the condition of self-financing. This completes the proof of Theorem 2.1. □

Figure 2.3. The resulting wealth X_k and the stock price S_k for the strategy applied with $\varepsilon = 1.5$ for ANZ Bank stocks during 100 days from 3 January 1995 to 30 May 1995. ——: values of $X_k \equiv S_k$ for "buy-and-hold" strategy; - - - - -: values of X_k with $X_0 = S_0$.



Proof of Theorem 2.2. Let $n_- \triangleq n - n_+$; then $n_+ = \nu n$ and $n_- = (1-\nu)n$. By (2.6),

$$\tilde{X}_n = \frac{1}{2}\left[(1+\varepsilon)^{\nu n}(1-\varepsilon)^{(1-\nu)n} + (1-\varepsilon)^{\nu n}(1+\varepsilon)^{(1-\nu)n}\right]. \qquad (2.18)$$

Hence $\tilde{X}_n \to \frac{1}{2}\left(e^{h(2\nu-1)} + e^{h(1-2\nu)}\right)$ as $n \to +\infty$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.4. The estimate (2.11) follows from (2.7), and (2.6) implies (2.10), since ξ_k are independent. Furthermore,

$$\begin{aligned}\mathbf{E}_\mu \gamma_k \tilde{S}_k &= \tfrac{X_0}{2}\mathbf{E}_\mu\left(\tfrac{\tilde{S}_k}{S_0} - \tfrac{v_k S_0}{\tilde{S}_k}\right)\\ &= \tfrac{X_0}{2}\left(\prod_{m=1}^k \mathbf{E}_\mu(1+\xi_m) - \prod_{m=1}^k \mathbf{E}_\mu \tfrac{c_m}{1+\xi_m}\right)\\ &= \tfrac{X_0}{2}\left(\prod_{m=1}^k (1+\mathbf{E}_\mu\xi_m) - \prod_{m=1}^k (1-\mathbf{E}_\mu\xi_m)\right).\end{aligned}$$

Hence, the inequality $\mathbf{E}_\mu\xi_{k+1}\mathbf{E}_\mu\gamma_k S_k \geq 0$ follows from (2.12). Then, by Remark 2.3, for a non-risk-neutral measure $\mathbf{P}_\mu$, there exists $k \in \{1,\ldots,n-1\}$

such that $\mathbf{E}_\mu\xi_{k+1}\mathbf{E}_\mu\gamma_k\tilde{S}_k > 0$. By Proposition 2.1,

$$\begin{aligned}\mathbf{E}_\mu\tilde{X}_k &= 1+\sum_{m=0}^{k-1}\mathbf{E}_\mu\tfrac{\gamma_m}{X_0}\left(\tilde{S}_{m+1}-\tilde{S}_m\right)\\ &= 1+\tfrac{1}{X_0}\sum_{m=0}^{k-1}\mathbf{E}_\mu\gamma_m\xi_{m+1}\tilde{S}_m\\ &= 1+\tfrac{1}{X_0}\sum_{m=0}^{k-1}\mathbf{E}_\mu\xi_{m+1}\mathbf{E}_\mu\gamma_m\tilde{S}_m.\end{aligned}$$

This completes the proof of Theorem 2.4. □

Theorem 2.3 is a special case of Theorem 2.4. □

Proof of Proposition 2.2. The proof follows from (2.13) and (2.7). □

Proof of Theorem 2.5. We have

$$\begin{aligned}\tilde{X}_{k+1}-\tilde{X}_k &= \tfrac{1}{2}\left(\tilde{z}_{1,k+1}-\tilde{z}_{1,k}+\tilde{z}_{2,k+1}-\tilde{z}_{2,k}\right)\\ &= \tfrac{1}{2}\left[\tilde{z}_{1,k+1}c_1\xi_{k+1}+\tilde{z}_{2,k+1}c_2\xi_{k+1}\right]\\ &= \tfrac{1}{2}\left[c_1\tilde{z}_{1,k+1}+c_2\tilde{z}_{2,k+1}\right]\tfrac{\tilde{S}_{k+1}-\tilde{S}_k}{\tilde{S}_k}\\ &= \gamma_k\left(S_{k+1}-S_k\right).\end{aligned}$$

Furthermore,

$$\begin{aligned}\tfrac{\tilde{X}_k}{\tilde{S}_k} &= \tfrac{1}{2}\left[\prod_{m=1}^k\tfrac{1+c_1\xi_m}{1+\xi_k}+\prod_{m=1}^k\tfrac{1+c_2\xi_m}{1+\xi_k}\right]\\ &= \tfrac{1}{2}\left[\prod_{m=1}^k\left(1+\tfrac{\varepsilon\xi_m}{1+\xi_k}\right)+\prod_{m=1}^k\left(1-\tfrac{\varepsilon\xi_m}{1+\xi_k}\right)\right]\geq v_k\end{aligned}$$

by the elementary estimate $y^{-1}+y\geq 2$ $(\forall y>0)$. It follows from Proposition 2.1 that X_k is the total wealth for the self-financing strategy (2.14), which is defined from the condition of self-financing. This completes the proof of Theorem 2.5. □

Proof of Theorem 2.6. Set

$$\alpha_k \triangleq \mathbf{E}_\mu\xi_k, \quad \varepsilon_k \triangleq \varepsilon\alpha_k(1+\alpha_k)^{-1}.$$

We have

$$\begin{aligned}\mathbf{E}_\mu\tilde{S}_k &= \prod_{m=1}^k(1+\alpha_m),\\ \mathbf{E}_\mu\tilde{X}_k &= \tfrac{1}{2}\left(\prod_{m=1}^k(1+c_1\alpha_m)+\prod_{m=1}^k(1+c_2\alpha_m)\right), \quad k=1,2,\dots,n.\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{E}_\mu\tilde{X}_k-\mathbf{E}_\mu\tilde{S}_k\\ &= \tfrac{1}{2}\left(\prod_{m=1}^k(1+(1+\varepsilon)\alpha_m)+\prod_{m=1}^k(1+(1-\varepsilon)\alpha_m)\right)\\ &\qquad -\prod_{m=1}^k(1+\alpha_m)\\ &= \prod_{m=1}^k(1+\alpha_m)\left[\tfrac{1}{2}\left(\prod_{m=1}^k(1+\varepsilon_m)+\prod_{m=1}^k(1-\varepsilon_m)\right)-1\right]\geq 0,\end{aligned}$$

and $\mathbf{E}_\mu\tilde{X}_k-\mathbf{E}_\mu\tilde{S}_k=0$ only if $\alpha_i=0$ $(\forall i\leq k)$. This completes the proof of Theorem 2.6. □

Chapter 3

STRATEGIES FOR INVESTMENT IN OPTIONS

Abstract We consider strategies for investment in options for the diffusion market model. We show that there exists a correct proportion between put and call options in the portfolio such that the average gain is almost always positive for a generic Black and Scholes model. This gain is zero if and only if the market price of risk is zero. A paradox related to the corresponding loss of option's seller is also discussed.

3.1. Introduction and definitions

We consider strategies for investment in options for a generic stochastic diffusion model of a financial market. It is assumed that there is a risky stock and a risk-free asset (bond), and that European put and call on that stock are available at the initial time. We consider only strategies for selecting an options portfolio at the initial time. The selection of this portfolio is the only action of the investor; after that, he or she waits until the expiration time to accept gain or loss.

We show that there exists a correct proportion between put and call options with the same expiration time, on the same underlying security (the so-called *long strangle* combination), such that the average gain is almost always positive. This gain is zero if and only if the market price of risk is zero, i.e., when the appreciation rate of the stock is equal to the interest rate of the risk-free asset (i.e., $a \neq r$ in (3.1)–(3.2) below). A paradox related to the corresponding loss of option's seller is also discussed.

Definitions

Consider the generic model of a financial market consisting of a risk-free asset (bond, or bank account) with price $B(t)$ and a risky asset (stock) with price $S(t)$. We are given a standard probability space with a probability measure **P**

N. Dokuchaev, *Dynamic Portfolio Strategies: Quantitative Methods and Empirical Rules for Incomplete Information*

and a standard Brownian motion $w(t)$. The bond and stock prices evolve as

$$B(t) = e^{rt}B_0, \tag{3.1}$$

$$dS(t) = aS(t)dt + \sigma S(t)dw(t). \tag{3.2}$$

Here $r \geq 0$ is the risk-free interest rate, $\sigma > 0$ is the volatility, and $a \in \mathbf{R}$ is the appreciation rate. We assume that $t \in [0, T]$, where $T > 0$ is a given terminal time. Equation (3.2) is Itô's equation and can be rewritten as

$$S(t) = S_0 \exp\left(at - \frac{\sigma^2 t}{2} + \sigma w(t)\right).$$

We assume that European put and call on that stock are available for that price defined by the Black–Scholes formula.

Further, we assume that $\sigma > 0$, $r \geq 0$, $B_0 > 0$, and $S_0 > 0$ are given, but the constant a is unknown.

Let $p_{BS}(S_0, K, r, T, \sigma)$ denote the Black–Scholes price for the put option, and $c_{BS}(S_0, K, r, T, \sigma)$ denote the Black–Scholes price for the call option. Here S_0 is the initial stock price, K is the strike price, r is the risk-free interest rate, σ is the volatility, and T is the expiration time.

We recall the Black–Scholes formula. Let

$$\Phi(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy,$$

$$d \triangleq \frac{\log(S_0/K) + T\left(r + \sigma^2/2\right)}{\sigma\sqrt{T}}, \quad d^- = d - \sigma\sqrt{T}. \tag{3.3}$$

Then

$$c_{BS}(S_0, K, r, T, \sigma) = S_0\Phi(d) - Ke^{-rT}\Phi(d^-), \tag{3.4}$$

$$p_{BS}(S_0, K, r, T, \sigma) = c_{BS}(S_0, K, r, T, \sigma) - S_0 + Ke^{-rT}. \tag{3.5}$$

Let X_0 be the initial wealth of an investor (i.e., at the initial $t = 0$), and let $X(T)$ be the wealth of the investor at the terminal time $t = T$.

Consider a vector (K_p, μ_p, K_c, μ_c) such that $K_p > 0, \mu_p \geq 0, K_c > 0, \mu_c \geq 0$. We shall use this vector to describe the following strategy: buy a portfolio of options that consists of μ_p put options with the strike price K_p and of μ_c call options with the strike price K_c, with the same expiration time T; thus,

$$X_0 = \mu_p p_{BS}(S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(S_0, K_p, r, T, \sigma) \tag{3.6}$$

(assume that the options are available for the Black–Scholes price). We have assumed that the investor does not take any other actions until the expiration time. In that case, the terminal wealth at time $t = T$ will be

$$X(T) = \mu_p(K_p - S(T))^+ + \mu_c(S(T) - K_c)^+. \tag{3.7}$$

DEFINITION 3.1 *The vector (K_p, μ_p, K_c, μ_c) is said to be a strategy.*

For the case of a risk-free "hold-only-bonds" strategy, $X(T) = e^{rT}X_0$. It is natural to compare the results of any investment with the risk-free investment.

DEFINITION 3.2 *The difference $\mathbf{E}X(T) - e^{rT}X_0$ is said to be the average gain.*

Note that the appreciation rate a in this definition is fixed but unknown. The average gain for a strategy depends on $a - r$. For example, for a call option holder, when $\mu_p = 0$, the average gain is positive if $a > r$.

REMARK 3.1 There are standard terms for option combinations. If you own both a put and a call with the same striking price, the same expiration date, on the same underlying security, you are *long a straddle*, i.e., you own a *straddle*). *Strangles* are similar to straddles, except the puts and calls have different striking prices (see Strong (1994)).

3.2. The winning strategy

Let d_p and d_c be defined by (3.3), where $d = d(K, S_0, T, r, \sigma)$ is defined after substituting $K = K_p$ or $K = K_c$, respectively.

THEOREM 3.1 *Let $\mu_p > 0$, $\mu_c > 0$ and*

$$\frac{\mu_c}{\mu_p} = \frac{1 - \Phi(d_p)}{\Phi(d_c)}. \tag{3.8}$$

Then the average gain for the strategy (K_p, μ_p, K_c, μ_c) is positive for any $a \neq r$, i.e.,

$$\mathbf{E}X(T) > e^{rT}X_0 \quad \forall a \neq r. \tag{3.9}$$

Moreover,

$$\mathbf{E}X(T) = e^{rT}X_0 \quad \text{if} \quad a = r. \tag{3.10}$$

For any $(K_p, K_c, S_0, T, r, \sigma)$, the proportion (3.10) is the only one which ensures (3.8): for any other proportion μ_c/μ_p, there exists $a \in \mathbf{R}$ such that the average gain is negative.

COROLLARY 3.1 *Let the variable a be random, independent of $w(\cdot)$, and such that $\mathbf{P}(a \neq r) > 0$. Then $\mathbf{E}X(T) > e^{rT}X_0$ for the strategy from Theorem 3.1.*

Set

$$Q(x,t) \triangleq \mu_p p_{BS}(x, K_p, r, T-t, \sigma) + \mu_c c_{BS}(x, K_c, r, T-t, \sigma),$$

$$\Delta(x,t) = \frac{dQ(x,t)}{dx}. \tag{3.11}$$

COROLLARY 3.2 *Let (K_p, μ_p, K_c, μ_c) be as in Theorem 3.1; then*

$$\Delta(S_0, 0) = 0. \tag{3.12}$$

3.3. Numerical examples

EXAMPLE 3.1 Consider a straddle, i.e., a combination of put and call options with $K_p = K_c = \$25$, $S_0 = \$30$, $T = 0.25$ (i.e. the expiration time is 3 months=25 years); $r = 0.05$ (i.e., 5% annual), and $\sigma = 0.45$ (i.e., 45% annual). Then $d = d_p = d_c = 0.978$, $\Phi(d) = 0.836$. We calculate $c_{BS}(S_0, K_p, r, T, \sigma) = 5.9625$ and $p_{BS}(S_0, K_p, r, T, \sigma) = 0.6519$; the winning proportion is

$$\frac{\mu_c}{\mu_p} = \frac{164}{836}.$$

Clearly, the pair (μ_c, μ_p) is uniquely defined from the system

$$\begin{cases} \frac{\mu_c}{\mu_p} = \frac{164}{836}, \\ \mu_p p_{BS}(S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(S_0, K_p, r, T, \sigma) = X_0. \end{cases}$$

To obtain the profit/loss diagram for the winning strategy from Example 3.1, we need to select X_0. Let

$$X_0 \triangleq p_{BS}(S_0, K_p, r, T, \sigma) + c_{BS}(S_0, K_p, r, T, \sigma).$$

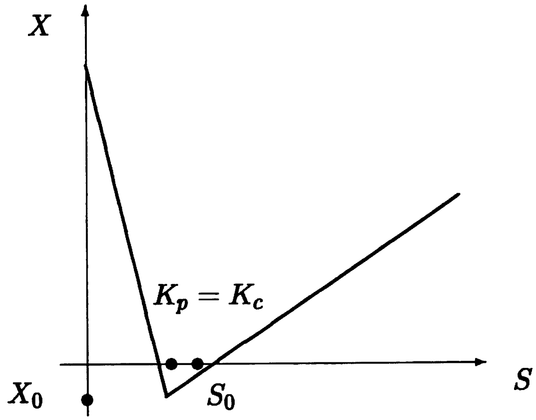
We have that

$$c_{BS}(S_0, K_p, r, T, \sigma) = 5.9625,$$
$$p_{BS}(S_0, K_p, r, T, \sigma) = 0.6519.$$

under the assumptions of Example 3.1. Then

$$X_0 = 5.9625 + 0.6519 = 6.6144,$$

Figure 3.1. Profit/loss diagram for a "winning" long straddle from Example 3.1: $X_0 = c_{BS} + p_{BS}$, where c_{BS} is the Black–Scholes price of the call option, p_{BS} is the Black-Scholes price of the put option, S is the stock price at the terminal time, and $K_p = K_c$ is the strike price for put and call.



and

$$(\mu_p, \mu_c) = (3.6311, 0.7123)$$

is the solution of the system

$$\begin{cases} 836\mu_c = 164\mu_p \\ 0.6519\mu_p + 5.9625\mu_c = 6.6144. \end{cases}$$

Figure 3.1 shows the profit/loss diagram for a winning long straddle from Example 3.1.

EXAMPLE 3.2 Let S_0 be arbitrary, $K_p = K_c = S_0$, $T = 0.25$, $r = 0.05$, and $\sigma = 0.45$. Then $d = d_p = d_c = 0.168$, $\Phi(d) = 0.567$, and the winning proportion is

$$\frac{\mu_c}{\mu_p} = \frac{433}{567}.$$

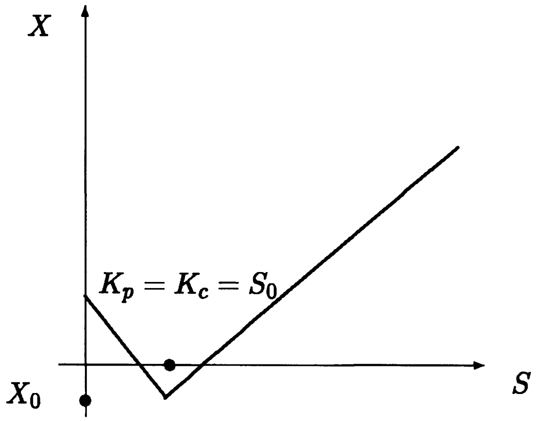
Again, let

$$X_0 \triangleq p_{BS}(S_0, K_p, r, T, \sigma) + c_{BS}(S_0, K_p, r, T, \sigma).$$

We have that $c_{BS}(S_0, K_p, r, T, \sigma) = 2.3841$, and $p_{BS}(S_0, K_p, r, T, \sigma) = 2.0736$ under assumptions of Example 3.2. Then

$$X_0 = 2.3841 + 2.0736 = 4.4577,$$

Figure 3.2. Profit/loss diagram for a "winning" long straddle from Example 3.2: $X_0 = c_{BS} + p_{BS}$, where c_{BS} is the Black–Scholes price of the call option, p_{BS} is the Black–Scholes price of the put option, S is the stock price at the terminal time, and $K_p = K_c = S_0$ is the strike price for put and call.



and

$$(\mu_p, \mu_c) = (1.1447, 0.8742)$$

is the solution of the system

$$\begin{cases} 567\mu_c = 433\mu_p \\ 2.0736\mu_p + 2.3841\mu_c = 4.4577. \end{cases}$$

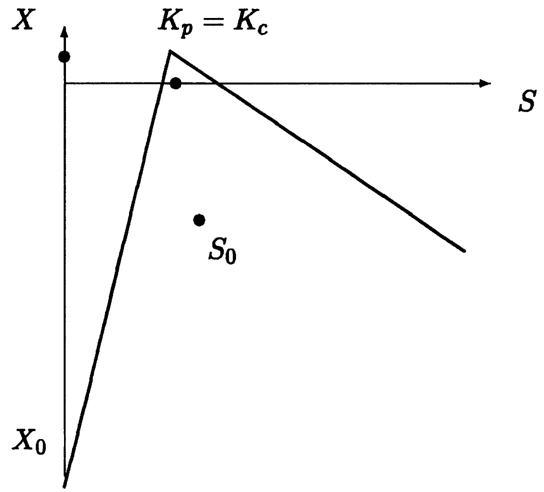
Figure 3.2 shows profit/loss diagram for a "winning" long straddle from Example 3.2.

3.4. A consequence for the seller and a paradox

> *What about the option writer? This person is on the "other side of the market" from the option buyer. Ignoring commissions, the options market is a zero sum game; aggregate gains and losses will always net to zero. If the call buyer makes money, the call writer is going to lose money, and vice versa.*
>
> —R. Strong, *Speculative markets*, 1994, p. 37

Consider now the result for a seller (writer) who has sold the options portfolio described in Theorem 3.1 (i.e., who is short the straddle described there). The seller receives the premium X_0 and must pay $X(T)$ at time $t = T$. Figure 3.3 shows profit/loss diagram for a short straddle from Example 3.1.

Figure 3.3. Profit/loss diagram for a short straddle from Example 3.1.



Let $Y(t)$ be the wealth, which was obtained by the seller from $Y_0 = X_0$ by some self-financing strategy. Let $\hat{Y}(T) = Y(T) - X(T)$ be the terminal wealth after paying obligations to options holder at the expiration time T.

Let us consider the possible actions of the sellers after receiving the premium. The following strategy is most commonly presented in textbooks devoted to the mathematical aspects of option pricing:

Strategy I: *To replicate the claim $X(T)$ using the replicating strategy.* As is known, the Black-Scholes price is defined as a minimum initial wealth such that the option's random claim can be replicated. For Strategy I, the number of shares is $\Delta(S(t), t)$ at any time $t \in [0, T]$, and $\hat{Y}(T) = 0$ a.s., i.e., *there is neither any risk nor any gain.* Thus, it is doubtful that the seller uses this strategy in practice.

Furthermore, it was mentioned in Strong (1994) [p. 53] that in practice the option writer just keeps the premium as real compensation for bearing the added risk of foregoing future price appreciation or depreciation. Thus, the second strategy is the following:

Strategy II: *To invest the premium X_0 into bonds, take no further actions and wait for the outcome of the price movement, similarly to the option's holder.*

The seller who sells only put (or only call) options and uses Strategy II puts his or her stake on the random events $K_p \le S(T)$ (or $S(T) \le K_c$ correspondingly). A gain by the holder implies a loss for the writer. The game looks fair in the case of selling either put or call options separately, because we know that the Black–Scholes price is fair and that the chances for gain should be equal for buyer and seller (otherwise, either the ask or the bid will prevail). But Theorem 3.1 implies that the seller will receive a nonpositive average gain which is negative on average for any $a \neq r$, if he or she sells *the combination described in Theorem 3.1.* In other words, we have a paradox:

A combination of two fair deals of selling put and call options gives an unfair deal.

The second paradox can be formulated as following:

We know that the Black–Scholes price is fair for buyers as well as for sellers (otherwise, either the ask or the bid will prevail). However, we have found an options combination such that buying is preferable, because the buyer has nonnegative (and almost always positive) average gain but the seller has a nonpositive (almost always negative) average gain.

In practice, unlike in for our generic model, brokers use sophisticated measures such as insurance, long positions in stocks or other options, etc. to reduce the risk, but that does not affect the core of the paradox.

A possible explanation is that the writer also use the premium X_0 to receive a gain from $a \neq r$ *by using some other strategies* such as Merton's strategies which are possibly more effective than the options portfolio. In other words, *a rational option seller does not use either risk-free replication of claims or a "keep-only-bonds" strategy; rather he or she uses strategies that are able to explore* $a \neq r$.

3.5. Proofs

Proof of Theorem 3.1. Let $\mathbf{P}_a$ be the conditional probability measure given a. Let $\mathbf{E}_a$ be the corresponding expectation. We denote by $\mathbf{E}_*$ the expectation that corresponds to the risk–neutral measure, when $a = r$. Set

$$h(a) \triangleq \mathbf{E}_a X(T).$$

By the definitions of $X(T)$, it follows that

$$h(a) = \mu_p \mathbf{E}_*(K_p - e^{(a-r)T} S(T))^+ + \mu_c \mathbf{E}_*(e^{(a-r)T} S(T) - K_c)^+. \quad (3.13)$$

As is known (see, e.g., Strong (1994) and Duffie (1988)), the Black-Scholes price can be presented as

$$\begin{aligned} p_{BS}(S_0, K, r, T, \sigma) &= e^{-rT}\mathbf{E}_*(K - S(T))^+, \\ c_{BS}(S_0, K, r, T, \sigma) &= e^{-rT}\mathbf{E}_*(S(T) - K)^+. \end{aligned} \quad (3.14)$$

Then

$$e^{-rT} h(a) = \mu_p p_{BS}(e^{(a-r)T} S_0, K_p, r, T, \sigma) + \mu_c c_{BS}(e^{(a-r)T} S_0, K_c, r, T, \sigma).$$

By the put and call parity formula, it follows that

$$\begin{aligned} e^{-rT}h(a) &= \mu_p\Big[c_{BS}(e^{(a-r)T}S_0, K_p, r, T, \sigma) - e^{(a-r)T}S_0 + K_p\Big] \\ &\quad +\mu_c c_{BS}(e^{(a-r)T}S_0, K_c, r, T, \sigma). \end{aligned}$$

The following proposition is well known (see, e.g., Strong (1994) [p. 100]).

PROPOSITION 3.1 *For any $T > 0, K > 0$, the following holds:*

$$\frac{\partial}{\partial x}c_{BS}(x, K, r, T, \sigma) = \Phi(d), \tag{3.15}$$

where d is defined by (3.3).

Let $d_c(a)$ and $d_p(a)$ be defined as d_p and d_c, respectively, with substituting S_0 for $e^{(a-r)T}S_0$. Set

$$y = y(a) \triangleq e^{(a-r)T}, \quad R(y) \triangleq h(T^{-1}\log y).$$

We have that

$$R(y(a)) \equiv h(a);$$

then

$$\frac{dR(y)}{dy} = e^{rT}S_0\left[(\mu_p(\Phi(d_p(a)) - 1) + \mu_c\Phi(d_c(a))\right].$$

By (3.8), it follows that

$$\frac{dR}{dy}(y)|_{y=1} = 0. \tag{3.16}$$

(Note that $y = 1$ if and only if $a = 1$.) It is known that

$$\frac{\partial^2}{\partial x^2}p_{BS}(x, K_p, 0, T, \sigma) > 0, \quad \frac{\partial^2}{\partial x^2}c_{BS}(x, K_p, 0, T, \sigma) > 0$$

and the derivatives exist. Then $R''(y) > 0$ $(\forall y)$, i.e., $R(\cdot)$ is strongly convex. It follows that $a = r$ is the only solition that minimizes h. By (3.6), (3.13), it follows that

$$R(1) = e^{rT}X_0.$$

The uniqueness of the proportion (3.8) follows from the uniqueness of μ_c/μ_p, which ensures (3.16). This completes the proof of Theorem 3.1. □

Corrolary 3.1 follows from (3.9) and (3.10). □

Corrolary 3.2 follows from (3.15). □

Chapter 4

CONTINUOUS-TIME ANALOGS OF "WINNING" STRATEGIES AND ASYMPTOTIC ARBITRAGE

Abstract In this chapter, we consider two continuous-time analogue of the model–free winning empirical strategy defined in Theorem 2.2, Chapter 2. These two continuous-time strategies ensure a positive average gain for any non-risk-neutral probability measure; the strategies bound risk and do not require forecasting of the volatility coefficient and appreciation rate estimation. As the number of the traded stocks increases, the strategies converge to arbitrage with a given positive gain that is ensured with probability arbitrarily close to 1.

4.1. Introduction

In this chapter, we study continuous-time analogue of the model–free strategy defined in Theorem 2.2, Chapter 2. Similar to the generic discrete-time model, these two continuous-time strategies achieve the following two aims:

(i) to bound risk; and

(ii) to give a positive average gain.

These two aims are achieved for all possible appreciation rates and volatilities from a wide class without volatility forecast and appreciation rate estimation. In fact, the strategy ensures a positive average gain for all volatilities and appreciation rates from a wide class that includes random bounded volatilities.

The strategy described in Section 4.3 below is such that a trader makes transactions at any time when the price deviation exceeds a given level. A number of transactions are known and finite, and the stopping time is random (but the expectation of the stopping time is finite).

The strategy described in Sections 4.4 and 4.5 requires continuous adjusting, but the time interval is nonrandom and given. The explicit formulas do not include the future values of volatility but include only the past stock prices and cumulative integral of the past historical volatility. The strategy ensures a positive average gain for all volatilities and appreciation rates from a wide

N. Dokuchaev, *Dynamic Portfolio Strategies: Quantitative Methods and Empirical Rules for Incomplete Information*

class that includes random bounded volatilities. As the number of traded stocks increases, the strategies converge to arbitrage with a given positive gain that is ensured with probability arbitrarily close to 1.

Our approach in Sections 4.4 and 4.5 can be summarized as follows. In the Black–Scholes model, the equation for wealth is a backward parabolic equation such that the corresponding boundary value problem is well posed with the Couchy condition at the terminal time and is ill posed with the Couchy condition at the initial time. In our solution, there are no specific claims at the terminal time that present a problem of option replication. We have found a special variant of the Couchy condition at the initial time such that this, generally speaking, ill posed boundary value problem has a solution, and this solution has the desired properties. In other words, we have freedom to select a strategy as well as a claim that has to be replicated, and we use this freedom to select a special claim given in (4.12) and (4.20) below such that the corresponding replication strategy has the desired properties.

4.2. Definitions

Consider the diffusion model of a securities market consisting of a risk–free bond or bank account with a price $B(t)$, $t \geq 0$, and risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, N$. We consider cases of both $N < +\infty$ and $N = +\infty$. The prices of the stocks evolve according to the stochastic differential equations

$$dS_i(t) = S_i(t)\Big(a_i(t)dt + \sigma_i(t)dw_i(t)\Big), \quad t > 0, \tag{4.1}$$

where $a_i(t)$ is the appreciation rate, $\sigma_i(t)$ is the volatility coefficient, and $w_i(t)$ is the standard Wiener process. The initial price $S_i(0) > 0$ is given and non-random. The price of the bond evolves according to

$$B(t) = e^{rt}B(0), \quad t \geq 0, \tag{4.2}$$

where $r \geq 0$ and $B(0)$ are given constants.

In practice, the volatility coefficients $\sigma_i(t)$ can be estimated from the measurement $S_i(t)$, but the task is more difficult for the appreciation rate $a_i(t)$, which is harder to estimate than $\sigma_i(t)$. Hence we assume that $\sigma_i(t)$ can be observed at the current time but cannot be forecasted and that $a_i(t)$ cannot be observed and forecasted.

We assume that $w_i(\cdot)$ are independent processes. Thus, this model corresponds to the model from Chapter 1 with a diagonal matrix volatilty.

Let $\mathcal{F}_t^{S,n}$ be the right-continuous monotony increasing filtration of complete σ-algebras of events generated by $\{S_i(t)\}_{i=1}^n$, $n \leq N$.

Let $\mathcal{F}_t^S \triangleq \mathcal{F}_t^{S,N}$.

Introduce the vector processes

$$
\begin{aligned}
&a^{(n)}(t) \triangleq (a_1(t), \dots, a_n(t)),\\
&\sigma^{(n)}(t) \triangleq (\sigma_1(t), \dots, \sigma_N(t)),\\
&S^{(n)}(t) \triangleq (S_1(t), \dots, S_n(t)),\\
&a(t) \triangleq a^{(N)}(t), \quad \sigma(t) \triangleq \sigma^{(N)}(t), \quad S(t) \triangleq S^{(N)}(t),\\
&\tilde{S}(t) = e^{-rt} S(t).
\end{aligned}
$$

Let $\mathcal{V}$ be the set of all $\sigma(\cdot)$ such that $\sigma_i(t)$ are bounded random processes that are progressively measurable with respect to $\mathcal{F}_t^S$, $i = 1, \dots, N$.

Let $\mathcal{A}$ be the set of all $a(\cdot)$ such that $a_i(t)$ are bounded random processes and $a(t)$ does not depend on $w_i(t+s) - w_i(t)$, $s > 0$, $i = 1, \dots, N$.

Let $\bar{\mathcal{A}} \subset \mathcal{A}$ be the set of $a(\cdot) \in \mathcal{A}$ such that all $a_i(t)$ are non-random processes, $i = 1, \dots, N$.

Let $X(0)$ be the initial wealth at time $t = 0$ and $X(t)$ be the wealth at time $t > 0$. Though the number of available assets is infinite, we assume that only a finite number of them are traded by the agent and that the wealth $X(t)$ at time $t \geq 0$ is

$$
X(t) = \beta(t)B(t) + \sum_{i=1}^{n} \gamma_i(t) S_i(t). \tag{4.3}
$$

Here $n \leq N$ and $n < +\infty$, $\beta(t)$ is the quantity of the bond portfolio, $\gamma_i(t)$ is the quantity of the i stock portfolio, and $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t. We call these pairs *strategies*.

We consider the problem of investment or choosing a strategy.

DEFINITION 4.1 *Let $(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \mathcal{A}$ be fixed. A pair $(\beta(\cdot), \gamma(\cdot)) = (\beta(\cdot), \gamma_1(\cdot), \gamma_2(\cdot), \dots, \gamma_n(\cdot))$ is said to be an admissible strategy if $n < +\infty$, $n \leq N$, and $\beta(t)$, $\gamma_i(t)$, $\gamma_i(t)S_i(t)$, $i = 1, \dots, n$, are random processes that are progressively measurable with respect to the filtration $\mathcal{F}_t^{S,n}$ and such that*

$$
\begin{aligned}
&\mathbf{E} \int_0^T |\beta(t)|^2 dt < +\infty,\\
&\mathbf{E} \int_0^T \left(|\gamma(t)|^2 + \sum_{i=1}^{n} S_i(t)^2 \gamma_i(t)^2\right) dt < +\infty \quad \forall T > 0.
\end{aligned} \tag{4.4}
$$

The main constraint in choosing a strategy is the so-called *condition of self-financing.*

DEFINITION 4.2 *A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be self-financing if*

$$
dX(t) = \beta(t)dB(t) + \sum_{i=1}^{n} \gamma_i(t) dS_i(t). \tag{4.5}
$$

In fact, any admissible self-financing strategy has the form

$$\begin{aligned}\gamma(t) &= (\gamma_1(t),\gamma_2(t),\dots,\gamma_n(t)) = \Gamma\left(t, S^{(n)}(\cdot)|_{[0,t]}\right),\\ \beta(t) &= \frac{X(t)-\sum_{i=1}^n \gamma_i(t)S_i(t)}{B(t)},\end{aligned} \tag{4.6}$$

where $\Gamma(t,\cdot) : C(0,t)^n \to \mathbf{R}^n$ is a functional, $t \geq 0$. For different $a(\cdot)$, the random processes $(\beta(t),\gamma(t))$ with the same $\Gamma(t,\cdot)$ in (4.6) may be different. Hence it will also be convenient to introduce strategies defined by $\Gamma(t,\cdot)$.

DEFINITION 4.3 *A functional* $\Gamma(t,\cdot) : C(0,t)^n \to \mathbf{R}^n$, $t \geq 0$, *is said to be an admissible CL-strategy (closed-loop strategy) if the corresponding pair* $(\beta(\cdot),\gamma(\cdot))$ *defined by (4.6) is admissible.*

DEFINITION 4.4 *The process* $\tilde{X}(t) \triangleq e^{-rt}\frac{X(t)}{X(0)}$ *is called the normalized wealth.*

Notice that $\tilde{X}(0) = 1$.

REMARK 4.1 *This definition is slightly different from Definition 1.2. To eliminate the difference, it suffices to assume that* $X_0 = 1$; *this can be done without loss of generality.*

4.3. Unbounded horizon: piecewise constant strategies

Consider the simplest case of the market model described above, with $n = N = 1$. We assume here that $a(t)$, $\sigma(t)$ are nonrandom processes, and that the process $a(t)$ is square integrable on all finite intervals $[0,T]$, $\delta_1 \leq \sigma(t)^2 \leq \delta_2$ $(\forall t)$ a.s., where δ_1, δ_2 are constants, $\delta_2 > \delta_1 > 0$.

Let $\mathcal{F}_t^S$ be the filtration generated by the process $S(t)$.

DEFINITION 4.5 *Let* $\mathcal{M}$ *be the set of Markov stopping times for the filtration* $\mathcal{F}_t^S$ *such that* $\mathbf{E}\theta < +\infty$, $\theta \geq 0$ *for* $\theta \in \mathcal{M}$.

REMARK 4.2 *It is important that* $\mathbf{E}\theta < +\infty$ *for the stopping times in this definition. It can be seen from Example 1.1 that there exists a trivial arbitrage without this restriction (a strategy that ensures positive gain with zero risk is said to be arbitrage).*

DEFINITION 4.6 *Let* $\theta \in \mathcal{M}$, *and let* $h(t)$ *be a random function. A pair* $(\beta(\cdot),\gamma(\cdot))$ *is said to be a bounded risk strategy with the bounds* $h(\cdot)$ *for the stopping time* θ *if*

$$\tilde{X}(t) \geq h(t) \quad \forall t \in [0,\theta] \quad a.s.$$

for all admissible $a(\cdot)$.

Let $\bar{n} \in \mu$, $\varepsilon \in (0,1)$, and $T > 0$ be given. We assume that $T \in (0, +\infty]$; in other words, the case of $T = +\infty$ is not excluded.

Let

$$\zeta(t,\theta) \triangleq e^{r(\theta-t)}\frac{S(t)}{S(\theta)} - 1, \quad 0 \le \theta \le t.$$

Then

$$S(t) = e^{r(t-\theta)}S(\theta)(1+\zeta(\theta,t)), \quad 0 \le \theta \le t, \quad \zeta(\theta,\theta) = 0.$$

Introduce the stopping times

$$\theta_0 \triangleq 0, \quad \theta_k \triangleq T \wedge \inf\{t :\ t > \theta_{k-1}, \quad |\zeta(t,\theta_{k-1})| = \varepsilon\}, \quad k = 1,\dots,\bar{n}.$$

Let

$$\begin{aligned}
&n \triangleq \max\{k : \theta_{k-1} < T, \quad k = 1,\dots,\bar{n}\},\\
&S_k \triangleq S(\theta_k), \quad B_k \triangleq B(\theta_k), \quad k = 0,1,\dots,n,\\
&\rho_k \triangleq \exp(r(\theta_k - \theta_{k-1})), \quad \xi_k \triangleq \tfrac{S(\theta_k)}{S(\theta_{k-1})\rho_k} - 1 = \zeta(\theta_k,\theta_{k-1}).
\end{aligned}$$

It is easy to see that $\mathbf{E}\theta_k < +\infty, \theta_k \in \mathcal{M}$ $(\forall k)$ and $|\xi_k| = \varepsilon$ for $k = 1,\dots,n-1$, $|\xi_n| \le \varepsilon$,

$$S_k = \rho_k S_{k-1}(1+\xi_k), \quad B_k = \rho_k B_{k-1}, \quad k = 1,\dots,n. \tag{4.7}$$

Let

$$v_k \triangleq (1-\varepsilon^2)^k, \quad k = 0,1,\dots,n-1, \quad v_n \triangleq (1-\varepsilon^2)^{n-1}(1-\xi_n^2).$$

THEOREM 4.1 *Let*

$$\begin{cases}
X_k \triangleq \frac{X(0)}{2}\left(\frac{S_k}{S_0} + e^{2r\theta_k}\frac{v_k S_0}{S_k}\right),\\
\gamma_k \triangleq \frac{X(0)}{2}\left(\frac{1}{S_0} - e^{2r\theta_k}\frac{v_k S_0}{S_k^2}\right),\\
\beta_k \triangleq \frac{X_k - \gamma_k S_k}{B_k}, \quad k = 0,1,2,\dots,n.
\end{cases} \tag{4.8}$$

Introduce the random function $h(t)$ such that $h(t) = (1-\varepsilon)(1-\varepsilon^2)^{k/2}$ for $t \in [\theta_k, \theta_{k+1})$. Furthermore, introduce the functions $\gamma(t)$, $\beta(t)$, $X(t)$ such that

$$\beta(t) = \beta_k, \quad \gamma(t) = \gamma_k, \quad t \in (\theta_k, \theta_{k+1}], \quad k = 0,1,2,\dots,n,$$

$$X(t) = \beta(t)B(t) + \gamma(t)S(t), \quad t \ge 0.$$

Then the pair $(\beta(\cdot),\gamma(\cdot))$ is an admissible and self-financing bounded risk strategy with the bounds $h(t)$, the corresponding wealth $X(t)$, the normalized wealth $\tilde{X}(t) \triangleq e^{-rt}\frac{X(t)}{X(0)}$, and

$$\tilde{X}(\theta_k) = \frac{1}{2}\left(\prod_{m=1}^{k}(1+\xi_m) + \prod_{m=1}^{k}(1-\xi_m)\right), \quad k = 1,2,\dots,n, \tag{4.9}$$

$$\tilde{X}(\theta_k) \geq (1-\varepsilon^2)^{k/2} \quad a.s, \quad k = 1, 2, \dots, n, \tag{4.10}$$

$$\tilde{X}(t) \geq (1-\varepsilon)(1-\varepsilon^2)^{k/2} \quad \forall t \in [\theta_k, \theta_{k+1}], \quad k = 1, 2, \dots, n. \tag{4.11}$$

THEOREM 4.2 *Let $a(t) \equiv a$, $\sigma(t) \equiv \sigma$ be nonrandom constants, $T = +\infty$, and $n \equiv \bar{n}$. Then $\mathbf{E}\theta_n < +\infty$. Furthermore, $\mathbf{E}\tilde{X}(\theta_n) > 1$ if and only if $a \neq r$, and $\mathbf{E}\tilde{X}(\theta_n) = 1$ if and only if $a = r$.*

4.4. Continuous-time strategies for a single stock market with a finite horizon

In this Section, we assume that $N = n = 1$. Introduce the random function

$$v(t) \triangleq \int_0^t \sigma^2(s)ds.$$

We shall employ the notation $\cosh(y) \triangleq (e^y + e^{-y})/2$.

THEOREM 4.3 *Let*

$$X(t) \triangleq \frac{X(0)}{2}\left(\frac{S(t)}{S(0)} + \exp\{2rt - v(t)\}\frac{S(0)}{S(t)}\right), \tag{4.12}$$

$$\gamma(t) \triangleq \frac{X(0)}{2}\left(\frac{1}{S(0)} - \exp\{2rt - v(t)\}\frac{S(0)}{S(t)^2}\right), \tag{4.13}$$

$$\beta(t) \triangleq \frac{X(t) - \gamma(t)S(t)}{B(t)}. \tag{4.14}$$

Then, for any $(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \mathcal{A}$, the pair $(\beta(\cdot), \gamma(\cdot))$ is an admissible and self-financing strategy with the corresponding wealth $X(t)$. The formula (4.13) gives a bounded risk strategy with the bound $C(t) = \exp\{-v(t)/2\}$, and

$$\tilde{X}(t) \geq \exp\left\{-\frac{v(t)}{2}\right\} \quad \forall(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \mathcal{A}, \quad \forall t \geq 0 \quad a.s., \tag{4.15}$$

where $\tilde{X}(t)$is the corresponding normalized wealth. Moreover,

$$\begin{aligned} \mathbf{E}\tilde{X}(t) &= \cosh\left(\textstyle\int_0^t a(s)ds - rt\right) \\ &\quad \forall(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \bar{\mathcal{A}}, \quad \forall t \geq 0, \end{aligned} \tag{4.16}$$

and if $(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \bar{\mathcal{A}}$, then

$$E\tilde{X}(t) > 1 \quad \textit{if and only if} \quad \frac{1}{t}\int_0^t a(s)ds \neq r. \tag{4.17}$$

REMARK 4.3 *The bounded risk strategy defined in Theorem 4.3 does not depend on $a(t)$ and the future values of volatility.*

REMARK 4.4 *If for all $t > 0$ the second inequality (4.17) does not hold, i.e., $\int_0^t a(s)ds = r$ ($\forall t > 0$), then $a(s) \equiv r$. The strategy defined in Theorem 4.3 is arbitrage (i.e., risk-free profit that is positive with nonzero probability) if and only if (4.17) holds and $\sigma \equiv 0$.*

4.5. Strategies for a multi-stock market

In this section, we assume that $N \le +\infty$, $1 \le n < +\infty$. Set

$$v_i(t) \triangleq \int_0^t \sigma_i^2(s)ds, \quad \nu(t) \triangleq \frac{1}{n^2}\sum_{i=1}^{n} v_i(t). \tag{4.18}$$

Furthermore, for $k = 0, \dots, 2^n - 1$, $t > 0$, and $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $x_i > 0$ ($\forall i$), introduce the functions

$$\begin{aligned}
&A_1(t) \triangleq \tfrac{1}{n}\textstyle\sum_{i=1}^n \left(\int_0^t a_i(s)ds - rt\right), \quad A_2(t) \triangleq -A_1(t),\\
&\hat{G}_1(x) \triangleq (x_1 x_2 \cdots x_n)^{1/n}, \quad \hat{G}_2(x) \triangleq (x_1 x_2 \cdots x_n)^{-1/n},\\
&G_k(x) \triangleq \tfrac{\hat{G}_k(x)}{\hat{G}_k(S(0))}, \quad k = 1, 2,\\
&u_1(t) \triangleq \tfrac{1}{2}\left(\tfrac{1}{n} - 1\right)\textstyle\sum_{i=1}^n v_i(t), \quad u_2(t) \triangleq \tfrac{1}{2}\left(\tfrac{1}{n} + 1\right)\sum_{i=1}^n v_i(t),\\
&\xi_k(t) \triangleq e^{-u_k(t)}, \quad k = 1, 2.
\end{aligned} \tag{4.19}$$

Set

$$\begin{aligned}
&\hat{H}_k(x,t) \triangleq \xi_k(t) G_k(x),\\
&H(x,t) \triangleq \tfrac{1}{2}\left(\hat{H}_1(x,t) + \hat{H}_2(x,t)\right).
\end{aligned} \tag{4.20}$$

THEOREM 4.4 *Let*

$$\begin{aligned}
&\tilde{X}(t) \triangleq H(e^{-rt}\tilde{S}(t), t),\\
&X(t) \triangleq e^{rt} X(0)\tilde{X}(t) = H(\tilde{S}(t), t),\\
&\gamma_i(t) \triangleq \tfrac{X(0)}{2}\tfrac{1}{nS_i(t)}\left(\hat{H}_1(\tilde{S}(t), t) - \hat{H}_2(\tilde{S}(t), t)\right),\\
&\gamma(t) \triangleq (\gamma_1(t), \dots, \gamma_n(t)), \quad \beta(t) \triangleq \tfrac{X(t) - \sum_{i=1}^n \gamma_i(t) S_i(t)}{B(t)}.
\end{aligned} \tag{4.21}$$

Then, for any $(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \mathcal{A}$, the pair $(\beta(t), \gamma(t))$ is an admissible and self-financing strategy with the corresponding wealth $X(t)$ and the normalized wealth $\tilde{X}(t)$. The formula (4.21) gives a bounded risk strategy with the bound $C(t) = e^{-\nu(t)}$, and

$$\tilde{X}(t) \geq e^{-\nu(t)} \quad \forall (\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \mathcal{A} \quad \forall t > 0 \quad a.s., \tag{4.22}$$

where $\tilde{X}(t)$is the corresponding normalized wealth. Moreover,

$$\begin{aligned} \mathbf{E}\tilde{X}(t) &= \textstyle\sum_{k=1,2} \cosh\left(A_k(t)\right) \\ &\quad \forall (\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \bar{\mathcal{A}}, \quad \forall t > 0, \end{aligned} \tag{4.23}$$

and if $(\sigma(\cdot), a(\cdot)) \in \mathcal{V} \times \bar{\mathcal{A}}$, then

$$\mathbf{E}\tilde{X}(t) > 1 \quad \textit{if and only if} \quad \exists k \in \{1, 2\} : \quad A_k(t) \neq 0. \tag{4.24}$$

For $T > 0$, $\bar{v} > 0$, introduce the class $\mathcal{V}(\bar{v}, T) \subset \mathcal{V}$ of all $\sigma(\cdot)$ such that

$$\frac{1}{n} \sum_{i=1}^{n} v_i(T) \leq \bar{v} \quad \forall n \quad \text{a.s.}, \tag{4.25}$$

where $v_i(t)$ are as defined in (4.18).

COROLLARY 4.1 *For any $T > 0$, $\bar{v} > 0$,*

$$\begin{aligned} \tilde{X}(t) \geq e^{-\bar{v}/2n} &= (1 - \varepsilon_n) \\ &\quad \forall (\sigma(\cdot), a(\cdot)) \in \mathcal{V}(\bar{v}, T) \times \mathcal{A}, \quad \forall t \in [0, T] \quad a.s., \end{aligned} \tag{4.26}$$

where

$$\varepsilon_n \triangleq 1 - \exp\left\{-\frac{\bar{v}}{2n}\right\} \simeq \frac{\bar{v}}{2n} \to 0 \quad as \quad n \to +\infty.$$

In other words, the maximum loss for the strategies defined in Theorem 4.4 converges to zero as the number n of traded stocks increases.

4.6. Definitions for asymptotic arbitrage

It can be concluded that, as the number of stocks increases, the proposed strategies converge to arbitrage . A risk-free profitable strategy is said to be arbitrage. Harrison and Pliska (1981) have shown that arbitrage does not exist in the diffusion stochastic market model. But some types of arbitrage as a limit or as asymptotic arbitrage do exist for models with an infinite number of assets. One definition of asymptotic arbitrage was introduced by Kabanov and Kramkov (1994). Another related definition is that of the so-called "free lunch" (Harrison and Kreps (1979)). There are many results concerning the existence or nonexistence of "free lunches" and asymptotic arbitrage. For example, it is known that free lunches do not exist in a diffusion market model with sequences of strategies that are piecewise constant with a bounded number of switchings, and free lunches do exist in the case of an unlimited number of switchings and unlimited borrowing (see, e.g., Dalang *et al.* (1990), Duffie and Huang (1986), Frittelli and Lakner (1992), Harrison and Kreps (1979), Jouini and Kallal (1995), Jouini (1996), Kreps (1981), Kabanov and Kramkov (1998), Klein and Schachermayer (1996)). The strategies introduced above are continuously changing; they do not require unlimited borrowing and information about future values of volatility, and it will be shown that they ensure asymptotic arbitrage of the first kind introduced by Kabanov and Kramkov (1994). Moreover, these strategies ensure a strengthened version of asymptotic arbitrage: a fixed positive gain is ensured with probability $1-\varepsilon$ for arbitrarily small $\varepsilon > 0$ for a wide class of volatilities and appreciation rates that includes all bounded random volatilities.

DEFINITION 4.7 *Let $T > 0$ be fixed, and let $C(t)$ be a random process such that $C(t) \in (0,1]$ for all t a.s. An admissible CL-strategy $\Gamma(t,\cdot)$ is said to be a bounded risk strategy with the bound $C(\cdot)$ if*

$$\tilde{X}(t) \geq C(t) \quad \forall(\sigma(\cdot),a(\cdot)) \in \mathcal{V}\times\mathcal{A} \quad \forall t\in[0,T] \quad a.s.$$

The following definition is a particular case of the classical definition of arbitrage (see Harrison and Pliska (1981)).

DEFINITION 4.8 *Let $(\sigma(\cdot),a(\cdot)) \in \mathcal{V}\times\mathcal{A}$ be given, $(\beta(t),\gamma(t))$ be an admissible self-financing strategy, and $\tilde{X}(t)$ be the corresponding normalized wealth. Let $T > 0$ be a given nonrandom time. Let*

$$\mathbf{P}(\tilde{X}(T)\geq 1) = 1, \quad \mathbf{P}(\tilde{X}(T) > 1) > 0.$$

Then this strategy is said to be arbitrage.

The following definition is a particular case of the definition of asymptotic arbitrage from Kabanov and Kramkov (1994, 1998).

DEFINITION 4.9 *Let $(\sigma(\cdot),a(\cdot)) \in \mathcal{V}\times\mathcal{A}$ and $T > 0$ be given. Let $(\beta^{(m)}(t),\gamma^{(m)}(t))$, $m = 1,2,\ldots$ be a sequence of admissible self-financing strategies, $X(t)$ be the corresponding total wealth, $\tilde{X}^{(m)}(t)$ be the corresponding normalized wealth, and $X^{(m)}(0) = X(0)$ $(\forall m)$. Suppose that there exist real numbers $\kappa > 1$, $p_0 > 0$ such that for any $\varepsilon > 0$ there exists a number $\bar{m}$*

such that

$$\mathbf{P}(\tilde{X}^{(m)}(T) \geq \kappa) \geq p_0,$$
$$\tilde{X}^{(m)}(t) \geq 1 - \varepsilon \quad \forall m \geq \bar{m}, \qquad \forall t \in [0, T] \quad a.s.$$

Then the sequence $(\beta^{(m)}(t), \gamma^{(m)}(t))$ *is said to be asymptotic arbitrage of the first kind.*

The following definitions strengthen the requirements of Definition 4.9; they assume a positive gain with probability arbitrarily close to 1, and a class of $(\sigma(\cdot), a(\cdot))$ is included in the consideration.

DEFINITION 4.10 *Let* $\mathcal{B} \subseteq \mathcal{V} \times \mathcal{A}$ *be a given subset of the set* $\mathcal{V} \times \mathcal{A}$. *Let* $\Gamma^{(m)}(t, \cdot)$, $m = 1, 2, \ldots$ *be a sequence of admissible CL-strategies,* $X(t)$ *be the corresponding total wealth,* $\tilde{X}^{(m)}(t)$ *be the normalized wealth,* $X^{(m)}(0) = X(0)$ $(\forall m)$. *Let* $T > 0$ *be given. Suppose that there exists a real number* $\kappa > 1$ *such that for any* $(\sigma(\cdot), a(\cdot)) \in \mathcal{B}$, $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, *there exists a number* $\hat{m}$ *such that*

$$\mathbf{P}(\tilde{X}^{(m)}(T) \geq \kappa - \varepsilon_1) \geq 1 - \varepsilon_2,$$
$$\tilde{X}^{(m)}(t) \geq 1 - \varepsilon \quad \forall m \geq \hat{m}, \quad \forall t \in [0, T] \qquad a.s.$$

Then the sequence $\Gamma^{(m)}(t, \cdot)$ *is said to be asymptotic arbitrage that almost guarantees the gain* κ *for the class* $\mathcal{B}$.

DEFINITION 4.11 *Let* $\mathcal{B} \subseteq \mathcal{V} \times \mathcal{A}$ *be a given subset of the set* $\mathcal{V} \times \mathcal{A}$. *Let* $\Gamma^{(m)}(t, \cdot)$, $m = 1, 2, \ldots$ *be a sequence of admissible CL-strategies,* $X(t)$ *be the corresponding total wealth,* $\tilde{X}^{(m)}(t)$ *be the normalized wealth, and* $X^{(m)}(0) = X(0)$ $(\forall m)$. *Let* $T > 0$ *be given. Suppose that there exists a real number* $\kappa > 1$ *such that for any* $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *there exists a number* $\hat{m}$ *such that*

$$\mathbf{P}(\tilde{X}^{(m)}(T) \geq \kappa - \varepsilon_1) \geq 1 - \varepsilon_2,$$
$$\tilde{X}^{(m)}(t) \geq 1 - \varepsilon \quad \forall m \geq \hat{m}, \quad \forall (\sigma(\cdot), a(\cdot)) \in \mathcal{B}, \quad \forall t \in [0, T] \qquad a.s.$$

Then the sequence $\Gamma^{(m)}(t, \cdot)$ *is said to be asymptotic arbitrage that almost guarantees the gain* κ *uniformly on the class* $\mathcal{B}$.

We cannot conclude yet that (4.24) and (4.26) ensure asymptotic arbitrage as it defined in Definitions 4.9 - 4.11 because a lower boundary of gain is not established. In the following section, we give some sufficient conditions that ensure asymptotic arbitrage.

4.7. Asymptotic arbitrage for the strategy (4.21)

In this section, we assume that $N = +\infty$. Let $T > 0$ be a fixed time. Let

$$\alpha_i(T) \triangleq \int_0^T a_i(t)dt - rT, \quad i = 1, 2, \dots .$$

For $\theta > 0$, introduce the set $\mathcal{A}(\theta, T) \subset \mathcal{A}$ such that for any $a(\cdot) \in \mathcal{A}(\theta, T)$, there exists a number $\hat{n}$ such that

$$\left| \frac{1}{n} \sum_{i=1}^{n} \alpha_i(T) \right| \geq \theta \quad \forall n > \hat{n} \quad \text{a.s.} \tag{4.27}$$

THEOREM 4.5 *Let $T > 0$, $\bar{v} > 0$, $\theta > 0$ be fixed. Consider the sequence of the strategies $(\beta(t), \gamma(t)) = (\beta^{(n)}(t), \gamma^{(n)}(t))$, defined in Theorem 4.4. Let $X^{(n)}$ be the corresponding total wealth. Then*

(i) *For any $(\sigma(\cdot), a(\cdot)) \in \mathcal{V}(\bar{v}, T) \times \mathcal{A}(\theta, T)$, $\varepsilon > 0$, there exists a number $\bar{n}$ such that*

$$\begin{aligned} &\mathbf{E}\tilde{X}^{(n)}(T) \geq X(0)\cosh(\theta), \\ &\tilde{X}^{(n)}(t) \geq 1 - \varepsilon \quad \forall n \geq \bar{n} \quad \forall t \in [0, T] \quad a.s. \end{aligned} \tag{4.28}$$

(ii) *For any $(\sigma(\cdot), a(\cdot)) \in \mathcal{V}(\bar{v}, T) \times \mathcal{A}(\theta, T)$, $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a number $\bar{n}$ such that*

$$\begin{aligned} &\mathbf{P}\left(\tilde{X}^{(n)}(T) \geq \cosh(\theta) - \varepsilon_1\right) \geq 1 - \varepsilon_2, \\ &\tilde{X}^{(n)}(t) \geq 1 - \varepsilon \quad \forall n \geq \bar{n} \quad \forall t \in [0, T] \quad a.s. \end{aligned} \tag{4.29}$$

(iii) *Let $\mathcal{A}_u \subset \mathcal{A}(\theta, T)$ be a set such that there exists a number $\hat{n}$ such that (4.27) holds for all $a(\cdot) \in \mathcal{A}_u$. Then for any $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a number $\bar{n}$ such that*

$$\begin{aligned} &\mathbf{P}\left(\tilde{X}^{(n)}(T) \geq X(0)(\cosh(\theta) - \varepsilon_1)\right) \geq 1 - \varepsilon_2, \\ &\tilde{X}^{(n)}(t) \geq 1 - \varepsilon \\ &\quad \forall n \geq \bar{n} \quad \forall (\sigma(\cdot), a(\cdot)) \in \mathcal{V}(\bar{v}, T) \times \mathcal{A}_u \quad \forall t \in [0, T] \quad a.s. \end{aligned} \tag{4.30}$$

COROLLARY 4.2 (i) *For any fixed $(\sigma(\cdot), a(\cdot)) \in \mathcal{V}(\bar{v}, T) \times \mathcal{A}(\theta, T, \bar{n})$, the sequence of strategies in Theorem 4.5 is asymptotic arbitrage of the first kind for time T (Definition 4.9).*

(ii) *This sequence is asymptotic arbitrage that almost guarantees the gain $\cosh(\theta) > 1$ for the class $\mathcal{V}(\bar{v}, T) \times \mathcal{A}(\theta, T)$ for time T (Definition 4.10).*

(iii) *This sequence is asymptotic arbitrage that almost guarantees the gain* $\cosh(\theta) > 1$ *uniformly on the class* $\mathcal{V}(\bar{v}, T) \times \mathcal{A}_u$ *for time* T *(Definition 4.11).*

REMARK 4.5 *The strategy* $(\beta^{(n)}(t), \gamma^{(n)}(t))$ *may be approximated by strategies* $(\beta^{(n,m)}(t), \gamma^{(n,m)}(t))$, $m = 1, 2, 3, \ldots$, *which are constant at the intervals* $(lT/m, (l+1)T/m)$, $l = 0, 1, \ldots, m-1$, *and such that* $\mathbf{E}|X^{(n,m)}(T) - X^{(n)}(T)|^2 \to 0$ *as* $m \to +\infty$ *for the corresponding values of wealth. Hence the first inequalities in (4.28)–(4.30) may be ensured as a limit for these piecewise constant strategies, but the second inequalities there can be guaranteed only with a probability close to 1, but not almost surely.*

4.8. Proofs

Proof of Theorem 4.1. It is easy to see that $e^{r\theta_k} = \prod_{m=1}^{k} \rho_m$. Consider the strategy defined in Theorem 2.1 applied to the discrete-time market model (4.7). The result of the strategy (4.8) coincides with the result of the strategy (2.5). Hence (4.9) and (4.10) hold. The corresponding total wealth is continuous, and therefore the underlying piecewise constant admissible strategy (4.8) is self-financing.

By (4.8),

$$X_k + \gamma_k \tilde{S}_k \geq 0, \quad X_k - \gamma_k S_k \geq 0.$$

Furthermore,

$$|\zeta(t, t_{k-1})| \leq \varepsilon \quad \forall t \in (\theta_k, \theta_{k+1}).$$

By Proposition 2.1,

$$\begin{aligned}
\tilde{X}(t) &= \tilde{X}(\theta_k) + \tfrac{\gamma_k}{X(0)}[\tilde{S}(t) - \tilde{S}(\theta_k)] \\
&= \tilde{X}(\theta_k) + \tfrac{\gamma_k}{X(0)}\tilde{S}(\theta_k)\zeta(t, t_{k-1}) \\
&= \tilde{X}(\theta_k)(1-\varepsilon) + \zeta(t, t_{k-1})\tfrac{\gamma_k}{X(0)}\tilde{S}_k + \varepsilon\tilde{X}_k \\
&\geq \tilde{X}(\theta_k)(1-\varepsilon).
\end{aligned}$$

Therefore, (4.11) holds. This completes the proof of the theorem. □

Proof of Theorem 4.2. We have that

$$\xi_k = y(\theta_k, \theta_{k-1}) - 1,$$

where $y(t, \theta)$ is the solution of the Itô's equation

$$\begin{cases} dy(t,\theta) = (a - r)y(t,\theta)dt + \sigma y(t,\theta)dw(t), \\ y(\theta,\theta) = 1. \end{cases}$$

Hence $\theta_k - \theta_{k-1}$ are independent because of the Markov property of the process $y(\cdot)$. Also, $w(\theta_k) - w(\theta_{k-1})$ are independent. Hence ξ_k are independent. We have that $y(t,\theta) \geq 0$ a.s. for all t,θ, and

$$\begin{aligned}\mathbf{E}\xi_k &= \mathbf{E}y(\theta_{k-1},\theta_{k-1}) + (a-r)\mathbf{E}\int_{\theta_{k-1}}^{\theta_k} y(t,\theta_{k-1})dt - 1 \\ &= (a-r)\mathbf{E}\int_{\theta_{k-1}}^{\theta_k} y(t,\theta_{k-1})dt.\end{aligned}$$

Hence $\mathbf{E}\xi_k > 0$ $(\forall k)$ if $a > r$, $\mathbf{E}\xi_k < 0$ $(\forall k)$ if $a < r$, and $\mathbf{E}\xi_k = 0$ $(\forall k)$ if $a = r$. All the assumptions of Theorem 2.4 hold. This completes the proof of the theorem. □

Let $Q \triangleq \{(x,t) = (x_1,\dots,x_n,t) : t > 0,\ x_i > 0\ (\forall i)\}$, $1 \leq n \leq N$, $n < +\infty$.

PROPOSITION 4.1 *Let $(\sigma(\cdot),a(\cdot)) \in \mathcal{V}\times\mathcal{A}$ be fixed. Let $\bar{H}(x,t)$ be a random continuous function of $(x,t) \in \mathbf{R}^{n+1}$ which has derivatives $\bar{H}'_x \in C(Q_0)$, $\bar{H}'_{xx} \in C(Q_0)$, and $\bar{H}'_t \in L_\infty(Q_0)$ for any bounded domain $Q_0 \subset Q$. Assume that $\bar{H}(x,t)$ is progressively measurable with respect to $\mathcal{F}_t^{S,N}$ and that for all $T > 0$, there exist constants $C > 0$, $c > 0$ such that*

$$\begin{aligned}&|\bar{H}'_t(x,t)| + |\bar{H}'_x(x,t)| + |\bar{H}'_{xx}(x,t)| \leq C(|x|^c + |x|^{-c} + 1) \\ &\forall (x,t) \in Q :\ t \leq T.\end{aligned} \tag{4.31}$$

Furthermore, assume that the following equality holds in the space $L_\infty(Q)$:

$$\frac{\partial \bar{H}}{\partial t}(x,t) + \sum_{i=1}^{n} \frac{\sigma_i^2(t)}{2} x_i^2 \frac{\partial^2 \bar{H}}{\partial x_i^2}(x,t) = 0. \tag{4.32}$$

Introduce the processes

$$\bar{X}(t) \triangleq \bar{H}(e^{-rt}S(t),t) = \bar{H}(\tilde{S}(t),t), \tag{4.33}$$

$$\bar{\gamma}_i(t) \triangleq \frac{\partial \bar{H}}{\partial x_i}(\tilde{S}(t),t), \quad \bar{\gamma}(t) \triangleq (\bar{\gamma}_1(t),\dots,\bar{\gamma}_n(t)), \tag{4.34}$$

$$\bar{\beta}(t) = \frac{e^{rt}\bar{X}(t) - \sum_{i=1}^{n} \bar{\gamma}_i(t)S_i(t)}{B(t)}. \tag{4.35}$$

Then the pair $(\bar{\beta}(t),\bar{\gamma}(t))$ is an admissible and self-financing strategy with the corresponding normalized wealth $\bar{X}(t)$.

Proof. It can be easily seen that the process $S_i^{-1}(t)$ evolves as

$$dS_i^{-1}(t) = \left(-a_i(t) + \sigma_i(t)^2\right) S_i^{-1}(t)dt + \sigma_i(t)S_i^{-1}(t)dw_i(t), \quad t > 0. \tag{4.36}$$

From (4.1) and (4.36), we have that

$$\sup_{t\le T}\mathbf{E}|S_i(t)|^p<+\infty,\quad \sup_{t\le T}\mathbf{E}|S_i(t)^{-1}|^p<+\infty\quad \forall p>1,\ \forall i=1,\dots,n.$$

By (4.31), the processes $\bar\beta(t)$, $\bar\gamma(t)$, $S_i(t)\bar\gamma_i(t)$ are square integrable and (4.4) holds; hence $(\bar\beta(t),\bar\gamma(t))$ is an admissible self-financing strategy.

To complete the proof of Proposition 4.1, we have to show that (4.5) holds for $e^{rt}\bar X(t)$, $\bar\beta(t)$, $\bar\gamma(t)$. To do so it is sufficient to find $d\bar X(t)$. The function $\bar H(x,t)$ is random, and hence Itô's formula is not applicable formally. But $d\bar X(t)$ may be obtained from the Itô-Ventssel formula, which is similar to the standard Itô's formula for this case when $\bar H(x,t)$ is absolutely continuous by t (for a case of continuous $\bar H_t'$, see Rozovskii (1990); for a case of noncontinuous $\bar H_t'$ see Dokuchaev (1994)). This completes the proof of Proposition 4.1. □

PROPOSITION 4.2 *Let $G(x)$ be a deterministic function that has continuous derivatives G_x',G_{xx}' in the open domain $\{x=(x_1,\dots,x_n)\colon\ x_i>0\ (\forall i)\}$. Let $\xi(t)$ be a random absolutely continuous function that has a derivative $\xi_t'\in L_\infty(0,T)$ for any $0<T<+\infty$. Assume that $\xi(t)$ is progressively measurable with respect to $\mathcal{F}_t^{S,n}$. Furthermore, let*

$$\sum_{i=1}^n\frac{\sigma_i^2(t)}{2}x_i^2\frac{\partial^2G}{\partial x_i^2}(x)\xi(t)=-G(x)\frac{\partial\xi}{\partial t}(t)\tag{4.37}$$

in $L_\infty(0,T)$ for all $x=(x_1,\dots,x_n)$, $x_i>0$ $(\forall i)$, $T>0$. Then the function $\bar H(x,t)=G(x)\xi(t)$ satisfies all the assumptions of Proposition 4.2.

PROPOSITION 4.3 *The functions $G(x)=G_k(x)$ and $\xi(x)=\xi_k(x)$ defined in (4.19) satisfy all the assumptions of Proposition 4.2. The functions $\bar H(x,t)=\hat H_k(x,t)$ and $\bar H(x,t)=H(x,t)$ defined in (4.20) satisfy all the assumptions of Proposition 4.1.*

The proofs Propositions 4.2–4.3 are straightforward and will be omitted.

Theorem 4.3 is a special case of Theorem 4.4 with $n=N=1$.

Proof of Theorem 4.4. Applying Propositions 4.1–4.3, we obtain the strategy (4.21).

It can be easily seen that

$$\begin{aligned}u_1(t)+u_2(t)&=\tfrac12\left[\tfrac1n\left(\tfrac1n-1\right)+\tfrac1n\left(\tfrac1n+1\right)\right]\textstyle\sum_{i=1}^n v_i(t)\\&=\tfrac{1}{n^2}\textstyle\sum_{i=1}^n v_i(t)=2\nu(t).\end{aligned}$$

This yields $\nu - u_m(t) = -\nu - u_k(t)$. We have that $G_1(x) = G_2(x)^{-1}$. By (4.21),

$$\begin{aligned}\tilde{X}(t) &= \tfrac{1}{2}\left(G_1(\tilde{S}(t))e^{-u_k(t)} + G_2(\tilde{S}(t))e^{-u_m(t)}\right)\\ &= e^{-\nu(t)}\tfrac{1}{2}\Big(G_1(\tilde{S}(t))\exp\{\nu(t) - u_1(t)\}\\ &\qquad + \exp\{\nu(t) - u_2(t)\}G_2(\tilde{S}(t))^{-1}\Big).\end{aligned}$$

Apply the elementary inequality $y^{-1} + y \geq 2$ ($\forall y > 0$) for all k. It yields $\tilde{X}(t) \geq e^{-\nu(t)}$. Hence (4.22) holds.

By (4.1), we have that

$$S_i(t) = S_i(0)\exp\left(\int_0^t a_i(s)ds + \int_0^t \sigma_i(s)dw_i(s) - \frac{v_i(t)}{2}\right), \quad i = 1, \dots, N.$$

Furthermore, the corresponding equations (4.19) may be rewritten as

$$u_1(t) = \left(\frac{1}{2n^2} - \frac{1}{2n}\right)\sum_{i=1}^{n} v_i(t).$$

If $a(\cdot) \in \mathcal{A}$, then $\alpha_i(t)$ is nonrandom for all i. In this case, (4.19) yields

$$\begin{aligned}&\mathbf{E}G_k(\tilde{S}(t))e^{-u_k(t)}\\ &= \mathbf{E}\exp\textstyle\sum_{i=1}^n\left(\frac{\alpha_i(t)}{n} - \frac{v_i(t)}{2n} + \frac{1}{n}\int_0^t \sigma_i(s)dw_i(s) - u_k(t)\right)\\ &= \exp\textstyle\sum_{i=1}^n \frac{\alpha_i}{n}\mathbf{E}\exp\sum_{i=1}^n\left(\frac{1}{n}\int_0^t \sigma_i(s)dw_i(s) - \frac{1}{n^2}\frac{v_i(t)}{2}\right)\\ &= \exp\textstyle\sum_{i=1}^n \frac{\alpha_i}{n} = e^{A_k(t)}.\end{aligned}$$

The expectation there is a standard expectation of an exponent of a summa of stochastic integrals (see, e.g., Karatzas and Shreve (1988), p. 191). We have that $A_2(t) = -A_1(t)$. Hence

$$\begin{aligned}\mathbf{E}\tilde{X}(t) &= \tfrac{1}{2}\mathbf{E}\left(G_1(\tilde{S}_t)e^{-u_1(t)} + G_2(\tilde{S}_t)e^{-u_2(t)}\right)\\ &= \tfrac{1}{2}\left(e^{A_1(t)} + e^{-A_1(t)}\right),\end{aligned}$$

and (4.23) holds. This completes the proof of Theorem 4.4. □

Proof of Theorem 4.5. The second inequalities in (4.28)–(4.30) hold by Corollary 4.2. Recall that by (4.19)

$$u_1(T) = \left(\tfrac{1}{2n^2} - \tfrac{1}{2n}\right)\textstyle\sum_{i=1}^n v_i(T), \quad u_2(T) = \left(\tfrac{1}{2n^2} + \tfrac{1}{2n}\right)\sum_{i=1}^n v_i(T).$$

Set

$$\eta_i \triangleq \int_0^T \sigma_i(t)dw_i(t).$$

We have that

$$\begin{aligned}\tilde{X}^{(n)}(T) = &\ \tfrac{1}{2}\Bigg[\left(\tfrac{S_1(T)S_2(T)\cdots S_n(T)}{S_1(0)S_2(0)\cdots S_n(0)}\right)^{1/n}\exp\{-u_{2^k-1}(T)-rT\} \\ &+\left(\tfrac{S_1(0)S_2(0)\cdots S_n(0)}{S_1(T)S_2(T)\cdots S_n(T)}\right)^{1/n}\exp\{-u_0(T)+rT\}\Bigg] \\ &=\tfrac{1}{2}\Bigg[\exp\textstyle\sum_{i=1}^n\left(\tfrac{\alpha_i(T)}{n}-\tfrac{v_i(T)}{2n^2}+\eta_i\right) \\ &\qquad+\exp\textstyle\sum_{i=1}^n\left(-\tfrac{\alpha_i(T)}{n}-\tfrac{v_i(T)}{2n^2}-\eta_i\right)\Bigg].\end{aligned} \tag{4.38}$$

Hence

$$\mathbf{E}\tilde{X}^{(n)}(T) = \cosh\left(\frac{1}{n}\sum_{i=1}^n \alpha_i(T)\right).$$

The expectation here is a standard expectation of products of exponents of stochastic integrals η_i. This completes the proof of Theorem 4.5 (i).

Furthermore, by (4.38),

$$\tilde{X}^{(n)}(T) \geq e^{-\nu(T)}\cosh(\psi_n+\varphi_n),$$

where

$$\psi_n \triangleq \frac{1}{n}\sum_{i=1}^n \alpha_i(T), \quad \varphi_n \triangleq \frac{1}{n}\sum_{i=1}^n \eta_i.$$

By (4.25),

$$\begin{cases}\nu(T)\to 0, \\ \mathbf{E}|\varphi_n|^2 = \frac{1}{n^2}\mathbf{E}\left|\sum_{i=1}^n\int_0^T\sigma_i(t)dw_i(t)\right|^2 \leq \frac{\bar{v}}{n}\to 0\end{cases} \quad \text{as} \quad n\to+\infty. \tag{4.39}$$

Let $\bar{n}>0$, $\varepsilon_3\in(0,\theta)$ be such that

$$\begin{cases} e^{\nu(T)}(\cosh(\theta)-\varepsilon_1) \leq \cosh(\theta-\varepsilon_3), \\ \mathbf{P}(|\varphi_n|\geq\varepsilon_3) < \varepsilon_2, \\ |\psi_n| \geq \theta \end{cases} \quad \text{a.s.} \quad \forall n>\bar{n}.$$

These $\bar{n}$, ε_3 do exist by (4.39). For Theorem 4.5 (ii), $\bar{n}$ depends on $a(\cdot)$. For Theorem 4.5 (iii), $\bar{n}$ depends on $\mathcal{A}_u$. We have that

$$\begin{aligned}
&\mathbf{P}\left(\tilde{X}^{(n)}(T) \ge (\cosh(\theta) - \varepsilon_1)\right) \\
&\ge \mathbf{P}\left(e^{-\nu(T)} \cosh(\psi_n + \varphi_n) \ge \cosh(\theta) - \varepsilon_1\right) \\
&\ge \mathbf{P}\left(\cosh(\psi_n + \varphi_n) \ge \cosh(\theta - \varepsilon_3)\right) \\
&\ge \mathbf{P}\left(|\psi_n + \varphi_n| \ge \theta - \varepsilon_3\right) \\
&\ge \mathbf{P}\left(|\psi_n| \ge \theta,\ |\varphi_n| < \varepsilon_3\right) \\
&\ge \mathbf{P}\left(|\psi_n| \ge \theta\right) - \mathbf{P}\left(|\varphi_n| \ge \varepsilon_3\right) \\
&\ge 1 - \varepsilon_2.
\end{aligned}$$

This completes the proof of Theorem 4.5. □

III

OPTIMAL STRATEGIES FOR THE DIFFUSION MARKET MODEL WITH OBSERVABLE PARAMETERS

Chapter 5

OPTIMAL STRATEGIES WITH DIRECT OBSERVATION OF PARAMETERS

Abstract We consider an optimal investment problem for a market consisting of a risk-free bond or bank account and a finite number of risky stocks with correlated stock prices. It is assumed that the stock prices evolve according to an Itô's stochastic differential equation. The parameters (interest rate, appreciation rate, and volatilities) need not be adapted to the driving Brownian motion, so the market is incomplete. The main assumption here is that all the market parameters, including the appreciation rates, are directly observable. That the condition is restrictive.

5.1. The market model

Consider the continuous-time market model introduced in Section 1.3, Chapter 1. This market consists of a risk-free bond or bank account with the price $B(t), t \geq 0$, and n risky stocks with prices $S_i(t), t \geq 0, i = 1, 2, \dots, n$, where $n < +\infty$ is given. The prices of the stocks evolve according to

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right), \quad t > 0, \tag{5.1}$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given nonrandom constant. The price of the bond evolves according to

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right), \tag{5.2}$$

where $B(0)$ is a given constant that we take to be 1 without loss of generality, and $r(t)$ is a random process of risk-free interest rate.

We assume that $\{w(t)\}_{0 \leq t \leq T}$ is a standard Wiener process and that $a(t)$, $r(t)$, and $\sigma(t)$ are uniformly bounded, measurable random process, independent of future increments of w, such that $c_1\mathbf{I}_n \leq \sigma(t)\sigma(t)^\top$, where $c_1 > 0$ is a constant and $\mathbf{I}_n$ is the identity matrix in $\mathbf{R}^{n\times n}$. Under these assumptions the solution of (5.1) is well defined, but the market is incomplete.

In Section 1.3, we introduced the processes $\tilde{a}(t) \triangleq a(t) - r(t)\mathbf{1}$, $\mu(t) \triangleq (r(t), \tilde{a}(t), \sigma(t))$ and the filtration $\{\mathcal{F}_t^a\}_{0\leq t\leq T}$, generated by the process $(S(t), \mu(t))$ completed with the null sets of $\bar{\mathcal{F}}$. By (1.12), it follows that $\{\mathcal{F}_t^a\}$ coincides with the filtration generated by the processes $(w(t), \mu(t))$.

It is easy to see that $\mathcal{F}_t^a$ coincides with the filtration generated by the processes $(\tilde{S}(t), \mu(t))$, where

$$p(t) \triangleq \exp\left(-\int_0^t r(s)ds\right) = B(t)^{-1}, \quad \tilde{S}(t) \triangleq p(t)S(t).$$

Let $w_*(t)$, $\mathcal{Z}_*(t)$, $\mathcal{Z}(t)$ and $\mathbf{P}_*$ be such as in Section 1.3. Note that $2\mathbf{E}_* \log \mathcal{Z}(T) = -J$.

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$, $X(0) = X_0$. We assume that

$$X(t) = \pi_0(t) + \sum_{i=1}^n \pi_i(t), \tag{5.3}$$

where the pair $(\pi_0(t), \pi(t))$ describes the portfolio at time t. The process $\pi_0(t)$ is the investment in the bond, $\pi_i(t)$ is the investment in the ith stock, and $\pi(t) = (\pi_1(t), \dots, \pi_n(t))^\top$, $t \geq 0$.

The process $\tilde{X}(t) \triangleq p(t)X(t)$ is called the normalized wealth.

DEFINITION 5.1 *Let $\mathcal{G}_t$ be a filtration. Let $\tilde{\Sigma}(\mathcal{G}_\cdot)$ be the class of all $\mathcal{G}_t$-adapted processes $\pi(\cdot)$ such that for a sequence of stopping times, $\{T_k\}$ with $T_k \uparrow T$ a.s.*

- $\int_0^{T_k} \left(|\pi(t)^\top \tilde{a}(t)| + |\pi(t)^\top \sigma(t)|^2\right) dt < \infty$ *a.s.*
- $X(T) \triangleq \lim_{k\to\infty} X(T_k)$ *exists a.s.*
- $\mathbf{E}_* \tilde{X}(T) = X_0$.

A process $\pi(\cdot) \in \tilde{\Sigma}(\mathcal{G}_\cdot)$ is said to be an *admissible* strategy with corresponding wealth $X(\cdot)$. Of course if the first condition in Definition 8.2 holds with $T_k = T$, then the other two are redundant. It turns out that the replicating strategies we use are given by the first spatial derivative of the solution of the heat equation and therefore may not be sufficiently regular at $t = T$ to allow us to take $T_k = T$. For an admissible strategy $\pi(\cdot)$, $X(t, \pi(\cdot))$ denotes the corresponding total wealth and $\tilde{X}(t, \pi(\cdot))$ the corresponding normalized total wealth.

Optimal investment problem

Let $T > 0$, $\hat{D} \subset \mathbf{R}$, and $X_0 \in \hat{D}$ be given. Let $U(\cdot) : \hat{D} \to \mathbf{R} \cup \{-\infty\}$ such that $U(X_0) > -\infty$.

We may state our problem as follows: Find an admissible self-financing strategy $\pi(\cdot)$ that solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T, \pi(\cdot))) \quad \text{over} \quad \pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}^a_\cdot) \tag{5.4}$$

$$\text{subject to} \begin{cases} \tilde{X}(0, \pi(\cdot)) = X_0, \\ \tilde{X}(T, \pi(\cdot)) \in \hat{D} \quad \text{a.s.} \end{cases} \tag{5.5}$$

5.2. Solution via dynamic programming

We solve here the optimal portfolio selection problem, but in our incomplete market. We find that in our setting there is no hedging of the coefficients. Let us explain. In the setting generally assumed in finance, cf. Merton (1990), Sec. 15.5, the coefficients, $\mu = (r, \tilde{a}, \sigma)$, are assumed to satisfy an Itô equation with driving Wiener process (Brownian motion) $(w(\cdot), \tilde{w}(\cdot))$, where $\tilde{w}(\cdot)$ is a Wiener process that is independent on $w(\cdot)$, i.e.

$$\begin{aligned} d\mu(t) = \beta(B(t), S(t), \mu(t), t)dt \quad &+\sigma^{\mu,S}(B(t), S(t), \mu(t), t)dw(t) \\ &+\sigma^{\mu}(B(t), S(t), \mu(t), t)d\tilde{w}(t). \end{aligned} \tag{5.6}$$

To Markovianize the problem, we use the state variables $X(t), B(t), S(t)$, and $\mu(t)$.

DEFINITION 5.2 *Let $\tilde{\Sigma}_M$ be the class of all $\mathcal{F}^a_t$-adapted processes $\pi(\cdot)$ such that there exists a measurable function $f : [0, T] \times \mathbf{R} \times (\mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n}) \to \mathbf{R}^n$ such that $\pi(t) = f(t, X(t), B(t), S(t), \mu(t))$.*

A process $\pi(\cdot) \in \tilde{\Sigma}_M$ is said to be a *Markov* strategy. Set

$$\begin{aligned} &J(x, b, s, \mu, t) \triangleq \\ &\sup\nolimits_{\pi(\cdot) \in \tilde{\Sigma}_M} \mathbf{E}\{U(B(T)^{-1}X(T)) | (X(t), B(t), S(t), \mu(t)) = (x, b, s, \mu)\}. \end{aligned}$$

Let $\hat{D} = \mathbf{R}$. Then the Bellman equation, satisfied formally by the value function (derived utility function) $J(x, b, s, \mu, t)$ is (if we denote the matrix diag $(s_1, \ldots, s_n)$ by $\mathbf{S}$)

$$\begin{aligned} \max_{\pi}\{&J_t(x, b, s, \mu, t) + J_x(x, b, s, \mu, t)[rx + \pi^\top \tilde{a}] + J_b(x, b, s, \mu, t)rb \\ &+J_s(x, b, s, \mu, t)^\top \mathbf{S}\tilde{a}) + J_\mu(x, b, s, \mu, t)^\top \beta(x, b, s, \mu, t) \end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}J_{x,x}(x,b,s,\mu,t)\pi^\top\sigma\sigma^\top\pi+\tfrac{1}{2}\mathrm{tr}\,[J_{s,s}(x,b,s,\mu,t)\mathbf{S}\sigma\sigma^\top\mathbf{S}]\\
&+\tfrac{1}{2}\mathrm{tr}\,[J_{\mu,\mu}(x,b,s,\mu,t)(\sigma^{\mu,S}(x,b,s,\mu,t)\sigma^{\mu,S}(x,b,s,\mu,t)^\top\\
&+\sigma^{\mu}(x,b,s,\mu,t)\sigma^{\mu}(x,b,s,\mu,t)^\top)]+J_{x,s}(x,b,s,\mu,t)\mathbf{S}\sigma\sigma^\top\pi\\
&+J_{x,\mu}(x,b,s,\mu,t)\sigma^{\mu,S}\sigma^\top\pi\\
&+\mathrm{tr}\,[J_{s,\mu}(x,b,s,\mu,t)\sigma^{\mu,S}(x,b,s,\mu,t)\sigma^\top\mathbf{S}]\}=0,
\end{aligned}
$$

$$J(x,b,s,\mu,T)=U(x/b).$$

Then the optimal π is (formally)

$$
\begin{aligned}
\pi(t)= \; & -\tfrac{J_x(X(t),B(t),S(t),\mu(t),t)}{J_{x,x}(X(t),B(t),S(t),\mu(t),t)}Q(t)\tilde{a}(t)\\
& -\mathbf{S}(t)\tfrac{J_{s,x}(X(t),B(t),S(t),\mu(t),t)}{J_{x,x}(X(t),B(t),S(t),\mu(t),t)}\\
& -Q(t)\sigma(t)\sigma^{\mu,S}(X(t),B(t),S(t),\mu(t),t)^\top\tfrac{J_{\mu,x}(X(t),B(t),S(t),\mu(t),t)}{J_{x,x}(X(t),B(t),S(t),\mu(t),t)}.
\end{aligned}
$$

The first term on the right-hand side gives the usual mean-variance type of strategy, the second, due to correlation between wealth and stock prices, is absent if $S(t)$ is not required as a state variable, e.g., if a Mutual Fund theorem hold; and the third depends on the correlation between S (or w) and μ and is considered to represent a hedge against future unfavorable behavior of the coefficients. Note that the Bellman equation is degenerate: the coefficient matrix for the second-order derivatives has a rank of at most $1+2n+n^2$, whereas there are $3+2n+n^2$ variables. The difference of 2 in the numbers arises from including $B(t)$ as a state variable (this might be avoided in some cases) and from the fact that the noise driving $X(t)$ is the same as that driving $S(t)$. This outcome is unavoidable. Hence there may not exist a solution J with second-order derivatives. If μ is independent of w, then $\sigma^{\mu,S}=0$ and B,S can be dropped as state variables. In this case the coefficients are said to be unhedgeable and the policy "myopic".

5.3. Solution via optimal claim

Set

$$
\begin{aligned}
&\theta(t)\triangleq\sigma(t)^{-1}\tilde{a}(t),\\
&R(t)\triangleq\tilde{a}(t)^\top Q(t)\tilde{a}(t)=|\theta(t)|^2,\\
&R\triangleq\textstyle\int_0^T R(t)dt,\\
&\bar{R}\triangleq\tfrac{R}{T},\\
&J\triangleq\mathbf{E}R,\\
&\tau(t)\triangleq\bar{R}^{-1}\textstyle\int_0^t R(s)\,ds.
\end{aligned}
\tag{5.7}
$$

We shall present a solution of the investment problem that allows us allow to deal with coefficients that are not necessarily generated by a diffusion. We show rigorously that in the isoelastic utility case, i.e., when $U(x) = \delta^{-1}x^{\delta}$ ($\delta < 1$, $\delta \neq 0$) or $U(x) = \log x$, the correlation terms are absent *provided* (in the power case) that the random variable R is independent of w. We do not require that the coefficients μ be independent of w. Moreover, *if R is constant*, then again these terms are absent without specification of the form of U.

5.3.1 Some additional assumptions

CONDITION 5.1 *There exists a measurable set $\Lambda \subseteq \mathbf{R}$, and a measurable function $F(\cdot,\cdot) : (0,\infty) \times \Lambda \to \hat{D}$ such that for each $z > 0$, $\hat{x} = F(z,\lambda)$ is a solution of the optimization problem*

$$\text{Maximize} \quad zU(x) - \lambda x \quad \text{over } x \in \hat{D}. \tag{5.8}$$

Moreover, this solution is unique for a.e. $z > 0$.

CONDITION 5.2 *There exist $\hat{\lambda} \in \Lambda$, $C > 0$, and $c_0 \in (0, 1/(2J))$ such that $F(\cdot,\hat{\lambda})$ is piecewise continuous on $(0,\infty)$, $F(\mathcal{Z}(T),\hat{\lambda})$ is $\mathbf{P}_*$-integrable, and*

$$\begin{cases} \mathbf{E}_*\{F(\mathcal{Z}(T),\hat{\lambda})\} = X_0, \\ |F(z,\hat{\lambda})| \leq C z^{c_0 \log z} \quad \forall z > 0. \end{cases} \tag{5.9}$$

CONDITION 5.3 *At least one of the following conditions holds:*

(i) $U(x) \equiv \log(x)$ *and* $\hat{D} = [0,+\infty)$; *or*

(ii) The random variable R is constant.

CONDITION 5.4 $F(x,\lambda) = C_1 \left(\frac{x}{\lambda}\right)^{\nu} + C_0$, *where $C_1 \neq 0$, C_0 and $\nu \neq 0$ are constants, and the random variable R and the process $w(\cdot)$ are independent.*

REMARK 5.1 *It is clear that Condition 5.1 is required to allow maximization of the Lagrangian. Condition 5.2 ensures that the optimal terminal wealth is replicable, and Condition 5.3 allows us to find the optimal, i.e., replicating, strategy explicitly. Condition 5.4 is a useful weakening of Condition 5.3(ii) for special utility functions.*

5.3.2 Special cases

It is easy to see that these conditions are satisfied in many examples.

LEMMA 5.1 *Conditions 5.1 and 5.2 hold in the following cases:*

(i) *Logarithmic utility.* $\hat{D} = [0, +\infty)$, $U(x) = \ln(x)$, $X_0 > 0$, $\Lambda = (0, \infty)$, $F(z, \lambda) = z/\lambda$, $\hat{\lambda} = 1/X_0$.

(ii) *Power utility. Assume* R *is constant (so* $R = J$*).* $\hat{D} = [0, +\infty)$, $U(x) = \frac{1}{\delta}x^\delta$, $X_0 > 0$, *where* $\delta < 1$, $\delta \neq 0$, $\Lambda = (0, \infty)$, $F(z, \lambda) = (z/\lambda)^{\frac{1}{1-\delta}}$, $\hat{\lambda} = X_0^{\delta-1} \exp\{\frac{\delta}{1-\delta}\frac{R}{2}\}$.

(iii) *Mean-variance utility. Assume* R *is constant.* $\hat{D} = \mathbf{R}$, $U(x) = -kx^2 + cx$, *where* $k \in \mathbf{R}$ *and* $c \geq 0$, $X_0 > 0$, $F(z, \lambda) = (c - \lambda/z)/(2k)$, $\hat{\lambda} = (c - 2kX_0)e^{-R}$.

(iv) *Assume* R *is constant.* $\hat{D} = [0, +\infty)$, $U(x) = -x^\delta + x$, *where* $\delta = 1 + 1/l$, *and* $l > 0$ *is an integer,* $X_0 > \delta^{-l}$, $\Lambda = (-\infty, 0]$, $F(z, \lambda) = (1 - \lambda/z)^l \delta^{-l}$, $\hat{\lambda}$ *is a (negative) zero of a polynomial of degree* l.

(v) *Goal-achieving utility. Assume* R *is constant.* $\hat{D} = [0, \infty)$ *and*

$$U(x) = \begin{cases} 0 & \text{if } 0 \leq x < \alpha, \\ 1 & \text{if } x \geq \alpha, \end{cases}$$

$0 < X_0 < \alpha$, $R > 0$, $\Lambda = (0, \infty)$ *and*

$$F(z, \lambda) = \begin{cases} = \alpha & \text{if } 0 < \lambda < z/\alpha, \\ \in \{0, \alpha\} & \text{if } \lambda = z/\alpha, \\ = 0 & \text{if } \lambda > z/\alpha, \end{cases}$$

and $\hat{\lambda}$ *is the solution of*

$$\Phi\left(\frac{\log \lambda}{\sqrt{R}} + \frac{\log \alpha}{\sqrt{R}} + \frac{\sqrt{R}}{2}\right) = 1 - \frac{X_0}{\alpha},$$

where Φ *is the cumulative of the normal distribution.*

For case (v) above, Condition 5.2 fails if $R = 0$. Note that the boundedness of the coefficients μ implies that R is bounded and hence $\mathbf{E}_* \mathcal{Z}(T)^q < \infty$ for any $q \in \mathbf{R}$. This outcome is sufficient for the integrability of $F(\mathcal{Z}(T), \hat{\lambda})$ in the above cases. Note also that Condition 5.4 is satisfied in cases (i)-(iii).

5.3.3 Replicating special claims

We show that claims of a certain kind (the kind that we need for our optimization problem) are among those that can be replicated in our incomplete market. Moreover, the assumption that the L^2 norm (in time) of the market price of risk is non-random allows us to exhibit the replicating strategy explicitly using a transformation of the heat equation.

LEMMA 5.2 *Let $R > 0$ be constant and let $f(\cdot) : (0, \infty) \to \mathbf{R}$ be a piecewise continuous function such that $|f(x)| \le C x^{c_0 \log x}$ ($\forall x > 0$), where $C > 0$ and $c_0 \in (0, (2J)^{-1})$ are constants.*

(a) The Cauchy problem

$$\begin{cases} \frac{\partial H}{\partial t}(x,t) + \frac{\bar{R}}{2} x^2 \frac{\partial^2 H}{\partial x^2}(x,t) = 0, \\ H(x,T) = f(x). \end{cases} \tag{5.10}$$

has a unique solution $H(\cdot) \in C^{2,1}((0,\infty) \times (0,T))$, with $H(x,t) \to f(x)$ a.e. as $t \to T-$.

(b) If in addition $f(\mathcal{Z}(T))$ is $\mathbf{P}_*$-integrable, then there exists a self-financing admissible strategy $\pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}^a_\cdot)$, with corresponding wealth $X(t)$, that replicates the claim $B(T)f(\mathcal{Z}(T))$. π and X are given by

$$\begin{aligned} \pi(t)^\top &= B(t)\frac{\partial H}{\partial x}(\mathcal{Z}(t), \tau(t))\mathcal{Z}(t)\tilde{a}(t)^\top Q(t), \\ X(t) &= B(t)H(\mathcal{Z}(t), \tau(t)), \end{aligned} \tag{5.11}$$

where the function $H(\cdot,\cdot) : (0,\infty) \times [0,T] \to \mathbf{R}$ is the solution of (5.10). Moreover

$$X_0 = \mathbf{E}_* f(\mathcal{Z}(T)), \tag{5.12}$$

Note that $C^{2,1}((0,\infty) \times (0,T))$ denotes the set of functions defined on $(0,\infty) \times (0,T)$, which are continuous and have two continuous derivatives in the first variable and one in the second.

5.3.4 Calculating the optimal strategy

Let $U^+(x) \triangleq \max(0, U(x))$ and let $F(\cdot,\cdot)$ be as in Condition 5.1.

THEOREM 5.1 (i) *Let $R = 0$. Then the trivial strategy $\pi(t) \equiv 0$ is the unique optimal strategy for the problem (5.4)–(5.5).*

(ii) *Assume Conditions 5.1, 5.2, and 5.3, and let $R > 0$.*

(a) *If Condition 5.3(i) holds, let $H(x,t) \triangleq x/\hat{\lambda}$; and*

(b) *if Condition 5.3(ii) holds, let $H(x,t)$ be the solution of (5.10) for $f(\cdot) = F(\cdot, \hat{\lambda})$.*

Then the strategy $\pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}^a_\cdot)$, defined as

$$\pi(t)^\top = B(t)\frac{\partial H}{\partial x}(\mathcal{Z}(t), \tau(t))\mathcal{Z}(t)\tilde{a}(t)^\top Q(t), \tag{5.13}$$

is optimal for the problem (5.4)-(5.5), and $X(t) = B(t)H(\mathcal{Z}(t), \tau(t))$ *is the corresponding wealth with* $X(0) = X_0$. *This strategy replicates the claim* $B(T)F(\mathcal{Z}(T), \hat{\lambda})$. *If* $\mathbf{E}\, U^+(F(\mathcal{Z}(T), \hat{\lambda})) < +\infty$ *and either the uniqueness in Condition 5.1 holds for all z or the law of* $\mathcal{Z}_*(T)$ *has a density, then the optimal strategy is unique.*

REMARK 5.2 Even under the imposed restrictions, Theorem 5.1 is used effectively for calculation of optimal strategy for *random* R in *maximin* setting (Theorem 7.1 below).

COROLLARY 5.1 *Let* $\mu(\cdot)$ *and* $w(\cdot)$ *be independent. Then the set of all* $\hat{\lambda}$ *such that Condition 5.2 is satisfied is uniquely defined by R, and the function* $F(\cdot, \hat{\lambda})$ *does not depend on the choice of* $\hat{\lambda}$ *from this set. Moreover, the probability distribution of the optimal normalized wealth is uniquely defined by R (i.e., does not depend on* $\mu(\cdot)$ *given R).*

5.3.5 The case of myopic strategies

PROPOSITION 5.1 *Assume Conditions 5.1, 5.2, 5.3(ii) and 5.4. Then*

$$H(x,t) = C_1(\hat{\lambda})x^{\nu} \exp\left\{\frac{1}{2}\nu(\nu-1)(T-t)\bar{R}\right\} + C_0$$

is the solution of (5.10) with $f(x) = F(x, \hat{\lambda})$, *and the optimal strategy has the form*

$$\pi(t)^{\top} = \nu B(t)(\tilde{X}(t) - C_0)\tilde{a}(t)^{\top} Q(t). \qquad (5.14)$$

It now follows that the solution of the problem (5.10), and hence the optimal strategy, can be written explicitly for cases (i)-(iv) in Lemma 5.1. For cases (ii)-(iv) we require also that *R be non-random* although we relax this for cases (ii) and (iii) in the next Corollary. For case (iv), $\pi = \sum_j \pi^j$, where the π^j are expressions of the form (5.14) with corresponding $\tilde{X}^j$. In fact, $\tilde{X} = \sum_j \tilde{X}^j$ and this decomposition depends on $\hat{\lambda}$. For case (v), H can be written in terms of the normal cumulative distribution function, so again the optimal strategy can be solved explicitly provided that *R is non-random.*

COROLLARY 5.2 *Assume Conditions 5.1 and 5.4, and assume Condition 5.2* under the conditional probability given *R. Then the optimal strategy is*

$$\pi(t)^{\top} = \nu B(t)(\tilde{X}(t) - C_0)\tilde{a}(t)^{\top} Q(t), \qquad (5.15)$$

where the normalized wealth is given by

$$\begin{cases} d\tilde{X}(T) = \nu(\tilde{X}(t) - C_0)\tilde{a}(t)^{\top} Q(t)\mathbf{S}(t)^{-1} dS(t), \\ \tilde{X}(0) = X_0. \end{cases}$$

This result is a generalization of the case of "totally unhedgeable" coefficients, cf. Karatzas and Shreve (1998), Chapter 6, Example 7.4. We see that the result holds for a larger class of utility functions than just the power utility functions, and the independence of the parameters and the Brownian motion can be relaxed considerably (when utility is only derived from terminal consumption). In fact, the corollary applies to cases (i)-(iii) in Lemma 5.1.

Here is another example where our theory applies, i.e., the optimal investment strategy is myopic, although this is not apparent from the corresponding Bellman equation. Assume that $\tilde{a}(t) \equiv \tilde{a}$ is constant in t and independent of $w(\cdot)$. Let $\bar{\sigma}_i$, $i = 1, 2$, be given random matrices in $\mathbf{R}^{n\times n}$ that are independent of $w(\cdot)$, and let $\varepsilon > 0$ be fixed. Let τ' be any Markov time with respect to $\mathcal{F}_t^a$. Assume that

$$\sigma(t) \triangleq \begin{cases} \bar{\sigma}_1, & \text{if } t \notin [\tau, \tau+\varepsilon) \\ \bar{\sigma}_2, & \text{if } t \in [\tau, \tau+\varepsilon) \end{cases}, \quad \text{where} \quad \tau \triangleq \tau' \wedge (T-\varepsilon).$$

Then R does not depend on $w(\cdot)$ (though $\mu(\cdot)$ does). For suitable U, we may apply Corollary 5.2 to obtain a strategy depending only on current wealth. If $\tilde{a}$ and $\bar{\sigma}_i$ are nonrandom, then R is nonrandom, the market is still incomplete, and according to Theorem 5.1 the strategy is myopic.

REMARK 5.3 *Our result is still valid without Condition 5.3, if our nonrandom T in (5.4)-(5.5) is replaced by $T' = \inf\{t: \; R(t) = \bar{R}\}$.*

5.4. Proofs

Proof of Lemma 5.2. By assumption, $\bar{R}$ and $J = T\bar{R}$ are nonrandom. Consider the following Cauchy problem:

$$\begin{cases} \frac{\partial V}{\partial t}(y,t) + \frac{\bar{R}}{2}\frac{\partial^2 V}{\partial y^2}(y,t) = 0, \\ V(y,T) = f(e^{y-J/2}), \quad y \in \mathbf{R}. \end{cases} \tag{5.16}$$

Let

$$p(y,t) \triangleq \frac{1}{\sqrt{2\pi(T-t)\bar{R}}} \exp\left(\frac{-y^2}{2\bar{R}(T-t)}\right). \tag{5.17}$$

This is the the fundamental solution of (5.16). By assumption, we have that

$$\begin{aligned} &|f(e^y)| \le Ce^{c_0 y^2} \quad \forall y \in \mathbf{R}, \\ &c_0 < \tfrac{1}{2J} \le \tfrac{1}{2\bar{R}(T-t)} \quad \forall t \in [0,T). \end{aligned}$$

Then the integral

$$V(y,t) = \int_{-\infty}^{+\infty} p(y-y_0,t) f(e^{y_0 - J/2}) dy_0 \tag{5.18}$$

converges, and $V(y,t) \in C^{2,1}(\mathbf{R} \times (0,T))$ is a solution of the problem (5.16) such that $V(y,t) \to f(e^{y-J/2})$ a.s as $t \to T-$. Then

$$H(x,t) \triangleq V\left(\log x + \frac{\bar{R}t}{2}, t\right)$$

is a solution of (5.10) in the desired sense.

Note that $\tau(T) = T$ and $d\tau(t)/dt = R(t)/\bar{R}$. Set

$$\bar{H}(x,t) \triangleq H(x, \tau(t)).$$

Then

$$\begin{cases} \frac{\partial \bar{H}}{\partial t}(x,t) + \frac{R(t)}{2} x^2 \frac{\partial^2 \bar{H}}{\partial x^2}(x,t) = 0, \\ \bar{H}(x,T) = f(x). \end{cases} \tag{5.19}$$

We define

$$\pi(t)^\top \triangleq B(t)\tfrac{\partial \bar{H}}{\partial x}(\mathcal{Z}(t),t)\mathcal{Z}(t)\tilde{a}(t)^\top Q(t),$$
$$X(t) \triangleq B(t)\bar{H}(\mathcal{Z}(t),t).$$

Let us show that X is the wealth, i.e. $X(t) = X(t, \pi(\cdot))$. Applying Itô's formula to $B(t)\bar{H}(\mathcal{Z}(t),t)$ and using (1.16), gives

$$dX(t) = r(t)X(t)\,dt + B(t)\frac{\partial \bar{H}}{\partial x}(\mathcal{Z}(t),t)\mathcal{Z}(t)\theta(t)^\top\,dw_*(t),$$

which is equivalent to (8.5) under our definition of π; hence X is the wealth corresponding to π. If π is admissible, then it replicates the claim $B(T)f(\mathcal{Z}(T)$, with initial wealth X_0.

The integrability of π, cf. Definition 8.2, follows if we take $T_k = T - 1/k$ and observe that $\frac{\partial \bar{H}}{\partial x}(\mathcal{Z}(t),t)$ is bounded pathwise for $t \le T_k$, since $\frac{\partial \bar{H}}{\partial x}(x,t)$ is continuous on $(0,\infty) \times (0,T)$. The second requirement of Definition 8.2 follows from the continuity of $X(t) = B(t)\bar{H}(\mathcal{Z}(t),t)$, and the last follows from (5.12). So π is admissible.

It remains to establish (5.12). Let

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{H}_k(x,t)$ be the corresponding solution of (5.19) and let $X_k(t)$ be the corresponding wealth. Then $\tilde{X}_k(t) = \bar{H}_k(\mathcal{Z}(t),t)$. Since

$$\lim_{t \uparrow T} \bar{H}_k(\mathcal{Z}(t),t) = \bar{H}_k(\mathcal{Z}(T),T) = f_k(\mathcal{Z}(T)) \quad \text{a.s.},$$

and since $|\bar{H}_k(x,t)| \leq k$, then by dominated convergence,

$$\lim_{t \uparrow T} \mathbf{E}_* \tilde{X}_k(t) = \mathbf{E}_* f_k(\mathcal{Z}(T)).$$

Since $\tilde{X}_k$ is a martingale for $t < T$, we then have $X_k(0) = \mathbf{E}_* f_k(\mathcal{Z}(T))$. Further,

$$|\bar{H}(1,0) - \bar{H}_k(1,0)| = \int_{\{z:|f(e^{z-J/2})|>k\}} p(-z,0)|f(e^{z-J/2})|dz \to 0,$$

since $\bar{H}(1,0) = \int p(-z,0)|f(e^{z-J/2})|dz < \infty$, cf. (5.17)-(5.18). Hence

$$\begin{aligned} X(0) &= \bar{H}(1,0) = \lim_k \bar{H}_k(1,0) \\ &= \lim_k X_k(0) = \lim_k \mathbf{E}_* f(\mathcal{Z}(T)) = \mathbf{E}_* f(\mathcal{Z}(T)). \end{aligned}$$

This completes the proof of Lemma 5.2. □

Let us establish a similar result in the case where Condition 5.3(i) holds.

LEMMA 5.3 *If Condition 5.3(i), is satisfied then*

$$\begin{aligned} &\hat{\xi} \triangleq F(\mathcal{Z}(T), \hat{\lambda}) = \mathcal{Z}(T) X_0, \\ &\hat{\lambda} = X_0^{-1}, \\ &\mathbf{E} U(\hat{\xi}) = J/2 + \log X_0, \end{aligned}$$

and $B(T)\hat{\xi}$ is replicated by π, where

$$\pi(t) = B(t) X_0 \mathcal{Z}(t) \tilde{a}(t)^\top Q(t).$$

Proof. $F(z,\lambda) = z/\lambda$. From (5.9), we have $\hat{\lambda} = X_0^{-1}$. Now $\mathbf{E}U(\hat{\xi}) = \mathbf{E}\log[X_0 \mathcal{Z}(T)] = J/2 + \log X_0$. Let $\tilde{X}(t) \triangleq \mathcal{Z}(t) X_0$. Then

$$\tilde{X}(t) = X_0 + \int_0^T X_0 \mathcal{Z}(s) \theta(s)^\top dw_*(s),$$

and this is the normalized wealth corresponding to π as given, cf. (1.20). □

Let $U^-(x) \triangleq \max(0, -U(x))$; define a set of claims by

$$\Psi \triangleq \{\xi : \xi \text{ is } \mathcal{F}_T\text{-measurable}, \mathbf{E}_*|\xi| < +\infty, \ \mathbf{E}U^-(\xi) < \infty\},$$

and define $\mathcal{J}_i : \Psi \to \mathbf{R}$, $i = 0, 1$ by

$$\mathcal{J}_0(\xi) \triangleq \mathbf{E}U(\xi), \quad \mathcal{J}_1(\xi) \triangleq \mathbf{E}_*\xi - X_0.$$

Let us now define the claims *attainable in* $\hat{D}$,

$$\Psi_0 \triangleq \{\xi \in \Psi : \xi = \tilde{X}(T, \pi(\cdot)) \in \hat{D} \text{ a.s., } \pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}_\cdot^a)\}.$$

Consider the problem

$$\text{Maximize} \quad \mathcal{J}_0(\xi) \quad \text{over} \quad \xi \in \Psi_0. \tag{5.20}$$

PROPOSITION 5.2 *Assume Conditions 5.1, 5.2, and 5.3. The optimization problem (5.20) has the solution* $\hat{\xi} = F(\mathcal{Z}(T), \hat{\lambda})$.

Proof. From Condition 5.2, it follows that $\mathbf{E}_*|\hat{\xi}| < \infty$ and $\mathbf{E}_*\hat{\xi} = X_0$. Let us show that $\mathbf{E}U^-(\hat{\xi}) < \infty$. For $k = 1, 2, \ldots$, introduce the random events

$$\Omega^{(k)} \triangleq \{-k \le U(\hat{\xi}) \le 0\},$$

along with their indicator functions $\chi^{(k)}$, respectively. The number $\hat{\xi}$ achieves the unique maximum of the function $\mathcal{Z}(T)U(\xi) - \hat{\lambda}\xi$ over the set $\hat{D}$, and $X_0 \in \hat{D}$. Hence for all $k = 1, 2, \ldots$,

$$\begin{aligned} \mathbf{E}\chi^{(k)}\left(U(\hat{\xi}) - \mathcal{Z}_*(T)\hat{\lambda}\hat{\xi}\right) &\ge \mathbf{E}\chi^{(k)}\left(U(X_0) - \mathcal{Z}_*(T)\hat{\lambda}X_0\right) \\ &\ge -|U(X_0)| - |\hat{\lambda}X_0|. \end{aligned}$$

Since $\mathbf{E}\mathcal{Z}_*(T)|\hat{\xi}| = \mathbf{E}_*|\hat{\xi}| < +\infty$, then $\mathbf{E}U^-(\hat{\xi}) < \infty$, and hence $\hat{\xi} \in \Psi$.

Let $L(\xi, \lambda) \triangleq \mathcal{J}_0(\xi) - \lambda\mathcal{J}_1(\xi)$, where $\xi \in \Psi$ and $\lambda \in \mathbf{R}$. We have

$$L(\xi, \lambda) = \mathbf{E}\left(U(\xi) - \lambda\mathcal{Z}_*(T)\xi\right) + \lambda X_0. \tag{5.21}$$

Let

$$\eta(\lambda) \triangleq F(\mathcal{Z}(T), \lambda) = F(\mathcal{Z}_*(T)^{-1}, \lambda). \tag{5.22}$$

From Condition 5.1, it follows that for any $\omega \in \Omega$, the random number $\eta(\lambda)$ provides the maximum in the set $\hat{D}$ for the function under the expectation in (5.21).

Lemmas 5.2 and 5.3 imply the attainability of $\hat{\xi}$, so $\hat{\xi} \in \Psi_0$. Furthermore,

$$L(\xi, \hat{\lambda}) \le L(\hat{\xi}, \hat{\lambda}) \quad \forall \xi \in \Psi. \tag{5.23}$$

Let $\xi \in \Psi_0$ be arbitrary. We have that $\mathcal{J}_1(\xi) = 0$ and $\mathcal{J}_1(\hat{\xi}) = 0$. Then

$$\begin{aligned} \mathcal{J}_0(\xi) - \mathcal{J}_0(\hat{\xi}) = \mathcal{J}_0(\xi) + \hat{\lambda}\mathcal{J}_1(\xi) \quad &-\mathcal{J}_0(\hat{\xi}) - \hat{\lambda}\mathcal{J}_1(\hat{\xi}) \\ &= L(\xi, \hat{\lambda}) - L(\hat{\xi}, \hat{\lambda}) \le 0. \end{aligned}$$

Hence $\hat{\xi}$ is an optimal solution of the problem (5.20). □

The following Lemma will prepare us for the proof of Theorem 5.1.

LEMMA 5.4 *Assume Conditions 5.1, 5.2, and 5.3. Let* $\hat{\xi} \triangleq F(\mathcal{Z}(T), \hat{\lambda})$. *Then*

(i) $\mathbf{E}\, U^-(\hat{\xi}) < \infty$, $\hat{\xi} \in \hat{D}$ *a.s.; and*

(ii) *if* $\mathbf{E}\, U^+(\hat{\xi}) < +\infty$ *and either the uniqueness in Condition 5.1 holds for all z or the law of* $\mathcal{Z}(T)$ *has a density, then* $\hat{\xi}$ *is unique (i.e., even for different* $\hat{\lambda}$*, the corresponding* $\hat{\xi}$ *agree a.s.).*

Proof. Part (i) is seen to hold from the proof of Proposition 5.2.

To show (ii), note that if $\mathbf{E}U^+(\hat{\xi}) < +\infty$, then $L(\hat{\xi}, \hat{\lambda}) < +\infty$. Let $\xi' \in \Psi_0$ be an optimal solution of the problem (5.20). Let $\hat{\lambda}$ be any number such that (5.9) holds. It is easy to see that

$$L(\xi', \hat{\lambda}) = \mathcal{J}_0(\xi') \geq \mathcal{J}_0(\hat{\xi}) = L(\hat{\xi}, \hat{\lambda}).$$

From Condition 5.1 it follows that $\hat{\xi} = \eta(\hat{\lambda})$ provides the maximum in the set $\hat{D}$ of the function under the expectation in (5.21) with $\lambda = \hat{\lambda}$. Hence both ξ' and $\hat{\xi}$ maximize $L(\cdot, \hat{\lambda})$. It follows that ξ' must also maximize the function under the expectation in (5.21) a.s., and hence $\xi' = \eta(\hat{\lambda})$ a.s. from the uniqueness assertion in Condition 5.1. Thus (ii) is satisfied. This completes the proof of Lemma 5.4. □

Proof of Theorem 5.1. If $R = 0$, then $\mathcal{Z}_*(T) = 1$, and hence the optimal claim $\hat{\xi}$ is nonrandom. The only strategy that replicates the nonrandom claim is the trivial risk-free strategy, and (i) follows. From Lemmas 5.3 and 5.2, it follows that π is admissible and replicates $B(T)\hat{\xi}$. The optimality follows from Proposition 5.2.

Let us show the uniqueness of the optimal strategy. By Lemma 5.4 (ii), $\hat{\xi}$ is the unique solution of the auxiliary problem (5.20). Hence if π and π' are two optimal strategies, they must both lead to the same wealth at time T. If we set $Y(t) = X(t, \pi'(\cdot)) - X(t, \pi(\cdot))$, then from (8.5) we obtain

$$\begin{cases} dY(t) = r(t)Y(t)\,dt + (\pi'(t) - \pi(t))^\top \sigma(t)\,dw_*(t), \\ Y(T) = 0. \end{cases}$$

Hence, given μ, $(Y(t), (\pi'(t) - \pi(t))^\top \sigma(t))$ is a solution of the corresponding backward stochastic differential equation

$$\begin{cases} dY(t) = r(t)Y(t)\,dt + y(t)\,dw_*(t), \\ Y(T) = 0. \end{cases}$$

The theory of such equations cf. Yong and Zhou (1999), Theorem 2.2, p. 349 implies that the equation has a unique solution, $(Y, y) \equiv (0, 0)$. It follows that $\pi' = \pi$ a.e. a.s. This completes the proof of Theorem 5.1. □

Proof of Corollary 5.1. Under Condition 5.3 (i) the minimum is unique, and under Condition 5.3 (ii), $\mathcal{Z}_{*\hat{f}}(T)$ under $\mathbf{P}$ is conditionally log-normal given μ with parameters depending only on the constant R, and hence is unconditionally log-normal, with parameters depending only on the constant R. Then the proof follows. □

Proof of Proposition 5.1. Direct verification shows that H is a solution of (5.10) and

$$\frac{\partial H(x,t)}{\partial x}x = \nu[H(x,t) - C_0].$$

The result follows. □

Proof of Corollary 5.2. Let $\tilde{\Sigma}^R(\mathcal{F}^a_\cdot)$ be the enlargement of $\tilde{\Sigma}(\mathcal{F}^a_\cdot)$ produced by replacing the filtration $\mathcal{F}^a_t$ by $\mathcal{F}^R_t$ generated by $\mathcal{F}^a_t$ and R in the definition. With $\mathbf{P}(\cdot)$ replaced by $\mathbf{P}(\cdot\,|\mathcal{F}^R_0)$, we may apply Proposition 5.1 to obtain the optimal π in the feedback form (5.14). It follows that π lies in the smaller control set $\tilde{\Sigma}(\mathcal{F}^a_\cdot)$; therefore, π is optimal in this class, and hence optimal for the original problem. □

Chapter 6

OPTIMAL PORTFOLIO COMPRESSION

Abstract In this chapter, we consider the problem of optimal portfolio compression. By this term we mean that admissible strategies may include no more than m different stocks concurrently, where m may be less than the total number n of available stocks.

6.1. Problem statement and definitions

Consider the market model from Chapters 1 and 5. We shall use all definitions and notations from these chapters. We study now the portfolio compression problem when admissible strategies may include no more than m different stocks concurrently, where m may be less than the total number n of available stocks. Although this problem has not been treated extensively in the literature, it is of interest to the investor. It is obviously not realistic to include *all* available stocks in the portfolio; the total number of assets in the market is too large. In fact, the number of stocks in the portfolio should be limited by the equity in the account because of the need to have a large enough position in each stock so that management fees and commissions are only a small proportion of the value of the portfolio. There is no point in having too many stocks in a small portfolio. Even in a large portfolio, it makes sense to limit the number of stocks.

Furthermore, it should be pointed out that the optimal solution of the investment problem with fixed and finite number of assets may be useless for a large market with infinite or very large number of assets. The following example demonstrates that the optimal solution of investment problem from Chapter 5 need to be revised for a case when $n \to +\infty$.

EXAMPLE 6.1 *Consider the problem (5.4)–(5.5) with* $U(x) \equiv \log x$ *for an extending set of underlying assets, i.e. when* $n \to +\infty$. *Let* $\tilde{a}_i(t) = \tilde{a}_i^{(n)}(t)$ *and* $\sigma(t) = \sigma^{(n)}(t)$ *be the corresponding appreciation rates and volatilities.*

N. Dokuchaev, *Dynamic Portfolio Strategies: Quantitative Methods and Empirical Rules for Incomplete Information*

Assume that

$$\sum_{i=1}^{n} \tilde{a}_i(0) \to +\infty \quad as \quad n \to +\infty \quad a.s. \tag{6.1}$$

(It should be pointed out that the assumption (6.1) is quite natural since the market price of risk is usually positive.) For the sake of simplicity, assume that $\sigma(t) \equiv \delta I_n$, *where* $\delta > 0$ *is a constant and* I_n *is the identity matrix in* $\mathbf{R}^{n\times n}$. *By Proposition 5.1, the optimal strategy for any* n *is* $\pi(t)^\top = (\pi_1(t), \dots, \pi_n(t))$, *where* $\pi_i(t) = \delta^{-2}X(t)\tilde{a}_i(t)$ $(\forall i)$ *and* $X(t) = X^{(n)}(t)$ *is the corresponding wealth. The process*

$$\pi_0(t) \triangleq X(t) - \sum_{i=1}^{n} \pi_i(t)$$

is the corresponding investment in the risk-free bond for the optimal strategy. By (6.1),

$$\sum_{i=1}^{n} \pi_i(0) = \delta^{-2}X(0)\sum_{i=1}^{n} \tilde{a}_i(0) \to +\infty \quad as \quad n \to +\infty \quad a.s.$$

Hence

$$\pi_0(0) = X(0)\left(1 - \delta^{-2}\sum_{i=1}^{n} \tilde{a}_i(0)\right) \to -\infty \quad as \quad n \to +\infty \quad a.s.$$

This property of the optimal strategy leads to a paradox and contradicts market practice.

We define classes of strategies where the portfolios at any one time may contain no more than a predetermined number of securities. Let m be a given integer, $0 \le m \le n$, and let $\mathcal{M}_m$ be the collection of subsets of $\{1, \dots, n\}$, each of which contains at most m elements.

Let $\mathcal{G}_t$ be a filtration, and let $\Sigma(\mathcal{G}.)$ be a class of admissible strategies $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_n(\cdot))$.

DEFINITION 6.1 *Denote by* $\mathcal{I}_m(\mathcal{G}.)$ *the set of random* $\mathcal{G}_t$*-adapted functions* $I : [0,T] \times \Omega \mapsto \mathcal{M}_m$.

We shall denote $\mathcal{I}_m^a = \mathcal{I}_m(\mathcal{F}^a_\cdot)$. Let $\mathcal{G}'_t$ be a filtration such that $\mathcal{G}'_t \subseteq \mathcal{F}^a_t$ $(\forall t)$.

DEFINITION 6.2 *Let* $I \in \mathcal{I}_n(\mathcal{G}.)$. *Denote by* $\Sigma(I(\cdot), \mathcal{G}'_\cdot)$ *the set of all strategies* $\pi(\cdot)$ *in* $\Sigma(\mathcal{G}'_\cdot)$ *for which* $\pi_i(t) = 0$ *if* $i \notin I(t)$.

DEFINITION 6.3 *Let* $\Sigma(m, \mathcal{G}., \mathcal{G}'_\cdot)$ *be the set of all strategies* $\pi(\cdot)$ *in* $\Sigma(\mathcal{G}'_\cdot)$ *such that there exists* $I \in \mathcal{I}_m(\mathcal{G}.)$ *for which* $\pi_i(t) = 0$ *if* $i \notin I(t)$.

Problem statement

Let $1 \leq m \leq n$, $T > 0$. Let $T > 0$ and let $m > 0$ be an integer. Let $\hat{D} \subset \mathbf{R}$, and let $X_0 \in \hat{D}$ be given. Let $\mathcal{G}_t$ be a filtration such that $\mathcal{G}_t \subseteq \mathcal{F}_t^a$ $(\forall t)$. Let $U(\cdot) : \hat{D} \to \mathbf{R} \cup \{-\infty\}$ such that $U(X_0) > -\infty$.

Our general statement of the problem follows: Find an admissible self-financing strategy $\pi(\cdot)$ which solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T, \pi(\cdot))) \quad \text{over} \quad \pi(\cdot) \in \tilde{\Sigma}(m, \mathcal{G}_\cdot, \mathcal{F}_\cdot^a) \tag{6.2}$$

$$\text{subject to} \begin{cases} \tilde{X}(0, \pi(\cdot)) = X_0, \\ \tilde{X}(T, \pi(\cdot)) \in \hat{D} \quad \text{a.s.} \end{cases} \tag{6.3}$$

The condition $\tilde{X}(T, \pi(\cdot)) \geq 0$ may represent a requirement for a minimal normalized terminal wealth.

In Chapter 5, we studied the problem for a case of $m = n$.

Some additional definitions

Let m be an integer, $0 \leq m \leq n$. We introduce more notation.

For a given $\bar{I} \in \mathcal{M}_m$, we denote by $L(\bar{I})$ the linear subspace of $\mathbf{R}^n$ such that $x = (x_1, \ldots, x_n) \in L(\bar{I})$ if and only if $x_i = 0$ for all $i \notin \bar{I}$.

Let $P(\bar{I}) \in \mathbf{R}^{n \times n}$ be the projection of $\mathbf{R}^n$ onto $L(\bar{I})$. In other words, $P(\bar{I}) = \{P^{(i,j)}(\bar{I})\}_{i,j=1}^n$, where

$$P^{(i,j)}(\bar{I}) \triangleq \begin{cases} 1 & \text{if } i = j, i \in \bar{I} \\ 0 & \text{if } i \neq j \text{ or } i \notin \bar{I}. \end{cases}$$

It follows that

$$b^\top P(\bar{I})V(t)P(\bar{I})b \geq c_1|P(\bar{I})b|^2 \quad \forall b \in \mathbf{R}^n,$$

so $P(\bar{I})V(t)P(\bar{I})$ is invertible on $L(\bar{I})$. Hence there exists a unique matrix $Q(\bar{I}, t) = \{Q^{(i,j)}(\bar{I}, t)\}_{i,j=1}^n \in \mathbf{R}^{n \times n}$ such that

$$Q^{(i,j)}(\bar{I}, t) = 0 \quad \text{if} \quad i \notin \bar{I} \quad \text{or} \quad j \notin \bar{I}$$

and

$$P(\bar{I})V(t)P(\bar{I})Q(\bar{I}, t)x = P(\bar{I})V(t)Q(\bar{I})x = x \quad \forall x \in L(\bar{I}).$$

If $m = 0$ and $\bar{I} = \emptyset \in \mathcal{M}_m$, then we assume that $Q(\bar{I}, t)$ is the zero matrix in $\mathbf{R}^{n \times n}$.

Further, set

$$\tilde{a}(\bar{I}, t) \triangleq V(t)Q(\bar{I}, t)\tilde{a}(t). \tag{6.4}$$

It can be seen that

$$\tilde{a}(\bar{I},t)^\top Q(t)\tilde{a}(\bar{I},t) = \tilde{a}(t)^\top Q(\bar{I},t)\tilde{a}(t)$$

and

$$\tilde{a}(\bar{I},t)^\top Q(t)\tilde{a}(\bar{I},t) = \hat{a}(t)^\top Q(\bar{I},t)\hat{a}(t).$$

For a given $I(\cdot) \in \mathcal{I}_n$, set

$$\begin{aligned}
&\tilde{a}_I(t) \triangleq \tilde{a}(I(t),t),\\
&L_I(t) \triangleq L(I(t)), \quad P_I(t) \triangleq P(I(t)),\\
&Q_I(t) \triangleq Q(I(t),t).
\end{aligned}$$

We shall use the notation $\tilde{a}_I$, $\mathbf{P}_{*I}$, $\mathbf{E}_{*I}$, and $\bar{\mathcal{Z}}_I$ for $\tilde{a}$, $\mathbf{P}_*$, $\mathbf{E}_I$, and $\bar{\mathcal{Z}}$ defined for the corresponding $\tilde{a}(\cdot) = \tilde{a}_I(\cdot)$, but with $\sigma(\cdot)$, $r(\cdot)$ unchanged.

Auxiliary market

It will be useful for the proof of the results below to introduce an auxiliary market.

Set $\mathcal{I}_m \triangleq \mathcal{I}_m(\mathcal{G}.)$. Consider an arbitrary $I = I(\cdot) \in \mathcal{I}_m$ and an auxiliary market defined by (5.1)-(5.2) with substitution $a(\cdot) = a_I(\cdot)$; we shall call it the *I–market.*

LEMMA 6.1 *For any $I(\cdot) \in \mathcal{I}_m$ and any strategy $\pi(\cdot) \in \tilde{\Sigma}(I(\cdot),\mathcal{G}.)$, the wealth obtained with the strategy $\pi(\cdot)$ is the same for both – the original market and the I–market.*

6.2. Optimal strategy for portfolio compression

Consider the optimal investment problem (6.2)-(6.3) and assume that Conditions 5.1, 5.2, and 5.3 are satisfied.

An individual investor may feel that she can only reasonably keep track of a limited number of stocks, or with finite capital wishes not to spread the investments too thinly, hence decides to hold at most m stocks in her portfolio at any one time. A function $I \in \mathcal{I}_m$, defined above, will specify which m stocks she holds at any time. If we restrict $\mathcal{I}_m$ to consist of constant functions only (i.e., $\mathcal{G}_t \equiv \mathcal{F}_0^a$), then this restriction amounts to choosing the m "best" stocks initially and then trading in the market consisting of these m stocks only. On the other hand, we may take $\mathcal{I}_m$ to consist of μ-adapted processes taking values in $\mathcal{M}_m$, where $\mu(t) \triangleq (r(t),\tilde{a}(t),\sigma(t))$. This form of $\mathcal{I}_m$ is not unreasonable. In fact, the rational investor, when choosing her portfolio, will want to maximize potential return while minimizing risk. Since these factors depend only on the coefficient processes, μ, it is reasonable to assume that I is μ adapted. In this case, if the parameters μ are nonrandom, then the functions $I \in \mathcal{I}_m$ will be nonrandom but possibly time varying.

Recall that we write $\mu_I(t)$ for $(r(t), a_I(t), \sigma(t))$. We strengthen Conditions 5.2 and 5.4 somewhat.

CONDITION 6.1 *For all $I \in \mathcal{I}_m$, there exist $\hat{\lambda} \triangleq \lambda_I \in \Lambda$, $C > 0$ and $c_0 \in (0, 1/(2J_I))$, such that $F(\cdot, \lambda_I)$ is piecewise continuous on $(0, \infty)$, $F(\mathcal{Z}_I(T), \lambda_I)$ is $\mathbf{P}_{*I}$-integrable, and*

$$\begin{cases} \mathbf{E}_{*I}\{F(\mathcal{Z}_I(T), \lambda_I)\} = X_0, \\ |F(z, \lambda_I)| \le C z^{c_0 \log z} \quad \forall z > 0. \end{cases} \tag{6.5}$$

CONDITION 6.2 *(i) For all $I \in \mathcal{I}_m$, there exist $\lambda_{R_I} \triangleq \lambda_I \in \Lambda$ a.s., $C > 0$ and $c_{R_I} \triangleq c_I \in (0, 1/(2R_I))$ a.s., such that $F(\cdot, \lambda_I)$ is a.s. piecewise continuous on $(0, \infty)$, $F(\mathcal{Z}_I(T), \lambda_I)$ is a.s. $\mathbf{P}_{*I}(\cdot \,|\, R_I)$-integrable, and*

$$\begin{cases} \mathbf{E}_{*I}\{F(\mathcal{Z}_I(T), \lambda_I) \,|\, R_I\} = X_0 \ a.s., \\ |F(z, \lambda_I)| \le C z^{c_I \log z} \quad \forall z > 0 \ a.s. \end{cases} \tag{6.6}$$

(ii) $F(x, \lambda) = C_1(\lambda) x^\nu + C_0$, where $C_1(\lambda) \neq 0$, C_0 and $\nu \neq 0$ are constants.

We say that $\hat{I}$ *dominates* I if $R_{\hat{I}} \ge R_I$, $a.s.$ and $\mathbf{P}\{R_{\hat{I}} > R_I\} > 0$.

THEOREM 6.1 *Let $\hat{I} \in \mathcal{I}_m$. Assume Conditions 5.1 and 6.1 and that $U^+(x) \le$ const $(|x| + 1)$, μ_I and w are independent for all $I \in \mathcal{I}_m$, and the random variable $R_{\hat{I}}$ is constant. Then the strategy $\pi_{\hat{I}}(\cdot)$, defined in Theorem 5.1 (with $a(\cdot) = a_{\hat{I}}(\cdot)$) belongs to the class $\tilde{\Sigma}(m, \mathcal{F}^a_\cdot)$, and*

$$\mathbf{E}U(\tilde{X}(T, \pi_{\hat{I}}(\cdot)) > \mathbf{E}U(\tilde{X}(T, \pi(\cdot))$$

for any strategy $\pi \in \tilde{\Sigma}(m, \mathcal{F}^a_\cdot)$ such that $\hat{I}$ dominates the corresponding I.

Observe that μ_I amd w are independent, in particular, if $\mathcal{I}_m$ contains only μ-adapted processes, and $\mu(\cdot)$ and $w(\cdot)$ are independent.

COROLLARY 6.1 *In Theorem 6.1, the assumption in (ii) that $R_{\hat{I}}$ be nonrandom can be replaced by Condition 6.2, and then Condition 6.1 can be dropped.*

COROLLARY 6.2 *Assume the hypotheses of either Theorem 6.1 or Corollary 6.1. If there exists $\hat{I} \in \mathcal{I}_m$ such that $R_{\hat{I}} = \max_{I \in \mathcal{I}_m} R_I$ a.s., then $\pi_{\hat{I}}(\cdot)$ is optimal for the problem (5.4)-(5.5). It is unique if $\hat{I}$ is.*

An interesting consequence is that the optimal $\hat{I} \in \mathcal{I}_m$ does not depend on $U(\cdot)$ or $\hat{D}$ — just choose the m stocks that provide a.s. the largest (in the L_2 sense) market price of risk. Because of the almost sure maximization requirement, this cannot always be done. However, if $\mathcal{I}_m$ contains at least the μ-adapted functions, then

$$\hat{I}(t) \in \arg \max_{M \in \mathcal{M}_m} \tilde{a}(t)^\top Q_M(t) \tilde{a}(t),$$

where $Q_M \triangleq Q_I$ with $I(\cdot) \equiv M$.

REMARK 6.1 Even under the imposed restrictions, Theorem 6.1 is used effectively for calculation of optimal strategy for *random* R in the *maximin* setting (Theorem 7.1 below).

6.3. A bond market: compression of the bond portfolio

We can also apply our theory to a zero-coupon bond market based on a generalization of the Vasicek interest rate model. In the Vasicek model, the market price of risk is constant, cf. Lamberton and Lapeyre (1996), Section 6.2.1. Our θ is their $-q$. We can generalize to θ a nonrandom function of t, so R is non-random. Given a progressively measurable $u(\cdot) : [0,T] \times \Omega \to [0,T]$ such that $u(t) \geq t$, a.s., we can construct the rolling bond $P(t, u(t))$, which expires at time $u(t)$. If $u(t) \equiv t$, this gives the usual bond-based construction of B. If $u(t) = [t] + 1$, this consists of a sequence of one-year bonds rolled over at expieration. In a market consisting of the bank account B and a finite number such bonds and with a utility that satisfies Conditions 5.1 and 5.2 as well as $U^+(x) \leq \text{const}\,(|x| + 1)$, we can deduce that optimal strategies exist and that one of them requires only one bond (with nonzero volatility) to be held in the optimal portfolio. Since there is only one driving Brownian motion, this result is not surprising.

Consider a model of a bond market such the one described in Lamberton and Lapeyre (1996), Section 6.2.1. Let $\mathcal{F}_t^w$ be a filtration generated by a scalar Wiener process $w(t)$. Let $r(t)$ be adapted to $\mathcal{F}_t^w$, and let $B(t)$ be the "risk-free" asset defined by (5.2). For each $u \in (0,T)$ and $t < u$, this asset is available for buying and selling a zero-coupon bond with price $P(t,u)$ such that $P(u,u) = 1$. We assume that the investor can buy and sell bonds on this market. It is shown cf. Lamberton and Lapeyre (1996), Section 6.2.1 that if this bond market is arbitrage–free, then there exists an $\mathcal{F}_t^w$-adapted process $q(t)$ such that

$$P(t,u) = \mathbf{E}\left\{\exp\left(-\int_t^u r(s)ds + \int_t^u q(s)dw(s) - \frac{1}{2}\int_t^u q(s)^2 ds\right) \Big| \mathcal{F}_t^w\right\}. \tag{6.7}$$

On the other hand, under some mild conditions, any $\mathcal{F}_t^w$-adapted process $q(t)$ defines an arbitrage–free bond market with prices (6.7).

We consider a special case when $q(t)$ is a deterministic process. This case is a modification of the Vasicec model, where $q(t)$ is a constant (see Lamberton and Lapeyre (1996), p. 127).

By Proposition 6.1.3 from Lamberton and Lapeyre (1996), for any u, there exists an $\mathcal{F}_t^w$–adapted process $\sigma_u(t)$ such that

$$d_tP(t,u) = P(t,u)([r(t) - q(t)\sigma_u(t)]dt + \sigma_u(t)dw(t)).$$

Then we can treat this market as a modification of the stock market, where the set of risky assets is $\{P(\cdot,u),\ u \in (0,T)\}$. We will call it the *bond market*.

Introduce the set $\bar{\Sigma}_B(m)$ of self-financing strategies $\pi(t) = (\pi_{u_1(t)}(t), \dots, \pi_{u_m(t)}(t))$ for the bond market which allows to contain in portfolio no more than m bonds at time t and which contains strategies with similar properties as strategies from $\bar{\Sigma}(m, \mathcal{F}_\cdot^a, \mathcal{F}_\cdot^a)$ for stocks market. Let $\tilde{X}(t) = \tilde{X}(t, \pi(\cdot))$ be the corresponding normalized wealth. As for the stock market, we have that

$$d\tilde{X}(t) = \sum_{i=1}^{m} B(t)^{-1}\pi_{u_i(t)} \left(-q(t)\sigma_{u_i(t)}dt + \sigma_{u_i(t)}dw(t)\right). \qquad (6.8)$$

Consider the problem (5.4)—(5.5) for the bond market.

Let $u(\cdot) : [0,T] \times \Omega \to [0,T]$ be a given function that is progressively measurable with respect to $\mathcal{F}_t^w$ and such that $u(t) \geq t$. Consider the set $\bar{\Sigma}_B(1, u(\cdot)) \subset \bar{\Sigma}_B(1)$ of strategies that allows the portfolio to contain at time t only bonds with maturity $u(t)$. Introduce an auxiliary "stock" with the price $S_u(t)$ defined by the equations

$$dS_u(t) = S_u(t)\left(q(t)\sigma_u(t)dt + \sigma_u(t)dw(t)\right), \quad t > 0, \qquad (6.9)$$

where $\sigma_u(t) \triangleq \sigma_{u(t)}(t)$. Consider an auxiliary market that consists of a risk-free asset $B(t)$ and the stock $S_u(t)$. We shall call this market the (B, S_u) market.

LEMMA 6.2 *Let* $U^+(x) \leq \text{const}\,(|x| + 1)$. *Assume Conditions 5.1 and 5.2 for the problem (5.4)-(5.5) stated for the* (B, S_u) *market. Then there exists a unique optimal strategy in the class* $\bar{\Sigma}_B(1, u(\cdot))$ *for this problem, and optimal normalized wealth* $\tilde{X}(T)$ *does not depend on* $u(\cdot)$ *or on the optimal value of* $\mathbf{E}U(\tilde{X}(T))$.

THEOREM 6.2 *Under the assumptions of Lemma 6.2, there exists a unique optimal strategy in the class* $\bar{\Sigma}_B(1, u(\cdot))$ *for the problem (5.4)-(5.5) stated for the bond market. The optimal strategy and the probability distribution of optimal* $\tilde{X}(T)$ *and* $\mathbf{E}U(\tilde{X}(T))$ *are same as in Lemma 6.2, where they are defined for the* (B, S_u) *market.*

COROLLARY 6.3 *For any integer $m > 0$, there exists an optimal strategy in the class $\tilde{\Sigma}_B(m)$ for the problem (5.4)-(5.5) for the bond market. The optimal $\tilde{X}(T)$ and $\mathbf{E}U(\tilde{X}(T))$ are same as in Theorem 6.2, where they are defined for $m = 1$, and as in Lemma 6.2, where they are defined for the (B, S_u) market; i.e., the optimal $\tilde{X}(T)$ and $\mathbf{E}U(\tilde{X}(T))$ do not depend on m.*

REMARK 6.2 In Lemma 6.2, the optimal strategy is uniquely defined by $u(\cdot)$. In Corollary 6.3, the optimal claim $\tilde{X}(T)$ can be replicated by different strategies if $m > 1$; in that case, an optimal strategy is not unique.

In fact, solution of optimal investment problems without compression is known for many different models of the bond market, for example, with several driving Brownian motions (see, e.g., Rutkowski (1997), Bielecki and Pliska (2001)).

6.4. Proofs

Proof of Lemma 6.1. It is easy to see that

$$P_I(t)\tilde{a}_I(t) = P_I(t)\tilde{a}(t), \quad P_I(t)\hat{a}_I(t) = P_I(t)\hat{a}(t). \tag{6.10}$$

We have from (6.10) that $P_{\hat{I}}(t)\tilde{a}(t) = P_{\hat{I}}(t)\tilde{a}_{\hat{I}}(t)$, so

$$\pi(t)^\top P_{\hat{I}}(t)[\sigma(t)\,dw(t) + \tilde{a}_{\hat{I}}(t)\,dt] = \pi(t)^\top P_{\hat{I}}(t)[\sigma(t)\,dw(t) + \tilde{a}(t)\,dt].$$

Then the proof follows from from (1.20). □

Proof of Theorem 6.1. For the given $\hat{I}$, let us introduce a new market. Consider the auxiliary market defined by (5.1)-(5.2) with $a(\cdot)$ replaced by $a_{\hat{I}}(\cdot)$; we shall call it the *$\hat{I}$–market*. For $\pi \in \tilde{\Sigma}(\hat{I}(\cdot), \mathcal{F}^a_\cdot)$, we have $\pi(t)^\top = \pi(t)^\top P_{\hat{I}}(t)$, and from (6.10) we have $P_{\hat{I}}(t)\tilde{a}(t) = P_{\hat{I}}(t)\tilde{a}_{\hat{I}}(t)$, so

$$\pi(t)^\top P_{\hat{I}}(t)[\sigma(t)\,dw(t) + \tilde{a}_{\hat{I}}(t)\,dt] = \pi(t)^\top P_{\hat{I}}(t)[\sigma(t)\,dw(t) + \tilde{a}(t)\,dt].$$

It follows from (1.20) that the wealth which is obtained with the strategy $\pi(\cdot)$ is the same for both markets — for the original market and for the $\hat{I}$–market – even though w_* is different in the two markets.

Note that the assumptions of Theorem 6.1 suffice for uniqueness of the optimal strategy, cf. Theorem 5.1, because under (i) the minimum is unique, and under (ii), $\mathcal{Z}_{*\hat{I}}(T)$ under $\mathbf{P}$ is conditionally log-normal given μ with parameters depending only on the constant $R_{\hat{I}}$, and therefore is unconditionally log-normal, and hence has a density. Then we can apply Theorem 5.1 to the $\hat{I}$–market to obtain the unique optimal strategy $\pi_{\hat{I}} \in \tilde{\Sigma}(\mathcal{F}^a_\cdot)$. We show first that

$\pi_{\hat{I}} \in \tilde{\Sigma}(m, \mathcal{F}^a_\cdot)$.

$$\begin{aligned}\pi_{\hat{I}}(t) &= B(t)\tfrac{\partial H}{\partial x}\big(\mathcal{Z}_{\hat{I}}(t), \tau_{\hat{I}}(t)\big)\mathcal{Z}_{\hat{I}}(t)Q(t)\tilde{a}_{\hat{I}}(t) \\ &= B(t)\tfrac{\partial H}{\partial x}\big((\mathcal{Z}_{\hat{I}}(t), \tau_{\hat{I}}(t)\big)\mathcal{Z}_{\hat{I}}(t)Q(t)V(t)Q_{\hat{I}}(t)P_{\hat{I}}(t)\tilde{a}(t) \\ &= B(t)\tfrac{\partial H}{\partial x}\big(\mathcal{Z}_{\hat{I}}(t), \tau_{\hat{I}}(t)\big)\mathcal{Z}_{\hat{I}}(t)Q_{\hat{I}}(t)P_{\hat{I}}(t)\tilde{a}(t).\end{aligned} \tag{6.11}$$

We have used that $Q(t) = V(t)^{-1}$. Since $Q_{\hat{I}}(t)$ maps $L_{\hat{I}}(t)$ into $L_{\hat{I}}(t)$, then $\pi_{\hat{I}}(t) \in L_{\hat{I}}(t)$ for all t, so $\pi_{\hat{I}}(\cdot) \in \tilde{\Sigma}(m, \mathcal{F}^a_\cdot)$, i.e., $P_{\hat{I}}(t)\pi_{\hat{I}}(t) \equiv \pi_{\hat{I}}(t)$. Let $\tilde{X}_{\hat{I}}(t)$ be the corresponding normalized wealth. By Theorem 5.1, there exists $\lambda_{J_{\hat{I}}} \triangleq \lambda_{\hat{I}} \in \Lambda$ such that $\tilde{X}_{\hat{I}}(T) = F(\mathcal{Z}_{\hat{I}}(T), \lambda_{\hat{I}})$.

Now assume $\hat{I}$ dominates I, and consider a new auxiliary market that we shall call the *I^+–market*: we assume that this market consists of the bond $B(t)$ and the stocks $S_1(t), \ldots, S_n(t)$, $S_{n+1}(t)$, where the stock prices $S_1(t), \ldots, S_n(t)$ are defined by (5.1), replacing $a(\cdot)$ by $a_I(\cdot)$, and where $S_{n+1}(t)$ is defined by the equation

$$dS_{n+1}(t) = S_{n+1}(t)\left((r(t) + \alpha)dt + dw_{n+1}(t)\right), \tag{6.12}$$

with

$$\alpha \triangleq \sqrt{\frac{R_{\hat{I}} - R_I}{T}}, \tag{6.13}$$

and with $w_{n+1}(t)$ a scalar Wiener process independent of $(w(\cdot), \mu(\cdot))$. Of course, the filtration for this market, $\{\mathcal{F}^+_t\}$, will be larger than $\{\mathcal{F}^a_t\}$ (it includes information on w_{n+1} and α). It is easy to see that the corresponding numbers J_{I^+} and R_{I^+} for the I^+–market are

$$R_{I^+} = R_I + \alpha^2 T = R_{\hat{I}}, \quad J_{I^+} = J_I + \alpha^2 T = J_{\hat{I}}. \tag{6.14}$$

It follows that if (ii) holds, then the distribution of $\mathcal{Z}_{I^+}(T)$ under $\mathbf{P}_{*I^+}$ and of $\mathcal{Z}_{\hat{I}}(T)$ under $\mathbf{P}_{*\hat{I}}$ are both log-normal with mean equal to 1 and variance of the log equal to $R_{I^+} = R_{\hat{I}}$, hence the same. Thus

$$\mathbf{E}_{*I^+}\{|F(\mathcal{Z}_{I^+}, \lambda_{\hat{I}})|\} = \mathbf{E}_{*\hat{I}}\{|F(\mathcal{Z}_{\hat{I}}, \lambda_{\hat{I}})|\} < +\infty,$$

and by Lemma 5.2,

$$\mathbf{E}_{*I^+}\{F(\mathcal{Z}_{I^+}, \lambda_{\hat{I}})\} = \mathbf{E}_{*\hat{I}}\{F(\mathcal{Z}_{\hat{I}}, \lambda_{\hat{I}})\} = X_0.$$

Lemma 5.3 implies the same result if (i) holds. Then Conditions 5.1—5.3 are satisfied for the I^+–market. By Theorem 5.1 applied to the I^+–market, there exists a unique optimal strategy for the problem (5.4)-(5.5) for the class

$\tilde{\Sigma}^+(m, \mathcal{G}., \mathcal{F}^a_\cdot)$ for this market (which is the analogue of the class $\tilde{\Sigma}(m, \mathcal{G}., \mathcal{F}^a_\cdot)$ for the original market). Let $\tilde{X}_{I^+}(t)$ be the corresponding normalized wealth for the I^+–market.

In case (i), Lemma 5.3 and (6.14) imply $\mathbf{E}U(\tilde{X}_{I^+}(T)) = J_{\hat{I}}/2 + \log X_0 = \mathbf{E}U(\tilde{X}_{\hat{I}}(T))$. Similarly, in case (ii), (6.14) implies

$$\begin{aligned} \mathbf{E}U(\tilde{X}_{I^+}(T)) &= \mathbf{E}_{*I^+}\{\mathcal{Z}_{I^+}(T)U(F(\mathcal{Z}_{I^+}(T), \lambda_{\hat{I}}))\} \\ &= \mathbf{E}_{*\hat{I}}\{\mathcal{Z}_{\hat{I}}(T)U(F(\mathcal{Z}_{\hat{I}}(T), \lambda_{\hat{I}}))\} \\ &= \mathbf{E}U(\tilde{X}_{\hat{I}}(T)). \end{aligned} \tag{6.15}$$

Now consider an arbitrary strategy $\pi(\cdot) \in \tilde{\Sigma}(m, \mathcal{G}., \mathcal{F}^a_\cdot)$, with corresponding I dominated by $\hat{I}$, as a strategy in the I^+–market, with the investment in stock $(n+1)$ equal to zero identically. By Theorem 5.1 the unique optimal strategy in the I^+–market holds a nonzero multiple of α in stock $(n+1)$; hence $\pi(\cdot)$ is not optimal. Then by (6.15),

$$\mathbf{E}U(\tilde{X}(T, \pi)) < \mathbf{E}U(\tilde{X}_{I^+}(T)) = \mathbf{E}U(\tilde{X}_{\hat{I}}(T)).$$

This completes the proof of Theorem 6.1. □

Proof of Corollary 6.1. For given $I \in \mathcal{I}_m$, introduce the filtration $\mathcal{F}^I_t$ generated by μ, S, and R_I; it is an enlargement of the filtration $\{\mathcal{F}^a_t\}$. Let $\tilde{\Sigma}^I_m \triangleq \tilde{\Sigma}^I(m, \mathcal{G}., \mathcal{F}^I_\cdot)$. Recall that w is still Brownian motion because of the independence of w and μ_I. We will work with the conditional probability, $\mathbf{P}(\cdot\,|\mathcal{F}^I_0)$, rather than with $\mathbf{P}$. Uniqueness of the optimal strategy still holds under this measure.

As in the proof of Corollary 5.2 applied to the $\hat{I}$-market, the strategy $\pi_{\hat{I}}(t) = \nu B(t)[\tilde{X}(t) - C_0]Q(t)\tilde{a}_{\hat{I}}(t) \in \tilde{\Sigma}(\mathcal{F}^a_\cdot)$ is optimal in $\tilde{\Sigma}^{\hat{I}}_n \supset \tilde{\Sigma}(\mathcal{F}^a_\cdot)$. Moreover, as in (6.11), $\pi_{\hat{I}} \in \tilde{\Sigma}(m, \mathcal{G}., \mathcal{F}^a_\cdot)$.

Proceeding as in the proof of Theorem 6.1, we can define the I^+-market. Then $\mathcal{F}^+_t$ is generated by $(\mu_I(\cdot), \alpha, w(\cdot), w_{n+1}(\cdot))$ and $\mathcal{F}^{I^+}_t$ by $(\mu_I(\cdot), \alpha, w(\cdot), w_{n+1}(\cdot), R_{I^+})$. The corresponding set of policies is denoted by $\tilde{\Sigma}^{I^+}_{n+1}$. Again, $R_{I^+} = R_{\hat{I}}$, and the conditional distribution of $\mathcal{Z}_{I^+}$ under $\mathbf{P}_{*I^+}$ given R_{I^+} is the same as the conditional distribution of $\mathcal{Z}_{\hat{I}}$ under $\mathbf{P}_{*\hat{I}}$ given $R_{\hat{I}}$. Then (6.15) becomes

$$\begin{aligned} \mathbf{E}\{U(\tilde{X}_{I^+}(T)) \mid R_{I^+}\} &= \mathbf{E}_{*I^+}\{\mathcal{Z}_{I^+}(T)U(F(\mathcal{Z}_{I^+}(T), \lambda_{\hat{I}})) \mid R_{I^+}\} \\ &= \mathbf{E}_{*\hat{I}}\{\mathcal{Z}_{\hat{I}}(T)U(F(\mathcal{Z}_{\hat{I}}(T), \lambda_{\hat{I}})) \mid R_{\hat{I}}\} \\ &= \mathbf{E}\{U(\tilde{X}_{\hat{I}}(T)) \mid R_{\hat{I}}\}. \end{aligned}$$

Taking expectations gives

$$\mathbf{E}\{U(\tilde{X}_{I^+}(T))\} = \mathbf{E}\{U(\tilde{X}_{\hat{I}}(T))\}. \tag{6.16}$$

Any $\pi \in \tilde{\Sigma}(\mathcal{F}^a)$, with corresponding I dominated by $\hat{I}$, can be considered as a (nonoptimal) element of $\tilde{\Sigma}_{n+1}^{I+}$, and so (6.16) implies

$$\mathbf{E}U(\tilde{X}(T,\pi)) < \mathbf{E}U(\tilde{X}_{I^+}(T)) = \mathbf{E}U(\tilde{X}(T,\pi_{\hat{I}})). \qquad \square$$

Proof of Lemma 6.2. The corresponding variable R is constant; then Conditions 5.3(ii) holds. It can be seen that Theorem 5.1 holds for $n = 1$ if the condition of boundedness of $\tilde{a}(t)$ and $\sigma(t)^{-1}$ is replaced by the less restrictive condition of boundedness of $\theta(t)$ only. Then there exists the unique optimal strategy in the class $\tilde{\Sigma}_B(1, u(\cdot))$, and the optimal $\tilde{X}(T)$ depends only on $\theta(\cdot)$, i.e., on $q(\cdot)$. $\square$

Proof of Theorem 6.2. It is easy to see that the wealth processes for the bond market and for the auxiliary (B, S_u) market are driven by the same equation. $\square$

Proof of Corollary 6.3. It suffices to prove that if Corollary 6.3 holds for m, then by implication it should hold for $m + 1$. To prove this, it suffices to show that for any claim attainable in $\tilde{\Sigma}_B(m + 1)$, there exists a replicating strategy from $\tilde{\Sigma}_B(m)$. By (6.8), for any time t, the wealth will be not changed if $\pi_{m+1}(t)$ is replaced by zero, and, at the same time, $\pi_k(t)$ is replaced by $\pi_k(t) + \pi_{m+1}(t)\sigma_{u_{m+1}}(t)\sigma_{u_k}(t)^{-1}$, where $k \in \{1, \ldots, m\}$ is such that $\sigma_{u_k}(t) \neq 0$. This completes the proof. $\square$

The proof of Corollary 6.2 is obvious.

Chapter 7

MAXIMIN CRITERION FOR OBSERVABLE BUT NONPREDICTABLE PARAMETERS

Abstract In this chapter, it is assumed that the risk-free rate, the appreciation rates, and the volatility rates of the stocks are all random; they are not adapted to the driving Brownian motion, and their distributions are unknown, but they are currently observable. Admissible strategies are based on current observations of the stock prices and the aforementioned parameters. The optimal investment problem is stated as a problem with a *maximin* performance criterion. This criterion is to ensure that a strategy is found such that the minimum of utility over all distributions of parameters is maximal. It is shown that the duality theorem holds for the problem and that the maximin problem is reduced to the *minimax* problem, with minimization *over a single scalar parameter* (even for a multistock market). This interesting effect follows from the result of Chapter 6 for the optimal compression problem. Using this effect, the original *maximin* problem is solved explicitly; the optimal strategy is derived explicitly via solution of a linear parabolic equation.

7.1. Definitions and problem statement

Similarly to Chapter 5, we consider the market model from Section 1.3. The market consists of a risk–free bond or bank account with price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \dots, n$, where $n < +\infty$ is given. The prices of the stocks evolve according to

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right), \quad t > 0, \qquad (7.1)$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given nonrandom constant. The price of the bond evolves according to

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right), \qquad (7.2)$$

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where $B(0)$ is a given constant that we take to be 1 without loss of generality, and $r(t)$ is the random process of the risk-free interest rate.

As usual, we assume that $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

Set $\mu(t) \triangleq (r(t), \tilde{a}(t), \sigma(t))$, where $\tilde{a}(t) \triangleq a(t) - r(t)\mathbf{1}$.

We describe now distributions of $\mu(\cdot)$ and what we suppose to know about them.

We assume that there exist a finite-dimensional Euclidean space $\bar{E}$, a compact subset $\mathcal{T} \subset \bar{E}$, and a measurable function

$$M(t,\cdot) = (M_r(t,\cdot), M_a(t,\cdot), M_\sigma(t,\cdot)),$$
$$M(\cdot) : [0,T] \times \mathcal{T} \to \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}$$

that is uniformly bounded and such that $M(t,\alpha)$ is continuous in $\alpha \in \mathcal{T}$ for all t, and the matrix $M_\sigma(\cdot)^{-1}$ is uniformly bounded.

DEFINITION 7.1 *Let $\mathcal{A}(\mathcal{T})$ be a set of all random processes $\mu'(t) = (r'(t), \tilde{a}'(t), \sigma'(t))$ such that there exists a random vector $\Theta : \Omega \to \mathcal{T}$ independent of $w(\cdot)$ and such that*

$$\begin{cases} r'(t) \equiv M_r\,(t,\Theta) \\ \tilde{a}'(t) \equiv M_a\,(t,\Theta) \\ \sigma'(t) \equiv M_\sigma\,(t,\Theta). \end{cases} \tag{7.3}$$

Let

$$\theta_\mu(t) \triangleq \sigma(t)^{-1}\tilde{a}(t) \tag{7.4}$$

be the *risk premium process* given $\mu(\cdot)$. Set

$$R_\mu \triangleq \int_0^T |\theta_\mu(t)|^2 dt. \tag{7.5}$$

Set

$$R_{min} \triangleq \inf_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} R_\mu.$$

We assume that $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ and this is the only information available about distribution on $\mu(\cdot)$. *Moreover, we are not supposed to know $\bar{E}, \mathcal{T}$ and $M(\cdot)$; we know only the fact of their existence and the value of R_{min}.*

EXAMPLE 7.1 Let $n = 1, \bar{E} = \mathbf{R}^N$, where $N > 0$ is an integer, $\mathcal{T} = [0,1]^N$, and

$$\begin{aligned} &(M_r(\alpha,t), M_\sigma(\alpha,t)) \equiv (r,\sigma), && \alpha = (\alpha_1,\dots,\alpha_N) \in \mathcal{T}, \\ &M_a(\alpha,t) = \alpha_k, \quad t \in \left[\tfrac{(k-1)T}{N}, \tfrac{kT}{N}\right), && k = 1,\dots,N. \end{aligned}$$

where r, σ are constants. Then $\mathcal{A}(\mathcal{T})$ is the set of all processes $\mu(t) = (r(t), \tilde{a}(t), \sigma(t))$ such that

$$\begin{aligned} &r(t) \equiv r, \quad \sigma(t) \equiv \sigma, \\ &\tilde{a}(t) = \Theta_k, \quad t \in \left[\tfrac{(k-1)T}{N}, \tfrac{kT}{N}\right), \quad k = 1,\dots,N, \end{aligned}$$

where $\Theta = (\Theta_1,\dots,\Theta_N)$ is a N-dimensional random vector independent of $w(\cdot)$, $|\Theta_k| \le 1$.

REMARK 7.1 It is easy to see that our description of the class of admissible $\mu(\cdot)$ covers a setting when the minimum of R_μ over the class is given, or when the class of admissible $\mu(\cdot)$ is defined by a condition $R_\mu \in [R_1, R_2]$, where R_1, R_2 are given, $0 \le R_1 < R_2 \le +\infty$. (It suffices to choose an appropriate pair $(\Theta, M(\cdot))$.)

Notice that the solution of (7.1) is well defined for any $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, but the market is incomplete.

For $\alpha \in \mathcal{T}$, set

$$\mu_\alpha(t) \triangleq (M_r(t,\alpha), M_a(t,\alpha), M_\sigma(t,\alpha)),$$

where $M_\sigma(t,\alpha)$ and $M_a(t,\alpha)$ are as in Definition 7.1.

Let $\mathcal{F}_t^a \subset \mathcal{F}$ be the filtration of complete σ-algebras of events generated by the process $(S(t), \mu(t))$, $t \ge 0$. Let $\tilde{\Sigma}(\mathcal{F}_\cdot^a)$ be the class of admissible strategies introduced in Definition 5.1, Chapter 5.

Let $X(0) > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$. Let $\tilde{X}(t)$ be the normalized wealth.

By the definitions of $\tilde{\Sigma}(\mathcal{F}_\cdot^a)$ and $\mathcal{F}_t^a$, any admissible self-financing strategy is of the form

$$\pi(t) = \Gamma(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}), \tag{7.6}$$

where $\Gamma(t,\cdot) : B([0,t]; \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}) \to \mathbf{R}^n$ is a measurable function, $t \ge 0$.

Clearly, the random processes $\pi(\cdot)$ with the same $\Gamma(\cdot)$ in (7.6) may be different for different $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot))$. Hence we also introduce strategies defined by $\Gamma(\cdot)$: the function $\Gamma(\cdot)$ in (7.6) is said to be a CL-strategy (closed-loop strategy).

DEFINITION 7.2 *Let $\mathcal{C}$ be the class of all functions $\Gamma(t,\cdot) : B([0,t];\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}) \to \mathbf{R}^n$, $t \geq 0$ such that the corresponding strategy $\pi(\cdot)$ defined by (7.6) belongs to $\tilde{\Sigma}(\mathcal{F}^a_\cdot)$ for any $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot)) \in \mathcal{A}(\mathcal{T})$ and*

$$\sup_{\mu(\cdot)=\mu_\alpha(\cdot):\, \alpha\in\mathcal{T}} \mathbf{E}\int_0^T |\pi(t)|^2\, dt < \infty.$$

A function $\Gamma(\cdot) \in \mathcal{C}$ is said to be an admissible CL-strategy.

Let the initial wealth $X(0)$ be fixed. For an admissible self-financing strategy $\pi(\cdot)$ such that $\pi(t) = \Gamma(t, [S(\cdot), \mu(\cdot)]|_{[0,t]})$, the process $(\pi(t), X(t))$ is uniquely defined by $\Gamma(\cdot)$ and $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot))$ given $w(\cdot)$. We shall use the notation $X(t, \Gamma(\cdot), \mu(\cdot))$ and $\tilde{X}(t, \Gamma(\cdot), \mu(\cdot))$ to denote the corresponding total wealth and normalized wealth. Furthermore, we shall use the notation $S(t) = S(t, \mu(\cdot))$ and $\tilde{S}(t) = \tilde{S}(t, \mu(\cdot))$ to emphasize that the stock price is different for different $\mu(\cdot)$.

For any $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, introduce the process $\mathcal{Z}(t, \mu(\cdot)) = \mathcal{Z}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]})$ defined by the equation

$$\begin{cases} d\mathcal{Z}(t,\mu(\cdot)) = \mathcal{Z}(t,\mu(\cdot))\tilde{a}(t)^\top Q(t)\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t) \\ \mathcal{Z}(0,\mu(\cdot)) = 1. \end{cases} \tag{7.7}$$

Then

$$\mathcal{Z}(t,\mu(\cdot)) = \exp\left(\int_0^t \theta_\mu(s)^\top dw(s) + \frac{1}{2}\int_0^t |\theta_\mu(s)|^2 ds\right).$$

Our standing assumptions imply that $\mathbf{E}\mathcal{Z}(T, \mu_\alpha(\cdot))^{-1} = 1$ for all $\alpha \in \mathcal{T}$. Define the (equivalent martingale) probability measure $\mathbf{P}^\alpha_*$ by

$$\frac{d\mathbf{P}^\alpha_*}{d\mathbf{P}} = \mathcal{Z}(T, \mu_\alpha(\cdot))^{-1}.$$

Let $\mathbf{E}^\alpha_*$ be the corresponding expectation.

Problem statement

Let $T > 0$ and the initial wealth X_0 be given. Let $U(\cdot) : \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ be a given measurable function such that $U(X_0) < +\infty$. Let $\hat{D} \subset \mathbf{R}$ be a given convex set, $X_0 \in \hat{D}$.

We may state our general problem as follows: Find an admissible CL-strategy $\Gamma(\cdot)$ and the corresponding self-financing strategy $\pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}^a_\cdot)$ that solves the following optimization problem:

$$\text{Maximize} \quad \min_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \tag{7.8}$$

$$\text{subject to} \begin{cases} X(0,\Gamma(\cdot),\mu(\cdot)) = X_0, \\ \tilde{X}(T,\Gamma(\cdot),\mu(\cdot)) \in \hat{D} \quad \text{a.s.} \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}). \end{cases} \tag{7.9}$$

DEFINITION 7.3 *Let $\mathcal{C}_0$ be the set of all admissible CL-strategies $\Gamma(\cdot) \in \mathcal{C}$ such that*

$$\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)) \in \hat{D} \quad a.s. \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}).$$

The problem (7.8)-(7.9) can be rewritten as

$$\text{Maximize} \quad \min_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \in \mathcal{C}_0. \tag{7.10}$$

Clearly, the maximin setting has no sense if, for example, $\mu(t) \equiv \Theta$, where Θ is a random element of $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}$ which is constant in time; one can identify Θ instantly. However, the optimal solution for a more general case needs knowledge about distribution of future values of $\mu(\cdot)$.

EXAMPLE 7.2 Let $n = 1, \mathcal{T} \triangleq \{\alpha_1, \alpha_2\}$, where $\alpha_i \in \mathbf{R}$. Let

$$(M_r(\alpha,t), M_a(\alpha,t)) \equiv (r,\tilde{a}), \qquad M_\sigma(\alpha,t) = \begin{cases} \alpha_1, & t < T/2 \\ \alpha, & t \geq T/2, \end{cases}$$

i.e.,

$$(r(t),\tilde{a}(t)) \equiv (r,\tilde{a}), \qquad \sigma(t) = \begin{cases} \alpha_1, & t < T/2 \\ \Theta, & t \geq T/2, \end{cases}$$

where $r, \tilde{a}$ are constants, and Θ is a random variable independent of $w(\cdot)$ which can have only two values, α_1 and α_2. Let $\kappa \in [0,1)$ and $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ be given. Consider the problem

$$\text{Maximize} \quad \mathbf{E}\log\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot)$$

$$\text{subject to} \begin{cases} X(0,\Gamma(\cdot),\mu(\cdot)) = X_0, \\ \tilde{X}(T,\Gamma(\cdot),\mu(\cdot)) \geq \kappa X_0 \quad \text{a.s.} \end{cases}$$

By Theorem 5.1, it follows that if $\Theta \equiv \alpha_1$ or $\Theta \equiv \alpha_1$, then the optimal strategy exists, and if $\kappa \neq 0$, then the corresponding optimal strategies for these two cases differs at the time interval $[0, T/2)$ (see, e.g., Lemma 7.1 below). Hence the optimal strategy can not be obtained from observations of historical $\tilde{a}(t)$ and $S(t)$ without knowledge of future distributions. The only exception is the case $\kappa = 0$, when the optimal strategy given $\mu(\cdot)$ is myopic.

The case of myopic strategies

PROPOSITION 7.1 *Let $X_0 = X(0) > 0$ and let one of the following conditions is satisfied:*

(i) $U(x) = \log(x)$, $\hat{D} = [0, +\infty)$;

(ii) $U(x) = x^\delta$, $\hat{D} = [0, +\infty)$, *where* $\delta < 1$, $\delta \neq 0$;

(iii) $U(x) = -kx^2 + cx$, $\hat{D} = \mathbf{R}$, *where* $k \in \mathbf{R}$ *and* $c \geq 0$.

Then there exists $C_0, C_1, \nu \in \mathbf{R}$ such that $C_1 \neq 0$, $\nu \neq 0$ are constants, and that the optimal strategy $\pi(\cdot) \in \tilde{\Sigma}(\mathcal{F}^\mu)$ for the problem (7.8)-(7.9) has the form

$$\pi(t)^\top = \nu B(t)(\tilde{X}(t) - C_0)\tilde{a}(t)^\top Q(t),$$

where $\tilde{X}(t)$ is the corresponding normalized wealth. This solution is optimal for the problem

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \tag{7.11}$$

for any $\mu(\cdot)$.

Some additional assumptions

To proceed further, we assume that Condition 5.1 remains in force throughout this chapter. Moreover, we impose the following additional conditions.

CONDITION 7.1 *For any $\alpha \in \mathcal{T}$, there exist $\hat{\lambda}_\alpha \in \Lambda$, $C = C_\alpha > 0$, and $c_0 = c_{0,\alpha} \in (0, 1/(2\mathbf{R}_{\mu_\alpha}))$ such that $F(\cdot, \hat{\lambda})$ is piecewise continuous on $(0, \infty)$, $F(\mathcal{Z}(T, \mu_\alpha(\cdot)), \hat{\lambda}_\alpha)$ is $\mathbf{P}_*^\alpha$-integrable, and*

$$\begin{cases} \mathbf{E}_*^\alpha\{F(\mathcal{Z}(T, \mu_\alpha(\cdot)), \hat{\lambda}_\alpha)\} = X_0, \\ |F(z, \hat{\lambda}_\alpha)| \leq C z^{c_0 \log z} \quad \forall z > 0. \end{cases}$$

CONDITION 7.2 *The function $U(x) : \mathbf{R} \to \mathbf{R}$ is either concave or convex in $x \in \hat{D}$, and there exist constants $c > 0$, $p \in (1, 2]$, $q \in (0, 1]$ such that*

$$\begin{aligned} &|U(x)| \leq c(|x|^p + 1), \\ &|U(x) - U(x_1)| \leq c(1 + |x| + |x_1|)^{2-q} |x - x_1|^q \quad \forall x, x_1 \in \hat{D}. \end{aligned} \tag{7.12}$$

Notice that condition (7.12) is not restrictive if $\hat{D}$ is bounded.

CONDITION 7.3 *At least one of the following conditions holds:*

(i) The set $\mathcal{T}$ is either finite or countable, i.e., $\mathcal{T} = \{\alpha_1, \alpha_2, \ldots\}$, where $\alpha_i \in \bar{E}$;

(ii) The function $(M_r(t,\alpha), M_\sigma(t,\alpha))$ does not depend on α, i.e., $M_r(t,\alpha) \equiv M_r(t)$ and $M_\sigma(t,\alpha) \equiv M_\sigma(t)$, and Condition 7.2 is satisfied with $p \in (1,2)$.

Note that Condition 7.3(i) looks restrictive, but in fact it is rather technical, since the total number of elements of $\mathcal{T}$ may be unbounded.

EXAMPLE 7.3 Let $k > 0$ be a integer. Let $n = 1$,

$$(r(t), \tilde{a}(t)) \equiv (r, \tilde{a}), \qquad \sigma(t) = \begin{cases} \bar{\sigma}, & t < \tau \\ \Theta, & t \geq \tau, \end{cases}$$

where $r > 0$, $\tilde{a}$, and $\bar{\sigma}$ are constants, τ and Θ are random variables such that the pair (τ, Θ) is independent of $w(\cdot)$, and such that

$$|\Theta| \in \left\{0, \tfrac{1}{k}, \ldots, \tfrac{k-1}{k}, 1\right\}, \qquad \tau \in \left\{0, \tfrac{1}{k}, \ldots, \tfrac{k-1}{k}, 1\right\}.$$

Then Condition 7.3(i) is satisfied with $\bar{E} = \mathbf{R}^2$,

$$\mathcal{T} = \left\{\alpha = (\alpha_1, \alpha_2): \ |\alpha_1| \in \left\{0, \tfrac{1}{k}, \ldots, \tfrac{k-1}{k}, 1\right\}, \ \alpha_2 \in \left\{0, \tfrac{1}{k}, \ldots, \tfrac{k-1}{k}, 1\right\}\right\},$$

$$(M_r(\alpha,t), M_a(\alpha,t)) \equiv (r, \tilde{a}), \qquad M_\sigma(\alpha, t) = \begin{cases} \bar{\sigma}, & t < \alpha_2 \\ \alpha_1, & t \geq \alpha_2. \end{cases}$$

7.2. Optimal solution of the maximin problem

For given $R > 0, \lambda \in \Lambda$, let the function $H(\cdot) = H(\cdot, R, \lambda) : \mathbf{R}_+ \times [0,T] \to \mathbf{R}$ be the solution of the following Cauchy problem:

$$\begin{cases} \frac{\partial H}{\partial t}(x,t,R,\lambda) + \frac{R}{2T}x^2 \frac{\partial^2 H}{\partial x^2}(x,t,R,\lambda) = 0, \\ H(x,T,R,\lambda) = F(x,\lambda). \end{cases} \tag{7.13}$$

Introduce a function $\tilde{\Gamma}(t,\cdot) : B([0,t]; \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}) \times (0,+\infty) \times \Lambda \to \mathbf{R}^n$ such that

$$\begin{aligned} &\tilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R, \lambda) \\ &\qquad \triangleq B(t) \tfrac{\partial H}{\partial x}\left[\mathcal{Z}(t,\mu(\cdot)), \tau_\mu(t,R), R, \lambda\right] \mathcal{Z}(t,\mu(\cdot)) \tilde{a}(t)^\top Q(t), \end{aligned}$$

where the process $\mathcal{Z}(t,\mu(\cdot))$ is defined by (7.7) and where

$$\tau_\mu(t,R) = \tau(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R) \triangleq \frac{T}{R}\int_0^t |\theta_\mu(s)|^2 ds.$$

Further, for a given $\alpha \in \mathcal{T}$, $R \geq 0$, let CL-strategy $\hat{\Gamma}_\alpha(\cdot, R)$ be defined as

$$\hat{\Gamma}_\alpha(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R) \triangleq \begin{cases} \tilde{\Gamma}(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R, \hat{\lambda}_\alpha) & \text{if } R > 0 \\ 0 & \text{if } R = 0, \end{cases}$$

where $\theta_\mu(\cdot)$ is defined by (7.4).

To formulate our main result, we shall need some results from Chapters 5-6, and these are summarized in the following lemma.

LEMMA 7.1 *(i) For any $R > 0$, $\lambda \in \Lambda$, the problem (7.13) has a unique solution $H(\cdot, R, \lambda) \in C^{2,1}((0,\infty) \times (0,T))$, with $H(x,t,R,\lambda) \to F(x,\lambda)$ a.e. as $t \to T-$.*

(ii) For any $\alpha \in \mathcal{T}$, the strategy

$$\hat{\Gamma}_\alpha(t,[S(\cdot),\mu(\cdot)]|_{[0,t]}, R_{\mu_\alpha})$$
$$\triangleq B(t)\frac{\partial H}{\partial x}\left[\mathcal{Z}(t,\mu(\cdot)), \tau_\mu(t,R_{\mu_\alpha}), R_{\mu_\alpha}, \hat{\lambda}_\alpha\right] \mathcal{Z}(t,\mu(\cdot))\tilde{a}(t)^\top Q(t)$$

belongs to $\mathcal{C}_0$ and

$$\begin{aligned}\mathbf{E}U(\tilde{X}(T,\hat{\Gamma}_\alpha(\cdot,R_{\mu_\alpha}),\mu_\alpha(\cdot))) \quad &\geq \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot))) \\ &\forall\Gamma(\cdot) \in \mathcal{C}_0, \ \forall\alpha \in \mathcal{T}.\end{aligned} \tag{7.14}$$

(iii) The functions $F(\cdot,\hat{\lambda}_\alpha)$, $H(\cdot,R_{\mu_\alpha},\hat{\lambda}_\alpha)$, $\hat{\Gamma}_\alpha(\cdot,R_{\mu_\alpha})$ as well as the probability distribution of the optimal normalized wealth $\tilde{X}(T,\hat{\Gamma}_\alpha(\cdot),\mu_\alpha(\cdot)))$ are uniquely defined by R_{μ_α}.

(iv) Let $\alpha_i \in \mathcal{T}$, $i = 1,2$ be such that $R_{\mu_1} < R_{\mu_2}$, where $\mu_i \triangleq \mu_{\alpha_i}$. Then

$$\mathbf{E}U(\tilde{X}(T,\hat{\Gamma}_{\alpha_1}(\cdot,R_{\mu_1}),\mu_1(\cdot))) < \mathbf{E}U(\tilde{X}(T,\hat{\Gamma}_{\alpha_2}(\cdot,R_{\mu_2}),\mu_2(\cdot))).$$

THEOREM 7.1 *Let R_{min} be known.*

(i) *If $R_{min} = 0$, then the trivial strategy, $\Gamma(\cdot) \equiv 0$, is the unique optimal strategy in $\mathcal{C}$ for the problem (7.8)–(7.9).*

(ii) *Let $R_{min} > 0$, and let $\hat{\alpha} \in \mathcal{T}$ be such that $R_{\hat{\mu}} = R_{min}$, where $\hat{\mu} \triangleq \mu_{\hat{\alpha}}$. Then the strategy*

$$\hat{\Gamma}_{\hat{\alpha}}(t,[S(\cdot),\mu(\cdot)]|_{[0,t]},R_{\hat{\mu}}) = \tilde{\Gamma}(t,[S(\cdot),\mu(\cdot)]|_{[0,t]},R_{min},\hat{\lambda}_{\hat{\alpha}}) \tag{7.15}$$

belongs to $\mathcal{C}_0$ and is optimal in $\mathcal{C}$ for the problem (7.8)-(7.9).

COROLLARY 7.1 *The optimal strategy for the problem (7.8)-(7.9) does not depend on $(\mathcal{T},\mathcal{M}(\cdot))$, if R_{min} is fixed.*

7.3. Proofs

Proof of Proposition 7.1. We have that Condition 5.1 is satisfied with $F(x,\lambda) = C_1\left(\frac{x}{\lambda}\right)^\nu + C_0$, where $C_1 \neq 0$, C_0 and $\nu \neq 0$ are constants. Then the proof follows from Corollary 5.2. □

Proof of Lemma 7.1. Statements (i)–(iii) follow immediately from Lemma 5.2, Theorem 5.1, and Corollary 5.1. Let us show that statement (iv) holds.

Let $\alpha_1 \in \mathcal{T}$ and $\alpha_2 \in \mathcal{T}$ be such that $R_{\mu_1} < R_{\mu_2}$, where $\mu_i \triangleq \mu_{\alpha_i}$. Further, let $\hat{\mu}_{\alpha_2}(\cdot)$ be a process that is independent of $(\mu_{\alpha_1}(\cdot), w(\cdot))$ and has the same distribution as $\mu_{\alpha_2}(\cdot) \in \mathcal{A}(\mathcal{T})$. Consider a new auxiliary market with $2n$ stocks that consists of two independent groups of stocks that correspond to $\mu_{\alpha_1}(\cdot)$ and $\hat{\mu}_{\alpha_2}(\cdot)$ (their driving Brownian motions and coefficients are independent). Then statement (iv) is a special case of Theorem 6.1, applied for the new market. □

Additional definitions

Without loss of generality, we describe the probability space as follows: $\Omega = \mathcal{T} \times \Omega'$, where $\Omega' = C([0,T];\mathbf{R}^n)$. We are given a σ-algebra $\mathcal{F}'$ of subsets of Ω' generated by cylindrical sets, and a σ-additive probability measure $\mathbf{P}'$ on $\mathcal{F}'$ generated by $w(\cdot)$. Furthermore, let $\mathcal{F}_{\mathcal{T}}$ be the σ-algebra of all Borel subsets of $\mathcal{T}$, and $\mathcal{F} = \mathcal{F}_{\mathcal{T}} \otimes \mathcal{F}'$. We assume also that each $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ generates the σ-additive probability measure ν_μ on $\mathcal{F}_{\mathcal{T}}$ (this measure is generated by Θ which corresponds to $\mu(\cdot)$).

Let $\tilde{\Sigma}^R(\mathcal{F}^a_\cdot)$ be the enlargement of $\tilde{\Sigma}(\mathcal{F}^a_\cdot)$ produced by replacing the filtration $\mathcal{F}^a_t$ by $\mathcal{F}^R_t$ generated by $\mathcal{F}^a_t$ and R_μ in the definition. (Note that the corresponding strategies are not adapted to $\mu(t)$.)

By the definitions of $\tilde{\Sigma}^R(\mathcal{F}^a_\cdot)$, any admissible self-financing strategy from this class is of the form

$$\pi(t) = \Gamma(t, [S(\cdot), \mu(\cdot)]|_{[0,t]}, R_\mu),$$

where $\Gamma(t,\cdot) : B([0,t];\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^N)\times\mathbf{R} \to \mathbf{R}^n$ is a measurable function, $t \geq 0$. With $\mathbf{P}(\cdot)$ replaced by $\mathbf{P}(\cdot \,|\, \mathcal{F}^R_0)$, we may apply Theorem 5.1 to obtain the optimal π in the class $\tilde{\Sigma}^R(\mathcal{F}^a_\cdot)$ for any $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ (the optimal strategy depends on random R_μ).

For a function $\Gamma(t,\cdot) : C([0,t];\overset{\circ}{\mathbf{R}}{}^n_+)\times B([0,t];\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^{n\times n})\times\mathbf{R}_+ \to \mathbf{R}^n$, introduce the following norm:

$$\|\Gamma(\cdot)\|_{\mathbf{X}} \triangleq \sup_{\mu(\cdot)=\mu_\alpha(\cdot):\ \alpha\in\mathcal{T}} \left(\sum_{i=1}^n \int_0^T \Gamma_i(t, [S(\cdot),\mu(\cdot)]|_{[0,t]}, R_\mu)^2 dt\right)^{1/2}. \tag{7.16}$$

Definition 7.4 *Let C^R_0 be the set of all admissible CL-strategies $\Gamma(t,\cdot) : B([0,t];\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^N)\times\mathbf{R} \to \mathbf{R}^n$ such that $\pi(t) = \Gamma(t,[S(\cdot),\mu(\cdot)]|_{[0,t]},R_\mu) \in \tilde{\Sigma}^R(\mathcal{F}^a_\cdot)$ for any $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, $\|\Gamma(\cdot)\|_{\mathbf{X}} < +\infty$ and*

$$\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)) \in \hat{D} \quad a.s. \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}).$$

In fact, C^R_0 is a subset of a linear space of functions with the norm (7.16).

7.3.1 A duality theorem

To prove Theorem 7.1, we need the following duality theorem.

THEOREM 7.2 *The following holds:*

$$\begin{aligned}&\sup_{\Gamma(\cdot)\in C_0^R}\inf_{\mu\in\mathcal{A}(\mathcal{T})}\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)))\\ &\qquad=\inf_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})}\sup_{\Gamma(\cdot)\in C_0^R}\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))).\end{aligned}\tag{7.17}$$

To prove Theorem 7.2, we need several preliminary results, which are presented below as lemmas. The first of which is

LEMMA 7.2 *The function $\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))$ is linear in $\Gamma(\cdot)$.*

Proof. By (1.20), it follows that $\tilde{X}(t)=\tilde{X}(t,\Gamma(\cdot),\mu(\cdot))$ satisfies

$$\begin{aligned}\tilde{X}(t)=X(0)+\textstyle\sum_{i=1}^n\int_0^t\ \ p(\tau)\Gamma_i(\tau,[S(\cdot,\mu(\cdot)),\mu(\cdot))]_{[0,\tau]},R)\Big(\tilde{a}_i(t)dt\\ +\textstyle\sum_{j=1}^n\sigma_{ij}(t)dw_j(\tau)\Big).\end{aligned}\tag{7.18}$$

It is easy to see that $\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))$ is linear in $\Gamma(\cdot)$. This completes the proof. □

LEMMA 7.3 *The set C_0^R is convex.*

Proof. Let $p\in(0,1)$, $\mu(\cdot)\in\mathcal{A}(\mathcal{T})$, $\Gamma^{(i)}(\cdot)\in C_0^R$, $i=1,2$, and

$$\Gamma(\cdot)\triangleq(1-p)\Gamma^{(i)}(t)+p\Gamma^{(i)}(\cdot).$$

By Lemma 7.2, it follows that

$$\tilde{X}(T,\Gamma(\cdot),\mu(\cdot))=(1-p)\tilde{X}(T,\Gamma^{(1)}(\cdot),\mu(\cdot))+p\tilde{X}(T,\Gamma^{(2)}(\cdot),\mu(\cdot)).$$

Furthermore, the set $\hat{D}$ is convex; then $\tilde{X}(t,\Gamma(\cdot),\mu(\cdot))\in\hat{D}$ a.s. This completes the proof. □

LEMMA 7.4 *There exists a constant $c>0$ such that*

$$\mathbf{E}|\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot))|^2\le c(\|\Gamma(\cdot)\|_{\mathbf{X}}^2+X_0^2)\quad\forall\Gamma(\cdot)\in C_0^R,\quad\forall\alpha\in\mathcal{T}.$$

Proof. For a $\Gamma(\cdot)\in C_0^R$, let

$$x(t)\triangleq\tilde{X}(t,\Gamma(\cdot),\mu_\alpha(\cdot)),\quad \pi(t)\triangleq\Gamma(t,[S(\cdot,\mu_\alpha(\cdot)),\mu(\cdot))]_{[0,t]},R_\mu),$$
$$\pi(t)=(\pi_1(t),\dots,\pi_n(t)).$$

By (7.18), it follows that

$$\begin{cases} dx(t) = p(t)\sum_{i=1}^{n}\pi_i(t)\left(\sum_{j=1}^{n}\sigma_{ij}dw_j(t) + \tilde{a}(t)dt\right), \\ x(0) = X_0. \end{cases}$$

This is a linear Itô stochastic differential equation, and it is easy to see that the desired estimate is satisfied. This completes the proof. □

LEMMA 7.5 *For a given $\alpha \in \mathcal{T}$, the function*

$$\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)))$$

is continuous in $\Gamma(\cdot) \in \mathcal{C}_0^R$.

Proof. Let $\Gamma^{(i)}(\cdot) \in \mathcal{C}_0^R$ and $\tilde{X}^{(i)}(t) \triangleq \tilde{X}(t,\Gamma^{(i)}(\cdot),\mu_\alpha(\cdot))$, $i = 1,2$. By Lemmas 7.2 and 7.4, it follows that

$$\mathbf{E}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^2 \le c\|\Gamma^{(1)}(\cdot) - \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}^2,$$

where $c > 0$ is a constant. Then

$$\begin{aligned} &\left|\mathbf{E}U\left(\tilde{X}^{(1)}(T)\right) - \mathbf{E}U\left(\tilde{X}^{(2)}(T)\right)\right| \\ &\le c_1\mathbf{E}\left[(1 + |\tilde{X}^{(1)}(T)| + |\tilde{X}^{(2)}(T)|)^{2-q}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^q\right] \\ &\le c_1\left[\mathbf{E}\left(1 + |\tilde{X}^{(1)}(T)| + |\tilde{X}^{(2)}(T)|\right)^2\right]^{1/k'}\left[\mathbf{E}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^2\right]^{1/k} \\ &\le c_2\left(1 + \|\Gamma^{(1)}(\cdot)\|_{\mathbf{X}} + \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}\right)^{1/k'}\|\Gamma^{(1)}(\cdot) - \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}^{1/k}, \end{aligned}$$

where $c_i > 0$ are constants, q is as defined in Condition 7.2, $k \triangleq 2/q$, $k' \triangleq k/(k-1) = 2/(2-q)$. This completes the proof. □

Let

$$z_*(\alpha,T) \triangleq \exp\left(\int_0^T \theta_{\mu_\alpha}(t)^\top dw(t) - \frac{1}{2}\int_0^T |\theta_{\mu_\alpha}(t)|^2 dt\right).$$

For $\alpha \in \mathcal{T}$, set

$$J'(\Gamma(\cdot),\alpha) \triangleq \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot))).$$

LEMMA 7.6 *Let Condition 7.3(ii) holds. Then, for a given $\Gamma(\cdot) \in \mathcal{C}_0^R$, the function $J'(\Gamma(\cdot),\alpha)$ is continuous in $\alpha \in \mathcal{T}$.*

Proof. Let $\Gamma(\cdot) \in \mathcal{C}_0^R$ and $\alpha_i \in \mathcal{T}$, $i = 1,2$. Set

$$Y_\alpha \triangleq \tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot)), \quad \alpha \in \mathcal{T}, \qquad Y_* \triangleq \tilde{X}(T,\Gamma(\cdot),\mu_*(\cdot)),$$

where $\mu_*(t) \triangleq [r(t), 0, \sigma(t)]$. By Girsanov's Theorem (see, e.g., Gihman and Skorohod (1979)), it follows that

$$\begin{aligned}|\mathbf{E}U(Y_{\alpha_1}) &- \mathbf{E}U(Y_{\alpha_2})| = |\mathbf{E}[z_*(\alpha_1,T) - z_*(\alpha_2,T)]U(Y_*)| \\ &\leq c_1\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|(|Y_*|^p + 1) \\ &\leq c_2 \left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'} (\mathbf{E}|Y_*|^p + 1)^q)^{1/q} \\ &\leq c_3 \left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'} (\mathbf{E}|Y_*|^2 + 1)^q)^{1/q} \\ &\leq c_4 \left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'} (\|\Gamma(\cdot)\|^2_{\mathbf{X}} + 1)^{1/q},\end{aligned}$$

where $p \in (1,2)$ is as defined in Conditions 7.2 and 7.3(ii),

$$q \triangleq \frac{2}{p}, \quad q' = \frac{q}{q-1}$$

and $c_i > 0$ are constants.

Furthermore, it is easy to see that for an $\alpha \in A$, we have $z_*(\alpha, T) = y(T)$, where $y(t) = y(t, \alpha)$ is the solution of the equation

$$\begin{cases} dy(t) = y(t)M_a(t,\alpha)^\top M_\sigma(t)^{-1^\top} dw(t), \\ y(0) = 1. \end{cases}$$

It is well known that $y(T)$ depends on $\alpha \in \mathcal{T}$ continuously in $L^{q'}(\Omega, \mathcal{F}, \mathbf{P})$ (see, e.g., Krylov (1980, Ch.2)). Hence

$$\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'} \to 0 \quad \text{as} \quad \alpha_1 \to \alpha_2.$$

This completes the proof. □

Let $\mathcal{V}$ be the set of all σ-additive probability measures on $\mathcal{F}_{\mathcal{T}}$. We consider $\mathcal{V}$ as a subset of $C(\mathcal{T}; \mathbf{R})^*$. (If the set $\mathcal{T}$ is at most countable, then we mean that $C(\mathcal{T}; \mathbf{R})$ is $B(\mathcal{T}; \mathbf{R})$.) Let $\mathcal{V}$ be equipped with the weak* topology in the sense that

$$\nu_1 \to \nu_2 \quad \Leftrightarrow \quad \int_{\mathcal{T}} \nu_1(d\alpha)f(\alpha) \to \int_{\mathcal{T}} \nu_2(d\alpha)f(\alpha) \quad \forall f(\cdot) \in C(\mathcal{T}; \mathbf{R}).$$

LEMMA 7.7 *The set $\mathcal{V}$ is compact and convex.*

Proof. The convexity is obvious. It remains to show the compactness of the set $\mathcal{V}$. In our case, $\mathcal{T}$ is a compact subset of finite–dimensional Euclidean space. Now we note that the Borel σ-algebra of subsets of $\mathcal{T}$ coincides with the Baire σ-algebra (see, e.g., Bauer (1981)). Hence, $\mathcal{V}$ is the set of Baire probability measures. By Theorem IV.1.4 from Warga (1972), it follows that $\mathcal{V}$ is compact. This completes the proof. □

We are now in the position to give a proof of Theorem 7.2.

Proof of Theorem 7.2. For a $\Gamma(\cdot) \in \mathcal{C}_0^R$, we have $J'(\Gamma(\cdot), \cdot) \in C(\mathcal{T}; \mathbf{R})$ and

$$\begin{aligned}\mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))) &= \int_{\mathcal{T}} d\nu_\mu(\alpha) \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu_\alpha(\cdot))) \\ &= \int_{\mathcal{T}} d\nu_\mu(\alpha) J'(\Gamma(\cdot), \alpha),\end{aligned}$$

where $\nu_\mu(\cdot)$ is the measure on $\mathcal{T}$ generated by Θ, which corresponds to $\mu(\cdot)$. Hence, $\mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)))$ is uniquely defined by ν_μ. Let

$$J(\Gamma(\cdot), \nu_\mu) \triangleq \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))).$$

By Lemma 7.7, $J(\Gamma(\cdot), \nu)$ is linear and continuous in $\nu \in \mathcal{V}$ given $\Gamma(\cdot)$.

To complete the proof, it suffices to show that

$$\sup_{\Gamma(\cdot) \in \mathcal{C}_0^R} \inf_{\nu \in \mathcal{V}} J(\Gamma(\cdot), \nu) = \inf_{\nu \in \mathcal{V}} \sup_{\Gamma(\cdot) \in \mathcal{C}_0^R} J(\Gamma(\cdot), \nu). \tag{7.19}$$

We note that $J(\Gamma(\cdot), \nu) : \mathcal{C}_0^R \times \mathcal{V} \to \mathbf{R}$ is linear in ν. By Lemmas 7.2 and 7.5–7.6, it follows that $J(\Gamma(\cdot), \nu)$ is either concave or convex in $\Gamma(\cdot)$ and that $J(\Gamma(\cdot), \nu) : \mathcal{C}_0^R \times \mathcal{V} \to \mathbf{R}$ is continuous in ν for each $\Gamma(\cdot)$ and continuous in $\Gamma(\cdot)$ for each ν. Furthermore, $\mathcal{C}_0^R$ and $\mathcal{V}$ are convex and $\mathcal{V}$ is compact. By the Sion Theorem (see, e.g., Parthasarathy and Ragharan (1971, p.123)), it follows that (7.19), and hence (7.17), are satisfied. This completes the proof of Theorem 7.2. □

We are now in the position to give a proof of Theorem 7.1.

7.3.2 Proof of Theorem 7.1

Let $\hat{\alpha} \in \mathcal{T}$ be such that $R_{\hat{\mu}} = R_{min}$, where $\hat{\mu}(\cdot) \triangleq \mu_{\hat{\alpha}}(\cdot)$. By Lemma 7.1(iii)-(iv), it follows that

$$\mathbf{E}U(\tilde{X}(T, \hat{\Gamma}_{\hat{\alpha}}(\cdot, R_{\hat{\mu}}), \hat{\mu}(\cdot))) \le \mathbf{E}U(\tilde{X}(T, \hat{\Gamma}_\alpha(\cdot, R_{\mu_\alpha}), \mu_\alpha(\cdot))) \quad \forall \alpha \in \mathcal{T}. \tag{7.20}$$

(If $R_{\hat{\mu}_1} = R_{\mu_\alpha}$, then statement (iii) is applicable; if $R_{\hat{\mu}} < R_{\mu_\alpha}$, then statement (iv) is applicable).

Let $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ be arbitrary, and let $\nu_\mu(\cdot)$ be the measure on $\mathcal{T}$ generated by Θ, which corresponds to $\mu(\cdot)$. By (7.20), it follows that

$$\begin{aligned}\mathbf{E}U(\tilde{X}(T, \hat{\Gamma}_{\hat{\alpha}}(\cdot, R_{\hat{\mu}}), \hat{\mu}(\cdot))) &\le \int_{\mathcal{T}} d\nu_\mu(\alpha) \mathbf{E}U(\tilde{X}(T, \hat{\Gamma}_\alpha(\cdot, R_{\mu_\alpha}), \mu_\alpha(\cdot))) \\ &= \sup_{\Gamma(\cdot) \in \mathcal{C}_0^R} \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))) \\ &\qquad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}).\end{aligned} \tag{7.21}$$

By (7.14), (7.21), and Theorem 7.2 it follows that the pair $(\hat{\mu}(\cdot), \hat{\Gamma}_{\hat{\alpha}}(\cdot))$ is a saddle point for the problem (7.8)–(7.9). This completes the proof of Theorem 7.1. □

IV

OPTIMAL STRATEGIES BASED ON HISTORICAL DATA FOR MARKETS WITH NONOBSERVABLE PARAMETERS

Chapter 8

STRATEGIES BASED ON HISTORICAL PRICES AND VOLUME: PROBLEM STATEMENT AND EXISTENCE RESULT

Abstract We consider the investment problem in the class of strategies that do not use direct observations of the appreciation rates of the stocks but rather use historical market data (i.e., stock prices and volume of trade) and prior distributions of the appreciation rates. We formulate the problem statement and prove the existence of optimal strategy for a general case.

8.1. The model

We consider the market model from Section 1.3. The market consists of a risk-free bond or bank account with price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, where $n < +\infty$ is given. The prices of the stocks evolve according to

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right), \quad t > 0, \tag{8.1}$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given non-random constant. The price of the bond evolves according to

$$B(t) = B(0)\exp\left(\int_0^t r(s)ds\right), \tag{8.2}$$

where $B(0)$ is a given constant that we take to be 1 without loss of generality, and $r(t)$ is the random process of the risk-free interest rate.

We are given a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

We shall use again the vector processes

$$\begin{aligned}
&w(t) = (w_1(t), \dots, w_n(t))^\top, \quad S(t) = (S_1(t), \dots, S_n(t))^\top, \\
&a(t) = (a_1(t), \dots, a_n(t))^\top, \quad \tilde{a}(t) = a(t) - r(t)\mathbf{1}, \\
&p(t) \triangleq \exp\left(-\int_0^t r(s)ds\right) = B(t)^{-1}, \quad \tilde{S}(t) \triangleq p(t)S(t)
\end{aligned}$$

and the matrix process $\sigma(t) = \{\sigma_{ij}(t)\}_{i,j=1}^n$.

We assume that $w(\cdot)$ is a standard Wiener process on the given probability space and that $a(t)$, $r(t)$, and $\sigma(t)$ are measurable random processes such that

$$\sigma(t)\sigma(t)^\top \geq c\mathbf{I}_n,$$

where $c > 0$ is a constant and $\mathbf{I}_n$ is the identity matrix in $\mathbf{R}^{n\times n}$. Furthermore, we assume that $w(\cdot)$ and $(r(\cdot), \sigma(\cdot))$ are independent processes and that the process $(r(t), \sigma(t))$ is uniformly bounded.

By Remark 1.1, it follows that the volatility coefficients can be effectively estimated from $S_i(t)$. However, (1.11) may not produce satisfactory results, because actual computations require time discretization. Another approach is to use the implied volatility, but then the possible illiquidity of the option market and time discretization for numerical purposes again pose complications. In any case, we shall ignore these difficulties here. It is more difficult to estimate the appreciation rate $a_i(t)$; in fact, an estimator of $a_i(t)$ is not satisfactory when the volatility is sufficiently large. In view of this, we assume that $r(t)$ and $S(t)$ are directly observable (which is natural), and we consider two cases — when $a(t)$ is currently directly observable and when $a(t)$ is not directly observable but the distribution of $\tilde{a}(t)$ is known.

An additional observable process

We assume that there is a random process that describes some additional available information about the market (in addition to stock prices and the interest rate); this process is directly observable. More precisely, we assume that there exists an integer $N > 0$ and a random process $\eta(t) = (\eta_1(t), \dots, \eta_N(t))$ that is currently observable. For example, $\eta(t)$ can describe prices at external markets, the difference between bid and ask prices for underlying assets, the size of deviation of prices of options on underlying assets near Black-Scholes price, weather, level of unemployment, or any other factors, which can be correlated with the appreciation rates of the stocks. However, the most important example is when $N = n$ and $\eta_i(t)$ is the trading volume for the ith stock at time t. It will be shown below in numerical experiments with real data that the joint distribution of prices and volume contains an important information; we improve the performance of a strategy by including volume in our consideration.

In this chapter and in Chapters 9 and 12, we shall consider the general case of random $\eta(\cdot)$; in the rest of this book, we shall assume that $\eta(t) \equiv 0$ (i.e., only the stock prices and the interest rate are available).

The prior distributions of parameters

To describe the prior distribution of $a(\cdot)$ and $\eta(\cdot)$, we assume that there exist linear Euclidean spaces E and E_0, a measurable set $\mathcal{T} \subseteq E$, random vectors $\Theta : \Omega \to \mathcal{T}$ and $\Theta_0 : \Omega \to E_0$, and measurable functions

$$A(t,\cdot) : \mathcal{T} \times C([0,t];\mathbf{R}^n) \times B([0,t];\mathbf{R} \times \mathbf{R}^N) \to \mathbf{R}^n$$

and

$$F_0(t,\cdot) : E_0 \times B([0,t];\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^N$$

such that

$$\tilde{a}(t) \equiv A\left(t, \Theta, [S(\cdot), r(\cdot), \eta(\cdot)]|_{[0,t]}\right),$$
$$\eta(t) = F_0(t, \Theta_0, [r(\cdot), S(\cdot)]|_{[0,t]}).$$

We are given a probability measure $\nu(\cdot)$ on $\mathcal{T}$ that describes the probability distribution of Θ. We assume that the probability distribution of Θ_0 is also known. Further we assume that

- Θ, Θ_0, $w(\cdot)$, $(r(\cdot), \sigma(\cdot))$ are mutually independent; and
- $\Theta, \Theta_0, A(\cdot), F_0(\cdot)$ are such that the solution of (8.1) is well defined as the unique strong solution of Itô's equation, and

$$\sup_t |\tilde{a}(t)| < \infty \ a.s., \quad \sup_t \mathbf{E}|\tilde{a}(t)|^2 < +\infty.$$

The dependence on $S(\cdot)$ is included for a technical reason, because it is important for a special problem of portfolio compression.

Under these assumptions, the market is incomplete.

Notice that for the simplest model, it suffices to set $\tilde{a}(\cdot) = \Theta(\cdot)$ for a process $\Theta(t)$ independent of $(r(\cdot), w(\cdot), \sigma(\cdot), \eta(\cdot))$. However, our setting allows important models such as

$$a(t) \equiv \Theta(t) f(\eta(t)) + r(t),$$

where $\eta(t)$ is the trade volume and $f(\cdot)$ is a function (for example, $f(y) = \operatorname{arctg} y$).

Another example of a reasonable model is

$$a(t) \equiv \Theta(t) + \frac{2}{t}\int_0^t r(s)\mathbf{1}ds - r(t)\mathbf{1},$$

where a sharp increase of $r(t)$ implies a short-time decrease of $a(t)$.

The wealth and strategies

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$, $X(0) = X_0$. We assume that

$$X(t) = \pi_0(t) + \sum_{i=1}^{n} \pi_i(t), \tag{8.3}$$

where the pair $(\pi_0(t), \pi(t))$ describes the portfolio at time t. The process $\pi_0(t)$ is the investment in the bond, $\pi_i(t)$ is the investment in the ith stock, and $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))^\top, t \geq 0$.

Let $\mathbf{S}(t) \triangleq \operatorname{diag}(S_1(t), \ldots, S_n(t))$ and $\tilde{\mathbf{S}}(t) \triangleq \operatorname{diag}(\tilde{S}_1(t), \ldots, \tilde{S}_n(t))$ be diagonal matrices with the corresponding diagonal elements. The portfolio is said to be self-financing if

$$dX(t) = \pi(t)^\top \mathbf{S}(t)^{-1} dS(t) + \pi_0(t) B(t)^{-1} dB(t). \tag{8.4}$$

It follows that for such portfolios,

$$dX(t) = r(t)X(t)\,dt + \pi(t)^\top \left(\tilde{a}(t)\,dt + \sigma(t)\,dw(t)\right), \tag{8.5}$$

$$\pi_0(t) = X(t) - \sum_{i=1}^{n} \pi_i(t),$$

so π alone suffices to specify the portfolio; it is called a *self-financing* strategy.

DEFINITION 8.1 *The process $\tilde{X}(t) \triangleq p(t)X(t)$ is called the normalized wealth.*

It satisfies

$$\begin{aligned} \tilde{X}(t) &= X(0) + \int_0^t p(s)\pi(s)^\top (\tilde{a}(s)ds + \sigma(s)dw(s)) \\ &= X(0) + \int_0^t B(s)^{-1}\pi(s)^\top \tilde{\mathbf{S}}(s)^{-1} d\tilde{S}(s). \end{aligned} \tag{8.6}$$

Let $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the filtration generated by the process $(r(t), S(t), \eta(t))$ completed with the null sets of $\mathcal{F}$.

Further, let $\{\mathcal{F}_t^a\}_{0 \leq t \leq T}$ be the filtration generated by the process $(a(t), r(t), \sigma(t), S(t))$ completed with the null sets of $\mathcal{F}$, and let $\{\mathcal{F}_t^{a,\eta}\}_{0 \leq t \leq T}$ be the filtration generated by the process $(a(t), r(t), \sigma(t), S(t), \eta(t))$ completed with the null sets of $\mathcal{F}$.

Let $\mathcal{G}_t$ be a filtration.

DEFINITION 8.2 *Let $\bar{\Sigma}(\mathcal{G}.)$ be the class of all $\mathcal{G}_t$-adapted processes $\pi(\cdot)$ such that*

- $\int_0^T \left(|\pi(t)^\top \tilde{a}(t)| + |\pi(t)^\top \sigma(t)|^2\right) dt < \infty$ *a.s., and*
- *there exists a constant q_π such that* $\mathbf{P}\{\tilde{X}(t) - X_0 \geq q_\pi, \forall t \in [0,T]\} = 1.$

A process $\pi(\cdot) \in \bar{\Sigma}(\mathcal{G}.)$ is said to be an *admissible* strategy with corresponding wealth $X(\cdot)$.

For an admissible strategy $\pi(\cdot)$, $X(t, \pi(\cdot))$ denotes the corresponding total wealth, and $\tilde{X}(t, \pi(\cdot))$ the corresponding normalized total wealth.

Note that by definition, admissible strategies from $\bar{\Sigma}(\mathcal{F}.)$ use observations of $(r(t), S(t), \eta(t))$ only. For these strategies, the processes $X(t)$ and $\tilde{X}(t)$ are $\mathcal{F}_t$ adapted.

8.2. A general problem and special cases

In this section, we describe a general optimal investment problem and several important special cases.

8.2.1 The general problem with constraints

Let $T > 0$ and X_0 be given. Let $m > 0$ be an integer. Let $U(\cdot,\cdot)$: $\mathbf{R} \times C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+) \to \mathbf{R} \cup \{-\infty\}$ and $G(\cdot,\cdot) : \mathbf{R} \times C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+) \to \mathbf{R}^m$ be given measurable functions such that $\mathbf{E}U(X_0, \tilde{S}(\cdot)) < +\infty$.

We may state our general problem as follows: Find an admissible self-financing strategy that solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T, \pi(\cdot)), \tilde{S}(\cdot)) \quad \text{over} \quad \pi(\cdot) \tag{8.7}$$

$$\text{subject to} \begin{cases} X(0, \pi(\cdot)) = X_0, \\ G(\tilde{X}(T, \pi(\cdot)), \tilde{S}(\cdot)) \leq 0 \quad \text{a.s.} \end{cases} \tag{8.8}$$

8.2.2 Special cases of constraints and costs functions

As can be seen from the following examples, the general setting (8.7)–(8.8) covers many important special problems.

Optimization of a portfolio with pre-determined positions

The general setting of the investment problem (8.7)–(8.8) covers a special, but quite realistic case, namely,when a portfolio consists of several different types of investments. For example, consider a case when the total portfolio includes the following: (i) a given portfolio of options (for example, selected by the rule introduced in Chapter 3), and (ii) a dynamically adjusted portfolio of stocks. In other words, we assume that the total wealth is divided into two parts (portfolios) $X(t) = X'(t)+X''(t)$, where $X'(t)$ is the price of the dynamically adjusted stock portfolio and $X''(t)$ is the price of the fixed portfolio of options. The problem is to select a strategy for e adjustment.

Let the portfolio of options be such that $\tilde{X}''(T) = \phi(\tilde{S}(\cdot))$, where $\phi(\cdot) : C([0,T];\mathbf{R}^n)$ is a deterministic function that describes the set of payoff functions for the options. (For example, if the portfolio consists of European call options with strike price K and the expiration time T, then $\phi(\tilde{S}(\cdot)) = N \cdot (S(T) - K)^+$, where N is the total amount of the options). Let $X'(0)$ and $X''(0)$ be given. Let $U(\cdot) : \mathbf{R} \to \mathbf{R}$ be a utility function.

We may state a problem of maximization of $\mathbf{E}U(\tilde{X}(T))$ via variations of the dynamically adjusted portfolio:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}'(T,\pi(\cdot)) + \phi(\tilde{S}(\cdot))) \quad \text{over} \quad \pi(\cdot) \tag{8.9}$$

$$\text{subject to} \quad X'(0,\pi(\cdot)) = X'(0). \tag{8.10}$$

Clearly, this problem is a special case of the problem (8.7)–(8.8).

Criterion with a synthetic numéraire

Let $\pi'(\cdot) \in \bar{\Sigma}(\mathcal{F}.)$ be a given strategy such that the corresponding normalized wealth is being considered as a numéraire. This situation can occur when the strategy $\pi'(\cdot)$ is given and an investor wishes to examine a performance of small deviations of this strategy using the wealth $\tilde{X}(t,\pi'(\cdot))$ as a numéraire. The corresponding optimization problem can be stated as the follows. Let $U(\cdot,\cdot) : \mathbf{R}\times\mathbf{R} \to \mathbf{R}\cup\{-\infty\}$ and $G(\cdot,\cdot) : \mathbf{R}\times\mathbf{R} \to \mathbf{R}^m$ be given measurable functions. Consider the problem

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T,\pi(\cdot)),\tilde{X}(T,\pi'(\cdot))) \quad \text{over} \quad \pi(\cdot) \tag{8.11}$$

$$\text{subject to} \begin{cases} X(0,\pi(\cdot)) = X_0, \\ G(\tilde{X}(T,\pi(\cdot)),\tilde{X}(T,\pi'(\cdot))) \le 0 \quad \text{a.s.} \end{cases} \tag{8.12}$$

It can be easily seen that this problem is a special case of the problem (8.7)–(8.8).

Hedging with bounds on risk

Let $h_i(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$ be given functions, $i = 1,2,\ldots$. such that $0 \le h_1(y) \le h_2(y) \le +\infty$ ($\forall y \in C([0,T];\mathbf{R}^n)$). Consider the constraints

$$h_1(\tilde{S}(\cdot)) \le \tilde{X}(T,\pi(\cdot)) \le h_2(\tilde{S}(\cdot)) \quad \text{a.s.} \tag{8.13}$$

These constraints correspond to $G(x,y) = 1 - \chi\{h_1(y) \le x \le h_2(y)\}$, where χ denotes the indicator function.

Hedging with logical constraints

Let $G_i(\cdot,\cdot) : \mathbf{R}\times \overset{\circ}{\mathbf{R}}{}^n_+ \to \mathbf{R}$, $i = 1,2$, be given functions. The following special case of the constraints (8.8) is called a logical-type constraint:

$$\text{If} \quad G_1(\tilde{X}(T,\pi(\cdot)),\tilde{S}(\cdot)) \le 0, \quad \text{then} \quad G_2(\hat{X}(T,\pi(\cdot)),\tilde{S}(\cdot)) \le 0 \quad \text{a.s.} \tag{8.14}$$

This constraint corresponds to

$$G(x,y) = \chi\{G_1(x,y) \le 0\} - \chi\{G_2(x,y) \le 0\}.$$

Hedging of claims with different criterions

Let $h(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$ be a given function. Let $\zeta \triangleq h(\tilde{S}(\cdot))$. (For example, $\zeta = \bar{h}(\tilde{S}(T))$ or $\zeta = \bar{h}\left(\int_0^T \tilde{S}(t)dt\right)$, where $\bar{h} : \mathbf{R}^n \to \mathbf{R}$ is a given function). Let $V(\cdot,\cdot) : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ be a given function. The following is a special case of the general cost function (8.7):

$$\text{Maximize} \quad \mathbf{E}V(\tilde{X}(T,\pi(\cdot)),\zeta). \tag{8.15}$$

In particular, if $V(x,y) = \chi\{y \ge x\}$, then (8.15) reduces to

$$\text{Maximize} \quad \mathbf{P}(\tilde{X}(T,\pi(\cdot)) \ge \zeta).$$

On the other hand, if $V(x,y) = -|x-y|^\delta$, $\delta > 1$, then (8.15) reduces to

$$\text{Minimize} \quad \mathbf{E}|\tilde{X}(T,\pi(\cdot)) - \zeta|^\delta.$$

Conditional criteria

Let $V_1(\cdot,\cdot) : \mathbf{R}\times C([0,T];\mathbf{R}^n) \to \mathbf{R}$, $V_2(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$, $i = 1,2$, be given functions. The following is a special case of the problem (8.7), where the following conditional probability criterion is its cost function:

$$\text{Maximize} \quad \mathbf{P}\left(V_1(\tilde{X}(T,\pi(\cdot)),\tilde{S}(\cdot)) \le 0 \,\middle|\, V_2(\tilde{S}(\cdot)) \le 0\right). \tag{8.16}$$

This criterion corresponds to $U(x,y) = \chi\{V_1(x,y) \le 0\}\cdot\chi\{V_2(x) \le 0\}$.

8.3. Solution via dynamic programming

In this section, we describe briefly the classical approach of dynamical programming to the optimal investment problems with nonobservable appreciation rates (see, e.g., Williams (1977), Detempte (1986), Dothan and Feldman (1986), Gennotte (1986), Brennan (1998)).

Let

$$\hat{a}(t) \triangleq \mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t\},$$

$$\hat{w}(t) \triangleq w(t) - \int_0^t \sigma(s)^{-1}[\tilde{a}(s) - \hat{a}(s)]ds.$$

The following lemma is well known (see Liptser and Shyryaev (1977, p. 278, Theorem 7.12).

LEMMA 8.1 *Assume that the process $\sigma(t)$ is deterministic. Then $\hat{w}(t)$ is a Wiener process, and the equation for stock prices (8.1) can be rewritten as*

$$dS_i(t) = S_i(t)\left([\hat{a}_i(t) + r(t)]dt + \sum_{j=1}^n \sigma_{ij}(t)d\hat{w}_j(t)\right), \quad t > 0. \quad (8.17)$$

COROLLARY 8.1 *The filtration generated by $(S(t), r(t), \hat{a}(t))$ coincides with the filtration $\mathcal{F}_t$.*

Thus, the problem (8.7)–(8.8) is reduced to a problem with directly observable parameters, as described in Chapter 5. However, this problem can be solved explicitly in some special cases only (see Chapter 5). In particular, this problem can be solved via the dynamic programming approach described in Section 5.2, Chapter 5, if the following holds:

- $G(\cdot) \equiv 0, U(x,y) \equiv U(x,y(T)), y \in C([0,T];\mathbf{R}^n)$; and
- the process $\mu(t) \triangleq (r(t), \hat{a}(t), \sigma(t))$ evolves as described by eq. (5.6), Chapter 5, with $\tilde{\sigma}^\mu \equiv 0$, i.e.,

$$d\mu(t) = \beta(B(t), S(t), \mu(t), t)dt + \sigma^{\mu,S}(B(t), S(t), \mu(t), t)dw(t). \quad (8.18)$$

In other words, the dynamic programming approach can be applied when the problem can be Markovianized and there are no constraints. We present below the optimal strategy for this case (which is in fact defined by (5.6), Chapter 5 with $\tilde{\sigma}^\mu \equiv 0$).

DEFINITION 8.3 *Let $\hat{\Sigma}_M$ be the class of all processes $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}_\cdot)$ such that there exists a measurable function $f : [0,T] \times \mathbf{R} \times (\mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n}) \to \mathbf{R}^n$ such that $\pi(t) = f(t, X(t), B(t), S(t), \mu(t))$, where $\mu(t) \triangleq (r(t), \hat{a}, \sigma(t))$.*

Set

$$J(x,b,s,\mu,t) \triangleq$$
$$\sup_{\pi(\cdot)\in\hat{\Sigma}_M} \mathbf{E}\{U(\tilde{X}(T), \tilde{S}(T)) \mid (X(t), B(t), S(t), \mu(t)) = (x,b,s,\mu)\}.$$

Then the Bellman equation, satisfied formally by the value function (derived utility function), $J(x,b,s,\mu,t)$, is (denoting the matrix diag $(s_1,\ldots,s_n)$ by $\mathbf{S}$)

$$\begin{aligned}
&\max_\pi\{J_t(x,b,s,\mu,t)+J_x(x,b,s,\mu,t)(rx+\pi^\top\hat{a})\\
&+J_b(x,b,s,\mu,t)rb+J_s(x,b,s,\mu,t)^\top\mathbf{S}\hat{a}\\
&+J_\mu(x,b,s,\mu,t)^\top\beta(x,b,s,\mu,t)\\
&+\tfrac{1}{2}J_{x,x}(x,b,s,\mu,t)\pi^\top\sigma\sigma^\top\pi+\tfrac{1}{2}\mathrm{tr}\,[J_{s,s}(x,b,s,\mu,t)\mathbf{S}\sigma\sigma^\top\mathbf{S}]\\
&+\tfrac{1}{2}\mathrm{tr}\,[J_{\mu,\mu}(x,b,s,\mu,t)\sigma^{\mu,S}(x,b,s,\mu,t)\sigma^{\mu,S}(x,b,s,\mu,t)^\top]\\
&+J_{x,s}(x,b,s,\mu,t)\mathbf{S}\sigma\sigma^\top\pi+J_{x,\mu}(x,b,s,\mu,t)\sigma^{\mu,S}\sigma^\top\pi\\
&+\mathrm{tr}\,[J_{s,\mu}(x,b,s,\mu,t)\sigma^{\mu,S}(x,b,s,\mu,t)\sigma^\top\mathbf{S}]\}=0,\\
&J(x,b,s,\mu,T)=U\left(\tfrac{x}{b},\tfrac{s}{b}\right).
\end{aligned} \tag{8.19}$$

Then the optimal π is (formally)

$$\pi(t)=-\frac{J_x(X(t),B(t),S(t),\mu(t),t)}{J_{x,x}(X(t),B(t),S(t),\mu(t),t)}Q(t)\hat{a}(t)-\mathbf{S}(t)\frac{J_{s,x}(X(t),B(t),S(t),\mu(t),t)}{J_{x,x}(X(t),B(t),S(t),\mu(t),t)}.$$

The first term on the right-hand side gives the usual mean-variance type of strategy; the second, due to correlation between wealth and stock prices, is absent if $S(t)$ is not required as a state variable, e.g., if a Mutual Fund theorem holds; and the third depends on the correlation between S (or w) and μ and is considered to represent a hedge against future unfavorable behavior of the coefficients. Note that the Bellman equation is degenerate: there are only n driving Wiener processes and $2+n+n^2$ variables. Hence there may not exist a solution J with second-order derivatives.

As an example, consider an important special case when $\tilde{a}=\tilde{a}(t)$ is a constant Gaussian vector. In tis case, the problem can be Markovianized: the process $\mathbf{E}\{(\tilde{a},\tilde{a}\tilde{a}^\top)|\mathcal{F}_t\})$ satisfies a special case of Itô's equation (8.18), which describes the *Kalman–Bucy filter* (see Liptser and Shyryaev (1977). For example, let $n=1$, $\tilde{a}(t)\equiv a$, and let $\tilde{a}$ be Gaussian, $\mathrm{Var}\,\tilde{a}=v_0^2$, and $\mathbf{E}\tilde{a}=0$. Further, let $\sigma(t)\equiv\sigma$ be a nonrandom constant. Then the Kalman–Bucy filter gives

$$\begin{cases} d\hat{a}(t)=\frac{v(t)}{\sigma^2}\left[\frac{d\tilde{S}(t)}{\tilde{S}(t)}-\hat{a}(t)dt\right],\\ dv(t)=-\frac{v(t)^2}{\sigma^2}dt,\\ \hat{a}(0)=a_0,\quad v(0)=v_0.\end{cases}$$

Here $v(t)=\mathbf{E}\{(\tilde{a}-\hat{a})^2|\mathcal{F}_t\}$.

We present below a more convenient version of (8.19). Assume that $(r(t),\sigma(t))$ is deterministic. Let the prior distribution of $\tilde{a}$ be such that there exist an integer $M>0$ and measurable functions $\Phi(\cdot):\mathbf{R}^{M\times n}\to\mathbf{R}^M$, $f(\cdot):$

$\mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^M$, $\sigma^y(\cdot) : \mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^{M\times n}$, $\tilde{\sigma}^y(\cdot) : \mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^{M\times M}$ and an observable process $y(t)$ such that

$$\hat{a}(t) \equiv \Phi(y(t), t),$$

and $y(t)$ is the solution of the Itô equation

$$dy(t) = f(y(t), t)dt + \sigma^y(y(t), t)dw(t) + \tilde{\sigma}^y(y(t), t)d\tilde{w}(t),$$

where $\tilde{w}(\cdot)$ is a Wiener process of dimension N that is independent on $w(\cdot)$. In particular, this setting may cover the case when $y(t) = (\hat{a}(t), S(t), \eta(t))$, where $\eta(t)$ is the process of market data introduced above.

DEFINITION 8.4 *Let $\Sigma_{M,Y}$ be the class of all processes $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}.)$ such that there exists a measurable function $\phi : [0,T] \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}^n$ such that $\pi(t) = f(t, X(t), y(t))$.*

Assume that

$$U(x, \tilde{S}(\cdot)) \equiv U(x, y(T)), \quad G(\cdot) \equiv 0.$$

Set

$$V(x, y, t) \triangleq \sup_{\pi(\cdot)\in\hat{\Sigma}_{M,Y}} \mathbf{E}\{U(\tilde{X}(T), y(T)) | (\tilde{X}(t), y(t)) = (x, y)\}.$$

Then the Bellman equation, satisfied formally by the value function (derived utility function), $V(x, y, t)$, is (denoting the matrix diag $(s_1, \dots, s_n)$ by $\mathbf{S}$)

$$\begin{aligned} \max_\pi \Big\{ & V_t(x,y,t) + p(t)V_x(x,y,t)p\pi^\top\Phi(y,t) + V_y(x,y,t)^\top f(x,y,t) \\ & + \tfrac{1}{2}p(t)^2 V_{x,x}(x,y,t)\pi^\top\sigma\sigma^\top\pi \\ & + \tfrac{1}{2}\operatorname{tr}\left[V_{y,y}(x,y,t)\left[\sigma^y(y,t)\sigma^y(y,t)^\top + \tilde{\sigma}^y(x,y,t)\tilde{\sigma}^\mu(x,y,t)^\top\right]\right] \\ & + p(t)\operatorname{tr}\left[V_{x,y}(x,y,t)\mathbf{S}\sigma\sigma^y(y,t)^\top\pi\right]\Big\} = 0, \\ & V(x,y,T) = U(x,y). \end{aligned}$$

Then the optimal π is (formally)

$$\pi(t) = -\frac{V_x(\tilde{X}(t), y(t), t)}{V_{x,x}(\tilde{X}(t), y(t), t)}Q(t)\hat{a}(t) - \mathbf{S}(t)\frac{V_{x,y}(\tilde{X}(t), y(t), t)}{V_{x,x}(\tilde{X}(t), y(t), t)}.$$

Unfortunately, this dynamic programming approach cannot be applied to problems with constraints and to problems that can not be Markovianized. Furthermore, it can be difficult to solve numerically nonlinear degenerate Bellman equations. We describe below some alternative methods.

8.4. Additional definitions

Set

$$\mathcal{Z} \triangleq \exp\left(\int_0^T (\sigma(t)^{-1}\tilde{a}(t))^\top dw(t) + \frac{1}{2}\int_0^T |\sigma(t)^{-1}\tilde{a}(t)|^2 dt\right). \quad (8.20)$$

Since $\sigma(\cdot)^{-1}\tilde{a}(\cdot)$ is independent of $w(\cdot)$ with bounded trajectories, then $\mathbf{E}\mathcal{Z}^{-1} = 1$.

Define the (equivalent) probability measure $\mathbf{P}_*$ by

$$\frac{d\mathbf{P}_*}{d\mathbf{P}} = \mathcal{Z}^{-1}.$$

Let $\mathbf{E}_*$ be the corresponding expectation.

REMARK 8.1 *It follows that for* $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}_\cdot)$, *the normalized wealth* $\tilde{X}(t,\pi(\cdot))$ *is a* $\mathbf{P}_*$*-supermartingale with* $\mathbf{E}_*\tilde{X}(t,\pi(\cdot)) \leq X_0$ *and* $\mathbf{E}_*|\tilde{X}(t,\pi(\cdot))| \leq |X_0| + 2|q_\pi|$.

Rather than employing usual technique of changing the Brownian motion using an equivalent martingale measure, we will change the stock price process to one that has the distribution of a risk-neutral price under the original measure. The technique relies on the law-uniqueness of the solution of the S.D.E. Let $S_*(t) \triangleq (S_{*1}(t),\ldots,S_{*n}(t))$ be the solution of the equation

$$\begin{cases} dS_*(t) = \mathbf{S}_*(t)\left(r(t)\mathbf{1}dt + \sigma(t)dw(t)\right), & t > 0, \\ S_*(0) = S(0), \end{cases} \quad (8.21)$$

where $\mathbf{S}_*(t) \triangleq \operatorname{diag}(S_{*1}(t),\ldots,S_{*n}(t))$ is the corresponding diagonal matrix.

Let $\tilde{S}_*(t) \triangleq p(t)S_*(t)$, and let

$$\tilde{\mathbf{S}}_*(t) \triangleq \operatorname{diag}(\tilde{S}_{*1}(t),\ldots,\tilde{S}_{*n}(t)), \quad \mathbf{S}_*(t) \triangleq \operatorname{diag}(S_{*1}(t),\ldots,S_{*n}(t))$$

be diagonal matrices with the corresponding diagonal elements.

For $\alpha \in \mathcal{T}$, set

$$\begin{aligned} &\eta_*(t) = (t,\Theta_0,[r(\cdot),S_*(\cdot)]|_{[0,t]}), \\ &\bar{A}(t,\alpha) \triangleq A(t,\alpha,[S(\cdot),r(\cdot),\eta(\cdot)]_{[0,t]}), \\ &\bar{A}_*(t,\alpha) \triangleq A(t,\alpha,[S_*(\cdot),r(\cdot),\eta_*(\cdot)]_{[0,t]}). \end{aligned}$$

For each $\alpha \in \mathcal{T}$, introduce the process $z(\alpha,t)$ as a solution of the equations

$$\begin{cases} dz(\alpha,t) = z(\alpha,t)\bar{A}(t,\alpha)^\top Q(t)\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \\ z(\alpha,0) = 1. \end{cases} \quad (8.22)$$

Set

$$\bar{\mathcal{Z}} \triangleq \int_{\mathcal{T}} d\nu(\alpha) z(\alpha, T).$$

Set

$$\mathcal{X}_t = C([0,t];\mathbf{R}^n) \times B([0,t];\mathbf{R}) \times B([0,t];\mathbf{R}^N),$$
$$\bar{\mathcal{X}}_t = C([0,t];\mathbf{R}^n) \times B([0,t];\mathbf{R}^n) \times B([0,t];\mathbf{R}) \times B([0,t];\mathbf{R}^N).$$

Set

$$\tilde{a}_*(t) \triangleq \bar{A}_*(t,\Theta) = A(t,\Theta,[S_*(\cdot),r(\cdot),\eta_*(\cdot)]_{[0,t]}),$$
$$a_*(t) = \tilde{a}_*(t) + r(t),$$
$$\mathcal{Z}_* \triangleq \exp\left(\int_0^T (\sigma(t)^{-1}\tilde{a}_*(t))^\top dw(t) - \frac{1}{2}\int_0^T |\sigma(t)^{-1}\tilde{a}_*(t)|^2 dt\right). \quad (8.23)$$

By definition, for each $\mathcal{F}_T$-measurable random variable ξ, there exists a measurable function $\phi : \mathcal{X}_T \to \mathbf{R}$ such that $\xi = \phi(S(\cdot), r(\cdot), \eta(\cdot))$. We shall use the notation ξ_* for the random number $\xi_* \triangleq \phi(S_*(\cdot), r(\cdot), \eta_*(\cdot))$.

PROPOSITION 8.1 *There exists a measurable function $\psi(\cdot) : \bar{\mathcal{X}}_T \to \mathbf{R}$ such that $\mathcal{Z}_* = \psi(S_*(\cdot), \tilde{a}_*(\cdot), r(\cdot))$ and $\mathcal{Z} = \psi(S(\cdot), \tilde{a}(\cdot), r(\cdot))$ a.s. Moreover, $z(\alpha, T) = \psi(S(\cdot), \bar{A}(\cdot, \alpha), r(\cdot))$.*

Let

$$\bar{\mathcal{Z}}_* \triangleq \int_{\mathcal{T}} d\nu(\alpha)\psi(S_*(\cdot), \bar{A}_*(\cdot,\alpha), r(\cdot)) \triangleq \bar{\psi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)).$$

Moreover, it follows from Proposition 8.1 that $\bar{\mathcal{Z}} = \bar{\psi}(S(\cdot), r(\cdot), \eta(\cdot))$. Finally, since Θ is independent of $(r(\cdot), w(\cdot), \sigma(\cdot), \eta(\cdot))$ and hence of $(S_*, r, \eta_*(\cdot))$, it follows that

$$\bar{\mathcal{Z}}_* = \mathbf{E}(\mathcal{Z}_* | S_*(\cdot), r(\cdot), \eta_*(\cdot)) \quad (8.24)$$

and

$$\bar{\mathcal{Z}} = \int_{\mathcal{A}} d\nu(\alpha)\psi(S(\cdot), \bar{A}(\cdot,\alpha), r(\cdot), \eta(\cdot)).$$

8.5. Existence result for the general case

8.5.1 Auxiliary problem and additional assumptions

Auxiliary optimization problem. Our approach is to investigate the problem (8.7)–(8.8) via the following finite-dimensional optimization problem: For $q \in \mathbf{R}$, $\lambda \in \mathbf{R}$, $y \in C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+)$,

$$\text{Maximize} \quad qU(x,y) - \lambda x \quad \text{over} \quad x \in \mathbf{R} : G(x,y) \le 0. \quad (8.25)$$

This problem will be used in the following way. We obtain an optimal claim as a solution of the problem (8.25) with the random number q depending on $\tilde{S}(\cdot)$. Then, the corresponding admissible self-financing strategy, which replicates the claim, is obtained readily.

Let

$$J(y) \triangleq \{x \in \mathbf{R} : G(x,y) \leq 0\}, \quad y \in C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+). \tag{8.26}$$

To proceed further, we assume that the following conditions are satisfied.

CONDITION 8.1 *At least one of the following conditions holds:*

(i) $U(x,y) \equiv \log x$, $X_0 > 0$ *and* $G(\cdot) \equiv 0$; *or*

(ii) *The process* $\sigma(t)$ *is nonrandom and known, and the processes* $\tilde{a}(\cdot)$ *and* $r(\cdot)$ *are independent.*

CONDITION 8.2 *There exists a measurable set* $\Lambda \subseteq \mathbf{R}$ *and a measurable function* $F(\cdot) : (0,\infty) \times C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+) \times \Lambda \to \mathbf{R}$ *such that for each* $z > 0$, $y \in C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+)$, $\lambda \in \Lambda$,

$$\hat{x} = F(z,y,\lambda)$$

is a solution of the optimization problem (8.25).

CONDITION 8.3 *There exist* $\hat{\lambda} \in \Lambda$ *such that* $\mathbf{E}_*|F(\bar{\mathcal{Z}},\tilde{S}(\cdot),\hat{\lambda})| < +\infty$ *and*

$$\mathbf{E}_* F(\bar{\mathcal{Z}},\tilde{S}(\cdot),\hat{\lambda}) = X_0. \tag{8.27}$$

CONDITION 8.4 *The function* $G(\cdot)$ *is such that there exists a measurable function* $f(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$ *such that* $\mathbf{E}|f(\tilde{S}_*(\cdot))|^2 < +\infty$, $\mathbf{E}f(\tilde{S}_*(\cdot)) = X_0$ *and* $G(f(\tilde{S}(\cdot)),\tilde{S}(\cdot)) \leq 0$ *a.s.*

Conditions 8.1–8.4 are satisfied for the special cases described in Chapters 9-11 with appropriately chosen parameters.

8.5.2 Existence result

We solve our original problem in two steps. First we show that $\mathbf{E}U(F(\bar{\mathcal{Z}},\hat{S}(\cdot),\hat{\lambda}))$ is an upper bound for the expected utility of normalized terminal wealth for $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}_\cdot)$. Then we find sufficient conditions for this claim to be attainable. This establishes the optimality of a strategy that replicates the claim (if attainable).

Let $F(\cdot)$ be as in Condition 8.2.

THEOREM 8.1 *With $\hat{\lambda}$ as in Condition 8.3, let*

$$\hat{\xi} \triangleq F(\bar{\mathcal{Z}}, \tilde{S}(\cdot), \hat{\lambda}). \tag{8.28}$$

Then

(i) $\mathbf{E}\,U^{-}(\hat{\xi}, \tilde{S}(\cdot)) < \infty$, $G(\hat{\xi}, \tilde{S}(\cdot)) \le \infty$ *a.s.*

(ii) Let $F(z, x, \hat{\lambda})$ *be the unique solution of the optimization problem (8.25) with* $\lambda = \hat{\lambda}$, *and let* $\mathbf{E}\,U^{+}(\hat{\xi}, \tilde{S}(\cdot)) < +\infty$. *Then* $\hat{\xi}$ *is unique (i.e. even for different* $\hat{\lambda}$, *the corresponding* $\hat{\xi}$ *agree up to equivalency).*

(iii) $\mathbf{E}U(\hat{\xi}, \tilde{S}(\cdot)) \ge \mathbf{E}U(\tilde{X}(T, \pi(\cdot)), \tilde{S}(\cdot))$, $\forall \pi(\cdot) \in \bar{\Sigma}(\mathcal{F}.)$.

(iv) Let one of the following two conditions be satisfied:

(a) $U(x, y) \equiv \log x$, $G(\cdot) \equiv 0$; *or*

(b) $\eta(t) \equiv 0$ *(i.e., there is now additional available information).*

Then the claim $B(T)\hat{\xi}$ *is attainable in* $\bar{\Sigma}(\mathcal{F}.)$, *and there exists a unique replicating strategy in* $\bar{\Sigma}(\mathcal{F}.)$. *This strategy is optimal for the problem (8.7)-(8.8).*

In other words, $B(T)\hat{\xi}$ is an optimal contingent claim for the problem (8.7)-(8.8) (if $\hat{\xi}$ is attainable). Thus, the optimal investment problem is reduced to the replication of the claim $B(T)\hat{\xi}$.

8.6. The optimal strategy as a conditional expectation

In this section, we assume that the optimal claim $\hat{\xi}$ defined in Theorem 8.1 is $\hat{\xi} = \hat{F}(\bar{\mathcal{Z}})$, where $\hat{F}(\cdot) : \mathbf{R} \to \mathbf{R}$ is a measurable deterministic function (i.e., $G(x, y)$ and $U(x, y)$ do not depend on x). In addition, up to the end of this section, we assume that

$$E = B([0, T]; \mathbf{R}^n),$$
$$A(t, \Theta, [S(\cdot), r(\cdot), \eta(\cdot)]_{[0,t]}) \equiv \Theta(t).$$

We give a solution for a case of differentiable $\hat{F}(\cdot)$. We obtain the optimal strategy only as a conditional expectation of a given random variable. Though this approach does not give an explicit solution, it opens the way for using Monte Carlo simulation. Let Condition 8.1 (ii) be satisfied so that $\theta_\alpha(t) \triangleq \sigma(t)^{-1}\alpha(t)$ is nonrandom. Set

$$\begin{aligned} z_*(\alpha, t) &\triangleq \exp\left(\int_0^t \theta_\alpha(s)^\top dw(s) - \tfrac{1}{2}\int_0^t |\theta_\alpha(s)|^2 ds\right), \\ \bar{\mathcal{Z}}_*(t) &\triangleq \int_{\mathcal{A}} d\nu(\alpha) z_*(\alpha, t). \end{aligned} \tag{8.29}$$

THEOREM 8.2 *Assume*

(i) there exists $\hat{F}'(x) \triangleq d\hat{F}(x)/dx$;

(ii) there exists $\varepsilon > 0$ *such that* $\mathbf{E}_*|\hat{F}'(\bar{\mathcal{Z}})|^{2+\varepsilon} < +\infty$;

(iii) the function $\left(\hat{F}(x), \hat{F}'(x)\right)$ *is either bounded or Hölder; and*

(iv) the functions $\theta_\alpha(\cdot)$ *are bounded, uniformly in* $\alpha \in \mathcal{A}$, *right continuous, and of bounded variation.*

Let $f(t,\cdot) : B([0,t];\mathbf{R}^n) \to \mathbf{R}$ *be a measurable function defined as*

$$f(t, w(\cdot)|_{[0,t]}) \triangleq \mathbf{E}\left\{\hat{F}'(\bar{\mathcal{Z}}_*(T))\Psi(t,T)\middle|\mathcal{F}_t\right\},$$

where

$$\Psi(t,T) \triangleq \int_{\mathcal{A}} d\nu(\alpha) z_*(\alpha,T)\theta_\alpha(t).$$

Then $\mathbf{E}\int_0^T |f(t, w(\cdot)|_{[0,t]})|^2 dt < +\infty$, *and*

$$\hat{\xi}_* = \mathbf{E}\hat{\xi}_* + \int_0^T f(t, w(\cdot)|_{[0,t]})^\top dw(t).$$

COROLLARY 8.2 *Under the assumptions of Theorem 8.2, there exists a measurable function* $f_0(t,\cdot) : B([0,t];\mathbf{R}^n) \to \mathbf{R}$ *such that* $f(t, w(\cdot)|_{[0,t]}) \equiv f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})$. *For such* $f_0(\cdot)$, *the following holds:*

$$\hat{\xi} = X_0 + \int_0^T f_0(t, \tilde{S}(\cdot)|_{[0,t]})^\top \sigma(t)^{-1}\tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t).$$

The strategy $\hat{\pi}(t) = B(t) f_0(t, \tilde{S}(\cdot)|_{[0,t]})^\top \sigma(t)^{-1}$ *replicates* $B(T)\hat{\xi}$, *belongs to* $\bar{\Sigma}(\mathcal{F}_\cdot)$, *and is optimal for the problem (8.7)–(8.8) in the class* $\bar{\Sigma}(\mathcal{F}_\cdot)$.

8.7. Proofs

Proof of Proposition 8.1. It follows from (1.11) that there exists a measurable function $\mathcal{V}(t,\cdot) : C([0,t];\mathbf{R}^n) \to \mathbf{R}^{n\times n}$ such that

$$V(t) = \mathcal{V}(t, S(\cdot)|_{[0,t]}) \equiv \mathcal{V}(t, S_*(\cdot)|_{[0,t]}) \qquad (8.30)$$

up to equivalency. By (8.30), it follows that

$$Q(t) = \mathcal{Q}(t, S(\cdot)|_{[0,t]}) \equiv \mathcal{Q}(t, S_*(\cdot)|_{[0,t]})$$

up to equivalency, where $Q(t,\cdot) : C([0,t];\mathbf{R}^n) \to \mathbf{R}^{n\times n}$ is a measurable function. Then

$$\log \mathcal{Z} = \int_0^T \tilde{a}(t)^\top Q(t, S(\cdot)|_{[0,t]}) \left[\mathbf{S}(t)^{-1} dS(t) - r(t)\mathbf{1}dt - \frac{1}{2}\tilde{a}(t)dt\right],$$

and

$$\log \mathcal{Z}_* = \int_0^T \tilde{a}_*(t)^\top Q(t, S_*(\cdot)|_{[0,t]}) \left[\mathbf{S}_*(t)^{-1} dS_*(t) - r(t)\mathbf{1}dt - \frac{1}{2}\tilde{a}_*(t)dt\right].$$

This completes the proof of Proposition 8.1.□

PROPOSITION 8.2 *Let $\phi : \bar{\mathcal{X}}_T \to \mathbf{R}$ be a measurable function such that $\mathbf{E}\phi^-(S(\cdot), \tilde{a}(\cdot), r(\cdot), \eta(\cdot)) < +\infty$, and let $\hat{\phi}$ be a similar function but with no dependence on $\tilde{a}$. Then*

$$\mathbf{E}\phi(S(\cdot), \tilde{a}(\cdot), r(\cdot), \eta(\cdot)) = \mathbf{E}\mathcal{Z}_*\phi(S_*(\cdot), \tilde{a}_*(\cdot), r(\cdot), \eta_*(\cdot)), \tag{8.31}$$

$$\mathbf{E}\hat{\phi}(S(\cdot), r(\cdot), \eta(\cdot)) = \mathbf{E}\bar{\mathcal{Z}}_*\hat{\phi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)), \tag{8.32}$$

$$\mathbf{E}_*\hat{\phi}(S(\cdot), r(\cdot), \eta(\cdot)) = \mathbf{E}\hat{\phi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)). \tag{8.33}$$

Proof. By assumptions, $(\Theta, \Theta_0, r(\cdot), \sigma(\cdot))$ does not depend on $w(\cdot)$. Then to prove (8.31), it suffices to prove

$$\begin{aligned} &\mathbf{E}\left\{\phi(S(\cdot), \tilde{a}(\cdot), r(\cdot), \eta(\cdot)) \middle| \Theta, \Theta_0, r(\cdot), \sigma(\cdot)\right\} \\ &= \mathbf{E}\left\{\mathcal{Z}_*\phi(S_*(\cdot), \tilde{a}_*(\cdot), r(\cdot), \eta_*(\cdot)) \middle| \Theta, \Theta_0, r(\cdot), \sigma(\cdot)\right\} \quad \text{a.s.} \end{aligned} \tag{8.34}$$

Thus, for the next paragraph, without loss of generality, we will suppose that $(\Theta, \Theta_0, r(\cdot), \sigma(\cdot))$ is deterministic.

By Girsanov's Theorem, the process

$$\hat{w}(t) \triangleq w(t) - \int_0^t \sigma(s)^{-1}\tilde{a}_*(s)ds$$

is a Wiener process on the the probability spaces defined by the probability measure $\hat{\mathbf{P}}$ such that $d\hat{\mathbf{P}}/d\mathbf{P} = \mathcal{Z}_*$. Furthermore, equations (8.1) and (8.21) can be rewritten as

$$dS(t) = \mathbf{S}(t)\left[a(t)dt + \sigma(t))dw(t)\right],$$
$$dS_*(t) = \mathbf{S}_*(t)\left[a_*(t)dt + \sigma(t))d\hat{w}(t)\right].$$

Since $(\Theta, \Theta_0, r(\cdot), \sigma(\cdot))$ are taken as deterministic, the processes $S(\cdot)$ and $S_*(\cdot)$ have the same distribution on the probability spaces defined by $\mathbf{P}$ and $\hat{\mathbf{P}}$, respectively, and (8.34), and hence (8.31) follow. Further, (8.32) follows by taking conditional expectation in (8.31). Finally, using Proposition 8.1 and (8.31),

$$\begin{aligned}
&\mathbf{E}_*\hat{\phi}(S(\cdot), r(\cdot), \eta(\cdot)) = \mathbf{E}\mathcal{Z}^{-1}\hat{\phi}(S(\cdot), r(\cdot), \eta(\cdot)) \\
&= \mathbf{E}\psi(S(\cdot), \tilde{a}(\cdot), r(\cdot), \eta(\cdot))^{-1}\hat{\phi}(S(\cdot), r(\cdot), \eta(\cdot)) \\
&= \mathbf{E}\mathcal{Z}_*\psi(S_*(\cdot), \tilde{a}_*(\cdot), r(\cdot), \eta_*(\cdot))^{-1}\hat{\phi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)) \\
&= \mathbf{E}\hat{\phi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)) \\
&= \mathbf{E}\hat{\phi}(S_*(\cdot), r(\cdot), \eta_*(\cdot)).
\end{aligned}$$

□

Introduce a class Φ of random numbers (claims) ξ that are $\mathcal{F}_T$-measurable with $\mathbf{E}U^-(\xi, \tilde{S}(\cdot)) < \infty$ and $\mathbf{E}_*(\xi)^- < \infty$.

Set

$$\Phi_0 \triangleq \{\xi \in \Phi : \quad \mathbf{E}_*\xi = X_0, \quad G(\xi, \tilde{S}(\cdot)) \leq 0 \text{ a.s.}\}.$$

By definition, for each $\xi \in \Phi$, there exists a measurable function $\phi : \mathcal{X}_T \to \mathbf{R}$ such that $\xi = \phi(S(\cdot), r(\cdot), \eta(\cdot))$. Recall that we use the notation ξ_* for the random number $\xi_* \triangleq \phi(S_*(\cdot), r(\cdot), \eta(\cdot))$.

Now define $\hat{\xi}_* \triangleq F(\bar{\mathcal{Z}}_*, \tilde{S}_*(\cdot), \hat{\lambda})$. If we define ϕ by $\hat{\xi} = \phi(S(\cdot), r(\cdot), \eta(\cdot))$, then $\hat{\xi}_* = \phi(S_*(\cdot), r(\cdot), \eta_*(\cdot))$.

By Proposition 8.2, for $\xi \in \Phi_0$,

$$\begin{aligned}
X_0 = \mathbf{E}_*\xi \quad &= \mathbf{E}\mathcal{Z}^{-1}\phi(S(\cdot), r(\cdot), \eta(\cdot)) \\
&= \mathbf{E}\psi(S(\cdot), \tilde{a}(\cdot), r(\cdot), \eta(\cdot))^{-1}\phi(S(\cdot), r(\cdot), \eta(\cdot)) \\
&= \mathbf{E}\mathcal{Z}_*\psi(S_*(\cdot), \tilde{a}(\cdot), r(\cdot), \eta_*(\cdot))^{-1}\phi(S_*(\cdot), r(\cdot), \eta_*(\cdot)) \\
&= \mathbf{E}\phi(S_*(\cdot), r(\cdot), \eta_*(\cdot)) = \mathbf{E}\xi_*.
\end{aligned} \tag{8.35}$$

Define $\mathcal{J}_i : \Phi \to \mathbf{R}$, $i = 0, 1$ by

$$\mathcal{J}_0(\xi) \triangleq \mathbf{E}U(\xi, \tilde{S}(\cdot)), \quad \mathcal{J}_1(\xi) \triangleq \mathbf{E}_*\xi - X_0.$$

Consider the problem

$$\text{Maximize} \quad \mathcal{J}_0(\xi) \quad \text{over} \quad \xi \in \Phi_0. \tag{8.36}$$

PROPOSITION 8.3 *The optimization problem (8.36) has solution* $\hat{\xi} \triangleq F(\bar{\mathcal{Z}}, \tilde{S}(\cdot), \hat{\lambda})$, *where* $\hat{\lambda}$ *is given by Condition 8.3.*

Proof. By Condition 8.3 and Proposition 8.2, it follows that $\mathbf{E}_*|\hat{\xi}| = \mathbf{E}|\hat{\xi}_*| < +\infty$. By Condition 8.3 again, $\mathbf{E}_*\hat{\xi} = X_0$.

Let $L(\xi, \lambda) \triangleq \mathcal{J}_0(\xi) - \lambda\mathcal{J}_1(\xi)$, where $\xi \in \Phi$ and $\lambda \in \mathbf{R}$. By Proposition 8.2,

$$L(\xi, \lambda) = \mathbf{E}\Big(\mathcal{Z}_*U(\xi_*, \tilde{S}_*(\cdot)) - \lambda\xi_*\Big) + \lambda X_0$$

We have that the process $S_*(\cdot)$ satisfies (8.21), where the processes $\hat{r}(t)$ and $\sigma(t)$ are independent of Θ. Hence ξ_* does not depend on Θ for any $\xi \in \Phi_0$. Then

$$L(\xi, \lambda) = \mathbf{E}\Big(\bar{\mathcal{Z}}_*U(\xi_*, \tilde{S}_*(\cdot)) - \lambda\xi_*\Big) + \lambda X_0.$$

By Condition 8.2, it follows that for any $\omega \in \Omega$, the random number $\hat{\xi}_*$ provides the maximum over $y \in J(\tilde{S}_*(\cdot))$ for the function $\bar{\mathcal{Z}}_*U(y, \tilde{S}_*(\cdot)) - \hat{\lambda}y$.

Let us show that $\mathbf{E}U^-(\hat{\xi}, \tilde{S}(\cdot)) < \infty$. For $k = 1, 2, \ldots$, introduce the random events

$$\Omega_*^{(k)} \triangleq \{-k \leq U(\hat{\xi}_*, \tilde{S}_*(\cdot)) \leq 0\}, \quad \Omega^{(k)} \triangleq \{-k \leq U(\hat{\xi}, \tilde{S}_*(\cdot)) \leq 0\},$$

along with their indicator functions, $\chi_*^{(k)}$ and $\chi^{(k)}$, respectively. The number $\hat{\xi}_*$ provides the unique maximum of the function $\bar{\mathcal{Z}}_*U(\xi_*, \tilde{S}_*(\cdot)) - \hat{\lambda}\xi_*$ over $J(\tilde{S}_*(\cdot))$, and $G(X_0, \tilde{S}_*(\cdot)) \leq 0$. Hence by Girsanov's Theorem again, we have, for all $k = 1, 2, \ldots$,

$$\begin{aligned}
\mathbf{E}\chi^{(k)}U(\hat{\xi}, \tilde{S}(\cdot)) - \mathbf{E}\chi_*^{(k)}\hat{\lambda}\hat{\xi}_* &= \mathbf{E}\chi_*^{(k)}\Big(\bar{\mathcal{Z}}_*U(\hat{\xi}_*, \tilde{S}_*(\cdot)) - \hat{\lambda}\hat{\xi}_*\Big) \\
&\geq \mathbf{E}\chi_*^{(k)}\Big(\bar{\mathcal{Z}}_*U(X_0, \tilde{S}_*(\cdot)) - \hat{\lambda}X_0\Big) \\
&= \mathbf{E}\chi^{(k)}U(X_0, \tilde{S}(\cdot)) - \hat{\lambda}X_0\mathbf{P}(\Omega_*^{(k)}) \\
&\geq -|U(X_0, \tilde{S}(\cdot))| - |\hat{\lambda}X_0| \\
&> -\infty.
\end{aligned}$$

Furthermore, we have that $\mathbf{E}_*|\hat{\xi}| = \mathbf{E}|\hat{\xi}_*| < +\infty$. Hence $\mathbf{E}U^-(\hat{\xi}, \tilde{S}(\cdot)) < \infty$, leading to $\hat{\xi} \in \Phi$.

Further,

$$L(\xi, \hat{\lambda}) \leq L(\hat{\xi}, \hat{\lambda}) \quad \forall\xi \in \Phi. \tag{8.37}$$

Let $\xi \in \Phi_0$ be arbitrary. We have that $\mathcal{J}_1(\xi) = 0$ and $\mathcal{J}_1(\hat{\xi}) = 0$; then

$$\mathcal{J}_0(\xi) - \mathcal{J}_0(\hat{\xi}) = \mathcal{J}_0(\xi) + \hat{\lambda}\mathcal{J}_1(\xi) - \mathcal{J}_0(\hat{\xi}) - \hat{\lambda}\mathcal{J}_1(\hat{\xi}) = L(\xi, \hat{\lambda}) - L(\hat{\xi}, \hat{\lambda}) \leq 0.$$

Hence $\hat{\xi}$ is an optimal solution of the problem (8.36). □

Proof of Theorem 8.1. Parts (i) and (iii) follow from Proposition 8.3.

To show (ii), note that if $\mathbf{E}U^+(\hat{\xi}, \tilde{S}(\cdot)) < +\infty$, then $L(\hat{\xi}, \hat{\lambda}) < +\infty$. Let $\xi' \in \Phi_0$ be an optimal solution of the problem (8.36). Let $\hat{\lambda}$ be any number such that Condition 8.3 holds. It is easy to see that

$$L(\xi', \hat{\lambda}) = \mathcal{J}_0(\xi') \geq \mathcal{J}_0(\hat{\xi}) = L(\hat{\xi}, \hat{\lambda}).$$

By assumptions, it follows that $\hat{\xi}_*$ provides the unique maximum in the set $J(\tilde{S}_*(\cdot))$ of $\bar{\mathcal{Z}}_* U(\xi_*, \tilde{S}_*(\cdot)) - \hat{\lambda}\xi_*$. Thus, $\xi'_* = \hat{\xi}_*$, $\xi' = \hat{\xi}$ and $\xi' = \eta(\hat{\lambda})$ a.s. for any $\hat{\lambda}$ from Condition 8.3. Thus (ii) is satisfied.

To show (iv), note that if Condition 8.1(i) is satisfied, then the claim is attainable (see Theorem 9.2). Let Condition 8.1(ii) be satisfied. Then $\hat{\xi} = \phi(w(\cdot))$, where $\phi(\cdot) : B([0,T]; \mathbf{R}^n) \to \mathbf{R}$ is a measurable functions. By the martingale representation theorem,

$$\hat{\xi}_* = \mathbf{E}\hat{\xi}_* + \int_0^T f(t, w(\cdot)|_{[0,t]})^\top dw(t),$$

where $f(t, \cdot) : B([0,t]; \mathbf{R}^n) \to \mathbf{R}$ is a measurable function such that $\int_0^T |f(t, w(\cdot)|_{[0,t]})|^2 dt < +\infty$ a.s. There exists a unique measurable function $f_0(t, \cdot) : B([0,t]; \mathbf{R}^n) \to \mathbf{R}$ such that $f(t, w(\cdot)|_{[0,t]}) \equiv f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})$. Thus,

$$\begin{aligned} \hat{\xi}_* &= \mathbf{E}\hat{\xi}_* + \textstyle\int_0^T f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})^\top dw(t) \\ &= \mathbf{E}\hat{\xi}_* + \textstyle\int_0^T f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})^\top \sigma(t)^{-1} \tilde{\mathbf{S}}_*(t)^{-1} d\tilde{S}_*(t). \end{aligned}$$

But $\mathbf{E}_*\hat{\xi} = X_0$. It follows that

$$\hat{\xi} = X_0 + \int_0^T f_0(t, \tilde{S}(\cdot)|_{[0,t]})^\top \sigma(t)^{-1} \tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t).$$

By (8.5), the strategy $\hat{\pi}(t)^\top = B(t) f_0(t, \tilde{S}(\cdot)|_{[0,t]})^\top \sigma(t)^{-1}$ replicates $B(T)\hat{\xi}$, and it can be seen that it belongs to $\bar{\Sigma}(\mathcal{F}_\cdot)$. This completes the proof of Theorem 8.1. □

Proof of Theorem 8.2. It is easy to see that

$$\forall k > 0 \quad \exists c > 0 : \quad k^{-1} \leq x \leq k \quad \Rightarrow |F(x, \hat{\lambda})| + |F'(x, \hat{\lambda})| \leq c.$$

If $T_k \triangleq T \wedge \inf\{t > 0 : z_*(\alpha, t) \notin [k^{-1}, k] \text{ for some } \alpha \in \mathcal{A}\}$, $k > 1$, then $T_k \to +\infty$ as $k \to +\infty$ a.s. because of the uniform bound on θ_α.

Since

$$\begin{aligned} z_*(\alpha, t) &= \exp\Big(- \textstyle\int_{[0,t]} d\theta_\alpha(s)^\top w(s) + \theta_\alpha(t)^\top w(t) \\ &\qquad - \theta_\alpha(0)^\top w(0) - \frac{1}{2} \textstyle\int_0^t |\theta_\alpha(s)|^2 ds \Big), \end{aligned} \tag{8.38}$$

then for a given α, $z_*(\alpha, T_k) = \phi_k(w(\cdot), \alpha)$, where $\phi_k(\cdot, \alpha) : C([0,T]; \mathbf{R}^n) \to \mathbf{R}$ is a measurable function that has the Frechet derivative

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\phi_k(h(\cdot) + \varepsilon\rho(\cdot), \alpha) - \phi_k(h(\cdot), \alpha)] = \int_0^{T_k} d\mu_k(h(\cdot), t, \alpha)\rho(t) \quad (8.39)$$

for any $\rho(\cdot) \in C([0,T]; \mathbf{R}^n)$, where $\mu_k(h(\cdot), t, \alpha)$ is a right-continuous, $1 \times n$ valued function of bounded variation in t. From (8.38) and (8.39), we find

$$\mu_k(h(\cdot), t, \alpha) = \begin{cases} 0 & \text{if } t < 0 \\ -\phi_k(\cdot, \alpha)\theta_\alpha(t)^\top & \text{if } 0 \le t < T_k \\ 0 & \text{if } t \ge T_k. \end{cases}$$

Let $t \in (0, T)$ be fixed. Then

$$\int_{\mathcal{A}} d\nu(\alpha) \int_{(t,T_k]} d\mu_k(w(\cdot), s, \alpha) = \Psi(t, T_k)^\top.$$

Set $\hat{\xi}_{*k} = F(\bar{\mathcal{Z}}_*(T_k), \hat{\lambda})$. By Clark's formula,

$$\hat{\xi}_{*k} = \mathbf{E}\,\hat{\xi}_{*k} + \int_0^{T_k} f_k(t, w(\cdot)|_{[0,t]})^\top dw(t),$$

where $f_k(t, \cdot) : B([0,t]; \mathbf{R}^n) \to \mathbf{R}$ is a measurable function defined by

$$f_k(t, w(\cdot)|_{[0,t]}) = \mathbf{E}\left\{ F'(\bar{\mathcal{Z}}_*(T_k), \hat{\lambda})\Psi(t, T_k) \middle| \mathcal{F}_t \right\}$$

(Clark (1970)). It can be seen that $\bar{\mathcal{Z}}_*(T_k)$ and $\hat{\xi}_{*k}$ converge in $L^q(\Omega, \mathbf{P}, \mathcal{F}, \mathbf{R})$ to $\bar{\mathcal{Z}}_*(T)$ and $\hat{\xi}_*$ respectively, as $k \to +\infty$ for any $q > 1$. Moreover, $\Psi(t, T_k)$ converges in $L^q(\Omega, \mathbf{P}, \mathcal{F}, \mathbf{R})$ to $\Psi(t, T)$ as $k \to +\infty$ for any $q > 1$ uniformly in $t < T$. Then

$$\mathbf{E}\int_0^T |f_k(t, w(\cdot)|_{[0,T]}) - f(t, w(\cdot)|_{[0,T]})|^2 dt \to 0 \quad \text{as} \quad k \to +\infty.$$

The proof follows. □

Proof of Corollary 8.2. We have that $\mathbf{E}_*\hat{\xi} = X_0$, and

$$\begin{aligned} \hat{\xi}_* &= \mathbf{E}\,\hat{\xi}_* + \textstyle\int_0^T f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})^\top dw(t) \\ &= X_0 + \textstyle\int_0^T f_0(t, \tilde{S}_*(\cdot)|_{[0,t]})^\top \sigma(t)^{-1}\tilde{\mathbf{S}}_*(t)^{-1} d\tilde{S}_*(t). \end{aligned}$$

Hence the strategy $\hat{\pi}(t) = B(t)f_0(t, \tilde{S}(\cdot)|_{[0,t]})^\top \sigma(t)^{-1}$ replicates $B(T)\hat{\xi}$. It can be seen that $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}_\cdot)$. □

Chapter 9

SOLUTION FOR LOG AND POWER UTILITIES WITH HISTORICAL PRICES AND VOLUME

Abstract We present the explicit solution of an optimal investment problem without additional constraints for log and power utility functions. Results are shown results for numerical experiments with historical data.

Assume that the conditions imposed in Chapter 8 are satisfied. Up to the end of this chapter, we assume that the following additional condition is satisfied.

CONDITION 9.1 *At least one of the following conditions holds:*
(i) $U(x,y) \equiv \log x$, $X_0 > 0$ *and* $G(\cdot) \equiv 0$; *or*
(ii) *the process* $\sigma(t)$ *is nonrandom and known, the processes* $\tilde{a}(\cdot)$ *and* $r(\cdot)$ *are independent, and*

$$\begin{aligned} &E = B([0,T];\mathbf{R}^n), \\ &\eta(t) \equiv 0; \\ &A(t,\Theta,[S(\cdot),r(\cdot),\eta(\cdot)]_{[0,t]}) \equiv \Theta(t), \end{aligned}$$

i.e., $\bar{A}(t,\alpha) = A(t,\alpha,[S(\cdot),r(\cdot),\eta(\cdot)]_{[0,t]}) \equiv \alpha(t)$.

9.1. Replicating special polynomial claims

We now find the replicating strategy for a special claim $B(T)\sum_{i=1}^N \bar{\mathcal{Z}}_i^m$, where $\bar{\mathcal{Z}}$ is as defined in Chapter 8 and where m_i are inregers.

Set

$$\gamma(\alpha_1,\ldots,\alpha_m) \triangleq \exp\Bigg\{\sum_{\substack{i,j=1 \\ i<j}}^m \int_0^T \bar{A}(t,\alpha_i)^\top Q(t)\bar{A}(t,\alpha_j)dt\Bigg\}.$$

N. Dokuchaev, *Dynamic Portfolio Strategies: Quantitative Methods and Empirical Rules for Incomplete Information*

Note that in case $m = 1$, we have $\gamma(\alpha_1) \equiv 1$. If Condition 9.1 (ii) holds, then

$$\gamma(\alpha_1, \dots, \alpha_m) \triangleq \exp\left\{ \sum_{\substack{i,j=1 \\ i<j}}^{m} \int_0^T \alpha_i(t)^\top Q(t) \alpha_j(t) dt \right\}.$$

Define

$$G(m) \triangleq \int_{\mathcal{A}^m} d\nu(\alpha_1) \cdots d\nu(\alpha_m) \gamma(\alpha_1, \dots, \alpha_m).$$

LEMMA 9.1 *Let $\zeta \triangleq \bar{\mathcal{Z}}^m$, where $m > 0$ is an integer, and assume that either $m = 1$ or Condition 9.1 (ii) holds. Further, assume that $G(m) < \infty$. Then*

$$\zeta = \int_{\mathcal{A}^m} d\nu(\alpha_1) \cdots d\nu(\alpha_m) \gamma(\alpha_1, \dots, \alpha_m) z \left(\sum_{k=1}^{m} \alpha_k, T \right), \tag{9.1}$$

$$\mathbf{E}_* \zeta = G(m). \tag{9.2}$$

If the strategy

$$\hat{\pi}(t)^\top = B(t) \int_{\mathcal{A}^m} d\nu(\alpha_1) \cdots d\nu(\alpha_m) \gamma(\alpha_1, \dots, \alpha_m) \times z \left(\textstyle\sum_{k=1}^{m} \alpha_k, t \right) \sum_{k=1}^{m} \bar{A}(t, \alpha_k)^\top Q(t) \tag{9.3}$$

belongs to $\bar{\Sigma}(\mathcal{F}.)$, then this strategy replicates $B(T)\zeta$ if and only if

$$X(0) = \mathbf{E}_* \zeta. \tag{9.4}$$

Note that $G(m) < \infty$ and the strategy (9.3) belong to $\bar{\Sigma}(\mathcal{F}.)$ if, for example, $\tilde{a}(t)$ is uniformly bounded or $\tilde{a}(t)$ is Gaussian with small enough variance.

LEMMA 9.2 *Let $\zeta \triangleq \hat{F}(\bar{\mathcal{Z}})$, where $\hat{F}(z) \triangleq \sum_{i=1}^N C_i z^{m_i} + C_0$, and $m_i > 0$ are integers, $C_i \in \mathbf{R}$. Let $G(m_i) < \infty$ and the strategy (9.3) belong to $\bar{\Sigma}(\mathcal{F}.)$ for all $m = m_i$. Then a strategy that belongs to $\bar{\Sigma}(\mathcal{F}.)$ and replicates $B(T)\zeta$ exists if and only if $X(0) = \mathbf{E}_* \zeta$. If the replicating strategy exists, then it is*

$$\hat{\pi}(t) = \sum_{i=1}^{N} C_i \hat{\pi}_i(t), \tag{9.5}$$

where $\hat{\pi}_i(t)$ is the strategy defined by (9.3) with $m = m_i$.

THEOREM 9.1 *Let $\hat{\xi} = \sum_{i=1}^N C_i \bar{\mathcal{Z}}^{m_i} + C_0$, where $m_i > 0$ are integers, $C_i \in \mathbf{R}$. Let $G(m_i) < \infty$ and the strategy (9.3) belong to $\bar{\Sigma}(\mathcal{F}.)$ for all $m = m_i$. Then the optimal strategy for the problem (8.7)–(8.8) in the class $\bar{\Sigma}(\mathcal{F}.)$ exists, is unique, replicates $B(T)\hat{\xi}$, and is defined by (9.5).*

In the next sections, we shall apply this result to several utility functions.

9.2. Log utility and minimum variance estimation of a

In this section, we assume that Condition 8.1 (i) holds, i.e., we *do not* assume that $\sigma(t)$ is nonrandom.

THEOREM 9.2 *Let* $U(x) \equiv \log x$, $X_0 > 0$ *and* $(0, +\infty) \subseteq \hat{D}$. *Then*

(i) *The unique optimal solution in the class* $\bar{\Sigma}(\mathcal{F}^a_\cdot)$ *of the problem (8.7)–(8.8) is*

$$\pi(t)^\top \triangleq X(t)\tilde{a}(t)^\top Q(t), \tag{9.6}$$

where $X(t) = X(t, \pi(\cdot))$ *is the corresponding wealth.*

(ii) *The optimal solution in the class* $\bar{\Sigma}(\mathcal{F}_\cdot)$ *of the problem (8.7)-(8.8) is*

$$\hat{\pi}(t)^\top \triangleq X_0 B(t) \int_{\mathcal{A}} d\nu(\alpha) z(\alpha, t) \bar{A}(t, \alpha)^\top Q(t). \tag{9.7}$$

In that case,

$$X(t, \hat{\pi}(\cdot)) = X_0 B(t) \int_{\mathcal{A}} d\nu(\alpha) z(\alpha, t), \tag{9.8}$$

and $X(t, \hat{\pi}(\cdot)) > 0$ *a.s. for all t.*

9.3. Power utility

Now we assume that Condition 8.1 (ii) holds. From (9.2) we have $\mathbf{E}_*\{\bar{\mathcal{Z}}^m\} = G(m)$, but we can also compute an expectation under $\mathbf{P}$ as follows. Since

$$E\left\{z\left(\sum_{k=1}^m \alpha_k, T\right) \middle| \tilde{a}(\cdot)\right\} = \exp\left\{\int_0^T \sum_{k=1}^m \alpha_k(t)^\top Q(t)\tilde{a}(t)\,dt\right\},$$

then (9.1) with $m = l - 1$ implies the following.

LEMMA 9.3 *For any integer* $l > 0$,

$$\mathbf{E}\{\bar{\mathcal{Z}}^{l-1}\} = G(l). \tag{9.9}$$

Let $F(\cdot)$ be as defined in Condition 8.2.

THEOREM 9.3 *Let* $(0, +\infty) \subseteq \hat{D}$, $X_0 > 0$, $U(x) \equiv x^\delta$, $\delta = (l-1)/l$ *for some integer* $l > 1$, *and* $G(l) < \infty$. *Then*

(i) $F(z, y, \lambda) \equiv z^l(\delta/\lambda)^l$, $\hat{\lambda} = \delta X_0^{-1/l}(\mathbf{E}_*\bar{\mathcal{Z}}^l)^{1/l}$, $N = 1$, $C_0 = 0$, $C_1 = (\delta/\hat{\lambda})^l$, $m_1 = l$.

(ii) *The unique optimal solution in the class* $\bar{\Sigma}(\mathcal{F}^a_\cdot)$ *of the problem (8.7)–(8.8) is*

$$\pi(t)^\top \triangleq lX(t, \pi(\cdot))\tilde{a}(t)^\top Q(t), \tag{9.10}$$

where $X(t, \pi(\cdot))$ *is the corresponding wealth.*

(iii) *The optimal solution in the class $\bar\Sigma(\mathcal{F}_\cdot)$ of the problem (8.7)–(8.8) is*

$$\hat\pi(t)^\top \triangleq X_0 B(t)\frac{\int_{A^l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)z(\sum_{k=1}^l \alpha_k,t)\sum_{k=1}^l \alpha_k^\top Q(t)}{\int_{A^l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)}. \tag{9.11}$$

In that case,

$$X(t,\hat\pi(\cdot)) = X_0 B(t)\frac{\int_{A^l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)z\left(\sum_{k=1}^l \alpha_k,t\right)}{\int_{A^l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)}, \tag{9.12}$$

and $X(t,\hat\pi(\cdot)) > 0$ a.s. for all t. Moreover,

$$\mathbf{E}U(\tilde X(T,\hat\pi(\cdot))) = X_0^\delta G(l)^{1-\delta}. \tag{9.13}$$

9.4. Filters (estimators) for the appreciation rate

The optimal solution of the problem (8.7)–(8.8) under Conditions 5.1–8.3 in the class $\bar\Sigma(\mathcal{F}_\cdot^a)$ is presented in Chapter 5 under some additional conditions on the utility function. By definition of $\bar\Sigma(\mathcal{F}_\cdot^a)$, this solution has the form $\pi(t) = \Gamma(t,[S(\cdot),\tilde a(\cdot),r(\cdot),\eta(\cdot)]|_{[0,t]})$, where $\Gamma(t,\cdot) : \bar{\mathcal{X}}_t \to \mathbf{R}^n$ is a measurable function.

Definition 9.1 *Let $\pi(t) = \Gamma(t,[S(\cdot),\tilde a(\cdot),r(\cdot),\eta(\cdot)]|_{[0,t]})$ be an optimal solution of the problem (8.7)–(8.8) in the class $\bar\Sigma(\mathcal{F}_\cdot^a)$, where $\Gamma(t,\cdot) : \bar{\mathcal{X}}_t \to \mathbf{R}^n$ is a measurable function. Further, let $\hat\pi(t)$ be an optimal solution of the problem (8.7)–(8.8) in the class $\bar\Sigma(\mathcal{F}_\cdot)$, and let there exists an n-dimensional $\mathcal{F}_t$-adapted random vector process $\hat a(t)$ such that $\hat\pi(t) \equiv \Gamma(t,[S(\cdot),\hat a(\cdot),r(\cdot),\eta(\cdot)]|_{[0,t]})$. Then $\hat a(t)$ is said to be the U-optimal filter of $\tilde a(t)$ with respect to the problem (8.7)–(8.8).*

We then say that $\hat a(t) + r(t)\mathbf{1}$ is the U-optimal filter of the appreciation rate $a(t)$.

Note that we *do not* assume that $\hat a(t)$ is a function of the current conditional distribution $\mathcal{P}_{\tilde a(t)}(\,\cdot\,|S(\tau),\tau < t)$ of $\tilde a(t)$.

Corollary 9.1 (i) *Under the conditions of Theorem 9.2, when $U(x) \equiv \log x$, the U-optimal filter of $\tilde a(t)$ with respect to the problem (8.7)–(8.8) is*

$$\hat a(t) = \frac{X_0 B(t)}{X(t,\hat\pi(\cdot))}\int_A d\nu(\alpha)z(\alpha,t)A\left(t,\alpha,[S(\cdot),r(\cdot),\eta(\cdot)]_{[0,t]}\right), \tag{9.14}$$

where $X(t,\hat\pi(\cdot))$ is the wealth defined by (9.8).

(ii) *The process $\hat{a}(t)$ defined by (9.14) is such that $\hat{a}(t) = \mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t\}$, i.e., it is the minimum variance estimate in the class of estimates based on observations of $S(t), r(t), \eta(t)$ given $(\mathcal{A}, \nu(\cdot))$. The optimal expected utility is*

$$\mathbf{E}\log \tilde{X}(T, \hat{\pi}(\cdot)) = \frac{1}{2}\mathbf{E}\int_0^T \hat{a}(t)^\top Q(t)\hat{a}(t)dt + \log X_0. \qquad (9.15)$$

We will see below that the estimate $\hat{a}(t)$ under the assumptions of Theorem 9.3, when $U(x) = x^{l-1/l}$, *differs* from $\mathbf{E}\{\tilde{a}(t)|S(\tau), \tau < t\}$ and, in general, is not a function of the current conditional distribution $\mathcal{P}_{\tilde{a}(t)}(\cdot|S(\tau), \tau < t)$ of $\tilde{a}(t)$. However, we can write $\hat{a}$ as a conditional expectation of $\tilde{a}$ if we change measure. Let us do so.

Let $b(t) = \{b_i(t)\}_{i=1}^l \in \mathcal{A}^l$ be a process such that for any measurable $B \subset \mathcal{A}^l$,

$$\mathbf{P}(b(\cdot) \in B) = \frac{1}{G(l)}\int_B d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1, \ldots, \alpha_l).$$

Let

$$\mathcal{A}_l \triangleq \left\{\sum_{i=1}^l \alpha_i(\cdot): \quad \alpha_i(\cdot) \in \mathcal{A}\right\} \subseteq B([0,T]; \mathbf{R}^n),$$

and let $\nu_l(\cdot)$ be the probability distribution on $\mathcal{A}_l$ of the process $\sum_{i=1}^l b_i(t)$. To obtain $\mathbf{P}_l$, we replace ν and $\mathcal{A}$ by ν_l and $\mathcal{A}_l$ respectively.

COROLLARY 9.2 *Under the conditions of Theorem 9.3, the U-optimal filter of $\tilde{a}(t)$ with respect to the problem (8.7)–(8.8) is*

$$\hat{a}(t) = \frac{Q(t)^{-1}\hat{\pi}(t)}{lX(t, \hat{\pi}(\cdot))} = l^{-1}\mathbf{E}_l\{\tilde{a}(t)|\mathcal{F}_t\}, \qquad (9.16)$$

where $X(t, \hat{\pi}(\cdot))$ is the wealth defined by (9.12).

In particular, if $\nu(\cdot)$ describes the distribution of a Gaussian random vector $\Theta(t) = \tilde{a}(t) \equiv \tilde{a}$ that does not depend on time, then $b(t) \equiv b$ is a Gaussian vector and has the probability density function

$$\frac{1}{G(l)}\varphi(x_1)\cdots\varphi(x_l)\exp\sum_{\substack{i,j=1\\i<j}}^l \int_0^T x_i^\top Q(t)x_j dt,$$

where $\varphi(\cdot)$ is the probability density function of $\tilde{a}$. In other words, $\nu_l(\cdot)$ is the distribution defined by the random vector $\sum_{i=1}^l b_i$, where b_i are correlated Gaussian variables with the same distribution as $\tilde{a}$; the correlation can be calculated from $\gamma(\cdot)$. This means that Kalman–Bucy filters can be employed to calculate $\hat{a}(t)$.

9.5. Portfolio compression for log utility

We use the notations of Chapter 6 here. Let m be given, $0 < m < n$. Consider the following problem:

$$\text{Maximize} \quad \mathbf{E} \log \tilde{X}(T, \pi(\cdot))) \quad \text{over} \quad \pi(\cdot) \in \bar{\Sigma}(m, \mathcal{F}_\cdot, \mathcal{F}_\cdot) \tag{9.17}$$

$$\text{subject to} \begin{cases} \tilde{X}(0, \pi(\cdot)) = X_0, \\ \tilde{X}(T, \pi(\cdot)) \geq 0 \quad \text{a.s.} \end{cases} \tag{9.18}$$

Set

$$\hat{a}(\bar{I}, t) \triangleq V(t)Q(\bar{I}, t)P(\bar{I})\hat{a}(t) = V(t)Q(\bar{I}, t)\hat{a}(t). \tag{9.19}$$

THEOREM 9.4 *Assume that $U(x) \equiv \log x$, $X_0 > 0$ and $(0, +\infty) \subseteq \hat{D}$, i.e. they are as in Theorem 9.2. Let $\hat{I}(\cdot) \in \mathcal{I}_m(\mathcal{F}_\cdot)$ be such that*

$$\hat{I}(t) \in \arg\max_{\bar{I} \in \mathcal{M}_m} \hat{a}(\bar{I}, t)^\top Q(t)\hat{a}(\bar{I}, t) \quad \textit{for a.e. } t \textit{ a.s.} \tag{9.20}$$

Then the strategy

$$\hat{\pi}(t)^\top = X(t, \hat{\pi}(\cdot))\hat{a}_{\hat{I}}(t)^\top Q(t), \tag{9.21}$$

where $X(t, \hat{\pi}(\cdot))$ is the corresponding wealth, belongs to the class $\bar{\Sigma}(m, \mathcal{F}_\cdot, \mathcal{F}_\cdot)$ and is optimal for the original problem (9.17)–(9.18) in the class $\bar{\Sigma}(m, \mathcal{F}_\cdot, \mathcal{F}_\cdot)$.

9.6. Some experiments with historical data

We have carried out the following experiment. Using daily price data from 1984 to 1997 for 16 leading Australian stocks (AMC, ANZ, LEI, LLC, LLN, MAY, MLG, MMF, MWB, MIM, NAB, NBH, NCM, NCP, NFM and NPC), we generated samples of price data for one synthetic stock for a time window of length T as $\{S(t)\}_{t \in [t_0, t_0+T]} \triangleq \{S_i(t)/S_i(t_0)\}_{t \in [t_0, t_0+T]}$ where $S_i(t)$ is the price of the ith stock, $i = 1, 2, \ldots, 16$, and where the various samples correspond to the possible values of i and of t_0. We accepted the hypothesis that the stock price $S(\cdot)$ for this model is a continuous process on the interval $[0, T]$ that satisfies (8.1) with $n = 1$. Thus, we have a set of paths. The distribution is described by (8.1) with two unknown processes $a(t)$ and $\sigma(t)$ as parameters.

In theory, the distribution of $(a(t), \sigma(t))$ can be estimated from the given data, but this is a difficult problem, since typically the effect of $a(\cdot)$ is overshadowed by that of the diffusion term. Instead, we examine the impact on the optimal expected utility of three different prior hypotheses about the distribution of a for the simple case of log utility.

We consider the case of daily adjustment of the portfolio and trading with 100 periods, which corresponds to $T = 100/365 \times 7/5 = 0.4$ (since on average there are at most five trading days in a week). In fact, the full 13 years of data was not available for all the stocks, but with $T = 100$ (trading) days ~ 0.4 years, we have 38,737 trials. Expectations are taken as averages over this collection of samples. Since the evaluation of a strategy can only be made after one collects the results of using it many times (i.e., either for different stocks or for different time intervals), we claim that our model and our experiment are not unreasonable.

By Corollary 9.1, the optimal strategy in the class $\bar{\Sigma}(\mathcal{F}.)$ for the log utility function is

$$\begin{cases} \hat{\pi}(t) = \hat{k}(t)X(t), \\ \hat{k}(t) \stackrel{\Delta}{=} \mathbf{E}\{k(t)|\mathcal{F}_t\}, \\ k(t) \stackrel{\Delta}{=} \sigma(t)^{-2}\tilde{a}(t). \end{cases} \tag{9.22}$$

It is seen that $\hat{k}(t) = \sigma(t)^{-2}\mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t\}$ depends on the prior of $\tilde{a}$, i.e., on $(\mathcal{A}, \nu(\cdot))$. We assumed that $r(t) \equiv 0.07$. Using the data with (1.11) gives $\sigma(t)$, which oscillates around 0.29, so we take $\sigma(t) \equiv 0.29$.

Let us describe the three hypotheses that we examined in our experiment.

Hypothesis 1. $\tilde{a}(t) = \tilde{a}$ does not depends on t and is Gaussian with parameters $\operatorname{Var}\tilde{a} = v_0$ and $\mathbf{E}\tilde{a} = \bar{a}_0$.

Hypothesis 2. There exist numbers k_1 and k_2 such that $\mathbf{P}(k(t) \equiv k_i) = 0.5$, $i = 1, 2$ (i.e., $k(t)$ can take only two values).

Hypothesis 3. There exist numbers k_1 and k_2 such that

$$\mathbf{P}(\,k(t) \equiv k_i f(\eta(t))\,) = 0.5, \qquad i = 1, 2,$$

where

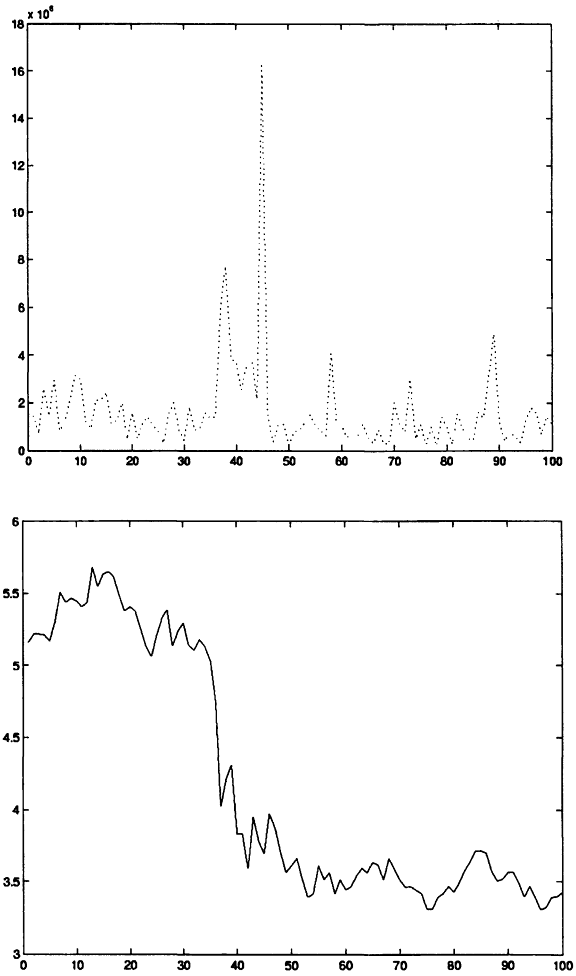
$$\eta(t) \stackrel{\Delta}{=} \frac{y(t)}{Y(t)}, \qquad Y(t) \stackrel{\Delta}{=} \frac{1}{t}\int_0^t y(s)ds, \qquad f(x) \stackrel{\Delta}{=} \frac{\operatorname{arctg}(Cx)}{\operatorname{arctg}(C)},$$

and where $y(t)$ is the trading volume at the time t for the underlying stock, with $C > 0$ as a parameter.

Note that Hypothesis 3 takes into account trading volume and assumes that the appreciation rate is positively correlated with trading volume. The function arctg here was chosen empirically, mostly because of its role in the neural network approach (and it appears that this choice ensures a good performance).

Figure 9.6 shows an example of the daily trading volume and daily prices for ANZ (Australia New Zealand) Bank stocks from September 1, 1987, to January 21, 1988 (i.e., for 100 trading days, including the October 1987 market crash).

Figure 9.1. The trading volume and the stock price for ANZ Bank stocks during 100 trading days from 1 September 1987 to 21 January 1988: ——: values of stock price; · · ·: values of trading volume.



Under Hypothesis 1, $\mathbf{E}\{\tilde{a}|\mathcal{F}_t\} = \hat{a}(t)$ can be found using Kalman–Bucy filtering (see, e.g., Brennan (1998)), and then the optimal strategy is defined by (9.22). In fact,

$$\begin{cases} d\hat{a}(t) = \frac{v(t)}{\sigma^2}\left[\frac{d\tilde{S}(t)}{\tilde{S}(t)} - \hat{a}(t)dt\right], \\ dv(t) = -\frac{v(t)^2}{\sigma^2}dt, \\ \hat{a}(0) = \bar{a}_0, \quad v(0) = v_0. \end{cases}$$

Here $v(t) = \mathbf{E}\{(\tilde{a} - \hat{a})^2|\mathcal{F}_t\}$. If $v_0 = 0$, then $\tilde{a}$ is a nonrandom constant.

Under Hypotheses 2, the equation for the optimal strategy (9.22) can be rewritten as

$$\hat{\pi}(t) = \frac{1}{2} X_0 B(t) \sum_{i=1,2} z_i(t) k_i, \tag{9.23}$$

where $z_i(t) = z(k_i, t)$ are defined by (8.22), which, after Euler approximation, is

$$z_i(t_{j+1}) = z_i(t_j) + z_i(t_j) k_i \frac{\tilde{S}(t_{j+1}) - \tilde{S}(t_j)}{\tilde{S}(t_j)},$$

where $\tilde{S}(t_j)$ is the normalized stock price at time t_j.

Under Hypotheses 3, the equation for the optimal strategy (9.22) can be rewritten as

$$\hat{\pi}(t) = \frac{1}{2} X_0 B(t) \sum_{i=1,2} z_i(t) k_i f(\eta(t)), \tag{9.24}$$

where $z_i(t) = z(k_i, t)$ are defined by (8.22), which, after Euler approximation, is

$$z_i(t_{j+1}) = z_i(t_j) + z_i(t_j) k_i f(\eta(t_j)) \frac{\tilde{S}(t_{j+1}) - \tilde{S}(t_j)}{\tilde{S}(t_j)}.$$

Finally, we found that the average of $\log[\tilde{S}(T)/S(0)]$ is 0.0091. This is the expected utility of the "buy-and-hold" strategy.

In Tables 9.1, 9.2, and 9.3, we exhibit the impact of these hypotheses on the performance of the strategy (9.22). For Table 9.1, we assume Hypothesis 1 and give the optimal expected utility for various values of the first two moments $(\bar{a}_0, v_0)$ of the Gaussian distribution of $\tilde{a}$. The best value, i.e., maximum expected utility, in Table 9.1 is 0.0098 and is attained at

$$\bar{a}_0 = 0.04, \quad v_0 = 0.05.$$

For Tables 9.2 and 9.3, we assume Hypotheses 2 and 3 (respectively) with various values for the parameters δ and $\bar{k}$, where $\bar{k} \triangleq (k_1 + k_2)/2$ and $\delta \triangleq (k_2 - k_1)/2$ (clearly, the pair $(\bar{k}, \delta)$ uniquely defines (k_1, k_2)). In Table 9.2, $\max \mathbf{E} \log \tilde{X}(T) = 0.0140$ is attained at

$$\bar{k} = 0.75, \qquad \delta = 1.5, \quad \text{i.e.,} \quad k_1 = 2.25, \quad k_2 = -0.75. \tag{9.25}$$

In Table 9.3, $\max \mathbf{E} \log \tilde{X}(T) = 0.0404$ is attained at

$$\bar{k} = 0.75, \quad \delta = 1.5, \quad C = 0.5 \tag{9.26}$$

It is interesting to note that Hypothesis 3 outperforms Hypothesis 1 under most of the parameter values tested, and all three outperform the buy-and-hold strategy. Moreover, this result is quite robust under variations: we obtain

Table 9.1. Average $\log \tilde{X}(T)$ for strategies based on Gaussian Hypothesis 1 for various parameter values of Gaussian distribution of $\tilde{a}$.

	$v_0 = 0$	$v_0 = 0.01$	$v_0 = 0.05$	$v_0 = 0.15$	$v_0 = 0.25$
$\bar{a}_0 = 0.01$	0.0030	0.0034	0.0044	0.0050	0.0042
$\bar{a}_0 = 0.04$	0.0091	0.0093	**0.0098**	0.0096	0.0082
$\bar{a}_0 = 0.07$	0.0088	0.0091	0.0097	0.0093	0.0076
$\bar{a}_0 = 0.10$	-0.0160	-0.0119	0.0003	0.0029	0.0012

Table 9.2. Average $\log \tilde{X}(T)$ for strategies based on two-point Hypothesis 2 for various parameter values.

	δ =0.5	δ =1.0	δ =1.5	δ =2.0	δ = 2.5
$\bar{k}$=0.65	0.0117	0.0134	0.0137	0.0099	-0.0173
$\bar{k}$=0.75	0.0119	0.0136	**0.0140**	0.0099	-0.0195
$\bar{k}$=0.85	0.0115	0.0134	0.0139	0.0093	-0.0219
$\bar{k}$=0.95	0.0107	0.0128	0.0135	0.0082	-0.0247
$\bar{k}$=1.05	0.0095	0.0117	0.0126	0.0062	-0.0277

similar results when we change T or the frequency of portfolio adjustment (with adjustment every 2, 3, and 10 days). Also, the result remains basically unchanged when we exclude some of the stocks from the trials. For example, for strategies based on Hypothesis 3 for daily transactions during 50 trading days, average $\log \tilde{X}(T)$ is 0.0177 with $C = 0.5$, $\delta = 0.5$, and $\bar{k} = 0.75$.

In summary, we show that a relaxation of the Gaussian hypothesis on the distribution of $\tilde{a}$ and taking into account trading volume can give a stable gain with strategies that basically are as simple as the classic strategies where it is assumed that $a(t)$ is given.

REMARK 9.1 Now it can be seen that the simplest model-free empirical strategies defined in Theorems 2.1 and 4.3 appear to be optimal for the investment problem with $U(x) \equiv \log x$ under a special case of Hypothesis 2, when $k_1 + k_2 = 1$. Similarly, the strategies defined in Theorem 2.5 appear to be

Table 9.3. Average $\log \tilde{X}(T)$ for strategies based on Hypothesis 3 considering trading volume with various parameter values for $(C, \delta, \bar{k})$.

	$\delta = 0.2$ $\bar{k} = 0.75$	$\delta = 0.5$ $\bar{k} = 0.75$	$\delta = 1.5$ $\bar{k} = 0.65$	$\delta = 1.5$ $\bar{k} = 0.75$
$C = 0.25$	-0.0045	0.0186	–	–
$C = 0.5$	0.0385	**0.0404**	0.0219	0.0248
$C = 0.75$	0.0361	0.0377	0.0277	0.0280
$C = 1$	0.0371	0.0347	-0.0036	0.0238
$C = 2$	–	–	0.0140	0.0238
$C = 3$	–	–	–	0.0256
$C = 4$	–	–	–	0.0241

optimal for the investment problem with $U(x) \equiv \log x$ under a special case of Hypothesis 2, when $k_1 + k_2 = 2$. The strategies do not require the probability distributions of the market parameters, but, as can be seen from our experiments, a good performance is achieved for parameters that correspond to a realistic prior distribution (in particular, the strategy from Theorem 2.5 with $\varepsilon = 1.5$ corresponds to $k_1 = 2.5$, $k_2 = 0.5$).

9.7. Proofs

Proof of of Lemma 9.1. It is easy to see that

$$z(\alpha, T) = \exp\Bigg(\int_0^T \left(\sigma^{-1}(t)\bar{A}(t,\alpha)\right)^\top \sigma(t)^{-1}\tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t) - \frac{1}{2}\int_0^T \left|\sigma^{-1}(t)\bar{A}(t,\alpha)\right|^2 dt \Bigg).$$

Then

$$\bar{z}^m = \int_{A^m} d\nu(\alpha_1)\cdots d\nu(\alpha_m) \exp\Bigg(\sum_{k=1}^m \int_0^T \left(\sigma(t)^{-1}\bar{A}(t,\alpha_k)\right)^\top \times \sigma(t)^{-1}\tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t) - \frac{1}{2}\sum_{k=1}^m \int_0^T \left|\sigma(t)^{-1}\bar{A}(t,\alpha_k)\right|^2 dt \Bigg),$$

and (9.1) holds.

Further, the assumptions imply that γ is nonrandom for any given (nonrandom) $\{\alpha_k\}_{k=1}^m \subset \mathcal{A}$. Then

$$dz\left(\sum_{k=1}^m \alpha_k, t\right) = z\left(\sum_{k=1}^m \alpha_k, t\right)\left[\sigma(t)^{-1}\bar{A}(t, \sum_{k=1}^m \alpha_k)\right]^{\top} \sigma(t)^{-1}\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \tag{9.27}$$

$$dz_*\left(\sum_{k=1}^m \alpha_k, t\right) = z_*\left(\sum_{k=1}^m \alpha_k, t\right)\left[\sigma(t)^{-1}\bar{A}_*(t, \sum_{k=1}^m \alpha_k)\right]^{\top} dw(t),$$

since $dw(t) = \sigma(t)^{-1}\tilde{\mathbf{S}}_*(t)^{-1}d\tilde{S}_*(t)$. But $\sigma^{-1}A$ is independent of $w(\cdot)$, so

$$\mathbf{E}_*z\left(\sum_{k=1}^m \alpha_k, T\right) = \mathbf{E}z_*\left(\sum_{k=1}^m \alpha_k, T\right) = 1. \tag{9.28}$$

Now (9.2) follows by taking expectation in (9.1).

Since

$$\begin{aligned} d\tilde{X}(t) &= B(t)^{-1}\tilde{\pi}(t)^{\top}\tilde{\mathbf{S}}(t)d\tilde{S}(t) \\ &= \int_{\mathcal{A}^m} d\nu(\alpha_1)\cdots d\nu(\alpha_m)\gamma(\alpha_1,\ldots,\alpha_m)dz\left(\sum_{k=1}^m \alpha_k, t\right), \end{aligned}$$

then

$$\begin{aligned} \tilde{X}(T) &= X(0) \\ &+ \int_{\mathcal{A}^m} d\nu(\alpha_1)\cdots d\nu(\alpha_m)\gamma(\alpha_1,\ldots,\alpha_m)\left[z\left(\sum_{k=1}^m \alpha_k, T\right) - 1\right] \\ &= X(0) - \mathbf{E}_*\zeta + \zeta \end{aligned}$$

and the replication result follows. □

Proofs of Lemma 9.2 and Theorem 9.1 follow immediately from Lemma 9.1 and Theorem 8.1.

Proof of Theorem 9.2. By (1.11), the process $Q(t)$ is $\mathcal{F}_t$ adapted, and therefore the strategy (9.6) belongs to the class $\bar{\Sigma}(\mathcal{F}^a\cdot)$. It follows from results presented in Chapter 5 that this strategy is optimal in this class. Then (i) follows.

For the log utility function, we have

$$\hat{\xi} = F(\bar{\mathcal{Z}}, \hat{\lambda}) = \frac{\bar{\mathcal{Z}}}{\hat{\lambda}} = \frac{\bar{\mathcal{Z}}X_0}{\mathbf{E}_*\bar{\mathcal{Z}}} = \bar{\mathcal{Z}}X_0,$$

since

$$\begin{aligned}\mathbf{E}_*\bar{\mathcal{Z}} &= \mathbf{E}_* \int_{\mathcal{A}} d\nu(\alpha) z(\alpha, T) \\ &= \int_{\mathcal{A}} d\nu(\alpha)\mathbf{E}_* z(\alpha, T) = \int_{\mathcal{A}} d\nu(\alpha)\mathbf{E} z_*(\alpha, T) = 1,\end{aligned}$$

cf. (9.28). So $m = 1$ and $\gamma = 1$.

Now (9.7), (1.20), and (8.22) imply that

$$\tilde{X}(t) = X_0 \int_{\mathcal{A}} d\nu(\alpha) z(\alpha, t),$$

and hence (9.8) follows. Since $z(\alpha, t) > 0$ for all α, then $X(t) > 0$ a.s for all t.

Finally, $\hat{\pi} \in \bar{\Sigma}(\mathcal{F}_\cdot)$, so the optimality of $\hat{\pi}$ follows from Theorem 9.1.□

Proof of Corollary 9.1. By (9.14) and (9.8),

$$\hat{\pi}(t)^\top \triangleq X(t, \hat{\pi}(\cdot))\hat{a}(t)^\top Q(t),$$

and by (9.6) the optimal strategy in $\bar{\Sigma}(\mathcal{F}^a_\cdot)$ is

$$\pi(t)^\top = X(t, \pi(\cdot))\tilde{a}(t)^\top Q(t).$$

Hence there exists a measurable function $\Gamma(t, \cdot) \ : \ B([0, t]; \mathbf{R}^n \times \mathbf{R} \times C([0, t]; \mathbf{R}^n) \to \mathbf{R}^n$ such that

$$\begin{aligned}\pi(t) &\equiv \Gamma(t, \tilde{a}(\cdot)|_{[0,t]}, r(\cdot)|_{[0,t]}, S(\cdot)|_{[0,t]}), \\ \hat{\pi}(t) &\equiv \Gamma(t, \hat{a}(\cdot)|_{[0,t]}, r(\cdot)|_{[0,t]}, S(\cdot)|_{[0,t]}),\end{aligned}$$

and by Definition 9.1, $\hat{a}(t)$ is the required estimate of $\tilde{a}(t)$.

Let us show (ii). We shall employ the notation

$$Y(t, \pi(\cdot)) \triangleq \log \frac{X(t, \pi(\cdot))}{B(t)X_0}.$$

Let $\mathcal{B}_2$ be the set of all processes $\bar{a}(t) \ : \ [0, T] \to \mathbf{R}^n$ that are progressively measurable with respect to $\mathcal{F}_t$ and such that $\mathbf{E} \int_0^T |\bar{a}(t)|^2 dt < +\infty$. For any $\bar{a}(\cdot) \in \mathcal{B}_2$, define

$$\bar{\pi}(t)^\top \triangleq X(t, \bar{\pi}(\cdot))\bar{a}(t)^\top Q(t),$$

where $X(t, \bar{\pi}(\cdot)) \triangleq B(t)\tilde{X}(t)$ and $\tilde{X}(\cdot)$ is found from (1.20) using $\pi^\top = B\tilde{X}\,\bar{a}^\top Q$. Then

$$Y(t, \bar{\pi}(\cdot)) = \int_0^t \left(\bar{a}(s)^\top Q(s)\tilde{\mathbf{S}}(s)^{-1} d\tilde{S}(s) - \frac{1}{2}\int_0^t \bar{a}(s)^\top Q(s)\bar{a}(s) ds\right),$$

and

$$\begin{aligned}&\mathbf{E}Y(T,\bar{\pi}(\cdot))\\&=\tfrac{1}{2}\mathbf{E}\int_0^T\left(-|\sigma(t)^{-1}(\bar{a}(t)-\tilde{a}(t))|^2+\tilde{a}(t)^\top Q(t)\tilde{a}(t)\right)dt.\end{aligned}\tag{9.29}$$

Set $a'(t)\triangleq\mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t\}$. By the Jensen inequality, $a'(\cdot)\in\mathcal{B}_2$. Consider the corresponding strategy

$$\pi'(t)^\top\triangleq X(t,\pi'(\cdot))a'(t)^\top Q(t).\tag{9.30}$$

It is well known that $\mathbf{E}Y(T,\pi'(\cdot))\geq\mathbf{E}Y(T,\bar{\pi}(\cdot))$ for all $\bar{\pi}(\cdot)$ that correspond to $\bar{a}(\cdot)\in\mathcal{B}_2$, so the strategy (9.30) is optimal over all these $\bar{\pi}(\cdot)$. Then (9.15) and the proof of Corollary 9.1 follow if $\hat{a}(\cdot)\in\mathcal{B}_2$.

Let us show that $\hat{a}(\cdot)\in\mathcal{B}_2$. For any $K>0$, set

$$T_K\triangleq\inf\left\{t\in[0,T]:\quad\int_0^t|\hat{a}(s)|^2ds>\int_0^t|a'(s)|^2ds+K\right\}.$$

As usual, we take $T_K=T$ if the set is empty. Note that

$$\mathbf{E}\log X(T_K,\hat{\pi}(\cdot))\geq\mathbf{E}\log X(T_K,\pi'(\cdot))\quad\forall K>0,\tag{9.31}$$

because if (9.31) fails, then

$$\mathbf{E}Y(T,\pi_K(\cdot))>\mathbf{E}Y(T,\hat{\pi}(\cdot)),$$

where

$$\pi_K(t)\triangleq\begin{cases}\pi'(t) & t\leq T_K\\ \hat{\pi}(t) & t>T_K.\end{cases}$$

Further, let $\chi_K(t)$ denote the indicator function of the event $\{t<T_K\}$, and let

$$\bar{a}_K(\cdot)\triangleq\chi_K(\cdot)\bar{a}(\cdot)\in\mathcal{B}_2,\quad\tilde{a}_K(t)\triangleq\chi_K(t)\tilde{a}(t).$$

As in (9.29), we have

$$\begin{aligned}&\mathbf{E}Y(T_K,\bar{\pi}(\cdot))\\&=\tfrac{1}{2}\mathbf{E}\int_0^{T_K}\left(-|\sigma(t)^{-1}(\bar{a}(t)-\tilde{a}(t))|^2+\tilde{a}(t)^\top Q(t)\tilde{a}(t)\right)dt\\&=\tfrac{1}{2}\mathbf{E}\int_0^T\left(-|\sigma(t)^{-1}(\bar{a}_K(t)-\tilde{a}_K(t))|^2+\tilde{a}_K(t)^\top Q(t)\tilde{a}_K(t)\right)dt.\end{aligned}\tag{9.32}$$

Then the process $\mathbf{E}\{\tilde{a}_K(t)|\mathcal{F}_t\}=\chi_K(t)\mathbf{E}\{\tilde{a}(t)|\mathcal{F}_t\}=\chi_K(t)a'(t)$ gives the maximum of $\mathbf{E}Y(T_K,\bar{\pi}(\cdot))$. It follows from (9.31) that

$$\chi_K(t)\hat{a}(t)=\chi_K(t)a'(t)$$

for $t \in [0,T]$ and $K > 0$. Thus, $T_K = T$ a.s. for any $K > 0$, and

$$a'(\cdot) = \hat{a}(\cdot), \quad \hat{a}(\cdot) \in \mathcal{B}_m.$$

Then (9.15) and (ii) follow. □

Proof of Theorem 9.3. Part (i) is straightforward, and (ii) follows from Chapter 5, Corollary 5.1. Part (iii) follows from Lemma 9.1. Since

$$\tilde{X}(T,\hat{\pi}(\cdot)) = F(\bar{\mathcal{Z}},\hat{\lambda}) = \frac{X_0\bar{\mathcal{Z}}^l}{G(l)},$$

then (9.13) follows from Lemma 9.3. □

Proof of Corollary 9.2. The first equality in (9.16) follows from (9.10). This and (9.11) and (9.12) imply

$$l\hat{a}(t) = \frac{\int_{\mathcal{A}^l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)z(\sum_1^l \alpha_k,t)\sum_1^l \alpha_k(t)^\top}{\int_{\mathcal{A}_l} d\nu(\alpha_1)\cdots d\nu(\alpha_l)\gamma(\alpha_1,\ldots,\alpha_l)z(\sum_1^l \alpha_k,t)},$$

i.e.,

$$l\hat{a}(t) = \frac{\int_{\mathcal{A}_l} d\nu_l(b)z(b,t)b(t)^\top}{\int_{\mathcal{A}_l} d\nu_l(b)z(b,t)}.$$

Comparing this with Corollary 9.1 (i) and (9.8), we see that $l\hat{a}(t)$ is the certainty equivalent estimate for the problem with $U(x) \equiv \log x$ and with the prior distribution of $\Theta = \tilde{a}(\cdot)$ described by $\mathcal{A}_l$ and $\nu_l(\cdot)$. By Corollary 9.1 (iii), $\hat{a}(t) = l^{-1}\mathbf{E}_l\{\tilde{a}(t)|\mathcal{F}_t\}$. This completes the proof of Corollary 9.2.□

Proof of Theorem 9.4. Let $I(\cdot) \in \mathcal{I}_m(\mathcal{F}_\cdot)$. Consider the problem (8.7)-(8.8) for the I-market. By Theorem 9.2, the unique optimal strategy is

$$\begin{aligned}\pi_I(t) &\triangleq X(t,\pi_I(\cdot))Q(t)\mathbf{E}\{\tilde{a}_I(t)|\mathcal{F}_t\}\\ &= X(t,\pi_I(\cdot))Q(t)\mathbf{E}\{\tilde{a}_I(t)|\mathcal{F}_t\} = X(t,\pi_I(\cdot))Q(t)\hat{a}_I(t).\end{aligned}$$

We have

$$\begin{aligned}\pi(t) &= X(t,\pi_I(\cdot))Q(t)\hat{a}_I(t)\\ &= X(t,\pi_I(\cdot))Q(t)V(t)Q_I(t)P_I(t)\hat{a}(t) = X(t,\pi_I(\cdot))Q_I(t)P_I(t)\hat{a}(t).\end{aligned}$$

We have used that $Q(t) = V(t)^{-1}$. Since $Q_I(t)$ maps $L_I(t)$ into $L_I(t)$, then $\pi_I(t) \in L_I(t)$ for all t, so

$$\hat{\pi}(\cdot) \in \bar{\Sigma}(m,\mathcal{G}_\cdot),$$

i.e., $P_{\hat{I}}(t)\hat{\pi}(t) = \hat{\pi}(t)$.

Further,

$$\tilde{X}(T,\pi_I(\cdot))) = X(0)\exp\left(\int_0^T(\sigma(t)^{-1}\hat{a}_I(t))^\top\sigma(t)^{-1}\tilde{S}(t)^{-1}\tilde{S}(t)\right.$$
$$\left. - \frac{1}{2}\int_0^T \hat{a}_I(t)^\top Q(t)\hat{a}_I(t)dt\right),$$

and

$$\mathbf{E}U(\tilde{X}(t,\pi_I(\cdot))) = \frac{1}{2}\mathbf{E}\int_0^T \hat{a}_I(t)^\top Q(t)\hat{a}_I(t)dt + \log X_0 = \frac{J_I}{2} + \log X_0.$$

It is easy to see that optimal $I(\cdot)$ satisfy (9.20) and the strategy (9.21) is optimal. This completes the proof of Theorem 9.4. □

Chapter 10

SOLUTION FOR GENERAL UTILITIES AND CONSTRAINTS VIA PARABOLIC EQUATIONS

Abstract We present the solution of an optimal investment problem with additional constraints and utility functions of a very general type, including discontinuous functions. Optimal portfolios are obtained in the class of strategies based on historical prices, when $a(t)$ is random and unobservable, but under some additional restrictions on the prior distributions of market parameters. Optimal investment strategies are expressed via solution of a linear deterministic parabolic backward equation.

10.1. The model

In this chapter, we consider a special case of the model described in Section 8.1. We assume that all conditions imposed in Sections 8.1–8.5 are satisfied and that the process $\eta(t) \equiv 0$, i.e., the filtration $\mathcal{F}_t$ is generated by $(S(t), r(t))$. In addition, we impose the following restriction on prior distribution of the appreciation rate $a(t)$: we assume that there exist an integer $L > 0$, deterministic and known vector processes $e_i(\cdot) \in B([0,T]; \mathbf{R}^n)$, and random variables $\theta_i(\cdot)$, $i = 1, \dots, L$, such that

$$\tilde{a}(t) \triangleq a(t) - r(t)\mathbf{1} = \sum_{i=1}^{L} \theta_i e_i(t). \tag{10.1}$$

To describe the prior distribution of $a(\cdot)$, we assume that $\mathcal{T} = \mathbf{R}^L$. Let $\Theta : \Omega \to \mathbf{R}^L$ be defined as $\Theta = (\theta_1, \dots, \theta_L)$. We are given a probability measure $\nu(\cdot)$ on $\mathcal{T}$ that describes the probability distribution of Θ.

We assume that the following conditions are satisfied:

- $\sigma(t)$ is deterministic;
- Θ, $w(\cdot)$, $r(\cdot)$ are mutually independent; and
- the process $(r(t), \sigma(t))$ is uniformly bounded.

Under these assumptions, the solution of (8.1) is well defined, but the market is incomplete.

10.2. Problem statement

Let $M \geq L$ be an integer. Consider the following system:

$$\begin{cases} dY(t) = f(Y(t),t)dt + b(Y(t),t)\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \\ Y(0) = 0. \end{cases} \tag{10.2}$$

Here $f(\cdot) : \mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^M$ and $b(\cdot) : \mathbf{R}^M \times \mathbf{R} \to \mathbf{R}^{M\times n}$ are deterministic functions. We assume that the functions $b(y,t)$, $\partial b(y,t)/\partial y$, $\partial^2 b(y,t)/\partial y^2$, $f(y,t)$, $\partial f(y,t)/\partial y$, and $\partial^2 f(y,t)/\partial y^2$ are uniformly bounded and Hölder.

Let $T > 0$, and $X(0)$ be fixed. Let $M > 0$ be an integer. Let $U(\cdot,\cdot) : \mathbf{R} \times \mathbf{R}^M \to \mathbf{R}$ and $G(\cdot,\cdot) : \mathbf{R} \times \mathbf{R}^M \to \mathbf{R}^m$ be given measurable functions.

We may state our general problem as follows: Find an admissible self-financing strategy $\pi(\cdot)$ that solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T,\pi(\cdot)),Y(T)) \tag{10.3}$$

$$\text{subject to} \begin{cases} X(0,\pi(\cdot)) = X(0), \\ G(\tilde{X}(T,\pi(\cdot)),Y(T)) \leq 0 \quad \text{a.s.} \end{cases} \tag{10.4}$$

Clearly, this problem is a special case of the problem (8.7)–(8.8).

EXAMPLE 10.1 Let $\varphi(\cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$, $U(\cdot,\cdot) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ and $G(\cdot,\cdot) : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^m$ be given measurable functions. Consider the following problem:

$$\text{Maximize} \quad \mathbf{E}U\left(\tilde{X}(T,\pi(\cdot)), \int_0^T \varphi(\tilde{S}(t),t)dt\right) \tag{10.5}$$

$$\text{subject to} \begin{cases} X(0,\pi(\cdot)) = X(0), \\ G\left(\tilde{X}(T,\pi(\cdot)), \int_0^T \varphi(\tilde{S}(t),t)dt\right) \leq 0 \quad \text{a.s.} \end{cases} \tag{10.6}$$

Consider processes $P_i(t)$, $i = 1,2$ such that

$$\begin{cases} dP_1(t) = \varphi\left(\left\{S_j(0)\exp\left[P_2^j(t) - \frac{1}{2}\sum_{k=1}^n \int_0^t \sigma_{jk}(s)^2 ds\right]\right\}_{j=1}^n, t\right), \\ dP_2(t) = \tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \\ P_i(0) = 0, \quad i = 1,2. \end{cases}$$

Clearly,

$$P_1(t) = \int_0^t \varphi\left(\tilde{S}(s),s\right)ds, \quad P_2^j(t) = \log\frac{\tilde{S}_j(t)}{S_j(0)} + \frac{1}{2}\sum_{k=1}^n \int_0^t \sigma_{jk}(s)^2 ds.$$

It is easy to see that the problem (10.5)–(10.6) is a special case of the problem (10.3)–(10.4), where $Y(t) = (P_1(t), P_2(t))$, $M = 2n$.

EXAMPLE 10.2 Let $T_i \in (0, T]$, $i = 1, \dots, K$, $T_1 < T_2 < \cdots < T_K$. Let $U(\cdot,\cdot) : \mathbf{R} \times (\mathbf{R}^n)^K \to \mathbf{R}$ and $G(\cdot,\cdot) : \mathbf{R} \times (\mathbf{R}^n)^K \to \mathbf{R}^m$ be given measurable functions. Consider the following problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T, \pi(\cdot)), \tilde{S}(T_1), \dots, \tilde{S}(T_K)) \tag{10.7}$$

$$\text{subject to} \begin{cases} X(0, \pi(\cdot)) = X(0), \\ G(\tilde{X}(T, \pi(\cdot)), \tilde{S}(T_1), \dots, \tilde{S}(T_K)) \le 0 \quad \text{a.s.} \end{cases} \tag{10.8}$$

Consider processes $P_i(t) = \{P_i^j\}_{j=1}^n$ such that

$$dP_i(t) = \begin{cases} \tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t), & t < T_i \\ 0, & t > T_i \end{cases}, \quad P_i(0) = 0, \quad i = 1, \dots, K.$$

Clearly,

$$P_i^j(t) = \begin{cases} \log \frac{\tilde{S}_j(t)}{\tilde{S}_j(0)} + \frac{1}{2} \sum_{k=1}^n \int_0^t \sigma_{jk}(s)^2 ds, & t < T_i \\ P_i^j(T_i), & t > T_i \end{cases}, \quad i = 1, \dots, K.$$

It is easy to see that the problem (10.7)–(10.8) is a special case of the problem (10.3)–(10.4), where $Y(t) = (P_1(t), \dots, P_K(t))$, $M = K \cdot n$.

10.3. Additional assumptions

Auxiliary optimization problem. Similarly to Chapter 8, we shall investigate the optimal investment problem via the following finite-dimensional optimization problem:
For $z \in \mathbf{R}$, $y \in \mathbf{R}^M$, $\lambda \in \mathbf{R}$,

$$\text{Maximize} \quad zU(x, y) - \lambda x \quad \text{over} \quad x \in \mathbf{R} : G(x, y) \le 0. \tag{10.9}$$

This problem will be used in the following way. We obtain an optimal claim as a solution of the problem (10.9) with the random variable z depending on $Y(T)$. Then, the corresponding admissible self-financing strategy, which replicates the claim, is obtained readily.

Let

$$J(y) \triangleq \{x \in \mathbf{R} : \ G(x, y) \le 0\}, \quad y \in \mathbf{R}^M. \tag{10.10}$$

To proceed further, we assume that the following conditions are satisfied.

CONDITION 10.1 *The function $G(\cdot)$ is such that there exists a measurable function of polynomial growth $F'(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$ such that $\mathbf{E}F'(\tilde{S}_*(\cdot)) = X(0)$ and $G(F'(\tilde{S}(\cdot)), y(T)) \le 0$ a.s.*

CONDITION 10.2 *There exists a measurable set $\Lambda \subseteq \mathbf{R}$, and a measurable function $F(\cdot,\cdot,\cdot) : (0,\infty)\times\mathbf{R}^M \times \Lambda \to \mathbf{R}$ such that for each $z > 0$, $y \in \mathbf{R}^M$, $\lambda \in \Lambda$,*

$$\hat{x} = F(z, x, \lambda)$$

is the unique solution of the optimization problem (10.9).

CONDITION 10.3 *There exist $\hat{\lambda} \in \Lambda$ such that $\mathbf{E}_*|F(\bar{\mathcal{Z}}, \tilde{S}(\cdot), \hat{\lambda})| < +\infty$ and*

$$\mathbf{E}_* F(\bar{\mathcal{Z}}, \tilde{S}(\cdot), \hat{\lambda}) = X_0. \qquad (10.11)$$

CONDITION 10.4 *The functions $f(\cdot)$, $\beta(\cdot)$ are such that*

$$dY_i(t) = e_i(t)^\top Q(t)\tilde{\mathbf{S}}(t)^{-1} d\tilde{S}(t), \quad i = 1, \dots, L,$$

where L and $e_i(t)$ are as in (10.1).

Note that the last condition does not implies a loss of generality (since the dimension M of $Y(\cdot)$ can be arbitrarily increased).

10.4. A boundary problem for parabolic equations

Introduce the Banach space $\mathcal{Y}^1$ of functions $u(\cdot) : \mathbf{R}^M \times [0,T] \to \mathbf{R}$ with the norm

$$\|u(\cdot)\|_{\mathcal{Y}^1} \triangleq \left(\sup_t \mathbf{E}|u(Y(t),t)|^2 + \mathbf{E}\int_0^T \left|\frac{\partial u}{\partial x}(Y(t),t)\right|^2 dt \right)^{1/2}.$$

PROPOSITION 10.1 *Let $C(\cdot) : \mathbf{R}^M \to \mathbf{R}$ be a measurable function such that $\mathbf{E}C(Y_*(T))^2 < +\infty$ and $\mathbf{E}C(Y(T))^2 < +\infty$. Then there exists an admissible strategy $\pi(\cdot)) = (\pi_1(t), \dots, \pi_n(t)) \in \bar{\Sigma}(\mathcal{F}_\cdot)$ that replicates the claim $B(T)C(Y(T))$ if and only if $\mathbf{E}C(Y_*(T)) = X(0)$, where $X(0)$ is the initial wealth. Furthermore,*

$$\pi_i(t) = p(t)^{-1}\frac{\partial V}{\partial x_i}(Y(t),t)b(Y(t),t), \quad \tilde{X}(t) = V(Y(t),t),$$

where $\tilde{X}(t)$ is the corresponding normalized wealth at time $t > 0$ and the function $V(\cdot,\cdot) : \mathbf{R}^M \times [0,t] \to \mathbf{R}$ is such that

$$\begin{cases} \frac{\partial V}{\partial t}(x,t) & + \sum_{i=1}^M \frac{\partial V}{\partial x_i}(x,t) f_i(x,t) \\ & + \frac{1}{2}\mathrm{Tr}\left\{ \frac{\partial^2 V}{\partial x^2}(x,t) b(x,t)\sigma(t)\sigma(t)^\top b(x,t) \right\} = 0, \\ V(x,T) & = C(x). \end{cases} \qquad (10.12)$$

The problem (10.12) admits a solution in the class $\mathcal{Y}^1$.

10.5. The optimal strategy

Let $\mathcal{Z}$, $\mathcal{Z}_*$, $\bar{\mathcal{Z}}$, and $\bar{\mathcal{Z}}_*$ be as defined in Chapter 8. Let $\Psi(\cdot) : \mathbf{R}^M \to \mathbf{R}$ be such that

$$\Psi(x) \triangleq \int_T d\nu(\alpha) \exp\left[\left(\sum_{i=1}^L \alpha_i x_i - \frac{\alpha_i^2}{2}\int_0^T |\sigma(t)^{-1}\sum_{i=1}^L e_i(t)|^2 dt\right)\right].$$

(We recall that the measure $d\nu(\alpha) = d\nu(\alpha_1, \dots, \alpha_L)$ describes the prior distribution of $(\theta_1, \dots, \theta_L)$ in (10.1).)

LEMMA 10.1 *The following holds:*

$$\bar{\mathcal{Z}} = \Psi(Y(T)), \quad \bar{\mathcal{Z}}_* = \Psi(Y_*(T)).$$

Let $\phi(x, \lambda) : \mathbf{R}^M \times \Lambda \to \mathbf{R}$ be such that

$$\phi(x, \lambda) \triangleq F(\Psi(x), x, \lambda). \tag{10.13}$$

THEOREM 10.1 *With $\hat{\lambda}$ as in Condition 10.3, there exists an admissible self-financing strategy $\pi(\cdot) = (\pi_1(t), \dots, \pi_n(t)) \in \bar{\Sigma}(\mathcal{F}_\cdot)$ that replicates the claim $\phi(Y(T), \hat{\lambda})$. This strategy is an optimal solution of the problem (10.7)–(10.8), and*

$$\pi_i(t) = p(t)^{-1}\frac{\partial V}{\partial x_i}(Y(t), t) b(Y(t), t), \quad \tilde{X}(t) = V(Y(t), t), \tag{10.14}$$

where $\tilde{X}(t)$ is the corresponding normalized wealth at time $t > 0$, and where $V(x, t) : \mathbf{R}^M \times [0, T] \to \mathbf{R}$ is the solution of the partial differential equation (10.12) with the condition

$$V(x, T) = \phi(x, \hat{\lambda}).$$

Moreover, if $F(z, x, \hat{\lambda})$ is the unique solution of the optimization problem (10.9) with $\lambda = \hat{\lambda}$, and if $\mathbf{E}\, U^+(\hat{\xi}, Y(T)) < +\infty$, then the optimal strategy is unique: all optimal processes $\tilde{X}(t)$ and $\pi(t)$ are same (equivalent) for all $\hat{\lambda}$, provided that Condition 10.3 is satisfied.

The following proposition may be useful.

PROPOSITION 10.2 *Let there exist constants* $c_0 > 0$, $c > 0$, *such that*

$$|F(z,y,\hat\lambda)| \le c_0(z^c + z^{-c} + |y|^c + 1),$$

$$U(x,y) \le c_0\Big(x^2 + |y|^c + 1\Big),$$

$$\forall y = (y_1,\dots,y_M) \in \mathbf{R}^M,\ \forall x \in J(y),\ \forall z \in \mathring{\mathbf{R}}_+ .$$

Then $\mathbf{E}\,U^+(\hat\xi, Y(T)) < +\infty$.

Notice that

$$V(x,t) \triangleq \int_{\mathbf{R}^M} \bar P_*(dy,T,x,t)\phi(y,\hat\lambda) = \mathbf{E}\Big\{\phi(Y_*(T),\hat\lambda)|Y_*(t) = x\Big\}, \tag{10.15}$$

where $\bar P_*(dy,\tau,x,t)$ as a function of dy is the conditional probability distribution for the vector $Y_*(\tau)$ given the condition $Y_*(t) = x$, where $0 \le t \le \tau$. In particular, the condition (10.11) has the form

$$\int_{\mathbf{R}^M} P_*(dx,T,0,0)\phi(x,\hat\lambda) = X(0).$$

Notice that for most important particular cases to be considered below, $\int_{\mathbf{R}^M} P_*(dx,T,0,0)\phi(x,\lambda)$ is a monotonic decreasing function of λ. Thus, it is not difficult to carry out the calculation of $\hat\lambda$ from (10.11) if $\phi(\cdot)$ is known. Chapter 11 will provide an explicit formula for $\phi(\cdot)$ for several special problems.

10.6. Proofs

Proof of Lemma 10.1. It suffices to prove that $\bar{\mathcal{Z}}_* = \psi(Y_*(T))$. By the definitions,

$$\begin{aligned}\mathcal{Z}_* &= \exp\Big(\textstyle\sum_{i=1}^L \int_0^T (\sigma(t)^{-1}e_i\theta_i(t))^\top dw(t) \\ &\qquad\qquad - \tfrac12 \int_0^T |\sigma(t)^{-1}\sum_{i=1}^L \theta_i e_i(t)|^2 dt\Big) \\ &= \exp\Big(\textstyle\sum_{i=1}^L \theta_i Y_{i*}(T) - \frac{\theta_i^2}{2}\int_0^T |\sigma(t)^{-1}\sum_{i=1}^L e_i(t)|^2 dt\Big).\end{aligned} \tag{10.16}$$

Then the proof follows. □

Proof of Proposition 10.1. Let an admissible self-financing strategy $\pi(\cdot)$ be such that $\tilde X(T,\pi(\cdot)) = C(Y(T))$ a.s. It is required to show that $X(0) = \mathbf{E}C(Y_*(T))$. Let $\tilde X(\cdot) = \tilde X(\cdot,\pi(\cdot))$. For an $\alpha \in \mathcal{T}$, it follows from the application of Girsanov's Theorem (see, e.g., Karatzas and Shreve (1991)) to the conditional probability space given the condition $a = \alpha$ that, for any $B \in \mathcal{B}$,

$$\mathbf{P}(\tilde S(\cdot) \in B) = 0 \quad \text{if and only if} \quad \mathbf{P}(\tilde S_*(\cdot) \in B) = 0. \tag{10.17}$$

In other words, the sets of samples for the processes $\tilde{S}_*(\cdot)$ and $\tilde{S}(\cdot)$ are $\mathbf{P}$-indistinguishable. Thus there exists a probability measure $\mathbf{P}_*$ such that $\tilde{S}(t)$ is a martingale. Let $\mathbf{E}_*$ be the corresponding mathematical expectation. We have

$$d\tilde{X}(t) = p(t)\pi(t)\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t),$$
$$\tilde{X}(0) = X(0), \qquad \tilde{X}(T) = C(Y(T)).$$

The process $\tilde{S}(t)$ is a martingale with respect to the probability measure $\mathbf{P}_*$. Hence $X(0) = \mathbf{E}_*C(Y(T)) = \mathbf{E}C(Y_*(T))$.

Let $X(0) = \mathbf{E}C(Y_*(T))$. It is required to show that the strategy defined in the Proposition 10.1 does exists and is admissible.

Assume that $C(\cdot)$ has a finite support inside an open domain in $\mathbf{R}^M$, and let the function $C(\cdot)$ be smooth enough. Then the problem (10.12) has a classical solution. Thus, $V(x,t)$ is a classical solution of (10.12). Using Itô's formula, we obtain again

$$d\tilde{X}(t) = p(t)\pi(t)^\top\tilde{\mathbf{S}}(t)^{-1}d\tilde{S}(t), \quad \tilde{X}(T) = C(Y(T)). \tag{10.18}$$

To continue, let $Z(t) \triangleq \sigma(t)^\top\pi(t)$. Consider the conditional probability space given $(r(\cdot),\Theta)$. With respect to the conditional probability space, it follows from (10.18) that

$$\begin{cases} d\tilde{X}(t) = p(t)Z(t)^\top dw(t) + p(t)Z(t)^\top\sigma(t)^{-1}\mathbf{E}\{\tilde{a}|\,r(\cdot),\Theta\}dt, \\ \tilde{X}(T) = C(Y(T)). \end{cases} \tag{10.19}$$

Note that equation (10.19) is linear. The solution $(Z(t),\tilde{X}(t))$ of the stochastic backward equation (10.19) is a square integrable process (see, e.g., Theorem 2.2 of Yong and Zhou (1999), Chapter 7, or Proposition 2.2 of El Karoui *et al.* (1997)).Thus, it can be shown that there exists a constant c_0, independent of $C(\cdot)$, such that

$$\sup_t \mathbf{E}\left\{|\tilde{X}(t)|^2|\,r(\cdot),\Theta\right\} + \mathbf{E}\left\{\int_0^T |Z(t)|^2dt\,\middle|\,r(\cdot),\Theta\right\}$$
$$\leq c_0\mathbf{E}\left\{C(Y(T))^2|\,r(\cdot),\Theta\right\} \quad \text{a.s.}$$

for all $\alpha \in \mathcal{T}$. Hence

$$\sup_t \mathbf{E}|\tilde{X}(t)|^2 + \mathbf{E}\int_0^T |Z(t)|^2dt \leq c_0\mathbf{E}C(Y(T))^2. \tag{10.20}$$

Let $C(\cdot)$ be a general measurable function satisfying the conditions specified in the proposition. Then, there exists a sequence $\{C^{(i)}(\cdot)\}$, where $C^{(i)}(\cdot)$ has finite support inside the open domain $\mathbf{R}^M$ and is smooth enough, such that

$$\mathbf{E}|C^{(i)}(Y(T)) - C(Y(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.$$

Let $\tilde{X}^{(i)}(\cdot)$, $\pi^{(i)}(\cdot)$, $V^{(i)}(\cdot)$ be the corresponding processes and functions. It can be shown that there exist a solution $V(\cdot)$ of (10.12) as a limit of $V^{(i)}(\cdot)$ in $\mathcal{Y}^1$.

By (10.20) and the linearity of (10.19), it follows that

$$\begin{aligned}\sup_t \mathbf{E}|\tilde{X}^{(i)}(t) & -\tilde{X}^{(j)}(t)|^2 + \mathbf{E}\int_0^T |\pi^{(i)}(t) - \pi^{(j)}(t))|^2 dt \\ & \le c_0 \mathbf{E}|C^{(i)}(Y(T)) - C^{(j)}(Y(T))|^2 \to 0 \quad \text{as} \quad i \to \infty.\end{aligned}$$

Thus, $\{\tilde{X}^{(i)}(\cdot)\}$, $\{\pi^{(i)}(\cdot)\}$ are Cauchy sequences in the space of square integrable processes, and hence it can be shown that the corresponding limits $\tilde{X}(\cdot)$, $\pi(\cdot)$ exist and are square integrable processes. This completes the proof. □

Proof of Theorem 10.1 follows from Proposition 10.1 and Theorem 8.1. □

Proof of Proposition 10.2 follows from Lemma 10.2 below.

LEMMA 10.2 *For* $k = 0, 1$, $\sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \mathcal{Z}^c < +\infty \quad (\forall c \in \mathbf{R})$.

Proof. It is easy to see that

$$\sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^c < +\infty \quad \forall c \in \mathbf{R},$$

and

$$\begin{aligned}&\sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \mathcal{Z}^c \\ &= \sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \left(\int_{\mathcal{T}} d\nu(\alpha_1)\, z(\alpha_1, T) \right)^c < +\infty \quad (\forall c > 0).\end{aligned}$$

For any $c < 0$, the function y^c is convex and

$$\begin{aligned}\sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \mathcal{Z}^c & = \sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \left(\int_{\mathcal{T}} d\nu(\alpha_1) z(\alpha_1, T) \right)^c \\ & \le \sup_{\alpha \in \mathcal{T}} \mathbf{E} z(\alpha, T)^k \int_{\mathcal{T}} d\nu(\alpha_1) z(\alpha_1, T)^c \\ & < +\infty.\end{aligned}$$

This completes the proof. □

Chapter 11

SPECIAL CASES AND EXAMPLES: REPLICATING WITH GAP AND GOAL ACHIEVING

Abstract In this chapter, the optimal portfolio is obtained for the class of strategies based on historical prices under some additional conditions that ensure that the optimal normalized wealth $\tilde{X}(t)$ and the optimal strategy $\pi(t)$ are functions of the current vector $\tilde{S}(t)$ of the normalized stock prices. In particular, these conditions are satisfied if $\sigma(t)$ is deterministic and σ, $\tilde{a}$ are time independent. A solution of a goal achieving problem and a solution of a problem of optimal replication of a European put option with a possible gap will be given among others. Explicit formulas for optimal claims and numerical examples are provided.

11.1. Additional assumptions and problem statement

We shall consider the following special case of the problem (10.3)-(10.4) from Chapter 10. Let $T > 0$ and X_0 be given. Let $m > 0$ be an integer. Let $U(\cdot,\cdot) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ and $G(\cdot,\cdot) : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^m$ be given measurable functions such that $\mathbf{E}U(X_0, \tilde{S}(T)) < +\infty$.

We may state our general problem as follows: Find an admissible self-financing strategy that solves the following optimization problem:

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T,\pi(\cdot)), \tilde{S}(T)) \quad \text{over} \quad \pi(\cdot) \tag{11.1}$$

$$\text{subject to} \begin{cases} X(0,\pi(\cdot)) = X(0), \\ G(\tilde{X}(T,\pi(\cdot)), \tilde{S}(T)) \le 0 \quad \text{a.s.} \end{cases} \tag{11.2}$$

We shall assume in this chapter that the matrix $\sigma(t)$ is deterministic and that the vector

$$\hat{\theta} \triangleq (\sigma(t)^{-1})^\top \theta(t)$$

does not depend on time, where

$$\theta(t) \triangleq \sigma(t)^{-1}\tilde{a}(t) = \sigma(t)^{-1}[a(t) - r(t)\mathbf{1}].$$

Clearly, this condition is satisfied if $r = r(t)$, $\sigma = \sigma(t)$ and $a = a(t)$ are constant in time. To describe the prior distribution of $a(\cdot)$, we assume that $\mathcal{T} = \mathbf{R}^n$ and $\Theta = \hat{\theta}$. We take as given a probability measure $\nu(d\alpha)$ on $\mathcal{T} = \{\alpha\}$ that describes the probability distribution of $\Theta = \hat{\theta}$.

PROPOSITION 11.1 *The problem (11.1)–(11.2) is a special case of the problem (10.7)–(10.8), where $L = M = n$ and*

$$Y(t) = \{Y_i(t)\}_{i=1}^n, \quad Y_i(t) \triangleq \log \frac{\tilde{S}_i(t)}{\tilde{S}_i(0)} + \frac{1}{2}\sum_{j=1}^n \int_0^t \sigma_{ij}(s)^2 ds,$$
$$dY_i(t) = \tilde{S}_i(t)^{-1} d\tilde{S}_i(t).$$

Let $\mathcal{F}_t$, $\mathcal{Z}$, $\bar{\mathcal{Z}}$, $F(\cdot)$, $\hat{\lambda}$ and $\hat{\xi} \triangleq F(\bar{\mathcal{Z}}, Y(T), \hat{\lambda})$ be such as defined in Chapter 10 with $L = M = n$. Let $\tilde{X}(t)$ be the optimal normalized wealth defined in Theorem 10.1.

COROLLARY 11.1 *There exist functions $\psi(\cdot) : \mathbf{R}^n \to \mathbf{R}$, $f(\cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $H(\cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ such that*

$$\bar{\mathcal{Z}} = \psi(\tilde{S}(T)), \quad \hat{\xi} = f(\tilde{S}(T), \hat{\lambda}),$$
$$\tilde{X}(t) = H(\tilde{S}(t), t), \quad \pi(t)^\top = p(t)^{-1}\frac{\partial H}{\partial x}(\tilde{S}(t), t)\tilde{\mathbf{S}}(t) = \frac{\partial H}{\partial x}(\tilde{S}(t), t)\mathbf{S}(t). \tag{11.3}$$

Clearly, $f(x, \lambda) = F(\psi(x), x, \lambda)$.

It is easy to see that

$$\log \tilde{S}_i(t) = \log S_i(0) + \int_0^t \tilde{a}_i(s) ds - \frac{t}{2}\sum_{j=1}^n \int_0^t \sigma_{ij}(s)^2 ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(s) dw_j(s),$$

$$\log S_{i*}(t) = \log S_i(0) - \frac{t}{2}\sum_{j=1}^n \int_0^t \sigma_{ij}^2(s) ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(s) dw_j(s).$$

From these formulas, it follows that $\tilde{S}(t)$ has a conditional log-normal probability density function given $\tilde{a}(\cdot)$, while $\tilde{S}_*(t)$ has an unconditional log-normal probability density function. In fact, $\tilde{S}(t)$ also has a probability density function. Let $p(x, t)$ be the probability density function for the vector $\tilde{S}(t)$, and let $p_*(x, t)$ be the probability density function for the vector $\tilde{S}_*(t)$.

PROPOSITION 11.2 *The following holds:*

$$\psi(x) = \frac{p(x, T)}{p_*(x, T)}, \quad \bar{\mathcal{Z}} = \frac{p(\tilde{S}_*(T), T)}{p_*(\tilde{S}_*(T), T)}. \tag{11.4}$$

PROPOSITION 11.3 *The function* $H(\cdot)$ *is such that*

$$H(x,t) \triangleq \int_{\mathbf{R}^n_+} \bar{p}_*(y,T,x,t) f(y,\hat{\lambda})dy = \mathbf{E}\left\{ f(\tilde{S}_*(T),\hat{\lambda}) | \tilde{S}_*(t) = x \right\}, \tag{11.5}$$

where $\bar{p}_*(y,\tau,x,t)$ *as a function of* y *is the conditional probability density function for the vector* $\tilde{S}_*(\tau)$ *given the condition* $\tilde{S}_*(t) = x$, *where* $0 \le t \le \tau$. *In particular,* $p_*(x,t) = \bar{p}_*(x,t,S(0),0)$, *and (10.11) has the form*

$$\int_{\mathbf{R}^n_+} p_*(x,T) f(x,\hat{\lambda})dx = X(0).$$

An explicit formula for $\psi(x)$ is given below for the case of noncorrelated stocks, i.e., $\sigma_{ij} = 0$ $(\forall i \neq j)$. In that case, it is known that the explicit formula for $\bar{p}_*(y,T,x,t)$ is given by

$$\bar{p}_*(y,\tau,x,t) = \prod_{i=1}^{n} p_*^{(i)}(y_i,\tau,x_i,t), \tag{11.6}$$

where $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ and

$$p_*^{(i)}(y_i,\tau,x_i,t) \triangleq \frac{1}{x_i\sigma_{ii}\sqrt{2\pi(t-\tau)}} \exp \frac{-\left(\ln(y_i)-\ln(x_i)+\sigma_{ii}^2(t-\tau)/2\right)^2}{2\sigma_{ii}^2(t-\tau)}. \tag{11.7}$$

For $i = 1,\ldots,n$, let

$$\bar{\sigma}_i \triangleq \sqrt{T}\sigma_{ii}, \quad \mu_i \triangleq a_i T, \quad \theta_i \triangleq \frac{\mu_i\bar{\sigma}_i^2 - \mu_i^2}{2\bar{\sigma}_i^2}. \tag{11.8}$$

THEOREM 11.1 *Let* $\sigma_{ij} = 0$ $(\forall i \neq j)$. *Then*

$$\psi(x) = \mathbf{E}\prod_{i=1}^{n}\left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2} e^{\theta_i}, \quad x = (x_1,\ldots,x_n). \tag{11.9}$$

11.2. Explicit formulas for optimal claims for special cases

In this section, we study some particular cases of $U(\cdot)$ and $G(\cdot)$.

11.2.1 Goal-achieving problem

Let k_1, k_2 be such that $-\infty < k_1 < k_2 < +\infty$. For any admissible strategy $\pi(\cdot)$, introduce the stopping times

$$\tau_1 \triangleq T \wedge \inf\{t : \tilde{X}(t,\pi(\cdot)) = k_1\}, \quad \tau_2 \triangleq \inf\{t : \tilde{X}(t,\pi(\cdot)) = k_2\}.$$

Let $X(0)$ be an initial wealth with $k_1 < X(0) < k_2$.

Consider the following goal-achieving problem:

$$\text{Maximize} \quad \mathbf{P}(\tau_1 \geq \tau_2) \quad \text{over} \quad \pi(\cdot) \tag{11.10}$$

$$\text{Subject to} \quad X(0, \pi(\cdot)) = X(0). \tag{11.11}$$

We now show that this problem is a particular case of the problem (10.3).

PROPOSITION 11.4 *Let $G(x,y) = \chi_{\{k_1 \leq x \leq k_2\}}$, $U(x,y) = \chi_{\{k_2 \leq x\}}$, where χ is the indicator function. Then Conditions 10.1–10.2 hold, with*

$$F(z,x,\lambda) = \begin{cases} k_2 & \text{if } 1 - qk_2 \geq -qk_1 \\ k_1 & \text{if } 1 - qk_2 < -qk_1, \end{cases} \tag{11.12}$$

where $q \stackrel{\Delta}{=} \lambda/z$. Furthermore, the assumptions of Theorem 10.1 hold.

Thus, by definition,

$$f(x,\lambda) = \begin{cases} k_2 & \text{if } \psi(x) \geq \lambda(k_2 - k_1) \\ k_1 & \text{if } \psi(x) < \lambda(k_2 - k_1) \end{cases} \tag{11.13}$$

and

$$\begin{aligned} H(x,t) &= k_1 + (k_2 - k_1)\mathbf{P}\left(\psi(\tilde{S}_*(T)) \geq \hat{\lambda}(k_2 - k_1) \middle| \tilde{S}_*(t) = x\right) \\ &= k_1 + (k_2 - k_1) \int_{\mathbf{R}^n_+} \bar{p}_*(y,T,x,t)\chi_{\{\psi(y) \geq \hat{\lambda}(k_2 - k_1)\}} dy, \end{aligned} \tag{11.14}$$

where $\bar{p}_*(y,T,x,t)$ is the probability density function for $\tilde{S}_*(T)$ conditional on $\tilde{S}_*(t) = x$. For the case when $\sigma_{ij} = 0$ for $i \neq j$, the functions $\psi(\cdot)$ and $\bar{p}_*(y,T,x,t)$ are defined explicitly in (11.6) and (11.9).

THEOREM 11.2 *The problem (11.1) with parameters specified in Proposition 11.4 and the problem (11.10)–(11.11) have the same optimal value of the functionals to be maximized. Moreover, an optimal strategy (as given in Theorem 10.1) for the problem (11.1) with parameters specified in Proposition 11.4 is also optimal for the problem (11.10)–(11.11).*

The above proof also leads to the following result immediately.

COROLLARY 11.2 *Under the assumptions of Proposition 11.4, any optimal strategy for the problem (11.10)–(11.11) must satisfy* $\mathbf{P}(\tau_1 \wedge \tau_2 < T) = 0$.

Corollary 11.2 shows that, for an optimal strategy, the first time when the wealth achieves k_1 or k_2 occurs only at $t = T$. In other words, stopping the investment before the expiration time T cannot be optimal.

11.2.2 Mean-variance criteria

The following proposition is devoted to the problem that is close to the Markowitz formulation of mean-variance optimal portfolio selection (see Markowitz (1952)), where the expectation of a return is to be maximized and the dispersion of the return is to be minimized.

PROPOSITION 11.5 *Let* $G(x,y) \equiv 0$, $U(x,y) = -kx^2 + cx$, *where* $c \in \mathbf{R}$, $k \in \mathbf{R}$, $k > 0$, $c \geq 0$. *Then Conditions 10.1–10.2 hold, with*

$$F(z,y,\lambda) = \frac{c-q}{2k}, \quad q \triangleq \frac{\lambda}{z}.$$

In this case, equation (10.11) has an unique solution

$$\begin{aligned} \hat{\lambda} &= 2k\left(\tfrac{c}{2k} - X(0)\right)\left(\int_{\mathbf{R}^n_+} \tfrac{p_*(x,T)^2}{p(x,T)}dx\right)^{-1} \\ &= 2k\left(\tfrac{c}{2k} - X(0)\right)\left(\mathbf{E}\tfrac{1}{\psi(\tilde{S}_*(T))}\right)^{-1}. \end{aligned} \tag{11.15}$$

Again, the above result can be verified directly. Moreover, it follows by Lemma 10.2 that the integrand in (11.15) is integrable.

11.2.3 Nonlinear concave utility functions

Let $G(x,y) = \chi_{\{h_1(y)\leq x \leq h_2(y)\}}$, where $h_i : \overset{\circ}{\mathbf{R}}{}^n_+ \to [-\infty, +\infty]$. Further, let $U(x,y) \equiv U(x) : \overset{\circ}{\mathbf{R}}_+ \to \mathbf{R}$ be a concave differentiable function. Then the problem (11.1)–(11.2) can be rewritten as

$$\text{Maximize} \quad \mathbf{E}U(\tilde{X}(T,\pi(\cdot))) \quad \text{over} \quad \pi(\cdot) \tag{11.16}$$

$$\text{subject to} \begin{cases} X(0,\pi(\cdot)) = X(0), \\ h_1(\tilde{X}(T,\pi(\cdot))) \leq \tilde{X}(T,\pi(\cdot)) \leq h_2(\tilde{X}(T,\pi(\cdot))) \quad \text{a.s.} \end{cases} \tag{11.17}$$

PROPOSITION 11.6 *Assume that* h_i, U *are such that the following holds:*

(i) *whenever* $h \; : \overset{\circ}{\mathbf{R}}{}^n_+ \to \mathbf{R}$ *is a function of polynomial growth, so are* $\max(h_1(y), h(y))$ *and* $\min(h_2(y), h(y))$*;*

(ii) $-\infty \leq h_1(y) \leq h_2(y) \leq +\infty$ *for all* y*;*

(iii) $\mathbf{E}h_1(\tilde{S}_*(T)) < X(0) < \mathbf{E}h_2(\tilde{S}_*(T))$*;*

(iv) $U'(x) : (0, +\infty) \to (0, +\infty)$ *is a bijection (i.e., a one-to-one mapping), and there exist constants* $C > 0$, $0 < c_1 < 1$, *and* $c_2 > 0$ *satisfying*

$$|U(x)| \le C(x^{c_1} + x^{-c_2} + 1), \quad |V(x)| \le C(x^{c_2} + x^{-c_2} + 1), \quad \forall x > 0, \tag{11.18}$$

where $V(x)$ *is the converse function of* $U'(x)$.

Then Assumption 10.2 holds, with

$$F(z, y, \lambda) = \begin{cases} h_1(y) & \text{if } V(q) \le h_1(y) \\ V(q) & \text{if } h_1(y) < V(q) < h_2(y) \\ h_2(y) & \text{if } V(q) \ge h_2(y), \end{cases} \tag{11.19}$$

where $q \triangleq \lambda/z$. *In this case, equation (10.11) has a unique solution. Furthermore, if* $h_1(y) \le 0$ $(\forall y)$, $h_2(y) \equiv +\infty$, *and* $V(x) = Kx^{-k}$, *where* $K > 0$, $k > 0$ *are constants, then*

$$\hat{\lambda} = \left(e^{rT} X(0)\right)^{-1/k} \left(K \int_{\mathbf{R}_+^n} p_*(x, T)^{1-k} p(x, T)^k dx\right)^{1/k}. \tag{11.20}$$

The proof is omitted here, since it can be checked directly.

Notice if $U(x) = \ln(x)$, then $V(x) = x^{-1}$; if $U(x) = x^{1/\delta}$, $\delta > 1$, then $V(x) = (\delta x)^{-\delta'}$, where $\delta' = \delta(\delta - 1)^{-1}$.

Also, it is a direct consequence of Lemma 10.2 that the integrand in (11.20) is integrable.

11.2.4 Nonconnected $J(y)$

Assume that for each $y \in \overset{\circ}{\mathbf{R}}{}_+^n$ there exist an integer $N(y) > 0$ and real numbers $a_i(y), b_i(y)$ such that

$$J(y) = \cup\left(\cup_{i=1}^{N(y)} [a_i(y), b_i(y)]\right) \cup (a_0(y), +\infty),$$

where $J(y)$ is defined in (10.10), and

$$a_0(y) \le +\infty, \quad -\infty < a_i(y) \le b_i(y) < +\infty, \quad i = 1, \dots, N(y).$$

Let

$$\mathcal{M}(y) \triangleq \{a_0(y), a_1(y), \dots, a_{N(y)}(y), b_1(y), \dots, b_{N(y)}(y)\},$$

and let int J be the interior of the set J.

PROPOSITION 11.7 (i) *Let* $U(x,y) \equiv -kx^2 + cx$, *where* $k \in \mathbf{R}$, $c \geq 0$. *Then Condition 10.2 holds, with*

$$F(z,y,\lambda) = \begin{cases} \bar{F}(q) & \text{if } \bar{F}(q) \in \operatorname{int} J(y) \\ \arg\max_{x \in \mathcal{M}(y)} U(x,y) - qx & \text{if } \bar{F}(q) \notin \operatorname{int} J(y), \end{cases} \tag{11.21}$$

where $\bar{F}(q) \equiv (c-q)/(2k)$ *and* $q \triangleq \lambda/z$.

(ii) *Let* $U(x,y) \equiv \ln x$. *Then Condition 10.2 holds with* $F(\cdot)$ *defined by (11.21) and with* $\bar{F}(q) \equiv 1/q$.

(iii) *Let* $U(x,y) \equiv x^{1/\delta}$, *where* $\delta > 1$. *Then Condition 10.2 holds, with* $F(\cdot)$ *defined by (11.21) and with* $\bar{F}(q) \equiv (\delta q)^{-\delta'}$, *where* $\delta' \triangleq \delta(\delta-1)^{-1}$ *and* $q \triangleq \lambda/z$.

(iv) *Let* $U(x,y) \equiv -y^{\delta} + y$, *where* $\delta = 1 + 1/m$ *and* $m > 0$ *is an integer. Then Condition 10.2 holds, with* $F(\cdot)$ *defined by (11.21) and with* $\bar{F}(q) \equiv (1-q)^m \delta^{-m}$ *and* $q \triangleq \lambda/z$.

(v) *Let* $U(x,y) \equiv -|h(y) - x|^{\delta}$, *where* $\delta > 1$, $\mu \in (1,\delta)$ *and where* $h(\cdot) : \overset{\circ}{\mathbf{R}}{}^n_+ \to \mathbf{R}$ *is a function of polynomial growth. Then Condition 10.2 holds, with* $F(\cdot)$ *defined by (11.21) and with* $\bar{F}(q) \equiv -\operatorname{sign} q |q/\delta|^{1/(\delta-1)} + h(y)$ *and* $q \triangleq \lambda/z$.

Notice that the case considered in (v) corresponds to the following problem of claim hedging with nonquadratic criterion:

$$\text{Minimize} \quad \mathbf{E}|\tilde{X}(T) - \varphi(\tilde{S}(T))|^{\delta}.$$

The following theorem is for the problem of claim hedging with bounds on risk (see (8.13)).

THEOREM 11.3 *Let* $h_i(\cdot) : \overset{\circ}{\mathbf{R}}{}^n_+ \to \mathbf{R}$, $i = 1,2$ *be functions of polynomial growth such that* $G(x,y) = 1 - \chi\{h_1(y) \leq x \leq h_2(y)\}$ *and* $0 \leq h_1(y) < h_2(y)$ $(\forall y)$. *Let* $F(z,y,\lambda)$ *be as defined in Condition 10.2. Furthermore, let either*

$$\begin{cases} F(z,y,\lambda) \to h_1(y) & \text{as} \quad q \to 0, \\ F(z,y,\lambda) \to h_2(y) & \text{as} \quad q \to +\infty \quad (\forall x) \end{cases}$$

or

$$\begin{cases} F(z,y,\lambda) \to h_2(y) & \text{as} \quad q \to 0 \\ F(z,y,\lambda) \to h_1(y) & \text{as} \quad q \to +\infty \quad (\forall y), \end{cases}$$

where $q \triangleq \lambda/z$. *Then there exists a* $\hat{\lambda} > 0$ *such that (10.11) holds.*

11.3. Numerical examples

11.3.1 Solution of the goal–achieving problem

We present a numerical solution of the goal achieving problem (11.10)-(11.11), with the following parameters:

$$\begin{aligned}
&n = 1, \quad S(0) = 1.6487, \quad X(0) = 1, \quad T = 1,\\
&r(t) \equiv 0, \quad \sigma = \sigma(t) \equiv 0.5,\\
&k_2 = 1.2, \quad k_1 = 1/1.2 = 0.8333,\\
&\mathbf{P}(a(t) \equiv \alpha_1) = \mathbf{P}(a(t) \equiv \alpha_2) = 1/2,
\end{aligned}$$

where

$$\alpha_1 = 0.2, \quad \alpha_2 = \log(2 - e^{0.2}).$$

Note that under this assumption, $\mathbf{E}S(T) = S(0)$.

With these parameters, the optimal claim $f(x, \hat{\lambda})$ is given by the formula

$$f(x, \hat{\lambda}) = \begin{cases} 1/1.2 & \text{if } x \in (1.1070,\ 2.3490) \\ 1.2 & \text{if } x \notin (1.1070,\ 2.3490), \end{cases}$$

with $\hat{\lambda} = 2.5915$.

Furthermore, by (11.3) and (11.5), we have

$$X(t) = H(S(t), t), \tag{11.22}$$

where

$$H(x,t) = \tfrac{1}{1.2} + \left[1.2 - \tfrac{1}{1.2}\right]\left(\int_0^{1.1070} \bar{p}_*(y,T,t,x)dy + \int_{2.3490}^{+\infty} \bar{p}_*(y,T,t,x)dy \right), \tag{11.23}$$

$$\bar{p}_*(y,T,x,t) = \frac{1}{y\sigma\sqrt{2\pi(T-t)}} \exp \frac{-\left(\ln(y) - \ln(x) - r(T-t) + \sigma^2(T-t)/2\right)^2}{2(T-t)\sigma^2}. \tag{11.24}$$

The strategy can be easily calculated from (11.3), (11.23), and (11.24).

The wealth process associated with the optimal strategy is given by the following:

$$X(t) = 0.8 + 0.3666 \left[1 - \mathbf{P}\left(S_*(T) \in (1.1070,\ 2.3490) \middle| S(t)\right)\right].$$

Figure 11.1 shows $H(x, 0)$ and the optimal claim $f(x, \hat{\lambda}) = H(x, T)$. Figure 11.2 shows the profit/loss diagram for the corresponding optimal strategy.

Figure 11.1. The optimal claim $f(x, \hat{\lambda})$ and $H(x, 0)$ for goal achieving with $k_1 = 1/1.2$, $k_2 = 1.2$. —: values of $H(x, 0)$; - - - -: values of $f(x, \hat{\lambda}) = H(x, T)$.

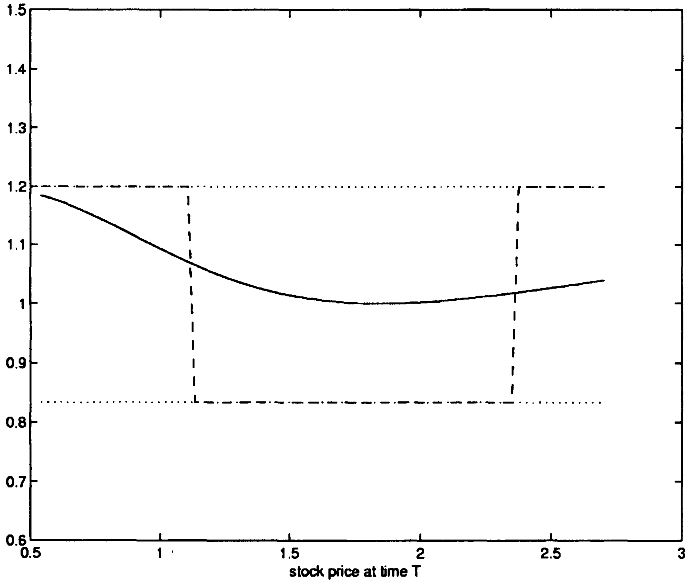
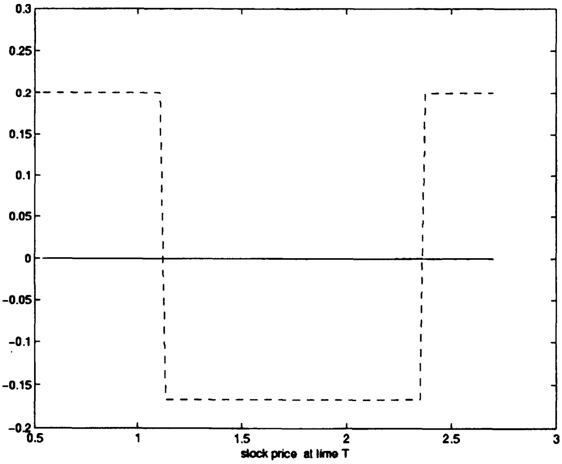


Figure 11.2. Profit/loss diagram for the optimal solution of the goal-achieving problem: - - - -: values of $X(T) - X(0) = H(S(T), T) - X(0)$.



11.3.2 Optimal replication of a put option with a possible gap

Consider a problem of optimal replication of a European put option with a possible gap. This problem is a particular case of the problem (11.16)–(11.17).

We present a numerical solution with the following parameters:

$$
\begin{aligned}
&n = 1, \quad U(x) = \ln(x), \quad S(0) = 1.6487, \quad X(0) = 1, \quad T = 1,\\
&r(t) \equiv 0, \quad \sigma = \sigma(t) \equiv 0.5,\\
&h_1(x) \equiv (S(0) - x)^+,\\
&h_2(x) = \begin{cases} (2S(0) - x)^+ & \text{if } x \le 1.8S(0) \\ 0.2S(0) & \text{if } x > 1.8S(0), \end{cases}\\
&\mathbf{P}(a(t) \equiv \alpha_1) = \mathbf{P}(a(t) \equiv \alpha_2) = 1/2,
\end{aligned}
$$

where

$$\alpha_1 = 0.2, \quad \alpha_2 = \log(2 - e^{0.2}).$$

With these parameters, we have

$$\mathbf{E}h_1(S_*(T)) = 0.3255, \quad \mathbf{E}h_2(S_*(T)) = 1.717,$$

and the optimal claim $f(x, \hat{\lambda})$ is given by the formula

$$f(x, \hat{\lambda}) = \begin{cases} h_1(x) & \text{if } \psi(x)/\hat{\lambda} \le h_1(x) \\ \psi(x)/\hat{\lambda} & \text{if } h_1(x) < \psi(x)/\hat{\lambda} < h_2(x) \\ h_2(x) & \text{if } \psi(x)/\hat{\lambda} \ge h_2(x), \end{cases}$$

with $\hat{\lambda} = 0.8923$. The function $\psi(x)$ is defined in Theorem 11.1:

$$\psi(x) = \frac{1}{2} \sum_{i=1,2} \left(\frac{x}{S(0)} \right)^{4\alpha_i} \exp\left(\frac{\alpha_i}{2} - 2\alpha_i^2 \right).$$

Therefore, the strategy can be calculated by virtue of (11.3).

Figure 11.3 depicts $H(x, 0)$ and the optimal claim $f(x, \hat{\lambda}) = H(x, T)$. Figure 11.4 shows the profit/loss diagram for the corresponding optimal strategy.

11.3.3 Solution with logical constraints

Consider the following optimal investment problem involving a single-stock market with logical constraints:

$$\text{Maximize} \quad \mathbf{P}(X(T, \pi(\cdot)) \ge 1.1 \cdot X(0))$$

$$\text{subject to} \begin{cases} X(0, \pi(\cdot)) = X(0), \\ X(T, \pi(\cdot)) \ge 0.5 \cdot X(0), \\ \text{if} \quad |S(0) \quad - S(T)| < 0.2 \cdot S(0) \\ \qquad\qquad \text{then} \quad X(T, \pi(\cdot)) \ge 0.9 \cdot X(0). \end{cases} \tag{11.25}$$

Figure 11.3. The optimal claim $f(x,\hat{\lambda})$ and $H(x,0)$ for replication of a put option with a gap. .....: values of $h_1(x)$, $h_2(x)$; —: values of $H(x,0)$; - - - -: values of $f(x,\hat{\lambda}) = H(x,T)$.

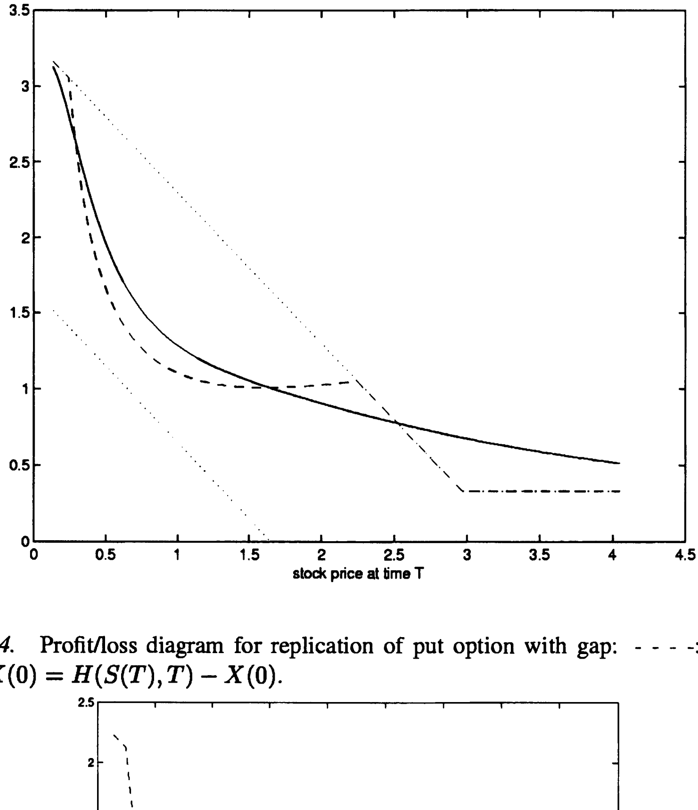
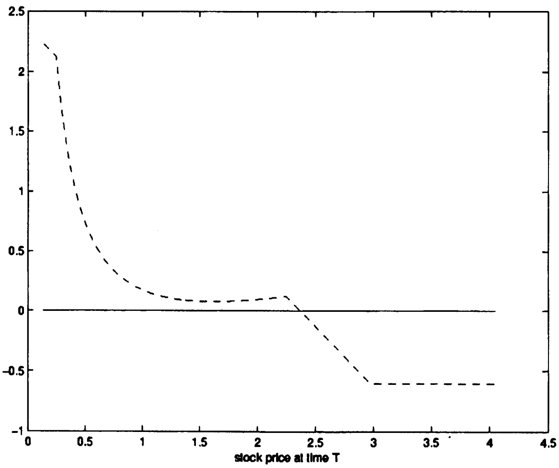


Figure 11.4. Profit/loss diagram for replication of put option with gap: - - - -: values of $X(T) - X(0) = H(S(T),T) - X(0)$.



This problem is a special case of the problem (11.1)–(11.2). We present a numerical solution with the following parameters:

$$n = 1, \quad S(0) = X(0) = 1.6487, \quad T = 1,$$
$$r(t) \equiv 0, \quad \sigma = \sigma(t) \equiv 0.5,$$
$$\mathbf{P}(a \equiv \alpha_1) = \mathbf{P}(a \equiv \alpha_2) = 1/2,$$

where

$$\alpha_1 = 0.2, \quad \alpha_2 = \log(2 - e^{0.2}).$$

With these parameters, the optimal claim $f(x, \hat{\lambda})$ is given by

$$f(x,\hat{\lambda}) = \begin{cases} 0.5 \cdot X(0) & \text{if } \hat{\lambda}/\psi(x) > 1,\ |x - S(0)| > 0.2S(0) \\ 0.9 \cdot X(0) & \text{if } \hat{\lambda}/\psi(x) > 1,\ |x - S(0)| \le 0.2S(0) \\ 1.1 \cdot X(0) & \text{if } \hat{\lambda}/\psi(x) \le 1, \end{cases}$$

where $\hat{\lambda} = 2.751$ is obtained numerically. The function $\psi(x)$ defined in Theorem 11.1 is

$$\psi(x) = \frac{1}{2} \sum_{i=1,2} \left(\frac{x}{S(0)} \right)^{4\alpha_i} \exp\left(\frac{\alpha_i}{2} - 2\alpha_i^2 \right). \qquad (11.26)$$

Figure 11.5 shows the initial wealth, $H(x, 0)$, given $S(0) = x$, and the optimal claim $f(x, \hat{\lambda})$, i.e., the final wealth given $\tilde{S}(T) = x$ (for the definition of H, see (11.5)). It is observed that although $H(x, 0)$ is smooth, the optimal claim is discontinuous and piecewise constant as a consequence of the logical-type constraints. With the obtained optimal claim, we can calculate the admissible self-financing strategy at each given time $t \in [0, T]$ depending only on the stock price at that given time as follows:

$$\pi(t)^\top = \frac{\partial H}{\partial x}(\tilde{S}(t), t)\mathbf{S}(t), \quad t \in [0, T].$$

The corresponding wealth at each time $t \in [0, T]$ is $X(t) = H(S(t), t)$, which also depends only on the current stock price.

Notice that the optimal strategy is well defined in accordance with Definition 8.2 (i.e. it is a square integrable process). However, it has a property in common with the goal-achieving problem studied in Karatzas (1997), Dokuchaev and Zhou (2001): $\mathbf{E}|\pi(t)|^2 \to +\infty$ as $t \to T - 0$.

11.4. Proofs

The proof of Propositions 11.4 and 11.1 is straightforward and will be omitted here.

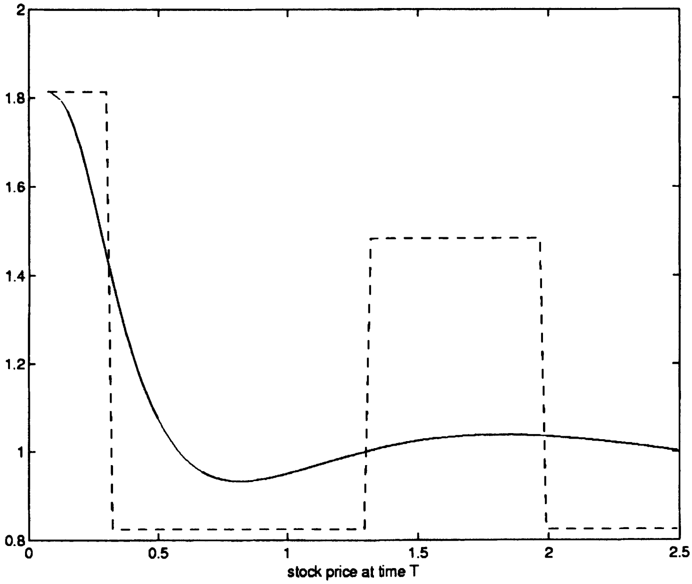
Proof of Corollary 11.1. We have that

$$z_*(\alpha, T) = \exp\left(\alpha^\top \int_0^T \sigma(t) dw(t) - \frac{1}{2} \int_0^T |\sigma(t)^\top \alpha| dt \right),$$
$$\int_0^T \sigma(t) dw(t) = v(\tilde{S}_*(T)),$$

where the function $v(\cdot) = \{v_i(\cdot)\}_{i=1}^n : \overset{\circ}{\mathbf{R}}{}^n_+ \to \mathbf{R}^n$ is such that

$$v_i(x) \triangleq \log \frac{x_i}{S_i(0)} + \frac{1}{2} \sum_{j=1}^n \int_0^T \sigma_{ij}(s)^2 ds, \quad x = \{x_i\}_{i=1}^n.$$

Figure 11.5. Optimal claim $f(x, \hat{\lambda})$ and $H(x, 0)$ for the problem with logical constraints. —: values of $H(x,0)$; - - - -: values of $f(x, \hat{\lambda}) = H(x,T)$.



Then there exists a measurable function $\psi(\cdot) : \mathbf{R}^n \to \mathbf{R}$ such that $\bar{\mathcal{Z}}_* = \psi(\tilde{S}_*(T))$. To complete the proof, it suffices to set $H(x,t) \triangleq V(v(x),t)$. □

Proof of Proposition 11.3. It is easy to see that

$$\begin{cases} \frac{\partial H}{\partial t}(x,t) + \frac{1}{2}\sum_{i,j=1}^n x_i x_j \frac{\partial^2 H}{\partial x_i \partial x_j}(x,t) \sum_{k=1}^n \sigma_{ik}(t)\sigma_{jk}(t) = 0, \\ H(x,T) = f(x, \hat{\lambda}). \end{cases} \tag{11.27}$$

It can be seen that (11.27) is the backward Kolmogorov equation for the distribution of $\hat{\mathbf{S}}_*(t)$. Then the proof follows. □

Proof of Proposition 11.2. We have that $\bar{\mathcal{Z}} = \psi(\tilde{S}_*(T))$, where $\psi : \mathbf{R}^n \to \mathbf{R}$ is a measurable function. By Girsanov's Theorem, it follows that for any bounded measurable function $\phi(\cdot) : \mathbf{R}^n \to \mathbf{R}$, we have

$$\begin{aligned} &\int_{\mathbf{R}^n_+} p(x,T)\phi(x)dx = \int_{\mathcal{T}} d\nu(\alpha)\mathbf{E}\{\phi(S(T)) \,|\, \tilde{a} = \alpha\} \\ &= \int_{\mathcal{T}} d\nu(\alpha) \int_{\mathbf{R}^n_+} \mathbf{E}\left\{ z(\alpha,T)\phi(\tilde{S}_*(T)) \,\middle|\, \tilde{a} = \alpha, \tilde{S}_*(T) = x \right\} p_*(x,T)dx \\ &= \int_{\mathcal{T}} d\nu(\alpha) \int_{\mathbf{R}^n_+} \mathbf{E}\left\{ z(\alpha,T) \,\middle|\, \tilde{S}_*(T) = x \right\} \phi(x)p_*(x,T)dx \\ &= \int_{\mathbf{R}^n_+} \psi(x)\phi(x)p_*(x,T)dx. \end{aligned}$$

This yields

$$\psi(x) = \frac{p(x,T)}{p_*(x,T)}.$$

This completes the proof. □

Proof of Theorem 11.3. By the definition of $f(\cdot)$, we have $h_1(x) \le f(x,\lambda) \le h_2(x)$. By assumptions, $\mathbf{E}|h_i(\tilde{S}_*(T))| < +\infty$. Let

$$g(\lambda) \triangleq \mathbf{E} f(\tilde{S}_*(T),\lambda).$$

By the Lebesgue Dominated Convergence Theorem, it follows that either

$$\begin{cases} g(\lambda) \to \mathbf{E}h_1(\tilde{S}_*(T)) & \text{as} \quad \lambda \to +\infty, \\ g(\lambda) \to \mathbf{E}h_2(\tilde{S}_*(T)) & \text{as} \quad \lambda \to 0+, \end{cases}$$

or

$$\begin{cases} g(\lambda) \to \mathbf{E}h_2(\tilde{S}_*(T)) & \text{as} \quad \lambda \to +\infty, \\ g(\lambda) \to \mathbf{E}h_1(\tilde{S}_*(T)) & \text{as} \quad \lambda \to 0+. \end{cases}$$

By Condition 10.1, there exists a measurable function of polynomial growth $F'(\cdot) : C([0,T];\mathbf{R}^n) \to \mathbf{R}$ such that $\mathbf{E}F'(\tilde{S}_*(\cdot)) = X(0)$ and

$$h_1(\tilde{S}(T)) \le F'(\tilde{S}(\cdot)) \le h_2(\tilde{S}(T)) \quad \text{a.s.}$$

Hence

$$h_1(\tilde{S}_*(T)) \le F'(\tilde{S}_*(\cdot)) \le h_2(\tilde{S}_*(T)) \quad \text{a.s.}$$

and

$$\mathbf{E}h_1(\tilde{S}_*(T)) \le X(0) \le \mathbf{E}h_2(\tilde{S}_*(T)).$$

Thus, there exists a $\lambda > 0$ such that (10.11) holds. This completes the proof. □

Proof of Theorem 11.1. For any fixed $a \in \mathcal{T}$, let $p_a^{(i)}(x_i,t)$ be the probability density function for the stock price $S_i(t)$. Furthermore, let $p_*^{(i)}(x_i,t)$ be the probability density function for the stock price $S_i(t)$ with $a_i(\cdot) \equiv r$. It is easy to see that

$$S_i(T) = S_i(0)\exp\left\{\mu_i - \frac{\bar{\sigma}_i^2}{2} + \xi_i\right\},$$

where μ_i, $\bar{\sigma}_i$ are defined in (11.8) and ξ_i are independent Gaussian random variables such that

$$\mathbf{E}\xi_i = 0, \quad \mathbf{E}\xi_i^2 = \bar{\sigma}_i^2.$$

Hence

$$\begin{aligned} p_*^{(i)}(x_i,T) &= \tfrac{1}{x_i\sqrt{2\pi}\bar{\sigma}_i}\exp\tfrac{-\left(\ln(x_i)-\ln(S_i(0))+\bar{\sigma}_i^2/2\right)^2}{2\bar{\sigma}_i^2}, \\ p_a^{(i)}(x_i,T) &= \tfrac{1}{x_i\sqrt{2\pi}\bar{\sigma}_i}\exp\tfrac{-\left(\ln(x_i)-\ln(S_i(0))-\mu_i-rT+\bar{\sigma}_i^2/2\right)^2}{2\bar{\sigma}_i^2}. \end{aligned} \tag{11.28}$$

Let

$$\varepsilon_i \triangleq \ln(x_i) - \ln(S_i(0)) + \frac{\bar{\sigma}_i^2}{2}.$$

Then

$$\exp \frac{-\left((\varepsilon_i-\mu_i)^2-\varepsilon_i^2\right)}{2\bar{\sigma}_i^2} = \exp \frac{\varepsilon_i\mu_i}{\bar{\sigma}_i^2} \exp \frac{-\mu_i^2}{2\bar{\sigma}_i^2} = \left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2} \exp \frac{\mu_i(\bar{\sigma}_i^2-2rT)-\mu_i^2}{2\bar{\sigma}_i^2}.$$

This yields

$$\frac{p_a^{(i)}(x_i,T)}{p_*^{(i)}(x_i,T)} = \left(\frac{x_i}{S_i(0)}\right)^{\mu_i/\bar{\sigma}_i^2} e^{\theta_i}, \tag{11.29}$$

where θ_i is defined in (11.8). Furthermore, we have

$$p_a(x,t) = \prod_{i=1}^{n} p_a^{(i)}(x_i,t), \quad p_*(x,t) = \prod_{i=1}^{n} p_*^{(i)}(x_i,t), \quad x = (x_1,\dots,x_n). \tag{11.30}$$

From (11.28)–(11.30), we obtain (11.9). This completes the proof. □

Proof of Theorem 11.2. Let G' be the set of admissible $\pi(\cdot)$ such that the constraints in (11.2) hold. Denote by $J'(\pi(\cdot))$ and $J''(\pi(\cdot))$ the functionals to be maximized in (11.1) and (11.10), respectively.

Suppose $\pi(\cdot)$ is an optimal strategy for the problem (11.10)–(11.11). Construct the following strategy:

$$\hat{\pi}(t) = \begin{cases} \pi(t), & t \le \tau, \\ 0, & t > \tau, \end{cases}$$

where

$$\tau \triangleq \tau_1 \wedge \tau_2.$$

Clearly, $\hat{\pi}(\cdot) \in G'$ and $J'(\hat{\pi}(\cdot)) = J''(\pi(\cdot))$. Hence

$$\sup_{\pi(\cdot)\in G'} J'(\pi(\cdot)) \ge \sup_{\pi(\cdot)} J''(\pi(\cdot)).$$

On the other hand, let $\bar{\pi}(\cdot)$ be the optimal strategy for the problem (11.1). This strategy is unique and is given by (11.5), (11.3), and (11.13). The corresponding optimal normalized wealth process $\tilde{X}(t)$ is given by (11.3) and (11.14). It can be easily seen from these equations that $\tilde{X}(t) \in (k_1, k_2)$ $(\forall t < T)$ a.s., where $\tilde{X}(t)$ is the corresponding optimal normalized wealth process. Hence $J'(\bar{\pi}(\cdot)) = J''(\bar{\pi}(\cdot))$, leading to

$$\sup_{\pi(\cdot)\in G'} J'(\pi(\cdot)) \le \sup_{\pi(\cdot)} J''(\pi(\cdot)).$$

This completes the proof. □

Chapter 12

UNKNOWN DISTRIBUTION: MAXIMIN CRITERION AND DUALITY APPROACH

Abstract In this chapter, a case is studied in which the appreciation rates, volatilities, and their prior distributions are unknown. The optimal investment problem is stated as a problem with a *maximin* performance criterion. This criterion is to ensure that a strategy is found such that the utility minimum over all distributions of parameters is maximal. It is shown that the duality theorem holds for the problem. Thus, the maximin problem is reduced to the *minimax* problem. This *minimax* problem is computationally a much easier problem.

12.1. Definitions and problem statement

Similarly to Chapter 8, we consider the market model from Section 1.3. The market consists of a risk free bond or bank account with price $B(t)$, $t \geq 0$, and n risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, where $n < +\infty$ is given. The prices of the stocks evolve according to

$$dS_i(t) = S_i(t)\left(a_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right), \quad t > 0, \qquad (12.1)$$

where the $w_i(t)$ are standard independent Wiener processes, $a_i(t)$ are appreciation rates, and $\sigma_{ij}(t)$ are volatility coefficients. The initial price $S_i(0) > 0$ is a given nonrandom constant. The price of the bond evolves according to the following equation

$$B(t) = B(0) \exp\left(\int_0^t r(s)ds\right), \qquad (12.2)$$

where $B(0)$ is a given constant that we take to be 1 without loss of generality, and $r(t)$ is the random process of the risk-free interest rate.

As usual, we assume that $w(\cdot)$ is a standard Wiener process on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\Omega = \{\omega\}$ is a set of elementary events, $\mathcal{F}$ is a complete σ-algebra of events, and $\mathbf{P}$ is a probability measure.

Set $\mu(t) \triangleq (r(t), \tilde{a}(t), \sigma(t))$, where $\tilde{a}(t) \triangleq a(t) - r(t)\mathbf{1}$.

Similarly Chapter 8, we assume that there exists an integer $N > 0$ and a random process $\eta(t) = (\eta_1(t), \dots, \eta_N(t))$ that is correlated with stock prices and that is currently observable. We assume that all the paths of $\eta(t)$ are bounded and that there exist a linear Euclidean space E_0, a random vector $\Theta_\eta : \Omega \to E_0$, and a measurable function

$$F_0(t, \cdot) : E_0 \times B([0,t]; \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n}) \to \mathbf{R}^N$$

such that

$$\eta(t) = F_0(t, \Theta_0, [S(\cdot), \mu(\cdot)]_{[0,t]}) \quad \forall t.$$

We assume that the distribution of Θ_0 is known. (An important example is a model when $\eta(t)$ describes trade volume; see Chapter 8).

We describe now distributions of $\mu(\cdot)$ and what we suppose to know about them.

We assume that there exist a finite-dimensional Euclidean space $\bar{E}$, a compact subset $\mathcal{T} \subset \bar{E}$, and a measurable function

$$M(t, \cdot) : \mathcal{T} \times \mathbf{E}_0 \times C\left([0,t]; \mathbf{R}^n\right) \to \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n \times n},$$

which is uniformly bounded and such that $M(t, \alpha, \xi)$ is continuous in $\alpha \in \mathcal{T}$ for all t and $\xi \in C([0,t]; \mathbf{R}^)$. Let

$$M(t, \cdot) = \left(M_r(t, \cdot), M_a(t, \cdot), M_\sigma(t, \cdot)\right),$$

where

$$M_r(t, \cdot) : \mathcal{T} \times \mathbf{E}_0 \times C\left([0,t]; \mathbf{R}^n\right) \to \mathbf{R},$$
$$M_a(t, \cdot) : \mathcal{T} \times \mathbf{E}_0 \times C\left([0,t]; \mathbf{R}^n\right) \to \mathbf{R}^n,$$
$$M_\sigma(t, \cdot) : \mathcal{T} \times \mathbf{E}_0 \times C\left([0,t]; \mathbf{R}^n\right) \to \mathbf{R}^{n \times n}.$$

We assume that the matrix $M_\sigma(\cdot)^{-1}$ is uniformly bounded.

We assume that Θ_0, $\mathcal{T}$, $F_0(\cdot)$ and $M(\cdot)$ are such that the solution of (12.1) for $\mu(t) = M(t, \alpha, \Theta_0, S(\cdot)|_{[0,t]})$ is well defined for any $\alpha \in \mathcal{T}$ as the unique strong solution of Itô's equation. Let $S_\alpha(\cdot)$ denote the corresponding solution.

For $\alpha \in \mathcal{T}$, set

$$\bar{M}_r(t, \alpha) \triangleq M_r\left(t, \alpha, \Theta_0, S_\alpha(\cdot)|_{[0,t]}\right),$$
$$\bar{M}_a(t, \alpha) \triangleq M_a\left(t, \alpha, \Theta_0, S_\alpha(\cdot)|_{[0,t]}\right),$$
$$\bar{M}_\sigma(t, \alpha) \triangleq M_\sigma\left(t, \alpha, \Theta_0, S_\alpha(\cdot)|_{[0,t]}\right).$$

For an $\alpha \in \mathcal{T}$, set

$$\mu_\alpha(t) \triangleq (\bar{M}_r(t, \alpha), \bar{M}_a(t, \alpha), \bar{M}_\sigma(t, \alpha)).$$

Furthermore, we shall use the notation $S(t) = S(t, \mu(\cdot))$ and $\tilde{S}(t) = \tilde{S}(t, \mu(\cdot))$ to emphasize that the stock price is different for different $\mu(\cdot)$. Clearly, $S_\alpha(t) = S(t, \mu_\alpha(\cdot))$.

DEFINITION 12.1 *Let $\mathcal{A}(\mathcal{T})$ be a set of all random processes $\mu'(t) = (r'(t), \tilde{a}'(t), \sigma'(t))$ such that there exists a random vector $\Theta : \Omega \to \mathcal{T}$ independent of $(w(\cdot), \Theta_0, r(\cdot), \sigma(\cdot))$ such that*

$$\begin{cases} r'(t) \equiv \bar{M}_r(t, \Theta), \\ \tilde{a}'(t) \equiv \bar{M}_a(t, \Theta), \\ \sigma'(t) \equiv \bar{M}_\sigma(t, \Theta). \end{cases} \tag{12.3}$$

We assume that $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ and that this is the only information available about the distribution on $\mu(\cdot)$.

Under these assumptions the solution of (12.1) is well defined, but the market is incomplete.

Let $\mathcal{F}_t \subset \mathcal{F}$ be the filtration of complete σ-algebras of events generated by the process $(r(t), S(t), \eta(t))$, $t \geq 0$. Let $\bar{\Sigma}(\mathcal{F}_\cdot)$ be the class of admissible strategies introduced in Chapter 8.

Let $X_0 > 0$ be the initial wealth at time $t = 0$, and let $X(t)$ be the wealth at time $t > 0$. Let $\tilde{X}(t)$ be the normalized wealth.

By the definitions of $\bar{\Sigma}(\mathcal{F}_\cdot)$ and $\mathcal{F}_t$, any admissible self-financing strategy is of the form

$$\pi(t) = \Gamma(t, [r(\cdot), S(\cdot), \eta(\cdot)]|_{[0,t]}), \tag{12.4}$$

where $\Gamma(t, \cdot) : B([0, t]; \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^N) \to \mathbf{R}^n$ is a measurable function, $t \geq 0$.

Clearly, the random processes $\pi(\cdot)$ with the same $\Gamma(\cdot)$ in (12.4) may be different for different $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot))$. Hence we introduce also strategies defined by $\Gamma(\cdot)$.

DEFINITION 12.2 *Let $\bar{C}$ be the class of all functions $\Gamma(t, \cdot) : C([0, t]; \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^N) \to \mathbf{R}^n$, $t \geq 0$, such that the corresponding strategy $\pi(\cdot)$ defined by (12.4) belongs $\bar{\Sigma}(\mathcal{F}^a_\cdot)$ for any $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot)) \in \mathcal{A}(\mathcal{T})$ and*

$$\sup_{\mu(\cdot) = \mu_\alpha(\cdot):\ \alpha \in \mathcal{T}} \mathbf{E} \int_0^T |\pi(t)|^2 \, dt < \infty.$$

A function $\Gamma(\cdot) \in \bar{C}$ is said to be an admissible CL-strategy (closed-loop strategy).

Let the initial wealth $X(0)$ be fixed. For an admissible self-financing strategy $\pi(\cdot)$ such that $\pi(t) = \Gamma(t, [r(\cdot), S(\cdot), \eta(\cdot)]|_{[0,t]})$, the process $(\pi(t), X(t))$ is uniquely defined by $\Gamma(\cdot)$ and $\mu(\cdot) = (r(\cdot), \tilde{a}(\cdot), \sigma(\cdot))$ given $\Theta_0, w(\cdot)$. We shall use the notation $X(t, \Gamma(\cdot), \mu(\cdot))$ and $\tilde{X}(t, \Gamma(\cdot), \mu(\cdot))$ to denote the corresponding total wealth and normalized wealth.

DEFINITION 12.3 *Let $\xi = \phi(S(\cdot, \mu(\cdot), \eta(\cdot))$, where $\phi : B([0,T]; \mathbf{R}^n\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times n} \times \mathbf{R}^N) \to \mathbf{R}$ is a measurable function. Let the initial wealth $X(0)$ and time $T > 0$ be fixed. An admissible self-financing strategy $\pi(\cdot)$ is said to replicate the claim ξ given the initial wealth $X(0)$ if*

$$X(T, \Gamma(\cdot), \mu(\cdot)) = \xi \quad a.s. \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}).$$

The claim ξ is said to be attainable.

Problem statement

Let $T > 0$ and X_0 be given. Let $m > 0$ be an integer. Let $U(\cdot, \cdot) : \mathbf{R} \times C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+) \to \mathbf{R} \cup \{-\infty\}$ and $G(\cdot, \cdot) : \mathbf{R} \times C([0,T] \to \overset{\circ}{\mathbf{R}}{}^n_+) \to \mathbf{R}^m$ be given measurable functions such that $\mathbf{E}U(X_0, \tilde{S}(\cdot)) < +\infty$.

We may state our general problem as follows: Find an admissible CL-strategy $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$ and the corresponding self-financing strategy $\pi(\cdot) \in \bar{\Sigma}(\mathcal{F}_\cdot)$ that solves the following optimization problem:

$$\text{Maximize} \quad \min_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \in \bar{\mathcal{C}} \tag{12.5}$$

$$\text{subject to} \begin{cases} X(0, \Gamma(\cdot), \mu(\cdot)) = X_0, \\ G(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \le 0 \quad \text{a.s.} \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}). \end{cases} \tag{12.6}$$

As usual, a vector inequality means component-wise inequalities.

DEFINITION 12.4 *Let $\bar{\mathcal{C}}_0$ be the set of all admissible CL-strategies $\Gamma(\cdot) \in \bar{\mathcal{C}}$ such that*

$$G(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \le 0 \quad a.s. \quad \forall \mu(\cdot) \in \mathcal{A}(\mathcal{T}).$$

The problem (12.5)–(12.6) can be rewritten as

$$\text{Maximize} \quad \min_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} \mathbf{E}U(\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \quad \text{over} \quad \Gamma(\cdot) \in \bar{\mathcal{C}}_0. \tag{12.7}$$

Let

$$J(y) \triangleq \{x \in \mathbf{R} : G(x, y) \le 0\}, \quad y \in C([0,T]; \overset{\circ}{\mathbf{R}}{}^n_+). \tag{12.8}$$

To proceed further, we assume that the following Conditions 12.1–12.3 below remain in force throughout this chapter.

CONDITION 12.1 *There exist constants* $p \in (1,2]$, $q \in (0,1]$, $c > 0$, $c_i > 0$, $i = 0,1,2$, *such that*

$$U(x,y) \le c_0\Big(x^2 + \textstyle\sum_{i=1}^n \sup_t \left(y_i(t)^{c_1} + y_i(t)^{-c_2}\right) + 1\Big),$$
$$|U(x,y)| \le c(|x|^p + 1),$$
$$|U(x,y) - U(x_1,y)| \le c\,(1 + |x| + |x_1|)^{2-q}\,|x - x_1|^q$$
$$\forall y = y(\cdot) \in C([0,T];\overset{\circ}{\mathbf{R}}{}^n_+),\ \forall x, x_1 \in J(y).$$

CONDITION 12.2 *At least one of the following conditions holds:*

(i) The set $\mathcal{T}$ *is at most countable, i.e.,* $\mathcal{T} = \{\alpha_1, \alpha_2, \ldots\}$, *where* $\alpha_i \in \bar{E}$;

(ii) The function $(M_r(t,\alpha,\zeta,\xi), M_\sigma(t,\alpha,\zeta,\xi))$ *does not depend on* α *given* $\zeta \in E_0$ *and* $\xi \in C([0,t];\mathbf{R}^n)$, *i.e.,* $M_r(t,\alpha,\zeta,\xi) \equiv M_r(t,\zeta,\xi)$ *and* $M_\sigma(t,\alpha,\zeta,\xi) \equiv M_\sigma(t,\zeta,\xi)$, *and Condition 12.1 is satisfied with* $p \in (1,2)$.

Condition 12.2(i) looks restrictive, but in fact it is rather technical, since the total number of elements of $\mathcal{T}$ is unbounded.

CONDITION 12.3 *The function* $G(\cdot)$ *is such that* $G(y,x) > 0$ $(\forall y < 0$, $\forall x \in C([0,T];\overset{\circ}{\mathbf{R}}{}^n_+))$, *and there exists an attainable claim* ξ *such that* $G(B(T)^{-1}\xi, \tilde{S}(T,\mu(\cdot))) \le 0$ *a.s. for all* $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$.

By Condition 12.3, it follows that $\bar{\mathcal{C}}_0 \neq \emptyset$: $\bar{\mathcal{C}}_0$ contains the CL-strategy which replicates the claim ξ.

12.2. A duality theorem

Without loss of generality, we describe the probability space as follows: $\Omega = \mathcal{T} \times \Omega'$, where

$$\Omega' = C([0,T];\mathbf{R}^n) \times E_0.$$

We are given a σ-algebra $\mathcal{F}'$ of subsets of Ω', and we assume that there is a σ-additive probability measure $\mathbf{P}'$ on $\mathcal{F}'$ generated by $(w(\cdot), \Theta_0)$. Furthermore, let $\mathcal{F}_\mathcal{T}$ be the σ-algebra of all Borel subsets of $\mathcal{T}$, and let $\mathcal{F} = \mathcal{F}_\mathcal{T} \otimes \mathcal{F}'$. We assume also that each $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$ generates the σ-additive probability measure ν_μ on $\mathcal{F}_\mathcal{T}$ (this measure is generated by Θ, which corresponds to $\mu(\cdot)$).

THEOREM 12.1 *Let the set* $J(y)$ *be convex for all* $y \in C([0,T];\overset{\circ}{\mathbf{R}}{}^n_+)$, *and let the function* $U(x,y) : \mathbf{R} \times C([0,T];\overset{\circ}{\mathbf{R}}{}^n_+) \to \mathbf{R}$ *be convex in* $x \in J(y)$ *for* *each* $y \in C([0,T]; \overset{\circ}{\mathbf{R}}{}^n_+)$. *Then*

$$\sup_{\Gamma(\cdot)\in\bar{\mathcal{C}}_0} \inf_{\mu\in\mathcal{A}(\mathcal{T})} \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot)) = \inf_{\mu(\cdot)\in\mathcal{A}(\mathcal{T})} \sup_{\Gamma(\cdot)\in\bar{\mathcal{C}}_0} \mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))). \tag{12.9}$$

12.3. Duality approach to the maximin problem

The problem (12.5)–(12.6) is a very difficult problem to solve. However, by virtue of Theorem 12.1, it is possible to replace the original *maximin* problem (12.5)–(12.6) by a *minimax* problem. This corresponding *minimax* problem is sometimes much easier to solve.

Suppose that, for a given $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, we can solve the following auxiliary problem:

$$\text{Maximize} \quad \mathbf{E}U\left(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))\right) \quad \text{over} \quad \Gamma(\cdot) \in \bar{\mathcal{C}}_0, \tag{12.10}$$

or, equivalently,

$$\text{Maximize} \quad \mathbf{E}U\left(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))\right) \quad \text{over} \quad \Gamma(\cdot)$$

$$\text{subject to} \begin{cases} X(0,\Gamma(\cdot),\mu(\cdot)) = X_0, \\ G(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))) \leq 0 \quad \text{a.s.} \end{cases}$$

Let $\hat{\Gamma}_\mu(\cdot)$ be an optimal solution for this problem. Consider the auxiliary problem

$$\text{Minimize} \quad \mathbf{E}U\left(\tilde{X}(T,\hat{\Gamma}_\mu(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))\right) \quad \text{over} \quad \mu(\cdot) \in \mathcal{A}(\mathcal{T}). \tag{12.11}$$

Let $\hat{\mu}(\cdot)$ be an optimal solution of the problem (12.11). Then, by Theorem 12.1, it follows that $\hat{\Gamma}_{\hat{\mu}}(\cdot)$ is an optimal solution of the original maximin problem (12.5)–(12.6).

Thus, an algorithm that solves the problem (12.5)–(12.6) can be described as follows:

- For an $\mu(\cdot)$, find the optimal solution $\hat{\Gamma}_\mu(\cdot)$ of the problem (12.10).
- Find the $\hat{\mu}(\cdot)$ that solves the problem (12.11).
- Then the strategy $\hat{\Gamma}_{\hat{\mu}}(\cdot)$ is optimal for the problem (12.5)–(12.6).

The solution of the problem (12.10) has been discussed in Chapters 8-11.

12.4. Minimizing with respect to $a(\cdot)$

We now consider the problem (12.11) under the assumptions of Corollary 11.1, Chapter 11. Let $F(\cdot)$ be such as defined in Chapter 11. For a given $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, we have

$$\begin{aligned}\mathbf{E} \quad & U\left(\tilde{X}(T,\hat{\Gamma}(\cdot),\mu(\cdot))),\tilde{S}(T,\mu(\cdot))\right) \\ & = \int_A \nu_\mu(d\alpha) \int_{\mathbf{R}^n_+} p_\alpha(x,T) U(f_\mu(x,\lambda_\mu),x)dx,\end{aligned}$$

where

$$f_\mu(x,\lambda) \triangleq F\left(\psi_\mu(x),x,\lambda\right), \quad \psi_\mu(x) \triangleq \int_{\mathcal{T}} \nu_\mu(d\alpha)\frac{p_\alpha(x,T)}{p_*(x,T)}, \qquad (12.12)$$

and where $p_\alpha(x,T)$ and $p_*(x,T)$ are, respectively, the probability density functions for $\tilde{S}(T)$ with given $\sigma(\cdot)$ and $a(t) \equiv M_a(t,\alpha)$, $r(t) \equiv M_r(t,\alpha)$. Let λ_μ be such that

$$\int_{\mathbf{R}^n_+} p_*(x,T)F(\psi_\mu(x),x,\lambda_\mu)dx = X_0.$$

Consider the case in which $\mathcal{T} = \{\alpha^{(1)},\ldots,\alpha^{(N)}\}$, where N is an integer. Let

$$\Delta_N \triangleq \left\{ l \in \mathbf{R}^N : \; l_i \in [0,1], \; \sum_{i=1}^N l_i = 1 \right\}.$$

Clearly, each $\nu_\mu(\cdot)$ can be described in terms of a vector of weights $l \triangleq (l_1,\ldots,l_N) \in \Delta_N$ such that

$$\mathbf{P}(a = \alpha^{(1)}) = l_i.$$

In this case, the problem (12.11) is reduced to a finite-dimensional optimization problem:

$$\begin{aligned}&\text{Maximize} \\ &\textstyle\sum_{i=1}^N l_i \int_{\mathbf{R}^n_+} p_{\alpha^{(i)}}(x,T) U\left[F\left(\sum_{i=1}^N l_i \frac{p_{\alpha^{(i)}}(x,T)}{p_*(x,T)},x,\lambda\right),x\right]dx \\ &\text{over} \quad (l,\lambda) \in \Delta_N \times \mathring{\mathbf{R}}_+ \qquad (12.13) \\ &\text{subject to} \\ &\textstyle\int_{\mathbf{R}^n_+} p_*(x,T)F\left(\sum_{i=1}^N l_i \frac{p_{\alpha^{(i)}}(x,T)}{p_*(x,T)},x,\lambda\right)dx = X_0.\end{aligned}$$

12.5. An illustrative example

Consider the following problem:

$$\text{Maximize} \quad \min_{\mu} \mathbf{E} \ln \hat{X}(T, \Gamma(\cdot), \mu(\cdot)) \tag{12.14}$$

$$\text{subject to} \begin{cases} X(0, \Gamma(\cdot), \mu(\cdot)) = X_0, \\ 0.95 \cdot X_0 \le \hat{X}(T, \Gamma(\cdot), \mu(\cdot)) \le 1.1 \cdot X_0. \end{cases} \tag{12.15}$$

This problem is a special case of the problem (12.5)–(12.6). We present a numerical solution with the following parameters:

$$n = 1, \quad S(0) = 1.6487, \quad X_0 = 1, \quad T = 1, \quad \sigma = \sigma(t) \equiv 0.5,$$

$$\mathcal{T} = \{\alpha^{(1)}, \alpha^{(2)}\}, \quad \text{where} \quad \alpha^{(1)} = 0.2, \quad \alpha^{(1)} = \log(2 - e^{0.2}),$$

and $\bar{M}_a(t, \alpha) \equiv \alpha$, $\bar{M}_\sigma(t, \alpha) \equiv 0.5$.

For any $\mu(\cdot) \in \mathcal{A}(\mathcal{T})$, the optimal claim is

$$X(T) = e^{rT} \tilde{X}(T) = e^{rT} f_{\hat{\mu}}(\tilde{S}(T), \hat{\lambda}),$$

where $f(x, \hat{\lambda}) = f(x, \hat{\lambda}, a, \sigma)$ is defined in Corollary 11.1, Chapter 11. For this special case,

$$f_\mu(x, \hat{\lambda}) = \begin{cases} 0.95 \cdot X_0 & \text{if } \psi_\mu(x)/\hat{\lambda} < 0.95 \cdot X_0 \\ \psi_\mu(x)/\hat{\lambda} & \text{if } 0.95 \cdot X_0 \le \psi_\mu(x)/\hat{\lambda} \le 1.1 \cdot X_0 \\ 1.1 \cdot X_0 & \text{if } \psi_\mu(x)/\hat{\lambda} > 1.1 \cdot X_0. \end{cases} \tag{12.16}$$

The corresponding function $\psi_\mu(x)$ defined in (12.12) is

$$\psi_\mu(x) = \sum_{i=1,2} l_i \left(\frac{x}{S(0)} \right)^{4\alpha^{(i)}} \exp\left(\frac{\alpha_i}{2} - 2\alpha_i^2 \right), \tag{12.17}$$

where $l_i = l_i(\mu(\cdot)) \triangleq \mathbf{P}(a = \alpha^{(i)})$, $i = 1, 2$.

It is not difficult to carry out the numerical calculation to obtain the optimal solution of the corresponding problem (12.13) $(\hat{l}_1, \hat{l}_2, \hat{\lambda})$, where

$$\hat{\lambda} = 0.9781, \quad \hat{l}_1 = 0.5455, \quad \hat{l}_2 = 1 - \hat{l}_1. \tag{12.18}$$

Hence, the strategy $\hat{\Gamma}(\cdot) = \hat{\Gamma}(\cdot, \hat{\mu}(\cdot), \hat{\lambda})$, defined in Corollary 11.1 with $f(x, \hat{\lambda}) = f_{\hat{\mu}}(x, \hat{\lambda})$, is optimal for the corresponding problem (12.14)–(12.15), and the "worst" $\hat{\mu}(\cdot)$ is such that $\mathbf{P}(\hat{a} = \alpha^{(i)}) = \hat{l}_i$. For this example, the "worst" distribution of a is pure stochastic (i.e., the $\hat{a}$ is not a deterministic vector). Thus, it can be concluded that the duality theorem does not hold for this problem if the class of random vectors a of the appreciation rates is replaced by the class of deterministic vectors.

We also obtain numerically that

$$\mathbf{E} \ln \tilde{X}(T, \hat{\Gamma}(\cdot), \hat{\mu}(\cdot)) = 0.004.$$

This means that

$$\mathbf{E} \ln \tilde{X}(T, \hat{\Gamma}(\cdot), \mu(\cdot)) \geq 0.004 \quad \forall \mu(\cdot) \in \mathcal{A}(T).$$

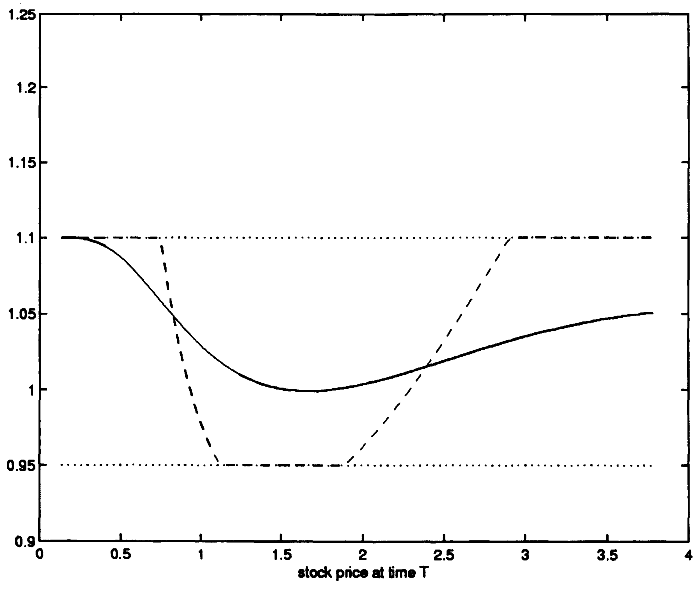
For comparison, note that

$$\mathbf{E} \ln \tilde{X}(T, \Gamma_0(\cdot), \mu(\cdot)) = 0 \quad \forall \mu(\cdot) \in \mathcal{A}(T),$$

where $\Gamma_0(\cdot) \equiv 0$ is the risk-free strategy.

Figure 12.1 shows the function $f_{\hat{\mu}}(x, \hat{\lambda})$ that describes the optimal claim. For the given optimal claim, it is not difficult to calculate the corresponding strategy and the corresponding normalized wealth $\tilde{X}(t) = H(\tilde{S}(t), t)$, where $H(\cdot)$ is the function defined by (11.5). Figure 12.1 shows $H(x, 0)$ which was calculated numerically.

Figure 12.1. The optimal claim for the example (12.14)-(12.15) with maximin criterion. ——: values of $H(x,0)$; - - - -: values of $f_{\hat{\mu}}(x, \hat{\lambda}) = H(x, T)$.



12.6. Proofs

To prove Theorem 12.1, we need several preliminary results, which are presented below as lemmas.

LEMMA 12.1 *The function* $\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))$ *is linear in* $\Gamma(\cdot)$.

Proof. By (1.20), it follows that $\tilde{X}(t) = \tilde{X}(t, \Gamma(\cdot), \mu(\cdot))$ satisfies

$$\tilde{X}(t) = X(0) + \sum_{i=1}^{n} \int_0^t p(\tau)\Gamma_i(\tau, [r(\cdot), S(\cdot, \mu(\cdot)), \eta(\cdot)]_{[0,\tau]}) \Big(\tilde{a}_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t) dw_j(\tau) \Big). \tag{12.19}$$

It is easy to see that $\tilde{X}(T, \Gamma(\cdot), \mu(\cdot))$ is linear in $\Gamma(\cdot)$. This completes the proof. □

LEMMA 12.2 *The set* $\bar{\mathcal{C}}_0$ *is convex.*

Proof. Let $p \in (0,1)$, $\mu(\cdot) \in \mathcal{A}$, $\Gamma^{(i)}(\cdot) \in \bar{\mathcal{C}}_0$, $i = 1, 2$, and

$$\Gamma(\cdot) \triangleq (1-p)\Gamma^{(i)}(t) + p\Gamma^{(i)}(\cdot).$$

By Lemma 12.1, it follows that

$$\tilde{X}(T, \Gamma(\cdot), \mu(\cdot)) = (1-p)\tilde{X}(T, \Gamma^{(1)}(\cdot), \mu(\cdot)) + p\tilde{X}(T, \Gamma^{(2)}(\cdot), \mu(\cdot)).$$

Furthermore, $G(\tilde{X}(t, \Gamma^{(i)}(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \le 0$ a.s., $i = 1, 2$. The set $J(y)$ is convex for all $y \in C([0,T]; \mathring{\mathbf{R}}^n_+)$; then $G(\tilde{X}(t, \Gamma(\cdot), \mu(\cdot)), \tilde{S}(\cdot, \mu(\cdot))) \le 0$ a.s. This completes the proof. □

For a function $\Gamma(t, \cdot) : B([0,t]; \mathbf{R}^+) \times C([0,t]; \mathring{\mathbf{R}}^n_+) \times B([0,t]; \mathbf{R}^N) \to \mathbf{R}^n$, introduce the following norm:

$$\|\Gamma(\cdot)\|_{\mathbf{X}} \triangleq \sup_{\mu = \mu_\alpha(\cdot),\ \alpha \in \mathcal{T}} \left(\sum_{i=1}^{n} \mathbf{E} \int_0^T \Gamma_i(t, [r(\cdot), S(\cdot, \mu(\cdot)), \eta(\cdot)]|_{[0,t]})^2 dt \right)^{1/2}. \tag{12.20}$$

By the definition of $\bar{\mathcal{C}}_0$, it follows that $\|\Gamma(\cdot)\|_{\mathbf{X}} < +\infty$ for all $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$. Thus, $\bar{\mathcal{C}}_0$ is a subset of a linear space of functions with the norm (12.20).

LEMMA 12.3 *There exists a constant* $c > 0$ *such that*

$$\mathbf{E}|\tilde{X}(T, \Gamma(\cdot), \mu_\alpha(\cdot))|^2 \le c\,(\|\Gamma(\cdot)\|^2_{\mathbf{X}} + X_0^2) \quad \forall \Gamma(\cdot) \in \bar{\mathcal{C}}_0, \quad \forall \alpha \in \mathcal{T}.$$

Proof. For a $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$, let $x(t) \triangleq \tilde{X}(t, \Gamma(\cdot), \mu_\alpha(\cdot))$, $\pi(t) \triangleq \Gamma(t, [r(\cdot), S(\cdot, \mu_\alpha(\cdot)), \eta(\cdot)]|_{[0,t]})$, $\pi(t) = (\pi_1(t), \dots, \pi_n(t))$. By (12.19), it follows that

$$\begin{cases} dx(t) = p(t)\sum_{i=1}^{n}\pi_i(t)\left(\sum_{j=1}^{n}\sigma_{ij}dw_j(t) + \tilde{a}(t)dt\right), \\ x(0) = X_0. \end{cases}$$

This is a linear Itô stochastic differential equation, and it is easy to see that the desired estimate is satisfied. This completes the proof. □

LEMMA 12.4 *For a given $\alpha \in \mathcal{T}$, the function*

$$\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot)),\tilde{S}(\cdot,\mu_\alpha(\cdot)))$$

is continuous in $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$.

Proof. Let $\Gamma^{(i)}(\cdot) \in \bar{\mathcal{C}}_0$ and $\tilde{X}^{(i)}(t) \triangleq \tilde{X}(t,\Gamma^{(i)}(\cdot),\mu_\alpha(\cdot))$, $i = 1,2$. By Lemmas 12.1 and 12.3, it follows that

$$\mathbf{E}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^2 \le c\|\Gamma^{(1)}(\cdot) - \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}^2,$$

where $c > 0$ is a constant. Then

$$\begin{aligned}
&\left|\mathbf{E}U\left(\tilde{X}^{(1)}(T),\tilde{S}(\cdot,\mu_\alpha(\cdot)))\right) - \mathbf{E}U\left(\tilde{X}^{(2)}(T),\tilde{S}(\cdot,\mu_\alpha(\cdot))\right)\right| \\
&\le c_1\mathbf{E}\left[(1+|\tilde{X}^{(1)}(T)| + |\tilde{X}^{(2)}(T)|)^{2-q}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^q\right] \\
&\le c_1\left[\mathbf{E}\left(1+|\tilde{X}^{(1)}(T)| + |\tilde{X}^{(2)}(T)|\right)^2\right]^{1/k'}\left[\mathbf{E}|\tilde{X}^{(1)}(T) - \tilde{X}^{(2)}(T)|^2\right]^{1/k} \\
&\le c_2\left(1+\|\Gamma^{(1)}(\cdot)\|_{\mathbf{X}} + \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}\right)^{1/k'}\|\Gamma^{(1)}(\cdot) - \Gamma^{(2)}(\cdot)\|_{\mathbf{X}}^{1/k},
\end{aligned}$$

where $c_i > 0$ are constants, q is as defined in Condition 12.1, $k \triangleq 2/q$, $k' \triangleq k/(k-1) = 2/(2-q)$. This completes the proof. □

Set

$$\begin{aligned}
\bar{M}_{a*}(t,\alpha) &\triangleq M_a\left(t,\alpha,\Theta_0,S_*(\cdot)|_{[0,t]}\right), \\
\bar{M}_{\sigma*}(t,\alpha) &\triangleq M_\sigma\left(t,\alpha,\Theta_0,S_*(\cdot)|_{[0,t]}\right).
\end{aligned}$$

For an $\alpha \in \mathcal{T}$, set

$$\theta_*(t,\alpha) \triangleq \bar{M}_{\sigma*}(t,\alpha)^{-1}\bar{M}_{a*}(t,\alpha),$$

where $\bar{M}_{\sigma*}(t,\alpha)$ and $\bar{M}_{a*}(t,\alpha)$ are as defined above. Let

$$z_*(\alpha,T) \triangleq \exp\left(\int_0^T \theta_*(t,\alpha)^\top dw(t) - \frac{1}{2}\int_0^T |\theta_*(t,\alpha)|^2 dt\right).$$

Set

$$J'(\Gamma(\cdot),\alpha) \triangleq \mathbf{E}U\left(\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot)),\tilde{S}(\cdot,\mu_\alpha(\cdot))\right).$$

LEMMA 12.5 *Let Condition 12.2(ii) holds. Then, for a given* $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$, *the function* $J'(\Gamma(\cdot),\alpha)$ *is continuous in* $\alpha \in \mathcal{T}$.

Proof. Let $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$ and $\alpha_i \in \mathcal{T}$, $i = 1,2$. Set

$$Y_\alpha \triangleq \tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot)), \quad \alpha \in \mathcal{T}, \qquad Y_* \triangleq \tilde{X}(T,\Gamma(\cdot),\mu_*(\cdot)),$$

where $\mu_*(t) \triangleq [r(t),0,\sigma(t)]$. By Girsanov's Theorem applied given Θ_0, it follows that

$$\begin{aligned}
|\mathbf{E} \quad & U(Y_{\alpha_1},\tilde{S}(\cdot,\mu_{\alpha_1}(\cdot))) - \mathbf{E}U(Y_{\alpha_2},\tilde{S}(\cdot,\mu_{\alpha_2}(\cdot)))| \\
&= |\mathbf{E}[z_*(\alpha_1,T) - z_*(\alpha_2,T)]U(Y_*,\tilde{S}_*(\cdot,\mu_*(\cdot)))| \\
&\le c_1\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|(|Y_*|^p + 1) \\
&\le c_2\left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'}(\mathbf{E}|Y_*|^p + 1)^q)^{1/q} \\
&\le c_3\left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'}(\mathbf{E}|Y_*|^2 + 1)^q)^{1/q} \\
&\le c_4\left(\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'}\right)^{1/q'}(\|\Gamma(\cdot)\|^2_{\mathbf{X}} + 1)^{1/q},
\end{aligned}$$

where $p \in (1,2)$ is as defined in Condition 12.1 and 12.2(ii),

$$q \triangleq \frac{2}{p}, \quad q' \triangleq \frac{q}{q-1},$$

and $c_i > 0$ are constants.

Furthermore, it is easy to see that for an $\alpha \in A$, we have $z_*(\alpha,T) = y(T)$, where $y(t) = y(t,\alpha)$ is the solution of the equation

$$\begin{cases} dy(t) = y(t)\bar{M}_{a*}(t,\alpha)^\top\bar{M}_{\sigma*}(t,\alpha)^{-1^\top}dw(t), \\ y(0) = 1. \end{cases}$$

It is well known that $y(T)$ depends on $\alpha \in \mathcal{T}$ continuously in $L^{q'}(\Omega,\mathcal{F},\mathbf{P})$ (see, e.g., Krylov (1980, Chapter 2)). Hence

$$\mathbf{E}|z_*(\alpha_1,T) - z_*(\alpha_2,T)|^{q'} \to 0 \quad \text{as} \quad \alpha_1 \to \alpha_2.$$

This completes the proof. □

Let $\mathcal{V}$ be the set of all σ-additive probability measures on $\mathcal{F}_\mathcal{T}$. We consider $\mathcal{V}$ as a subset of $C(\mathcal{T};\mathbf{R})^*$. (If the set $\mathcal{T}$ is at most countable, then we mean that $C(\mathcal{T};\mathbf{R})$ is $B(\mathcal{T};\mathbf{R})$.) Let $\mathcal{V}$ be equipped with the weak* topology in the sense that

$$\nu_1 \to \nu_2 \quad \Leftrightarrow \quad \int_{\mathcal{T}} \nu_1(d\alpha) f(\alpha) \to \int_{\mathcal{T}} \nu_2(d\alpha) f(\alpha) \quad \forall f(\cdot) \in C(\mathcal{T};\mathbf{R}).$$

LEMMA 12.6 *The set $\mathcal{V}$ is compact and convex.*

Proof. The convexity is obvious. It remains to show the compactness of the set $\mathcal{V}$. In our case, $\mathcal{T}$ is a compact subset of finite-dimensional Euclidean space. Now we note that the Borel σ-algebra of subsets of $\mathcal{T}$ coincides with the Baire σ-algebra (see, e.g., Bauer (1981)). Hence, $\mathcal{V}$ is the set of Baire probability measures. By Theorem IV.1.4 from Warga (1972), it follows that $\mathcal{V}$ is compact. This completes the proof. □

We are now in the position to give a proof of Theorem 12.1.

Proof of Theorem 12.1. For a $\Gamma(\cdot) \in \bar{\mathcal{C}}_0$, we have $J'(\Gamma(\cdot),\cdot) \in C(\mathcal{T};\mathbf{R})$ and

$$\begin{aligned}&\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot)))\\ &= \textstyle\int_{\mathcal{T}} d\nu_\mu(\alpha)\mathbf{E}U(\tilde{X}(T,\Gamma(\cdot),\mu_\alpha(\cdot)),\tilde{S}(\cdot,\mu_\alpha(\cdot))) = \int_{\mathcal{T}} d\nu_\mu(\alpha)J'(\Gamma(\cdot),\alpha),\end{aligned}$$

where $\nu_\mu(\cdot)$ is the measure on $\mathcal{T}$ generated by Θ that corresponds to $\mu(\cdot)$. Hence, $\mathbf{E}U\left(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(T,\mu(\cdot))\right)$ is uniquely defined by ν_μ given $\gamma(\cdot)$. Let

$$J(\Gamma(\cdot),\nu_\mu) \triangleq \mathbf{E}U\left(\tilde{X}(T,\Gamma(\cdot),\mu(\cdot)),\tilde{S}(\cdot,\mu(\cdot))\right).$$

By Lemma 12.6, $J(\Gamma(\cdot),\nu)$ is linear and continuous in $\nu \in \mathcal{V}$ given $\Gamma(\cdot)$.

To complete the proof, it suffices to show that

$$\sup_{\Gamma(\cdot)\in\bar{\mathcal{C}}_0} \inf_{\nu\in\mathcal{V}} J(\Gamma(\cdot),\nu) = \inf_{\nu\in\mathcal{V}} \sup_{\Gamma(\cdot)\in\bar{\mathcal{C}}_0} J(\Gamma(\cdot),\nu). \tag{12.21}$$

We note that $J(\Gamma(\cdot),\nu) : \bar{\mathcal{C}}_0 \times \mathcal{V} \to \mathbf{R}$ is linear in ν. By Lemmas 12.1, 12.4 and 12.5, it follows that $J(\Gamma(\cdot),\nu)$ is concave in $\Gamma(\cdot)$, and $J(\Gamma(\cdot),\nu) : \bar{\mathcal{C}}_0 \times \mathcal{V} \to \mathbf{R}$ is continuous in ν for each $\Gamma(\cdot)$ and continuous in $\Gamma(\cdot)$ for each ν. Furthermore, $\bar{\mathcal{C}}_0$ and $\mathcal{V}$ are convex and $\mathcal{V}$ is compact. By the Sion Theorem (see, e.g., Parthasarathy and Ragharan (1971, p. 123)), it follows that (12.21), and hence (12.9), are satisfied. This completes the proof of Theorem 12.1. □

Chapter 13

ON REPLICATION OF CLAIMS

Abstract In Chapters 5 and 8, the solution of the optimal investment problem was decomposed on two different problems: calculation of the optimal claim and calculation of a strategy to replicate the optimal claim. In this chapter, we discuss some aspects of replication of given claims. First, some possibilities are considered for replicating the desired claim by purchasing options. Second, an example is considered of an incomplete market with transactions costs and with nonpredictable volatility, when replication is replaced for rational superreplication.

13.1. Replication of claims using option combinations

In previous chapters, we studied strategies that use buying and selling stocks to replicate an optimal claim. For the real market, there exist some other possibilities: one can replicate the desired claim using combinations of *derivatives*, for example, put and call options.

Combinations of options

Combination are strategies in which the investor simultaneously holds long or short options of different types. If an investor combines different options, he or she can obtain different piecewise profit/loss diagrams, where the number of pieces is proportional to the number of different options.

Figures 1.1 and 1.2 in Chapter 1 present profit/loss diagrams for generic European put and call options, i.e., they show the wealth of the European call and put option holder as a function of the stock price at the terminal time. Figures 13.1 and 13.2 here present profit/loss diagrams for corresponding short positions for generic European put and call options, i.e., they show the wealth of the European call and put option seller as a function of the stock price at the terminal time. Different combinations of functions in Figures 1.1,1.2, 13.1, and 13.2 gives different piecewise profit/loss diagrams. There are some popular combinations:

Figure 13.1. Profit/loss diagram for a short call: c is the price of the call option, S is the stock price at the terminal time, and K is the strike price.

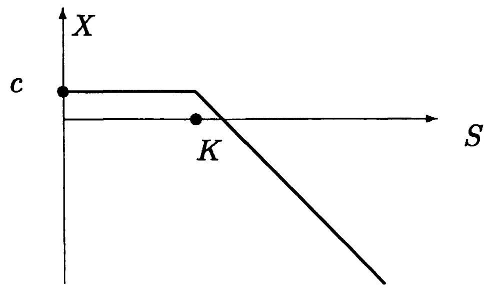
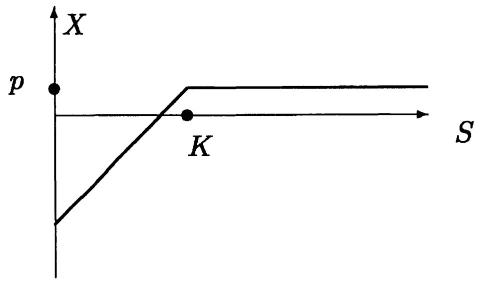


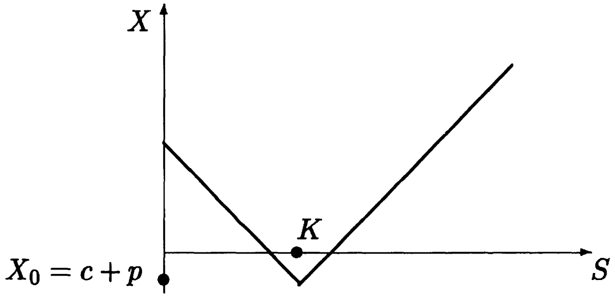
Figure 13.2. Profit/loss diagram for a short put: p is the price of the put option, S is the stock price at the terminal time, and K is the strike price.



- *covered call* = long stock + short call
- *protective put* = long stock + long put
- *long spread*= long call + short call
- *long straddle* = long call + long put
  (If you own both a put and a call with the same striking price and expiration date on the same underlying security, you are *long* a straddle, i.e., you own a straddle)
- *strangles* are similar to straddles, except the puts and calls have different striking prices

Recall that a winning combination of put and call options was presented in Chapter 3.

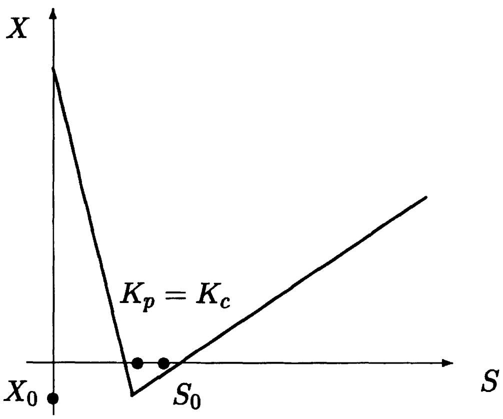
Figure 13.3. Profit/loss diagram for a standard long straddle. $X_0 = c + p$, where c is the price of the call option, p is the price of the put option, S is the stock price at the terminal time, and K is the strike price.



As an example, Figure 13.3 shows the profit/loss diagram for the long straddle, which consists of equal amounts of put and call options.

Figure 13.4 shows the profit/loss diagram for the "winning" long straddle with put and call options that was derived in Example 3.1, Chapter 3.

Figure 13.4. Profit/loss diagram for the "winning" long straddle from Example 3.1, Chapter 3.



Replication using option combinations

Consider optimal claims such as these presented in numerical examples in Chapter 11. Any of the functions $f(x, \hat{\lambda})$ can be approximated by piecewise functions that are payoff functions of combinations of European put and call options, together with long or short positions in the underlying stock. If an investor chooses to purchase the corresponding combination of options, then he or she will have the same terminal wealth as in the case when one replicates the corresponding claim by buying and selling stocks, with the same initial wealth.

EXAMPLE 13.1 Consider the example from Section 12.5, i.e., the following optimal investment problem:

$$\text{Maximize} \quad \min_{\tilde{a}} \mathbf{E} \ln \hat{X}(T, \Gamma(\cdot), \mu(\cdot)) \tag{13.1}$$

$$\text{subject to} \begin{cases} X(0, \Gamma(\cdot), \mu(\cdot)) = X(0), \\ 0.95 \cdot X(0) \le \hat{X}(T, \Gamma(\cdot), \mu(\cdot)) \le 1.1 \cdot X(0). \end{cases} \tag{13.2}$$

A numerical solution was presented with the following parameters:

$$n = 1, \quad S(0) = \$1.6487, \quad X(0) = \$1,$$
$$T = 1, \quad \sigma(t) \equiv 0.5,$$
$$\textstyle\sum_{i=1,2} \mathbf{P}(\tilde{a} \equiv \alpha^{(i)}) = 1, \quad \text{where} \quad \alpha^{(1)} = 0.2, \quad \alpha^{(1)} = \log(2 - e^{0.2}).$$

For any $\mu(\cdot) \in \mathcal{A}$, the optimal claim is

$$X(T) = e^{rT} \tilde{X}(T) = e^{rT} f_{\hat{\mu}}(\tilde{S}(T), \hat{\lambda}),$$

where

$$f_\mu(x, \hat{\lambda}) = \begin{cases} 0.95 \cdot X(0) & \text{if } \psi_\mu(x)/\hat{\lambda} < 0.95 \cdot X(0) \\ \psi_\mu(x)/\hat{\lambda} & \text{if } 0.95 \cdot X(0) \le \psi_\mu(x)/\hat{\lambda} \le 1.1 \cdot X(0) \\ 1.1 \cdot X(0) & \text{if } \psi_\mu(x)/\hat{\lambda} > 1.1 \cdot X(0). \end{cases} \tag{13.3}$$

The corresponding function $\psi_\mu(x)$ is defined by (12.12)(12.18).

Figure 12.1 shows the function $f_{\hat{\mu}}(x, \hat{\lambda})$, which describes the optimal claim given $\mathbf{S}(T) = x$. For the given optimal claim, it is not difficult to calculate the corresponding replicating strategy and the corresponding normalized wealth $\tilde{X}(t) = H(\tilde{S}(t), t)$, where $H(\cdot)$ is the function defined by (11.5).

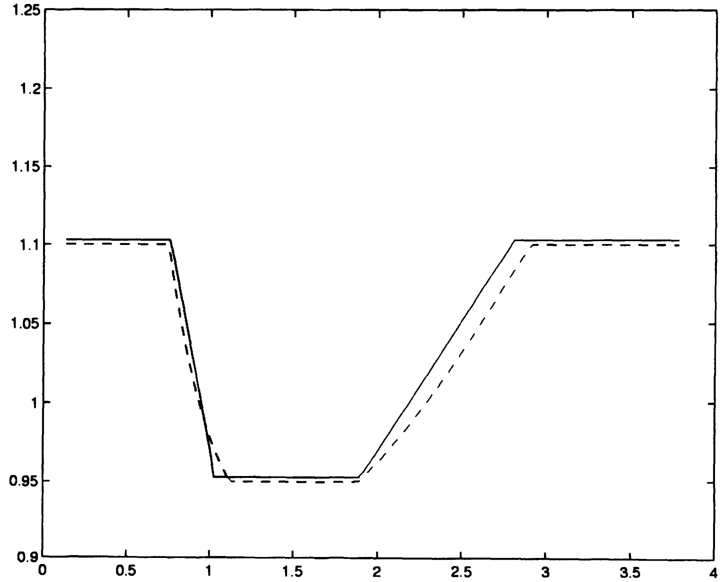
On the other hand, the optimal claim in Figure 12.1 can be approximated closely enough by a claim generated by a combination of two put options (long and short) and two call options (long and short) (the *long condor* combination, in terms of Strong (1994), p. 72). This combination can be constructed by the following way:

- buy 0.55 put options with strike price \$1.03;
- write 0.55 put options with strike price \$0.76;

- buy 0.16 call options with strike price \$1.95;
- write 0.16 call options with strike price \$2.9.

Figure 13.5 shows that the two claims are close. Thus, the option combination described above gives an approximate solution of the optimal investment problem (13.1)–(13.2).

Figure 13.5. Approximation of the optimal claim for the problem (12.14)–(12.15) by a combination of put and call options. —: values of the claim for the option combination; - - - -: values of the optimal claim.



13.2. Superreplication under uncertainty and transaction costs

In this section, the diffusion model of a financial market is modified and investigated under the assumption that the volatility coefficient may be time-varying, uncertain, and random. Moreover, in our modified model, transaction costs are taken into account. It is shown that there exists a superreplicating strategy for the European type claims. The strategy is obtained by solving a *nonlinear* parabolic partial differential equation.

On the impact of transaction costs and uncertainty for volatility

In the classic Samuelson and Black–Scholes model, the volatility is assumed to be given and fixed, and transaction costs are not taken into account. However, in any real financial market, transaction costs have to be taken into account.

Furthermore, empirical research shows that the real volatility is time-varying, random, and correlated with stock prices (see Black and Scholes (1973)).

Because the volatility coefficient appears in the formulas defining the structure of hedging and optimal strategies, the estimation of the volatility from usually incomplete statistical data of stock prices is of a special importance (see Day and Levis (1992), Kupiec (1996), Taylor and Xu (1994)). Many authors emphasize that the main difficulty in modifying the Black–Scholes model is taking into account the fact that the volatility does depend (as it is shown by statistics) on both time and stock prices. Christie (1982)) has shown that the volatility is correlated with stock prices. Lauterbach and Schultz (1990) note that the Black–Scholes option pricing model consistently misprices warrants (see also Hauser and Lauterbach (1997)).

In modified Black–Scholes models, a number of formulas and equations for volatility were proposed (see, e.g., Hull and White (1987) and also Christie (1982), Finucame (1989), Johnson and Shanno (1987), , Masi *et al.* (1994), Scott (1987)). Following Avellaneda *et al.* (1995), Avellaneda and Parás (1995), we assume that the bounds of the volatility are given.

Another problem arises out of the desire to take into account transaction costs. Black and Scholes (1972) noted that in real financial markets, transaction costs are quite large. Many authors remark that the return volatility is correlated with the trade volume, transactions costs and stock prices (Grossman and Zhou (1996), Kupiec (1996)). A number of mathematical models with transaction costs were proposed (see Bielecki and Pliska (1999), Davis and Norman (1990), Edirisinghe *et al.* (1993), Leland (1985), Taksar *et al.* (1988)). Similarly to Leland (1985) and Grossman and Zhou (1996), we investigate a financial market model where the costs of the high-frequency component of the portfolio are taken into account. In addition to the results obtained in the cited papers, we consider the sufficient and necessary conditions of superreplication. Further, we consider a model with the costs of jumps for the portfolio as well as with uncertain volatility.

13.2.1 Market model and problem setting

Consider the single-stock diffusion model of a financial market consists of two assets: the risk-free bond or bank account $B = (B(t))_{t\geq 0}$ and the risky stock $S = (S(t))_{t\geq 0}$. In this model, it is assumed that the dynamics of the stock is described by the stochastic differential equation

$$dS(t) = S(t)[adt + \sigma dw(t)], \quad t > 0, \tag{13.4}$$

where a is the appreciation rate, σ is the volatility coefficient, and $w(t)$ is the standard Wiener process. The initial price $S_0 > 0$ is a given nonrandom value. The dynamics of the bond is described by

$$B(t) = e^{rt}B_0, \tag{13.5}$$

where $r \geq 0$ and B_0 are given constants.

We assume that $a = a(t)$ and $\sigma = \sigma(t)$ are random processes that are square integrable and do not depend on the future. In other words, $a(t)$ and $\sigma(t)$ do not depend on $w(t+h) - w(t)$ for $h > 0$.

In the classic Black and Scholes model, σ is supposed to be known and fixed, and a is arbitrary and unknown. Our aim is to take into account transaction costs and the fact that the volatility coefficient σ does depend on both time t and the stock price $S(t)$. In our model, the main assumptions are related to the upper and lower bounds of the volatility coefficient and the nature of transaction costs.

ASSUMPTION 13.1 *The volatility coefficient $\sigma = \sigma(t)$ satisfies the following condition: $\sigma_1 \leq \sigma(t) \leq \sigma_2$ for some constants σ_1, σ_2, where $0 < \sigma_1 \leq \sigma_2$.*

Let $X_0 > 0$ be the initial wealth at time $t = 0$ of the investor. The total wealth of the investor at time $t > 0$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{13.6}$$

Here $\gamma(t)$ is the quantity of the stock and $\beta(t)$ is the quantity of the bond. The pair $(\beta(t), \gamma(t))$ describes the state of the securities portfolio at time t. We call such pairs *strategies*. Some constraints will be imposed later upon operations in the market, or, in other words, upon strategies.

REMARK 13.1 *In previous chapters, we have used the term strategy for the process of the investment in the stock $\pi(t) \triangleq \gamma(t)S(t)$. The vector π alone suffices to specify the portfolio for a self-financing strategy for a model without transaction costs.*

The main constraint in choosing a strategy in the classical problem without transaction costs is the so-called *condition of self-financing*.

DEFINITION 13.1 *A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be self-financing in a financial market model without transaction costs if*

$$dX(t) = \beta(t)dB(t) + \gamma(t)\,dS(t). \tag{13.7}$$

As usual, we denote $\tilde{S}(t) = e^{-rt}S(t)$ and $\tilde{X}(t) = e^{-rt}X(t)$; the process $\tilde{X}(t)$ is said to be the normalized wealth.

Our aim is to extend this definition and the corresponding results to the case of transaction costs and uncertain volatility.

DEFINITION 13.2 *A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy if the following conditions hold:*

(i) *$\gamma(t), \beta(t)$ are square integrable random processes that do not depend on the future (in other words, $\beta(t)$ and $\sigma(t)$ do not depend on $w(t+h) - w(t)$ for $h > 0$);*

(ii) *the process $\gamma(t)$ is piecewise continuous a.s. (almost surely);*

(iii) *there exists a set of open random time intervals $I_k \subset [0,T]$, $I_k = (\tau_k^-, \tau_k^+)$ such that τ_k^-, τ_k^+ are Markov time moments, $I_k \cap I_m = \emptyset$ for $k \neq m$ a.s.,* mes $\{[0,T] \setminus \cup_{k=1}^N I_k\} = 0$ *a.s., where $N \leq +\infty$ is a random number of intervals, and $\gamma(t)$ has the differential*

$$d\gamma(t) = \tilde{\gamma}(t)dt + \hat{\gamma}(t)dw(t) \quad \textit{for} \quad t \in I_k;$$

(iv) *there exists a function $G(x,t) : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that*

$$\gamma(t) = G(\tilde{S}(t), t), \tag{13.8}$$

and $G(x,t)$ is bounded on any bounded domain; and

(v) *the processes $a(t)\gamma(t)$ and $\gamma(t)S(t)$ are square integrable.*

Here mes denotes the Lebesque measure.

We also give a more constructive description of admissible strategies. For this, we notice that a strategy $(\beta(\cdot), \gamma(\cdot))$ is admissible if $\beta(t)$ satisfies all the above assumptions, $\gamma(t) = G(\tilde{S}(t), t)$, where $G(x,t) : \mathbf{R} \times [0,T] \to \mathbf{R}$ is a function bounded on any bounded domain and of a polynomial growth, and there exists a set of domains D_k, $k = 1, 2, \ldots$, with piecewise C^1-smooth boundaries ∂D_k, such that

$$\mathbf{R} \times [0,T] = \cup_{k \geq 1} \bar{D}_k, \quad D_k \cap D_m = \emptyset \ \ (k \neq m), \quad G|_{D_k} \in W_2^{2,1}(D_k).$$

REMARK 13.2 The corresponding intervals I_k are maximum connected open intervals $I_k = \{t : \ (\tilde{S}(t), t) \in D_m\}$. Further, the set $[0,T] \setminus \cup_{k=1}^N I_k$ is the set of time moments when $(\tilde{S}(t), t) \in \cup_{k \geq 1} \partial D_k$, and this set is an a.s. continuous (or noncountable) Kantor type set with zero Lebesque measure; $\bar{D}_k = D_k \cup \partial D_k$.

We introduce some transaction cost for the time interval $[0,t]$ as

$$\int_0^t \lambda(\tau)d\tau + \sum_{k:\ \tau_k^- < t} C_k,$$

where $\lambda(t)$ is a given nonnegative random function that depends on $(\beta(\cdot), \gamma(\cdot), S(\cdot))|_{[0,t]}$, and C_k are the costs for the jump in the stock portfolio quantity.

DEFINITION 13.3 *An admissible strategy* $(\beta(\cdot),\gamma(\cdot))$ *is said to be self-financing in a financial market with transaction costs if*

$$X(t) = X_0 + \int_0^t \beta(\tau)dB(\tau) \quad + \int_0^t \gamma(\tau)dS(\tau) - \int_0^t \lambda(\tau)d\tau - \sum_{k:\ \tau_k^- < t} C_k, \quad t > 0. \tag{13.9}$$

ASSUMPTION 13.2 *We assume that*

$$\lambda(t) = c(t)\,|\hat{\gamma}(t)S(t)|,$$

where $c(t)$ *is a random function and* $c(t) \in [0,\bar{c}]$ *for all* $t > 0$, *where* $\bar{c} \geq 0$ *is a given constant. Furthermore, we assume that*

$$C_k = \varphi(|\gamma(\tau_k^-) - \gamma(\tau_{k-1}^+)|),$$

where $\varphi(\cdot)$ *is a given nonnegative function.*

In other words, the transaction cost over the time period $(0,t]$ is

$$\int_0^t c(\tau)\,|\hat{\gamma}(\tau)S(\tau)|\,d\tau + \sum_{k:\ \tau_k^- < t} \varphi(|\gamma(\tau_k^-) - \gamma(\tau_{k-1}^+)|).$$

In this assumption, the continuous "slow" change of the quantity of the stock portfolio $\gamma(t)$ is not taken into account. A similar assumption was introduced by Leland (1985) for the analysis of the trade volume and the volatility in a financial market (see also Grossman and Zhou (1996)).

Note that the case of $\bar{c} = 0$, $\varphi \equiv 0$ corresponds to zero transaction cost.

We can now rewrite Definition 13.3.

DEFINITION 13.4 *An admissible strategy* $(\beta(\cdot),\gamma(\cdot))$ *is said to be self-financing in a financial market with transaction costs if*

$$X(t) = X_0 + \int_0^t \beta(\tau)dB(\tau) \quad + \int_0^t \gamma(\tau)dS(\tau) - \int_0^t c(\tau)\,|\hat{\gamma}(\tau)S(\tau)|\,d\tau - \sum_{k:\ \tau_k^- < t} \varphi(|\gamma(\tau_k^-) - \gamma(\tau_{k-1}^+)|).$$

Problem of superreplication

Consider the problem of replication of a given claim. Let $\xi = F(S(T))$ be a random claim, where $F(x) : \mathbf{R} \to \mathbf{R}$ is a given nonnegative function and $T > 0$ is a given time.

DEFINITION 13.5 *A strategy* $(\beta(\cdot), \gamma(\cdot))$ *is said to superreplicate a claim* $F(S(T))$ *for the market with transaction costs and uncertain volatility if the following conditions holds:*

(i) $(\beta(\cdot), \gamma(\cdot))$ *is admissible and self-financing, and the function* G *in (13.8) depends on parameters* σ_1, σ_1, $\bar{c}$, $\varphi(\cdot)$, T, $F(\cdot)$*; and*

(ii)

$$X(T) \geq F(S(T)) \quad a.s. \tag{13.10}$$

for all admissible $c(t)$, $\sigma(t)$.

In the approach of Black and Scholes, the option price is the initial wealth which may be raised to the option-writer obligation by some investment transactions. Following this approach, we define the fair (rational) price of a claim.

DEFINITION 13.6 *Let* Π *be the set of all values of the initial wealth* X_0 *such that there exists an admissible superreplicating strategy for the claim* $F(S(T)$. *Then, the fair (rational) price* $\hat{C}$ *for the claim in this class of admissible strategies is defined as*

$$\hat{C} = \inf_{X_0 \in \Pi} X_0.$$

DEFINITION 13.7 *A strategy* $(\gamma(\cdot), \beta(\cdot)$ *that superreplicates the claim* $F(S(T)$ *with the initial wealth* $X(0)$ *is said to be rational if* $X_0 = \hat{C}$, *where* $\hat{C}$ *is the fair (rational) price of the claim.*

13.2.2 Superreplicating strategy

We assume that $F(x)$ is piecewise smooth and $|F(x)| + |dF(x)/dx| \leq \text{const}(|x| + 1)$. Furthermore, we assume that one of the following conditions holds:

(i) The function $F(x)$ is a convex function and there are nonzero transaction costs (in other words, $\bar{c} \neq 0$, $\varphi \neq 0$).

(ii) The function $F(x)$ may be nonconvex, but the transaction costs are absent (in other words, $\bar{c} = 0$, $c(t) \equiv 0$, $\varphi(x) \equiv 0$).

Notice that the function $F(x) = (x - K)_+$ from the standard European call option is convex.

Suppose $H(x, t)$ is a solution of the boundary value problem for the nonlinear parabolic equation

$$\begin{cases} \frac{\partial H}{\partial t}(x,t) + \frac{1}{2} \max_{\sigma \in [\sigma_1, \sigma_2]} \left\{ \sigma^2 x^2 \frac{\partial^2 H}{\partial x^2}(x,t) \right\} + \bar{c}\sigma_2 \left| \frac{\partial^2 H}{\partial x^2}(x,t) \right| x^2 = 0, \\ H(x,T) = F(x). \end{cases} \tag{13.11}$$

in the domain $x > 0$, $t \in [0, T]$. It is known that this equation has a unique solution with locally square integrable derivatives (see Krylov (1987)).

Furthermore, let

$$\tilde{X}(t) = H(\tilde{S}(t), t) + \int_0^t \alpha(t)dt, \tag{13.12}$$

where

$$\begin{aligned}\alpha(t) \triangleq \max_{\sigma\in[\sigma_1,\sigma_2]} \ & \left\{\tfrac{\sigma^2-\sigma(t)^2}{2}\tilde{S}(t)^2\tfrac{\partial^2 H}{\partial x^2}(\tilde{S}(t),t)\right\} \\ & + (\bar{c}\sigma_2 - c(t)\sigma(t))\left|\tfrac{\partial^2 H}{\partial x^2}(\tilde{S}(t),t)\right|\tilde{S}(t)^2.\end{aligned} \tag{13.13}$$

Let $X(t) \triangleq e^{rt}\tilde{X}(t)$,

$$\gamma(t) = \frac{\partial H}{\partial x}(\tilde{S}(t), t), \quad \beta(t) = \frac{X(t) - \gamma(t)S(t)}{B(t)}. \tag{13.14}$$

Now we are in a position to present the main results of this chapter.

THEOREM 13.1 *The strategy (13.14) is a superreplicating strategy for the claim $e^{rt}F(\tilde{S}(T))$, and the corresponding normalized wealth $\tilde{X}(t)$ is defined in (13.12).*

THEOREM 13.2 *The rational price of the claim $e^{rt}F(\tilde{S}(T))$ is*

$$\hat{C} = H(S_0, 0). \tag{13.15}$$

COROLLARY 13.1 *The strategy (13.14) is a rational strategy that superreplicates the claim $e^{rt}F(\tilde{S}(T))$.*

Application to the Black and Scholes model

We shall extend the Black and Scholes results to the case of the uncertain volatility coefficient and transactions costs.

COROLLARY 13.2 *Let $F(x)$ be a convex function. Then*

$$H(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F\left(x\exp\left\{\hat{\sigma}y\sqrt{t} - \frac{t\hat{\sigma}^2}{2}\right\}\right)\exp\left(-\frac{y^2}{2}\right)dy, \tag{13.16}$$

where

$$\hat{\sigma} = \sqrt{\sigma_2^2 + 2\bar{c}\sigma_2}, \tag{13.17}$$

Moreover, if $F(x) = (x - K)_+$, *where* $K > 0$ *is a constant, then the rational price of the claim* $e^{rt}F(\tilde{S}(T))$ *is*

$$\hat{C} = H(S_0, 0) = S_0 N(d_+) - KN(d_-), \tag{13.18}$$

where

$$d_\pm = (\hat{\sigma}\sqrt{T})^{-1}\left(\ln\frac{S_0}{K} \pm \frac{T\hat{\sigma}^2}{2}\right).$$

$N(d_\pm)$ *is the cumulative standard normal distribution evaluated at* $d_\pm$,

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

13.3. Proofs

PROPOSITION 13.1 *Let* $F(x)$ *be a convex function. Then the solution of the equation (13.11) coincides with the solution of the equations*

$$\begin{cases} \frac{\partial H}{\partial t}(x,t) + \frac{1}{2}\hat{\sigma}^2 x^2 \frac{\partial^2 H}{\partial x^2}(x,t) = 0 \\ H(x,T) = F(x), \end{cases} \tag{13.19}$$

where $\hat{\sigma} = \sqrt{\sigma_2^2 + 2\bar{c}\sigma_2}$.

Proof. Let $H(x,t)$ be a solution of (13.19). Suppose that there exists $t_0 \in [0,T)$ such that the function $H(\cdot, t_0)$ is not convex. Since T is arbitrary and the coefficients of the equations are constants, it is enough to consider only $t_0 = 0$. Suppose the function $H(\cdot, 0)$ is not convex. Then there exist $x_1 > 0$, $x_2 > 0$ such that

$$H(x_1, 0) + H(x_1, 0) < 2H\left(\frac{x_1 + x_2}{2}, 0\right).$$

Consider the classical problem of the option pricing with the volatility coefficient $\sigma = \hat{\sigma}$ and without transaction costs. Let

$$\tilde{S}_0 \triangleq \frac{x_1+x_2}{2},$$
$$\tilde{S}^{(i)}(t) \triangleq \frac{2x_i\tilde{S}(t)}{x_1+x_2}, \quad \gamma^{(i)}(t) \triangleq \frac{\partial H}{\partial x}(\tilde{S}^{(i)}(t), t), \quad i = 1, 2.$$

Set

$$\bar{\gamma}(t) \triangleq \left(\gamma^{(1)}(t) + \gamma^{(2)}(t)\right)/2.$$

Clearly, there exists $\bar{\beta}(t)$ such that $(\bar{\beta}(t), \bar{\gamma}(t))$ is a self-financing strategy. Let $\bar{X}(t)$ be the corresponding wealth. It is easy to see that $(\bar{\beta}(t), \bar{\gamma}(t))$ is a super-replicating strategy and that $\bar{X}_0 < H(\tilde{S}_0, 0)$. However, this result contradicts function, $H(\cdot,t)$ is convex for any time t, and $H''_{xx}(x,t) \geq 0$. Hence, equation (13.11) holds. This completes the proof of Proposition 13.1. □

Proof of Theorem 13.1. Let $F(\cdot)$ be a convex function. From Proposition 13.1, equations (13.16) and (13.19) hold for H defined by (13.11). Let

$$G(x,t) = \frac{\partial H}{\partial x}(x,t).$$

The fundamental solution for (13.19) is known (see Proposition 11.3 and (11.6), and, e.g., Shyryaev *et al.* (1994)):

$$H(x,t) \triangleq \int_{\mathbf{R}^n_+} \bar{p}_*(y,T,x,t)F(y)dy,$$

where $\bar{p}_*(y,\tau,x,t)$ as a function of y is the conditional probability density function for the vector $\tilde{S}_*(\tau)$ given the condition $\tilde{S}_*(t) = x$, where $0 \leq t \leq \tau$. More precisely,

$$\bar{p}_*(y,\tau,x,t) \triangleq \frac{1}{x\hat{\sigma}\sqrt{2\pi(t-\tau)}} \exp \frac{-(\ln(y_i) - \ln(x) + \hat{\sigma}^2(t-\tau)/2)^2}{2\hat{\sigma}^2(t-\tau)}.$$

Using this solution, we can obtain the explicit formula for G and conclude that G has continuous derivatives G'_t, G'_x, G''_{xx} in Q for any domain $Q = D \times (0,T_*)$, where $D \subset \mathbf{R}^+$, $T_* \in (0,T)$ (or $G \in C_2^{2,1}(Q)$).

Clearly, this strategy is admissible with

$$\hat{\gamma}(t) = \frac{\partial G}{\partial x}(\tilde{S}(t),t)\sigma(t)\tilde{S}(t) = \frac{\partial^2 H}{\partial x^2}(\tilde{S}(t),t)\sigma(t)\tilde{S}(t),$$

$$\lambda(t) = c(t)\left|\frac{\partial^2 H}{\partial x^2}(\tilde{S}(t),t)\sigma(t)\right|\tilde{S}^2(t).$$

From Itô's formula and (13.12) and (13.13), we have that

$$\begin{aligned} d\tilde{X}(t) &= d_t H(\tilde{S}(t),t) + \alpha(t)dt \\ &= G(\tilde{S}(t),t)d\tilde{S}(t) \\ &\quad + \left(\tfrac{\partial H}{\partial t}(\tilde{S}(t),t) + \tfrac{1}{2}\sigma(t)^2\tilde{S}(t)^2\tfrac{\partial^2 H}{\partial x^2}(\tilde{S}(t),t)\right)dt + \alpha(t)dt \\ &= G(\tilde{S}(t),t)d\tilde{S}(t) - \lambda(t)dt. \end{aligned}$$

Hence the strategy is self-financing. Furthermore, it is easy to see that $\alpha(t) \geq 0$ and (13.10) hold.

In the case of zero transaction cost, we do not need the existence of derivatives G'_t, G'_x, G''_{xx}, and the proof is similar. This completes the proof of Theorem 13.1. □

Proof of Theorem 13.2. In the classic case of zero transaction costs and a known constant volatility (when $\bar{c} = 0$, $\varphi \equiv 0$, $\sigma_1 = \sigma_2$), we have $\tilde{X}(T) = F(\tilde{S}(T))$ for the replicating strategy, and fair price is $\hat{C} = \mathbf{E}^* F(\tilde{S}(T))$, where $\mathbf{E}^*$ is the expectation by such probability measure that $\tilde{S}(t)$ is a martingale and, hence, that $\hat{C}$ is the rational (fair) price. We cannot use this method in our case because we have only inequality $\tilde{X}(T) \geq F(\tilde{S}(T))$, and the values $\tilde{X}(T) - F(\tilde{S}(T))$ depend on strategies. However, we can use another approach that does not use martingale properties.

Let $(\bar{b}(t), \bar{\gamma}(t))$ be some other superreplicating strategy, $\bar{\gamma}(t) = \bar{G}(\tilde{S}(t), t)$, $\bar{X}(t)$ be the corresponding normalized wealth, and $\bar{C} = \bar{X}_0 < \hat{C}$. Suppose that $\sigma(t) \equiv \sigma_2$, $c(t) \equiv \bar{c}$. Introduce the following function:

$$\bar{H}(x,t) = \int_0^x \bar{G}(y,t)dy.$$

Let I_k be the random time intervals introduced above for admissible strategies $k = 1, \dots, N$. We have from Itô's formula that

$$\begin{aligned}\bar{H}(\tilde{S}(T),T) \quad & -\bar{H}(S_0,0) = \textstyle\int_0^T \bar{G}(\tilde{S}(t),t)d\tilde{S}(t) \\ & + \textstyle\sum_{k=1}^N \Big\{ \bar{H}(\tilde{S}(\tau_{k+1}^-), \tau_{k+1}^-) - \bar{H}(\tilde{S}(\tau_k^+), \tau_k^+) \\ & + \textstyle\int_{I_k} \left(\frac{\partial \bar{H}}{\partial t}(\tilde{S}(t),t) + \frac{1}{2}\sigma_2^2 \tilde{S}(t)^2 \frac{\partial^2 \bar{H}}{\partial x^2}(\tilde{S}(t),t) \right) dt \Big\}.\end{aligned}$$

Here we use some version of Itô's formula for a function with nonsmooth derivatives (see Krylov (1988) and Dokuchaev (1994)). The condition of self-financing and (13.10) give us that

$$\begin{aligned}& \textstyle\int_0^T \bar{G}(\tilde{S}(t),t)d\tilde{S}(t) \\ & = \bar{X}(T) - \bar{X}_0 + \textstyle\int_0^T \lambda(t)dt + \sum_k C_k = F(\tilde{S}(T)) + \xi + \int_0^T \lambda(t)dt - \bar{X}_0.\end{aligned}$$

Here $\xi \geq 0$ is some random value. Denote

$$\mathcal{L}\bar{H} \triangleq \frac{\partial \bar{H}}{\partial t} + \frac{1}{2}\sigma_2^2 x^2 \frac{\partial^2 \bar{H}}{\partial x^2} + \bar{c}\sigma_2 \left| \frac{\partial^2 \bar{H}}{\partial x^2} \right| x^2.$$

Then

$$\begin{aligned}& \textstyle\sum_{k=1}^N \Big\{ \int_{I_k} \mathcal{L}\bar{H}(\tilde{S}(t),t)dt + \bar{H}(\tilde{S}(\tau_{k+1}^-), \tau_{k+1}^-) - \bar{H}(\tilde{S}(\tau_k^+), \tau_k^+) \Big\} \\ & = \bar{H}(\tilde{S}(T),T) - \bar{H}(S_0,0) - F(\tilde{S}(T)) - \hat{\xi} + \bar{X}_0,\end{aligned}$$

where $\hat{\xi} \geq 0$ is some random value. Denote by $\mathcal{X}$ the space $W_2^{2,1}(Q)^*$ that is dual to the Sobolev space $W_2^{2,1}(Q)$, $Q = D \times [0,T]$, where $D \subset (0, +\infty)$ is an arbitrary interval. The element $\xi \in \mathcal{X}$ is said to be nonnegative if $\langle \xi, f \rangle \geq 0$ for every $f \in W_2^{2,1}(Q)$ such that $f(x,t) \geq 0$. In this sense, $\mathcal{L}\bar{H} \leq 0$ as the element of $\mathcal{X}$. Then $H(x,0) \leq \bar{H}(x,0)$ because of (13.11). This completes the proof of Theorem 13.2. □

Proof of Corollary 13.1. The proof is straightforward.

Proof of Corollary 13.2. The fundamental solution for equation (13.19) is known, and (13.19) holds for H defined by (13.12) (see, e.g., Shyryaev *et al.* (1994)). From Proposition 13.1, the equations (13.11) hold for this H. For $F(x) = (x - K)_+$, the formula for $\hat{C}$ is a consequence of the Black-Scholes result. This completes the proof. □

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